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Computational Geometric and Algebraic Topology

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Abstract. Computational topology is a young, emerging field of mathematics that seeks out practical algorithmic methods for solving complex and fundamental problems in geometry and topology. It draws on a wide variety of techniques from across pure mathematics (including topology, differential geometry, combinatorics, algebra, and discrete geometry), as well as applied mathematics and theoretical computer science. In turn, solutions to these problems have a wide-ranging impact: already they have enabled significant progress in the core area of geometric topology, introduced new methods in applied mathematics, and yielded new insights into the role that topology has to play in fundamental problems surrounding computational complexity.

At least three significant branches have emerged in computational topology: *algorithmic 3-manifold and knot theory*, *persistent homology* and *surfaces and graph embeddings*. These branches have emerged largely independently. However, it is clear that they have much to offer each other. The goal of this workshop was to be the *first significant step to bring these three areas together*, to share ideas in depth, and to pool our expertise in approaching some of the major open problems in the field.

Mathematics Subject Classification (2010): Primary: 05C10, 55-04, 57-04, 57M15, 57N10, 57N65, 62-07; Secondary: 55M25, 55U10, 68W05.

Introduction by the Organisers

The workshop *Computational Geometric and Algebraic Topology*, organised by Benjamin Burton, Herbert Edelsbrunner, Jeff Erickson and Stephan Tillmann was held 11–17 October 2015. The meeting was well attended with 53 participants from 13 different countries. Besides the 25 lectures, many given by young

and emerging researchers, the program included two evening sessions on mathematical software and an evening session on open problems.

The aim of the workshop was to bring together the following three branches of computational topology:

- *algorithmic 3–manifold and knot theory*, which seeks practical solutions to the core algorithmic problems that have driven low-dimensional topology, such as testing whether two 3–manifolds are homeomorphic, or whether two knots in $\mathbb{R}^3$ are isotopic;
- *persistent homology*, which developed into a key theory, extending traditional homology to facilitate effective topological data analysis to extract structure from noisy, incomplete, or sampled data;
- *surfaces and graph embeddings*, which have been a fertile breeding ground both for powerful algorithmic tools to attack natural topological problems in several different areas of computing, and for deep structural theorems about minor-closed families of graphs.

These branches have developed relatively independently. In recent years, however, they have intersected more frequently in both mathematics and computer science settings and their scope extends even further. For instance, persistent homology cannot be clearly classified as pure or applied mathematics, but nevertheless it clearly owes its existence to motivations from applications. It gives evidence to the broader claim that all of mathematics finds its justifications in a network of connections and applications, only that some paths are longer than others.

Bringing together key people from all three branches, in a setting that allows for expository talks, detailed research talks, a flexible programme and significant time for informal discussions resulted in a stimulating workshop, where much informal interaction, and many fruitful discussions developed between mathematicians who had just met for the first time. Throughout the programme, we included talks giving a high-level overview from experts in each of the three main branches of computational topology, so that participants from other branches become familiar with the central ideas (see the talks by Schleimer, Wang, Colin de Verdière and Bubenik). In the more specialised talks, common themes resonated throughout the week, as witnessed by the follwing examples:

- The study of *surfaces* is a classical, but extremely active field due to the ubiquity of Riemann surfaces. Algorithms for surfaces or discretisations of geometric properties of surfaces featured in talks by Chambers, Thurston, Sidiropoulos, Krane and Bell.
- *Triangulation* provide frameworks to do computational geometry and topology on spaces. We saw different approaches to estimate minimal number of simplices, or to build natural triangulations, in talks by Eckmann, Joswig, Spreer and Sharygin.

- The problem of *embedding complexes* in manifolds is also a classical topic, and new approaches and results were discussed by de Mesmay and U. Wagner.
- With computational advances, questions about *statistical patterns* in the study of invariants of objects taken from a suitable sample space become amenable to systematic study. This was addressed for knots by Dunfield and Hass and for configuration spaces by Kahle.
- The talks by Hiraoka, Knudson, Robins, Scolamiero, Pokorny, Carstens and H. Wagner on topics in *persistent homology* highlighted the breadth of mathematical areas with which computational topology interacts.

Computational topology is rooted in mathematics—its algorithms build upon deep and sophisticated mathematical theories, and its *software tools* now play an important role in modern mathematical research. For example, the software package *SnapPea*, introduced over 20 years ago, has had an enormous impact on 3–manifold theory through allowing researchers to explore and study in depth concrete examples of hyperbolic 3–manifolds. More recently, the software package *Regina* has been used to resolve Thurston's 30-year-old conjecture on the Weber-Seifert space through a combination of new topological theory and computer-assisted proof. Software tools are now beginning to penetrate 4–manifold topology, where they are already proving useful despite the abundance of theoretically undecidable problems in this area. See `computop.org` for a rich variety of additional examples.

The workshop had a computational focus, and the invitees included lead developers of several important topological software packages, which were presented in the software presentations. This gave developers the opportunity to shared experiences, difficulties and technical constraints that they have encountered and overcome through the development of their software. Short descriptions of the software presented in these sessions are included in this report.

Computational topology proved to be an appealing new topic for an Oberwolfach workshop. It forges links between mathematics and computer science, and between different branches of pure and applied mathematics. It also relates directly to two of the seven Clay Millennium problems (the Poincaré conjecture, and P-vs-NP). It is clear that the major open problems in each of the three fields require input from a broad base of mathematical expertise, and workshops such as this generate an excellent creative environment for generating progress on these problems.

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Workshop: Computational Geometric and Algebraic Topology

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Abstracts

Recognising three-manifolds

Saul Schleimer

We give an overview of some of the foundational computational problems in three-manifold topology.

1.1. **The homeomophism problem.** We begin with a decision problem that lies at the heart of classical topology. Given a pair of (PL) compact connected n–manifolds M and N, the problem HOMEOMORPHISM asks if M is (PL) homeomorphic to N. In dimension one, counting boundary points suffices. In dimension two, checking orientability, Euler characteristic, and number of boundary components suffices; this follows from the classification theorem for surfaces.

In dimensions four and higher, the problem is undecidable. This is because it can be reduced to the isomorphism for finitely presented groups [8]. Thus we are left with the case of dimension three. Kuperberg, in a lovely paper [7], proves the following "folklore" theorem.

Theorem 1. *If Thurston's geometrisation conjecture holds, then the* THREE-MANIFOLD HOMEOMORPHISM *problem is decidable.*

The geometrisation conjecture is now known to hold, due to Perelman's brilliant fulfilment of Hamilton's programme of Ricci flow [9]. Naturally, we still want to know how difficult the homeomorphism problem really is. In the same paper Kuperberg [7] shows that the tools of normal surface give an *elementary recursive* algorithm: the time required is bounded by a tower of exponentials.

1.2. **Recognition.** The homeomorphism problem can be simplified by fixing one of the manifolds, say M. Then, given a (PL) compact connected manifold N, the problem M RECOGNITION asks if $M \cong N$. Now we are allowed to use any knowledge we have about M to design our algorithms.

The first case of interest is recognising spheres; as usual, this is straight-forward in dimensions one and two. Work of Novikov [17] proves that recognising the five-sphere is impossible. The status of S^4 recognition is unknown [18, page 76]. In dimension three, the upper bound has steadily improved; the problem is now known to lie in **NP** [11, 16, 2, 12, 6]. All of this work again relies on normal surface theory.

Recently, Hass and Kuperberg [4] announced that three-sphere recognition lies in **coNP**, if the generalised Riemann hypothesis (GRH) holds. By the Poincaré conjecture, and since M (assumed closed) is not the three-sphere, the fundamental group $\pi_1(M)$ is non-trivial. The GRH then provides us with enough primes to ensure that $\pi_1(M)$ surjects a sufficiently small non-trivial finite group.

1.3. **Seeing three-manifolds.** There are a few universal methods to describe three-manifolds: triangulations and other polytopal schemes, special spines, and link surgery diagrams. These see the three-manifold "from below": they are descriptions which do not give us any *a priori* knowledge about the manifold specified. Given a manifold in this way, one often resorts to normal surface theory to figure out the basic properties of the manifold, such as its connect sum decomposition, JSJ decomposition or Hakenness.

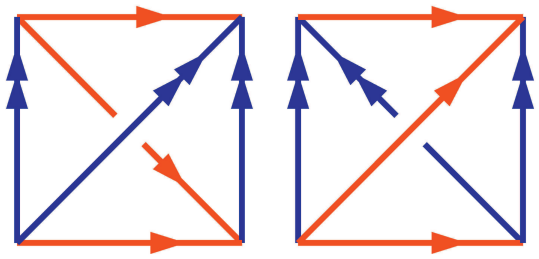

FIGURE 1. An ideal triangulation.

In Figure 1 we have an example of an *ideal triangulation*: after gluing faces as indicated by the edge pattern, and after removing the vertex, we obtain a non-compact three-manifold M^3 whose end is homeomorphic to a torus crossed with an open interval.

It is also possible to see three-manifolds "from the side": if the manifold is given via a Heegaard or knot diagram, or as surface bundle over the circle, then we know that the manifold has special properties: for example a bound on the rank of the first Betti number or knowing that the manifold is irreducible. In Figure 2 we have drawn a diagram of the figure-eight knot F. One step up the ladder of the homeomorphism problem is to see why $S^3 - F$, the three-sphere minus the figure-eight knot, is homeomorphic to the manifold M^3 given in Figure 1.

At the top of the ladder we will see manifolds "from above": that is, we hope to realise them as finite volume quotients of one of the eight Thurston geometries [15]. To give an example, we take $\mathbb{H}^3$ to be hyperbolic three-space and we recall that $\mathrm{Isom}^+(\mathbb{H}^3) \cong \mathrm{PSL}(2,\mathbb{C})$. Taking ω to be a third root of unity, form $\Gamma = \mathrm{PSL}(2,\mathbb{Z}[\omega])$. As first proved by Riley [10], we have the remarkable homeomorphism $\mathbb{H}^3/\Gamma \cong S^3 - F$.

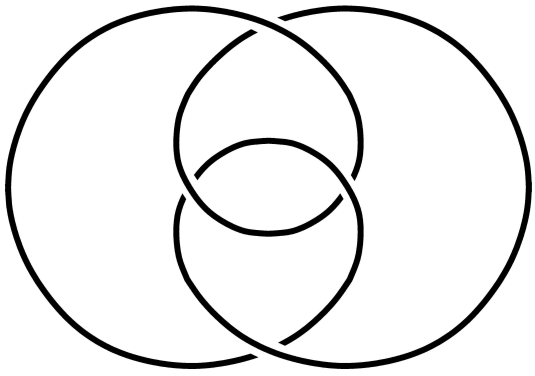

FIGURE 2. Diagram of the figure-eight knot.

This generalises: in an act of amazing insight, Thurston found that sometimes we can jump from the very bottom of the ladder to the very top. The programme SnapPea [3], written by Jeff Weeks, uses Newton's method to find hyperbolic shapes for the tetrahedra of an ideal triangulation. This can be used to prove that certain manifolds are indeed hyperbolic. For example, if we give all edges, of both tetrahedra in Figure 1, dihedral angle 60° then we recover the

above hyperbolic structure on $S^3 - F$. To explain the remarkable effectiveness of SnapPea is one of the great open problems in the theory of three-manifolds.

1.4. **Ever upwards.** What SnapPea can guess, we should be able to prove. Here is a theorem of Ben Burton and myself [14]:

Theorem 2. *For F the figure-eight knot: the problem of recognising $S^3 - F$ among ideal triangulations lies in* **NP**.

More generally, we can recognise when a three-manifold is a knot complement in the three-sphere. The problem of recognising the unknot among knot diagrams was already known to be in **NP** [5]. Following ideas of Agol, Lackenby has recently announced that unknot recognition also lies in **coNP**.

One problem that calls out to be resolved is how to certify (in **NP**) that a three-manifold has a particular Thurston geometry. For the geometries S^3, $S^2 \times \mathbb{R}$, $\mathbb{E}^3$, Nil, and Sol this is known [14]. The remaining cases $\mathbb{H}^2 \times \mathbb{R}$, $\mathrm{PSL}(2, \mathbb{R})$, and $\mathbb{H}^3$ are much more difficult. It is also very tempting to speculate which of the above problems are polynomial time. For a further survey of the computational aspects of three-manifold topology, including hardness results, see [1], [14], and [13].

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Comparing Graphs via Persistence Distortion

Yusu Wang
(joint work with Tamal Dey, Dayu Shi)

Metric graphs are ubiquitous in science and engineering. For example, many data are drawn from hidden spaces that are graph-like, such as the cosmic web. A metric graph offers one of the simplest yet still meaningful ways to represent the non-linear structure hidden behind the data. In this talk, we consider the problem of developing a notion of distance for comparing metric graphs.

Related work. To date, graph matching algorithms fall into two broad categories: exact graph matching methods and inexact graph matching (distances between graphs) methods. The exact graph matching, also called the graph isomorphism problem, checks whether there is a bijection between the node sets of two input graphs that also induces a bijection in their edge sets. While polynomial time algorithms exist for many special cases, e.g., [6, 10], for general graphs, it is not known whether the graph isomorphism problem is NP complete or not [3].

In real world applications, input graphs often suffer from noise and deformation, and it is highly desirable to obtain a *distance* between two input graphs beyond the binary decision of whether they are the same (isomorphic) or not. One line of work is based on graph edit distance which is NP-hard to compute [15]. Many heuristic methods, using for example A^* algorithms, have been proposed to address the issue of high computational complexity, see the survey [2] and references within. One of the main challenges in comparing two graphs is to determine how "good" a given alignment of graph nodes is in terms of the quality of the pairwise relations between those nodes. Hence matching two graphs naturally leads to an integer quadratic programming problem (IQP), which is a NP-hard problem. Several heuristic methods have been proposed to approach this optimization problem; e.g. [4, 9, 13, 14]. Finally, there have been several methods that formulate the optimization problem based on spectral properties of graphs; see e.g, [12, 1, 8, 7].

Proposed distance measure. Different from previous approaches, we view input graphs as *continuous* metric spaces. Intuitively, we assume that our input is a finite graph $G = (V, E)$ where each edge is assigned a positive length value. We now consider G as a metric space $(|G|, d_G)$ on the underlying space $|G|$ of G, with metric d_G being the shortest path metric in $|G|$. Given two metric graphs G_1

and G_2, a natural way to measure their distance is to use the so-called Gromov-Hausdorff distance [5, 11] to measure the metric distortion between these two metric spaces. Unfortunately, it is NP-hard to even approximate the Gromov-Hausdorff distance for graphs within a constant factor. Instead, we propose a new metric, called the *persistence-distortion distance* $\mathrm{d_{PD}}(G_1, G_2)$, which draws upon a topological idea and is computable in polynomial time with techniques from computational geometry. This provides a new angle to the graph comparison problem, and our distance has several nice properties:

(1) The persistence-distortion distance takes all points in the input graphs into account, while all previous graph matching algorithms align only graph nodes. Hence our persistence-distortion distance is insensitive to different discretization of the same graph.

(2) We show that our persistence-distortion distance $\mathrm{d_{PD}}(G_1, G_2)$ is stable w.r.t. changes to input metric graphs as measured by the Gromov-Hausdorff distance. In particular, let $\delta_{GH}(G, G')$ denote the Gromov-Hausdorff distance between two metric graphs (G, d_G) and $(G', d_{G'})$. We have that

$$\mathrm{d_{PD}}(G_1, G_2) \leq 6\delta_{GH}(G_1, G_2).$$

(3) Despite that our persistence-distortion distance is a *continuous* measure which considers all points in the input graphs, we show that it can be computed in polynomial time ($O(m^{12} \log m)$ where m is the total complexity of input graphs). We note that the *discrete* version of our persistence-distortion distance, where only graph nodes are considered (much like in previous graph matching algorithms), can be computed much more efficiently in $O(n^2 m^{1.5} \log m)$ time, where n is the number of graph nodes in input graphs.

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Some topological algorithms for graphs on surfaces

Éric Colin de Verdière

Introduction. Computational topology is a field at the interface between computer science and mathematics that looks at the interplay between algorithms and topology. In the last 15 years, there has been a lot of interest in revisiting problems for graphs on topological surfaces from an algorithmic viewpoint. Basic questions are decision problems (are these two curves homotopic?) and optimization problems (compute a shortest non-contractible closed curve, where the metric is specified in a discrete sense)—here we mostly focus on the latter. Not only does this give a new, fresh viewpoint to classical two-dimensional topology, but this also has an impact to several domains of computer science:

- in computer graphics, geometry processing, and computer-aided geometric design, because these areas use meshes to represent surfaces, which in general are topologically non-trivial. Examples include texture mapping, morphing, parameterization, remeshing, and compression;
- in graph algorithms, because a standard way to bypass the inherent complexity of problems in general graphs is to give more efficient algorithms for some classes of graphs. One such class is formed by the family of planar graphs; however this class is somewhat restricted, and many authors have developped algorithms for embedded graphs, which are efficient if the surface is fixed (and has small genus).

Algorithms manipulate combinatorial data: Specifying the embedding of a graph G on a surface S is (essentially) done by giving, for each vertex v, the cyclic ordering of the edges incident to v ("rotation system" [15]); implicitly it is assumed that all faces of G are topological disks, so this also defines the topology of S. In general we look for curves that are drawn on G ("walks"). Most often we use a discrete metric on S as follows: Every edge of G has a positive *weight*; the length of a curve c is the sum of the weights of the edges traversed by c, with multiplicity; this is called the *combinatorial surface model*.

Below we briefly survey some results in the area given by the author and/or other researchers.

Shortest non-contractible or non-separating closed curves. This is the most basic problem in this area: Cutting along a non-contractible or non-separating

closed curve simplifies the topology of the surface. Erickson and Har-Peled [11] provide an algorithm with running time $O(n^2 \log n)$ (here and in the sequel, n is the total number of vertices, edges, and faces of the embedded graph G that is the input of the algorithm). A different, conceptually simpler algorithm with the same running time uses the *cut locus* and the key property that a shortest non-contractible loop through a fixed baspoint b crosses the cut locus exactly once, and is thus essentially the concatenation of two shortest paths [4, 1]. Many refinements and extensions of this problem have been studied.

Shortest cut graph. A *cut graph* of a surface S is a graph embedded on S whose complement is an open disk. A well-studied problem is the computation of a short cut graph on a surface. Here, it is convenient to use a model dual to the combinatorial surface model, called *cross-metric surface model*: One is given as input an edge-weighted graph G embedded on a surface S, and looks for a graph H in *general position* with respect to G; the length of H is now defined as the sum of the weights of the edges of G *crossed* by H. Erickson and Har-Peled [11] prove that computing a shortest cut graph is NP-hard, and provide an $O(\log^2 g)$-approximation algorithm in small polynomial time, where g is the genus of S. Erickson and Whittlesey [12], however, give an $O(n \log n + gn)$-time algorithm to compute a shortest cut graph with a single, fixed vertex. More generally, Colin de Verdière [5] gives a similarly efficient algorithm to compute a shortest cut graph with prescribed vertex set.

Shortest homotopic curves. Another natural problem is the following: Given a weighted embedded graph G and a path (or walk) p on G, compute (in the combinatorial surface model) a shortest walk with the same endpoints as p that is homotopic to p. Colin de Verdière and Lazarus [8, 9] provide efficient iterative algorithms for this purpose assuming p is simple and the weights of G satisfy a mild condition. Colin de Verdière and Erickson [7] give an efficient general algorithm using cross-metric surfaces. To compute shortest homotopic paths, it suffices to compute shortest paths in the universal cover; by decomposing the surface S along an arrangement of suitable closed curves, meeting four at a vertex and cutting S along octagons, it actually suffices to compute shortest paths in a small (and in particular finite) region of the universal cover.

Multicut in bounded-genus graphs. Finally, as an application of the above techniques, we study a problem from combinatorial optimization. The *multicut problem* is defined as follows: Given an edge-weighted graph G, and specified pairs of vertices of G, called *terminals*, $(t_1, t_1'), \ldots, (t_k, t_k')$, compute a minimum-weight set E of edges such that, in $G - E$, no terminal pair has its two terminals in the same connected component. This is a standard extension of the *multiway cut problem*, where there is a set T of vertices, and the terminal pairs are all the pairs of distinct vertices in T.

Both problems are hard in general. For example, the multiway cut problem is NP-hard, even in planar graphs [10]; the multicut problem is APX-hard, even in unweighted stars [2]. However, if the graph is planar and the number t of

vertices that are terminals is bounded, then there are polynomial-time algorithms for multiway cut: The first published algorithm runs in $n^{O(t)}$ time [10], and a recent one by Klein and Marx [13] solves it in $2^{O(t)} \cdot n^{O(\sqrt{t})}$; Marx [14] actually proves that the $n^{\Omega(\sqrt{t})}$ dependence is unavoidable assuming the Exponential Time Hypothesis.

Using topological techniques, one can actually give [6] an algorithm for the more general multicut problem, also working for graphs embedded on surfaces, which has almost the same running time as the best algorithm for multiway cut in the planar case. Specifically, the algorithm runs in $(g+t)^{O(g+t)} \cdot n^{O(\sqrt{g^2+gt})}$ time. Working in the cross-metric surface defined by G, one computes a shortest cut graph K with vertex set the set of terminals. A minimum multicut corresponds to a shortest graph separating the specified pairs of terminals; one proves that this unknown graph H crosses K few times, enumerates all the combinatorial patterns for the overlay of H and K, and, for each such pattern, computes the shortest H with that pattern using shortest homotopic curves. This general strategy has been used earlier for other problems [3].

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Random knots: their properties and algorithmic challenges

Nathan Dunfield

(joint work with Malik Obeidin)

Malik Obeiden and I are studying random prime knots with 100 to 1,000 crossings, both to probe algorithmic complexity in practice and to better understand the properties of random knots in the spirt of [DT2, DT1]. Our model of random knot mimics the idea of "picking from the table of n-crossing prime knots" rather than a model where the bridge index is fixed, and is similar in spirit to [DEZ, DEHZ]. We started with a sample of 998 knot diagrams, one for each possible crossing number between 3 and 1,000. (These aren't all minimal-crossing projections, but they are close to being so.) It is surprisingly practical to compute certain invariants of these knots, even the one with 1,000 crossings. Our initial findings give compelling evidence of linear growth (with little spread) with respect to crossing number of the following invariants: hyperbolic volume (slope ≈ 2), knot genus (slope ≈ 0.25), and bridge number (slope ≈ 0.15). One pattern that demands explanation: these knots have triangulations where most of the tetrahedra are "fat" in the hyperbolic structure, that is, have volumes near that of the regular ideal one. So far, we can usually only approximate $g(K)$, but we have a new fast heuristic for finding candidate Seifert surfaces, motivated by [DH], which is quite promising. For this grant, we will first expand our sample and the invariants considered and then try to prove that these patterns really exist.

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Configuration spaces of disks

Matthew Kahle
(joint work with Baryshnikov, Bubenik, Carlsson, Gorham, Mason, MacPherson)

Let Config(n, r) denote the configuration space of n nonoverlapping disks in a region R. This is the phase space for a hard spheres gas, s it is of intrinsic interest in physics. It is also interesting mathematically, as the generalization of the configuration space of points. We are particularly interested in the topology of Config(n, r), for example Betti numbers. We discuss three papers investigating this setting - Morse theory is a comon theme throughout.

1. Min-type Morse theory

(joint w/ Baryshnikov and Bubenik). Let Config(n) denote the configuration space of n points in R. We assume that R is convex in $\mathbb{R}^2$, so it is homeomporphic to $\mathbb{R}^2$ and we understand the topology of Config(n) very well. Now define $F : \text{Config}(n) \to \mathbb{R}_{>0}$, the "tautological Morse function" to be the largest radius r such that the disks of radius r centered at the given n points are non-overlapping and completely contained in R. Then $\text{Config}(n, r) = F^{-1}(r, \infty)$. We show that the mechanically ballanced configurations play the role of critical points an smooth Morse theory: if $F^{-1}(r, r')$ contains no critical points, then Config(n, r) is homotopy equivalent to Config(n, r').

2. Computational topology approach

(joint w/ Carlsson, Gorham, Mason) We study Config(n, r) for $n \leq 5$ and all values of r when the ambient region R is a square. We use a variety of techniques including computational Morse theory (nudgd elastic band) to find all the mechanically balanced configurations in the sense above. We find dozens of these configurations when $n = 5$, suggesting that the topology changes many times. It seems that homology can be surprisingly large - we think, for example, that for $0.1686 < r < 0.1692$, $\beta_1 = 2761$.

3. Disks in a strip

(joint w/ MacPherson) We study Config(n, w), the configuration space of n disks of unit diameter in a strip of width w. We are able to understand the asymptotics of $\beta_j[\text{Config}(n, w)]$ for j, w fixed and $n \to \infty$. We find a stable regime where β_j grows polynomially fast, and an unstable regime where β_j grows exponentially fast. Again a Morse-theory view is useful: we get an upper bound on β_j in the unstable regime by a complicated discrete Morese theory argument. We discuss the boundary between the stable and unstable regimes as analogs to a gas-liquid phase transition.

Topological data analysis on materials science: statistics and continuation

Yasuaki Hiraoka

In this talk, I surveyed topological data analysis on materials science in WPI-AIMR, Tohoku university, and gave some mathematical new results in statistics and inverse problems. In TDA on materials science, structural analysis on amorphous solids was explained in detail, following the results in [1, 2]. The contents included the classification of liquid, glass, and crystals states, the detection of the real space origins of the first sharp diffraction peak (FSDP), and mechanical responses of glass structures, all of which are analyzed by using persistence diagrams (PD). Then, we proposed the persistence weighted Gaussian kernel (PWGK) and showed a stability result with respect to the distance defined by PWGK [3]. As an example, we statistically detected a glass transition temperature by using our kernel method. We also briefly explained a continuation method in [4] as a tool of inverse problems of persistence diagrams.

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Topological analysis of democracy data

Kevin Knudson
(joint work with Laura Sjoberg)

In this talk we discussed the topological structure of a data set built from empirical measures of democracy. The data combined two sets: the Polity IV data set [PIV] and the Boix–Miller–Rosato Dichotomous Coding of Democracy [BMR]. We studied the concatenation of these two sets for the year 2007, which yields a point cloud in $\mathbb{R}^1 4$.

The Polity IV set consists of 9 measurements related to the recruitment and selection of chief executives, freedom of participation in the electoral process, etc. These numbers are nonnegative integers with upper bounds ranging from 3 to 10, depending on the particular concept being measured. Political scientists involved with the Polity Project then use a step function on these data to associate to each country in the world its Polity score, an integer between -10 and $+10$. Stable functioning democracies such as the U.S. or most of Western Europe score a $+10$, while autocratic countries such as North Korea score a -10.

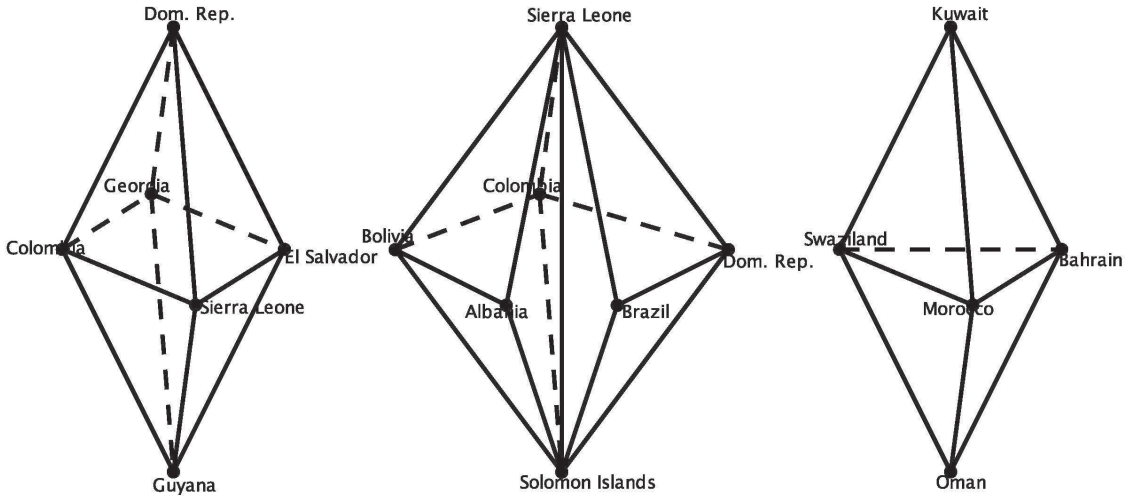

FIGURE 1. Some 2-cycles in the 14-dimensional data. Note that these are representations of the figures in 3-dimensional space; the actual cycles are embedded in $\mathbb{R}^{14}$.

The Boix–Miller–Rosato set consists of 5 measures of democracy, 4 of which are completely dichotomous (1 = democracy, 0 = non-democracy) along with a count of the number of previous democratic breakdowns (e.g., the U.S. counts as 0 in this category–the Civil War was not technically a democratic breakdown).

For the year 2007, even though there are more than 160 distinct countries listed, only 88 of them yield unique data points. For example, the United States, United Kingdom, most of Western Europe and Scandinavia, Japan, Australia, and a few others have the exact same Polity and dichotomous measures. We built the Čech complex on these 88 points using the Euclidean metric on $\mathbb{R}^{14}$ and computed its persistent homology. The set has 10 components, 8 of which are acyclic. The other two components contained nontrivial 2-cycles, six in one component represented by the Dominican Republic and one in the component represented by Swaziland. We show three of these in Figure 1.

Future work will add a temporal component to the study–the sets include data back to 1800 and so it would be interesting to see how the topology changes over time. Some audience members during the MFO talk suggested that the Euclidean metric may not be the best choice; we plan to investigate this as well. Another potential avenue of study would be to place discrete Morse functions on these sets to investigate how democracy "flows" or to track gradients over time by examining the Morse functions on a family of years.

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Persistent homology analysis of porous and granular materials

Vanessa Robins

(joint work with Olaf Delgado-Friedrichs, Mohammad Saadatfar, Adrian Sheppard, Katharine Turner)

Topology plays a fundamental role in many aspects of materials science from quantifying the geometric and topological structure of porous rocks, to describing configuration spaces of granular matter, identifying possible topological phases of matter and underpinning recent advances in mathematical crystallography. My past research in applied topology has contributed to the mathematical foundations of persistent homology [1], algorithms for building combinatorial complexes from digital images [2] and their application to image analysis [3], the enumeration of crystallographic graphs (periodic nets) via periodic surface tilings [4], and the geometric entanglement (knotting) of graphs [5, 6]. The talk focussed on our recent work using persistent homology to analyse the geometric structure of porous and granular materials.

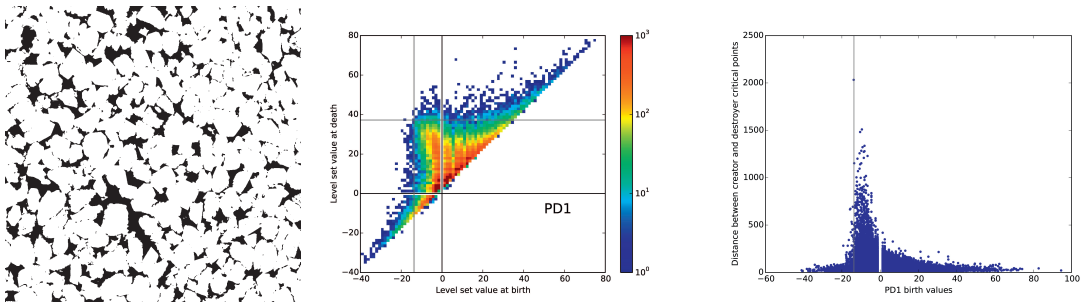

Figure 1. Single slice image of a sandstone. A histogram of birth and death values of the persistence diagram for dimension 1 (PD1). A scatter plot of PD1 distances between locations of the paired birth and death critical points versus their birth critical values. The pale grey vertical and horizontal lines mark percolation thresholds in the pore and grain phase respectively.

Porous materials are a wide class with geological, organic and engineering origins, e.g. sandstone, wood, synthetic foam. The common structure of these disparate materials is that they consist of *two phases*: a solid matrix and a gas or liquid-filled porespace. The phases are spatially intermixed with some amount of *disorder* in their distribution and are typically *bicontinuous*, i.e. both phases span the sample in some or all directions. We include granular materials as they are often the precursor to a porous material: sand grains are subjected to chemical

deposition and pressure to form sandstone, metal beads are sintered to form a porous solid, bubbles (gaseous grains) are introduced to a liquid polymer to create a foam. My colleagues in Applied Mathematics at ANU operate a state-of-the-art x-ray tomography facility for imaging porous materials at micrometer length scales [7]. They use these images to conduct virtual experiments on the samples via computational simulations of physical process and compare these results with laboratory measurements.

It has been long understood that the connectivity of different phases in a porous material (rock and water in an aquifer, for example) strongly influences physical properties of the system, particularly those related to transport (fluid flow or diffusion of contaminants in the aquifer). This relationship between topology and transport has been studied primarily through network models of the porespace, and percolation theory, i.e., the study of cluster formation and growth under the addition of randomly added components.

A simple percolation model relevant to porous materials is to grow balls of radius r about a random point pattern X (a Poisson-Boolean model). *Percolation* occurs at a radius r_c so that for $r > r_c$, the union of balls centred on X has a connected component that spans the domain: fluid can flow from one side to the other, and for $r < r_c$ fluid flow is not possible. Computational and probabilistic analyses have shown that percolation is a critical phenomenon. There is a well-defined value of r called the percolation threshold, r_*, so that in the limit of large domains and many repeated realisations of a random point pattern, the spanning cluster exists with probability 1 for $r > r_*$, and the largest cluster does not span with probability 1 for $r < r_*$. One indication of the physical significance of the percolation threshold is its correlation with the permeability of fluid flow through porous materials [8, 9].

The above simple geometric percolation process is translated to two-phase porous materials using the signed distance function $f(x)$ that measures the distance from x to the closest point on the interface between the two phases, with positive values in one phase and negative values in the other. The lower level sets of f now form the space of interest and we define the percolation threshold as the lowest value r_c, for which the lower level set $f(x) \leq r_c$ spans the domain. For subsets of $\mathbb{R}^3$ there is a complementary threshold, r_c', the largest value for which the upper level sets span: $f(x) \geq r_c'$.

Persistent homology quantifies the topology of lower (or upper) level sets of a function as the threshold parameter increases. Percolation and persistent homology are therefore seen to be two different topological measurements on the same underlying geometric filtration. This connection is elaborated on in our preprint [10]. The key observation is that persistence pairs are able to become spatially separated around the percolating threshold. This behaviour is seen in both the model system of random points with balls of radius r attached, and in real sandstones.

Our initial work has focussed on single data sets, but a full statistical analysis of many samples over large domains will be necessary to establish possible critical

behaviour and sharp transitions in persistence data. This statistical analysis is facilitated by work with Katharine Turner [11] where we have demonstrated that standard statistical methods can be applied to functional summaries of persistent homology such as the rank function. It is also worth mentioning that the Betti numbers by themselves are not sensitive to the percolation transition [12].

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Triangulating 3-dimensional balls

Jean-Pierre Eckmann

(joint work with Pierre Collet and Maker Younan)

I reported on the work with Pierre Collet and Maker Younan on reducing the question of the number of triangulations of the 3-ball with t tetrahedra. We construct triangulations from elementary pieces, called "atoms" and show that if the number of atoms with t tetrahedra is bounded by C^t then the number of triangulations of the 3-ball is bounnded by C_*^t. An atom here is a triangulation with only external nodes, internal faces hace at least 2 internal edges and for every

external edge there is a node n so that the triangle spanned by the 3 nodes is *not* a face.

Combinatorial topology with applications to neuroscience

Martina Scolamiero

(joint work with P. Dłotko, K. Hess, R. Levi, H. Markram, E. Muller, M. Nolte, S. Raynor, M. Reimann and K. F. Turner.)

Directed graphs offer a simple but efficient formalism to model relational information between agents, in particular causal relationships. In neuroscience directed graphs are widely used to study brain morphological and functional connectivity. In this talk I introduced some new methods to identify patterns and densely connected regions in directed graphs. By defining the oriented clique complex construction I associated Betti numbers and the Euler characteristic to a directed graph. Classical invariants in network theory, as for example the clustering coefficient, can be generalised to identify higher order connectivity. By detecting the number of directed k-simplices in the neighbourhood of each vertex we have a more accurate description of the local cohesiveness of the directed graph and a generalised measure of segregation.

Applications of our directed graph metrics to study the Blue Brain model were then presented. The Blue Brain Project built a biologically based digital reconstruction of the microcircuitry of the hind-limb somatosensory cortex of a two-week-old rat [1, 2]. I reported on an on-going project aimed at investigating topological properties of this model and emergent behaviour of the microcircuits through simulated neural activity. A column is a unit in a microcircuit consisting of $\sim 31,000$ neurons and $\sim 8.2 \times 106$ connections between them and represents a cortex region of 0.5 mm in diameter and 2 mm in hight. As an example of construction we use to study the spiking dynamics within a column I presented the so called successful transmission graphs $\{G_t\}_{t\in\mathbb{N}}$. Having fixed the size of a time bin, the vertices in graph G_t represent neurons in the column region and there is an edge $i \to j$ in G_t if and only if :

(1) i and j are structurally connected and
(2) neuron i fires in time bin t and neuron j in time bin t or $t+1$.

Finally I showed how the dynamics generated by different types of stimuli is reflected by the time series representing Betti numbers and Euler characteristic of the successful transmission graphs.

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Topological Motion Planning and Classification for Robotics

Florian T. Pokorny
(joint work with Ken Goldberg, Majd Hawasly, Lydia E. Kavraki, Danica Kragic, Subramanian Ramamoorthy)

In the past three decades tremendous progress has been made in the development of robot motion planning algorithms. While methods closely related to Morse theory and cell-decompositions from algebraic topology have lead to breakthroughs in the understanding of theoretical properties of the motion planning problem in the 80s and 90s [1, 2], a large number of current approaches to motion planning are based on sampling based approaches and construct randomized graph representations of the free configuration space of a robot [3, 4].

However, these graph-based methods only approximate the path-connectivity of the free configuration space $\mathcal{C}_f$ and are not able to capture information about the fundamental group of $\mathcal{C}_f$. Information about the fundamental group can however be beneficial to enable a robotic system to reason about homotopy classes of trajectories – both to *plan motion alternatives* and to *classify observed motions*. While classical approaches such as analytical cell-decompositions could be used in theory to compute information about homotopy classes, these approaches are currently difficult to scale to robotic systems with many degrees of freedom. Additionally, real robotic systems suffer from noisy perception and control. As a result, a noise-free and complete description of $\mathcal{C}_f$, for example in terms of semi-algebraic functions, is typically not available.

I will present our recent work on algorithms that capture information about homotopy classes in $\mathcal{C}_f$ using persistent homology. Our approach is based on building multi-scale *sampling-based* representations of $\mathcal{C}_f$. I will present experimental results [5, 6] utilizing collision free samples $X = \{x_1, \ldots, x_n\} \subset \mathcal{C}_f \subset \mathbb{R}^d$ from the collision-free part of a robot's configuration space to study filtrations of topological spaces $X_{r_1} \subseteq X_{r_2} \subseteq \ldots \subseteq X_{r_k}$, where $X_{r_i} = \bigcup_{i=1}^{n} \{x \in \mathbb{R}^d : |x - x_i| \leq r_i\}$ denotes the union of r_i balls around each sample and $r_1 \leq r_2 \leq \ldots \leq r_k$. We use the fact that X_{r_i} is homotopy equivalent to the Delaunay-Čech complex $DC_{r_i}(X)$ [7], to compute information about the first persistent homology groups of the filtration. I will discuss how persistent homology can be used to cluster motion trajectories into homology classes [5] and how persistent cohomology can be combined with a Dijkstra-based graph-search algorithm to determine homotopy inequivalent trajectories for robotic systems [6]. Our work provides a fully sampling-based approach to motion planning closely related to the recent work of [8] where differential one-forms are used for motion planning.

Finally, I will present recent work on scaling homotopy-aware motion planning algorithms to configuration spaces of dimension up to 10 [9]. Our approach here is based on 2D topological task projections (TTPs): mappings from the configuration space to $\mathbb{R}^2$, where simplicial complex filtrations and persistent homology can be efficiently computed. I will discuss how such projections can be used to determine homotopy inequivalent trajectories in the high-dimensional configuration space

$\mathcal{C}_f$. We in particular propose the Winding Augmented RRT and RRT* (WA-RRT/RRT*) algorithms using which homotopy inequivalent trajectories can be computed. I will present recent experimental results using this idea that enabled us to determine complex homotopy inequivalent 10 degree of freedom planar robot arm trajectories.

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Webs of stars or how to triangulate free sums of point configurations

Michael Joswig

(joint work with Benjamin Assarf, Julian Pfeifle)

A *triangulation* of a finite configuration of points $S \subset \mathbb{R}^d$ is a triangulation of the convex hull $\mathrm{conv}(S)$ which uses (some of) the points in S as its vertices. A special case occurs when that triangulation is induced by a height function; then it is called *regular*. Given S it is of major interest to be able to list all (regular) triangulations of S, possibly up to symmetry. This algorithmic problem is motivated, e.g., by applications in optimization, statistics or algebraic geometry; see [4, Chapter 1] for a general overview or [3, §14.4] for the special case of toric geometry. The standard software for enumerating triangulations is `TOPCOM` [8], whose algorithm is described in [7] and [4, §8.3].

Conceptually, enumerating all (regular) triangulations is fairly easy, but it is very hard in practice. To some extent this is due to the fact that the sheer numbers grow out of any reasonable bounds very quickly. Therefore, it is important to be able derive general information about the triangulations of point sets, even special ones. The paper [6] describes families of point sets for which it is easy to describe

all triangulations. In this talk we look into a specific construction which somehow marks the next step.

Let $S \subset \mathbb{R}^d$ and $T \subset \mathbb{R}^e$ be finite point configurations. Let us assume that the affine spans of S and T are full-dimensional. Further, assume that the origin of $\mathbb{R}^d$ is a point of S and that additionally, it is an interior point of $\operatorname{conv}(S)$; and we also assume the same properties for T. Then we can form the *free sum* $S \oplus T$ which is the union in $\mathbb{R}^{d+e}$ of the two point sets viewed as embedded into an orthogonal pair of linear subspaces. That union is disjoint up to the origins in $\mathbb{R}^d$ and $\mathbb{R}^e$, which get identified in $\mathbb{R}^{d+e}$. For example, when S is the set $\{-1, 0, 1\}$ in the real line, then $\operatorname{conv}(S \oplus T)$ is a bipyramid over $\operatorname{conv}(T)$. The free sum is the coproduct in the category of finite points sets with zero and linear maps as morphisms. Our main result describes the triangulations of $S \oplus T$ in terms of the triangulations of the two summands. In general, one triangulation Δ_S of S and one triangulation Δ_T of T give rise to more than one triangulation of $S \oplus T$. The choices are controlled by partially ordered sets which can be associated with Δ_S and Δ_T. These posets encode something like relative visibility information in those triangulations. A *web of stars*, which was mentioned in the title, (in the direction from S to T) is a poset homomorphism from the *stabbing order* on Δ_P to the *star-shaped balls* of Δ_Q. Now Δ_S and Δ_T yield one triangulation of $S \oplus T$ for each *compatible* pair of webs of stars, one from S to T and one in the opposite direction.

Theorem. *For each triangulation Δ_S of S, each triangulation Δ_T of T and each compatible pair of webs of stars there is a triangulation of the free sum $S \oplus T$. Conversely, each triangulation of $S \oplus T$ arises in this way.*

We have a conjecture to characterize the regular triangulations of free sums.

In addition to their categorical relevance studying free sums of point configurations is motivated by recent results on the classification of smooth Fano polytopes; see [1] and [2]. Fano polytopes occur naturally in toric geometry [3, §8.3]. Our method is implemented in `polymake` [5]; this feature will become available with the next release 3.0 (scheduled for January 2016).

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Collapsibility and 3-sphere recognition

Jonathan Spreer
(joint work with João Paixão)

A 3-manifold triangulation M is homeomorphic to the 3-sphere if there exists a sequence of elementary collapses from M with one tetrahedron removed onto a single vertex. Unfortunately, there exist many triangulations of the 3-sphere where such a collapsing sequence is difficult to find or even does not exist. We show that out of all 39 8-vertex triangulations of the 3-sphere, 22 admit non-collapsing sequences onto contractible non-collapsible 2-complexes. As a side product we classify all such 2-complexes with at most 18 triangles. In addition, we propose an easy-to-check heuristic characterisation for 3-sphere triangulations to admit very few collapsing sequences.

1. Introduction

Triangulations of manifolds come with two layers of complexity. One from the topology of the underlying manifold, one from combinatorial properties of the triangulation itself. While, typically, we cannot eliminate the former, there is hope that the latter can sometimes be avoided. Here we restrict ourselves to simplicial triangulations of the 3-sphere, i.e., simplicial complexes whose vertex links are triangulated 2-spheres and whose underlying space is homeomorphic to the 3-sphere. We thus focus on difficult combinatorial properties of triangulations.

One measure of difficulty of a triangulation is given by the framework of *collapsibility* [7]. Suppose a simplicial complex C contains an i-dimensional face $\delta \in C$ which is only contained in a single $(i+1)$-dimensional face $\Delta \in C$. We can then remove the pair of faces from C obtaining a homotopy equivalent complex $C \setminus \{\delta, \Delta\}$. This is called an *elementary collapse* of C, and C is said to be *collapsible* if there exist a sequence of elementary collapses from C down to a single vertex. If no such sequence exist it is called *non-collapsible*.

If C is a triangulation of a closed 3-manifold we can form its *dual graph* $\Gamma(C)$, whose vertices correspond to tetrahedra and whose edges correspond to pairs of tetrahedra sharing a triangle. Given a spanning tree $T \subset \Gamma(C)$, we can collapse all tetrahedra of C with one tetrahedron removed along T, yielding a 2-dim. complex C_T. If there exist a $T \subset \Gamma(C)$ such that C_T is collapsible, C must be homeomorphic to the 3-sphere. However, the converse is not true [2]. Thus, 3-sphere triangulations without such spanning trees are "more difficult than necessary".

We want to *quantify* the complicatedness of a 3-sphere triangulation by studying how many spanning trees $T \subset \Gamma(C)$ lead to collapsible complexes C_T. Closely related questions about complicated triangulations of manifolds and in particular

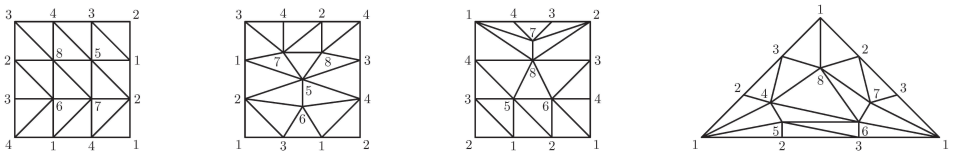

FIGURE 1. Minimal contractible non-collapsible 2-complexes.

spheres in arbitrary dimensions have recently been studied by Benedetti and Lutz [2], and Joswig, Lutz and Tsuruga [5].

2. Contractible non-collapsible 2-complexes in small 3-spheres

There are no non-collapsible 3-sphere triangulations in the classification of 3-spheres up to 11 vertices. To nevertheless be able to quantify complicatedness of 3-sphere triangulations S in a systematic way, we focus on single choices of $T \subset \Gamma(S)$ such that S_T is (contractible but) non-collapsible. In this case we have:

Lemma 1 (Lemma 4.2 in [6])**.** *Let C be a contractible non-collapsible simplicial 2-complex. Then C must have at least 8 vertices and 17 triangles.*

A minimal contractible non-collapsible 2-complex can be represented by a triangulated disk whose nine boundary edges are identified in triples. Such complexes can be classified with the help of the software `plantri` by Brinkmann and McKay [3]: There are seven combinatorially distinct 17-triangle triangulations of the dunce hat, and 80 contractible non-collapsible 18-triangle 2-complexes of four distinct topological types. See Figure 1 for a collection of different types of complexes, see [6] for a list of all complexes.

It follows that, for a given 3-sphere triangulation S to contain a contractible non-collapsible 2-complex, S must have at least 8 vertices and 16 tetrahedra. Otherwise, S_T would have < 17 triangles, a contradiction to Lemma 1.

From [1] we know that such minimal triangulations of dunce hats actually already exist in 8-vertex 3-spheres. But how common are these pathological combinatorial structures in small 3-sphere triangulations? To give a precise answer to this question we conduct a number of large scale experiments on small triangulations of the 3-sphere using the software package *simpcomp* [4] which can be found in detail in [6]. In the 8-vertex case we are able to give a complete answer, using the above classification of contractible non-collapsible 2-complexes.

Theorem 1. *There are 17 triangulations of the 3-sphere with eight vertices such that all spanning trees lead to a collapsible 2-complex. The remaining 22 8-vertex triangulations of the 3-sphere triangulations admit a collapsing sequence onto a contractible non-collapsible 2-complex.*

3. Predicting the complicatedness of 3-sphere triangulations

In this section we propose a heuristic to pre-evaluate whether or not a given 3-sphere triangulation S has complicated combinatorial properties (such as very few or no collapsing sequences). The prediction is based on the edge degrees $\deg_S(e)$, $e \in S$, of S, i.e., the number of tetrahedra of S containing e.

Definition 2. *Let S be an n-tetrahedron, v-vertex triangulation of the 3-sphere. In average the edges of S are contained in $\overline{\deg}_S = \frac{6n}{n+v}$ tetrahedra. We say that S has* edge variance

$$\operatorname{var}(S) = \frac{1}{n+v} \sum_{i=1}^{n+v} (\overline{\deg}_S - \deg_S(e_i))^2.$$

Given a 3-sphere triangulation S, computing $\operatorname{var}(S)$ is a simple procedure, but might nonetheless approximate quite accurately how complicated S is. This follows from the following two observations.

(i) Given a spanning tree $T \subset \Gamma(S)$, the number of free edges in S_T correlates with whether or not S_T is collapsible: triangles of S_T can be removed as long as there are free edges left, and the more free edges there are to begin with, the higher are the chances that in this process all triangles can be removed.
(ii) Given an edge $e \in S$, the minimum probability of e being free in S_T can be shown to decay exponentially with the degree $\deg_S(e)$ of e in S [6, Theorem 5.1].

Combining these two observations we expect S to be very difficult to collapse whenever S has very few degree three and four edges, that is, whenever $\operatorname{var}(S)$ is very small. Thus, decreasing the edge variance of a given initial 3-sphere triangulation using, for instance, bistellar moves, may produce difficult-to-collapse 3-sphere triangulations.

Indeed, following this idea, we are able to produce a 15-vertex 3-sphere triangulation S_{15} with far less collapsing sequences than known pathological input of similar size (eg. from [2], see [6] for details).

Moreover, this technique might translate to a useful tool in higher dimensions. Using a more general version of the edge variance for co-dimension two faces we can aim at increasing this quantity step by step to obtain easier-to-handle triangulations. This strategy of "intermediate objectives" for simplification heuristics might succeed where current simplification heuristics fail.

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Topological Measures of Similarity for Curves on Surfaces (mostly)

Erin Wolf Chambers

The question of how to measure similarity between curves in various settings has received much attention recently, motivated by applications in GIS data analysis, medical imaging, and computer graphics. While geometric measures such as the Hausdorff and Fréchet distance [1] have efficient algorithms, measures that take the underlying topology of the space are relatively new. Several candidates have been proposed in recent years, but many of these are only tractable in restricted settings, and surprisingly little is known about their practicality.

In this talk, we survey several variants which have algorithms for common applications settings. The two main settings considered will be curves in the plane minus point or polygonal obstacles, and the combinatorial surface model, where curves lie on some orientable 2-manifold which has a piecewise flat triangulation or graph embedding imposed on it. We will also briefly discuss further possible generalizations, including finding how similar two surfaces are when they are embedded either in a 3-manifold or in 3-dimensional Euclidean space.

For homotopy-based similarity measures, one can consider the width, height, or area of the homotopy swept by two input curves. In the case of homotopy width, often called homotopic Fréchet distance, a polynomial time algorithm is known for the plane minus polygonal obstacles [2]. However, for the case of curves on surfaces, it is not even known that the problem is in NP. If one then considers homotopy height [3], even less is known. Even for the case of two curves which bound an unweighted triangulated planar graph, it is unknown if the optimal homotopy will be monotonic, so we again do not know if the problem is in NP. The only algorithmic result known here is an $O(\log n)$-approximation algorithm for both problems [6]. Homotopy area is surprisingly more tractable than either height or width, and polynomial time algorithms are known for either the planar or surface-embedded setting [4].

A newer similarity measure is based on homology rather than homotopy. With homology area, one considers all 2-chains with boundary equal to the two input curves, and optimizes to find the one with the minimum area. In this case, there is an algorithm based on the fact that homology computation reduces to well known and highly optimized linear programming and optimization problems [5]. Unlike previous metrics, this method generalizes smoothly to arbitrary dimensional submanifolds of arbitrary dimensional cell decompositions.

Any of these measures could be considered in more general settings, as homotopy and homology are well defined on (for example) 3-manifolds. However,

homotopy in general is less tractable in these settings, although it is known that homology area and homotopy area are equal for dimension ≥ 3 [7]. How to apply these similarity measures and compute and use them efficiently in these higher dimensional settings remains a very interesting open question.

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Embeddability of 2-complexes

Arnaud de Mesmay

(joint work with Benjamin A. Burton, Uli Wagner)

In this talk, we investigate the complexity of various problems related to embeddings of 2-dimensional complexes, and in particular non-orientable surfaces, into 3-manifolds. Our motivation is two-fold:

- On the one hand, embeddings of graphs in the plane or on surfaces are a staple topic in graph theory, and 2-complexes are the direct generalization of graphs one dimension higher. While testing graph planarity or embeddability on a surface of given genus are well studied questions, the corresponding problems for 2-complexes are still very poorly understood. Matoušek, Sedgwick, Tancer and Wagner [5] showed recently that testing whether a 2-complex embeds in $\mathbb{R}^3$ is decidable, and the next natural steps are to improve on the complexity of their algorithm and to investigate what can still be carried out in other 3-manifolds.
- On the other hand, finding an interesting surface in a 3-manifold is pretty much the basic challenge underlying most of the decision problems in low-dimensional topology. This includes for instance unknot recognition, knot genus, 3-sphere recognition, Hakenness testing, or prime and JSJ decompositions (see for example the book of Matveev [4]). This motivates the

study of the embeddability of surfaces (which are the simplest examples of 2-complexes) in 3-manifolds, and since every 3-manifold contains all the orientable surfaces, the first example of a non-trivial surface for this problem is a non-orientable one.

We first show that deciding whether a non-orientable surface of odd Euler genus g embeds into a 3-manifold M is NP-hard. This implies that testing whether a 2-complex embeds in a 3-manifold is NP-hard as well – note that this problem is not known to be decidable[1]. The reduction is inspired by the one of Agol, Hass and Thurston [1] for Knot Genus, but the proof is significantly more involved in our case.

Our second result shows that the same problem is in NP, and is therefore NP-complete. However, our algorithm, based on normal surface theory, relies crucially on the hypothesis that the Euler genus g is odd. Even with no complexity requirement, we are not aware of *any* algorithm in the case of even Euler genus.

Lastly, we discuss the problem of *thickenability*, i.e., the problem of deciding whether there exists a 3-manifold M in which a given 2-complex C embeds. While the lower dimensional problem is trivial (every graph embeds on some surface!), there are examples of 2-complexes which do not embed in any 3-manifold, as is the case with the cone on the complete graph K_5 for example. In addition to being a natural problem to investigate, thickenability is also the first complexity hurdle to overcome in the aforementioned quest to improve the algorithm of Matoušek, Sedgwick, Tancer and Wagner. Following works of Neuwirth [7] and Skopenkov [8], we show that testing thickenability boils down to a combinatorial problem that is very similar to *Simultaneous Embeddings with Fixed Edges*, a cornerstone problem in the field of Graph Drawing (see Blasius, Kobourov and Rutter [2] for a survey on simultaneous embedding questions). This problem lies exactly at the edge of the current knowledge: it is not known to be NP-hard, and for some specific cases, polynomial-time algorithms are known. Leveraging on recent tools [3] on SPQR and PQ trees, we provide an algorithm to test thickenability in polynomial time if the link of every vertex of the input complex C is 2-connected.

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Discrete Measured Foliations

Dylan P. Thurston
(joint work with Steven Gortler, David Palmer)

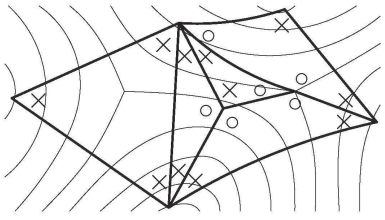
We introduce a notion of *discrete measured foliations* on a filling, elastic graph on a surface. This gives a discrete analogue of harmonic measured foliations and holomorphic quadratic differentials, and in particular a system of coordinates on measured foliations. This gives a single uniform system of coordinates for all measured foliations on a closed surface. It is also useful for computationally approximating actual harmonic measured foliations.

This work was partially supported by NSF grant DMS-1507244.

Definition 1. *A* filling elastic graph (Γ, k) *in a surface* Σ *is a graph* Γ *(a 1-dimensional CW complex) embedded in* Σ *that is* filling*, in the sense that the complementary regions are all disks, and* elastic*, in the sense that each edge* e *has an associated number* $k(e) \in \mathbb{R}_{\geq 0}$*, the* elastic spring constant*. An elastic graph has a natural dual elastic graph* (Γ^*, k^*)*, where* $k^*(e^*) = 1/k(e)$ *if* e^* *is the edge dual to* e*.*

Definition 2. *A* corner structure *on a filling graph* Γ *is an assignment of a marking* $m(c) \in \{\times, \circ\}$ *for each corner* c *of a face of* $\Sigma \setminus \Gamma$*. We require that each face have at least two* $\times$ *markings and each vertex have at least two* $\circ$ *markings. A* length structure (ℓ, m) *on* Γ *is a corner structure* m *and a length* $\ell(e) \in \mathbb{R}_{\geq 0}$ *for each edge* e *of* Γ*.*

To construct a length structure from a measured foliation F, overlay Γ on F so that the edges don't backtrack as on the right. Mark a corner with $\circ$ if there is a leaf running in to that corner, and $\times$ otherwise. The length $\ell(e)$ is the total measure of e with respect to the measured foliation.



Definition 3. *A sequence of non-negative real numbers* $(x_1, \ldots, x_k)$ *is said to* satisfy the triangle inequality *if no* x_i *is greater than the sum of the others. This implies that* $k \geq 2$*, and if* $k = 2$ *then* $x_1 = x_2$*.*

Definition 4. *A length structure* (ℓ, m) *on a graph* Γ *is* closed *if, around each face* f*, the lengths grouped between successive* $\times$*'s satisfy the triangle inequality.*

More precisely, suppose the edges around f are $e_1, \dots, e_n$, with k $\times$ corners between (e_{i_1}, e_{i_1+1}) through (e_{i_k}, e_{i_k+1}) (interpreted cyclically). Then we must have the triangle inequality for the sums

$$x_j = \sum_{t=i_{j-1}+1}^{i_j} \ell(e_t).$$

Definition 5. *A length structure (ℓ, m) on an elastic graph (Γ, k) induces a dual length structure (ℓ^*, m^*) on the dual graph by*

$$\ell^*(e^*) = k(e)\ell(e)$$
$$m^*(c^*) = \begin{cases} \times & m(c) = \circ \\ \circ & m(c) = \times. \end{cases}$$

Definition 6. *A length structure is* co-closed *if its dual is closed; that is, ℓ satisfies weighted triangle inequalities at each vertex.*

Definition 7. *A* harmonic *length structure or* discrete quadratic differential *on a filling elastic graph is a closed and co-closed length structure.*

One intuition is that a discrete quadratic differential (non-zero on each edge) gives a rectangle-tiled surface: each edge e gives a rectangle of aspect ratio $k(e)$, with length $\ell(e)$ and width $\ell^*(e^*) = k(e)\ell(e)$. The triangle inequalities guarantee that there is at least one way to sew the rectangles together around the vertices and around the faces.

There is a discrete version of the Poincaré-Hopf index theorem.

Definition 8. *Given a corner structure m, let $n_\times$ and $n_\circ$ be the number of $\times$'s or $\circ$'s in m around a face or vertex. For a face f and vertex v, define the index by*

$$\operatorname{ind}(f) \coloneqq n_\times(f) - 2$$
$$\operatorname{ind}(v) \coloneqq n_\circ(v) - 2.$$

We say that a vertex or face is non-singular *if its index is* 0.

Proposition 9. *For any length structure on a graph Γ in a surface Σ,*

$$\sum_{f \text{ a face}} \operatorname{ind}(f) + \sum_{v \text{ a vertex}} \operatorname{ind}(v) = -2\chi(\Sigma).$$

Definition 10. *A* discrete measured foliation *on a filling graph $\Gamma \subset \Sigma$ is a closed length structure on Γ, together with, for each face f of Γ, a choice of metric tree T_f and an onto map $\phi_f \colon \partial f \to T_f$ so that between adjacent $\times$'s on ∂f, the map ϕ_f is an isometry. If f is non-singular, then T_f is necessarily an interval. If f has index 1, then T_f is a tripod, with uniquely determined lengths. In general, the triangle inequalities guarantee there is at least one valid tree T_f. A discrete measured foliation canonically determines a partial measured foliation on Σ.*

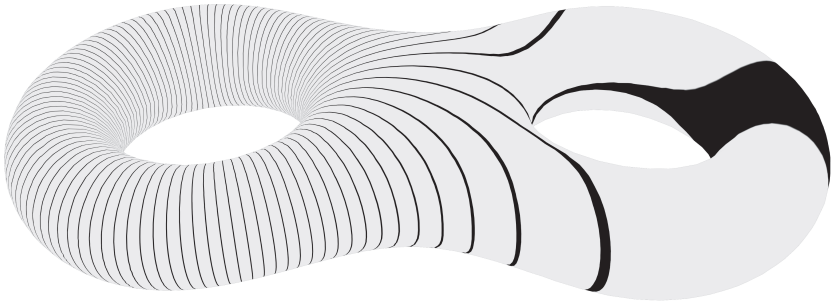


FIGURE 1. A harmonic measured foliations computed by this method, starting from a partial measured foliation supported in a neighborhood of a curve on the left of the surface.

Definition 11. *The* energy *of a length structure* (ℓ, m) *on* (Γ, k) *is*

$$E(\ell) \coloneqq \sum_e k(e)\ell(e)^2.$$

Proposition 12. *If* $\Gamma \subset \Sigma$ *is a filling elastic graph, then every Whitehead class of measured foliations on* Σ *is represented by an essentially unique harmonic measured foliation* (ℓ, m) *on* Γ. *Furthermore,* (ℓ, m) *is a global minimum for* E *within the Whitehead class.*

Here *essentially unique* means that the lengths are the same, and that the markings differ by only changing the markings in inessential ways without changing the lengths at all. Furthermore, there is a local update algorithm for finding the harmonic representative Proposition 12.

There are two potential types of applications for this model:

- Γ could be as coarse as possible, with very few edges, to give (e.g.) an efficient coding for the space of measured foliations.
- Γ could be fine, with the goal of approximating well the actual harmonic measured foliations on an underlying conformal surface.

For the second type of application, there is a long history of discrete approximations to harmonic functions or 1-forms on planar domains or Riemann surface Σ. In particular, if Σ is approximated by a triangulation T into Euclidean triangles with acute angles, an arbitrary function on Σ can be approximated by a simplex-wise linear function g. After a short calculation [1, 2], one finds that

$$\nabla^2(g) = \sum_{e \text{ edge of } T} \frac{\cot(\alpha) + \cot(\beta)}{2}\big(g(v) - g(w)\big), \tag{1}$$

where v and w are the two endpoints of e and α and β are the two angles opposite from e in the adjacent triangles. The same weights may be used to approximate harmonic measured foliations.

David Palmer, in a forthcoming thesis supervised by Steven Gortler, refined and implemented the above algorithm for triangulations with cotangent weights. Figure 1 shows an example of the output from his program.

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Combinatorial formulas for the characteristic classes of triangulated S^1-fibre bundles

Georgy Sharygin

As one knows, characteristic classes of a smooth vector bundles over manifolds can be determined either with the help of connections on these manifolds or as the pullbacks of cohomology classes of universal classifying spaces; the latter method is more functorial, but difficult from the point of view of computations. Given a smooth oriented manifold M, it has a natural vector bundle associated with it, i.e. its tangent bundle. Then one defines Pontryagin classes of M as characteristic classes of this vector bundle. These classes $p_k(M)$ lie in the integer cohomology $H^{4k}(M, \mathbb{Z})$.

Defined in this way, these classes ostensibly depend on the smooth structure, which is necessary to determine the tangent bundle. However, in 1958 R.Thom showed (see [1]) that rational Pontrjagin classes (i.e. images of $p_k(M)$ under the natural change of coefficients map $H^*(M, \mathbb{Z}) \to H^*(M, \mathbb{Q})$) of a triangulated manifold can be determined only from combinatoric data, encoded by the triangulation. Later, in a paper published in 1978 (see [2]) N.Levitt and C.P. Rourke showed, that there exist local formulas, that represent simplicial cocycles, corresponding to the rartional classes, in terms of the links of simplices in this triangulation. That is a class $p_k(M)$ can be represented by a simplicial cocycle

$$\mathcal{P}_k(M) = \sum_{\sigma^{4k} \in M} f_k(lk_M \sigma^{4k}) \delta_{\sigma^{4k}},$$

where σ^{4k} is a $4k$-dimensional simplex in the triangulation of M, $lk_M \sigma^{4k}$ is its link in the triangulation, and $\delta_{\sigma^{4k}}$ is the corresponding dual δ-cochain. Functions f_k are maps from the set of all triangulations of spheres of suitable dimensions into rational numbers (in the cited paper the authors deal with the Poincaré duals of Pontryagin classes in homology). However, these results are pure existence theorems, and the problem to find explicit expressions for such classes is still open (one should mention papers by Gelfand, Gabrielov, Losik, [3, 4], Gelfand and MacPherson, [5], and Gaifullin, [6], dealing with this question, in which various approach to it are used).

In my talk I am going to describe a solution of a similar, but simpler problem: suppose we have a triangulated principal S^1-bundle $E \to B$, i.e. a pair of triangulations: one of the base B and another of the total space E, and a simplicial map between them, such that geometric realization of this triad is homeomorphic to the bundle. I shall give explicit local formulas to express powers of its Chern class

in terms of this triangulation; similar formulas in the context of singularity theory were given by Kazarian, [7]. One can use this result to evaluate the number of simplices, necessary to triangulate a base of an S^1-bundle so that there will exist a triangulation of the total space, that agrees with it. The talk is based on a joint work with Nikolai Mnëv (POMI).

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Line Bundles in Geometry Processing

Keenan Crane

(joint work with Felix Knöppel, Ulrich Pinkall, and Peter Schröder)

Geometry processing is the natural extension of digital signal processing to geometric data, such as simplicial surfaces in R^3. However, it lacks many of the basic tools used in traditional signal processing (e.g., the fast Fourier transform) due to the fact that (i) geometric data is often irregularly sampled, and (ii) the metric is no longer homogeneous. We therefore seek to replace traditional signal processing tools with more flexible counterparts adapted from differential geometry; here we examine several problems which can be formulated in terms of discrete line bundles. In particular, the solution to each computational problem is obtained by computing the ground state of the associated Schrödinger operator.

Consider a triangulated manifold of any dimension with vertices V, edges E and triangles F (for our discussion, we will not need to refer to higher-dimensional simplices). A discrete line bundle consists of a one-dimensional complex vector space at each vertex, together with the following data:

(1) unit complex numbers r_{ij} for each oriented edge (i, j) that are antisymmetric in the sense that $r_{ji} = r_{ij}^{-1}$, and

(2) real values Ω_{ijk} for each triangle such that $r_{ki} r_{jk} r_{ij} = e^{\imath \Omega_{ijk}}$.

The values r_{ij} correspond to the connection 1-form; the values Ω_{ijk} correspond to the curvature 2-form. Unlike a smooth vector bundle, both pieces of data are needed in order to resolve the curvature ambiguity inherent in the complex representation of parallel transport.

This data can be used to measure the variation of a given section $\psi : V \to \mathbb{C}$ along any edge (i, j). In particular, the section is parallel if $r_{ij}\psi_i = \psi_j$. Therefore, the overall regularity of the section can be measured via a *Dirichlet energy*

$$E_D(\psi) := \sum_{ij \in E} w_{ij} |r_{ij}\psi_i - \psi_j|^2$$

where $w_{ij} \in \mathbb{R}$ are edge weights (on surfaces one might use the usual *cotangent weights*). This energy can be encoded via a matrix $A \in \mathbb{C}^{|V| \times |V|}$ which can be viewed as a discrete magnetic Schrödinger operator. We can likewise define a norm on sections given by

$$||\psi||^2 := \sum_{i \in V} m_i |\psi|_i^2,$$

where the values $m_i \in \mathbb{R}$ are positive weights, often corresponding to the volume of a cell associated with vertex i (e.g., one third the area of incident triangles). If we encode this norm by a matrix $M \in \mathbb{C}^{|V| \times |V|}$, then the ground state can be obtained by solving the sparse eigenvalue problem

$$A\psi = \lambda M \psi$$

for the section ψ associated with the smallest eigenvalue λ; in practice, this problem can be solved efficiently, making it attractive for geometry processing tasks involving large datasets. Notably, this machinery can easily be adapted to irregular meshes with variable curvature.

In this framework, different choices of line bundle naturally arise from the data available in different geometry processing tasks. For instance, when seeking the smoothest vector field—or more generally, *n-direction field*—on a surface, one simply uses the Levi-Civita connection [1]; when seeking a global continuous surface parameterization aligned with a given direction field X, one uses the corresponding 1-form $\langle X, \cdot \rangle$ [3]; when seeking a decomposition of fluid motion into vortex rings, the connection is induced by the fluid vorticity [2]. One can also use this machinery to compute constrained volume deformations in $\mathbb{R}^3$ that are close to conformal [4]. Further details can be found in [5].

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Invariants of Random Knots and Links

Joel Hass
(joint work with Chaim Even-Zohar, Joel Hass, Nati Linial, Tahl Nowik)

A *petal diagram* is a planar curve, comprised of $2n+1$ straight segments crossing at a single point, and arcs connecting consecutive pairs of segment tips. This representation is *universal*, so that all knots are realized by some petal diagram [1, Theorem 1]. A similar statement applies for links. Figure 1 shows petal diagrams for two knots and for a 2-component link.

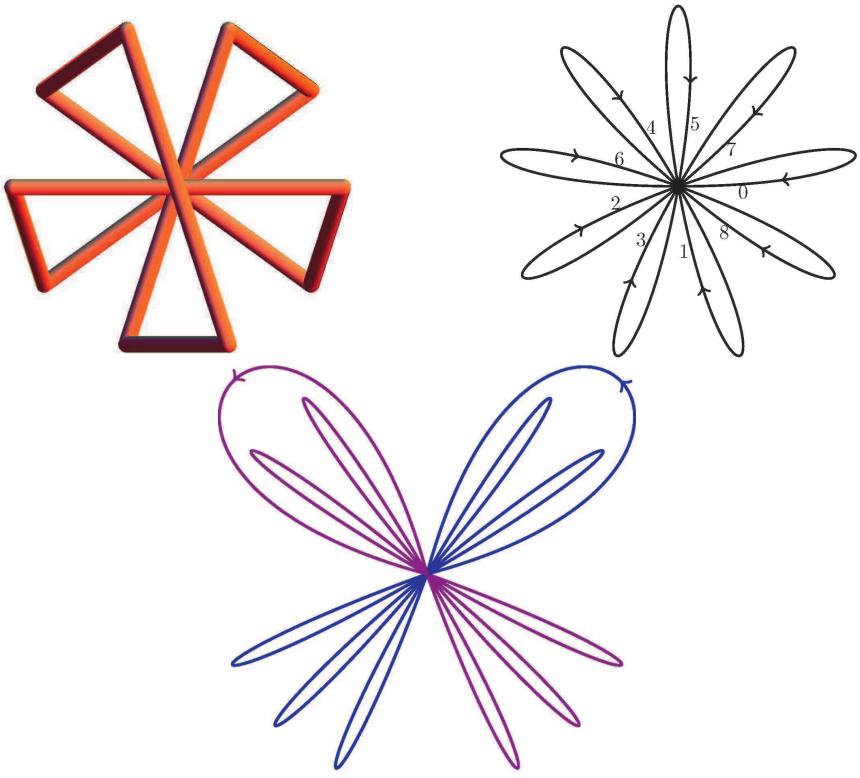

Figure 1. Petal diagrams of knots with 5 and 9 petals and a 2-component link with 12 petals. In the 9 petal diagram, the heights of the arcs are marked. The heights correspond to a permutation in S_9.

We introduce a new model for random knots and links in $\mathbb{R}^3$, called the *Petaluma model*. In this model a permutation determines a knot or a link, and every knot and link arises from some permutation. Our main results give formulas for the

distribution of the linking number of a random two-component link and for the moments of the Casson invariant and the order-3 knot invariant v_3.

Along with its projection, a petal diagram comes with information on how to construct a knot in $\mathbb{R}^3$ that projects to the diagram. The additional information specifies the height of the arcs passing above the single crossing. The ordering of these heights is specified by a permutation $\pi \in S_{2n+1}$, with $\pi(i)$ giving the knot's height as it passes over the center for the ith time. Each permutation determines a knot, and in the *Petaluma model* we define a random knot $K_{2n+1}(\pi)$ to be a knot with a $2n+1$ petal diagram and permutation $\pi \in S_{2n+1}$, drawn uniformly at random.

This construction extends to links. A *two-component link petal diagram* consists of two planar curves, each of which transversely passes $2n$ times through a single point, as shown in Figure 1 for $n = 3$. A two-component link is uniquely determined by a permutation $\pi \in S_{4n}$. The strands of the first component pass above the crossing at heights $\pi(1), \ldots, \pi(2n)$ and the strands of the second at heights $\pi(2n+1), \ldots, \pi(4n)$. This gives a universal model for two-component links [1, Theorem 2], and the Petaluma model for a random two-component link $L_{4n}(\pi)$, is obtained by drawing π uniformly at random from S_{4n}. This model can be adjusted to allow for unequal numbers of petals in the two components, or a higher number of components.

We study the behavior of finite type invariants of knots and links in the Petaluma model. A knot invariant is viewed as a random variable on the set of all diagrams with $2n+1$ petals, and we ask for its distribution and for its asymptotic growth as $n \to \infty$.

The kth *moment* of a random variable X is the expected value $E[X^k]$. The moments of an invariant indicate its value on a randomly sampled knot or link. To understand the distribution of an invariant as $n \to \infty$ we must determine how to normalize it as n grows. The following theorems give the order of growth of the finite type invariants lk, c_2, and v_3, corresponding to the linking number, Casson invariant (second coefficient of the Conway polynomial) and the third coefficient of the modified Jones polynomial ($J(e^t)$).

Theorem 1. *$E[(lk(L_{4n}))^k]$ is a polynomial in n of degree $\leq k$.*

Theorem 2. *$E[(c_2(K_{2n+1}))^k]$ is a polynomial in n of degree $2k$.*

Theorem 3. *$E[(v_3(K_{2n+1}))^k]$ is a polynomial in n of degree $\leq 3k$.*

Remark. In Theorems 1 and 3 there is equality for k even, while the odd moments are 0.

We also determine the leading term of the $E[c_2^k]$ polynomial. This yields the limits of the moments of the normalized invariant c_2/n^2. For $k = 1, 2, 3$ we find $\lim_{n\to\infty} E[c_2^k/n^{2k}] = 1/24$, $7/960$, and $5119/2419200$ respectively. Similarly we obtain the limiting variance $\lim_{n\to\infty} E[v_3^2/n^6] = 4649/2721600$.

In the case of the linking number of two-component links, we can exactly describe the limiting distribution of the normalized first order invariant $lk(L_{4n})/n$

as $n \to \infty$. Theorem 4 gives the first explicit description of the asymptotic probability distribution for any knot or link invariant.

Theorem 4. *The limiting probability distribution of $lk(L_{4n})/4n$ as $n \to \infty$ is given by*

$$P\left[\alpha < \frac{lk(L_{4n})}{4n} < \beta\right] \to \int_{\alpha}^{\beta} \frac{\pi}{\cosh^2(2\pi x)} dx \;=\; \frac{\tanh(2\pi\beta) - \tanh(2\pi\alpha)}{2} \quad .$$

The expression for the linking number turns out to be the same as an expression found in a problem studied by physicists, the flux through a random curve in the plane. Our proof of Theorems 1 and 4 is an adaptation and simplification of Mingo and Nica's derivation of this flux [2].

Theorem 2 shows that $c_2(K_{2n+1})$ typically grows as n^2, while Theorem 3 show that $v_3(K_{2n+1})$ grows as n^3. In combination with Theorem 4, these results suggest the following conjecture:

Conjecture 5. *Let v_m be a knot invariant of order $m > 0$. Then $v_m(K_{2n+1})/n^m$ weakly converges to a limit distribution as $n \to \infty$.*

The analysis of knot invariants in this paper is part of a project to apply the probabilistic method in topology. This methodology begins by defining a probability distribution on the objects of study. Parameters and invariants of interest then become random variables on this probability space. Tools of probability theory are then applied to investigate the distribution of these random variables. This general approach has yielded many unexpected results, often providing existence proofs of objects with unexpected properties. In many cases existence of such objects can be established using probabilistic methods, yet finding an explicit construction remains open. Our focus here is to bring this approach to random knots and links, and to associated knot and link invariants.

It is important to consider to what extent our results are model dependent, and to investigate what might happen if we switch to a different model. To test the extent of model dependency of our statistics, we ran numerical studies on the distribution of the c_2 and v_3 invariants in a other random models. Our numerical experiments yield distributions for c_2 that share many features with the distribution obtained for the Petaluma model. It remains unclear how the choice of a random model determines the statistics of a knot and link invariant, and which universality principles apply across a wide range of models.

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Approximation algorithms for topological graph properties

Anastasios Sidiropoulos

Topological graph theory has given rise to fundamental algorithms for computing several graph invariants. Some of the central problems in this area concern the computation of crossing number, graph genus, vertex/edge deletion number, and various other minor-closed properties. Many of these algorithms build upon and expand the seminal work of Robertson and Seymour on graph minors. Consequently many of these algorithms are applicable only on the exact setting. Since computing most topological graph invariants of interest is in general NP-hard, the running time of these exact algorithms has exponential dependence on the topological parameter in question.

This talk will present some recent efforts to extend the above results to the setting of approximation algorithms, thus removing the exponential dependence of the running time on the topological parameter. More precisely, we obtain algorithms for approximating the orientable and non-orientable genus and the crossing number of bounded degree graphs. We also extend our approximation algorithm for non-orientable genus to the case of general graphs. Finally, we obtain a poly-logarithmic approximation for minimum vertex deletion. This is the first approximation algorithm with a poly-logarithmic guarantee for any topological property of this kind.

Persistent homology and Hilbert spaces

Peter Bubenik

Persistent homology is the central technique for an approach to using algebraic topology to analyze data. To start, I will give an high-level overview of this method. Next I will discuss a number of ways in which this may be generalized. For the main part of the talk, I will discuss the fusion of topological data analysis with statistics and machine learning, and how moving to a Hilbert spaces can help one to combine these approaches. To end, I will give some biological applications.

1. Persistent homology

The standard pipeline in topological data analysis may be represented by the following steps.

(1) Raw data is obtained as the output of some experiment or observation.
(2) This is preprocessed and converted to a form amenable to mathematical analysis. For example, points in Euclidean space.
(3) From this nice data, a geometric construction is applied to obtain a *filtered complex*.
(4) Then one applies homology with coefficients in a field to obtain a *persistence module*.
(5) From this one extracts a *topological summary*.
(6) Finally, we analyze our summary to learn something from our data.

For now, we will consider steps 3, 4 and 5. We will consider 6 in Section 3. The filtered complex, K, has the form

$$K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots \subseteq K_n,$$

where each K_j is a (simplicial) complex and each map is an inclusion. To this we apply the functor $H_i(-, \mathbf{k})$, to obtain the persistence module, V, which has the form

$$V_1 \subseteq V_2 \subseteq V_3 \subseteq \cdots \subseteq V_n,$$

where each V_i is a vector space over $\mathbf{k}$ and each map is linear.

Each of the following points of view of V is fruitful.

- V is a graded $\mathbf{k}[x]$-module.
- V is a representation of the quiver (i.e. directed graph), $\vec{A}_n$, given by $1 \to 2 \to 3 \to \cdots \to n$.
- V is a functor from $\vec{A}_n$ to **vect**, the category of finite dimensional vector spaces.
- V is a module over $\mathbf{k}\vec{A}_n$ the algebra of paths in $\vec{A}_n$.

The following theorem from the representation theory of quivers is fundamental.

Theorem 1 (Gabriel, 1972)**.** *V is isomorphic to a direct sum of* interval modules*, $I_{[b,d)}$, given by $0 \to 0 \to \cdots 0 \to \mathbf{k} \to \mathbf{k} \cdots \to \mathbf{k} \to 0 \to \cdots \to 0$, where the first $\mathbf{k}$ is in position b, the last $\mathbf{k}$ is in position $d-1$, and each of the maps $\mathbf{k} \to \mathbf{k}$ is the identity map.*

As a result, V may be completely described by the multiset of intervals $\{[b_i, d_i)\}$, called a *barcode*, or the corresponding multiset of ordered points, $\{(b_i, d_i)\}$, called a *persistence diagram*.

2. Generalizations

There are a number of directions in which we may generalize the setup in the previous section.

- Replace $\vec{A}_n$ with other quivers. For example, $\vec{A}_m \times \vec{A}_n$.
- Replace $\vec{A}_n$ with another poset, preordered set, or small category. For example, $(\mathbb{R}, \leq)$, or $(\mathbb{R}^2, \leq)$.
- Replace $H_i(-, \mathbf{k})$ with other functors. For example, cohomology, some other generalized homology theory, or rational homotopy groups.
- Replace **vect** with some other abelian category. For example, $\mathbf{R} - \mathbf{mod}$, or $\mathbf{Shv}(\mathbf{X})$.

However, there are difficulties not apparent in ordinary persistence; in almost no other cases is there a persistence diagram that we can fully understand.

3. Hilbert space

To combine topological data analysis with statistics and machine learning we would like to test hypotheses, average topological summaries, understand their variance, calculate correlations, perform exploratory data analysis, cluster, and classify our

data. Some of these are quite hard to do directly with persistence diagrams. One solution is to map persistence diagrams to a Hilbert space. Such a map is called a *feature map*. Examples include the rank function, the stable multi-scale kernel [3], and the persistence-weighted Gaussian kernel. Another feature map is the *persistence landscape*, $\lambda : D \to L^2(\mathbb{N} \times \mathbb{R})$, [1]. It has the following advantages:

- it is stable;
- it is a piecewise-linear function;
- if P is the number of critical points of this PL function, then it can be calculated in $O(P)$;
- one can recover the persistence diagram from the persistence landscape; and
- it is easy to average: $\bar{\lambda}(k,t) = \frac{1}{n} \sum_{i=1}^{n} \lambda^{(i)}(k,t)$.

4. Applications

In this talk we show an application to protein data. We have the three dimensional structure of the maltose binding protein in various shapes/conformations. These were obtained by x-ray crystallography and are publicly available in the protein data bank. Using persistence landscapes, we are able to distinguish between the 'open' and 'closed' conformations of the maltose binding protein. Applying support vector machines, we find a separating hyperplane for homology in degrees 0, 1 and 2. For more details see [2].

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Classifying the phase behaviour of lipid formulations using persistent homology

Corrie Jacobien Carstens

(joint work with Dallas Warren, Craig Westerland)

There is a growing number of drugs that are poorly soluble in water. Such drugs are typically inefficiently absorbed by the human body. However, by dissolving these drugs in inactive substances (known as excipients) their bioavailability can be improved. Lipid formulations generally consist of a drug dissolved in two or more excipients [4]. It is the structure formed by these excipients that ensures the drug is in a dissolved state. The effectiveness of a lipid formulation for drug-delivery purposes depends on its behaviour when mixing with the aqueous environment in

the gut. This behaviour can be predicted using molecular dynamics simulations with increasing amounts of water.

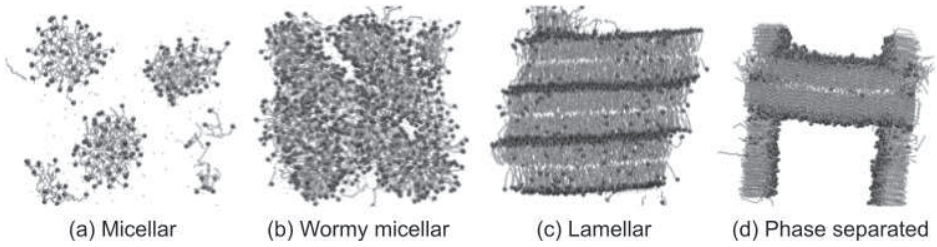

FIGURE 1. Four distinct types of phase behaviour that emerge when simulating the movement of excipients in water (not shown).

The behaviour of a lipid formulation can be classified into different phases (see Figure 1). This is done by examining the geometrical structure formed by excipients. Currently, the results of molecular dynamics simulations are inspected by eye and classified into different phases manually. This process is labour intensive and would benefit from more quantitative methods. We have been working on a method that uses persistent homology for the classification of the phase behaviour of lipid formulations.

Persistent homology [3] is a mathematical framework that can be used to quantify the shape of point clouds [2]. We initially use persistent homology as an exploratory data analysis tool. In order to do this we wrote a graphical user interface for JavaPlex. This software allows us to visualize 3-dimensional point-clouds and their associated Vietoris-Rips complex as well as the corresponding persistence barcode or diagram. We also use it to highlight the generators of 1-dimensional homology classes. We decided to use persistent homology to compare the geometrical and topological structure of a collection of lipid formulations. We did not attempt to describe each lipid formulation separately. Instead we cluster the lipid formulations using the landscape distances [1] between a collection of persistence barcodes that can be associated to them. Our initial results indicate that persistent homology can distinguish fairly well between different phase behaviours.

Ultimately our goal is to automatically label lipid formulations. This will assist practitioners in assigning the correct phase to each system. Understanding the phase behaviour in turn helps recognize under which conditions the drugs are likely to be absorbed best.

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Topological Data Analysis with Bregman Divergences

Hubert Wagner
(joint work with Herbert Edelsbrunner)

1. Introduction

Topological Data Analysis (TDA) brings the methods of *computational geometry and topology* into the practice of data analysis. In particular, these methods are used to gain insight into high-dimensional point-cloud data. We extend the theoretical and algorithmic framework of TDA to the setting of point-cloud data measured with Bregman divergences, which are commonly used in practical applications.

In TDA, the first step is to build combinatorial representation of data, using a Čech, Vietoris-Rips, or Delaunay (alpha) simplicial complex. These constructions capture the homotopy type of the union of balls centered around the input points. With *persistence* one studies the changes of homology groups as the radius of balls grows from zero to infinity. This gives a rich geometric-topological summary of the data, called the *persistence diagram* or *barcode* [3].

These constructions are typically used with the Euclidean distance. We demonstrate that TDA can be applied to data measured with arbitrary Bregman divergences, which are generally not metrics. They are commonly used in information retrieval, speech recognition, statistics, and other disciplines.

2. Bregman divergences

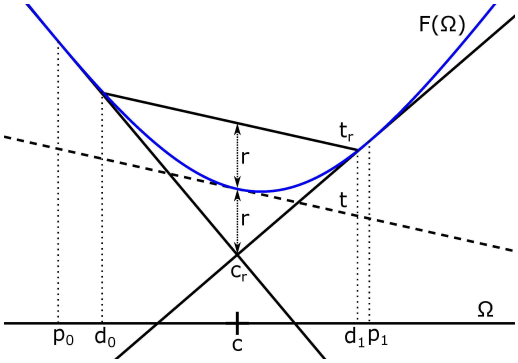
In applications, point-cloud data are often studied with dissimilarity measures that are not metrics. Bregman divergences form a family of such measures with some members commonly used in practice [4]. Each Bregman divergence is parametrized by a function F. Let $\Omega \subseteq \mathbb{R}^n$ be convex and $F\colon \Omega \to \mathbb{R}$ differentiable and strictly convex. For two points $x, y \in \Omega$, the *Bregman divergence* from x to y associated with F is the difference between the value of F at x and the first-order Taylor expansion of F around y evaluated at x [2]:

$$D_F(x\|y) = F(x) - [F(y) + \langle \nabla F(y), x - y\rangle]\,;$$

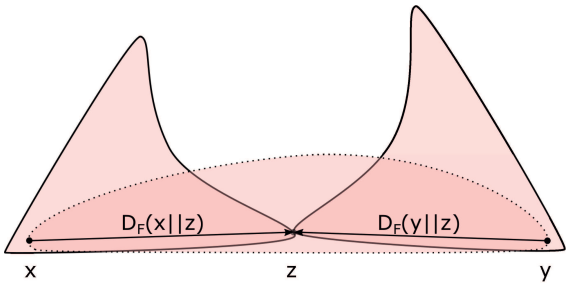
Since $D_F(x\|y)$ is not necessarily symmetric in its two arguments, we define two types of *Bregman balls*, *primal* and *dual*:

$$B_F(c;r) = \{x \in \Omega \mid D_F(c\|x) \leq r\},$$
$$B_F^{'}(c;r) = \{x \in \Omega \mid D_F(x\|c) \leq r\}.$$

The *Bregman-Voronoi domains*, $\mathbb{V}_F(x)$ and $\mathbb{V}'_F(x)$, have analogous definitions.



(A) Primal ball, $B_F(c;r) = [p_0, p_1]$, is determined by the subset of $F(\Omega)$ *illuminated* by light shining from point c_r; Dual ball, $B'_F(c;r) = [d_0, d_1]$, is *cut away* by plane t_r parallel to tangent plane t. Note $r = D_F(c\|p_0) = D_F(c\|p_1) = D_F(d_0\|c) = D_F(d_1\|c)$.



(B) Two *primal* Bregman balls, $B_F(x;r)$, $B_F(y;r)$, of equal radius intersect if their centers are contained in a *dual* Bregman ball of the same radius, $B'_F(c;r)$. We use the Itakura-Saito divergence; note the non-convexity of primal balls.

FIGURE 1. Properties of Bregman balls.

	$F(x)$	$D_F(x\|y)$	applicable
Kullback-Leibler	$\sum_i x_i \log(x_i)$	$\sum_i x_i \log(x_i/y_i)$	texts, images
Itakura-Saito	$-\sum_i \log(x_i)$	$\sum_i (x_i/y_i - \log(x_i/y_i) - 1)$	speech
Sq. Mahalanobis	$\frac{1}{2}x^T Q x$	$\frac{1}{2}(x-y)^T Q (x-y)$	statistics
Sq. Euclidean	$\|x\|^2$	$\|x-y\|^2$	real world

TABLE 1. Commonly used Bregman divergences.

See Figure 1a for geometric intuition and Table 1 for typical examples. Note that Bregman balls live in the input space, Ω. We stress that dual balls are convex by construction. Primal balls need not be convex.

3. Proximity complexes

Given a finite point-cloud, $X \subset \Omega$, we define three constructions in the Bregman setting, namely Čech, Delaunay, and Vietoris-Rips simplicial complexes:

$$\check{\text{C}}\text{ech}_F(X;r) = \{S \subseteq X \mid \bigcap_{x \in S} B_F(x;r) \neq \emptyset\},$$

$$\text{Del}_F(X;r) = \{S \subseteq X \mid \bigcap_{x \in S} [B_F(x;r) \cap \mathbb{V}_F(x)] \neq \emptyset\}.$$

A Vietoris-Rips complex approximates a Čech complex. $\mathrm{Rips}_F(X;r)$ is the flag complex whose 1-skeleton coincides with the 1-skeleton of $\check{\mathrm{C}}\mathrm{ech}_F(X;r)$. In other words, we put a $(k+1)$-simplex for each k-tuple of Bregman balls with *pairwise* nonempty intersection.

To see that the Bregman-Čech and Deluanay complexes have the homotopy type of the union of primal Bregman balls, we apply the Nerve Theorem [1] to the cover of the union formed by the balls. We can do this because the common intersection of any collection of primal balls is either empty of contractible. This property lies at the heart of our approach and can be proved using the conjugate function obtained from F with the Legendre transform. The same idea works if we clip the primal Bregman balls with the corresponding Voronoi domains.

4. Algorithms

We show a simple algorithm transforming a (Bregman) Vietoris-Rips complex into a (Bregman) Čech complex:

```
def cech_complex(points, divergence, max_r, max_d):
   complx = rips_complex(points, divergence, max_r, max_d)
   for s in complx.simplices():
      r = min_enclosing_dual_ball_radius(s.points, divergence)
      if r > max_r:
         complx.remove_simplex(s)
      else:
         s.filtration_value = r
   return complx
```

It exploits the following duality: A tuple of primal Bregman balls of radius r have nonempty intersection iff their centers are contained in a *dual* Bregman ball of the same radius. See Figure 1b for an illustration. Algorithms for smallest enclosing Bregman balls have been studied [5], giving hope for efficient implementations.

5. Discussion

In the full version of the paper, the theory sketched here is developed in details. TDA for data measured with Bregman divergences opens many interesting applications, but there remain open questions in this setting:

- Stability of persistent homology.
- Computing sparsified complexes.

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Encoding Surface Maps

Mark C. Bell

Let S be a closed, orientable surface of genus $g > 0$ with $n > 0$ marked points. We will look at an efficient technique for combinatorially representing mapping classes $h \in \mathrm{Mod}(S)$.

This representation is based off of the various ways of triangulating S using $\zeta = 6g+3n-6$ *arcs* and the marked points as *vertices*. We arrange these triangulations into the *flip graph* $G(S)$ based on their similarity. That is, $G(S)$ has a vertex for each triangulation of S and two triangulations are adjacent, written $\mathcal{T} - \mathcal{T}'$, if and only if they share $\zeta - 1$ arcs.

Theorem 1 ([4, Page 190], [5, Page 36])**.** *The flip graph $G(S)$ is connected.*

Now the mapping class group $\mathrm{Mod}(S)$ acts geometrically on $G(S)$. Thus the flip graph gives a combinatorial model for $\mathrm{Mod}(S)$ where we represent the mapping class h by a path from $\mathcal{T}$ to $h(\mathcal{T})$ [2, Section 2.2]. This can always be done by Theorem 1. This representation allows us to efficiently determine the images of simple closed curves under mapping classes and has been implemented as part of `flipper` [1].

Of particular interest is the *diameter* of the finite quotient

$$\mathcal{M}(S) = G(S)/\,\mathrm{Mod}(S)$$

[6] [3]. For certain surfaces we can use `flipper` to explicitly compute $\mathcal{M}(S)$, for example see Figure 1. For these surfaces we can then compute $\mathrm{diam}(\mathcal{M}(S))$ exactly. See Table 2.

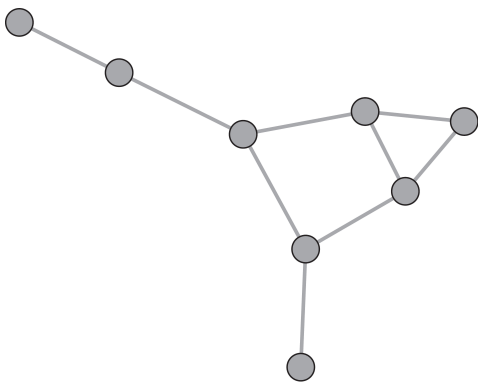

Figure 1. The graph $\mathcal{M}(S)$ when $g = 2$ and $n = 1$.

		vertices					
		1	2	3	4	5	6
genus	1	1_0	5_4	40_9	450_{14}	6370_{19}	104498_{24}
	2	8_4	232_{13}	7271_{19}	224055_{24}		
	3	927_{12}	95221_{21}				
	4	676445_{20}					

TABLE 2. $|\mathcal{M}(S)|_{\mathrm{diam}(\mathcal{M}(S))}$ for various surfaces.

As these graphs become extremely large as g and n grow, it quickly becomes impractical to construct them. The second half of this talk will be devoted to an entirely local technique for producing these paths for a *Dehn twist* D_γ where γ is a simple closed curve.

The idea of this technique is based off of the fact that if γ meets $\mathcal{T}$ in only two places then $\mathcal{T} - D_\gamma(\mathcal{T})$ and it is straightforward to find a path between them. We will discuss the following pair of lemmas that allow us to reduce a general curve back to this specific case.

Lemma 1. *Suppose that $\mathcal{T}$ is a triangulation and γ is a curve. If $\iota(\gamma, \alpha) > 2$ for some arc $\alpha \in \mathcal{T}$ then we have that $\mathcal{T} - \mathcal{T}'$ where $\iota(\gamma, \mathcal{T}') < \iota(\gamma, \mathcal{T})$.*

Lemma 2. *Suppose that $\mathcal{T}$ is a triangulation and γ is a non-separating curve. If $\iota(\gamma, \mathcal{T}) > 2$ then we have that $\mathcal{T} - \mathcal{T}' - \mathcal{T}''$ where $\iota(\gamma, \mathcal{T}'') < \iota(\gamma, \mathcal{T})$.*

These lemmas, and so the ability to perform Dehn twists along arbitrary curves, have also been implemented as part of `flipper`.

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Eliminating Multiple Intersections of Maps: Codimension 2

ULI WAGNER

(joint work with Sergey Avvakumov, Isaac Mabillard, Arkadiy Skopenkov)

Motivated by *Tverberg-type problems* in topological combinatorics and generalizing classical results about *embeddings* (maps without double points), we study conditions under which a finite simplicial complex K can be mapped to d-dimensional Euclidean space $\mathbb{R}^d$ without higher-multiplicity intersections.

Let K be a finite simplicial complex, and let $r \geq 2$ and $d \geq 1$. A map $f\colon K \to \mathbb{R}^d$ is an *almost r-embedding* if $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint simplices of K.[1,2]

The long-standing *topological Tverberg conjecture*, raised by Bajmoczy and Bárány [BB] and Tverberg [GS, Problem 84] asserts that for $N = (d+1)(r-1)$, the N-dimensional simplex σ^N does not admit an almost r-embedding in $\mathbb{R}^d$. This was proved in the case that r is a prime [BB, BShSz] or a prime power [Öz] in the 1980's, but the case of arbitrary r remained open and was considered a central problem in topological combinatorics.

Recently, we developed an approach to constructing counterexamples to the conjecture whenever r is not a prime power [MW14, MW]. Building on this approach, the first counterexamples were constructed by Frick [Fr] for $d \geq 3r+1$; a different construction for $d \geq 3r$ was given in [MW].

Here, we improve this as follows:

Theorem 1. *If r is not a prime power and $d \geq 2r$, then there is a PL almost r-embedding of the $(d+1)(r-1)$-simplex in $\mathbb{R}^d$.*

The smallest counterexample to the topological Tverberg conjecture obtained in this way is an almost 6-embedding of the 65-dimensional simplex in $\mathbb{R}^{12}$.

There is a well-known necessary condition for the existence of an almost r-embedding: Let $(K)^r_\Delta$ denote the *deleted product* of K, i.e., the subcomplex of the Cartesian product K^r whose cells are the products of r *pairwise disjoint simplices* of K.

Lemma 1. *If there exists an almost r-embedding $f : K \to \mathbb{R}^d$ then there exists an equivariant map*[3] $F\colon (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

In the case that r is a prime or a prime power, the topological Tverberg conjecture was proved by showing that there is no equivariant map $(\sigma^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ [BB, BShSz, Öz]. However, in the case that r is *not a prime power*, Özaydin [Öz] showed that there exists an equivariant map $(K) \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ whenever $\dim(K)^r_\Delta \leq d(r-1)$, in particular for $K = \sigma^N$.

Motivated by Özaydin's result, we proved in [MW14, MW] that the necessary condition in Lemma 1 is also sufficient for the existence of an almost r-embedding, provided $\dim K = (r-1)k$ and $d = rk$ for some $k \geq 3$ and $r \geq 2$ (we call k the *codimension*). Our proof of Theorem 1 is based on the following extension (for $r \geq 3$) of this criterion to *codimension* 2:

Theorem 2. *Let $r \geq 3$, and let K be a finite $2(r-1)$-dimensional simplicial complex. Then the following are equivalent:*

[1]We stress that this definition depends on the simplicial complex, i.e., a specified triangulation of the underlying polyhedron.

[2]Any sufficiently small perturbation of an almost r-embedding is again an almost r-embedding; thus, without loss of generality, it suffices to consider maps that are piecewise-linear (PL) and in general position.

[3]Here, $S^{d(r-1)-1} = \{(y_1, \ldots, y_r) \in (\mathbb{R}^d)^r \mid \sum_{i=1}^r y_i = 0, \sum_{i=1}^r \|y_i\|_2^2 = 1\}$, and the symmetric group $\mathfrak{S}_r$ acts on both spaces by permuting components.

(i) *There exists a PL almost r-embedding $f\colon K \to \mathbb{R}^{2r}$.*
(ii) *There exists a PL map $g\colon K \to \mathbb{R}^{2r}$ in general position such that the algebraic intersection number of the g-images of any r pairwise disjoint simplices of K is zero.*
(iii) *There exists an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{2r(r-1)-1}$.*

Remark 1. It follows from work of Freedman, Krushkal, and Teichner [FKT, KT] that the analogous criterion for $r = 2$ is false (more precisely, the implication (ii) $\Rightarrow$ (i) fails).

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Software Sessions

Marc Culler – SnapPy. SnapPy is a software suite designed for studying hyperbolic 3-manifolds. The computational core of SnapPy is a C library originally written by Jeff Weeks for use in his Macintosh application named SnapPea, which ran on OS 8 and 9. Weeks' original design separated the graphical user interface (written in ThinkC) from the computational kernel (written in ANSI C). This design choice made it possible for Nathan Dunfield, while he was a graduate student at the University of Chicago, to use SWIG to wrap the SnapPea kernel as a python module. Having access to the SnapPea kernel from python enabled automated experimentation over large families of manifolds. In its current form SnapPy provides access to the kernel (subsequently extended by Marc Culler and Nathan Dunfield with important contributions from Mark Bell, Mathias Goerner and Saul Schleimer) from python, or from Sage, with enhanced interaction with

Sage's data types, or as a standalone graphical application which runs on OS X, Windows and linux systems. The Cython language is used to construct python extension modules. Several other components have been written in python to support more flexible interaction with the SnapPea kernel. These include PLink, an interactive editor for link diagrams which can export the complement of the link as a SnapPea manifold; Spherogram, which provides algorithms for generating tangles and link diagrams formally and computing knot or link invariants; CyPari, which is a standalone version of Sage's Pari module, usable from standard python; and Twister which provides tools for constructing and studying surface bundles over the circle and surface mapping classes. Computational capabilities of the kernel have been significantly expanded by providing an option to extend the floating point arithmetic to quad-double precision. SnapPy can be used in rigorous mathematics, thanks to work of Goerner's. While these capabilities will be expanded in the future, SnapPy is now able to provide a mathematically rigorous proof that the topological manifold given by an ideal trianglulation admits a hyperbolic structure. For experimentation, SnapPy contains huge databases of 3-manifolds. These include all 61911 manifolds constructible from up to 9 ideal tetrahedra; Rolfsen's table of 1215 links with up to 10 crossings; the Hoste-Thistlethwaite tables of the 491,326 alternating knots with up to 16 crossings, the 1.210,608 non-alternating knots with up to 16 crossings, and the 180509 links with up to 14 crossings. For more information or downloads, please visit http://snappy.computop.org .

Mark C. Bell – Flipper. `flipper` is a program for computing properties of mapping classes and their actions on laminations. To do this it uses ideal triangulation coordinates on the space of measured laminations.

Among other things, `flipper` can effectively determine the Nielsen–Thurston type of a mapping class and decide whether two mapping classes are conjugate – whenever at least one of them is pseudo-Anosov. The algorithms that `flipper` uses are exact and in **NP**. Hence these can also be used to produce quickly verifiable certificates of these properties.

`flipper` can be run as a Python 2, Python 3 or Sage Python module and is available through the Python Package Index.

Monique Teillaud – CGAL. CGAL, the Computational Geometry Algorithms Library (www.cgal.org) provides industrial and academic users with useful and reliable geometric algorithms. CGAL is an open source project, through which contributors gain impact and visibility. CGAL is distributed under a double license scheme: GPL and commercial. The talk presents an overview of the contents of CGAL and of its characteristics: robustness, genericity, flexibility, and efficiency. Then it focuses on triangulation and meshes packages and finally shows some work in progress.

Dimitriy Morozov – Dionysus. Morozov presented Dionysus, a C++ library with Python bindings for computation of persistent homology. Besides persistent homology, the library supports computation of vineyards, image persistence, persistent cohomology, circular coordinates, and zigzag persistence. It provides

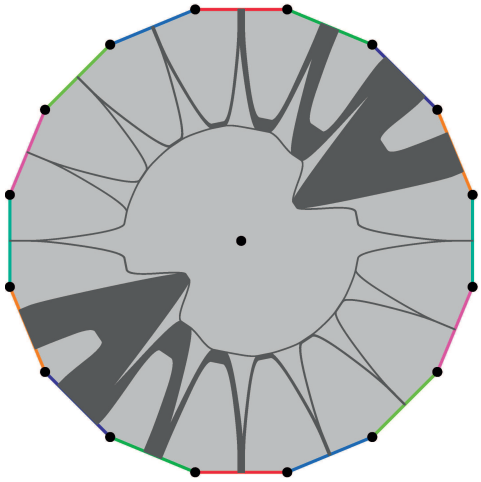

FIGURE 1. The stable lamination of a pseudo-Anosov, as found by `flipper`.

functionality to construct filtrations from data, including Vietoris-Rips filtrations, alpha complexes, and Cech complexes. Examples included with the library illustrate how to construct lower-star and extended filtrations. The library is open-source and is available from Morozov's website.

Ulrich Bauer – Phat and DIPHA. Phat is a C++ library for the computation of persistent homology by matrix reduction, targeted towards developers of software for topological data analysis. It aims for a simple generic design that decouples algorithms from data structures without sacrificing efficiency or user-friendliness. The library provide numerous different reduction strategies as well as data types to store and manipulate the boundary matrix. It is available at ¡https://bitbucket.org/phat-code/¿.

DIPHA is a C++ library for the distributed computation of persistent homology, implemented using OpenMPI. It supports input in the form of distance matrices for Vietoris–Rips filtrations, d-dimensional gray-scale image data for cubical lower star (sublevel set) filtrations, and general filtrations in the form of a boundary matrix. It is available at ¡https://bitbucket.org/dipha¿.

Michael Kerber – Distance between Persistence Diagrams. Our software considers the problem of computing distances between persistence diagrams, a problem that comes up frequently in topological data analysis. While exploiting geometric structure to improve the asymptotic complexity of the problem is a well-studied subject, the practical advantages of using geometry have not been explored. We implement geometric variants of the Hopcroft–Karp algorithm for bottleneck matching (based on previous work by Efrat el al.), and of the auction algorithm by Bertsekas for Wasserstein distance computation. Both implementations use k-d trees to replace a linear scan with a geometric proximity query. We show that our geometric matching algorithms lead to a substantial performance gain, both in running time and in memory consumption, over their purely combinatorial

counterparts and over other implementations available for comparing persistence diagrams.

Vitaliy Kurlin – HoPeS: 2D cloud segmentation and persistent skeletons. The input is a 2D cloud of any points with real coordinates. The first output is a hierarchical segmentation of the cloud based on 1D persistent homology. The second output is a homologically persistent skeleton (HoPeS), which optimally represents all persistent 1D structures in the cloud. The running time is O(n log n) for any n points in the plane. The slides are at [1]. The C++ code is at [2]. For any support, please e-mail vitaliy.kurlin@gmail.com.

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[2] http://kurlin.org/projects/cloud-analysis.cpp.

Jacobien Carstens – HomViz and Persistence Landscape Wrapper. HomViz [1] is a graphical user interface for persistent homology and powered by JavaPlex. It provides an interactive visualization of the Vietoris-Rips complex of a point cloud and its corresponding persistent diagram at different filtration values. Furthermore it has the capacity to highlight generators of homology classes. Persistence landscapes offer a convenient topological summary of persistent homology. The Persistence Landscape Wrapper [2] provides a Matlab interface for the C++ library 'persistent landscape toolkit' that was developed by Pawel Dlotko .

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[1] Binaries URL: http://jacobiencarstens.com
[2] Source code URL: https://github.com/queenBNE/Persistent-Landscape-Wrapper.

Michael Joswig – Polymake. `polymake` started out as tool to study the geometry and the combinatorics of convex polytopes and polyhedra. By now it also deals with combinatorial manifolds, matroids, toric and tropical varieties and other objects. In this presentation the versatility of the software was highlighted by referring to several previous talks at this workshop. This included the following.

- The construction of an 8-dimensional polytope from political science data discussed by Kevin Knudson.
- The visualization of a deltahedron as in Vanessa Robins' presentation.
- Statistics for random discrete Morse functions as studied by Jonathan Spreer.

`polymake` is an open source software tool which can be downloaded from `www.polymake.org`.

Jonathan Spreer – the *GAP* pacakge *simpcomp*. The combinatorial topology software *simpcomp* [1, 2, 3] is an extension – a so called package – to the open source computer algebra system *GAP* [4]. Its primary purpose is to provide functionality to deal with simplicial complexes within the *GAP* framework. The package enables the user to compute numerous properties of (abstract) simplicial

complexes, such as homology or automorphism groups. In addition it provides functions to compute discrete Morse functions, slicings, a combinatorial version of an algebraic blowup, vertex transitive triangulations, and many more. It can generate simplicial complexes from facet lists, orbit representatives, difference cycles, or isomorphism signatures. Moreover a comprehensive collection of sporadic examples and infinite series of triangulations of manifolds are included in the built-in library of the package.

A major focus of *simpcomp* is extendability. Most of its code is written in the easy-to-learn *GAP* scripting language. That is, functionality can be added or changed without re-compiling the source code. The software comes with a thoroughly developed 200+ page manual.

The software is jointly developed and maintained with Felix Effenberger.

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Benjamin A. Burton – Regina. Regina is a suite of mathematical software for 3-manifold topologists. It focuses on the study of 3-manifold triangulations, normal surfaces, and angle structures.

Regina can perform high-level tasks such as 3-sphere recognition, connected sum decomposition, census enumeration, combinatorial recognition of manifolds, and testing for properties such as Hakenness and hyperbolicity. Under the hood it uses exact computations, with algorithms that draw on polytope theory, integer programming, and tree decompositions.

Regina comes with a full graphical user interface for desktops and mobile devices, and also offers Python bindings and a low-level C++ programming interface. The software is freely available at `regina.sourceforge.net`.

Open Problem Session

Problem ONE. *Matthew Kahle*: Is there a constant $C > 0$ such that every 2-dimensional complex Δ with n vertices and $Cn^{\frac{5}{2}}$ faces contains an embedded torus?

Comments:

- Part of more general question.
- True for the sphere.
- This bound would be best possible.

- Known bound: $Cn^{\frac{8}{3}}$ due to simple double counting argument in the link.
- You can find surfaces of arbitrary genus embedded in random complexes on n vertices with $n^{\frac{5}{2}}$ faces almost surely as $n \to \infty$ (result due to Gundert Wagner).

Problem TWO. *Ulrich Bauer*: Poisson point process in $\mathbb{R}^d$, T its Delaunay triangulation. Consider the 1-skeleton of T, take its flag complex. How many of the d-simplices are Delaunay? More generally, how many of the k-simplices, $k \leq d$, are Delaunay?

Comments:

- More likely or less likely in higher dimensions?
- Motivation: $\exists$ new code in CGAL where only the 1-skeleton is stored.
- Note: there are some related results in stochastic geometry about the probability of Delaunay simplices of a certain shape.

Problem THREE. *Jeff Erickson*: Given a generic immersion of the circle in the plane $\gamma : S^1 \to \mathbb{R}^2$ with n double points. Is there an upper bound as a function of n in terms of Reidemeister type moves that are necessary to resolve double points of the immersion.

Comments:

- Trivial lower bound $\Omega(n)$
- Classical upper bound $O(n^2)$ [Steinitz, 1912]
- New bound $\Omega(n^{\frac{3}{2}})$ [Chang, E. 2015]
- This is proven by using an Arnold type curve invariant called *defect* which for certain curves is $\Omega(n^{\frac{3}{2}})$. On the other hand, every generic immersion with n vertices has defect $O(n^{\frac{3}{2}})$.
- New result $O(n^2/\log(n))$ [Chang, ∞]
- Are the bounds related to separator size? Related yes, but not used in the upper bound proof.
- Almost all moves are type three Reidemeister type moves
- $\exists$ preprint: `http://arxiv.org/abs/1510.00571`.

Problem FOUR. *Joel Hass*: Let $\gamma_1, \gamma_2, \gamma_3, \ldots$ be generic immersions of the circle in the plane. We say they universally carry knots if eventually every knot will appear in some curve in the sequence after resolving double points.

What characterises a universal sequence of plane curves?

Comments:

- General position is assumed.
- Example: Every knot has the same projection as the standard $(n, n+1)$ torus knot projection after some resolution.
- *Pot holder diagrams* might be a good starting example.
- Should the sequence be minimal? Extra question: It should be efficient.

- Are there necessary conditions known? There is a necessary condition that the width cannot be bounded.
- Treewidth might be a good place to start.

Problem FIVE. *Keenan Crane*: The 2-sphere can be turned inside out. What about discrete spheres? Given a simplicial immersion, that is, a locally injective simplicial map. Can we answer one of the following questions:

(1) Smallest sphere everted? Some triangulated 2-spheres cannot be turned inside out. Consider, for instance, the boundary of the tetrahedron.
(2) On a smooth surface there are 2^{2g} homotopy classes. What is the smallest genus g simplicial sphere such that you have 2^{2g} discrete regular homotopy classes?

Comments:

- Do you have an upper bound? The actual proof for the smooth sphere contains a polyhedral sphere, so there is an upper bound.
- Is there anything smaller than this?
- Jeff Erickson: Algorithmic formulation of this question. Given an embedded simplicial 2-sphere in $\mathbb{R}^3$, can it be everted?

Problem SIX. *Kevin Knudson*: $f, g : M \to \mathbb{R}$ Morse functions which are "close" (for your favourite notion of close). Is there a discrete Morse theory version of this? Can we perturb a discrete Morse function a little bit (possibly without changing the triangulation)?

Comments:

- What if you want to sample (discretise) a smooth family?
- If you have a smooth Morse function then there exists a discrete version of this function with the same properties [Benedetti].
- Cerf's theorem: For two Morse functions $f, g : M \to \mathbb{R}$, there exists a family of smooth functions $F : M \times I \to \mathbb{R}$ such that $f(\cdot, 0) = f$ and $f(\cdot, 1) = g$, and all but finitely many functions in between are Morse.
- Ulrich Bauer: For discrete Morse functions we can compare their discrete gradients.

Problem SEVEN. *Frank Lutz*: Question: What is $\chi_3(S^3) = ?$

The question goes back to Jesper M. Møller. We are considering vertex-colourings of triangulations of the 3-sphere S^3 such that monochromatic triangles are allowed, but monochromatic tetrahedra are not.

In general, this gives a hierarchy of colouring numbers, where χ_i means that monochromatic $(i-1)$-faces are allowed, but monochromatic i-faces are not.

It is known that

- $\chi_1(S^2) = 4$ (4-colour theorem)
- $\chi_1(M^2) =$ finite (map colour theorem [Ringel, Youngs, 1978])
- $\chi_1(M^3) = \infty$ (by existence of neighbourly triangulations [Walkup, 1970])
- $\chi_2(M^3) = \infty$ (L., Møller [2015])

Comments:

- We know that $\chi_3(S^3) \geq 2$.
- Is there an upper bound? No, might even be infinite.
- Question could be asked for any other 3-manifold as well as in higher dimensions.

Problem EIGHT. *Benjamin A. Burton*

Greater picture: can we solve unknot recognition in polynomial time?

More precisely, given a knot K whose minimal treewidth is bounded below by some constant. Does there exist a function $f : \mathbb{N} \to \mathbb{N}$ such that the minimal triangulation of $S^3 \setminus K$ has a dual graph with treewidth $\geq f(k)$ such that $\lim_{k\to\infty} f(k) = \infty$?

Background: One possible way to find a polynomial time algorithm for unknot recognition is to first prove that unknot recognition is in FPT and then find a way to retriangulate knot diagrams in order to decrease their treewidth. The converse of this question is a necessary condition for this plan to be feasible.

Comments:

- Saul Schleimer: The $(k, k + 1)$-candidate might have no diagram with low treewidth, but their knot complements admit triangulations with very small treewidth.

Reporter: Katharine Turner

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