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Mini-Workshop: Friezes

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Abstract. Frieze patterns were introduced in the early 1970s by Coxeter. They are infinite arrays of numbers in which every four neighbouring entries always satisfy the same arithmetic relation. Amazingly, friezes appear in many situations from various areas of mathematics: projective geometry, number theory, algebraic combinatorics, difference equations, integrable systems, representation theory, cluster algebras...

The mini-workshop aimed to gather researchers with diverse fields of expertise to present recent developments and to discuss new directions of investigation and open problems around friezes.

Mathematics Subject Classification (2010): 05E10, 13F60, 37J35, 39A70, 81R12.

Introduction by the Organisers

The mini-workshop *Friezes*, organised by Thorsten Holm, Peter Jørgensen and Sophie Morier-Genoud was attended by 17 participants coming from Canada (2), US (3), UK (1), Germany (2), Austria (3) and France (6). The programme of the workshop consisted in three mini-courses (by S. Tabachnikov, P.-G. Plamondon, M. Cuntz), 11 talks and a problem session. A large place was left to discussions and collaborations.

The three mini-courses given during the workshop offered different approaches to friezes. Serge Tabachnikov exposed friezes from geometric viewpoints. Pierre-Guy Plamondon lectured on friezes in the context of the representation theory of quivers. Michael Cuntz's courses connected friezes with discrete geometry coming from Nichols algebras and Weyl groupoids.

The participants presented in their talks recent results on friezes using new and inventive methods. Various combinatorial models and techniques have been brought out.

The last day of the workshop a session of open problems was organized. Participants were happy to share ideas and questions on friezes and related topics. Let us mention different problems that were suggested and discussed during this session.

- Does the continous analogue of 2-friezes correspond to the moduli space of periodic curves in the projective plane?
- Find the discrete version in terms of 2-friezes or SL_3-friezes of Richard Schwartz's inequality for the periodicity condition of projective curves satisfying a third order ODE.
- Do hyperplane arrangements in dimension greater than 2 give rise to friezes?
- Find direct links between Nichols algebras/Weyl groupoids and cluster algebras.
- Find combinatorial interpretations of the entries in a SL_k-friezes with positive integers.
- Do SL_k-friezes with positive integers correspond to integer points in the corresponding cluster varieties?
- How to interpret friezes with positive integers that do not come from cluster characters?
- The moduli space of hyperbolic structures on the disc with one marked point and two orbifold points has a generalised cluster structure: can the situation be extended by making sense of a "complex order" of an orbifold point?

The problems illustrate that friezes are an important nexus linking a number of major and otherwise disjoint areas of contemporary mathematics: moduli spaces of polygons and curves, Teichmüller theory, cluster algebras, quantum groups, combinatorics, representation theory, integrable systems...

Friezes and their generalisations have attracted much interest over the past five years. They are currently an active topic of fundamental research. The subject has roots in classical areas of mathematics, such as projective geometry, number theory, and grows in many directions connecting different domains. There are promising results and perspective for this topic.

Based on their own experience and the positive feedback received from the participants, the organisers are pleased to say that the mini-workshop was a great success. The meeting was extremely stimulating and enriching and everyody spent a very enjoyable week.

We are deeply grateful to MFO for providing us with all we needed and for offering outstanding conditions of work.

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Mini-Workshop: Friezes

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Abstracts

Three lectures on frieze patterns

Serge Tabachnikov

This mini-series of talks covers the following topics on and around frieze patterns:

- Definition and examples of Coxeter's frieze patterns;
- Relations with discrete Hill's equation with monodromy $-Id$ and with polygons in the projective line ([8]);
- Periodic rational maps from closed frieze patterns ([9]);
- Cluster coordinates, Laurent phenomenon, and arithmetic friezes;
- Conway-Coxeter theorem on arithmetic frieze patterns (after [3]);
- Continuous limits of frieze patterns as a PDE; parameterization of its solutions ([10]);
- Frieze patterns as discretizations of a coadjoint orbit of the Virasoro algebra ([10]);
- The "trinity": super-periodic linear difference equations of order $k+1$, the moduli space of projective polygons in $\mathbf{RP}^k$, and closed SL_{k+1}-friezes. Interaction with projective duality ([9]);
- Classical and combinatorial Gale transform ([9]);
- Gale transform and Krichever's theorem on commuting difference operators ([7]);
- Pentagram map, its complete integrability ([11, 12]);
- New configuration theorems of projective geometry ([13]);
- Continuous limit of the pentagram map as the Boussinesq equation ([11]);
- Cluster dynamics of the pentagram map, higher pentagram maps, their complete integrability (see [1, 2]);
- 1-dimensional pentagram map (leapfrog map) and circle patterns ([1]);
- Higher-dimensional pentagram maps, their integrability and non-integrability (see [4, 5, 6]).

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Friezes and representations of quivers

Pierre-Guy Plamondon

The aim of these lectures is to give an introduction to the links between the representation theory of quivers and friezes. The last decade or so has seen these links discovered and developed in a more general setting often referred to as "additive categorification of cluster algebras". As references for this short report, we will limit ourselves to the list of survey articles given at the end.

In the first lecture, we introduced the notion of quiver representation. After reviewing the basic definitions, we stated Gabriel's theorem, which asserts that a connected quiver admits only finitely many indecomposable representations (up to isomorphism) if and only if it is an orientation of a Dynkin diagram. We then focused on the example of a quiver of type A_3, where all indecomposable representations can be written down.

In the second lecture, we saw how "counting subrepresentations" gives rise to friezes. For a fixed dimension vector, we defined the quiver Grassmannian, which is a variety whose points parametrize subrepresentations of a given representation. We then stated a theorem linking quiver representations to friezes: the sum (over all dimension vectors) of the Euler characteristics of the quiver Grassmannians of a given indecomposable representation of a Dynkin quiver Q is an entry in a frieze of type Q. Moreover, all entries in the frieze are obtained in this way, except entries of value 1. It was hinted that Auslander–Reiten theory, and more precisely almost-split sequences, allow us to recover the equations defining friezes.

Finally, in the third lecture, a definition of the cluster category was given. The notion of derived category of an abelian category was recalled, and we stated Happel's theorem on the structure of the derived category of the category of representations of Dynkin quivers. The cluster category was then defined as an orbit category of the derived category under the action of a certain automorphism. We then saw how a special class of objects, called cluster-tilting objects, allow us to recover complete friezes. We ended the lectures with comments on the generalization of these results to friezes of any quiver type.

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Nichols algebras, Weyl groupoids, and friezes

Michael Cuntz

This abstract is for a short lecture consisting of two separate talks.

Summary. Weyl groups are important invariants attached for example to Lie algebras. The more general Weyl groupoids play a similar role in the theory of Nichols algebras. Although these Weyl groupoids historically were found in the context of Nichols algebras, their geometry and combinatorics also appear in other areas: they may for example be viewed as toric varieties and in the special case of rank two they correspond to (finite and infinite) frieze patterns.

Nichols algebras and Heckenberger's Cartan graph. In the first talk, I give a very short introduction to Nichols algebras and construct their Weyl groupoids in the case of diagonal type.

Let V be a vector space, $c : V \otimes V \to V \otimes V$ a linear isomorphism with $(c\otimes \mathrm{id})(\mathrm{id}\otimes c)(c\otimes \mathrm{id}) = (\mathrm{id}\otimes c)(c\otimes \mathrm{id})(\mathrm{id}\otimes c)$. Then c is a *braiding*, and (V, c) is a *braided vector space*. Any braiding defines a map $\rho : S_n \to \mathrm{End}(V^{\otimes n})$ in a natural way. Let $\mathfrak{S}_n := \sum_{\omega\in S_n} \rho(\omega)$. The Algebra $\mathfrak{B}(V) := \bigoplus_{n\geq 0} T^n(V)/\ker(\mathfrak{S}_n)$ is called the *Nichols algebra* of (V, c).

Now let $\{x_1, \dots, x_r\}$ be a basis of $V^a := V$, and assume that $c(x_i \otimes x_j) = q_{ij}x_j\otimes x_i$, $q_{ij} \in \mathbb{C}$. Then c and $\mathfrak{B}(V^a)$ are called of *diagonal type*. The numbers q_{ij}, $i, j = 1, \dots, r$ define a *bicharacter* $\chi^a : \mathbb{Z}^r \times \mathbb{Z}^r \to \mathbb{C}$, $((a_1, \dots, a_r), (b_1, \dots, b_r)) \mapsto \prod_{i,j=1}^r q_{ij}^{a_ib_j}$. Call (V^a, c) *locally finite* if there is a well defined matrix $(c^a_{ij})_{1\leq i,j\leq n} \in \mathbb{Z}^{n\times n}$ such that

$$c^a_{ij} = -\min\{m \in \mathbb{N}_0 \mid 1 + q_{ii} + q_{ii}^2 + \dots + q_{ii}^m = 0 \text{ or } q_{ii}^m q_{ij}q_{ji} = 1\}$$

if $i \neq j$ and $c^a_{ii} = 2$. We call such a matrix (c^a_{ij}) a *Cartan matrix*. We denote $\alpha_1, \dots, \alpha_n$ the standard basis of $\mathbb{Z}^n$. Let $\sigma^a_i \in \mathrm{End}(\mathbb{Z}^n)$ be defined by $\sigma^a_i(\alpha_j) = \alpha_j - c^a_{ij}\alpha_i$ for $j = 1, \dots, n$. Then define a new bicharacter $\chi^b = (\sigma^a_i)^*\chi^a$ by $\chi^b(\alpha, \beta) = \chi^a((\sigma^a_i)^{-1}(\alpha), (\sigma^a_i)^{-1}(\beta))$. Repeating this procedure, we obtain a so-called *Cartan graph* with vertices labeled by Cartan matrices of bicharacters and edges labeled by the reflections σ_i (see [7] for details). The Cartan graph defines a category called its *Weyl groupoid*.

Weyl groupoids and friezes. In the second talk, we investigate the Weyl groupoid independently from the Nichols algebra and explain the connection to frieze patterns.

A finite set $\mathcal{A} := \{H_1, \ldots, H_n\}$ of linear hyperplanes in $V = \mathbb{R}^r$ is called an *arrangement of hyperplanes*. Let $\mathcal{K}(\mathcal{A})$ be the set of connected components (*chambers*) of $V \setminus \bigcup_{H \in \mathcal{A}} H$. If every chamber K is an *open simplicial cone*, i.e. there exist $\beta_1, \ldots, \beta_r \in V$ such that $K = \{\sum_{i=1}^r a_i \beta_i \mid a_i > 0 \quad \text{for all} \quad i = 1, \ldots, r\}$, then $\mathcal{A}$ is called a *simplicial arrangement*.

Let $\mathcal{A}$ be simplicial. For each H_i, $i = 1, \ldots, n$ we choose an element $x_i \in V^*$ such that $H_i = x_i^\perp$ and let $R := \{\pm x_1, \ldots, \pm x_n\} \subseteq V^*$. For each chamber $K \in \mathcal{K}(\mathcal{A})$ let B^K be the set of normal vectors in R of the walls of K pointing to the inside. We call $(\mathcal{A}, R)$ a *crystallographic arrangement* if for all $K \in \mathcal{K}(\mathcal{A})$: $R \subseteq \sum_{\alpha \in B^K} \mathbb{Z}\alpha$. Crystallographic arrangements are in one-to-one correspondence with Weyl groupoids (see [1]).

A good motivation to consider these special arrangements is the following fact: If $\mathfrak{B}$ is a finite dimensional Nichols algebra of diagonal type, and R_+ is the set of degrees (in $\mathbb{Z}^r \subseteq \mathbb{R}^r$) of the PBW generators of $\mathfrak{B}$, then $\{\alpha^\perp \subseteq \mathbb{R}^r \mid \alpha \in R_+\}$ is a crystallographic arrangement.

Theorem 1 ([4], [3], [5], [6])**.** *There are exactly three families of crystallographic arrangements:*

(1) *The family of rank two parametrized by triangulations of a convex n-gon by non-intersecting diagonals.*
(2) *For each rank $r > 2$, arrangements of type A_r, B_r, C_r and D_r, and a further series of $r-1$ arrangements.*
(3) *Further 74 "sporadic" arrangements of rank r, $3 \leq r \leq 8$.*

In particular, finite Weyl groupoids of rank two correspond to Conway-Coxeter frieze patterns. Under this correspondence, the quiddity row of the frieze consists of the Cartan entries of the Weyl groupoid and the other entries of the frieze are the coordinates of the roots of the root systems of the Weyl groupoid (see [2] for details). Arbitrary (not necessarily finite) Weyl groupoids of rank two correspond to infinite friezes.

It turns out that the theory of Weyl groupoids has many similarities to the theory of cluster algebras. Thus it is natural to ask:

Open question. *Is there a deeper connection between Weyl groupoids (or perhaps Nichols algebras) and cluster algebras?*

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Rationality of friezes and string modules

David Smith
(joint work with I. Assem, C. Reutenauer and I. Assem, G. Dupont, R. Schiffler)

The goal of the talk was to report on some aspects of two papers written with I. Assem and C. Reutenauer [2] and I. Assem, G. Dupont and R. Schiffler [1]. Here, we restrict ourselves to the setting that was presented in the talk; see [1, 2] for more details and more general statements.

1. Rationality of friezes of quivers

Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver, where Q_0 is the set of vertices and Q_1 is the set of arrows. Define, for each vertex v in Q_0, a sequence $v(n)$ by the initial condition $v(0) = 1$ and the recursion

$$v(n+1) = \frac{1}{v(n)} \left(1 + \prod_{v \to w \text{ in } Q_1} w(n) \prod_{w \to v \text{ in } Q_1} w(n+1) \right).$$

Observe that it follows from the Laurent phenomenon, established by S. Fomin and A. Zelevinsky [5] (in the context of cluster algebras) that the sequence $(v(n))$ are sequences of positive integers.

As an example, for the quiver $v_1 \rightrightarrows v_2$, one gets the sequences $(v_1(n)) = 1, 2, 13, 89, 610, \ldots$ and $(v_2(n)) = 1, 5, 34, 233, 1597, \ldots$. One easily observes that these sequences satisfy the linear recurrence relations $v_i(n) = 7v_i(n-1) - v_i(n-2)$ for all $n \geq 2$ and $i = 1, 2$. Moreover, these sequences are $\mathbb{N}$-rational, in the sense that there exist an integer $d \geq 1$, vectors $\lambda \in \mathbb{N}^{1\times d}$, $\gamma \in \mathbb{N}^{d\times 1}$ and a matrix $M \in \mathbb{N}^{d\times d}$ such that $v_i(n) = \lambda M^n \gamma$ for all $n \in \mathbb{N}$. Indeed, for all $n \in \mathbb{N}$,

$$v_1(n) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}^n \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad v_2(n) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 5 & 3 \\ 3 & 2 \end{bmatrix}^n \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

It is a classical result that every $\mathbb{N}$-rational sequence $(a_n)_{n\in\mathbb{N}}$ satisfies some linear recurrence, in the sense that there exist an integer $k \geq 1$ and $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{Q}$ such that $a_{n+k} = \alpha_1 a_{n+k-1} + \alpha_2 a_{n+k-2} + \cdots + \alpha_k a_n$, for all $n \in \mathbb{N}$.

S. Fomin and A. Zelevinsky proved in [6] that all the sequences $(v(n))$ are periodic if and only if the quiver Q is of Dynkin type. In particular, the sequences are $\mathbb{N}$-rational. Based on this observation and the previous example, the following question was raised: is it possible to characterize the acyclic quivers giving rise to sequences satisfying linear recurrence relations?

Theorem 1. [2] *Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver. If the sequence $(v(n))_{n\in\mathbb{N}}$ is $\mathbb{N}$-rational for all $v \in Q_0$, then Q is of Dynkin or Euclidean type.*

Theorem 2. [2] *Let $Q = (Q_0, Q_1)$ be a finite acyclic quiver. If Q is of Euclidean type $\tilde{\mathbb{A}}$ or $\tilde{\mathbb{D}}$, then the sequence $(v(n))_{n\in\mathbb{N}}$ is $\mathbb{N}$-rational for all $v \in Q_0$.*

Observe that it was further demonstrated by B. Keller and S. Scherotzke [7] that if $Q = (Q_0, Q_1)$ is a finite acyclic quiver of any Euclidean type, then all sequences $(v(n))_{n\in\mathbb{N}}$, with $v \in Q_0$, satisfy some linear recurrence relations.

2. SL_2-tilings of the discrete plane

The general idea for the proof of Theorem 2 in the case $\tilde{\mathbb{A}}$ consists in considering the universal covering of the quiver of type $\tilde{\mathbb{A}}$ and embed this covering in the discrete plane $\mathbb{Z} \times \mathbb{Z}$ in order to extend this to an SL_2-tiling of the discrete plane. We will refrain from giving more details and refer to [2] for more precision.

Nevertheless, as a byproduct of this construction, we obtain a way to compute some entries in an SL_2-tiling in term of other entries on a so-called frontier.

Definition 1. *Let R be a commutative ring containing $\mathbb{Q}$.*

(a) *An SL_2-tiling of the discrete plane is a function $t : \mathbb{Z} \times \mathbb{Z} \to R$ such that*

$$\begin{vmatrix} t(i,j) & t(i+1,j) \\ t(i,j-1) & t(i+1,j-1) \end{vmatrix} = 1$$

for all $i, j \in \mathbb{Z}$. In other words, an SL_2-tiling of the plane is a filling of the discrete plane by elements in R such that the determinant of every adjacent 2×2-matrix is one.

(b) *Given an SL_2-tiling t, a frontier is a path $(a_k)_{k\in\mathbb{Z}} = (t(i_k, j_k))$ such that*
   (1) *For all $k \in \mathbb{Z}$,*
      (i) *$a_{k+1} = t(i_k + 1, j_k)$, or*
      (ii) *$a_{k+1} = t(i_k, j_k + 1)$,*
   (2) *As $k \to \pm\infty$, there are infinitely many shifts between 1(i) and 1(ii).*

To every entry $t(u, v)$ below a frontier (a_k), we associate a word given by its projection on the frontier. For instance, if

$$\begin{array}{llll} & a_4\, a_5 & & a_6 \\ a_2\, a_3 & & & \vdots \\ a_1 \cdot \cdots & \cdot & & t(u,v) \end{array}$$

then the associated word would be $w_(u, v) = a_1 y a_2 x a_3 y a_4 x a_5 x a_6$, where the x and y are used to keep track of the horizontal or vertical displacements.

Theorem 3. [2] *Let t be an SL_2-tiling. of the discrete plane. Suppose that $t(u,v)$ is an entry below a frontier (a_k) with associated word $w_(u,v) = a_m x_m a_{m+1} x_{m+1} \cdots a_n$, where $x_i \in \{x,y\}$ for all i. Then*

$$t(u,v) = \frac{\left[\begin{array}{cc} 1 & a_m \end{array}\right] \prod_{i=m+1}^{n-2} M(a_i, x_i, a_{i+1}) \left[\begin{array}{c} 1 \\ a_n \end{array}\right]}{a_m a_{m+1} \cdots a_n},$$

$$\text{where } M(a_i, x_i, a_{i+1}) = \begin{cases} \left[\begin{array}{cc} a_i & 1 \\ 0 & a_{i+1} \end{array}\right] & \text{if } x_i = x, \\ \left[\begin{array}{cc} a_{i+1} & 0 \\ 1 & a_i \end{array}\right] & \text{if } x_i = y. \end{cases}$$

3. String modules and cluster variables

For an acyclic quiver Q of type $\tilde{A}$, Theorem 3 can be exploited to provide a combinatorial formula for expressing some cluster variables in the (coefficient-free) cluster algebra associated with Q, but not all. In [1], Theorem 3 was adapted to compute more cluster variables (all in the case of a quiver of type $\tilde{A}$), and in a more general context. We present below a brief overview of this adaptation but refer to [1] for the complete details and the most general context.

In [3], Caldero and Chapoton noticed that cluster variables in simply-laced coefficient-free cluster algebras of finite type can be expressed as generating series of Euler-Poincaré characteristics of Grassmannians of submodules. Generalising this work, Caldero-Keller [4] and Palu [8] introduced the notion of a cluster character associating to each module M over a 2-Calabi-Yau tilted algebra B_T a certain Laurent polynomial X_M^T allowing one to compute a corresponding cluster variable. In general, cluster characters are hard to compute because one first needs to find the Euler characteristics of Grassmannians of submodules and dimensions of certain Hom-spaces in the corresponding 2-Calabi-Yau category.

The main result of [1] gives an explicit formula for the cluster character associated with a string module over a 2-Calabi-Yau tilted algebra. This can be stated as follows. Let T be a tilting object in a Hom-finite triangulated 2-Calabi-Yau category $\mathcal{C}$ and $B_T = \mathrm{End}_{\mathcal{C}}(T)$ be the corresponding 2-Calabi-Yau tilted algebra whose ordinary quiver is denoted by Q. To any string B_T-module M, we associate a tuple of integers $\mathbf{n}_M = (n_i)_{i \in Q_0}$, called the normalisation of M, and a Laurent polynomial L_M in the ring of Laurent polynomials in the indeterminates x_i indexed by the set Q_0 of points of Q, which can be expressed as a product of 2×2 matrices, inspired by the one given in Theorem 3, see [1, Section 1.3]. Using the notation $\mathbf{x}^{\mathbf{n}_M} = \prod_{i \in Q_0} x_i^{n_i}$, our main result [1, Theorem 5.11] can be stated as :

$$X_M^T = \frac{1}{\mathbf{x}^{\mathbf{n}_M}} L_M.$$

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All SL_2-tilings come from infinite triangulations

Peter Jørgensen
(joint work with Christine Bessenrodt and Thorsten Holm)

1. SL_2-tilings. An SL_2-tiling is a pattern assigning a positive integer to each integral point in the plane such that every adjacent 2×2-matrix has determinant 1. SL_2-tilings were introduced by Assem, Reutenauer and Smith [1] and shown to be useful for providing closed formulae for cluster variables (in types A and $\tilde{A}$). Note that SL_2-tilings can be seen as a natural extension of the classic Conway–Coxeter friezes of [3] to the entire plane. We give two examples of SL_2-tilings.

$$
\begin{array}{ccccccccccccc}
 & & & & & & \vdots & & & & & & \\
 & 10 & 23 & 13 & 3 & 5 & 2 & 3 & 7 & 11 & 4 & 1 & \\
 & 23 & 53 & 30 & 7 & 12 & 5 & 8 & 19 & 30 & 11 & 3 & \\
 & 13 & 30 & 17 & 4 & 7 & 3 & 5 & 12 & 19 & 7 & 2 & \\
 & 16 & 37 & 21 & 5 & 9 & 4 & 7 & 17 & 27 & 10 & 3 & \\
 & 3 & 7 & 4 & 1 & 2 & 1 & 2 & 5 & 8 & 3 & 1 & \\
\cdots & 5 & 12 & 7 & 2 & 5 & 3 & 7 & 18 & 29 & 11 & 4 & \cdots \\
 & 2 & 5 & 3 & 1 & 3 & 2 & 5 & 13 & 21 & 8 & 3 & \\
 & 5 & 13 & 8 & 3 & 10 & 7 & 18 & 47 & 76 & 29 & 11 & \\
 & 3 & 8 & 5 & 2 & 7 & 5 & 13 & 34 & 55 & 21 & 8 & \\
 & 4 & 11 & 7 & 3 & 11 & 8 & 21 & 55 & 89 & 34 & 13 & \\
 & 1 & 3 & 2 & 1 & 4 & 3 & 8 & 21 & 34 & 13 & 5 & \\
 & & & & & & \vdots & & & & & &
\end{array}
$$

Note the staircase-like appearance of 1's in this pattern. But an SL_2-tiling need not have any 1, as the following second example shows.

$$\begin{array}{ccccccccccc} & & & & & \vdots & & & & & \\ & 265 & 218 & 171 & 124 & 77 & 107 & 137 & 167 & 197 & \\ & 203 & 167 & 131 & 95 & 59 & 82 & 105 & 128 & 151 & \\ & 141 & 116 & 91 & 66 & 41 & 57 & 73 & 89 & 105 & \\ & 79 & 65 & 51 & 37 & 23 & 32 & 41 & 50 & 59 & \\ \cdots & 17 & 14 & 11 & 8 & 5 & 7 & 9 & 11 & 13 & \cdots \\ & 57 & 47 & 37 & 27 & 17 & 24 & 31 & 38 & 45 & \\ & 154 & 127 & 100 & 73 & 46 & 65 & 84 & 103 & 122 & \\ & 405 & 334 & 263 & 192 & 121 & 171 & 221 & 271 & 321 & \\ & & & & & \vdots & & & & & \end{array}$$

Assem, Reutenauer and Smith [1] have shown that certain infinite patterns of 1's (complete staircases) can always be completed to an SL_2-tiling. But their results would not cover any of the above two examples.

Our main result is that every SL_2-tiling can be obtained from a triangulation of certain combinatorial objects by a counting procedure.

2. Triangulations of the circle with accumulation points. We now describe the combinatorial objects whose triangulations will give SL_2-tilings. These 'circles with accumulation points' appeared first in work of Igusa and Todorov on certain new cluster categories [5]. For constructing SL_2-tilings it turns out that we only have to consider a special case, namely a circle with four accumulation points. Between any of the accumulation points we have a discrete set of vertices, indexed by the integers, see the left of Figure 1. We consider triangulations of these circles with four accumulation points, like in the example on the right of Figure 1. On such a triangulation we perform the following counting procedure: Pick a vertex i of the polygon, and assign 0 to it. Next, assign a 1 to each vertex sharing a triangle with i. Then, inductively, whenever two vertices of a triangle have already been assigned numbers, then the third vertex gets assigned their sum. It mainly follows from classic Conway-Coxeter theory that starting from the vertices at the top, the numbers you get at the bottom give the rows of an SL_2-tiling. In the above example, compare the red numbers at the bottom with the red row of the second example of an SL_2-tiling above. The following converse is our main result.

Theorem [2]. *For every SL_2-tiling there exists a triangulation of the circle with four accumulation points which produces the given SL_2-tiling by the counting procedure described above.*

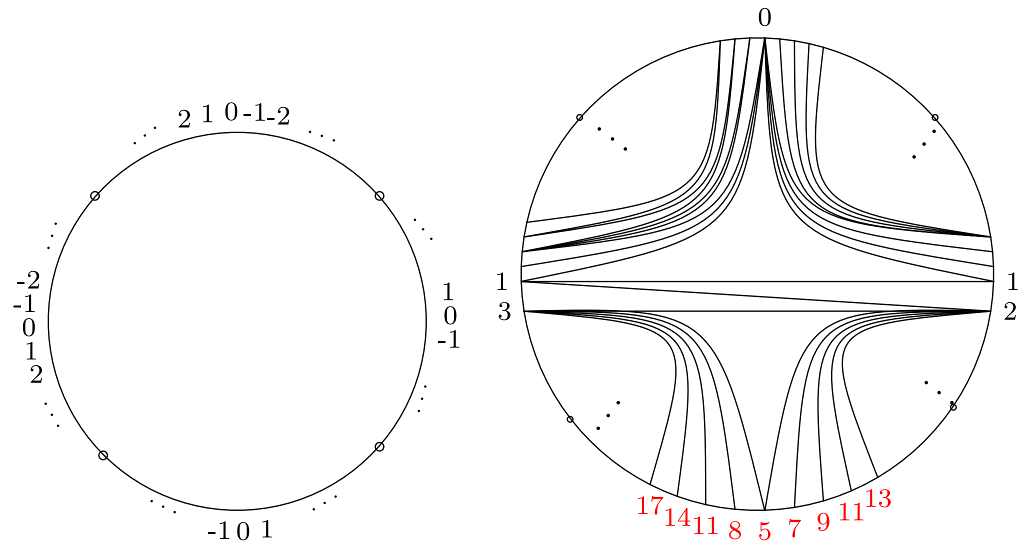

Figure 1. A circle with four accumulation points and BCI-counting on a triangulation

The proof of the theorem is constructive, i.e. the diagonals of the triangulation to be found can be read off from the entries of the given SL_2-tiling (but the way to do so is not obvious). The methods in the proof are completely different in the cases where the SL_2-tiling contains 1's and the much harder case where it does not. For the latter case a crucial observation for getting started is that any SL_2-tiling without 1's has a unique minimal entry.

Of course, one could also consider triangulations of circles with more accumulation points. Somewhat surprisingly, it turns out that four accumulation points are sufficient to produce all SL_2-tilings.

There are classes of SL_2-tilings where fewer than four accumulation points are sufficient. For instance, if there are infinitely many 1's in the given SL_2-tiling to both the south-west and north-east direction then two accumulation points suffice. This leads to the triangulations of the strip considered in [4]. If there are infinitely many 1's but only to one direction then circles with three accumulation points suffice.

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Snake graphs and generalised cluster algebras

Anne-Sophie Gleitz
(joint work with Gregg Musiker)

In 2008, Gregg Musiker and Ralph Schiffler [6] showed that in a cluster algebra from a surface, the monomials in the cluster expansion formulas for cluster variables (from the Laurent phenomenon) can be realised as perfect matchings of some special graphs called "snake graphs". Generalised cluster algebras have been introduced by Chekhov and Shapiro [1] in 2011; it is natural to wonder if snake graphs can also be used to visualise cluster expansion formulas. Early results (joint with G. Musiker) let us hope for a positive answer in general.

1. Cluster algebras and snake graphs

1.1. **Cluster algebras.** Cluster algebras were introduced by Fomin and Zelevinsky [3] in 2000, and have appeared in a growing number of mathematical research areas ever since. A *cluster algebra* is a commutative ring (domain) with a special set of generators, called *cluster variables*, grouped into overlapping subsets of the same finite cardinality (the *rank* of the algebra), called *clusters*.

Cluster variables are built inductively using an algorithm called *mutation*. More precisely, one starts with an *initial cluster* $\mathbf{x} = (x_1, \ldots, x_n)$; then mutation in direction $k \in [\![1, n]\!]$ leaves the variables x_i, $i \neq k$ unchanged, and maps x_k to a new cluster variable x'_k, using an exchange relation of the form

$$x_k x'_k = p_k^+ m_+^{(k)} + p_k^- m_-^{(k)},$$

where $m_\pm^{(k)}$ are monomials in the x_i, $i \neq k$, that depend on the k-th column of the skew-symmetrisable *exchange matrix* $B \in \mathcal{M}_n(\mathbb{Z})$. This matrix, along with the coefficients $p_i^\pm$, is also subject to mutation, as described in [3, 4].

The cluster algebra of *initial seed* $(\mathbf{x}, B)$ is the subring of $\mathbb{Q}(\mathbf{x}, (p_i^\pm)_{i\in[\![1,n]\!]})$, generated by all the (possibly infinitely many) cluster variables obtained by performing every possible sequence of mutations. If the cluster algebra contains finitely many cluster variables, then it is of *finite type*.

Theorem 1 ([4, Theorems 1.4,1.9])**.** *1.* Laurent phenomenon*: Every cluster variable is a Laurent polynomial in the initial cluster variables, with integer coefficients.*
2. Finite type classification*: Cluster algebras of finite type are parametrised by the Cartan matrices of types A_n to G_2.*
3. In finite type, there is a bijection from almost positive roots to cluster variables, which maps negative simple roots to the initial cluster variables.

The last property was extended by Keller [5, Theorem 3.1, §3.3] to seeds whose exchange matrices correspond to Coxeter elements in the appropriate Coxeter group. In particular, the result is true for (twisted) affine types.

1.2. **Cluster algebras from a surface and perfect matchings.** Recall that for any graph $G = (G_0, G_1)$, a *perfect matching* of G is a subgraph $\Gamma = (G_0, \Gamma_1)$ such that each vertex of Γ is the endpoint of exactly one edge.

Example 1. *A square graph has 2 perfect matchings: both pairs of opposite sides.*

Cluster algebras may arise from certain triangulated surfaces [2]; we briefly recall how to attach a snake graph to each cluster variable, as described in [6].

Let (S, M) be a surface S with a boundary and a set M of marked points on the boundary. Choose a triangulation $T = \{\tau_1, \ldots, \tau_{n+m}\}$ of S. Each internal arc τ_i, $i \in [\![1, n]\!]$, corresponds to an initial cluster variable x_i; frozen variables are attached to boundary arcs. To each oriented arc γ in S, a cluster variable x_γ is attached, depending on the triangulation arcs $\tau_{i_1}, \ldots, \tau_{i_d}$ crossed by γ. The expression of x_γ as a Laurent polynomial in $x_1, \ldots, x_r$ is given by the *snake graph* of γ. Each arc τ_{i_k} lies in two triangles, thus in the *tile* S_{i_k}, which is the graph obtained by glueing these two triangles along τ_{i_k}, weighted by the appropriate cluster (or frozen) variables. These tiles are then glued together following the orientation of γ, by their north or east side (this can be determined by (T, γ)-paths, see [6]). Removing the squares' diagonals yields the snake graph of γ.

Theorem 2 ([6, Theorem 3.1])**.** *Suppose that the arc γ crosses a_i times each internal arc τ_i, and yields the snake graph G. The cluster variable x_γ can then be written:*

$$x_\gamma = \frac{1}{x_1^{a_1} \ldots x_n^{a_n}} \sum_{\Gamma \text{ perf. mat. of } G} \left(\prod_{w \in \Gamma_1} \text{label}(w) \right).$$

For cluster algebras of finite type, all the non-initial cluster variables correspond to connected subgraphs of the snake graph attached to the cluster variable in bijection with the highest root. For a complete example, see [6, Section 7].

2. Generalising cluster algebras

In 2011, Chekhov and Shapiro [1] have introduced the notion of *generalised cluster algebra*, in a geometrical context. The notion of cluster variables, exchange matrix and seeds remain unaltered. The difference lies in the exchange relations: if the GCD of the entries in the k-th column of the exchange matrix, written d_k, is at least 2 (this is mutation-invariant), then the right-hand side of the k-th exchange relation becomes a homogeneous polynomial $\theta_k(m_+^{(k)}, m_-^{(k)})$ of degree d_k, that contains more than 2 monomials.

Theorem 3 ([1, Theorems 2.5,2.7])**.** *The Laurent phenomenon and the finite type classification theorem remain true for generalised cluster algebras.*

The combinatorial structure of the generalised cluster algebra is the same as its standard counterpart; only the Laurent polynomial expressions change, as they gain extra monomials. In fact, nullifying the extra coefficients yields the standard formulas. The bijection from Theorem 1(3.) also holds in the generalised case.

Note that cluster algebras of simply-laced type cannot be generalised, since $d_i = 1$ for all i. In finite type, only the C_n and G_2 cases can be generalised.

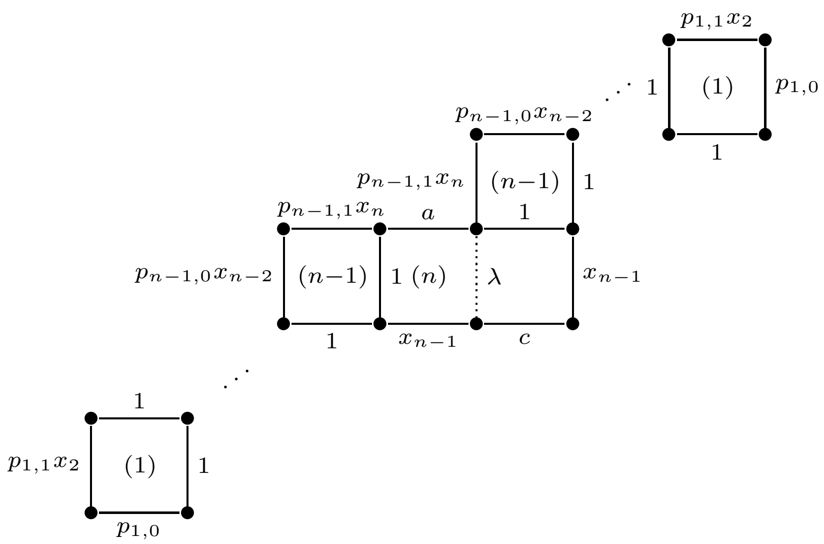
3. Generalised cluster algebras and perfect matchings

The combinatorial structure is kept intact when a cluster algebra is generalised. New monomials may appear within the cluster expansion formulas. The idea is thus to keep the same tiles as for a standard cluster algebra, enlarge the ones that correspond to generalised exchange relations, and add edges to yield the extra perfect matchings corresponding to the extra monomials.

Conjecture 1. *Let $\mathcal{A}_n$ be the generalised cluster algebra of type C_n, with initial cluster $(x_1, \ldots, x_n)$, initial exchange matrix $B = (b_{ij})$ such that $b_{ii} = 0$ $(i \in [\![1, n]\!])$, $b_{i+1,i} = -1$ $(i \in [\![1, n-1]\!])$, $b_{i,i+1} = 1$ $(i \in [\![1, n-2]\!])$, and $b_{n-1,n} = 2$, and initial exchange polynomials*

$$\theta_i^0(u, v) = p_{i,0}u + p_{i,1}v \ \ (i \in [\![1, n-1]\!]) \quad \textit{and} \quad \theta_n^0(u, v) = au^2 + \lambda uv + cv^2,$$

then every cluster variable corresponds to a connected subgraph of



We have similar conjectures for (twisted) affine types $C_n^{(1)}$, $A_2^{(2)}$, $\widetilde{BC}$, and $A_{2k-1}^{(2)}$. In each of these cases, the infinitely many non-initial cluster variables are conjectured to be attached to connected subgraphs of infinite periodic snake graphs.

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Counting D_n Friezes

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(joint work with Pierre-Guy Plamondon)

An observation credited to Caldero in [1] is that Fomin and Zelevinsky's cluster algebras [4] allow for a huge generalization of Coxeter and Conway's definition of frieze. In particular one way to define friezes is to say that they are ring homomorphisms from a cluster algebra to the ring of integers such that all cluster variables are sent to positive integers. In cluster algebras of Dynkin type, a cluster-free definition may be given as follows [1, Section 3]. Let $C = (C_{i,j})_{n\times n}$ be a Cartan matrix of Dynkin type Δ, and assume that we have an acyclic orientation of the associated Dynkin diagram. Then a *frieze of type* Δ is a collection of positive integers $a(j,m)$, with $j \in \{1,\dots,n\}$ and $m \in \mathbb{Z}$, such that

$$a(j,m)a(j,m+1) = 1 + \Big(\prod_{j\to i} a(i,m)^{|C_{i,j}|}\Big)\Big(\prod_{i\to j} a(i,m+1)^{|C_{i,j}|}\Big).$$

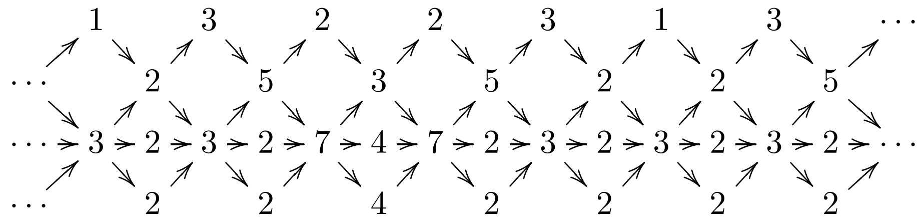

Figure 1. A frieze in type D_5.

For friezes of type D_n, there is a model developed by Schiffler [6] (see also [2] and [3]) involving tagged arcs in a punctured polygon, which is the main tool used in proof of the following theorem.

Theorem 1. *The number of D_n friezes is* $\sum_{m=1}^{n} d(m)\begin{pmatrix} 2n-m-1 \\ n-m\end{pmatrix}$, *where $d(m)$ is the number of divisors of m.*

In the case $n = 4$, the result is 51, agreeing with [5]. Using this result and others, we can count friezes in types B_n, C_n and G_2 by folding Dynkin diagrams:

Corollary 1. $\sum_{m\le\sqrt{n+1}}\begin{pmatrix} 2n-m^2+1 \\ n\end{pmatrix}$, $\begin{pmatrix} 2n \\ n\end{pmatrix}$ *and* 9, *are the number of friezes in types B_n, C_n and G_2 is respectively.*

The theory of cluster algebras provides a way to construct friezes, namely by specializing variables of a given cluster to 1. The D_5 frieze above does not arise in this fashion. Thus it is worth noting that in types B_n, D_n and G_2, the number of friezes is strictly greater than the number of clusters.

For the other Dynkin types, we propose the following

Conjecture 1. *The number of friezes of type E_6, E_7, E_8 and F_4 is* 868, 4400, 26952 *and* 112, *respectively.*

In the case of E_6, evidence for this number was obtained in [5] and for E_7 and E_8, these numbers also agree with some preliminary computer calculations by Michael Cuntz for SL_3 friezes.

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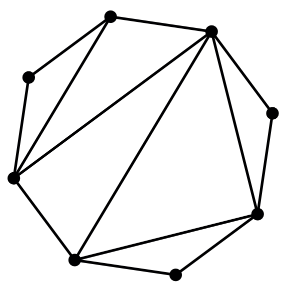
Multitriangulations, pseudotriangulations and friezes

Vincent Pilaud

(joint work with Michel Pocchiola)

Conway-Coxeter friezes are in bijection with triangulations of convex polygons [CC73]. Namely, a frieze $(a_{i,j})$ of width $n-3$ corresponds to a triangulation T of a convex n-gon as follows: the entry $a_{i,j}$ is the number of perfect matchings of the vertex-triangle graph of T (with nodes for triangles and vertices of T and edges connecting triangles to their vertices), where the nodes corresponding to vertices i and j are deleted. In particular, the 1's in the frieze correspond to the diagonals of T and the values $a_{i,i+1}$ in the first row of the frieze are given by the number of triangles incident to vertex i in T. See Figure 1.



...	0	0	0	0	0	0	0	0	0	...
...	1	1	1	1	1	1	1	1	1	...
...	2	1	3	3	1	3	1	4	2	...
...	7	1	2	8	2	2	2	3	7	...
...	3	1	5	5	3	1	5	5	3	...
...	2	2	2	3	7	1	2	8	2	...
...	1	3	1	4	2	1	3	3	1	...
...	1	1	1	1	1	1	1	1	1	...
...	0	0	0	0	0	0	0	0	0	...

Figure 1. A triangulation of the octagon (left) and its corresponding Conway-Coxeter frieze of width 5 (right).

The talk presents a similar duality between certain generalizations of the triangulations of a convex n-gon and certain pseudoline arrangements with a specific support. A *pseudotriangulation* of a point set P (in general position in the plane) is a maximal set of pointed and pairwise non-crossing edges between points of P. A k*-triangulation* of a convex n-gon is a maximal set of diagonals so that no $k+1$ of them mutually cross. Examples are given in Figure 2 and references can be found in [RSS08] and [PS09]. Note that both families contain the triangulations of a convex n-gon (when P is convex for pseudotriangulations, and when $k = 1$ for k-triangulations).

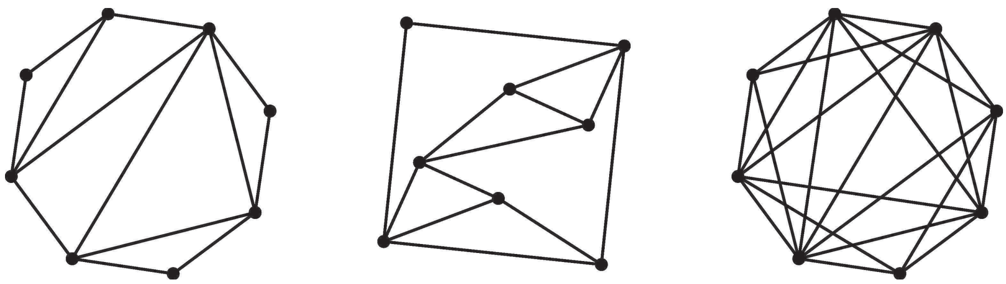


FIGURE 2. A triangulation of the octagon (left), a pseudotriangulation of an 8-point set (middle), and a 2-triangulation of the octagon (right).

These graphs have relevant combinatorial properties generalizing that of classical triangulations:

- As triangles for triangulations, these graphs can be seen as complexes of cells: *pseudotriangles* (*i.e.* simple closed polygons with three convex corners connected by three concave chains) for pseudotriangulations and k*-stars* (*i.e.* $2k+1$ vertices in convex position related by their k-edges) for k-triangulations. See Figure 3.

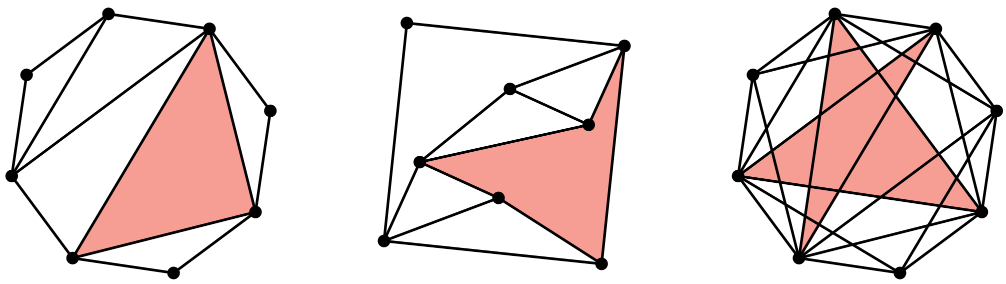


FIGURE 3. A triangle (left), a pseudotriangle (middle), and a 2-star (right).

- Any triangulation (resp. pseudotriangulation, resp. k-triangulations) of n points contains $n-2$ triangles (resp. $n-2$ pseudotriangles, resp. $n-2k$ k-stars) and $2n-3$ diagonals (resp. $2n-3$ edges, resp. $k(2n-2k-1)$ diagonals. See Figure 2.
- Any sufficiently internal edge can be flipped to a unique other edge. See Figure 4. The flip graph is regular and it is the graph of a simplicial sphere. It is even known that this sphere is polytopal for pseudotriangulations, but this question remains open for multitriangulations.

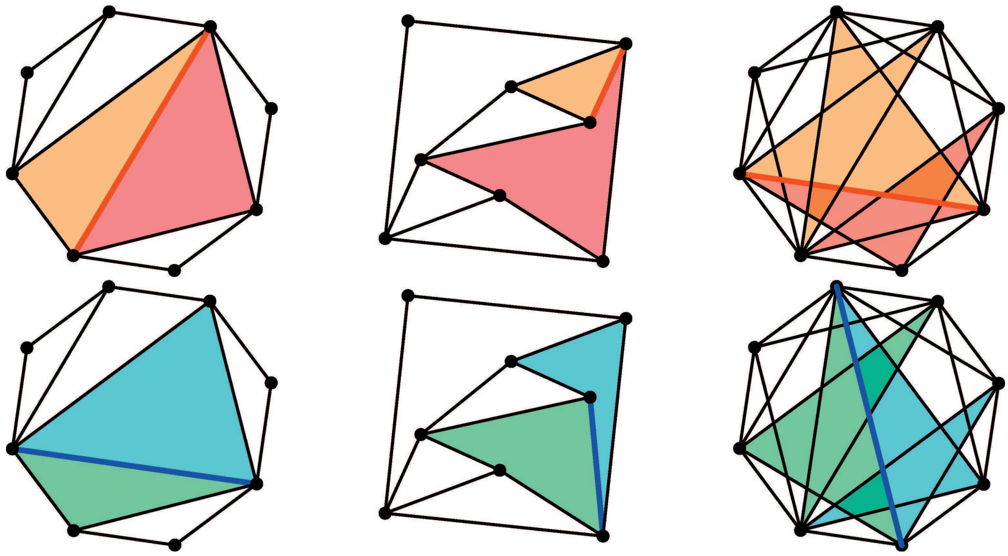


FIGURE 4. Flipping a diagonal in a triangulation (left), a pseudotriangulation (middle), and a 2-triangulation (right).

The talk presents a unified explanation for these properties using a duality introduced in [PP12]. For a point set P, let P^* denote its dual pseudoline arrangement in the Möbius strip (line space of the plane). Then the triangulations of a convex polygon C (resp. the pseudotriangulations of a point set P, resp. the k-triangulations of a convex polygon C) are in bijections with the pseudoline arrangements supported by C^* minus its first level (resp. P^* minus its first level, resp. C^* minus its first k levels). This correspondence is illustrated in Figure 5.

The talk presents this duality [PP12], the connection to subword complexes in finite Coxeter groups [KM04], and some related questions related to friezes.

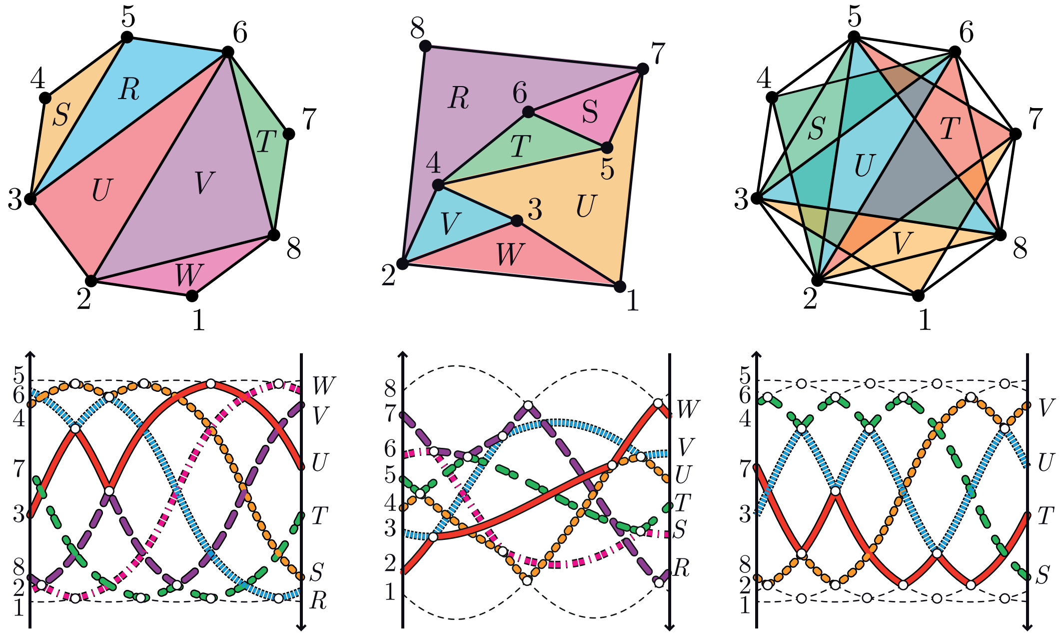


Figure 5. Duality between triangulations (resp. pseudotriangulations, resp. multitriangulations) and pseudoline arrangements with a given support.

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Higher-dimensional tropical friezes

Hugh Thomas

Frieze patterns being two-dimensional arrays of numbers, it seems natural to ask about a higher-dimensional generalization. I do not propose an answer to this problem. However, based on the approach taken in a paper with Steffen Oppermann [3], it is possible to define a higher-dimensional analogue of *tropical* friezes.

The tropical version of the frieze relation $EW = NS + 1$ is $E + W = \max(N + S, 0)$. This can be obtained by replacing multiplication by addition and addition by max; there is also a standard way in which the tropical relation arises as a limit of the classical relation. See [2] for more details on tropical frieze relations.

From that perspective, though, it is not clear how to generalize to higher dimensions. To see how one might do so, we turn to the approach to tropical cluster algebras described in [1]. Staying in the type A case, we have a surface with marked points on the boundary numbered $1, \dots, n$. A lamination L is a collection of non-crossing curves, with positive rational weights, beginning and ending on the boundary of the disk, avoiding the marked points. Let γ be a diagonal of the disk. Define c_γ to be the sum of the weights of the curves of L intersecting γ. Then, we observe that if $1 \leq i < j < k < \ell \leq n$, we have

$$c_{ik} + c_{j\ell} = \max\left(c_{ij} + c_{k\ell}, c_{jk} + c_{i\ell}\right). \tag{1}$$

If we assume that $c_{i,i+1} = 0$ for all i and set $j = i+1, \ell = k+1$, the above equation is exactly the tropical frieze relation we would like.

However, $c_{i,i+1} = 0$ for all i actually forces the lamination to be empty. The solution to this problem is to use $\mathcal{A}$-laminations, as explained in [1]. We say that a curve is "short" if cuts off just a single marked point. We note that (1) still holds for a lamination L even if we allow short curves of L to carry a negative weight. We therefore define a $\mathcal{A}$-lamination to be one in which short curves may carry a negative weight, and where we insist that the total weight of all curves intersecting any boundary component is zero. The resulting collection c_{ij} form a tropical frieze pattern (in $\mathbb{Q}$), and [1] shows that any tropical frieze pattern can be realized in this way.

We now turn a higher-dimensional analogue of the surface picture, replacing the disk by a $2d$-dimensional cyclic polytope with n vertices, as described in [3]. The frieze entries are indexed by the set $\mathcal{I}$ of increasing $d+1$-tuples from $\{1, \dots, n\}$ such that each successive pair of entries differs by at least 2 (including cyclically: i.e., a $d+1$-tuple in $\mathcal{I}$ may not contain both 1 and n). The $d+1$-tuples of $\mathcal{I}$ correspond to d-dimensional simplices not lying fully on the boundary of the cyclic polytope.

For A a subset of $\{1, \dots, d+1\}$, write e_A for the 0/1-vector with 1's in the positions specified by A. Write $\underline{1}$ for the all-ones vector. Let $I \in \mathcal{I}$. The higher tropical cluster exchange relation is:

$$c_{I+\underline{1}} = \max\left(\sum_{A \subsetneq \{1,\dots,d+1\}} (-1)^{d-|A|} c_{I+e_A}, (-1)^d c_I\right). \tag{2}$$

We interpret c_{I+e_A} as zero if $I+e_A \notin \mathcal{I}$. Addition of $d+1$-tuples is done componentwise and modulo n.

We refer to a collection of integers c_I satisfying (2) as a *higher tropical frieze pattern*. A notion of higher-dimensional laminations is defined in [3], and it can be extended to a notion of $\mathcal{A}$-laminations by analogy with the classical case. Weighted intersections with $\mathcal{A}$-laminations satisfy (2).

Questions. Can all higher tropical frieze patterns be realized by some $\mathcal{A}$-lamination?

Is there a nice description of the collection of all $\mathcal{A}$-laminations?

Is there a more explict description of the collection of higher tropical frieze patterns?

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Polynomially weighted walks around dissected polygons

Christine Bessenrodt

In 1973, Conway and Coxeter introduced arithmetical frieze patterns and classified the friezes via triangulated polygons. Broline, Crowe and Isaacs in 1974 explained the frieze matrix associated to the fundamental region of the frieze pattern as enumerating special arcs of the triangulated polygon. They also showed that the determinant of such a frieze matrix associated to a triangulation of an n-gon is just $-(-2)^{n-1}$; this formula was generalized 2012 in a cluster algebra setting by Baur and Marsh [1].

In joint work with Holm and Jørgensen, the classical situation was already generalized to d-angulations [2], and the determinantal result refined to a determination of the Smith normal form. In the talk we discussed a further generalization and refinement to polynomially weighted walks around polygons with arbitrary dissections (see [3]). Here, variable weights are put on the pieces; the weights of the walks capture the dissection in detail, by collecting the weights of the pieces chosen along the walk. Introducing also variable edge weights allows for crucial reduction arguments. The weight matrix associated to the walks between the vertices of the polygon then satisfies a complementary symmetry condition. Furthermore, its determinant is a multisymmetric multivariate polynomial given explicitly. Indeed, using elementary row and column operations over a ring of Laurent polynomials and reduction arguments, this matrix can be transformed into a diagonal form, with a contribution from each piece of the dissection, and trailing 1's. Going into the opposite direction and considering the generalized polynomial frieze associated to the weight matrix of a dissected polygon, the non-zero local determinants are explicitly given monomials; the non-vanishing condition is intricately related to the geometry of the dissection. In the case of a triangulation with an even number of triangles, it is easy to see that specializing the weights of all the pieces to -1 produces a Conway-Coxeter frieze without zeros but where also negative numbers occur; indeed, it may be obtained from a positive frieze by multiplying every second row by -1. This is also related to recent work by Bruce Fontaine who showed that there are no further non-zero integral friezes [4].

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Growth behaviour of infinite frieze patterns

Manuela Tschabold

(joint work with Karin Baur, Klemens Fellner, Mark Parsons)

In the first part of this talk, we present the notion of infinite frieze patterns of positive integers in the plane and provide a characterisation of periodic infinite friezes via triangulations of once-punctured discs and annuli. Part of this is joint work with Karin Baur and Mark Parsons [2].

An *infinite frieze pattern* of positive integers is an array $(m_{ij})_{i,j\in\mathbb{Z},j\geq i-2}$ of infinitely many shifted bi-infinite rows such that $m_{i,i-2} = 0$, $m_{i,i-1} = 1$ for all i and $m_{ij} \in \mathbb{Z}_{>0}$ for all $i \leq j$,

$$\begin{array}{ccccccccccccccc}
 & 0 & & 0 & & 0 & & 0 & & 0 & & & & & \\
\cdots & & 1 & & 1 & & 1 & & 1 & & 1 & & \cdots & & \\
 & & m_{-1,-1} & & m_{00} & & m_{11} & & m_{22} & & m_{33} & & & & \\
 & & \cdots & m_{-1,0} & & m_{01} & & m_{12} & & m_{23} & & m_{34} & & \cdots & \\
 & & & & m_{-1,1} & & m_{02} & & m_{13} & & m_{24} & & m_{35} & & \\
 & & & & & & & \ddots & & & & \ddots & & &
\end{array}$$

satisfying the *unimodular rule* everywhere: $m_{ij}m_{i+1,j+1} - m_{i+1,j}m_{i,j+1} = 1$ for all $i \leq j$. Such a frieze is called *n-periodic* if the first non-trivial row is periodic with period n.

The following is one of our main results.

Theorem 1 ([3],[2]). *Triangulations of once-punctured discs and annuli give rise to periodic infinite friezes, and all such friezes arise in this way. Every entry in a periodic infinite frieze is given as a matching number for any associated triangulation.*

Furthermore, each diagonal in a frieze arising from a triangulation of a once-punctured disc is made up of a collection of arithmetic sequences, the number of which depends on the period of the frieze ([3, Proposition 3.11]). Therefore such friezes can be considered to have linear growth.

In the second part, we then introduce the notion of growth coefficients for periodic infinite friezes and end by giving some recent results on this new subject, especially on the growth behaviour of periodic infinite friezes. This is work in progress with Karin Baur, Klemens Fellner and Mark Parsons [1].

For every n-periodic infinite frieze, the difference between the nth and the $(n-2)$th non-trivial rows is constant. More precisely, we have

Theorem 2 ([1]). *Given an n-periodic infinite frieze $(m_{ij})_{i,j}$, let $s = m_{1,n} - m_{2,n-1}$. Then $s = m_{k+1,k+n} - m_{k+2,k+n-1}$, for all $k \in \mathbb{Z}$.*

There is a linear recursion formula for the entries in a diagonal of a periodic infinite frieze that depends on this value.

Theorem 3 ([1]). *Let $(m_{ij})_{i,j}$ be an n-periodic infinite frieze and let $s = m_{1,n} - m_{2,n-1}$. Then we have $m_{i,j+2n} = sm_{i,j+n} - m_{ij}$, and $m_{i-2n,j} = sm_{i-n,j} - m_{ij}$.*

In the special case where n is the minimal period of the infinite frieze, we define the *kth growth coefficient* ($k \geq 0$) by $s_0 := 2$, and $s_k := m_{1,kn} - m_{2,kn-1}$, otherwise.

For the growth coefficients of a periodic infinite frieze, we have $s_k \geq 2$ and $s_k \leq s_{k+1}$ for all $k \geq 0$. Besides, they are recursively related to each other as follows $s_{k+2} = s_1 s_{k+1} - s_k$, for all $k \geq 0$. Moreover, we can give a closed formula for every growth coefficient in terms of s_1. For all $k \geq 1$, we have

$$s_k = s_1^k + k \sum_{l=1}^{\lfloor \frac{k}{2} \rfloor} (-1)^l \frac{1}{k-l} \binom{k-l}{l} s_1^{k-2l}. \tag{1}$$

These properties are used in establishing the following proposition.

Proposition 1 ([1]). *Given a periodic infinite frieze, the following are equivalent:*

(1) *There exists $k > 0$ such that $s_k = 2$.*
(2) *$s_k = 2$ for all $k \geq 0$.*
(3) *The frieze arises from a triangulation of a once-punctured disc.*

Thus we can conclude, using (1), that triangulations of once-punctured discs provide the only friezes of linear growth, while friezes arising from triangulations of annuli have exponential growth.

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Categorification of quantum cluster algebras and quantum groups

Dylan Rupel

Quantum cluster algebras [3] are non-commutative deformations of classical cluster algebras [4] and share much of their combinatorial structure. The main philosophy leading to this quantization can be given as follows:

- Each cluster should be replaced by a collection of quasi-commuting elements $\mathbf{X} = \{X_1, \ldots, X_m\}$ where $X_iX_j = v^{2\Lambda(\varepsilon_i,\varepsilon_j)}X_jX_i$ for some skew-symmetric bilinear form $\Lambda : \mathbb{Z}^m \times \mathbb{Z}^m \to \mathbb{Z}$. Since this must be true of every cluster obtained by mutations from an initial cluster we are forced to impose the condition $\Lambda(\mathbf{b}^k, \varepsilon_\ell) = 0$ for $k \neq \ell$ where $\mathbf{b}^k$ denotes the k^{th} column of the initial exchange matrix B. Miraculously this compatibility is mantained under arbitrary sequences of mutations.
- The quantum torus algebra $\mathcal{T} = \mathbb{Z}[v^{\pm1}][X_1^{\pm1}, \ldots, X_m^{\pm1}]$ generated by $\mathbf{X}$ admits an anti-involution which fixes each X_i and interchanges v with v^{-1}. Since each initial cluster variable is fixed by this *bar-involution* we ask the same to be true of every non-initial cluster variable as well, this forces a unique choice of scaling for each monomial in the exchange relations.

Much of the well-known theory from classical cluster algebras carries over to this setting, in particular the following analogue of the Laurent phenomenon.

Theorem 1. [3] *Let $\Sigma = (\mathbf{X}, B)$ denote an initial quantum seed. Then each quantum cluster variable obtained from Σ by a sequence of mutations is contained in $\mathcal{T}$.*

Thus we obtain the following natural question: what are the initial cluster Laurent expansions of the non-initial quantum cluster variables?

To answer this we view the prinicipal block of B as the adjacency matrix for a (valued) quiver Q, which we assume to contain no oriented cycles. The category of representations of Q forms a hereditary abelian category.

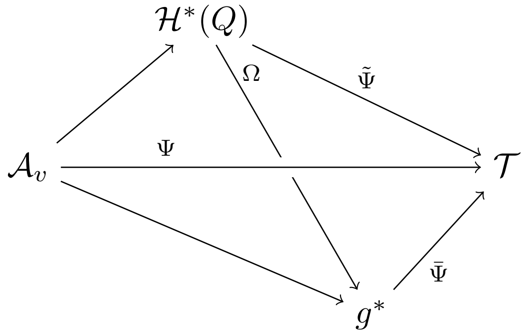
Theorem 2. [15, 11, 16] *For each indecomposable representation V of Q satisfying* $\mathrm{Ext}(V, V) = 0$*, the quantum Laurent polynomial*

$$X_V = \sum_{\mathbf{e}\in\mathbb{Z}^n_{\geq 0}} v^{-\langle \mathbf{e}, \underline{\dim}\, V - \mathbf{e}\rangle} \big| Gr_{\mathbf{e}}(V) \big| X^{-\mathbf{e}^* - {}^*(\underline{\dim}\, V - \mathbf{e})}$$

is a (non-initial) quantum cluster variable and each one arises in this way.

At the heart of the definition of a quantum cluster algebra is a deep conjecture that certain quantum groups admit quantum cluster algebra structures and that the recursively, combinatorially defined quantum cluster monomials are elements of the (dual) canonical basis. This has been proven in the acyclic, skew-symmetric case [9] using the theory of perverse sheaves on quiver varieties. We offer an alternative proof based on relating the categorification of Theorem 2 with the categorification of quantum groups [7, 8, 14] using irreducible characters of quiver Hecke algebra representations.

This is accomplished using several well-known algebra homomorphisms out of the quantum group $\mathcal{A}_v$. The first is an embedding [12] into the (dual) Hall algebra $\mathcal{H}^*(Q)$ with basis $\{[V]^*\}$ indexed by the isomorphism classes of representations of Q whose structure constants count extensions between representations. Taking a geometric interpretation for this map leads [10] to the construction of the (dual) canonical basis in terms of perverse sheaves. The second is a map Ψ to the quantum torus $\mathcal{T}$ (this is a special case of a more general setup used to establish the Gelfand-Kirillov conjectures for $\mathcal{A}_v$ [5, 6, 1]). Finally an embedding [13] into a quantum shuffle algebra g^*, this is the natural home for the characters of the quiver Hecke algebra. These can be completed to a commuting tetrahedron of algebra homomorphisms:



(1)

The next result combined with Theorem 2 shows how this relates to the quantum cluster algebra associated to Q.

Theorem 3. [2] *For any representation V of Q we have $\tilde{\Psi}([V]^*) = X_V$.*

The dual canonical basis conjecture states that (a localization of) $\Psi(\mathcal{A}_v)$ carries a quantum cluster algebra structure and that the quantum cluster monomials are contained in the image of the dual canonical basis. The following result establishes this conjecture in the case of acyclic, skew-symmetric quantum cluster algebras.

Theorem 4. [17] *Let Q be an acyclic quiver. For each indecomposable representation V of Q satisfying* $\mathrm{Ext}(V,V) = 0$, *$\Omega([V]^*)$ is contained in the image of $\mathcal{A}_v$ and coincides with an element of the dual canonical basis.*

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Asymptotic triangulations and Coxeter transformations of the annulus

Hannah Vogel

Asymptotic triangulations can be viewed as limits of triangulations under the action of the mapping class group. Asymptotic triangulations were introduced by Baur and Dupont with respect to unpunctured mark surfaces. These asymptotic triangulations can be mutated as usual triangulations, and they provide a natural way to compactify the usual exchange graph of the triangulations of an annulus.

An asymptotic triangulation is defined by the presence of strictly asymptotic arcs. Strictly asymptotic arcs are isotopy classes of arcs starting at a marked point on the boundary of a surface and spiraling either positively or negatively around a non-contractible closed curve in the surface. These strictly asmptotic arcs have infinite length. An asymptotic triangulation contains at least two strictly asymptotic arcs, and possibly non-asymptotic arcs as well.

To any asymptotic triangulation we can associate a quiver. Such a quiver may have loops and 2-cycles, and hence classical quiver mutation cannot be applied. We introduce a modified version of quivers of asymptotic triangulations in order to use the classical quiver mutation. We also introduce quivers with potentials associated to asymptotic triangulations, and can then work with quiver mutation on an algebraic level.

The process of going from a finite triangulation to an asymptotic triangulation constitutes applying the Dehn twist infinitely many times to a triangulation. Applying the Dehn twist infinitely many times causes some arcs of the triangulation to become identified, while breaking other arcs into two parts so that we are left with two triangulations – one based at each boundary component of our annulus. The Dehn twist provides us with a topological way of obtaining asymptotic triangulations.

A combinatorial method of obtaining asymptotic triangulation is by using a Coxeter transformation. A Coxeter transformation is done by applying a sequence of flips to the arcs of the triangulation. This sequence is determined by the associated quiver. We can obtain an admissible sequence of vertices of the associated quiver (going from sources to sinks), and then perform the sequence of corresponding flips in the triangulation. It turns out that the Coxeter transformation and Dehn twist behave the same in the limit. This allows us to use whichever process of obtaining an asymptotic triangulation that is the most useful for that setting. The benefit of having a combinatorial method to describe this process is that we can now study other variables and systems associated to the surface. Cluster variables associated to asymptotic triangulations were studied by Felikson and Tumarkin. Felikson and Tumarkin consider the induced triangulation of the double cover of the annulus, and then use the signed adjacency quiver, which is free of loops and 2-cycles, and can thus be mutated using classical quiver mutation. This procedure described for an annulus can be done for any triangulated hyperbolic surface.

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Superfriezes and superclusters

Valentin Ovsienko

A supersymmetric analog of Coxeter's frieze patterns was introduced in a joint work with S. Morier-Genoud and S. Tabachnikov [1]. A *superfrieze* is the following array

$$
\begin{array}{cccccccccccc}
 & \cdots & 0 & & & & 0 & & & & 0 & \\
\cdots & 0 & & 0 & & 0 & & 0 & & 0 & & \cdots \\
1 & & & & 1 & & & & 1 & & & \cdots \\
 & \varphi_{0,0} & & \varphi_{\frac{1}{2},\frac{1}{2}} & & \varphi_{1,1} & & \varphi_{\frac{3}{2},\frac{3}{2}} & & \varphi_{2,2} & & \cdots \\
 & & f_{0,0} & & & & f_{1,1} & & & & f_{2,2} & \\
 & \varphi_{-\frac{1}{2},\frac{1}{2}} & & \varphi_{0,1} & & \varphi_{\frac{1}{2},\frac{3}{2}} & & \varphi_{1,2} & & \varphi_{\frac{3}{2},\frac{5}{2}} & & \cdots \\
f_{-1,0} & & & & f_{0,1} & & & & f_{1,2} & & & \\
 & \iddots & & \iddots & & \ddots & & \ddots & & \ddots & & \ddots
\end{array}
$$

where $f_{i,j}$ are even and $\varphi_{i,j}$ are odd elements of a $\mathbb{Z}_2$-graded ring, and where every *elementary diamond*:

$$\begin{array}{ccccc} & & B & & \\ & \Xi & & \Psi & \\ A & & & & D \\ & \Phi & & \Sigma & \\ & & C & & \end{array}$$

satisfies the following conditions:

$$AD - BC = 1 + \Sigma\Xi, \qquad A\Sigma - C\Xi = \Phi, \qquad B\Sigma - D\Xi = \Psi,$$

called the *frieze rule*.

A superfrieze is *closed* if it is also bounded below by a row of 1's and two rows of 0's. The number of even rows between the rows of 1's is called the *width* of the superfrieze. The main properties of superfriezes are similar to those of classical friezes.

Theorem 1. [1] *A generic superfrieze of width m satisfies the following glide symmetry*

$$\begin{aligned} f_{i,j} &= f_{j-m-1,i-2}, \\ \varphi_{i,j} &= \varphi_{j-m-\frac{3}{2},i-\frac{3}{2}}, \\ \varphi_{i+\frac{1}{2},j+\frac{1}{2}} &= -\varphi_{j-m-1,i-1}, \end{aligned} \tag{1}$$

for all $i, j \in \mathbb{Z}$.

In particular, the rows of a superfrieze of width m satisfy the following periodicity property:

$$\varphi_{i+n,j+n} = -\varphi_{i,j}, \qquad f_{i+n,j+n} = f_{i,j},$$

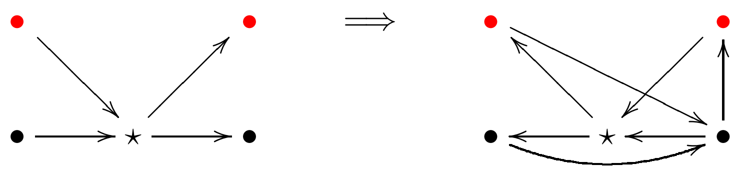
where $n = m + 3$.

The *Laurent phenomenon* is one of the main properties of the Coxeter friezes. Similar phenomenon occurs in the superfriezes.

Theorem 2. [1] *Entries of a superfrieze are Laurent polynomials in the entries of any diagonal.*

Similarly to the classical case, see [2], superfriezes are related with linear difference operators generalizing classical Hill's operators.

Using superfriezes as a starting example, I present an attempt [3] to develop the notion of cluster superalgebra. The main ingredients are modified exchange relations and quiver mutations. The vertices of the quiver are labeled by the even and odd coordinates. Essentially, the mutations of such a quiver (observe that the two upper vertices are colored while the three lower ones are black) are defined by the following modified rule:



An open question. Use superfriezes to define a structure of cluster supermanifold on the moduli spaces $\mathfrak{M}_{0.n}$ of n-gons in the contact projective line $\mathbb{P}^{1|1}$.

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