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Convex Geometry and its Applications

Organised by
Franck Barthe, Toulouse
Martin Henk, Berlin
Monika Ludwig, Wien

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Abstract. The past 30 years have not only seen substantial progress and lively activity in various areas within convex geometry, e.g., in asymptotic geometric analysis, valuation theory, the L_p-Brunn-Minkowski theory and stochastic geometry, but also an increasing amount and variety of applications of convex geometry to other branches of mathematics (and beyond), e.g. to PDEs, statistics, discrete geometry, optimization, or geometric algorithms in computer science. Thus convex geometry is a flourishing and attractive field, which is also reflected by the considerable number of talented young mathematicians at this meeting.

Mathematics Subject Classification (2010): 52A, 68Q25, 60D05.

Introduction by the Organisers

The meeting *Convex Geometry and its Applications*, organised by Franck Barthe, Martin Henk and Monika Ludwig, was held from December 6 to December 12, 2015. It was attended by 55 participants working in all areas of convex geometry. Of these 19% were female and about one third were younger participants. The programme involved 12 plenary lectures of one hour's duration, 18 shorter lectures and a problem session on Wednesday evening. Some highlights of the program were as follows.

In the opening lecture, Mark Rudelson gave a remarkable talk on the complexity of the family of all n-dimensional unconditional bodies. The given double exponentially lower bound (in n) not only answers a question by Pisier but it also shows that the class of n-dimensional unconditional convex bodies cannot be well approximated by projections of sections of an N-dimensional simplex as long as N

is sub-exponential in n. The later result is relevant for the algorithmic treatment of unconditional bodies.

Erwin Lutwak and Gaoyong Zhang shared a one hour lecture in order to introduce new fundamental measures of convex bodies, the dual curvature measures. These measures fill a longstanding gap in the (dual) Brunn-Minkowski theory and may be regarded as the differentials of the dual quermassintegrals and as the missing dual counterparts to Federer's area measures. The associated Minkowski problems miraculously join the classical (and completely solved) Aleksandrov problem and the modern (and wide open) logarithmic Minkowski problem.

Shiri Artstein-Avidan gave a beautiful inspiring talk on Godbersen's conjecture on the mixed volumes of K and $-K$ and recent developments. In particular, she showed by a very elegant argument that the conjectured bound is true "in the average".

In the first part of his plenary talk, Pierre Calka gave a highly stimulating introduction to the area of random polytopes. In the second part he introduced, among others, a new method for Poisson random polytopes by which (now) explicit calculations of limiting variances for certain geometric quantities (e.g., number of k-faces, the volume) are possible.

Seymon Alesker presented a delicate new construction of continuous valuations on convex sets which is based on quaternionic Monge-Ampère operators of convex functions. One key ingredient here is a quaternionic version of a result by Chern-Levine-Nierenberg in the complex case.

There were also several excellent talks by young researchers.

In a joint (plenary and short) lecture Eugenia Saorín Gomez and Judit Abardia gave interesting characterizations of convex bodies valued operators which satisfy certain types of inequalities such as Brunn-Minkowski and/or Rogers-Shephard type inequalities.

Lukas Parapatits presented a complete classification of Borel measurable $\mathrm{SL}(n)$-covariant symmetric tensor valuations on convex polytopes containing the origin in their interiors. This is a remarkable result that contains previous classification results of vector and matrix valuations as special cases.

Using tools from convex geometry, Ronen Eldan gave a new universal construction of a self-concordant barrier function. Due to the fundamental work of Nesterov&Nemirovski, these functions are of central interest for interior point methods.

Based on a functional adaption of Ball's approach to cube slicing, Galyna Livshyts presented optimal bounds for marginal densities of product measures and in this way also an alternate approach to a recent theorem by Rudelson and Vershynin.

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Workshop: Convex Geometry and its Applications

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Abstracts

Characterization of Minkowski valuations by means of bounds on the volume of the image

Judit Abardia
(joint work with A. Colesanti and E. Saorín Gómez)

Let $\mathcal{K}^n$ denote the set of convex bodies (compact and convex sets) in $\mathbb{R}^n$, endowed with the Hausdorff metric, and let $\mathcal{K}^n_s$ denote the set of convex bodies in $\mathbb{R}^n$ which are symmetric with respect to the origin. Let $(\mathcal{A},+)$ be an Abelian semigroup. An operator $\varphi : \mathcal{K}^n \longrightarrow \mathcal{A}$ is called a *valuation* if for any $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$,

$$\varphi(K \cup L) + \varphi(K \cap L) = \varphi(K) + \varphi(L).$$

An active area of valuation theory deals with the characterization of classical (and new) objects appearing in convex geometry. The first fundamental classification result in this context goes back to Hadwiger, who classified the continuous, translation invariant, real-valued valuations, i.e., $(\mathcal{A},+) = (\mathbb{R},+)$ in the above definition of valuation, which are also invariant under the rotations of the Euclidean space. Since then, many other classification results have been obtained.

Together with the case of real-valued valuations, there has been an increasing interest in the so-called Minkowski valuations. A valuation where $(A,+) = (\mathcal{K}^n,+)$ is called a *Minkowski valuation*, i.e., it is a valuation which takes values in the space of convex bodies endowed with the *Minkowski sum*, defined as

$$K + L = \{x + y \,:\, x \in K, y \in L\}, \quad K, L \in \mathcal{K}^n.$$

A systematic study of characterization results in the theory of Minkowski valuations was started by M. Ludwig (see [5, 6]). She obtained, for instance, the following characterization result for the *difference body operator* $D : \mathcal{K}^n \longrightarrow \mathcal{K}^n$ defined by

$$DK := K + (-K),$$

where $-K := \{x \in \mathbb{R}^n \,:\, -x \in K\}$.

Theorem A ([6])**.** *Let $n \geq 2$. An operator $\varphi : \mathcal{K}^n \longrightarrow \mathcal{K}^n$ is a continuous, translation invariant,* SL(n)*-covariant Minkowski valuation if and only if there is a $\lambda \geq 0$ such that $\varphi(K) = \lambda DK$.*

If instead of SL(n)-covariance in Theorem A, SL(n)-contravariance is imposed, then a characterization for the projection body operator is obtained (see [5]).

After these two seminal results of M. Ludwig, a lot of research on Minkowski valuations has been launched and characterization results for other groups acting on $\mathcal{K}^n$ and for certain subfamilies of $\mathcal{K}^n$ have been obtained.

A new classification result for the difference body operator, without using the Minkowski valuation property has been obtained recently by R. Gardner, D. Hug and W. Weil, in [3].

Theorem B ([3]). *Let $n \geq 2$. An operator $\varphi : \mathcal{K}^n \longrightarrow \mathcal{K}^n_s$ is a continuous, translation invariant and* GL(n)*-covariant o-symmetrization if and only if there is a $\lambda \geq 0$ such that $\varphi K = \lambda DK$.*

An *o-symmetrization* is an operator $\varphi : \mathcal{K}^n \longrightarrow \mathcal{K}^n_s$, taking values in the space of symmetric convex bodies with respect to the origin.

Theorem B was obtained as part of a systematic study of operations between convex sets (see also [4] and [7]).

With the aim of characterizing known operators in convex geometry by their essential properties, it is also natural to consider inequalities satisfied by the corresponding operators. However, it seems that the first paper to consider some inequality as a characterization property is [2], where the authors study continuous, translation invariant, real-valued valuations satisfying a Brunn-Minkowski type inequality. In the case of Minkowksi valuations, a characterization result for the difference body using its fundamental affine inequality relating the volume of the difference body DK and the volume of K - the so-called *Rogers-Shephard* or *difference body inequality* - has been recently obtained in [1] (see the abstract of the talk given by Eugenia Saorín Gómez).

In the next relation, the lower bound for the volume of DK is a direct consequence of the Brunn-Minkowski inequality and the upper bound is the Rogers-Shephard inequality.
For $K \in \mathcal{K}^n$, it holds

$$2^n \mathrm{vol}(K) \leq \mathrm{vol}(DK) \leq \binom{2n}{n} \mathrm{vol}(K).$$

Equality holds on the left-hand side if and only if K is centrally symmetric and on the right hand side precisely if K is a simplex.

As stated in the abstract of Eugenia Saorín Gómez, we say that an operator $\varphi : \mathcal{K}^n \longrightarrow \mathcal{K}^n$ *satisfies a Rogers-Shephard type inequality* (in short RS) if there exists a constant $C > 0$ such that for all $K \in \mathcal{K}^n$

$$\mathrm{vol}(\varphi K) \leq C \mathrm{vol}(K).$$

Analogously, φ *satisfies a Brunn-Minkowski type inequality* (in short BM) if there exists a constant $c > 0$ such that for all $K \in \mathcal{K}^n$

$$c \mathrm{vol}(K) \leq \mathrm{vol}(\varphi K).$$

In the talk, we presented the following result:

Theorem 1. *Let $n \geq 2$ and $\varphi : \mathcal{K}^n \longrightarrow \mathcal{K}^n_s$ be a continuous translation invariant Minkowski valuation satisfying Brunn-Minkowski and Rogers-Shephard type inequality. Then, either φ is homogenous of degree one or there exist a centered segment S and an o-symmetric $(n-1)$-dimensional convex body L, with $\dim(L + S) = n$ such that $\varphi K = L + \mathrm{vol}(K)S$ for every $K \in \mathcal{K}^n$.*

In order to completely characterize those operators, it remains to describe the 1-homogeneous valuations which satisfy the properties in the theorem above. However, the above result represents a first step towards a better understanding of the continuous, translation invariant Minkowski valuations without any equivariance under the action of some group $G \subseteq \mathrm{GL}(n)$.

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Constructing continuous valuations on convex sets via Monge-Ampère operators

Semyon Alesker

The class of continuous (in the Hausdorff metric) valuations on convex sets has been studied extensively since Hadwiger defined it explicitly in 1940's. This class turned out to be very rich in geometric examples, structures, and applications to integral geometry. Despite numerous investigations, there are known rather few methods to construct them. The goal of this talk is to describe a relatively new method of construction based on the use of various Monge-Ampère (MA) operators of convex functions. Real and complex MA operators are classically known and I explain to use them in order to construct valuations (in fact, in this talk I omit the description of the real case, since the method must be refined in order to produce non-trivial examples of valuations). Furthermore I describe the quaternionic MA operator, which I introduced some years ago, and use it to construct more examples of valuations. Along the similar lines I have also introduced the MA operator in two octonionic variables again with applications to construction of valuations.

Let us start with the reminder of the definition of the complex MA operator on smooth functions. We write a complex variable z in the standard form $z = x + iy$ with $x, y \in \mathbb{R}$. Then we have standard differential operators on any smooth $\mathbb{C}$-valued function F:

$$\frac{\partial F}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial F}{\partial x} + i\frac{\partial F}{\partial y}\right), \ \frac{\partial F}{\partial z} := \overline{\frac{\partial \bar{F}}{\partial \bar{z}}} = \frac{1}{2}\left(\frac{\partial F}{\partial x} - i\frac{\partial F}{\partial y}\right)$$

Let $\Omega \subset \mathbb{C}^n$ be an open subset. The complex Hessian of a C^2-smooth function $f\colon \Omega \to \mathbb{R}$ is defined by

$$\text{Hess}_{\mathbb{C}}(f) := \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right)_{i,j=1}^{n}. \tag{1}$$

Then the complex MA operator of f is defined

$$\text{MA}_{\mathbb{C}}(f) := \det \left(\frac{\partial^2 f}{\partial z_i \partial \bar{z}_j} \right). \tag{2}$$

In order to construct valuations on convex sets one has to extend the complex MA operator to arbitrary convex functions. This can be done due to the following theorem.

Theorem 1 (Chern-Levine-Nirenberg, 1969 [7])**.** *For any convex (more generally, continuous plurisubharmonic) function $f\colon \Omega \to \mathbb{R}$ one can define its complex MA operator, denoted by $MA_{\mathbb{C}}(f)$, which is a non-negative measure uniquely characterized by the following properties:*
(a) if $f \in C^2$ then $MA_{\mathbb{C}}(f)$ is as in (2);
(b) if a sequence of convex (more generally, continuous plurisubharmonic) functions $\{f_i\}$ converges to a function f uniformly on compact subsets of Ω then $MA_{\mathbb{C}}(f_i) \overset{w}{\to} MA_{\mathbb{C}}(f)$ weakly in sense of measures;
(c) for any open subset $U \subset \Omega$ one has $MA_{\mathbb{C}}(f|_U) = MA_{\mathbb{C}}(f)|_U$.

Remark 2. A version of this result for the real MA operator on convex functions was proven earlier by A. D. Alexandrov in 1958 [1].

Proposition 3. *Let $\psi \in C_c(\mathbb{C}^n)$. Then the functional on convex compact subsets of $\mathbb{C}^n$ given by*

$$K \mapsto \int_{\mathbb{C}^n} \psi \cdot MA_{\mathbb{C}}(h_K),$$

where h_K is the supporting functional of K, is a translation invariant continuous valuation.

Notice that translation invariance is obvious, continuity follows from Theorem 1(b), and the valuation property follows from the next non-trivial result.

Theorem 4 (Blocki, 2000[6])**.** *Let $f, g, \min\{f,g\}\colon \Omega \to \mathbb{R}$ be convex (more generally, continuous psh) functions. Then*

$$MA_{\mathbb{C}}(\max\{f,g\}) + MA_{\mathbb{C}}(\min\{f,g\}) = MA_{\mathbb{C}}(f) + MA_{\mathbb{C}}(g).$$

This theorem implies the valuation property since for $K, L, K \cup L \in \mathcal{K}(\mathbb{C}^n)$ one has

$$h_{K\cup L} = \max\{h_K, h_L\},\ h_{K\cap L} = \min\{h_K, h_L\}.$$

Let us discuss the quaternionic case. Any $q \in \mathbb{H}$ is written in the standard form

$$q = t + ix + jy + kz \text{ with } t, x, y, z \in \mathbb{R}.$$

We are going to define quaternionic MA operator. First for any smooth $\mathbb{H}$-valued function we define

$$\frac{\partial F}{\partial \bar q} := \frac{\partial F}{\partial t} + i\frac{\partial F}{\partial x} + j\frac{\partial F}{\partial t} + k\frac{\partial F}{\partial t},\ \frac{\partial F}{\partial q} := \overline{\frac{\partial \bar F}{\partial \bar q}} = \frac{\partial F}{\partial t} - \frac{\partial F}{\partial x}i - \frac{\partial F}{\partial t}j - \frac{\partial F}{\partial t}k.$$

In the case of several quaternionic variables the operators $\frac{\partial}{\partial q_i}$ and $\frac{\partial}{\partial \bar q_j}$ commute with each other. Now define the quaternionic Hessian of a C^2-smooth real valued function f in n quaternionic variables: $\mathrm{Hess}_{\mathbb{H}}(f) := \left(\frac{\partial^2 f}{\partial q_i \partial \bar q_j}\right)_{i,j=1}^n$.

To define the MA operator one needs a notion of determinant. For matrices with non-commuting entries there are several approaches, but no one of them is as good as in the commutative setting. Fortunately the quaternionic Hessian of a real valued function is a Hermitian matrix (i.e. $a_{ij} = \bar a_{ji}$), and for them there is a notion of the Moore determinant which seems to be as good as the usual determinant of real symmetric or complex Hermitian matrices. To define it, recall that for any $A \in Mat_n(\mathbb{H})$ one can define its realization $A^{\mathbb{R}} \in Mat_{4n}(\mathbb{R})$ as follows. A defines an $\mathbb{R}$-linear operator $\hat A\colon \mathbb{H}^n \to \mathbb{H}^n$ given by multiplication $\hat A(x) = Ax$. Identifying $\mathbb{H}^n \simeq \mathbb{R}^{4n}$, we denote by $A^{\mathbb{R}}$ the matrix of the corresponding operator $\mathbb{R}^{4n} \to \mathbb{R}^{4n}$. One has the following result (see the survey [5]):

Theorem 5. *On the space of quaternionic Hermitian matrices of size n there exists a real polynomial $\det_M$, called the Moore determinant, which is uniquely characterized by the following two properties:*

(a) for any quaternionic Hermitian matrix A one has $\det(A^{\mathbb{R}}) = (\det_M(A))^4$;

(b) $\det_M(I_n) = 1$.

I defined in [2] the quaternionic MA operator of a real valued C^2-smooth function f in a domain $\Omega \subset \mathbb{H}^n$ by

$$\mathrm{MA}_{\mathbb{H}}(f) = \det_M\left(\frac{\partial^2 f}{\partial q_i \partial \bar q_j}\right).$$

In [2] I have extended theorems of Alexandrov and Chern-Levine-Nirenberg to the quaternionic case thus defining $\mathrm{MA}_{\mathbb{H}}$ for convex (more generally, quaternionic psh) functions. In [3] I proved the Blocki type formula for $\mathrm{MA}_{\mathbb{H}}$ (which looks exactly the same as in the complex case). Hence it follows that for any $\psi \in C_c(\mathbb{H}^n)$ the functional

$$K \mapsto \int_{\mathbb{H}^n} \psi \cdot \mathrm{MA}_{\mathbb{H}}(h_K)$$

is a continuous translation invariant valuation.

In [4] I generalized the above constructions and results to the case of 2 octonionic variables.

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On Godbersen's conjecture and related inequalities

Shiri Artstein-Avidan
(joint work with K. Einhorn, D. I. Florentin and Y. Ostrover)

Recently, in the paper [2] we have shown that for any $\lambda \in [0,1]$ and for any convex body K one has that

$$\lambda^j(1-\lambda)^{n-j}V(K[j],-K[n-k]) \leq \mathrm{Vol}(K).$$

In particular, picking $\lambda = \frac{j}{n}$, we get that

$$V(K[j],-K[n-k]) \leq \frac{n^n}{j^j(n-j)^{n-j}}\mathrm{Vol}(K) \sim \binom{n}{j}\sqrt{2\pi\frac{j(n-j)}{n}}.$$

The conjecture for the tight upper bound $\binom{n}{j}$, which is what ones get for a body which is an affine image of the simplex, was suggested in 1938 by Godbersen [3] (and independently by Hajnal and Makai Jr. [4]).

Conjecture 1 (Godbersen's conjecture)**.** For any convex body $K \subset \mathbb{R}^n$ and any $1 \leq j \leq n-1$,

$$V(K[j],-K[n-j]) \leq \binom{n}{j}\mathrm{Vol}(K), \tag{1}$$

with equality attained only for simplices.

We mention that Godbersen proved the conjecture for certain classes of convex bodies, in particular for those of constant width. We also mention that the conjecture holds for $j = 1, n-1$ by the inclusion $K \subset n(-K)$ for bodies K with center of mass at the origin. The bound from [2] quoted above seems to be the currently smallest known upper bound for general j.

In this work we improve the aforementioned inequality and show

Theorem 2. *For any convex body $K \subset \mathbb{R}^n$ and for any $\lambda \in [0,1]$ one has*

$$\sum_{j=0}^{n}\lambda^j(1-\lambda)^{n-j}V(K[j],-K[n-j]) \leq \mathrm{Vol}(K).$$

The proof of the inequality will go via the consideration of two bodies, $C \subset \mathbb{R}^{n+1}$ and $T \subset \mathbb{R}^{2n+1}$. Both were used in the paper of Rogers and Shephard [6].

We shall show by imitating the methods of [6] that

Lemma 3. *Given a convex body $K \subset \mathbb{R}^n$ define $C \subset \mathbb{R} \times \mathbb{R}^n$ by*

$$C = \mathrm{conv}(\{0\} \times (1-\lambda)K \cup \{1\} \times -\lambda K).$$

Then we have

$$\mathrm{Vol}(C) \leq \frac{\mathrm{Vol}(K)}{n+1}.$$

With this lemma in hand, we may prove our main claim by a simple computation

Proof of Theorem 2.

$$\begin{aligned}
\mathrm{Vol}(C) &= \int_0^1 \mathrm{Vol}((1-\eta)(1-\lambda)K - \eta\lambda K)d\eta \\
&= \sum_{j=0}^{n} \binom{n}{j} (1-\lambda)^{n-j}\lambda^j V(K[j], -K[n-j]) \int_0^1 (1-\eta)^{n-j}\eta^j d\eta \\
&= \frac{1}{n+1} \sum_{j=0}^{n} (1-\lambda)^{n-j}\lambda^j V(K[j], -K[n-j]).
\end{aligned}$$

Thus, using the lemma, we have that

$$\sum_{j=0}^{n} (1-\lambda)^{n-j}\lambda^j V(K[j], -K[n-j]) \leq \mathrm{Vol}(K).$$

□

Integration with respect to the parameter λ yields

Corollary 4. *For any convex body $K \subset \mathbb{R}^n$*

$$\frac{1}{n+1} \sum_{j=0}^{n} \frac{V(K[j], -K[n-j])}{\binom{n}{j}} \leq \mathrm{Vol}(K),$$

which can be rewritten as

$$\frac{1}{n-1} \sum_{j=1}^{n-1} \frac{V(K[j], -K[n-j])}{\binom{n}{j}} \leq \mathrm{Vol}(K).$$

So, on average the Godbersen conjecture is true. Of course, the fact that it holds true on average was known before, but with a different kind of average, namely by Rogers Shephard inequality for the difference body

$$\sum_{j=0}^{n} \frac{\binom{n}{j}}{\binom{2n}{n}} V(K[j], -K[n-j]) \leq \mathrm{Vol}(K).$$

However, our new average is a uniform one, so we know for instance that the median of the sequence $\left(\frac{V(K[j],-K[n-j])}{\binom{n}{j}}\right)_{j=1}^{n-1}$ is less than one, so that at least for one half of the indices $j = 1, 2, \ldots, n-1$, the mixed volumes satisfy Godbersen's conjecture with factor 2.

Corollary 5. *Let $K \subset \mathbb{R}^n$ be a convex body with $\mathrm{Vol}(K) = 1$. For at least half of the indices $j = 1, 2, \ldots n-1$ it holds that*

$$V(K[j], -K[n-j]) \leq 2\binom{n}{j}.$$

We mention that from the inequality of Theorem 2 we get as a by-product that for K with $\mathrm{Vol}(K) = 1$ one has

$$\sum_{j=1}^{n-1} \lambda^{j-1}(1-\lambda)^{n-j-1}[V(K[j], -K[n-j]) - \binom{n}{j}] \leq 0$$

So that by taking $\lambda = 0, 1$ we see, once again, that $V(K, -K[n-1]) = V(K[n-1], -K) \leq n$.

Our next assertion is connected with the following conjecture regarding the unbalanced difference body

$$D_\lambda K = (1-\lambda)K + \lambda(-K).$$

Conjecture 6. For any $\lambda \in (0, 1)$ one has

$$\frac{\mathrm{Vol}(D_\lambda K)}{\mathrm{Vol}(K)} \leq \frac{\mathrm{Vol}(D_\lambda \Delta)}{\mathrm{Vol}(\Delta)}$$

where Δ is an n-dimensional simplex.

Reformulating, Conjecture 6 asks whether the following inequality on the numbers $V_j = V(K[j], -K[n-j])$ holds

$$\sum_{j=0}^{n} \binom{n}{j} \lambda^j (1-\lambda)^{n-j} V_j \leq \sum_{j=0}^{n} \binom{n}{j}^2 \lambda^j (1-\lambda)^{n-j}. \tag{2}$$

Clearly Conjecture 6 follows from Godbersen's conjecture. It holds for $\lambda = 1/2$ by the Rogers-Shephard difference body inequality, it holds for $\lambda = 0, 1$ as then both sides are 1, and it holds on average over λ by Lemma 3 (applied with $\lambda = 1/2$ to $2K$). We recall that we know the following two inequalities on the sequence V_j:

$$\sum_{j=0}^{n} \lambda^j (1-\lambda)^{n-j} V_j \leq \sum_{j=0}^{n} \binom{n}{j} \lambda^j (1-\lambda)^{n-j}. \tag{3}$$

$$\sum_{j=0}^{n} \binom{n}{j} V_j \leq \sum_{j=0}^{n} \binom{n}{j}^2. \tag{4}$$

In all three inequalities we may disregard the 0^{th} and n^{th} terms as they are equal in both sides. We may take advantage of the fact that the j^{th} and the $(n-j)^{th}$ terms are the same in each inequality, and ask of the sum only up to $(n/2)$ (but be careful, if n is odd then each term appears twice, and if n is even then the middle term appears only once).

Theorem 7. *For $n \leq 5$ Conjecture 6 holds.*

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Translation invariant Minkowski valuations on lattice polytopes

Károly J. Böröczky
(joint work with M. Ludwig)

Two classification theorems were critical in the beginning of the theory of valuations on convex sets: first, the Hadwiger theorem [8] for valuations on convex bodies (that is, compact convex sets) in $\mathbb{R}^n$ and second, the Betke & Kneser theorem [5] for valuations on lattice polytopes (that is, convex polytopes with vertices in $\mathbb{Z}^n$). In recent years, numerous classification results were established for convex-body valued valuations (see, for example, [9, 10, 12, 7, 3, 1, 2, 20, 19, 22, 6, 17, 16, 21, 13]). The aim of this talk is to establish classification results for convex-body valued valuations defined on lattice polytopes. The question leads us to define and classify the discrete Steiner point.

A function z defined on a family $\mathcal{F}$ of subsets of $\mathbb{R}^n$ with values in an abelian group (or more generally, an abelian monoid) is a valuation if

$$\mathrm{z}(P) + \mathrm{z}(Q) = \mathrm{z}(P \cup Q) + \mathrm{z}(P \cap Q) \qquad (1)$$

whenever $P, Q, P \cup Q, P \cap Q \in \mathcal{F}$ and $\mathrm{z}(\emptyset) = 0$.

An operator $\mathrm{Z} : \mathcal{F} \to \mathcal{K}(\mathbb{R}^n)$ is called a Minkowski valuation if Z satisfies (1) and addition on $\mathcal{K}(\mathbb{R}^n)$ is Minkowski addition; that is,

$$K + L = \{x + y : x \in K,\, y \in L\}.$$

An operator $\mathrm{Z} : \mathcal{F} \to \mathcal{K}(\mathbb{R}^n)$ is called $\mathrm{SL}_n(\mathbb{R})$ equivariant if $\mathrm{Z}(\phi P) = \phi\, \mathrm{Z}\, P$ for $\phi \in \mathrm{SL}_n(\mathbb{R})$ and $P \in \mathcal{F}$. Define $\mathrm{SL}_n(\mathbb{Z})$ equivariance of operators on $\mathcal{P}(\mathbb{Z}^n)$ analogously. For valuations $\mathrm{Z} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ that are $\mathrm{SL}_n(\mathbb{R})$ equivariant and translation invariant, a complete classification has been established. Let $n \geq 2$.

Theorem 1 ([11])**.** *An operator $\mathrm{Z} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ is an $\mathrm{SL}_n(\mathbb{R})$ equivariant and translation invariant Minkowski valuation if and only if there exists a constant $c \geq 0$ such that for every $P \in \mathcal{P}(\mathbb{R}^n)$, we have*

$$\mathrm{Z}\, P = c(P - P).$$

The aim of this talk is to classify certain types of Minkowski valuations on lattice polytopes. The following result is an analogue of Theorem 1. Let $n \geq 2$.

Theorem 2. *An operator* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}(\mathbb{R}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *equivariant and translation invariant Minkowski valuation if and only if there exist constants* a, $b \geq 0$ *such that for every* $P \in \mathcal{P}(\mathbb{Z}^n)$*, we have*

$$\mathrm{Z}\,P = a(P - \ell_1(P)) + b(-P + \ell_1(P)).$$

Here for a lattice polytope P, the point $\ell_1(P)$ is its discrete Steiner point that is a new notion. It is defined as the one-homogeneous part of the Ehrhart expansion of the discrete moment vector $\ell(P) = \sum_{x \in P \cap \mathbb{Z}^n} x$; namely,

$$\ell(\lambda P) = \sum_{i=0}^{n+1} l_i(P)\lambda^i \quad \text{for } \lambda \in \mathbb{N}.$$

That such an expansion exists follows from results by McMullen [14]. The discrete Steiner point is characterized in the following result.

Theorem 3. *A function* $\mathrm{z} : \mathcal{P}(\mathbb{Z}^n) \to \mathbb{R}^n$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *and translation equivariant valuation if and only if* $\mathrm{z} = \ell_1$.

Theorem 3 corresponds to the well-known characterization of the classical Steiner point due to Schneider [18].

An operator $\mathrm{Z} : \mathcal{F} \to \mathcal{K}(\mathbb{R}^n)$ is called $\mathrm{SL}_n(\mathbb{R})$ contravariant if $\mathrm{Z}(\phi P) = \phi^{-t}\,\mathrm{Z}\,P$ for $\phi \in \mathrm{SL}_n(\mathbb{R})$ and $P \in \mathcal{F}$, where ϕ^{-t} is the inverse of the transpose of ϕ. Define $\mathrm{SL}_n(\mathbb{Z})$ contravariance of operators on $\mathcal{P}(\mathbb{Z}^n)$ analogously. For $\mathrm{SL}_n(\mathbb{R})$ contravariant Minkowski valuations on $\mathcal{P}(\mathbb{R}^n)$, a complete classification has been established. Let $n \geq 2$.

Theorem 4 ([11])**.** *An operator* $\mathrm{Z} : \mathcal{P}(\mathbb{R}^n) \to \mathcal{K}(\mathbb{R}^n)$ *is an* $\mathrm{SL}_n(\mathbb{R})$ *contravariant and translation invariant Minkowski valuation if and only if there exists a constant* $c \geq 0$ *such that for every* $P \in \mathcal{P}(\mathbb{R}^n)$*, we have*

$$\mathrm{Z}\,P = c\,\Pi\,P.$$

Here $\Pi\,P$ is the so-called projection body of P. For operators on lattice polytopes, we obtain the following result (here we do not quote the slightly more complicated case $n = 2$).

Theorem 5. *For* $n \geq 3$*, an operator* $\mathrm{Z} : \mathcal{P}(\mathbb{Z}^n) \to \mathcal{K}(\mathbb{R}^n)$ *is an* $\mathrm{SL}_n(\mathbb{Z})$ *contravariant and translation invariant Minkowski valuation if and only if then there exists a constant* $c \geq 0$ *such that for every* $P \in \mathcal{P}(\mathbb{Z}^n)$*, we have*

$$\mathrm{Z}\,P = c\,\Pi\,P.$$

Open problems

(1) Characterize all $\mathrm{SL}_n(\mathbb{Z})$ equivariant Minkowski valuations on at most n-dimensional lattice polytopes.
(2) Characterize all $\mathrm{SL}_n(\mathbb{Z}+i\mathbb{Z})$ equivariant and translation invariant Minkowski valuations on at most $2n$-dimensional lattice polytopes where $\mathbb{Z} + i\mathbb{Z}$ stands for the Gauß integers.

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Random polytopes: scaling limits and variance asymptotics

Pierre Calka
(joint work with T. Schreiber and J. Yukich)

This talk is based on a joint work with Tomasz Schreiber and Joe Yukich and on several joint works with Joe Yukich, including a work in progress.

The study of so-called *random polytopes* defined as convex hulls of independent and identically distributed random points in $\mathbb{R}^d$, $d \geq 2$, has started more than 50 years ago. Focus has quickly turned to the description of their asymptotic behavior when the size of the input goes to infinity. In two seminal works published in 1963 and 1964 [9, 10], A. Rényi and R. Sulanke obtained explicit formulae for the asymptotics of the mean number of vertices, mean area and mean perimeter of a planar random polytope. In particular, the growth of the number of extreme points is polynomial in the case of a uniform distribution in a smooth convex body while it is logarithmic for both the uniform distribution in a polytope and the standard Gaussian distribution.

These results have been extended to higher dimensions in several subsequent works and more recently, attention hase been drawn to second-order results and in particular central limit theorems and variance estimates (see e.g. the survey [8]). Sharp lower and upper bounds for the variance of the number of k-dimensional faces and the volume were obtained by M. Reitzner [7], I. Bárány and V. H. Vu [4] and M. Reitzner and I. Bárány [3] for the uniform distribution in a smooth convex body, the standard Gaussian distribution and the uniform distribution in a polytope respectively.

Showing the existence of limiting variances has proved to be more intricate. When the size of the input is Poisson distributed, we present a new method which provides the explicit calculation of limiting variances for the number of k-dimensional faces, the volume and sometimes the intrinsic volumes in the cases of uniform points in a smooth convex body, Gaussian points and uniform points in a simple polytope. The technique is based on the introduction of a proper scaling transformation and the use of so-called stabilization methods in the rescaled space. This provides, as by-product, the convergence in distribution of the rescaled boundary of the random polytope. We illustrate it with the particular cases of uniform points in the unit-ball [6] and especially of uniform points in a simple polytope [5].

In the first case, we define a global scaling transformation and we show that the boundary of the convex hull and its associated so-called flower have explicit scaling limits in the rescaled product space which are particular types of hull and growth processes with parabolic grains.

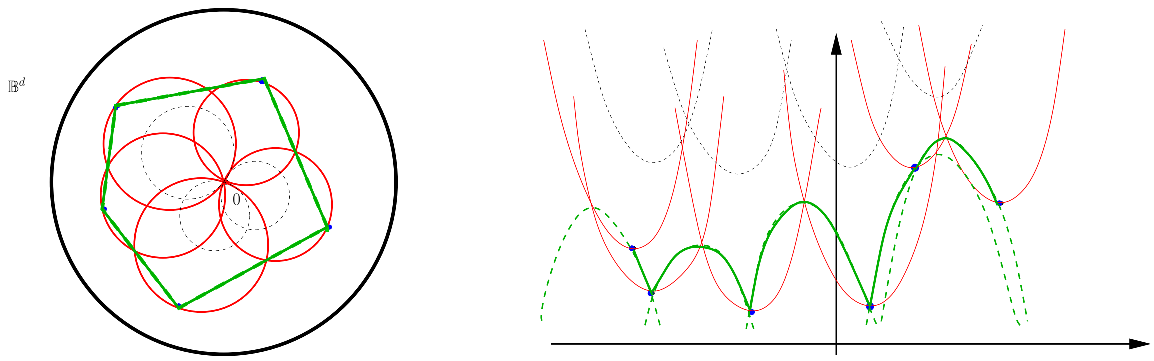

FIGURE 1. The random polytope and its flower in the d-dimensional unit-ball B^d (left); The scaling limits in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$ of its boundary and flower (right).

In the second case of uniform points in a simple polytope K, we first show that the contributions of the convex hull near the vertices of K decorrelate asymptotically and that the contribution of the convex hull far from the vertices is negligible. To do so, we use a technique introduced by I. Bárány and M. Reitzner [2] and based on the construction of dependency graphs. We then define a local scaling transformation in the vicinity of a particular vertex. The construction of such a function is based on the use of floating bodies of K which have proved on several occasions to be key objects in the asymptotic study of random polytopes [1]. We obtain dual scaling limits as hull and growth processes with cone-like grains for both the boundary and the associated flower of the convex hull in the vicinity of a vertex of K.

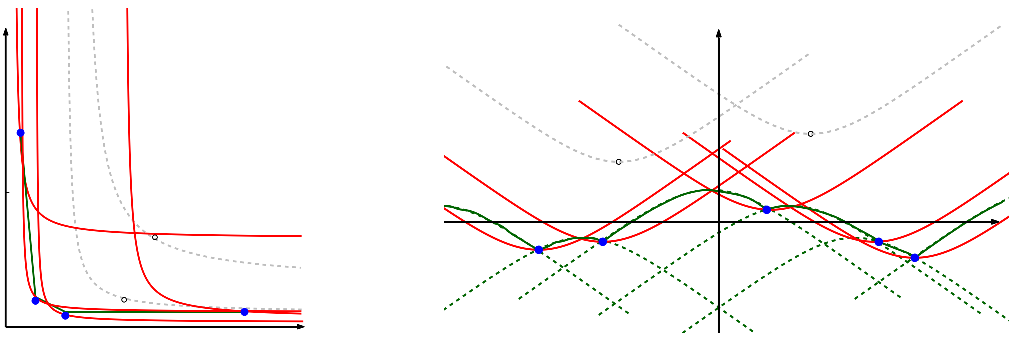

FIGURE 2. The random polytope and its flower in a d-dimensional cube (left); The scaling limits in the product space $\mathbb{R}^{d-1} \times \mathbb{R}$ of its boundary and flower (right).

This leads us to get explicit limiting variances as explained in the theorem below.

Theorem. Let K be a simple polytope of $\mathbb{R}^d$, $d \geq 2$, and K_λ be the convex hull of the intersection of a homogeneous Poisson point process of intensity $\lambda > 0$ with K. Then for every $1 \leq k \leq (d-1)$, the variance of the number of k-dimensional faces of K_λ divided by $\log^{d-1}(\lambda)$ (resp. of the volume of K_λ divided by $\log^{d-1}(\lambda)/\lambda^2$) converges to $c_{d,k} f_0(K)$ (resp. to $c'_d f_0(K)$) where $f_0(K)$ is the number of vertices of K, $c_{d,k}$ (resp. c_d) being an explicit constant depending only on d and k (resp. on d).

We have been unable up to now to show a similar result for a general polytope. Indeed, it is still unclear whether the variance would be asymptotically additive with respect to flags associated with a general polytope K. Nevertheless, this presentation makes a new parallel between the asymptotic analysis of several types of random polytopes and might pave the way for a unified treatment of the different models.

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Valuations on spaces of functions

Andrea Colesanti
(joint work with L. Cavallina and N. Lombardi)

Let X be a space of functions, all defined on a common domain, which will be $\mathbb{R}^n$ or the unit sphere $\mathbb{S}^{n-1}$ in all the examples that we will consider. A (real-valued) *valuation* on X is an application $\mu : X \to \mathbb{R}$ such that

$$\mu(f \vee g) + \mu(f \wedge g) = \mu(f) + \mu(g)$$

for every f and g in X such that $f \vee g, f \wedge g \in X$ (here $\vee$ and $\wedge$ denote the point-wise maximum and minimum).

The study of valuations on spaces of functions stems principally from the theory of valuations on convex bodies, which is currently one of the most active and prolific branches of convex geometry. In analogy with celebrated results concerning valuations of convex bodies (*e.g.* the Hadwiger theorem), the typical goal in the context of spaces of functions is to characterize all valuations on the space X which have some continuity, invariance and possibly monotonicity property.

The following is a very synthetic summary of what have been achieved in this area.

- X = space of *definable* functions; μ rigid motion invariant and continuous (w.r.t a suitable topology). A Hadwiger type theorem was obtained in [20] and [1].
- $X = L^p(\mathbb{R}^n)$ or $X = L^p(\mathbb{S}^{n-1})$; μ continuous and rigid motion invariant. A classification results was proved in [16]. See also [8] and [2] for related results.

- $X = W^{1,p}(\mathbb{R}^n)$ or $X = BV(\mathbb{R}^n)$ (functions of bounded variation); μ continuous and $\mathrm{SL}(n)$ invariant. Classification results were obtained in [19], [18], [14] and [13].
- $X =$ space of convex functions or of quasi-concave functions; μ rigid motion invariant, continuous w.r.t. a suitable topology and monotone. Classification results are proved in [3] and [4].

The results that we have mentioned so far concern *real-valued* valuations, but there are also studies regarding other types of valuations (e.g. matrix-valued valuations, or Minkowski and Blaschke valuations, etc.) that are interlaced with the results previously mentioned. A strong impulse to these studies have been given by Ludwig in the works [10], [11], [12] (see also [17] and [15]).

To illustrate briefly what happens in one specific example, we focus on quasi-concave functions. A function $f : \mathbb{R}^n \to [0, \infty)$ is called quasi-concave if for every level $t > 0$ the set

$$L_f(t) = \{x \in \mathbb{R}^n : f(x) \geq t\}$$

is a convex body (a compact convex subset of $\mathbb{R}^n$). This class of functions includes, for instance, characteristic functions of convex bodies as well as log-concave functions. Let X be the class of quasi-concave functions. Here is an easy way to construct a valuation on this space. For a fixed $t > 0$, given $f \in X$ consider the quantity

$$V_i(L_f(t))$$

where V_i is the i-th *intrinsic volume*. This is already a valuation, as we will see, but we can make the construction more articulate: we may sum over different levels and multiply each summand by a weight. Even more generally we may consider the quantity

$$\mu(f) = \int_0^\infty V_i(L_f(t))d\nu(t) \quad \forall\, f \in X, \tag{1}$$

where ν is an arbitrary Radon measure on $(0, \infty)$. To see that this is a valuation just notice that for every $f, g \in X$ and $t > 0$

$$L_{f\vee g}(t) = L_f(t) \cup L_g(t), \quad L_{f\wedge g}(t) = L_f(t) \cap L_g(t).$$

These relations and the valuation property of intrinsic volumes lead to

$$V_i(L_{f\vee g}(t)) + V_i(L_{f\wedge g}(t)) = V_i(L_f(t)) + V_i(L_g(t)).$$

If we now integrate both sides of the previous equality with respect to ν we obtain the valuation property for μ. It can be proved that (1) is finite for every $f \in X$ if and only if the support of ν is bounded away from zero, i.e. $\nu(0, \delta) = 0$ for some $\delta > 0$. Moreover μ is: (a) rigid motion invariant (i.e. $\mu(u) = \mu(f \circ T)$ for every $f \in X$ and rigid motion T); (b) increasing ($f \leq g$ implies $\mu(f) \leq \mu(g)$). Finally, if (and only if) ν is non-atomic then μ is continuous in the following sense: (c) if f_i, $i \in \mathbb{N}$, is a *monotone* sequence of elements of X converging point-wise in $\mathbb{R}^n$ to $f \in X$, then $\mu(f_i) \to \mu(f)$.

In [4] it is proved that properties (a)-(c) characterize valuations of type (1), up to linear combinations.

The perspectives of developments of the research in this area are wide. It is natural to investigate and possibly characterize continuous and invariant valuations on spaces like, for instance, the space of continuous functions or other familiar functions spaces. One possible direction of research is the classification of valuations defined on support functions (e.g. restricted to the unit sphere), under suitable conditions of continuity and invariance. This could lead, in principle, to analytic proofs of results like Hadwiger's theorem or McMullen decomposition theorem.

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New conjectures in the Geometry of Numbers

Daniel Dadush
(joint work with O. Regev)

1. Introduction

A k-dimensional Euclidean lattice $\mathcal{L} \subseteq \mathbb{R}^n$ is defined to be all integer linear combinations of k linearly independent vectors $B = (\vec{b}_1, \ldots, \vec{b}_k)$ in $\mathbb{R}^n$, where we call B a basis for $\mathcal{L}$. The determinant of $\mathcal{L}$ is defined as $\sqrt{\det(B^{\mathsf{T}}B)}$, which is invariant to the choice of basis for $\mathcal{L}$. The dual lattice of $\mathcal{L}$ is $\mathcal{L}^* = \{\vec{y} \in \mathrm{span}(\mathcal{L}) : \langle \vec{y}, \vec{x} \rangle \in \mathbb{Z}, \forall \vec{x} \in \mathcal{L}\}$. It is easy to verify that $B(B^{\mathsf{T}}B)^{-1}$ yields a basis for $\mathcal{L}^*$ and that $\det(\mathcal{L}^*) = 1/\det(\mathcal{L})$.

In this paper, we study the relationship between the questions:

(1) How can we bound the number of lattice points inside a centrally symmetric convex body?
(2) When can we guarantee that a convex body contains a lattice point?

It is generally understood that these questions are in essence dual to each other, and showing strong quantitative relationships between these questions is an important area within the geometry of numbers.

2. Reversing Minkowski's Theorem

Beginning with the first question, given a symmetric convex body $K \subseteq \mathbb{R}^n$ and lattice $\mathcal{L} \subseteq \mathbb{R}^n$, Minkowski's classical convex body theorem tells us that

$$|K \cap \mathcal{L}| \geq \lceil 2^{-n} \mathrm{vol}_n(K)/\det(\mathcal{L}) \rceil.$$

While the above bound is very useful, it is easy to come up with examples where is it very far from being tight. In particular, if the set of lattice points $K \cap \mathcal{L}$ lives a lower dimensional subspace, the volumetric bound can easily be confused by extraneous parts of the lattice. A natural attempt to fix this, is to not only compute Minkowski's bound on the lattice itself, but also on its lattice subspaces. A linear subspace $W \subseteq \mathbb{R}^n$, a lattice subspace of $\mathcal{L}$ if W admits a basis of vectors in $\mathcal{L}$. From here, if we define

$$M(K, \mathcal{L}) = \max_{\substack{W \text{ lat. sub. of } \mathcal{L} \\ 0 \leq d = \dim(W) \leq n}} \mathrm{vol}_d(K \cap W)/\det(\mathcal{L} \cap W),$$

Minkowski's convex body theorem implies that

$$|K \cap \mathcal{L}| \geq M(K/2, \mathcal{L}).$$

By convention, we define the determinant / volume of a 0-dimensional set (i.e. for $d = 0$ above) to be 1, and hence we note that $M(K, \mathcal{L}) \geq 1$ always.

With this strengthened volumetric lower bound, we may now ask again whether it is close to being tight. In this spirit, we prove the following theorem:

Theorem 1 (Weak Reverse Minkowski). *For a symmetric convex body K and n-dimensional lattice $\mathcal{L}$ in $\mathbb{R}^n$, we have that $|K \cap \mathcal{L}| \leq M(3nK, \mathcal{L})$. Futhermore, letting $\mathbb{B}_2^n$ denote the unit Euclidean ball, $|\mathbb{B}_2^n \cap \mathcal{L}| \leq M(3\sqrt{n}\mathbb{B}_2^n, \mathcal{L})$.*

The above theorem is in fact a relatively simple consequence of Minkowski's second theorem and a bound on the number of lattices points in terms of successive minima due to Henk [5]. However, it seems to be far very far from tight, in particular, it seems possible that the factor $3n$ could be replaced by a factor $O(\log n)$, i.e. exponentially better! Indeed, the worst current example we know of so far, corresponds to $K = \{x \in \mathbb{R}^n : \sum_{i=1}^n |x_i| \leq 1\}$ (the ℓ_1 ball) and $\mathcal{L} = \mathbb{Z}^n$, which shows that we cannot hope better than this. Given, we posit the following (optimistic) conjecture:

Conjecture 2 (Strong Reverse Minkowski). For a symmetric convex body K and n-dimensional lattice $\mathcal{L}$ in $\mathbb{R}^n$, $|K \cap \mathcal{L}| \leq M(O(\log n)K, \mathcal{L})$. Furthermore, $|\mathbb{B}_2^n \cap \mathcal{L}| \leq M(O(\sqrt{\log n})\mathbb{B}_2^n, \mathcal{L})$.

Here we note that the main interesting point in the above conjecture is that the required dilation factor may be *sub-polynomial* in the lattice dimension n, representing a potentially new phenomena in the geometry of numbers.

In the next section, we relate our main application of this conjecture.

3. The Kannan-Lovász Conjecture

Continuing to the second question, i.e. when can we guarantee that a convex body contains lattice points, one of the most elegant ways is to examine the covering radius of the body with respect to the lattice. Given a convex body K and n-dimensional lattice $\mathcal{L}$ in $\mathbb{R}^n$, we define the covering radius

$$\mu(K, \mathcal{L}) = \inf\{s \geq 0 : \mathcal{L} + sK = \mathbb{R}^n\} ,$$

or equivalently, the minimum scaling s of K for which $sK + \vec{t}$ contains a point of $\mathcal{L}$ for every translations $\vec{t} \in \mathbb{R}^n$. Note that if $\mu(K, \mathcal{L}) \leq 1$, then K contains a lattice point in *every translation*.

With this definition, we may rephrase the question as: when is the covering radius of a convex body smaller than 1? or more generally, is there a good alternate "dual" characterization of the covering radius? We note good answers to this question have been important in the context of discrete optimization. In particular, they have played a crucial role in the development of algorithms for the Integer Programming problem, where given a convex body K and lattice $\mathcal{L}$, the goal is to compute a point in $K \cap \mathcal{L}$ or decide that $K \cap \mathcal{L} = \emptyset$.

A first satisfactory answer in this context is known as Khinchine's Flatness theorem, which states that either $\mu(K, \mathcal{L}) \leq 1$ or K has small *lattice width*, i.e. K is "flat". Improving the quantitative estimates on how "flat" K must be has been a focus of much research [6, 1, 7, 8, 2, 3, 4]. Letting the width norm of K be $\mathrm{width}_K(\vec{z}) = \max_{\vec{x} \in K}\langle \vec{z}, \vec{x}\rangle - \min_{\vec{x} \in K}\langle \vec{z}, \vec{x}\rangle$, for $\vec{z} \in \mathbb{R}^n$, we define the lattice width of K w.r.t. to $\mathcal{L}$ as

$$\text{width}(K,\mathcal{L}) = \min_{y \in \mathcal{L}^* \setminus \{\vec{0}\}} \text{width}_K(\vec{y}).$$

The best current estimate on flatness can now be stated as follows:

Theorem 3 (Khinchine's Flatness Theorem)**.**

$$1 \leq \mu(K,\mathcal{L})\text{width}(K,\mathcal{L}) \leq \tilde{O}(n^{4/3}) \ .$$

We note that the estimate can be improved for $O(n)$ for ellipsoids [2] and $O(n \log n)$ for centrally symmetric convex bodies [3]. However, for any convex body, there exists a lattice for which the rhs is $\Omega(n)$, hence the relationship between lattice width and the covering radius cannot be made sub-polynomial in general.

Circumventing this problem, Kannan & Lovász [7] defined a volumetric generalization of flatness, proving the following bounds:

Theorem 4 ([7])**.**

$$1 \leq \mu(K,\mathcal{L}) \max_{\substack{W \text{ lat. sub. of } \mathcal{L}^* \\ 1 \leq d = \dim(W) \leq n}} \text{vol}_d(\pi_W(K))^{1/d} \det(\mathcal{L}^* \cap W)^{1/d} \leq n$$

where π_W is the orthogonal projection onto W.

We note that the standard flatness theorem corresponds to setting $d = 1$. In this context, there are no known examples for which the rhs need be larger than $O(\log n)$! This bound is in fact achieved for $K = \text{conv}(\vec{e}_1, 2\vec{e}_2, \ldots, n\vec{e}_n)$ and $\mathcal{L} = \mathbb{Z}^n$ ($\vec{e}_1, \ldots, \vec{e}_n$ denotes the standard basis). Kannan & Lovász asked whether this is indeed the worst case, thus we henceforth call an affirmative answer to this question as the Kannan-Lovász conjecture. Specializing the conjecture to the important special case $K = \mathbb{B}_2^n$, we get:

Conjecture 5 (The ℓ_2 Kannan-Lovász Conjecture)**.**

$$\Omega(1) \leq \mu(K,\mathcal{L}) \max_{\substack{W \text{ lat. sub. of } \mathcal{L}^* \\ 1 \leq d = \dim(W) \leq n}} \det(\mathcal{L}^* \cap W)^{1/d}/\sqrt{d} \leq O(\sqrt{\log n}),$$

where the upper bound is obtained by the lattice generated by the basis $B = (\vec{e}_1, \vec{e}_2/\sqrt{2}, \ldots, \vec{e}_n/\sqrt{n})$.

The main contribution of our paper is to show that the ℓ_2-version of the Kannan-Lovász conjecture is implied up to poly-logarithmic factors by a strong reverse Minkowski inequality:

Theorem 6. *Let $f(n)$ denote the least number such that for any n-dimensional lattice $\mathcal{L} \subseteq \mathbb{R}^n$,*

$$|\mathbb{B}_2^n \cap \mathcal{L}| \leq M(f(n)\mathbb{B}_2^n, \mathcal{L}).$$

Then the ℓ_2 Kannan-Lovász conjecture holds with bound $O(\log n f(n))$.

In fact, the above theorem holds assuming a weaker alternate characterization of the so-called lattice *smoothing parameter* [9], which is implied by 2. To achieve the reduction in Theorem 6, we rely on a novel convex relaxation for the covering radius and a rounding strategy for the corresponding dual program to extract the relevant subspace.

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The entropic barrier: a simple and optimal universal self-concordant barrier

Ronen Eldan

The objective of this talk is to demonstrate how to use elementary tools from convexity to introduce a new universal construction of a so-called *self-concordant barrier function*, an object of central importance in the theory of Interior Point Methods (IPMs). A self-concordant barrier over a convex body is a convex function going to infinity at the boundary of the body, and whose derivatives satisfy certain (quantitative) regularity conditions. The algorithms which use these functions (introduced by Nesterov and Nemirovski) have revolutionized mathematical optimization. Our construction is very simple to describe and turns out to be the first universal construction which attains optimal parameters. This talk will assume no prior knowledge in mathematical optimization or interior point methods.

To introduce the definition of a self-concordant barrier, we introduce some notation. For a C^3-smooth function $g : \mathbb{R}^n \to \mathbb{R}$, denote by $\nabla^2 g[\cdot,\cdot]$ its Hessian which we understand as a bilinear form over $\mathbb{R}^n$. Likewise, by $\nabla^3 g[\cdot,\cdot,\cdot]$ we denote its third derivative tensor. The definition of a self-concordant barrier is as follows.

Definition 1. A function $g : \mathrm{int}(\mathcal{K}) \to \mathbb{R}$ is a barrier for $\mathcal{K}$ if

$$g(x) \xrightarrow[x \to \partial\mathcal{K}]{} +\infty.$$

A C^3-smooth convex function $g : \mathrm{int}(\mathcal{K}) \to \mathbb{R}$ is self-concordant if for all $x \in \mathrm{int}(\mathcal{K}), h \in \mathbb{R}^n$,

$$\nabla^3 g(x)[h,h,h] \leq 2(\nabla^2 g(x)[h,h])^{3/2}. \tag{1}$$

Furthermore it is ν-self-concordant if in addition for all $x \in \mathrm{int}(\mathcal{K}), h \in \mathbb{R}^n$,

$$\nabla g(x)[h] \leq \sqrt{\nu \cdot \nabla^2 g(x)[h,h]}. \tag{2}$$

Let $\mathcal{K} \subset \mathbb{R}^n$ be a convex body, namely a compact convex set with a non-empty interior. Our main result is:

Theorem 2. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be defined for $\theta \in \mathbb{R}^n$ by*

$$f(\theta) = \log \left(\int_{x \in \mathcal{K}} \exp(\langle \theta, x \rangle) dx \right). \tag{3}$$

Then the Fenchel dual $f^* : \mathrm{int}(\mathcal{K}) \to \mathbb{R}$, defined for $x \in \mathrm{int}(\mathcal{K})$ by $f^*(x) = \sup_{\theta \in \mathbb{R}^n} \langle \theta, x \rangle - f(\theta)$, is a $(1+\epsilon_n)n$-self-concordant barrier on $\mathcal{K}$, with $\epsilon_n \leq 100\sqrt{\log(n)/n}$, for any $n \geq 80$.

From a theoretical point of view, one of the most important results in the theory of IPM is Nesterov and Nemirovski's construction of the *universal barrier*, which is a ν-self-concordant barrier that always satisfies $\nu \leq Cn$, for some universal constant $C > 0$. Theorem 2 is the first improvement (for convex bodies) over this seminal result: we show that in fact there always exists a barrier with self-concordance parameter $\nu = (1 + o(1))n$.

Our proof relies on elementary techniques from high dimensional convex geometry. In particular, the bound (1) relies on the analysis of extremal points on the set of log-concave measures, using the Krein-Milman theorem (and following a result of Fradelizi-Guédon). The main ingredient in the proof of the bound (2) is the Prekopa-Leindler inequality.

The Integral Geometry of indefinite orthogonal groups

Dmitry Faifman

(joint work with A. Bernig and S. Alesker)

1. Overview

1.1. **General theory.** Valuation theory bridges convex and integral geometry. For the classical theory, see [5]. For a survey of recent developments following Alesker's pivotal work [1], see [3]. For us, a continuous valuation on $V = \mathbb{R}^n$ will be a continuous map $\phi : \mathcal{K}(V) \to \mathbb{C}$ from the set of compact convex sets in V equipped with the Hausdorff metric, into the complex numbers, satisfying $\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L)$ whenever $K, L, K \cup L \in \mathcal{K}(V)$. The space of translation-invariant continuous valuations is denoted $\mathrm{Val}(V)$. It has a natural topology of a Banach space. We write $\mathrm{Val}_k^{\pm}(V)$ for the k-homogeneous even/odd valuations. McMullen's direct sum decomposition reads $\mathrm{Val}(V) = \oplus_{k=0}^n \mathrm{Val}_k(V)$.

Now $\mathrm{Val}(V)$ is a Banach representation of $GL(n)$, and as such has a dense subspace of smooth elements, denoted $\mathrm{Val}^{\infty}(V)$. The Alesker-Poincare duality is a non-degenerate pairing

$$\mathrm{Val}_k(V) \otimes \mathrm{Val}^{\infty}_{n-k}(V) \to \mathbb{C}$$

The elements of the weak dual of $\mathrm{Val}^{\infty}_{n-k}(V)$ are the generalized valuations, denoted $\mathrm{Val}^{-\infty}_k(V)$. One has the natural inclusions $\mathrm{Val}^{\infty}(V) \subset \mathrm{Val}(V) \subset \mathrm{Val}^{-\infty}(V)$. Generalized valuations can be thought of as valuations on smooth convex bodies.

A basic problem in valuation theory is to describe the G-invariant valuations for various subgroups $G \subset GL(n)$. For example, Hadwiger's famous theorem states that $\mathrm{Val}_k(V)^{SO(n)} = \mathrm{Span}(\mu_k)$, where μ_k is the k-th intrinsic volume. A theorem of Alesker describes the compact Lie groups that possess a finite-dimensional space of invariant valuations - those are precisely the groups acting transitive on S^{n-1}. Moreover, the G-invariant valuations are then smooth.

1.2. **Compact groups.** A central role in integral geometry is played by kinematic formulas. Those come in many flavors. In our setting we will consider two types of kinematic formulas - intersectional and additive. For example, for the special orthogonal group, the intersectional and additive kinematic formula are respectively

$$\int_{g\in SO(n)}\int_{x\in V}\mu_k(A\cap(gB+x))dxdg = \sum_{i+j=n+k} c_k^{ij}\mu_i(A)\mu_j(B)$$

$$\int_{g\in SO(n)}\mu_k(A\cap gB)dg = \sum_{i+j=k} d_k^{ij}\mu_i(A)\mu_j(B)$$

where c_k^{ij}, d_k^{ij} are certain explicit coefficients. The existence of such formulas is immediate from Hadwiger's theorem.

Given a Lie compact group $G \subset GL(n)$ as above one defines the kinematic operators, which are in fact co-products:

$$a_G, k_G : \mathrm{Val}(V)^G \to \mathrm{Val}(V)^G \otimes \mathrm{Val}(V)^G$$

$$k_G(\phi)(A,B) = \int_G\int_V \phi(A\cap(gB+x))dxdg$$

$$a_G(\phi)(A,B) = \int_G \phi(A+gB)dg$$

By a theorem of Bernig and Fu, those operators are conjugate through the Alesker-Fourier duality:

$$\mathbb{F}\otimes\mathbb{F}(k_G(\phi)) = a_G(\mathbb{F}\phi)$$

Using the rich algebraic structure on Val^{∞} introduced by Alesker, Bernig and Fu were able to determine the kinematic formulas explicity for $G = U(n)$ the unitary group, see [4].

2. Indefinite orthogonal group

2.1. **Invariant valuations.** What happens when G is non-compact? It turns out that to have an interesting theory, one has to consider generalized valuations instead of just smooth (or continuous). The first non-compact group to be considered was the Lorentz group in [2]. Generalizing the results therein to the indefinite orthogonal group $G = O(p,q)$, the Hadwider-type theorem is the following

Theorem 1 (Bernig-F.). *For $1 \leq k \leq n-1$, $\dim \mathrm{Val}_k^{-\infty}(V)^{O(p,q)} = 2$.*

An explicit description of those spaces can be provided through the Klain embedding. Non-formally, an even generalized k-homogeneous valuation is uniquely described by its values on k-dimensional parallelotopes. In these terms, we have the following result

Theorem 2 (Bernig-F.). *One has the basis $\mathrm{Val}_k^{-\infty}(V)^{O(p,q)} = Span\{\phi_k^0, \phi_k^1\}$, where, writing $E = \mathrm{Span}\{u_1, ..., u_k\}$ for independent vectors $u_1, ..., u_k$, and B_u the parallelotope they span,*

$$\phi_k^0(B_u) = \begin{cases} |\det Q(u_i,u_j)|^{\frac{1}{2}}, & \mathrm{sign}\, Q|_E = (k,0), (k-4,4), \dots \\ -|\det Q(u_i,u_j)|^{\frac{1}{2}}, & \mathrm{sign}\, Q|_E = (k-2,2), (k-6,6), \dots \\ 0 & \textit{otherwise} \end{cases}$$

and

$$\phi_k^1(B_u) = \begin{cases} |\det Q(u_i,u_j)|^{\frac{1}{2}}, & \mathrm{sign}\, Q|_E = (k-1,1), (k-5,5), \dots \\ -|\det Q(u_i,u_j)|^{\frac{1}{2}} & \mathrm{sign}\, Q|_E = (k-3,3), (k-7,7), \dots \\ 0 & \textit{otherwise} \end{cases}$$

Those valuations are not continuous, but they do in fact depend continuously on k-dimensional bodies. Such valuations form the Klain-continuous class, denoted $\mathrm{Val}^{KC}(V)$, and they play an important role in the theory - this class admits many of the operations available for smooth or continuous valuations, such as restrictions, kinematic operators and the Alesker-Fourier transform.

2.2. **Kinematic formulas.** Now to write kinematic formulas for $O(p,q)$, one has to fight the non-compactness of $O(p,q)$. The first step is to eliminate the group from the polarizing operators. For this, we need to consider the space of bivaluations, $\mathrm{BVal}(V)$, consisting of continuous functions $\mathcal{K}(V) \times \mathcal{K}(V) \to \mathbb{C}$ which are valuations in every variable. Most of the notions above extend naturally to bivaluations.

We define the kinematic operators $k_0, a_0 : \mathrm{Val}(V) \to \mathrm{BVal}(V)$ by $k_0(\phi)(A,B) := \int_V \phi(A \cap (B+x))dx$, $a_0(\phi)(A,B) := \phi(A+B)$. Let k_0^+, a_0^+ denote the bi-even component of the operators. It turns out that k_0^+, a_0^+ extend to operators between spaces of Klain-continuous (bi)valuations, satisfying the Bernig-Fu relation $(\mathbb{F} \otimes \mathbb{F})k_0^+(\phi) = a_0^+(\mathbb{F}\phi)$. Thus we may focus on one type of kinematic operators.

We still have to deal with the non-compactness of G. For simplicity we will consider only the even component of the kinematic formulas, equivalently, we will consider origin-symmetric bodies A, B. We will only consider the simplest degree of homogeneity which is non-trivial for additive kinematic operators, $k = 2$. Higher degrees remain to be understood. For $\phi \in \mathrm{Val}_2^{-\infty}(V)^{O(p,q)}$, origin symmetric smooth A, B and $g \in O(p,q)$, $\phi(A + gB) = \phi(A) + \phi(B) + a_0^+\phi(A, gB)$. Only the last summand depends on g and thus interesting for a kinematic formula. This term is not always integrable though, and a correction term is sometimes needed.

There is a natural bivaluation $\psi_Q \in \mathrm{BVal}_{1,1}(V)^{O(p,q)}$ associated with the (p,q)-quadratic form on V, which is denoted Q. It is given by its Klain embedding, namely, $\psi_Q(u, v) = |Q(u, v)|$ for $u, v \in V$. We then can prove the following:

Theorem 3 (F.). *The integrals*

$$\int_{O(p,q)} a_0^+\phi_2^0(A, gB)dg$$

$$\int_{O(p,q)} \left(a_0^+\phi_2^1(A, gB) - \psi_Q(A, gB)\right) dg$$

converges in the sense of Klain-continuous bivaluations. In particular, they converge for smooth convex bodies A, B.

When combined with the Hadwiger-type classification, we conclude that those integrals are given by a finite linear combination of the form $\sum_{i,j=0}^{1} c_{ij}\phi_1^i(A)\phi_1^j(B)$. The computation of the coefficients, and more importantly, determining to which extent the algebraic apparatus availalbe for compact groups extends to the non-compact setting, remain to be done.

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Do Minkowski averages get progressively more convex?

Matthieu Fradelizi

(joint work with M. Madiman, A. Marsiglietti and A. Zvavitch)

The question of the title has its origin in the formal analogy between the Information Theory and the Brunn-Minkowski theory, which comes from the statements of the two fundamental inequalities of each theory:

Theorem 1 (Brunn-Minkowski inequality)**.** *Let A and B be two compact sets in $\mathbb{R}^n$. Then*

$$|A+B|^{\frac{1}{n}} \geq |A|^{\frac{1}{n}} + |B|^{\frac{1}{n}}.$$

Theorem 2 (Entropy Power Inequality (EPI))**.** *Let X and Y be two independent random vectors in $\mathbb{R}^n$. Then*

$$N(X+Y) \geq N(X) + N(Y),$$

where for X with density f, $N(X) = \frac{1}{2\pi e} e^{\frac{2}{n}H(X)}$ and $H(X) = -\int f \log(f)$.

In this talk, we shall review some of the conjectures made to support this analogy and some of the results obtained to prove, disprove and modify those.

1) Blachman-Stam inequality. Elaborating on this analogy, Dembo-Cover-Thomas conjectured in [4] the following analogue of the Blachman-Stam inequality: for any convex bodies A, B

$$\frac{|A+B|}{\partial(A+B)} \geq \frac{|A|}{\partial(A)} + \frac{|B|}{\partial(B)}.$$

In [5] in 2003, with Giannopoulos and Meyer we established that the conjecture holds true in dimension 1 and 2 but it is false in dimension $n \geq 3$.

2) Concavity of entropy power. As an analogue of the concavity of entropy power, Costa and Cover conjectured in [3] that for any compact set A in $\mathbb{R}^n$,

$$t \mapsto |A + tB_2^n|^{\frac{1}{n}} \text{ is concave.}$$

They also established the inequality if A is convex. With Marsiglietti in 2014 in [6] we proved that the conjecture holds true in dimension 1, in dimension 2 for A connected and in dimension n for A finite and $t \geq t(A)$. But it is false in dimension $n \geq 2$ in general. We don't know if for any compact A there exists a $t(A) \geq 0$ so that the result holds true for $t \geq t(A)$.

3) Monotonicity of entropy. Artstein, Ball, Barthe and Naor have proved in 2004, in [1], the monotonicity of the entropy in the following sense:
Let $X_1, \dots, X_m, \dots$ be independent random vectors; then

$$m \mapsto H\left(\frac{X_1 + \cdots + X_m}{\sqrt{m}}\right) \quad \text{is non-decreasing.}$$

Pursuing the analogy, Bobkov, Madiman and Wang conjectured in 2011, in [2], that for any compact set A in $\mathbb{R}^n$, if we denote

$$A(m) = \frac{\overbrace{A + \cdots + A}^{m \text{ times}}}{m}$$

then $m \mapsto |A(m)|$ is non-decreasing. More generally we investigate in [7] if the Minkowski averages get progressively more convex, that is if the sequence $A(m)$

comes closer in a monotone way to $\mathrm{conv}(A)$, with respect to different distances. Considering the volume difference, the question reduces to Bobkov-Madiman-Wang's conjecture to which we give the following partial answer.

Theorem 3 (F., Madiman, Marsiglietti, Zvavitch [7]). *Let A be a compact set in $\mathbb{R}^n$. Then $m \mapsto |A(m)|$ is non-decreasing for $n = 1$ but there are counter-examples for $n \geq 12$.*

The counter-example in dimension $n = 12$ is built as the union of two convex sets in supplementary subspaces: $A = ([-1,1]^6 \times \{0\}) \cup (\{0\} \times [-1,1]^6)$. Then $\left|\frac{A+A}{2}\right| > \left|\frac{A+A+A}{3}\right|$.

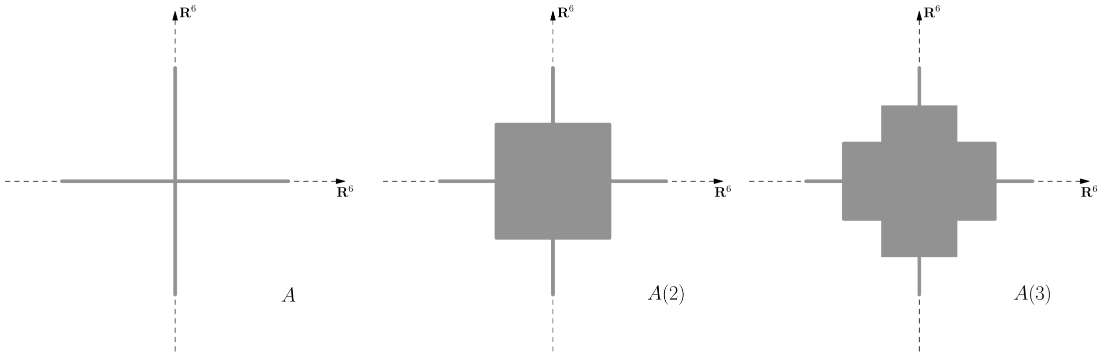

FIGURE 1. A counterexample in $\mathbb{R}^{12}$.

4) We also consider the distance to the convex hull measured in Hausdorff distance. Let A be compact in $\mathbb{R}^n$. We denote $d(A) = d_H(A, \mathrm{conv}(A))$:

$$d(A) = \inf\{r > 0; \mathrm{conv}(A) \subset A + rB_2^n\} = \sup_{x \in \mathrm{conv}(A)} \inf_{a \in A} |x - a|.$$

Shapley-Folkmann-Starr proved in [9] that for A compact in $\mathbb{R}^n$,

$$d(A(m)) \leq \min\left(\frac{\sqrt{n}}{m}, \frac{1}{\sqrt{m}}\right) R(A).$$

In [7], we observe that for $m \geq c(A)$, $d(A(m+1)) \leq \frac{m}{m+1} d(A(m))$, where $c(A)$ is an affine invariant measure of convexity defined by Schneider in [8] by $c(A) = \inf\{\lambda > 0; A + \lambda \mathrm{conv}(A) \text{ is convex}\}$. Moreover Schneider proved that $c(A) \leq n$ with equality if and only if A is a set of $n+1$ affinity independent points; and if A is connected then $c(A) \leq n - 1$. Using the above observation, we deduce:

Corollary 4 (F., Madiman, Marsiglietti, Zvavitch [7]). *Let A be a compact set in $\mathbb{R}^n$. Then $m \mapsto d(A(m))$ is non-increasing for $n = 1$ and $n = 2$, for $n = 3$ if A is connected, for $m \geq n$.*

We also prove the following theorem regarding Schneider's convexity index.

Theorem 5 (F., Madiman, Marsiglietti, Zvavitch [7]). *Let A be a compact set in $\mathbb{R}^n$. Then for any $m \in \mathbb{N}$,*

$$c(A(m+1)) \leq \frac{m}{m+1} c(A(m)).$$

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Operations between functions

Richard J. Gardner

(joint work with M. Kiderlen)

Throughout mathematics and wherever it finds applications, there is a need to combine two or more functions to produce a new function. The four basic arithmetic operations, together with composition, are so fundamental that there seems no need to question their existence or utility, for example in calculus. In more advanced mathematics, other operations make their appearance, but are still sometimes tied to simpler operations, as is the case for convolution, which via the Fourier transform becomes multiplication. Of course, a myriad of different operations have been found useful. One such is L_p addition $+_p$, defined for f and g in a suitable class of nonnegative functions by

$$(f +_p g)(x) = \left(f(x)^p + g(x)^p\right)^{1/p}, \tag{1}$$

for $0 < p < \infty$, and by $(f +_\infty g)(x) = \max\{f(x), g(x)\}$. Particularly for $1 \le p \le \infty$, L_p addition is of paramount significance in functional analysis and its many applications. It is natural to extend L_p addition to $-\infty \le p < 0$ by defining

$$(f +_p g)(x) = \begin{cases} \left(f(x)^p + g(x)^p\right)^{1/p}, & \text{if } f(x)g(x) \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

when $-\infty < p < 0$, and $(f +_{-\infty} g)(x) = \min\{f(x), g(x)\}$.

But what is so special about L_p addition, or, for that matter, ordinary addition? This is one motivation for our investigation, which focuses on operations

$*: \Phi(A)^m \to \Phi(A)$, $m \geq 2$, where $\Phi(A)$ is a class of real-valued (or extended-real-valued) functions on a nonempty subset A of n-dimensional Euclidean space $\mathbb{R}^n$. We offer a variety of answers, usually stating that an operation $*$ satisfying just a few natural properties must belong to rather special class of operations. What emerges is the beginning of a structural theory of operations between functions.

Our most general results need no restriction on $\Phi(A)$ other than that it is a cone (i.e., $rf \in \Phi(A)$ whenever $f \in \Phi(A)$ and $r \geq 0$) of real-valued functions, a property enjoyed by the classes of arbitrary, or continuous, or differentiable functions, among many others. For example, we prove (Theorem A) that if $m = 2$, $\Phi(A)$ is a cone containing the constant functions, and $*$ is pointwise, positively homogeneous, monotonic, and associative, then $*$ must be one of 40 types of operations. Of the properties assumed, the last three are familiar (monotonic means increasing). The first property, *pointwise*, means that there is a function $F: E \subset \mathbb{R}^m \to \mathbb{R}$ such that

$$(*(f_1, \dots, f_m))(x) = F(f_1(x), \dots, f_m(x)), \tag{2}$$

for all $f_1, \dots, f_m$ in $\Phi(A)$ and all $x \in A$, where $*(f_1, \dots, f_m)$ denotes the result of combining the functions $f_1, \dots, f_m$ via the operation $*$. The pointwise property is, to be sure, a quite restrictive one, immediately eliminating composition, for example. Nonetheless, (2) is general enough to admit a huge assortment of operations, and it is surprising that with just three other assumptions the possibilities can be narrowed to a relatively small number.

For nonnegative functions, the situation is easier, because a result of Pearson [5] now does most of the work. If $\Phi(A)$ is a cone of nonnegative functions and $*$ is pointwise, positively homogeneous, monotonic (or pointwise continuous), and associative, then $*$ must one of six types of functions (or three types of functions, respectively), including L_p addition, for some $-\infty \leq p \neq 0 \leq \infty$. Except for L_p addition, the various types of functions are either rather trivial or a trivial modification of L_p addition.

Applications to convex analysis stem from two key lemmas, one of which states that a pointwise operation $*: \Phi(A)^m \to \mathrm{Cvx}(A)$ must be monotonic, with an associated function F that is increasing in each variable, when $\Phi(A)$ is $\mathrm{Cvx}(A)$, $\mathrm{Cvx}^+(A)$, $\mathrm{Supp}(\mathbb{R}^n)$, or $\mathrm{Supp}^+(\mathbb{R}^n)$. Here $\mathrm{Cvx}(A)$ is the class of real-valued convex functions on a nontrivial convex set A in $\mathbb{R}^n$, $\mathrm{Supp}(\mathbb{R}^n)$ is the class of support functions of nonempty compact convex sets in $\mathbb{R}^n$, and the superscript $+$ denotes the nonnegative functions in these classes.

We prove (Theorem B) that $*: \mathrm{Cvx}(A)^m \to \mathrm{Cvx}(A)$ is pointwise and positively homogeneous if and only if there is a nonempty compact convex set $M \subset [0, \infty)^m$ such that

$$(*(f_1, \dots, f_m))(x) = h_M(f_1(x), \dots, f_m(x)), \tag{3}$$

for all $f_1, \dots, f_m \in \mathrm{Cvx}(A)$ and all $x \in A$, where h_M is the support function of M. We call an operation defined by (3), for an arbitrary nonempty subset M of $\mathbb{R}^m$, *M-addition*. As far as we know, such operations $*: \mathrm{Cvx}(A)^m \to \mathrm{Cvx}(A)$ have not be considered before. The same result also characterizes the pointwise and positively homogeneous operations $* : \mathrm{Cvx}^+(A)^m \to \mathrm{Cvx}^+(A)$ as those satisfying (3) for some 1-unconditional compact convex set M in $\mathbb{R}^m$, but in this case, at least for $m = 2$, such operations were first introduced by Volle [7]. He observed that if $\|\cdot\|$ is a monotone norm on $\mathbb{R}^2$ (i.e., $\|(x_1, y_1)\| \le \|(x_2, y_2)\|$ whenever $x_1 \le x_2$ and $y_1 \le y_2$), then the operation $(f +_{\|\cdot\|} g)(x) = \|\,(f(x), g(x))\,\|$ still preserves the convexity of nonnegative real-valued functions, where (1) corresponds to the L_p norm. From Theorem B it is easy to characterize Volle's operations as being, with trivial exceptions, precisely the pointwise and positively homogeneous operations $* : \mathrm{Cvx}^+(A)^2 \to \mathrm{Cvx}^+(A)$. Moreover, we show, again with trivial exceptions, that any pointwise, positively homogeneous, and associative operation $* : \mathrm{Cvx}^+(A)^2 \to \mathrm{Cvx}^+(A)$ must be L_p addition, for some $1 \le p \le \infty$. Another result (Theorem C) completely characterizes the pointwise, positively homogeneous, and associative operations $* : \mathrm{Cvx}(A)^2 \to \mathrm{Cvx}(A)$; with trivial exceptions, they are either ordinary addition or defined by

$$(4) \qquad (f * g)(x) = \begin{cases} (f +_p g)(x), & \text{if } f(x), g(x) \ge 0, \\ f(x), & \text{if } f(x) \ge 0, g(x) < 0, \\ g(x), & \text{if } f(x) < 0, g(x) \ge 0, \\ -\left(|f| +_q |g|\right)(x), & \text{if } f(x), g(x) < 0, \end{cases}$$

for all $f, g \in \mathrm{Cvx}(A)$ and $x \in A$ and for some $1 \le p \le \infty$ and $-\infty \le q \le 0$. Here Theorem A is used in an essential way, the function F associated with the operation $*$ defined by (4) (with $m = 2$) being one of the 40 listed in that result. Again, it appears that these operations have not been considered before. A further result (Theorem D) provides a somewhat surprising characterization of ordinary addition by showing it to be the unique pointwise operation $* : \mathrm{Cvx}(A)^2 \to \mathrm{Cvx}(A)$ satisfying the identity property, i.e., $f * 0 = 0 * f = f$, for all $f \in \mathrm{Cvx}(A)$. An example shows why does not seem to be a natural version of Theorem D that applies to the class $\mathrm{Cvx}^+(A)$.

All the results in the previous paragraph have counterparts for operations $* : \mathrm{Supp}(\mathbb{R}^n)^m \to \mathrm{Supp}(\mathbb{R}^n)$ or $* : \mathrm{Supp}^+(\mathbb{R}^n)^m \to \mathrm{Supp}^+(\mathbb{R}^n)$; indeed, the same results hold verbatim, if the condition of positive homogeneity is omitted. Such operations can be transferred in a natural manner to operations between compact convex sets, so they are in part anticipated by work of Gardner, Hug, and Weil [1], of which the present paper can be regarded as a sequel. Even in this context, however, part of Theorem C is new, giving a partial answer to the still unresolved question of the role of associativity in classifying operations between arbitrary compact convex sets.

In convex analysis it is essential to work not only with real-valued functions but also with extended-real-valued functions and we devote a section of the paper to this task.

We stress that for each of the previously described results, and indeed those throughout the paper, we provide a full set of examples showing that none of the assumptions we make can be omitted. In particular, the assumption that the operations are pointwise is essential. Nevertheless, in certain circumstances it is possible to classify operations that are not necessarily pointwise, *provided they* *are associative.* Our inspiration here is the work of Milman and Rotem [4] on operations between closed convex sets, and we lean heavily on their methods to achieve our results. In what follows, $\overline{\mathrm{Cvx}}^+(A)$ is the class of nonnegative extended-real-valued convex functions and $\overline{\mathrm{Supp}}^+(\mathbb{R}^n)$ is the class of support functions of nonempty closed convex sets in $\mathbb{R}^n$ containing the origin. We prove (Theorem E) that any operation $* : \overline{\mathrm{Cvx}}^+(A)^2 \to \overline{\mathrm{Cvx}}^+(A)$ or $* : \overline{\mathrm{Supp}}^+(\mathbb{R}^n)^2 \to \overline{\mathrm{Supp}}^+(\mathbb{R}^n)$ that is monotonic, associative, weakly homogeneous, and has the identity and δ-finite properties, must be L_p addition, for some $1 \le p \le \infty$. The δ-finite property is a certain weak technical condition. Weak homogeneity is introduced for the first time and serves two purposes: It directly relates to (and is much weaker than) positive homogeneity and it allows us to avoid the slightly artificial "homothety" property used in [4]. Indeed, we prove that in the presence of monotonicity and the identity property, the homothety property implies weak homogeneity. We establish a corresponding result (Theorem F) for operations $* : \mathrm{Supp}^+(\mathbb{R}^n)^2 \to \mathrm{Supp}^+(\mathbb{R}^n)$, which immediately yields a characterization of L_p addition as an operation between closed convex (or compact convex) sets containing the origin, that strengthens [4, The orems 2.2 and 6.1].

Returning to pointwise operations, we briefly mention two other contributions. The first is the introduction of Orlicz addition between functions. This is motivated by the recent discovery of Orlicz addition of sets, a generalization of L_p addition of sets; see [2, 8]. Orlicz addition $+_\varphi$ of functions turns out to be an operation $+_\varphi : \Phi(A)^m \to \Phi(A)$ in several useful instances, for example when $\Phi(A)$ is the class of nonnegative Borel or nonnegative continuous functions on A, or $\mathrm{Cvx}^+(A)$, or $\mathrm{Supp}^+(\mathbb{R}^n)$. It has the remarkable features that the function F associated with $+_\varphi$ (as in (2)) is implicit and that when $m = 2$, $+_\varphi$ is in general neither commutative nor associative.

The second contribution referred to above is a characterization of the Asplund sum among operations between log-concave functions on $\mathbb{R}^n$.

Problem. Find results in the spirit as those described above for operations that are neither necessarily pointwise nor associative.

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Brascamp-Lieb inequality and quantitative versions of Helly's theorem

Apostolos Giannopoulos
(joint work with S. Brazitikos)

We present new quantitative versions of Helly's theorem; recall that the classical result asserts that if $\mathcal{F} = \{F_i : i \in I\}$ is a finite family of at least $n+1$ convex sets in $\mathbb{R}^n$ and if any $n+1$ members of $\mathcal{F}$ have non-empty intersection then $\bigcap_{i\in I} F_i \neq \emptyset$. Variants of this statement have found important applications in discrete and computational geometry.

(A) Quantitative Helly-type results were first obtained by Bárány, Katchalski and Pach. In particular, they proved the following volumetric result:

Let $\{P_i : i \in I\}$ be a family of closed convex sets in $\mathbb{R}^n$ such that $\left|\bigcap_{i\in I} P_i\right| > 0$. There exist $s \leqslant 2n$ and $i_1, \dots, i_s \in I$ such that

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant n^{2n^2} \left|\bigcap_{i\in I} P_i\right|.$$

The example of the cube $[-1,1]^n$ in $\mathbb{R}^n$, expressed as an intersection of exactly $2n$ closed half-spaces, shows that one cannot replace $2n$ by $2n-1$ in the statement above. Naszódi has recently proved a volume version of Helly's theorem with a constant $\leqslant (cn)^{2n}$, where $c > 0$ is an absolute constant. In fact, a slight modification of Naszódi's argument leads to the exponent $\frac{3n}{2}$ instead of $2n$. In [5], relaxing the requirement that $s \leqslant 2n$ to the weaker one that $s = O(n)$, Brazitikos has improved the exponent to n:

Theorem 1 (Brazitikos)**.** *There exists an absolute constant $\alpha > 1$ with the following property: for every family $\{P_i : i \in I\}$ of closed convex sets in $\mathbb{R}^n$, such that $P = \bigcap_{i\in I} P_i$ has positive volume, there exist $s \leqslant \alpha n$ and $i_1, \dots, i_s \in I$ such that*

$$|P_{i_1} \cap \cdots \cap P_{i_s}| \leqslant (cn)^n \, |P|,$$

where $c > 0$ is an absolute constant.

The proof of Theorem 1 involves a theorem of Srivastava and the following approximate geometric Brascamp-Lieb inequality (we state below its reverse counterpart too).

Theorem 2 (Brazitikos)**.** *Let $\gamma > 1$. Assume that $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy*

$$I_n \preceq A := \sum_{j=1}^{s} c_j u_j \otimes u_j \preceq \gamma I_n$$

and set $\kappa_j = c_j\langle A^{-1}u_j, u_j\rangle > 0$, $1 \leqslant j \leqslant m$. *If* $f_1, \ldots, f_m : \mathbb{R} \longrightarrow [0, +\infty)$ *are integrable functions then*

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{\kappa_j}(\langle x, u_j\rangle)dx \leqslant \gamma^{\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j(t)dt\right)^{\kappa_j}.$$

Also, if $w, h_1, \ldots, h_m : \mathbb{R} \longrightarrow [0, \infty)$ *are integrable functions and* $w(x) \geqslant \sup\left\{\prod_{j=1}^m h_j^{\kappa_j}(\theta_j) : \theta_j \in \mathbb{R}\ ,\ x = \sum_{j=1}^m \theta_j c_j u_j\right\}$, *then*

$$\int_{\mathbb{R}^n} w(x)dx \geqslant \gamma^{-\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j(t)\,dt\right)^{\kappa_j}.$$

(B) A continuous version of Theorem 2 can be also obtained. We say that a Borel measure ν on S^{n-1} is a γ-*approximation of an isotropic measure* (for some $\gamma > 1$) if

$$I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u\, d\nu(u) \preceq \gamma I_n.$$

Following Barthe's argument for the isotropic case and a generalization of the so-called Ball-Barthe lemma (proved by Lutwak, Yang and Zhang for isotropic measures on the sphere) one can obtain a continuous Brascamp-Lieb inequality and its reverse form for a γ-approximation of an isotropic measure.

Theorem 3 (Brazitikos-Giannopoulos). *Let* ν *be a* γ*-approximation of an isotropic Borel measure on* S^{n-1} *and let* (f_u), $u \in S^{n-1}$ *be a family of functions* $f_u : \mathbb{R} \longrightarrow [0, +\infty)$ *that satisfy natural continuity conditions. Then,*

$$\int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log f_u(\langle x, u\rangle)\langle T_\nu^{-1}u, u\rangle d\nu(u)\right) dx$$
$$\leqslant \gamma^{\frac{n}{2}} \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right).$$

Also, if h *is a measurable function such that*

$$h\left(\int_{S^{n-1}} \theta(u)u\, d\nu(u)\right) \geqslant \exp\left(\int_{S^{n-1}} \log f_u(\theta(u))\langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right)$$

for every integrable function θ, *then*

$$\gamma^{\frac{n}{2}} \int_{\mathbb{R}^n} h(y)\, dy \geqslant \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1}u, u\rangle\, d\nu(u)\right).$$

(C) Bárány, Katchalski and Pach proved a quantitative Helly-type theorem for the diameter in place of volume:
Let $\{P_i : i \in I\}$ *be a family of closed convex sets in* $\mathbb{R}^n$ *such that* $\operatorname{diam}\left(\bigcap_{i\in I} P_i\right) = 1$. *There exist* $s \leqslant 2n$ *and* $i_1, \ldots, i_s \in I$ *such that*

$$\operatorname{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant (cn)^{n/2},$$

where $c > 0$ *is an absolute constant.*

In the same work the authors conjecture that the bound should be polynomial in n; in fact they ask if $(cn)^{n/2}$ can be replaced by $c\sqrt{n}$. Relaxing the requirement that $s \leqslant 2n$, and using a similar strategy as in [5], Brazitikos proved in [6] the following:

Theorem 4 (Brazitikos). *There exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{int}\left(\bigcap_{i\in I} P_i\right) \neq \emptyset$, then there exist $z \in \mathbb{R}^n$, $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$z + P_{i_1} \cap \cdots \cap P_{i_s} \subseteq cn^{3/2}\left(z + \bigcap_{i\in I} P_i\right),$$

where $c > 0$ is an absolute constant.

It is clear that Theorem 4 implies polynomial estimates for the diameter:

Theorem 5 (Brazitikos). *There exists an absolute constant $\alpha > 1$ with the following property: if $\{P_i : i \in I\}$ is a finite family of convex bodies in $\mathbb{R}^n$ with $\mathrm{diam}\left(\bigcap_{i\in I} P_i\right) = 1$, then there exist $s \leqslant \alpha n$ and $i_1, \dots i_s \in I$ such that*

$$\mathrm{diam}(P_{i_1} \cap \cdots \cap P_{i_s}) \leqslant cn^{3/2},$$

where $c > 0$ is an absolute constant.

(D) The proof of Theorem 4 is based on the following non-symmetric version of a lemma of Barvinok: There exists an absolute constant $\alpha > 1$ with the following property: if K is a convex body whose minimal volume ellipsoid is the Euclidean unit ball, then there is a subset $X \subset \mathrm{bd}(K) \cap S^{n-1}$ of cardinality $\mathrm{card}(X) \leqslant \alpha n$ such that

$$B_2^n \subseteq cn^{3/2}\mathrm{conv}(X),$$

where $c > 0$ is an absolute constant. The random analogue of this fact is given by the next theorem (see [8]).

Theorem 6 (Brazitikos-Chasapis-Hioni). *There exists an absolute constant $\beta > 1$ with the following property: if K is a convex body in $\mathbb{R}^n$ whose center of mass is at the origin, if $N = \lceil \beta n \rceil$ and if $x_1, \dots, x_N$ are independent random points uniformly distributed in K then, with probability greater than $1 - e^{-n}$ we have*

$$K \subseteq c_1 n\, \mathrm{conv}(\{x_1, \dots, x_N\}),$$

where $c_1 > 0$ is an absolute constant.

A consequence of Theorem 6 is the next estimate for the vertex index $\mathrm{vi}(K)$ (studied by Bezdek and Litvak in the symmetric case) of a not necessarily symmetric convex body K in $\mathbb{R}^n$.

Theorem 7 (Brazitikos-Chasapis-Hioni). *There exists an absolute constant $c_2 > 0$ such that for every $n \geqslant 2$ and for every convex body K in $\mathbb{R}^n$,*

$$\mathrm{vi}(K) \leqslant c_2 n^2.$$

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Random geometry in spherical space

Daniel Hug

(joint work with I. Bárány, G. Last, A. Reichenbacher, M. Reitzner, R. Schneider and W. Weil)

Random geometry in Euclidean space $\mathbb{R}^d$ has been studied extensively and much progress has been made within the last 20 years. Very recently, spherical space has come into focus in this context, which is quite natural from the pure mathematicians viewpoint but is also suggested by applications in stochastic geometry.

We start this talk with a study of random spherical polytopes generated as the spherically convex hull of random points sampled in a hemisphere. In contrast to the Euclidean case, we obtain closed form expressions (as well as asymptotic results) for some of the geometric characteristics of spherical polytopes (see [1]). In particular, we take the opportunity to introduce or recall some of the relevant geometric functionals such intrinsic volumes, quermassintegrals, number of k-faces, k-face contents and generalizations thereof in spherical space.

Second, we consider random tessellations of the unit sphere $\mathbb{S}^{d-1}$ generated by great subspheres of codimension 1 (the intersections of $\mathbb{S}^{d-1}$ with $(d-1)$-dimensional linear subspaces). Equivalently, we study conical tessellations of $\mathbb{R}^d$ by codimension 1 linear subspaces. For various geometric functionals, we obtain mean value formulas for certain random cones, the Schläfli-cone S_n and the Cover-Efron cone C_n, which are shown to be dual to each other. The Schläfli cone is obtained by picking at random (with equal chances) one of the (Schläfli) cones generated

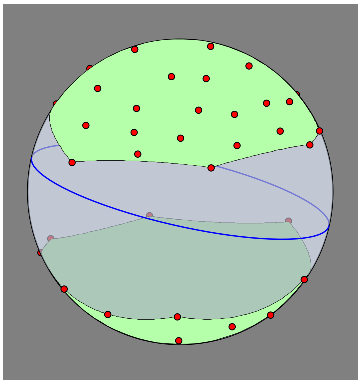

FIGURE 1. Random spherical polytope in a hemisphere together with its reflection in the origin. ©cinderella

by stochastically independent random linear subspaces $H_1, \dots, H_n \in G(d, d-1)$, which all follow the same distribution. In addition to mean value formulas, we also derive some explicit second order moments (and thus covariances) in the isotropic case (see [5]).

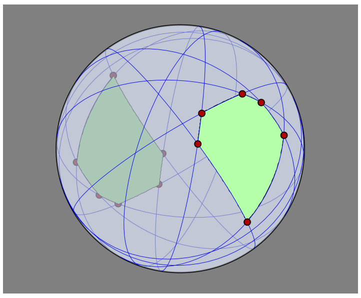

FIGURE 2. Tessellation of the sphere by great subspheres and one of the induced spherical cells together with its reflection in the origin. ©cinderella

In Euclidean space, the problem of determining the asymptotic or limit shape (if it exists) of large cells in Poisson driven random tessellations has become known as Kendall's problem (see [2, 3, 4]). In spherical space, the statement of the problem has to be modified since "large cells" cannot occur. We discuss and provide spherical analogues in the high intensity regime. This involves geometric inequalities of isoperimetric type and related stability results in spherical space (see [6]). We briefly point to applications for the study of the Boolean model on the sphere (see [7]). In this context it is interesting to note that an immediate spherical analogue of Hadwiger's famous characterization theorem for Minkowski functionals in Euclidean space is unknown in the spherical setting. Hence, rotational integral-geometric formulas for functionals on the sphere have to be established in a different way.

Figure 3. Boolean model of spherical caps (kindly provided by Michael Klatt).

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Sampling from a convex body using projected Langevin Monte-Carlo

Joseph Lehec
(joint work with S. Bubeck and R. Eldan)

Framework and main result. Let $K \subset \mathbb{R}^n$ be a convex body containing the Euclidean ball of radius 1, and contained in the Euclidean ball of radius R. Let $V : K \to \mathbb{R}$ be a convex function. We assume that V is Lipschitz with constant L and that its gradient is Lipschitz with constant β. We consider the probability measure μ given by

$$\mu(dx) = Z\, \mathrm{e}^{-V(x)} \mathbb{1}_{\{x \in K\}}\, dx,$$

where Z is just the normalization constant. Our goal is to generate a random sample from the log-concave measure μ.

Let η be a positive parameter and let $\xi_1, \xi_2, \ldots$ be an i.i.d. sequence of standard Gaussian random vectors in $\mathbb{R}^n$. We study the following Markov chain, which we call *Projected Langevin Monte Carlo*:

$$X_{k+1} = \mathcal{P}_K \left(X_k - \frac{\eta}{2} \nabla V(X_k) + \sqrt{\eta}\, \xi_{k+1} \right), \tag{1}$$

where $\mathcal{P}_K$ is the Euclidean projection on K. Our main result states as follows.

Theorem 1. *Let $\varepsilon > 0$. If the parameter η and the number of steps N satisfy:*

$$\eta \approx \frac{R^2}{N}, \quad N \approx \frac{R^6 \max(n, RL, R\beta)^{12}}{\varepsilon^{12}},$$

then we have

$$\mathrm{TV}(X_N, \mu) \leq \varepsilon.$$

Moreover, we have a slightly better result when V is constant, in other words when μ is the uniform measure on K: The same conclusion holds with

$$N \approx \frac{R^6 n^7}{\varepsilon^8}.$$

There is a long line of works in theoretical computer science proving similar results, starting with Dyer, Frieze, Kannan [2]. For the constant potential case, the best estimate is due to Lovasz and Vempala [3] who showed that essentially n^4 steps of the hit–and–run walk are enough to approximate the uniform measure on K. The starting point of this work is Dalalyan's article [1] which treats the unconstrained case ($K = \mathbb{R}^n$) and assumes furthermore that the potential V is uniformly convex. Let us sketch briefly his argument.

Dalalyan's argument. Let (W_t) be a standard Brownian motion on $\mathbb{R}^n$ and consider the Langevin SDE associated to the potential V:

$$dY_t = dW_t - \frac{1}{2}\nabla V(Y_t)\, dt. \tag{2}$$

It is well–known that the measure μ given by $\mu(dx) = Z\,\mathrm{e}^{-V(x)}\, dx$ is stationary and that the process is ergodic: $Y_t \to \mu$ in law as $t \to +\infty$. Now fix a positive parameter η and consider the following discretization of (2)

$$d\widetilde{Y}_t = dW_t - \frac{1}{2}\nabla f(\widetilde{Y}_{\lfloor t/\eta \rfloor \eta})\, dt. \tag{3}$$

It is easily seen that the law of the sequence $(\widetilde{Y}_{k\eta})$ coincides with that of the Markov chain given by (1) (when $K = \mathbb{R}^n$). Next write:

$$\mathrm{TV}(\widetilde{Y}_t, \mu) \leq \mathrm{TV}(\widetilde{Y}_t, Y_t) + \mathrm{TV}(Y_t, \mu).$$

Assuming that $\nabla^2 V \geq \alpha\, \mathrm{I}_n$ for some positive α, Bakry–Émery's theory easily yields an exponential decay for the second term. To bound the first term, rewrite (3) as

$$d\widetilde{Y}_t = d\widetilde{W}_t - \frac{1}{2}\nabla f(\widetilde{Y}_t)\, dt$$

where

$$d\widetilde{W}_t = dW_t + \frac{1}{2}\nabla V(\widetilde{Y}_t)\, dt - \frac{1}{2}\nabla V(\widetilde{Y}_{\lfloor t/\eta \rfloor \eta})\, dt.$$

This shows that it is enough to bound the total variation between W and $\widetilde{W}$, which is done using the hypothesis $\nabla^2 V \leq \beta\, \mathrm{I}_n$ and Girsanov's formula. Putting everything together Dalalyan shows that essentially n^3 steps of the algorithm are enough to approximate the measure μ.

Now we want to adapt this argument to the case where the measure μ is supported on a convex body K. For simplicity, we shall only consider the constant potential case. Thus the measure μ is uniform on K and the algorithm reads

$$X_{k+1} = \mathcal{P}_K\left(X_k + \sqrt{\eta}\,\xi_{k+1}\right), \tag{4}$$

The first step is to understand what is the underlying continuous process.
The reflected Brownian motion. Let $w\colon [0,T) \to \mathbb{R}^n$ be a path with $w(0) \in K$. We say that a couple of paths (x,φ) solves the Skorokhod problem associated to w if

- $x(t) \in K$, for all $t < T$.
- $x = w + \varphi$
- The path φ satisfies $\varphi(t) = -\int_0^t \nu_s\, L(ds)$ where L is a measure on $[0,T]$ supported on the set $\{t \in [0,T)\colon x(t) \in \partial K\}$ and ν_s is an outer unit normal at $x(s)$.

Tanaka [4] showed that for every piecewise continuous w there is a unique solution (x,φ) to the Skorokhod problem. The path x is called the reflection of w at the boundary of K and L is called the local time of x at the boundary.
Let (W_t) be a standard Brownian motion and let (Y_t) be its reflection at the boundary of K. It is not hard to see that (Y_t) is Markov and that μ (the uniform measure on K) is stationary. Moreover $Y_t \to \mu$ in law as $t \to \infty$. Now fix a parameter $\eta > 0$ and let $\widetilde{W}_t = W_{\lfloor t/\eta \rfloor \eta}$ be the discretized Brownian motion. It is not hard to see that the reflection $(\widetilde{Y}_t)$ of $(\widetilde{W}_t)$ at the boundary of K is constant on intervals $[k\eta, (k+1)\eta)$ and that the sequence $(\widetilde{Y}_{k\eta})$ has the same law as the Markov chain given by (4).
Analysis of the algorithm. Our goal is to bound $\mathrm{TV}(\widetilde{Y}_t, \mu)$. Following Dalalyan we write

$$\mathrm{TV}(\widetilde{Y}_t, \mu) \le \mathrm{TV}(\widetilde{Y}_t, Y_t) + \mathrm{TV}(Y_t, \mu).$$

We use a coupling argument to deal with the second term (Bakry–Émery does not apply anymore). Let (W_t) and (W_t') be two Brownian motions started from x and x' respectively and let (Y_t) and (Y_t') be their respective reflections at the boundary of K. We couple (W_t) and (W_t') in such a way that for each time t, the increment dW_t' is the reflection of dW_t with respect to the hyperplane median to $[Y_t, Y_t']$ (this is called *mirror coupling*). Then using the convexity of K it is pretty straightforward to show that

$$\mathbb{P}(Y \text{ and } Y' \text{ have not yet met at time } t) \le \frac{\|x - x'\|}{\sqrt{2\pi t}}.$$

This implies easily that $\mathrm{TV}(Y_t, \mu) \le \frac{R}{\sqrt{2\pi t}}$.
The first term cannot be dealt with as easily as before, just because no matter how small η is, the total variation between the Brownian motion W and its discretization $\widetilde{W}$ is always 1. On the other hand, using a deterministic inequality of Tanaka and an easy estimate on the local time of (Y_t) at the boundary, it is possible to bound the expected distance between Y_t and $\widetilde{Y}_t$:

$$\mathbb{E}\left[|Y_t - \widetilde{Y}_t|\right] \lesssim n^{3/4} t^{1/2} \eta^{1/4}.$$

The last step is to show that one can pass from this transport cost estimate to a total variation estimate. We shall not spell this out here. Let us just say that it uses the mirror coupling again, alongside with an estimate of the hitting time of the boundary of K for a Brownian motion started from a uniform point in K.

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On sharp bounds for marginal densities of product measures

Galyna Livshyts

(joint work with G. Paouris and P. Pivovarov)

We present an alternate approach to a recent theorem of Rudelson and Vershynin on marginal densities of product measures [18]. To fix the notation, if f is a probability density on Euclidean space $\mathbb{R}^n$ and E is a subspace, the marginal density of f on E is defined by

$$\pi_E(f)(x) = \int_{E^\perp + x} f(y)dy \quad (x \in E).$$

In [18], it is proved that if $f(x) = \prod_{i=1}^n f_i(x_i)$, where each f_i is a density on $\mathbb{R}$, bounded by 1, then for any $k \in \{1, \dots, n-1\}$, and any subspace E of dimension k,

$$\|\pi_E(f)\|_{L^\infty(E)}^{1/k} \leq C, \tag{1}$$

where C is an absolute constant.

In [18], it is pointed out that when $k = 1$, the constant C in (1) may be taken to be $\sqrt{2}$. This follows from a theorem of Rogozin [17], which reduces the problem to $f = \mathbb{1}_{Q_n}$ where $Q_n = [-1/2, 1/2]^n$ is the unit cube, together with Ball's theorem [1], [2] on slices of Q_n. More precisely, one can formulate Rogozin's Theorem as follows: if θ is a unit vector with linear span $[\theta]$, then

$$\left\|\pi_{[\theta]}(f)\right\|_{L^\infty([\theta])} \leq \left\|\pi_{[\theta]}(\mathbb{1}_{Q_n})\right\|_{L^\infty([\theta])} \tag{2}$$

for any f in the class

$$\mathcal{F}_n = \left\{ f(x) = \prod_{i=1}^n f_i(x_i) : \|f_i\|_{L^\infty(\mathbb{R})} \leq 1 = \|f_i\|_{L^1(\mathbb{R})}, \ i = 1, \dots, n \right\}.$$

By definition of the marginal density and the Brunn-Minkowski inequality,

$$\begin{aligned}\left\|\pi_{[\theta]}(\mathbb{1}_{Q_n})\right\|_{L^\infty([\theta])} &= \max_{x\in[\theta]}|Q_n\cap(\theta^\perp+x)|_{n-1}\\ &= |Q_n\cap\theta^\perp|_{n-1},\end{aligned}$$

where $|\cdot|_{n-1}$ denotes $(n-1)$-dimensional Lebesgue measure. Ball's theorem gives $|Q_n\cap\theta^\perp|_{n-1}\le\sqrt{2}$, which shows $C=\sqrt{2}$ works in (1).

Since Ball's theorem holds in higher dimensions, i.e.,

$$\max_{E\in G_{n,k}}|Q_n\cap E^\perp|_{n-k}^{1/k}\le\sqrt{2}\quad(k\ge 1),\tag{3}$$

where $G_{n,k}$ is the Grassmannian of all k-dimensional subspaces of $\mathbb{R}^n$, it is natural to expect that $C=\sqrt{2}$ works in (1) for all $k>1$. However, in the absence of a multi-dimensional analogue of Rogozin's result (2), the authors of [18] prove (1) with an absolute constant C via different means.

Our goal is to show that one can determine the optimal C for suitable $k>1$ directly by adapting Ball's arguments giving (3), and a related estimate, to the functional setting. The main result of this talk is the following theorem.

Theorem 1. *Let $1\le k<n$ and $E\in G_{n,k}$. Then there exists a collection of numbers $\{\gamma_i\}_{i=1}^n\subset[0,1]$ with $\sum_{i=1}^n\gamma_i=k$ such that for any bounded functions $f_1,\dots,f_n:\mathbb{R}\to[0,\infty)$ with $\|f_i\|_{L^1(\mathbb{R})}=1$ for $i=1,\dots,n$, the product $f(x)=\prod_{i=1}^n f_i(x_i)$ satisfies*

$$\|\pi_E(f)\|_{L^\infty(E)}\le\min\left(\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}},2^{k/2}\right)\prod_{i=1}^n\|f_i\|_{L^\infty(\mathbb{R})}^{\gamma_i}.\tag{4}$$

In particular, the theorem implies that if $f\in\mathcal{F}_n$ and $E\in G_{n,k}$, then

$$\|\pi_E(f)\|_{L^\infty(E)}\le\min\left(\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}},2^{k/2}\right).\tag{5}$$

As noted in [2], if $f=\mathbb{1}_{Q_n}$, the bound $\left(\frac{n}{n-k}\right)^{(n-k)/2}$ is achieved when $n-k$ divides n and $E_0\in G_{n,k}$ is chosen so that $Q_n\cap E_0^\perp$ is a cube of suitable volume; note that $\left(\frac{n}{n-k}\right)^{\frac{n-k}{2}}\le e^{k/2}$. When $k\le n/2$, the bound $2^{k/2}$ is sharp when $Q_n\cap E_0^\perp$ is a box of suitable volume. Thus for such k, Theorem 1 implies

$$\sup_{E\in G_{n,k}}\|\pi_E(f)\|_{L^\infty(E)}\le\sup_{E\in G_{n,k}}\|\pi_E(\mathbb{1}_{Q_n})\|_{L^\infty(E)}\quad(f\in\mathcal{F}_n).\tag{6}$$

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Dual curvature measures and their Minkowski problems

Erwin Lutwak, Gaoyong Zhang

(joint work with Y. Huang and D. Yang)

The Brunn-Minkowski theory centers around the study of geometric functionals of convex bodies as well as the differentials of these functionals. The fundamental geometric functionals in the Brunn-Minkowski theory are the quermassintegrals (which include volume and surface area as special cases). The differentials of the quermassintegrals are geometric measures called area measures and (Federer's) curvature measures.

There are two extensions of the Brunn-Minkowski theory: the dual Brunn-Minkowski theory, which emerged in the mid-1970s, and the L_p Brunn-Minkowski theory actively investigated since the early 1990s but dating back to the 1950s.

L_p surface area measure and its associated Minkowski problem in the L_p Brunn-Minkowski theory were introduced some two decades ago. The logarithmic Minkowski problem and the centro-affine Minkowski problem are unsolved singular cases of the L_p Minkowski problem.

Minkowski-type problems in the dual Brunn-Minkowski theory had not been previously encountered. While, over the years, the "duals" of many concepts and problems of the classical Brunn-Minkowski theory have been discovered and studied, the duals of Federer's curvature measures (and thus their associated Minkowski problems) within the dual Brunn-Minkowski theory have remained elusive. Behind this lay an inability to calculate the differentials of the dual quermassintegrals. It was the elusive nature of the duals of Federer's curvature measures that kept the PDEs of the dual theory well hidden. It turns out that the duals of Federer's curvature measures contain a number of surprises. Perhaps the biggest is that they connect known measures that were never imagined to be related.

The *quermassintegrals*, are the principal geometric functionals in the Brunn-Minkowski theory. In differential geometry, the quermassintegrals are the integrals of intermediate mean curvatures of closed smooth convex hypersurfaces. In integral geometry, the quermassintegrals are the means of the projection areas of convex body K in $\mathbb{R}^n$:

$$W_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G(n,i)} \mathrm{vol}_i(K|\xi)\, d\xi, \quad i = 1, \ldots, n, \tag{1}$$

where $\xi \in G(n,i)$, the Grassmann manifold of i-dimensional subspaces in $\mathbb{R}^n$, while $K|\xi$ is the image of the orthogonal projection of K onto ξ, where vol_i is just Lebesgue measure in ξ, and ω_i is the i-dimensional volume of the i-dimensional unit ball. The integration here is with respect to the rotation-invariant probability measure on $G(n,i)$.

Aleksandrov's variational formula for the Minkowski combination states that for each convex body K,

$$\frac{d}{dt} W_{n-j-1}(K+tL)\Big|_{t=0^+} = \int_{S^{n-1}} h_L(v)\, dS_j(K,v), \qquad j = 0, \ldots, n-1, \tag{2}$$

holds for each convex body L. Here, K and L are convex bodies and the Minkowski combination $K+tL$ is defined by $h_{K+tL} = h_K + t h_L$, where $h_Q : S^{n-1} \to \mathbb{R}$ is used to denote the support function of the convex body Q. The Borel measures $S_0(K,\cdot), \ldots, S_{n-1}(K,\cdot)$ on S^{n-1} defined by (2) are the classical *area measures* and were introduced by Fenchel & Jessen and Aleksandrov.

In addition to the area measures of Aleksandrov and Fenchel & Jessen, associated with a convex body K are the *curvature measures* of Federer, $\mathcal{C}_0(K,\cdot), \ldots, \mathcal{C}_{n-1}(K,\cdot)$. These measures are defined on $\mathbb{R}^n$ but are supported on ∂K. By restricting our attention to convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors, and using the fact that each ray emanating from the origin intersects a unique point on ∂K and a unique point on the unit sphere S^{n-1}, there is an obvious pull-back that sends the curvature measure $\mathcal{C}_j(K,\cdot)$ to a measure $C_j(K,\cdot)$

that is defined on the unit sphere. The measure $C_0(K,\cdot)$ was first defined by Aleksandrov, who called it the *integral curvature of* K. The total measures of both area measures and curvature measures give the quermassintegrals:

$$S_j(K,S^{n-1}) = C_j(K,S^{n-1}) = nW_{n-j}(K),$$

for $j = 0, 1, \dots, n-1$.

A theory dual to the theory of mixed volumes was introduced in 1970s. The duality, as a guiding principle, is conceptual in a heuristic sense and has motivated much investigation. The main geometric functionals in the dual Brunn-Minkowski theory are the *dual quermassintegrals*. The following integral geometric definition of the dual quermassintegrals, via the volume of the central sections of the body, shows their dual nature to the quermassintegrals defined in (1),

$$\widetilde{W}_{n-i}(K) = \frac{\omega_n}{\omega_i} \int_{G(n,i)} \operatorname{vol}_i(K \cap \xi)\, d\xi, \qquad i = 1, \dots, n. \tag{3}$$

The volume functional V is both the quermassintegral W_0 and the dual quermassintegral $\widetilde{W}_0$.

For each convex body K in $\mathbb{R}^n$ that contains the origin in its interior, we construct explicitly a set of Borel measures $\widetilde{C}_0(K,\cdot), \dots, \widetilde{C}_n(K,\cdot)$, on S^{n-1} that we call the *dual curvature measures* of K associated with the dual quermassintegrals, and such that

$$\widetilde{C}_j(K,S^{n-1}) = \widetilde{W}_{n-j}(K), \qquad j = 0, \dots, n.$$

These geometric measures can be viewed as the differentials of the dual quermassintegrals.

While the curvature measures of a convex body depend closely on the body's boundary, its dual curvature measures depend more on the body's interior, but yet have deep connections with classical concepts. In the case $j = n$, the dual curvature measure $\widetilde{C}_n(K,\cdot)$ turns out to be the *cone volume measure* of K (see, e.g., [1]). In the case $j = 0$, the dual curvature measure $\widetilde{C}_0(K,\cdot)$ is Aleksandrov's integral curvature of the polar body of K (divided by n).

We establish dual generalizations of Aleksandrov's variational formula. Suppose K is a convex body in $\mathbb{R}^n$ that contains the origin in its interior, $f : S^{n-1} \to \mathbb{R}$ is a continuous function. For a sufficiently small $\delta > 0$, define a family of *logarithmic Wulff shapes*,

$$[K,f]_t = \{x \in \mathbb{R}^n : x \cdot v \le h_t(v), \text{ for all } v \in S^{n-1}\},$$

for each $t \in (-\delta, \delta)$, where $h_t(v)$, for $v \in S^{n-1}$, is given by

$$\log h_t(v) = \log h_K(v) + tf(v) + o(t,v),$$

where $\lim_{t\to 0} o(t,\cdot)/t = 0$, uniformly on S^{n-1}.

We prove that for $1 \le j \le n$, and each convex body K that contains the origin in the interior, there exists a Borel measure $\widetilde{C}_j(K,\,\cdot\,)$ on S^{n-1} such that

$$\frac{d}{dt}\widetilde{W}_{n-j}([K,f]_t)\Big|_{t=0} = j \int_{S^{n-1}} f(v)\, d\widetilde{C}_j(K,v), \tag{4}$$

for each continuous $f : S^{n-1} \to \mathbb{R}$.

The main problem to be solved is the following *dual Minkowski problem*:

Suppose k is a fixed integer such that $1 \leq k \leq n$.[1] *If μ is a finite Borel measure on S^{n-1}, find necessary and sufficient conditions on μ so that μ is the k-th dual curvature measure $\widetilde{C}_k(K, \cdot)$ of some convex body K in $\mathbb{R}^n$.*

For $k = n$ the dual Minkowski problem is just the *logarithmic Minkowski problem* (also known as the *L_0-Minkowski problem*). See e.g., [1]. When the measure μ has a density function $g : S^{n-1} \to \mathbb{R}$, the partial differential equation that is the dual Minkowski problem is a Monge-Ampère type equation on S^{n-1}:

$$\frac{1}{n} h |\nabla h|^{k-n} \det(h_{ij} + h\delta_{ij}) = g. \tag{5}$$

where (h_{ij}) is the Hessian matrix of the (unknown) function h with respect to an orthonormal frame on S^{n-1}, and δ_{ij} is the Kronecker delta.

If $\frac{1}{n} h|\nabla h|^{k-n}$ were omitted in (5), then (5) would become the partial differential equation of the classical Minkowski problem. If only the factor $|\nabla h|^{k-n}$ were omitted, then equation (5) would become the partial differential equation associated with the logarithmic Minkowski problem. The gradient component in (5) significantly increases the difficulty of the problem when compared to the classical Minkowski problem or the logarithmic Minkowski problem. Existence of solutions to the PDE (5) when the "data" is a measure is much more complicated to prove and depends on "measure concentration".

Suppose k is a fixed integer such that $1 \leq k \leq n$. We say that a finite Borel measure μ on S^{n-1} satisfies the *k-subspace mass inequality*, if

$$\frac{\mu(S^{n-1} \cap \xi_i)}{\mu(S^{n-1})} < 1 - \frac{k-1}{k} \frac{n-i}{n-1},$$

for each $\xi_i \in G(n, i)$ and for each $i = 1, \ldots, n-1$.

The main theorem is:

Suppose k is a fixed integer such that $1 \leq k \leq n$. If the finite even Borel measure μ on S^{n-1} satisfies the k-subspace mass inequality, then there exists an origin-symmetric convex body K in $\mathbb{R}^n$ such that $\widetilde{C}_k(K, \cdot) = \mu$.

The case of $k = n$ was proved in [1]. Necessity is wide open, even when restricted to origin-symmetric bodies

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[1] Recall that the case where $k = 0$ is classical and was completely solved by Aleksandrov.

The isotropy constant and boundary properties of convex bodies

Mathieu Meyer
(joint work with S. Reisner)

Let $\mathcal{K}^n$ be the set of all convex bodies in $\mathbb{R}^n$ endowed with the Hausdorff distance. We prove that if $K \in \mathcal{K}^n$ has positive generalized Gauss curvature at some point of its boundary, then K is not a local maximizer for the isotropy constant L_K.

Geometric means of convex sets and functions and related problems

Vitali Milman, Liran Rotem

The first section of this report is based on the talk "Geometric means of convex sets and functions" delivered by Rotem during the workshop. The second section is based on questions asked by Milman during the problem session. This report is based on the content of two papers – [5] and [8].

1. Geometric means of convex sets and functions

Denote by $\mathrm{Cvx}(\mathbb{R}^n)$ the class of convex, lower semi-continuous functions $\varphi : \mathbb{R}^n \to (-\infty, \infty]$. For $\varphi_0, \psi_0 \in \mathrm{Cvx}(\mathbb{R}^n)$ we define their geometric mean $\rho = G(\varphi_0, \psi_0)$ by setting

$$\begin{aligned} \varphi_{n+1}(x) &= \frac{1}{2}\left(\varphi_n(x) + \psi_n(x)\right) \\ \psi_{n+1}(x) &= \frac{1}{2}\inf_{y\in\mathbb{R}^n}\left(\varphi_n(x+y) + \psi_n(x-y)\right) \end{aligned} \tag{1}$$

for all n, and defining

$$\rho(x) = \lim_{n\to\infty} \varphi_n(x) = \lim_{n\to\infty} \psi_n(x).$$

It is not difficult to prove that the sequences $\{\varphi_n\}_{n=0}^{\infty}$ and $\{\psi_n\}_{n=0}^{\infty}$ indeed converge pointwise to a common limit, under some weak assumptions on φ_0 and ψ_0. For example, it is enough to assume that φ_0 and ψ_0 are everywhere finite. In the special case where φ_0 and ψ_0 are 2-homogeneous this process was considered by Asplund ([1]).

A similar process may be carried out for convex bodies. Denote by $\mathcal{K}_0^n$ the class of closed, convex sets in $K \subseteq \mathbb{R}^n$ such that $0 \in K$. For a fixed $1 \le p < \infty$, and given $A_0, B_0 \in \mathcal{K}_0^n$, we set

$$\begin{aligned} A_{n+1} &= \frac{1}{2^{1/p}}\left(A_n +_p B_n\right) \\ B_{n+1} &= \left[\frac{1}{2^{1/p}}\left(A_n^\circ +_p B_n^\circ\right)\right]^\circ \end{aligned} \tag{2}$$

Here K° denotes the polar body of K, and $+_p$ denotes the p-addition of convex bodies (introduced by Firey in [4]). If A_0 and B_0 are compact then the common limit

$$G = \lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n$$

exists in the Hausdorff sense, and is called the p-geometric mean of A_0 and B_0. We denote $G = G_p(A_0, B_0)$.

In order to understand the name "geometric mean", one should think about the polarity map $K \mapsto K^\circ$ as an inversion, i.e. $K^\circ =$ "K^{-1}". Similarly, one should think about the Legendre transform φ^* of a function $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ as "φ^{-1}". The first result following this ideology was probably the inequality

$$\frac{K+T}{2} \supseteq \left(\frac{K^\circ + T^\circ}{2}\right)^\circ$$

proved by Firey in [3]. Firey called this result an arithmetic mean-harmonic mean inequality. And indeed, if one thinks of K° as the inverse of K, the right hand side is exactly the harmonic mean of K and T. Similarly, the harmonic mean of $\varphi, \psi \in \mathrm{Cvx}(\mathbb{R}^n)$ is $\left(\frac{\varphi^*+\psi^*}{2}\right)^*$, which is same as the inf-convolution that appears in (1).

Another manifestation of the same ideology is the following theorem proved in [8]:

Theorem 1. *For every $\varphi \in \mathrm{Cvx}(\mathbb{R}^n)$ one has*

$$(\varphi + \delta)^* + (\varphi^* + \delta)^* = \delta,$$

where $\delta(x) = \frac{1}{2}|x|^2$.

This theorem is the analogue of the trivial identity $\frac{1}{x+1} + \frac{1}{1/x+1} = 1$ for every $x > 0$. It has applications for Santaló type inequalities and for the theory of summands.

Once we accept the above principle, the name "geometric mean" becomes easy to explain. For fixed numbers $x_0, y_0 > 0$, define sequences $\{x_n\}$ and $\{y_n\}$ by

$$x_{n+1} = \frac{x_n + y_n}{2}, \qquad y_{n+1} = \left(\frac{x_n^{-1} + y_n^{-1}}{2}\right)^{-1}. \tag{3}$$

It is an easy exercise that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = \sqrt{x_0 y_0}$. The processes (1) and (2) are the natural analogues of (3), which justifies the use of the term "geometric mean".

We now list some of the properties of the geometric mean. We start with Theorem 2 that summarizes some of its basic properties. The first four parts of the theorem are analogues of trivial results about the geometric mean of numbers.

Theorem 2. *For every everywhere-finite $\varphi, \psi, \varphi', \psi' \in \mathrm{Cvx}(\mathbb{R}^n)$, every compact $K, T, K', T' \in \mathcal{K}_0^n$ and every $1 \le p < \infty$, the geometric mean has the following properties:*

- $G(\varphi,\varphi)=\varphi$ and $G_p(K,K)=K$.
- G is monotone in its arguments: If $K\subseteq K'$ and $T\subseteq T'$ then $G_p(K,T)\subseteq G_p(K',T')$. Similarly if $\varphi\le\varphi'$ and $\psi\le\psi'$ then $G(\varphi,\psi)\le G(\varphi',\psi')$.
- $G(\varphi,\psi)^*=G(\varphi^*,\psi^*)$ and $G_p(K,T)^\circ=G_p(K^\circ,T^\circ)$.
- $G(\varphi,\varphi^*)=\delta$ and $G_p(K,K^\circ)=D$ (the Euclidean unit ball).
- For every linear map u we have $G(\varphi\circ u,\psi\circ u)=G(\varphi,\psi)\circ u$ and $G_p(uK,uT)=u\cdot G_p(K,T)$.

For ellipsoids, we have the following result:

Theorem 3. *For any centered ellipsoids $\mathcal{E}_1$ and $\mathcal{E}_2$ the mean $G_p(\mathcal{E}_1,\mathcal{E}_2)$ is an ellipsoid, which is independent of p.*

Notice that for $p=2$, whenever A_0 and B_0 in the process (2) are ellipsoids all the sets A_n and B_n are ellipsoids as well, so it is not surprising that the limit $G_2(A_0,B_0)$ is an ellipsoid. However, for $p\ne 2$, the sets A_n and B_n are not ellipsoids, and still the limit $G_p(A_0,B_0)$ is an ellipsoid.

It is easy to check that for numbers, the function $G(x,y)=\sqrt{xy}$ is concave on $(\mathbb{R}_+)^2$. For functions we have the following analogous result (taken from [8]):

Theorem 4. *The function $(\varphi,\psi)\to G(\varphi,\psi)$ is concave in its arguments. More explicitly, fix everywhere-finite $\varphi_0,\varphi_1,\psi_0,\psi_1\in\mathrm{Cvx}(\mathbb{R}^n)$ and $0<\lambda<1$. Define $\varphi_\lambda=(1-\lambda)\varphi_0+\lambda\varphi_1$ and $\psi_\lambda=(1-\lambda)\psi_0+\lambda\psi_1$. Then*

$$G(\varphi_\lambda,\psi_\lambda)\supseteq(1-\lambda)\cdot G(\varphi_0,\psi_0)+\lambda G(\varphi_1,\psi_1).$$

This theorem implies a similar theorem for sets, where the addition taken is the 2-addition: For every convex bodies $K_0,K_1,T_0,T_1\in\mathcal{K}_0^n$ one has

$$G_2(K_\lambda,T_\lambda)\supseteq\sqrt{1-\lambda}G_2(K_0,T_0)+_2\sqrt{\lambda}G_2(K_1,T_1),$$

where $K_\lambda=\sqrt{1-\lambda}K_0+_2\sqrt{\lambda}K_1$ and $T_\lambda=\sqrt{1-\lambda}T_0+_2\sqrt{\lambda}T_1$. Perhaps surprisingly, however, it turns out that the geometric mean of sets is *not* concave with respect to the regular Minkowski addition.

In order to better understand the body $G_p(K,T)$, let us compare it with another known construction. Given $K,T\in\mathcal{K}_0^n$, Böröczky, Lutwak, Yang and Zhang ([2]) define the logarithmic mean (or 0-mean) of K and T to be

$$L(K,T)=\left\{x\in\mathbb{R}^n:\ \langle x,\theta\rangle\le\sqrt{h_K(\theta)h_L(\theta)}\text{ for all }\theta\right\},$$

where h_K is the support function of K. Let us also define $L^*(K,T)=(L(K^\circ,T^\circ))^\circ$. Notice that $L^*(K,T)$ is simply the smallest convex body such that $r_{L^*}(\theta)\ge\sqrt{r_K(\theta)r_T(\theta)}$, where r_K is the radial function of K.

Theorem 5. *For every $1\le p<\infty$ we have $L^*(K,T)\subseteq G_p(K,T)\subseteq L(K,T)$.*

In general the inclusions above may be strict. However, at least in some directions, we will always have equality:

Corollary 6. *Fix $K,T\in\mathcal{K}_0^n$. Assume that in direction η the bodies K and T have parallel supporting hyperplanes, with normal vector θ. Then $h_{G_p(K,T)}(\theta)=\sqrt{h_K(\theta)h_T(\theta)}$ and $r_{G_p(K,T)}(\eta)=\sqrt{r_K(\eta)r_T(\eta)}$ for all p.*

Notice there are always such directions η – the points where $\eta \mapsto \dfrac{r_K(\eta)}{r_T(\eta)}$ attains its extrema.

2. Related problems

If one interprets K° as "K^{-1}", most constructions we know in convexity are "rational constructions" – built by a finite number of additions and "inversions". It appears that the time has come for "irrational constructions" as well. The geometric mean described above is one example. Let us describe two more:

(1) In [6], Molchanov builds continued fractions of convex sets. In particular, if $K \supseteq D$ is a compact convex body then the process
$$\Big(K + \big(K + (K + \cdots)^\circ\big)^\circ\Big)^\circ$$
converges to a limit Z. This Z is the unique solution of the "quadratic equation" $Z^\circ = Z + K$. More generally, one may also consider periodic continued fractions with period > 1 to be solutions of more generalized quadratic equations.

(2) Once the geometric mean is defined, one may define the Gauss arithmetic-geometric mean in the same way it is done for numbers (see, e.g. [7]): Given A_0, B_0 we set
$$A_{n+1} = \frac{A_n + B_n}{2} \qquad B_{n+1} = G(A_n, B_n)$$
(where G is, say, G_1 defined above). The common limit $\lim_{n\to\infty} A_n = \lim_{n\to\infty} B_n$ is the arithmetic-geometric mean of A_0 and B_0, which we will denote by $M_{ag}(A_0, B_0)$. The geometric-harmonic mean, $M_{hg}(A_0, B_0)$, is defined similarly.

In order to further develop the "irrational" theory of convexity the following open problems need to be addressed:

(1) We want to think about the relation $G(A, B) = P$ as "A is polar to B with respect to P". To justify this intuition, the following questions need to be answered:
   (a) Does $G(A, B) = D$ imply that $B = A^\circ$?
   (b) A "rational" variant of the previous question is the following: Assume $A + B = A^\circ + B^\circ$. Does it follow that $B = A^\circ$? If the answer to this question is "no", the answer to the previous question is "no" as well.
   (c) Let P be a convex body such that for every A there exists a B with $G(A, B) = P$. Does it follow that P is an ellipsoid?

(2) (a) Is it true that $G_p(A, B) = G_1(A, B)$ for all A, B?
   (b) Define the upper elliptic envelope of $G(A, B)$ as
$$\mathcal{E}_u(A, B) = \bigcap \left\{ G(\mathcal{E}_1, \mathcal{E}_2) : \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 \text{ are ellipsoids with} \\ A \subseteq \mathcal{E}_1 \text{ and } B \subseteq \mathcal{E}_2 \end{array} \right\}.$$
   Is it true that $\mathcal{E}_u(A, B) = G(A, B)$?

(c) Do we have $G(A, \lambda B) = \sqrt{\lambda} G(A, B)$ for $\lambda > 0$?

A positive answer to (2b) will imply a positive answer for (2a) and (2c) as well.

(3) Under what conditions $|G(A, B)|^2 \geq |A|\,|B|$? Here $|\cdot|$ denotes the (Lebesgue) volume. We now that the answer is not "always", even if A and B are assumed to be origin-symmetric. By the Blaschke–Santaló inequality the inequality holds whenever A is origin-symmetric and $B = A^\circ$.

(4) (a) Does there exist an "exponential map", i.e. a map $E : \mathcal{K}_0^n \to \mathcal{K}_0^n$ such that $E(\{0\}) = D$ and

$$E\left(\frac{A+B}{2}\right) = G\left(E(A), E(B)\right)?$$

(b) In the opposite direction, does there exists a "logarithmic map" with the property

$$L\left(G(A, B)\right) = \frac{L(A) + L(B)}{2},$$

and what is its natural domain?

For numbers, it is proved in [7] that

$$M_{gh}(N, 1) = \frac{2}{\pi} \log 4N + O\left(1/N^2\right),$$

and perhaps a similar result will hold in our case as well. We would like to thank Hermann König for referring us to the paper [7].

(5) Is it true that $G(M_{ag}(A, B), M_{gh}(A, B)) = G(A, B)$? This property does hold for numbers.

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On randomized Dvoretzky's theorem for subspaces of L_p

Grigoris Paouris
(joint work with P. Valettas and J. Zinn)

In 1961 Dvoretzky [1] gave an affirmative answer to a question of Grothendieck by proving that every finite dimensional normed space has lower dimensional subspace which is almost Euclidean and the dimension grows with respect to the dimension of the ambient space. The optimal dependance on the dimension was proved 10 years later by V. Milman in his groundbreaking work [4]. Milman;s result states that for any $\varepsilon \in (0,1)$ there exists a function $c(\varepsilon) > 0$ with the following property: for every n-dimensional normed space X there exists $k \geq c(\varepsilon)\log n$ and linear map $T : \ell_2^k \to X$ with $\|x\|_2 \leq \|Tx\|_X \leq (1+\varepsilon)\|x\|_2$ for all $x \in \ell_2^k$ – we say that ℓ_2^k can be $(1+\varepsilon)$-embedded into X and we write: $\ell_2^k \overset{1+\varepsilon}{\hookrightarrow} X$. The example of $X = \ell_\infty^n$ shows that this result is best possible with respect to n. Milman also showed a "randomized" version of the above theorem where a "random" subspace in a "critical" dimension is almost Euclidean. The best bounds on the dependance on ε ($c(\varepsilon) = \varepsilon^2$) on the "randomized" version of the above theorem is due to Gordon [3] (see also [7]). We have investigate the problem of the dependance on ε in the "randomized Dvoretzky's Theorem" in the case of the classical spaces ℓ_p^n. Here $B_p^n := \{x \in \mathbb{R}^n : \|x\|_p \leq 1$ where $\|x\|_p := (\sum_{i=1}^n |x_i|^p)^{\frac{1}{p}}$.

Theorem 1. [5] *Let $1 \leq p \leq \infty$. Then, for each n and for any $0 < \varepsilon < 1$ the random k-dimensional section of B_p^n with dimension $k \leq k(n,p,\varepsilon)$ is $(1+\varepsilon)$-Euclidean with probability greater than $1 - C\exp(-ck(n,p,\varepsilon))$, where $k(n,p,\cdot)$ is defined as:*

i. *If $1 \leq p < 2$, then*

$$k(n,p,\varepsilon) \simeq \varepsilon^2 n,\ 0 < \varepsilon < 1. \tag{1}$$

ii. *If $2 < p < \varepsilon_0 \log n$, then*

$$k(n,p,\varepsilon) \simeq \begin{cases} (Cp)^{-p}\varepsilon^2 n, & 0 < \varepsilon \leq (Cp)^{p/2} n^{-\frac{p-2}{2(p-1)}} \\ p^{-1}\varepsilon^{2/p} n^{2/p}, & (Cp)^{p/2} n^{-\frac{p-2}{2(p-1)}} < \varepsilon \leq 1/p \\ \varepsilon p n^{2/p}/\log\frac{1}{\varepsilon}, & \frac{1}{p} < \varepsilon < 1 \end{cases}. \tag{2}$$

In fact for $p < \varepsilon_0 \log n$ and $p \simeq \log n$ we have:

$$k(n,p,\varepsilon) \simeq \log n/\log\frac{1}{\varepsilon}. \tag{3}$$

iii. *If $p \geq \varepsilon_0 \log n$, then*

$$k(n,p,\varepsilon) \simeq \varepsilon \log n/\log\frac{1}{\varepsilon},\ 0 < \varepsilon < 1. \tag{4}$$

where $C, c, \varepsilon_0 > 0$ are absolute constants.

The proof of the above theorem depends sharp concentration inequalities for the p-norms with respect to the Gaussian measure. To achieve this we are using

functional inequalities as log-Sobolev inequality and "$L_1 - L_2$ Talagrand's inequality".
Using these ideas we were able to investigate the dependance on ε in the case of subspaces of L_p. The above result extends the aforementioned result in the case where p is "fixed". This new results improves a previous result of Figiel, Linderstrauss and Milman [2].

Theorem 2. [6] *For any $p > 2$ there exists a constant $c(p) > 0$ with the following property: for any n-dimensional subspace X of L_p and for any $\varepsilon \in (0,1)$ there exists $k \geq c(p)\min\{\varepsilon^2 n, (\varepsilon n)^{2/p}\}$ so that ℓ_2^k can be $(1+\varepsilon)$-embedded into X.*

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Centro-affine tensor valuations

Lukas Parapatits
(joint work with C. Haberl)

Let p be a nonnegative integer and define

$$M^{p,0}(K) = (n+p)\int_K x^{\otimes p}\ dx$$

for all $K \in \mathcal{K}^n$. The map $M^{p,0}$ is a continuous valuation from $\mathcal{K}^n$ to $\mathrm{Sym}^p(\mathbb{R}^n)$, i.e. to the space of symmetric tensors of order p. Furthermore, $M^{p,0}$ is $\mathrm{SL}(n)$-covariant in the sense that

$$M^{p,0}(\vartheta K) = \vartheta \cdot M^{p,0}(K)$$

for all $\vartheta \in \mathrm{SL}(n)$. The action of the $\mathrm{SL}(n)$ on $(\mathbb{R}^n)^{\otimes p}$ is uniquely determined by

$$\vartheta \cdot (v_1 \otimes \cdots \otimes v_p) = (\vartheta v_1) \otimes \cdots \otimes (\vartheta v_p)$$

for all $v_1, \ldots, v_p \in \mathbb{R}^n$.

Define

$$M^{0,p}(K) = \int_{S^{n-1}} u^{\otimes p}\ h_K(u)^{1-p}\ dS_K(u)$$

for all $K \in \mathcal{K}_o^n$, i.e. for all convex bodies containing the origin in the interior. The map $M^{0,p}$ is a continuous valuation from $\mathcal{K}_o^n$ to $\mathrm{Sym}^p(\mathbb{R}^n)$. Furthermore, $M^{0,p}$ is $\mathrm{SL}(n)$-contravariant in the sense that

$$M^{0,p}(\vartheta K) = \vartheta^{-t} \cdot M^{0,p}(K).$$

In fact, these two examples are part of a larger family. Let r, s be nonnegative integers and define

$$\hat{M}^{r,s}(K) = \int_{\partial K} x^{\otimes r} \otimes u_K(x)^{\otimes s} \langle x, u_K(x)\rangle^{1-s} \, d\mathcal{H}^{n-1}(x)$$

for all $K \in \mathcal{K}_o^n$. The map $\hat{M}^{r,s}$ is a continuous valuation from $\mathcal{K}_o^n$ to $(\mathbb{R}^n)^{\otimes(r+s)}$. It is compatible with the $\mathrm{SL}(n)$ in the sense that

$$\hat{M}^{r,s}(\vartheta K) = \left(\vartheta^{\otimes r} \otimes \left(\vartheta^{-t}\right)^{\otimes s}\right) \hat{M}^{r,s}(K).$$

The following classification result for maps on $\mathcal{P}_o^n$, i.e. on the space of convex polytopes containing the origin in the interior, is proved in [3].

Theorem 1. *Let $n \geq 3$, $p \geq 2$ and $\mu \colon \mathcal{P}_o^n \to \mathrm{Sym}^p(\mathbb{R}^n)$. The map μ is an $\mathrm{SL}(n)$-covariant measurable valuation if and only if it is a linear combination of $M^{p,0}$ and $M^{0,p} \circ {}^*$, where * denotes the polar body.*

In the plane additional examples show up. Denote by ρ the rotation about an angle of $\frac{\pi}{2}$. Define

$$M_\rho^{r,s}(K) = \int_{\partial K} x^{\odot r} \odot (\rho u_K(x))^{\odot s} \langle x, u_K(x)\rangle^{1-s} \, d\mathcal{H}^1(x)$$

for all $K \in \mathcal{K}_o^2$. The map $M_\rho^{r,s}$ is an $\mathrm{SL}(2)$-covariant continuous valuation from $\mathcal{K}_o^2$ to $\mathrm{Sym}^{r+s}(\mathbb{R}^2)$. The result in the plane reads as follows.

Theorem 2. *Let $p \geq 2$ and $\mu \colon \mathcal{P}_o^2 \to \mathrm{Sym}^p(\mathbb{R}^2)$. The map μ is an $\mathrm{SL}(2)$-covariant measurable valuation if and only if it is a linear combination of $M_\rho^{i,p-i}$, $i \in \{0, \dots, p\} \setminus \{p-1\}$, and $\rho \cdot M^{p,0} \circ {}^*$.*

Similar results for $p = 0$ and $p = 1$ were already established in [1, 2] but are also a consequence of the work in [3]. The first results of this type were established by Monika Ludwig, see e.g. [4, 5, 6].

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Random ball-polyhedra and inequalities for intrinsic volumes

Peter Pivovarov
(joint work with G. Paouris)

I discussed inequalities for intrinsic volumes and associated randomized versions. Recall that the intrinsic volumes $V_1, \ldots, V_n$ can be defined via the Steiner formula: for any convex body $K \subseteq \mathbb{R}^n$ and $\varepsilon > 0$,

$$|K + \varepsilon B| = \sum_{j=0}^{n} \omega_{n-j} V_j(K) \varepsilon^{n-j},$$

where $|\cdot|$ denotes n-dimensional Lebesgue measure, $B = B_2^n$ is the unit Euclidean ball in $\mathbb{R}^n$, ω_{n-j} is the volume of B_2^{n-j}, and $V_0 \equiv 1$; V_1 is a multiple of the mean-width, $2V_{n-1}$ is the surface area and $V_n = |\cdot|$ is the volume. The V_j's satisfy the extended isoperimetric inequality: for $1 \leqslant j < n$,

$$\left(\frac{V_n(K)}{V_n(B)}\right)^{1/n} \leqslant \left(\frac{V_j(K)}{V_j(B)}\right)^{1/j}; \tag{1}$$

as well as the generalized Urysohn inequality: for $1 < j \leqslant n$,

$$\left(\frac{V_j(K)}{V_j(B)}\right)^{1/j} \leqslant \frac{V_1(K)}{V_1(B)}. \tag{2}$$

The classical isoperimetric inequality corresponds to $j = n-1$ in (1); Urysohn's inequality to $j = n$ in (2) (or $j = 1$ in (1)). The Alexandrov-Fenchel inequality for mixed volumes implies both (1) and (2).

Known inequalities from stochastic geometry for the expected intrinsic volumes of random polytopes in convex bodies can be seen as randomized versions of (1). Such inequalities have their roots in the classical Sylvester's problem and build on work of Busemann, Groemer, Rogers-Shephard, Pfiefer, Campi-Gronchi, Hartzoulaki-Paouris, among others; see the references in [3]. Drawing on [3], one can formulate a type of stochastic dominance as follows. Assume that $|K| = |B|$ and sample independent random vectors $X_1, \ldots, X_N$ according to the uniform density $\frac{1}{|K|}\mathbb{1}_K$, i.e., $\mathbb{P}(X_i \in A) = \frac{1}{|K|}\int_A \mathbb{1}_K(x)dx$ for Borel sets $A \subseteq \mathbb{R}^n$. Additionally, sample independent random vectors $Z_1, \ldots, Z_N$ according to $\frac{1}{|B|}\mathbb{1}_B$. Then for all $1 \leqslant j \leqslant n$ and $s > 0$,

$$\mathbb{P}\left(V_j(\mathrm{conv}\{X_1, \ldots, X_N\}) > s\right) \geqslant \mathbb{P}\left(V_j(\mathrm{conv}\{Z_1, \ldots, Z_N\}) > s\right), \tag{3}$$

where conv denotes the convex hull. Integrating in s yields

$$\mathbb{E}V_j(\mathrm{conv}\{X_1, \ldots, X_N\}) \geqslant \mathbb{E}V_j(\mathrm{conv}\{Z_1, \ldots, Z_N\}). \tag{4}$$

By the law of large numbers, the latter convex hulls converge to their respective ambient bodies and thus when $N \to \infty$, $V_j(K) \geqslant V_j(B)$ whenever $V_n(K) = V_n(B)$, which is equivalent to (1). Thus (1) can be seen as a global inequality which arises through a random approximation procedure in which stochastic domination holds at each stage. Kindred results in [3] are related to inequalities in L_p-Brunn-Minkowski theory due to Lutwak, Yang and Zhang. The inequalities above involve volume and can be proved using Steiner symmetrization for convex bodies and rearrangement inequalities in the setting with probability densities.

A randomized version of (2) arises through a different random model but shares similar characteristics, despite the fact that it need not involve volume. The model for such random sets is motivated by work of Bezdek, Lángi, Naszódi and Papez [1] on ball-polyhedra, which are intersections of finitely many congruent Euclidean balls. To fix the notation, let $B(x,R)$ be the closed Euclidean ball centered at $x \in \mathbb{R}^n$ with radius R. Let f be the density of a continuous probability distribution on $\mathbb{R}^n$ and assume that f is bounded and, for simplicity, that $\|f\|_\infty \leq 1$. Sample independent random vectors $X_1,\dots,X_N$ according to f and $Z_1,\dots,Z_N$ according to the density $\mathbb{1}_{B(0,r_n)}$, where r_n satisfies $|B(0,r_n)| = 1$. Our main result in [4] is that for $1 \leqslant j \leqslant n$ and $s > 0$,

$$\mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(X_i,R)\right) > s\right) \leqslant \mathbb{P}\left(V_j\left(\bigcap_{i=1}^N B(Z_i,R)\right) > s\right). \tag{5}$$

By sampling the X_i's in a particular star-shaped set, (5) leads to a stochastic dominance that underlies (2). We find this fact surprising since (5) deals with sets (or uniform densities f on those sets) of a given volume and compares V_j - V_1 is not singled out in the formulation. Given a convex body K with support function h_K and $R > 0$, define a star-shaped set $A(K,R)$ by specifying its radial function:

$$\rho_{A(K,R)}(-\theta) = R - h_K(\theta) \qquad (\theta \in S^{n-1}).$$

We prove that, in the Hausdorff metric,

$$K = \lim_{R\to\infty} \bigcap_{x \in A(K,R)} B(x,R)$$

and writing $r(K,R,n) = \omega_n^{-1/n}|A(K,R)|^{1/n}$ (which is the radius of a Euclidean ball with the same volume as $A(K,R)$),

$$R - r(K,R,n) \geqslant \int_{S^{n-1}} h_K(\theta)d\sigma(\theta);$$

and equality holds as $R \to \infty$. Thus if we sample independent random vectors $X_1, X_2, \dots$ according to $f = \frac{1}{|A(K,R)|}\mathbb{1}_{A(K,R)}$ and $Z_1, Z_2, \dots$ according to $\frac{1}{|A(K,R)|}\mathbb{1}_{r(K,R,n)B}$, inequality (5) (with a suitable renormalization) implies

$$\mathbb{E}V_j\left(\bigcap_{i=1}^N B(X_i,R)\right) \leqslant \mathbb{E}V_j\left(\bigcap_{i=1}^N B(Z_i,R)\right). \tag{6}$$

As $N \to \infty$,

$$V_j\left(\bigcap_{x \in A(K,R)} B(x,R)\right) \leqslant V_j\left(\bigcap_{z \in rB} B(z,R)\right). \tag{7}$$

As $R \to \infty$, we ultimately arrive at

$$V_j(K) \leq V_j((w(K)/2)B),$$

where $w(K)$ is the mean width of K, which is (2). Thus (5) can be seen as a randomized version of the generalized Urysohn inequality in which the extremal sets are not Euclidean balls but random ball-polyhedra generated using the uniform measure on the Euclidean ball.

As mentioned above, (1) and (2) share a common result - Urysohn's inequality. Since we have two different randomzied inequalities that lead to Urysohn's inequality, namely for random ball-polyhedra by taking $j = n$ in (6), and for random convex hulls by taking $j = 1$ in (4), it is natural to investigate the relationship between the two randomized forms. It turns out that the random ball-polyhedra version implies the random convex hull version. This is a consequence of a result of Gorbovickis [2], which has been used to establish the Kneser-Poulsen conjecture for large radii.

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The centroid body: algorithms and statistical estimation for heavy-tailed distributions

Luis Rademacher

(joint work with J. Anderson, N. Goyal and A. Nandi)

Independent component analysis (ICA) is the problem of efficiently recovering a matrix $A \in \mathbb{R}^{n \times n}$ from i.i.d. observations of $X = AS$ where $S \in \mathbb{R}^n$ is a random vector with mutually independent coordinates. This problem has been intensively studied, but all existing efficient algorithms with provable guarantees require that the coordinates S_i have finite fourth moments. We consider the heavy-tailed ICA problem where we do not make this assumption, about the second moment. This problem also has received considerable attention in the applied literature. In the present work, we first give a provably efficient algorithm that works under the assumption that for constant $\gamma > 0$, each S_i has finite $(1+\gamma)$-moment, thus substantially weakening the moment requirement condition for the ICA problem to be solvable. We then give an algorithm that works under the assumption that matrix A has orthogonal columns but requires no moment assumptions. Our techniques exploit standard properties of the multivariate spherical Gaussian distribution in a novel way and draw ideas from convex geometry. In particular, a contribution of this work is the algorithmic use of the centroid body from convex geometry to play a role analogue to the covariance matrix but for a heavy-tailed distribution.

This abstract is based on [1].

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On the complexity of the set of unconditional convex bodies

Mark Rudelson

This is a report on paper [5].

In [1] Barvinok and Veomett posed a question whether any n-dimensional convex symmetric body can be approximated by a projection of a section of a simplex whose dimension is subexponential in n. The importance of this question stems from the fact that the convex bodies generated this way allow an efficient construction of the membership oracle. The question of Barvinok and Veomett has been answered in [3], where it was shown that for all $1 \le n \le N$, there exists an n-dimensional symmetric convex body B such that for every n-dimensional convex body K obtained as a projection of a section of an N-dimensional simplex one has

$$d(B, K) \ge c\sqrt{\frac{n}{\ln \frac{2N\ln(2N)}{n}}},$$

where $d(\cdot, \cdot)$ denotes the Banach-Mazur distance and c is an absolute positive constant. Moreover, this result is sharp up to a logarithmic factor.

One of the main steps in the proof of this result was an estimate of the complexity of the set of all convex symmetric bodies in $\mathbb{R}^n$, i. e., the Minkowski or Banach–Mazur compactum. The complexity is measured in terms of the maximal size of a t-separated set with respect to the Banach–Mazur distance

$$d(K, D) = \inf\{l \ge 1 \mid D \subset TK \subset lD\},$$

where the infimum is taken over all linear operators $T : \mathbb{R}^n \to \mathbb{R}^n$. A set A in a metric space (X, d) is called t-separated if the distance between any two distinct points of A is at least t. It follows from [3] that for any $1 \le t \le cn$, the set of all n-dimensional convex bodies contains a t-separated subset of cardinality at least

$$\exp(\exp(cn/t)). \tag{1}$$

Note that for $t = O(1)$, the estimate above shows that the complexity of the Minkowski compactum is doubly exponential in terms of the dimension. This fact has been independently established by Pisier [4], who asked whether a similar statement holds for the set of all unconditional convex bodies and for the set of all completely symmetric bodies. We show that the answer to the first question is affirmative, and to the second one negative.

Consider unconditional convex bodies first. A convex symmetric body $K \subset \mathbb{R}^n$ is called unconditional if it symmetric with respect to all coordinate hyperplanes. This property can be conveniently reformulated in terms of the norm generated by K. For $x \in \mathbb{R}^n$, set

$$\|x\|_K = \min\{a \geq 0 \mid x \in aK\}.$$

The body K is unconditional if thew norm generated by it is a function of the absolute values of the coordinates.

Our main result shows that the complexity of the set $\mathcal{K}_n^{unc}$ of unconditional convex bodies at the scale t is doubly exponential as long as $t = O(1)$. More precisely, we prove the following theorem.

Theorem 1. *Let $1 \leq t \leq \tilde{c} n^{1/2} \log^{-5/2} n$. The set of n-dimensional unconditional convex bodies contains a t-separated set of cardinality at least*

$$\exp\left(\exp\left(\frac{c}{t^2 \log^4(1+t)} n\right)\right).$$

Here, $\tilde{c}$ and c are positive absolute constants.

Note that unlike the estimate (1), which is valid for $1 \leq t \leq cn$, the estimate above holds only in the range $1 \leq t \leq \tilde{c} n^{1/2} \log^{-5/2} n$. By a theorem of Lindenstrauss and Szankowski [2], the maximal Banach–Mazur distance between two n-dimensional unconditional bodies does not exceed $Cn^{1-\varepsilon_0}$ for some $\varepsilon_0 \geq 1/3$. This means that a non-trivial estimate of the cardinality of a t-separated set in $\mathcal{K}_n^{unc}$ is impossible whenever $t > n^{1-\varepsilon_0}$.

Following the derivation of Theorem 1.1 [1], one can show that Theorem 1 implies a result on the hardness of approximation of an unconditional convex body by a projection of a section of a simplex refining the solution of the problem posed by Barvinok and Veomett.

Corollary 2. *Let $n \leq N$. There exists an n-dimensional unconditional convex body B, such that for every n-dimensional convex body K obtained as a projection of a section of an N-dimensional simplex one has*

$$d(B, K) \geq c \left(\frac{n}{\log N}\right)^{1/4} \cdot \log^{-1}\left(\frac{n}{\log N}\right),$$

where c is an absolute positive constant.

In particular, Corollary 2 means that to be able to approximate all unconditional convex bodies in $\mathbb{R}^n$ by projections of sections of an N-dimensional simplex within the distance $O(1)$, one has to take $N \geq \exp(cn)$.

Consider now the set of completely symmetric bodies. We will call an n-dimensional convex body completely symmetric if it is unconditional and invariant under all permutations of the coordinates. This term is not commonly used. In the language of normed spaces, completely symmetric convex bodies correspond to the spaces with 1-symmetric basis. However, since the term "symmetric convex bodies" has a different meaning, we will use "completely symmetric" for this class of bodies.

The set of completely symmetric convex bodies is much smaller than the set of all unconditional ones. This manifests quantitatively in the fact that the cardinality of a t-separated set of completely symmetric bodies is significantly lower. Namely, we prove the following proposition.

Proposition 3. *Let $t \geq 2$. The cardinality of any t-separated set in $\mathcal{K}^{cs}$ does not exceed*

$$\exp\left(\exp\left(C\frac{\log^2 n}{\log t}\right)\right).$$

This proposition means, in particular, that the complexity of the set of completely symmetric convex bodies is not doubly exponential in the dimension, which answers the second question of Pisier.

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On bodies with directly congruent projections

Dmitry Ryabogin

(joint work with M. A. Alfonseca and M. Cordier)

In this talk we address the following problem (see [3, Problem 3.2, page 125]).

Problem 1. Suppose that $2 \leq k \leq n-1$ and that K and L are convex bodies in $\mathbb{R}^n$ such that the projection $K|H$ is congruent to $L|H$ for all $H \in \mathcal{G}(n,k)$. Is K a translate of $\pm L$?

Here we say that $K|H$, the projection of K onto H, is congruent to $L|H$ if there exists an orthogonal transformation $\varphi \in O(k,H)$ in H such that $\varphi(K|H)$ is a translate of $L|H$; $\mathcal{G}(n,k)$ stands for the Grassmann manifold of all k-dimensional subspaces in $\mathbb{R}^n$.

If the corresponding projections are translates of each other the answer to Problem 1is known to be affirmative [3, Theorems 3.1.3], (see also [1], [7]). Besides, for Problem 1, with $k = n-1$, Hadwiger established a more general result and showed that it is not necessary to consider projections onto all $(n-1)$-dimensional subspaces; the hypotheses need only be true for one fixed subspace H, together with all subspaces containing a line orthogonal to H. In other words, one requires only a "ground" projection on H and all corresponding "side" projections. Moreover, Hadwiger noted that in $\mathbb{R}^n$, $n \geq 4$, the ground projection might be dispensed with (see [5], and [3, pages 126–127]).

If the corresponding projections of convex bodies are rotations of each other, the results in the case $k = 2$ were obtained by the third author in [6].

In the general case of rigid motions, Problem 1 is open for any k and n. In the special case of *direct rigid motions*, *i.e.*, when the general orthogonal group $O(k)$ is replaced by the special orthogonal group $SO(k)$, the problem is open as well.

Golubyatnikov [4] obtained several interesting results related to the cases $k = 2, 3$ [4, Theorem 2.1.1, page 13; Theorem 3.2.1, page 48]. In particular, he gave an affirmative answer to Problem 1 in the case $k = 2$ if the projections of K and L are directly congruent and have no direct rigid motion symmetries.

If the bodies are symmetric, then the answer to Problems 1 is known to be affirmative and it is a consequence of the Aleksandrov Uniqueness Theorem about convex bodies, having equal volumes of projections (see [3, Theorem 3.3.1, page 111]).

In this talk we follow the ideas from [4] and [6] to obtain several Hadwiger-type results related to Problems 1 in the case $k = 3$. In order to formulate these results we introduce some notation and definitions.

Let $n \geq 4$ and let S^{n-1} be the unit sphere in $\mathbb{R}^n$. We will use the notation $w^\perp$ for the $(n-1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to $w \in S^{n-1}$. We denote by $d_K(\zeta)$ the diameter of a convex body K, which is parallel to the direction $\zeta \in S^{n-1}$. We will also denote by $\mathcal{O} = \mathcal{O}_\zeta \in O(n)$ the orthogonal transformation satisfying $\mathcal{O}|_{\zeta^\perp} = -I|_{\zeta^\perp}$, and $\mathcal{O}(\zeta) = \zeta$.

We define the notion of *rigid motion symmetry* for sets. Let D be a subset of $H \in \mathcal{G}(n, k)$, $3 \leq k \leq n-1$. We say that D has a rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-identical orthogonal transformation $\varphi \in O(k, H)$ in H. Similarly, D has a direct rigid motion symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some non-trivial rotation $\varphi \in SO(k, H)$. In the case when D is a subset of $H \in \mathcal{G}(n, 3)$, and $\xi \in (H \cap S^{n-1})$, we say that D has a $(\xi, \alpha\pi)$-symmetry if $\varphi(D) = D + a$ for some vector $a \in H$ and some rotation $\varphi \in SO(3, H)$ by the angle $\alpha\pi$, $\alpha \in (0, 2)$, satisfying $\varphi(\xi) = \xi$. If, in particular, the angle of rotation is π, we say that D has a (ξ, π)-symmetry.

1. Results about directly congruent projections

We start with the following 4-dimensional result.

Theorem 1. *Let K and L be two convex bodies in $\mathbb{R}^4$ having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$, such that the "side" projections $K|w^\perp$, $L|w^\perp$ onto all subspaces $w^\perp$ containing ζ are directly congruent, see Figure 1. Assume also that these projections have no (ζ, π)-symmetries and* *no (u,π)-symmetries for any $u \in (\zeta^\perp \cap w^\perp \cap S^3)$. Then $K = L + b$ or $K = \mathcal{O}L + b$ for some $b \in \mathbb{R}^4$.*

If, in addition, the "ground" projections $K|\zeta^\perp$, $L|\zeta^\perp$, are directly congruent and do not have rigid motion symmetries, then $K = L + b$ for some $b \in \mathbb{R}^4$.

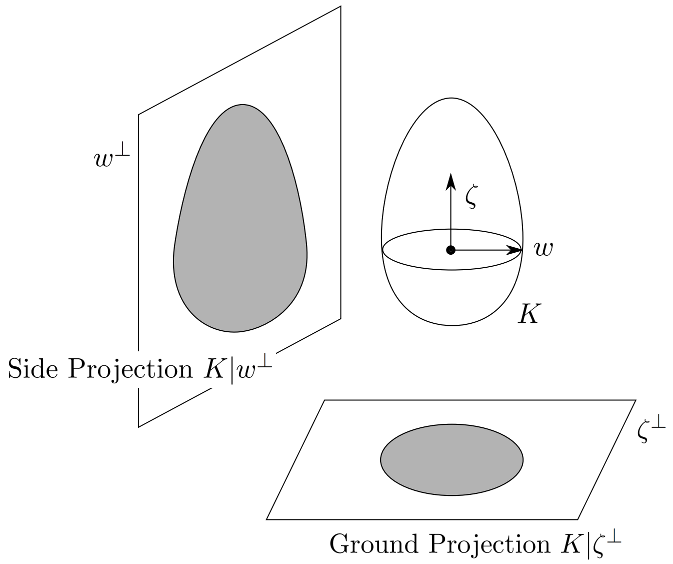

FIGURE 1. Diameter $d_K(\zeta)$, side projection $K|w^\perp$ and ground projection $K|\zeta^\perp$.

We state a straight n-dimensional generalization of Theorem 1 as a corollary.

Corollary 2. *Let K and L be two convex bodies in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters. Assume that there exists a diameter $d_K(\zeta)$ such that the "side" projections $K|H$, $L|H$ onto all 3-dimensional subspaces H containing ζ are directly congruent. Assume also that these projections have no (ζ,π)-symmetries and no (u,π)-symmetries for any $u \in (\zeta^\perp \cap H \cap S^{n-1})$. Then $K = L + b$ or $K = \mathcal{O}L + b$ for some $b \in \mathbb{R}^n$.*

If, in addition, the "ground" projections $K|G$, $L|G$ onto all 3-dimensional subspaces G of $\zeta^\perp$, are directly congruent and have no rigid motion symmetries, then $K = L + b$ for some $b \in \mathbb{R}^n$.

In particular, we see that if K and L are convex bodies in $\mathbb{R}^n$, $n \geq 4$, having countably many diameters, and directly congruent projections onto **all** 3-dimensional subspaces, and if the "side" and "ground" projections related to one of the diameters satisfy the conditions of the above corollary, then K and L are translates of each other.

This statement was proved by Golubyatnikov [4, Theorem 3.2.1, page 48] under the stronger assumptions that the "side" projections have no direct rigid motion symmetries. Theorem 1 and Corollary 2 under the same stronger assumptions are implicitly contained in his proof. To weaken the symmetry conditions on the "side" projections we replace the topological argument from [4] with an analytic one based on ideas from [6].

We note that the assumption about countability of the sets of the diameters of K and L can be weakened. Instead, one can assume, for example, that these sets are subsets of a countable union of the great circles containing ζ. We also note that the set of bodies considered in the above statements contains the set of all polytopes whose three dimensional projections do not have rigid motion symmetries. This set of polytopes is an everywhere dense set with respect to the Hausdorff metric in the class of all convex bodies in $\mathbb{R}^n$, $n \geq 4$.

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The role of the Rogers-Shephard inequality in the classification of the difference body

Eugenia Saorín Gómez

(joint work with J. Abardia)

Let $\mathcal{K}^n$ denote the set of convex bodies (compact and convex sets) in $\mathbb{R}^n$ and $h(K, v) = h_K(v)$ the support function of $K \in \mathcal{K}^n$ in the direction $v \in \mathbb{R}^n$. If $A \subset \mathbb{R}^n$ is measurable, we write $\mathrm{Vol}(A)$ for its volume (n-dimensional Lebesgue measure) and $\mathrm{GL}(n)$ and $\mathrm{SL}(n)$ to denote the general and special linear groups in $\mathbb{R}^n$.

The difference body DK of $K \in \mathcal{K}^n$ is the Minkowski sum of K and its reflection in the origin, i.e.,

$$DK := K + (-K). \tag{1}$$

The *Rogers-Shephard inequality* ([3]) constitutes the fundamental (affine) inequality relating the volume of the difference body DK and the volume of K. It is usually introduced together with a lower bound, which is a consequence of the Brunn-Minkowski inequality:

Let $K \in \mathcal{K}^n$. Then

$$2^n \operatorname{Vol}(K) \leq \operatorname{Vol}(DK) \leq \binom{2n}{n} \operatorname{Vol}(K). \tag{2}$$

As an operator on convex bodies

$$\begin{array}{rccc} D: & \mathcal{K}^n & \longrightarrow & \mathcal{K}^n \\ & K & \mapsto & DK, \end{array}$$

the difference body enjoys several properties. It is continuous in the Hausdorff metric, $\mathrm{SL}(n)$-covariant and homogeneous of degree 1. Further, $K \mapsto DK$ is a translation invariant Minkowski valuation.

An operator $\phi : \mathcal{K}^n \longrightarrow \mathcal{K}^n$ is said to be *G-covariant* for a group of transformations G if for any $K \in \mathcal{K}^n$ it holds

$$\phi(gK) = g\phi K \text{ for any } g \in G,$$

and it is *homogeneous of degree $k \in \mathbb{R}$* if for any $K \in \mathcal{K}^n$,

$$\phi(\lambda K) = \lambda^k \phi K \text{ for any } \lambda > 0.$$

The operator ϕ is a *Minkowski valuation* if for any $K, L \in \mathcal{K}^n$ with $K \cup L \in \mathcal{K}^n$,

$$\phi(K \cup L) + \phi(K \cap L) = \phi(K) + \phi(L),$$

where the addition on $\mathcal{K}^n$ is the vectorial addition. An operator ϕ is *translation invariant* if

$$\phi(K + t) = \phi(K) \text{ for any } t \in \mathbb{R}^n.$$

Indeed, in [2] M. Ludwig proved that already continuity, translation invariance, Minkowski valuation and $\mathrm{SL}(n)$-covariance are enough to determine the difference body operator.

Theorem C ([2])**.** *Let $n \geq 2$. An operator $\phi : \mathcal{K}^n \longrightarrow \mathcal{K}^n$ is continuous, translation invariant and $\mathrm{SL}(n)$-covariant Minkowksi valuation if and only if there is a $\lambda \geq 0$ such that $\phi K = \lambda DK$.*

If the image of the operator ϕ is restricted to origin symmetric convex bodies, a characterization in the same direction is provided by R. Gardner, D. Hug and W. Weil in [1]. Following their notation, let the subclass $\mathcal{K}_s^n$ of convex bodies symmetric with respect to the origin (for short, o-symmetric), an operator $\phi : \mathcal{K}^n \longrightarrow \mathcal{K}_s$ is called an *o-symmetrization.*

Theorem D ([1])**.** *Let $n \geq 2$. An operator $\phi : \mathcal{K}^n \rightarrow \mathcal{K}_s^n$ is a continuous, translation invariant and $\mathrm{GL}(n)$-covariant o-symmetrization if and only if there is a $\lambda \geq 0$ such that $\phi K = \lambda DK$.*

However, none of the above classifications makes use of the fundamental affine isoperimetric inequalities attached to it, namely, (2).

It is our goal to understand whether these two inequalities may play a role in classifying the difference body operator. To this aim we introduce the following two definitions.

We say that an operator $\phi : \mathcal{K}^n \to \mathcal{K}^n$ satisfies a *Rogers-Shephard type inequality* (in short RS) if there exists a constant $C > 0$ such that for all $K \in \mathcal{K}^n$,

$$\operatorname{Vol}(\phi K) \leq C \operatorname{Vol}(K). \tag{3}$$

Analogously, ϕ satisfies a *Brunn-Minkowski type inequality* (in short BM) if there exists a constant $c > 0$ such that for all $K \in \mathcal{K}^n$,

$$c \operatorname{Vol}(K) \leq \operatorname{Vol}(\phi K). \tag{4}$$

In general, if any of the assumptions in Theorems C and D is replaced by Rogers-Shephard inequality, there is, in general, no possibility of getting close to a characterization of the difference body. The following examples are intended to illustrate this issue.

Example 1. Let $L \in \mathcal{K}_s^n$ have dimension at most $n - 1$. Then, the operator

$$\begin{array}{cccc} \phi : & \mathcal{K}^n & \longrightarrow & \mathcal{K}^n \\ & K & \mapsto & L \end{array}$$

is a continuous, Minkowski valuation which is also an o-symmetrization and translation invariant. It satisfies RS but it is not $\mathrm{GL}(n)$-covariant. Further, it does not satisfy BM.

Example 2. Let $p \in \mathbb{R}^n$. The operator

$$\begin{array}{cccc} \phi_p : & \mathcal{K}^n & \longrightarrow & \mathcal{K}^n \\ & K & \mapsto & K - p. \end{array}$$

is a continuous Minkowski valuation, which clearly satisfies RS and BM. However, ϕ_p is neither an o-symmetrization, nor $\mathrm{GL}(n)$-covariant or translation invariant.

The last example shows also that the three conditions continuity, Minkowski valuation and RS together, neither characterize the difference body nor imply $\mathrm{GL}(n)$-covariance.

Example 3. Let $a(K)$ denote the center of gravity (centroid) of K. The operator

$$K \mapsto \operatorname{conv}\left((K - a(K)) \cup (-K + a(-K))\right)$$

satisfies BM and RS. Moreover, it is a $\mathrm{GL}(n)$-covariant o-symmetrization, but this is (because of $a(K)$) not continuous on $\mathcal{K}^n$.

Example 4. Let $L \in \mathcal{K}_s^n$ have dimension at most $n - 1$. Then, the operator

$$\phi K = \begin{cases} DK, & \text{if } \dim K = n \\ L, & \text{otherwise} \end{cases}$$

is an o-symmetrization, translation invariant and satisfies both RS and BM. It is however, not continuous.

If L is chosen to be the origin, then it is also $\mathrm{GL}(n)$-covariant, monotonic and one homogeneous.

Example 5. Let $\omega(K)$ denote the mean width of K (see e.g. [4, (1.30)]). Let us consider the operator

$$\begin{array}{rccc} \phi : & \mathcal{K}^n & \longrightarrow & \mathcal{K}^n \\ & K & \mapsto & B_{\omega(K)}, \end{array}$$

where $B_{\omega(K)}$ denotes the ball centered at the origin and of radius $\omega(K)$.

The operator ϕ is a continuous, o-symmetrization, translation invariant and Minkowski valuation which satisfies BM. It is also monotonic and homogeneous of degree 1. However, it neither satisfies RS nor is $\mathrm{GL}(n)$-covariant.

Example 6. Let $\phi K = \mathrm{Vol}(K)^{1/n} B_n$. It is a continuous o-symmetrization satisfying BM and RS. Further, it is translation invariant and homogeneous of degree one. It is clearly not a Minkowski valuation.

The following example is particularly important for us. It connects the actual extended abstract with the one by J. Abardia.

Example 7.

$$\phi K = L + \mathrm{Vol}(K)S$$

where S is a centered segment and L is an o-symmetric $(n-1)$-dimensional convex body so that $\dim(S+L) = n$. The operator ϕ is a continuous, translation invariant Minkowski valuation, and also an o-symmetrization which satisfies a Rogers-Shephard and a Brunn-Minkowski type inequality. Brunn-Minkowski and Rogers-Shephard inequality hold since

$$\mathrm{Vol}(L + \mathrm{Vol}(K)S) = \mathrm{Vol}(K)V(L[n-1], S).$$

In the talk by J. Abardia, it is proven that a continuous, translation invariant Minkowski valuation, which is an o-symmetrization and satisfies a Rogers-Shephard and a Brunn-Minkowski type inequality is either of the above type or one-homogeneous.

In view of the above examples, a characterization of the difference body using RS and BM, if possible, needs several assumptions.

Our main result in this direction is the following:

Theorem 8. *Let $n \geq 2$. Let $\phi : \mathcal{K}^n \to \mathcal{K}^n$ be a continuous, $\mathrm{GL}(n)$-covariant operator. Then, the following statements are equivalent*

(1) *ϕ satisfies a Rogers-Shephard type inequality*
(2) *$\dim \phi K = \dim K$ for every $K \in \mathcal{K}^n$*
(3) *there exists $K \in \mathcal{K}^n$, $\dim K \leq n-1$, $0 \notin \mathrm{aff}\, K$ with $\dim \phi K = \dim K$*
(4) *ϕ is additive*
(5) *there exist $a, b \geq 0$ such that $\phi K = aK + b(-K)$*

As a consequence of this we obtain the following corollary.

Corollary 9. *Let $n \geq 2$. An o-symmetrization $\phi : \mathcal{K}^n \to \mathcal{K}^n_s$ is continuous, $\mathrm{GL}(n)$-covariant and satisfies a Rogers-Shephard type inequality if and only if there exists $\lambda \geq 0$ such that $\phi K = \lambda DK$.*

The main tools to prove the above statements are the next two results. The first is a slight modification of a result by R. Gardner, D. Hug and W. Weil:

Theorem 10. [1, Lemma 7.4, Lemma 8.1 and Theorem 8.2] *Let $n \geq 2$. The operator $\phi : \mathcal{K}^n \to \mathcal{K}^n$ is continuous and* GL(n)*-covariant if and only if there is a planar convex body $M \subset \mathbb{R}^2$ such that*

$$h(\phi K, x) = h_M \left(h_K(x), h_{-K}(x)\right), \tag{5}$$

for all $K \in \mathcal{K}^n$ and all $x \in \mathbb{R}^n$. In this case we say that M is an associated planar convex body to ϕ.

The second one is the following proposition:

Proposition 11. *Let $n \geq 2$ and $\phi : \mathcal{K}^n \to \mathcal{K}^n$ be continuous and* GL(n)*-covariant with $M \in \mathcal{K}^2$ as associated convex body and let $\omega(K, u)$ denote the width of $K \in \mathcal{K}^n$ in the direction $u \in S^{n-1}$. Then,*

(1) $\phi K = \{0\}$ *for every $K \in \mathcal{K}^n$ iff* $h_M(1, 1) = 0$
(2) $\omega(K, u)h_M(1, 1) \leq \omega(\phi K, u), \quad \forall u \in S^{n-1}$
(3) *if $\phi \not\equiv 0$, then* $\operatorname{aff} K \subseteq \operatorname{aff} \phi K$
(4) *If $K, L \in \mathcal{K}^n$ satisfy $\omega(K, u) \leq \omega(L, u)$ for every $u \in S^{n-1}$, then* $\operatorname{Vol}(K) \leq 2^{-n}\binom{2n}{n} \operatorname{Vol}(L)$

The role of a Brunn-Minkowski type inequality in classifying the difference body happens not to be relevant accompanied of GL(n)-covariance and continuity, as we prove in the following result.

Theorem 12. *Let $n \geq 2$. If $\phi : \mathcal{K}^n \to \mathcal{K}^n$, $\phi \not\equiv 0$, is a continuous and* GL(n)*-covariant operator, then it satisfies a Brunn-Minkowski type inequality.*

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Iterations of the projection body operator and a remark on Petty's conjectured projection inequality

Christos Saroglou
(joint work with A. Zvavitch)

The projection body of a convex body K in $\mathbb{R}^d$ is defined as the body with support function

$$h_{\Pi K}(x) = |K|x^{\perp}|, \text{ for all } x \in \mathbb{S}^{d-1},$$

where $K|x^\perp$ denotes the orthogonal projection of K onto the subspace $x^\perp = \{y \in \mathbb{R}^d : \langle x, y\rangle = 0\}$. The direct application of Cauchy projection formula gives us

$$h_{\Pi K}(x) = \frac{1}{2}\int_{\mathbb{S}^{d-1}} |\langle x, y\rangle| dS_K(y),\ x \in \mathbb{S}^{d-1},$$

where S_K is the surface area measure of K, viewed as a measure on $\mathbb{S}^{d-1}$. When S_K is absolutely continuous (with respect to the Lebesgue measure on the sphere), its density f_K is called curvature function of K.

It is very interesting to study the iterations of projection body operator. It is trivial to see that the projection body of Euclidean ball B_2^d is again, up to a dilation, B_2^d, moreover the same property is true for a unit cube B_∞^d. Weil proved that if K is a polytope in $\mathbb{R}^d$, then $\Pi^2 K$ is homothetic to K if and only if K is a linear image of cartesian products of planar symmetric polygons or line segments. But no other description of fixed points of projection body operator is known as well as no much known about possible convergence of the sequence $\Pi^m K$. Clearly, Weil's result tell us that one cannot expect in general that $\Pi^m K \to B_2^d$, with respect to the Banach-Mazur distance. It seems more plausible, however, that $\Pi^m K \to B_2^d$, if K has absolutely continuous surface area measure and $d \geq 3$ (for $d = 2$, if K is symmetric, then $\Pi^2 K = 4K$). We prove the following:

Theorem 1. *Let $d \geq 3$. There exists an $\varepsilon_d > 0$ with the following property: For any convex body K in $\mathbb{R}^d$, with absolutely continuous surface area measure and the curvature function f_K satisfying $\|f_{TK} - 1\|_\infty < \varepsilon_d$, for some $T \in GL(d)$, we have $\Pi^m K \to B_2^d$, in the sense of the Banach-Mazur distance.*

The idea of the above theorem follows from the study of the properties of intersection body operator done by Fish, Nazarov, Ryabogin and Zvavitch. The authors proved that B_2^d is a local attractor:

Theorem E. *Let $d \geq 3$. There exists an $\varepsilon_d > 0$ with the following property: For any star body K in $\mathbb{R}^d$, which satisfies $\|\rho_{TK} - 1\|_\infty < \varepsilon_d$, for some $T \in GL(d)$ (in other words, K is close, in Banach-Mazur distance, to B_2^d), we have $I^m K \to B_2^d$, in the sense of the Banach-Mazur distance.*

Here, IK denotes the intersection body of K. The *intersection body* IK of a star body K was defined by Lutwak using the radial function of the body IK:

$$\rho_{IK}(u) = |K \cap u^\perp|, \quad \text{for } u \in \mathbb{S}^{d-1}.$$

Again it is trivial to see that IB_2^d is a dilate of B_2^d and $I^2 K = 4K$ for symmetric $K \subset \mathbb{R}^2$, but no much information is known about other fixed points of I

Another reason to consider Theorem 1 is that it can be applied to study of Petty's conjectured inequality. Indeed, it was shown by Petty that the quantity $P(K) := |\Pi K|/|K|^{d-1}$ is affine invariant. Petty also conjectured the following:

Conjecture 2. Let $d \geq 3$. The affine invariant $P(K)$ is minimal if and only if K is an ellipsoid.

The restriction $d \geq 3$ is because in the plane it is well known that $|\Pi K| \geq 4|K|$, with equality if and only if K is symmetric Petty's conjecture, if true, would be a very strong inequality, as it would imply a number of important isoperimetric inequalities, such as the classical isoperimetric inequality, the Petty projection inequality (a remarkable functional form of the latter was established by G. Zhang), and the affine isoperimetric inequality.

Very little seem to be known about the conjecture of Petty. For instance, as shown by Saroglou, Steiner symmetrization fails for this problem. A useful fact, established by Schneider, is that

$$P(K) \geq P(\Pi K), \tag{1}$$

with equality if and only if K is homothetic to $\Pi^2 K$ In particular, it follows that every solution to the Petty problem must be a zonoid (a body which is a limit of Minkowski sum of segments).

Although bodies with minimal surface area significantly larger than the surface area of the ball (of the same volume) are known to satisfy the Petty conjecture no natural class of convex bodies was known to satisfy the Petty conjecture (natural class means that is connected with respect to the Banach-Mazur distance and contains the ball). Below, we have a result towards this direction.

Theorem 3. *Let $d \geq 3$. There exists an $\varepsilon_d > 0$ with the following property: For any non-ellipsoidal convex body K in $\mathbb{R}^d$, which has absolutely continuous surface area measure and satisfies $\|f_{TK} - 1\|_\infty < \varepsilon_d$, for some $T \in GL(d)$, we have $P(K) > P(B_2^d)$.*

Denote by W_i the i-th quermassintegral functional in $\mathbb{R}^d$, $i = 0, 1, \ldots, d-1$, which is the mixed volume of $d-i$ copies of a convex body K with i copies of B_2^d Recall the Aleksandrov-Fenchel inequalities for quermassintegrals:

$$W_{i+1}^{d-i}(K) \geq \omega_d W_i^{d-i-1}(K), \qquad i = 0, \ldots, d-2, \tag{2}$$

where K is any convex body and $\omega_d = |B_2^d|$ proved that if Petty's conjecture was proven to be true, then a family of inequalities that are stronger than (2) would have been established. These conjectured inequalities involve the notion of the i-th projection body $\Pi_i K$ of K, whose support function is given by:

$$h_{\Pi_i K}(u) = W_{i|u^\perp}(K|u^\perp), \qquad i = 0, \ldots, d-2,$$

where $W_{i|u^\perp}$ stands for the i-th quermassintegral in $u^\perp$. Note that $\Pi K = \Pi_0 K$. Actually, Lutwak established a certain member of this family of inequalities:

$$W_{d-2}(\Pi_{d-2} K) \geq \omega_{d-1}^2 W_{d-2}(K), \tag{3}$$

where $d \geq 3$, with equality if and only if K is a ball. To see that (3) is stronger than (2) (in the sense that it interpolates (2)), for $i = d-1$, note that since $W_{d-1}(K)$ is proportional to the mean width of K, we get:

$$W_{d-1}(\Pi_{d-2} K) = \omega_{d-1} W_{d-2}(K), \tag{4}$$

Thus by (2) we obtain:

$$\frac{\omega_d}{\omega_{d-1}^2} W_{d-2}(\Pi_{d-2}K) \leq W_{d-1}^2(K),$$

with equality if and only if $\Pi_{d-2}K$ is a ball.

Theorem 3 allows us to prove a stronger version of Lutwak's inequality:

Theorem 4. *Let K be a convex body in $\mathbb{R}^d$, $d \geq 3$. Then,*

$$W_{d-2}(\Pi_{d-2}K) \geq \frac{d(d-2)\omega_{d-1}^2}{(d-1)^2\omega_d} W_{d-1}^2(K) + \frac{\omega_{d-1}^2}{(d-1)^2} W_{d-2}(K).$$

This inequality is sharp for the ball. Moreover, if K is not centrally symmetric, then the inequality is strict.

Projection functions, area measures and the Alesker–Fourier transform

Franz E. Schuster
(joint work with F. Dorrek)

The Busemann–Petty problem was one of the most famous problems in convex geometric analysis of the last century. It asks whether the volume of an origin-symmetric convex body K in $\mathbb{R}^n$ is smaller than that of another such body L, if all central hyperplane sections of K have smaller volume than those of L. After a long list of contributions it was eventually shown that the answer is affirmative if $n \leq 4$ and negative otherwise (see [6, 7, 20] and the references therein). A crucial step in the final solution was taken by Lutwak [12] who showed that the answer to the Busemann–Petty problem is affirmative if and only if every origin-symmetric convex body in $\mathbb{R}^n$ is an intersection body. This class of bodies first appeared in Busemann's definition of area in Minkowski geometry and has attracted considerable attention within different subjects since the seminal paper by Lutwak.

Since its final solution, various generalizations of the original Busemann–Petty problem have been investigated. Each of these variants is related to a certain generalization of the notion of intersection body in a similar way that Lutwak's intersection bodies are related to the Busemann–Petty problem. Of particular interest is the following notion of j-intersection bodies introduced by Koldobsky.

Definition ([8]) *Let $1 \leq j \leq n-1$ and let D and M be origin-symmetric star bodies in $\mathbb{R}^n$. Then D is called the* j-intersection body *of M if*

$$\mathrm{vol}_j(D \cap E^{\perp}) = \mathrm{vol}_{n-j}(M \cap E)$$

for every $n-j$ dimensional subspace E of $\mathbb{R}^n$.

Note that if $j = 1$, then j-intersection bodies coincide with Lutwak's intersection bodies. The class of j-intersection bodies was investigated by several authors, in particular, in connection with the lower dimensional Busemann–Petty problem (see, e.g., [8, 15, 16, 19]). The fundamental result about j-intersection bodies is the following Fourier analytic characterization by Koldobsky.

Theorem 1 ([9]). *Let $1 \leq j \leq n-1$ and let D and M be origin-symmetric star bodies in $\mathbb{R}^n$. Then D is the j-intersection body of M if and only if*

$$\mathbf{F}_{-j}\, \rho(D,\cdot)^j = \frac{(2\pi)^{n-j} j}{n-j}\, \rho(M,\cdot)^{n-j}. \tag{1}$$

Here and in the following, $\mathbf{F}_{-j} g$ denotes the restriction to S^{n-1} of the usual Fourier transform in $\mathbb{R}^n$ of the $-j$-homogeneous extension of $g \in L^2(S^{n-1})$. We refer to [10] for more information.

As a natural dual to Koldobsky's notion of j-intersection bodies, we introduce the class of j-projection bodies.

Definition *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is called the* j-projection body *of L if*

$$\mathrm{vol}_j(K|E^{\perp}) = \mathrm{vol}_{n-j}(L|E)$$

for every $n-j$ dimensional subspace E of $\mathbb{R}^n$.

Note that if $j = 1$, then j-projection bodies coincide with the classical projection bodies of Minkowski. Examples of j-projection bodies of intermediary degree were given by McMullen [13, 14], Schnell [18], and Schneider [17]. However, apart from a few examples very little seems to be known about this class of convex bodies. Our goal was therefore to start a more systematic investigation and, in particular, to prove the following analogue of Koldobsky's characterization of j-intersection bodies, Theorem 1.

Theorem 2 ([5]). *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is the j-projection body of L if and only if*

$$\mathbf{F}_{-j}\, S_j(K,\cdot) = \frac{(2\pi)^{n-j} j}{n-j}\, S_{n-j}(L,\cdot). \tag{2}$$

Here $S_j(K,\cdot)$ and $S_{n-j}(L,\cdot)$ are the area measures of order j and $n-j$ of the convex bodies K and L, respectively.

Theorem 2 is a generalization of a well known Fourier analytic characterization of Minkowski's projection bodies (see, e.g., [11]). One way to prove this result exploits a connection to the theory of valuations. Recall that a map $\phi : \mathcal{K}^n \to \mathbb{R}$ is called a *valuation* if

$$\phi(K) + \phi(L) = \phi(K \cup L) + \phi(K \cap L)$$

whenever $K \cup L$ is convex. Let $\mathbf{Val}^{\infty}_{(j)}$, denote the space of smooth translation invariant valuations (of degree $j \in \{0, \dots, n\}$). As part of their modern reconceptualization of integral geometry, Alesker [1] and Bernig and Fu [3] discovered natural product and convolution structures on the space $\mathbf{Val}^{\infty}$. More recently, Alesker [2] showed that there exists a Fourier type transform $\mathbb{F} : \mathbf{Val}^{\infty}_j \to \mathbf{Val}^{\infty}_{n-j}$ which relates these structures in the same way the usual product and convolution of functions on $\mathbb{R}^n$ are related by the classical Fourier transform. The following result connects the Alesker-Fourier transform on even *spherical* valuations with the class of j-projection bodies.

Theorem 3 ([5]). *Let $1 \leq j \leq n-1$ and let K and L be origin-symmetric convex bodies with non-empty interior in $\mathbb{R}^n$. Then K is the j-projection body of L if and only if*

$$\phi(K) = (\mathbb{F}\phi)(L)$$

for all even $\phi \in \mathbf{Val}_j^{\infty,\mathrm{sph}}$.

Here $\mathbf{Val}_j^{\infty,\mathrm{sph}}$ denotes the subspace of smooth spherical valuations. These valuations correspond to *spherical representations* of the group $\mathrm{SO}(n)$ with respect to $\mathrm{SO}(n-1)$ (see, e.g., [4]).

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On the thin-shell conjecture for the Schatten classes

Beatrice-Helen Vritsiou
(joint work with J. Radke)

We study whether the thin-shell conjecture holds true for the unit balls of the Schatten classes. Let S_p^n denote the space of $n \times n$ real or complex matrices endowed with the norm that sends each matrix T to the ℓ_p norm of its singular-values-vector, or in other words to the norm

$$\|s(T)\|_p := \left(\sum_{i=1}^{n} |s_i(T)|^p \right)^{1/p}$$

of the vector $s(T) = (s_1(T), \ldots, s_n(T))$ of the eigenvalues of $\sqrt{T^*T}$ (ordered in an non-increasing way). The unit balls K_p of S_p^n, $p \in [1, \infty]$, have been studied in the past with respect to other important conjectures or questions in Convex Geometry as well: we build on some of the techniques appearing in [5], where König, Meyer and Pajor showed that the Schatten classes satisfy the hyperplane conjecture, or in [4], where Guédon and Paouris proved Paouris' theorem on concentration of volume for the Schatten classes (this was before Paouris' theorem (see [6]) was established for all isotropic convex bodies, as are the balls K_p, through more general methods).

One of the key ideas that we as well employ is to reduce estimates about moments of the Euclidean norm with respect to the uniform measure on the balls K_p, which are convex bodies of an n^2 or a $2n^2$-dimensional real space, to estimates about moments of the Euclidean norm with respect to a density f_p on $\mathbb{R}^n$ now; the new density is no longer a uniform, or even a log-concave, density, but it is invariant under permutations of the coordinates of vectors in $\mathbb{R}^n$. We then take advantage of these symmetry properties of the new density f_p to get very precise recursive identities that involve the variance of the Euclidean norm, which is what, in the case of the thin-shell conjecture, we need to bound. In a little more detail, we obtain identities that involve the quantities

$$\int_{\mathbb{R}^n} \|x\|_2^4 \cdot f_p - \left(\int_{\mathbb{R}^n} \|x\|_2^2 \cdot f_p \right)^2 =$$

(1)

$$n \cdot \left[\int_{\mathbb{R}^n} x_1^4 \cdot f_p - \left(\int_{\mathbb{R}^n} x_1^2 \cdot f_p \right)^2 \right] + n(n-1) \cdot \left[\int_{\mathbb{R}^n} x_1^2 x_2^2 \cdot f_p - \left(\int_{\mathbb{R}^n} x_1^2 \cdot f_p \right)^2 \right],$$

where the equality here follows from the aforementioned symmetry properties of the density f_p.

We are able, through these identities, to get tight estimates for the above quantities when p is really large, that is, when p is at least as large as the dimension of the balls K_p. This allows us to establish the thin-shell conjecture for the Schatten classes S_p^n when $p \gtrsim n^2 \log n$, and in particular for the case of the operator (or spectral) norm ($p = \infty$) in all dimensions.

Our second main result is the following: given any $p \geqslant 1$, if the thin-shell conjecture is true for K_p, then we must have some rather strong negative correlation property; namely, the cross term in (1) must be negative to counteract the first term in (1). Recall that, in the case of the ℓ_p balls, the thin-shell conjecture follows immediately from a similar negative correlation property which Ball and Perissinaki established in [1]; in that case however, all one needs to know is that the cross terms are non-positive. On the contrary, in the case of the Schatten classes the cross terms have to be negative (and sufficiently large in absolute value) if we want to conclude that the conjecture is true. An immediate consequence of this is that this negative correlation property holds for all p for which we have already verified the conjecture, and, in particular, for the operator norm.

It would be of course very interesting to see what happens for the Schatten classes corresponding to the remaining p: the estimates that we have get progressively worse as p gets smaller; still, up to $n \log n$ say, they continue to give something better than the best thus far known bound for the Schatten classes with respect to the thin-shell conjecture (this bound is due to Barhe and Cordero-Erausquin [2], and is slightly better than the best bound known for all isotropic convex bodies (see [3])). One more consequence of these estimates is that the abovementioned negative correlation property remains true for the Schatten classes corresponding to these smaller p as well. Unfortunately, just showing that the cross term in (1) is non-positive (or even that it is negative, but without getting precise estimates on its magnitude) for the remaining Schatten classes S_p^n would yield no better bound for the variance than the one in [2]. This suggests that the thin-shell conjecture for the remaining Schatten classes might be a strictly more difficult problem than establishing (even rather strong) negative correlation properties, and perhaps to deal with it one needs the introduction of quite different methods as well; that said, both questions seem intriguing in their own right, and it would be perhaps useful to even try to treat the latter one independently of the former one.

Finally, it seems worthwhile to explore whether the methods mentioned here can also be employed in the study of other problems from Convex Geometry or Probability that concern the Schatten classes.

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The floating body in real space forms

Elisabeth M. Werner

(joint work with F. Besau)

1. Introduction

Two important and closely tied notions in affine convex geometry are the floating body and the affine surface area of a convex body. The affine surface area was introduced by Blaschke in 1923 [6] and is now omnipresent in geometry. Much of this can be attributed to the important and strong properties of the affine surface area which make it an effective and powerful tool. For instance, it is at the very core of the rapidly developing L_p and Orlicz Brunn–Minkowski theory (see e.g. [10, 11, 13, 26, 27, 30, 36, 42, 43]). A characterization of affine surface area was achieved by Ludwig and Reitzner [24]. It had a profound impact on valuation theory of convex bodies starting a strong line of research (see e.g. [17, 22, 23, 25, 32, 35]) leading up to the very recent characterization of all centro-affine valuations by Haberl and Parapatits [18]. There is a natural inequality associated with affine surface are, the affine isoperimetric inequality, which states that among all convex bodies with fixed volume, affine surface area is maximized for ellipsoids. This inequality has led to a rich theory (see e.g. [2, 3, 18, 19, 26, 27, 29, 44, 45]). There are numerous other applications for affine surface area. We only mention a few cornerstones, such as, the approximation theory of convex bodies by polytopes [7, 8, 14, 15, 21, 33, 37, 39], affine curvature flows [1, 20, 40], information theory [2, 9, 31] and partial differential equations [28].

We introduce the floating bodies for spaces of constant curvature. Those admit a natural and intrinsic definition for floating bodies similar to Euclidean space. Our considerations lead to a seminal new surface area measure for convex bodies, which we call the floating area. This floating area is intrinsic to the constant curvature space and not only coincides with affine surface area in the flat case, but also has similar properties in the general case. Namely, the floating area is a valuation and upper semi-continuous. However, the group of transformations that leave it invariant is inherited from the space of constant curvature, in which it is intrinsic.

We draw a complete picture of this new notion of floating bodies and the floating area related to them in constant curvature spaces. An emphasis will be put on hyperbolic space. For all proofs and more details we refer to [5].

2. The μ-Floating Body

We introduce the μ-floating body, or weighted floating body, which serves as a unifying framework for dealing with Euclidean, spherical and hyperbolic floating bodies. In the following we recall facts from Euclidean convex geometry. For general references we refer to [12, 16, 34].

A convex body is a compact convex subset and the set of convex bodies is denoted by $\mathcal{K}(\mathbb{R}^n)$. The subset of convex bodies with non-empty interior is $\mathcal{K}_0(\mathbb{R}^n)$. We denote the Euclidean volume by vol_n and integration is simply denoted by dx. If a σ-finite Borel measure μ is absolutely continuous to another σ-finite Borel measure ν on an open set $D \subseteq \mathbb{R}^n$, then this is denoted by $\mu \ll_D \nu$ and μ is equivalent to ν on D, $\mu \sim_D \nu$, if and only if $\mu \ll_D \nu$ and $\nu \ll_D \mu$. Evidently, by the Radon–Nikodym Theorem, for a σ-finite Borel measure μ we have that $\mu \sim_D \mathrm{vol}_n$ if and only if there is Borel function $f_\mu \colon D \to \mathbb{R}$ such that $d\mu(x) = f_\mu(x)dx$ and $\mathrm{vol}_n(\{f_\mu = 0\}) = 0$. For a convex body $K \in \mathcal{K}_0(\mathbb{R}^n)$ we consider σ-finite measures μ such that $\mu \sim_{\mathrm{int}\, K} \mathrm{vol}_n$, where $\mathrm{int}\, K$ denotes the interior of K. Thus without loss of generality we may assume μ to be a σ-finite Borel measure on $\mathbb{R}^n$ with support K and for any measurable set A we have

$$\mu(A) = \int_{A \cap \mathrm{int}\, K} f_\mu(x)\, dx.$$

We denote the set of σ-finite measures μ on $\mathbb{R}^n$ with support K and which are equivalent to vol_n on $\mathrm{int}\, K$ by $\mathcal{M}(K, \mathrm{vol}_n)$. The subset of measures that are non-negative, that is, positive almost everywhere on $\mathrm{int}\, K$, is denoted by $\mathcal{M}_+(K, \mathrm{vol}_n)$.

Definition 1 (μ-Floating Body)**.** Let $K \in \mathcal{K}_0(\mathbb{R}^n)$ and let $\mu \in \mathcal{M}_+(K, \mathrm{vol}_n)$. For $\delta > 0$, we define the *μ-floating body* $\mathcal{F}^\mu_\delta\, K$, by

$$\mathcal{F}^\mu_\delta\, K = \bigcap \{H^- : \mu(H^+ \cap K) \leq \delta^{\frac{n+1}{2}}\},$$

where H^+ is an arbitrary closed half-space of $\mathbb{R}^n$ and H^- denotes the complementary *closed* half-space.

The μ-Floating Body is very similar to the notion of *weighted floating bodies* which was introduced by the second author in [41].

It is shown in [5] that the μ-floating body exists (i.e. is non-empty) if δ is small enough and, since it is an intersection of closed half-spaces, it is a convex body contained in K.

3. Statement of Principal Results

For $\lambda \in \mathbb{R}$ we denote the simply connected complete real space form with constant sectional curvature λ by $\mathrm{Sp}^n(\lambda)$. These include the special cases of the sphere $\mathbb{S}^n = \mathrm{Sp}^n(1)$, hyperbolic space $\mathbb{H}^n = \mathrm{Sp}^n(-1)$ and Euclidean space $\mathbb{R}^n = \mathrm{Sp}^n(0)$. The set of convex bodies in a space form is denoted by $\mathcal{K}_0(\mathrm{Sp}^n(\lambda))$, $\mathcal{K}_0(\mathbb{S}^n)$ or $\mathcal{K}_0(\mathbb{H}^n)$.

A totally geodesic hypersurface H in a real space form $\mathrm{Sp}^n(\lambda)$ is isometric to $\mathrm{Sp}^{n-1}(\lambda)$ and splits the space into two open and connected parts which are called half-spaces. We denote the closed half-spaces corresponding to a totally geodesic hypersurface H by H^+ and H^-. The standard Riemannian volume measure on $\mathrm{Sp}^n(\lambda)$ is vol_n^λ, i.e., we have

$$d\mathrm{vol}_n^\lambda(x) = \left(1+\lambda\|x\|^2\right)^{-\frac{n+1}{2}}\,dx.$$

According to Definition 1, we then define the λ-Floating Body with $\mu = \mathrm{vol}_n^\lambda(x)$ as follows.

Definition 2 (λ-Floating Body)**.** Let $\lambda \in \mathbb{R}$ and $K \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$. For $\delta > 0$ the *λ-floating body* $\mathcal{F}_\delta^\lambda\, K$ is defined by

$$\mathcal{F}_\delta^\lambda\, K = \bigcap\left\{H^- : \mathrm{vol}_n^\lambda\left(K\cap H^+\right) \le \delta^{\frac{n+1}{2}}\right\}.$$

The particular cases $\lambda = 1$ and $\lambda = -1$ give the Spherical Floating Body $\mathcal{F}_\delta^s\, K$ [4] and the Hyperbolic Floating Body $\mathcal{F}_\delta^h\, K$ [5].

In the main theorem we prove the following:

Theorem 3. *Let $K \in \mathcal{K}_0(\mathrm{Sp}^n(\lambda))$. Then the right-derivative of $\mathrm{vol}_n^\lambda(\mathcal{F}_\delta^\lambda\, K)$ at $\delta = 0$ exists. That is,*

$$\lim_{\delta\to 0^+} \frac{\mathrm{vol}_n^\lambda(K) - \mathrm{vol}_n^\lambda(\mathcal{F}_\delta^\lambda\, K)}{\delta} = c_n \Omega^\lambda(K),$$

where $c_n = \frac{1}{2}\left(\frac{n+1}{\kappa_{n-1}}\right)^{\frac{2}{n+1}}$. $\Omega^\lambda(K)$ is called the λ-floating area *of K and we have*

$$\Omega^\lambda(K) = \int_{bd\,K} H_{n-1}^\lambda(K,x)^{\frac{1}{n+1}}\, d\mathrm{vol}_{\mathrm{bd}\,K}^\lambda(x).$$

Here we consider $\mathrm{bd}\,K$ as a immersed submanifold of $\mathrm{Sp}^n(\lambda)$ and denote by $d\mathrm{vol}_{\mathrm{bd}\,K}^\lambda$ the intrinsic volume form and by $H_{n-1}^\lambda(K,x)$ the intrinsic generalized Gauss-Kronecker curvature on the boundary inherited by $\mathrm{Sp}^n(\lambda)$. κ_{n-1} is the $(n-1)$- dimensional volume of the $(n-1)$- dimensional Euclidean unit ball.

For $\lambda = 0$, i.e. for the Euclidean Space, Theorem 3 was first established in this complete form by E. Werner and C. Schütt in [38]. For $\lambda = 1$ the theorem was proved only very recently by the authors in [4].

We prove the complete form for all $\lambda \in \mathbb{R}$ by a new unifying approach which consists in considering geodesic Euclidean models for the space forms $\mathrm{Sp}^n(\lambda)$.

Corollary 4. *Given a convex body $K \in \mathcal{K}_0(\mathbb{B}^n)$ we have that*

$$\lim_{\delta\to 0^+} \frac{\mathrm{vol}_n^h\left(K \setminus \mathcal{F}_\delta^h K\right)}{\delta} = c_n \int_{bd\,K} H_{n-1}^h(K,x)^{\frac{1}{n+1}}\, d\mathrm{vol}_{bd\,K}^h(x)$$

Theorem 3 leads us to introduce the Floating Area or Equi-affine surface area.

Definition 5 (λ-Floating Area)**.** For a convex body K in a space form of sectional curvature $\lambda \in \mathbb{R}$ we define the λ-floating area

$$\Omega_\lambda(K) = \frac{1}{c_n} \lim_{\delta\to 0^+} \frac{\mathrm{vol}_n^\lambda\left(K \setminus \mathcal{F}_\delta^\lambda K\right)}{\delta}.$$

Some of the properties of the Floating Area are collected in the next proposition.

Proposition 6 (Properties of the Floating Area)**.** *Let K be a convex body in a space form of sectional curvature $\lambda \in \mathbb{R}$. Then we have*
(i) Ω_λ can be localized to a measure on bdK: For a Borel subset $B \subset bdK$,

$$\Omega_\lambda(B) = \int_B H_{n-1}^\lambda(K,x)^{\frac{1}{n+1}}\, d\mathrm{vol}_{bd\,K}^\lambda(x).$$

(ii) Ω_λ is upper semi-continuous with respect to the Hausdorff metric.
(iii) Ω_λ is a valuation, that is, if K, L and $K \cup L$ are convex bodies then

$$\Omega_\lambda(K) + \Omega_\lambda(L) = \Omega_\lambda(K \cup L) + \Omega_\lambda(K \cap L).$$

(iv) Ω_λ vanishes on polytopes.

Again, for all proofs and more details we refer to [5]

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A discrete version of Koldobsky's slicing inequality

Artem Zvavitch
(joint work with M. Alexander and M. Henk)

As usual, we will say that $K \subset \mathbb{R}^d$ is a convex body if K is a convex, compact subset of $\mathbb{R}^d$ equal to the closure of its interior. We say that K is origin-symmetric if $K = -K$, where $\lambda K = \{\lambda \boldsymbol{x} : \boldsymbol{x} \in K\}$, for $\lambda \in \mathbb{R}$. For a set K we denote by $\dim(K)$ its dimension, that is, the dimension of the affine hull of K. We will also denote by vol_d the d-dimensional Hausdorff measure, and if the body K is d-dimensional we will call $\mathrm{vol}_d(K)$ the volume of K. Finally, let us denote by $\boldsymbol{\xi}^\perp$ a hyperplane perpendicular to a unit vector $\boldsymbol{\xi}$, i.e.

$$\boldsymbol{\xi}^\perp = \{\boldsymbol{x} \in \mathbb{R}^d : \boldsymbol{x} \cdot \boldsymbol{\xi} = 0\}.$$

The slicing problem of Bourgain is, undoubtedly, one of the major open problems in convex geometry asking if a convex, origin-symmetric body of volume one must have a large (in volume) hyperplane section. More precisely, it asks whether there exists an absolute constant $\mathcal{L}_1$ so that for any origin-symmetric convex body K in $\mathbb{R}^d$

$$\mathrm{vol}_d(K)^{\frac{d-1}{d}} \le \mathcal{L}_1 \max_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \mathrm{vol}_{d-1}(K \cap \boldsymbol{\xi}^\perp). \tag{1}$$

The problem is still open, with the best-to-date estimate of $\mathcal{L}_1 \le O(d^{1/4})$ established by Klartag, who improved the previous estimate of Bourgain. Recently, Koldobsky proposed an interesting generalization of the slicing problem: Does there exists an absolute constant $\mathcal{L}_2$ so that for every even measure μ on $\mathbb{R}^d$, with a positive density, and for every origin-symmetric convex body K in $\mathbb{R}^d$ such that

$$\mu(K) \le \mathcal{L}_2 \max_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \mu(K \cap \boldsymbol{\xi}^\perp) \mathrm{vol}_d(K)^{\frac{1}{d}}? \tag{2}$$

Koldobsky was able to solve the above question for a number of special cases of the body K and provide a general estimate of $O(\sqrt{d})$. The most amazing fact here is that the constant $\mathcal{L}_2$ in (2) can be chosen independent of the measure μ under the assumption that μ has even positive density. In addition, Koldobsky and the speaker were able to prove that $\mathcal{L}_2$ is of order $O\left(d^{1/4}\right)$ if one assumes that the measure μ is s-concave. We note that the assumption of positive density is essential for the above results and (2) is simply not true if this condition is dropped. Indeed, to create a counterexample consider an even measure μ on $\mathbb{R}^2$ uniformly distributed over $2N$ points on the unit circle, then the constant $\mathcal{L}_2$ in (2) will depend on N.

During the 2013 AIM workshop on “Sections of convex bodies” Koldobsky asked if it is possible to provide a discrete analog of inequality (2): Let $\mathbb{Z}^d$ be the standard integer lattice in $\mathbb{R}^d$, define $\#K = \mathrm{card}(K \cap \mathbb{Z}^d)$, the number of points of $\mathbb{Z}^d$ in K.

Question: *Does there exist a constant* $\mathcal{L}_3$ *such that*

$$\#K \leq \mathcal{L}_3 \max_{\boldsymbol{\xi} \in \mathbb{S}^{d-1}} \left(\#(K \cap \boldsymbol{\xi}^\perp)\right) \mathrm{vol}_d(K)^{\frac{1}{d}},$$

for all convex origin-symmetric bodies $K \subset \mathbb{R}^d$ *containing* d *linearly independent lattice points?*

We note here that we require that K contains d linearly independent lattice points, i.e., $\dim(K \cap \mathbb{Z}^d) = d$, in order to eliminate the degenerate case of a body (for example, take a box $[-1/n, 1/n]^{d-1} \times [-20, 20]$) whose maximal section contains all lattice points in the body, but whose volume may be taken to 0 by eliminating a dimension.

Koldobsky’s question is yet another example of an attempt to translate questions and facts from classical Convexity to more general settings including Discrete Geometry. The properties of sections of convex bodies with respect to the integer lattice were extensively studied in Discrete Tomography by Gardner, Gritzmann, Gronchi, Zhong and many others. Many interesting new properties were proved and a series of exciting open questions were proposed. It is interesting to note that after translation many questions become quite non-trivial and counterintuitive, and the answer may be quite different from the continuous case. In addition, finding the relation between the geometry of a convex set and the number of integer points contained in the set is always a non-trivial task. One can see this, for example, from the history of Khinchin’s flatness theorem and facts around it.

The main goal of this talk is to present main step towards a solution of Koldobsky’s question. First we discuss a solution for the 2-dimensional case. The solution is based on the classical Minkowski’s First and Pick’s theorems from the Geometry of Numbers and gives a general idea of the approach to be used for high dimensional case. Next, we apply a discrete version of the theorem of F. John due to T. Tao and V. Vu to give a partial answer to Koldobsky’s question and show that the constant $\mathcal{L}_3$ can be chosen independent of the body K and as small as $O(d)^{7d/2}$. We also present a simple proof that in the case of unconditional bodies (i.e. bodies symmetric with respect to coordinate hyperplanes) $\mathcal{L}_3$ can be chosen of order $O(d)$ which is best possible, as can be seen from the example of a cross-polytope. Finally, we prove the discrete analog of Brunn’s theorem and use it to show that the constant $\mathcal{L}_3$, for the general case, can be chosen as small as $O(1)^d$. In fact, our work contains a slightly more general result that

$$\#K \leq O(1)^d d^{d-m} \max\left(\#(K \cap H)\right)\ \mathrm{vol}_d(K)^{\frac{d-m}{d}},$$

where the maximum is taken over all m-dimensional linear subspaces $H \subset \mathbb{R}^d$. We also provide a short observation that $\mathcal{L}_1 \leq \mathcal{L}_3$.

Reporter: Jesús Yepes Nicolás

Participants

Dr. Judit Abardia
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-10
60325 Frankfurt am Main
GERMANY

Prof. Dr. Semyon Alesker
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. David Alonso-Gutierrez
Departamento de Matematicas
Universidad de Zaragoza
Facultad de Ciencias
C/Pedro Cerbuna 12
50009 Zaragoza
SPAIN

Prof. Dr. Juan-Carlos Alvarez Paiva
Laboratoire Paul Painleve
UFR de Mathématiques
Université Lille I
59655 Villeneuve d'Ascq Cedex
FRANCE

Dr. Gergely Ambrus
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
13-15 Realtanoda u.
1053 Budapest
HUNGARY

Prof. Dr. Shiri Artstein-Avidan
School of Mathematical Sciences
Tel Aviv University
Ramat Aviv
Tel Aviv 69978
ISRAEL

Prof. Dr. Guillaume Aubrun
Institut Camille Jordan
Université Claude Bernard Lyon 1
43 blvd. du 11 novembre 1918
69622 Villeurbanne Cedex
FRANCE

Prof. Dr. Imre Barany
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. Franck Barthe
Institut de Mathématiques
Université Paul Sabatier
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Andreas Bernig
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-10
60325 Frankfurt am Main
GERMANY

Prof. Dr. Gabriele Bianchi
Dipto. di Matematica e Informatica
"U.Dini"
Università di Firenze
Viale Morgagni, 67/A
50134 Firenze
ITALY

Prof. Dr. Karoly Böröczky, Jr.
Alfred Renyi Institute of Mathematics
Hungarian Academy of Sciences
P.O.Box 127
1364 Budapest
HUNGARY

Prof. Dr. Pierre Calka
Laboratoire de Mathématiques
Raphael Salem, UMR-CNRS 6085
Université de Rouen
Avenue de l'Université, BP 12
76801 Saint-Etienne-du-Rouvray
FRANCE

Dr. Andrea Colesanti
Dipto. di Matematica e Informatica
"U.Dini"
Università di Firenze
Viale Morgagni 67/A
50134 Firenze
ITALY

Dr. Daniel Dadush
Centrum Wiskunde & Informatica
(CWI)
Postbus 94079
1090 GB Amsterdam
NETHERLANDS

Dr. Susanna Dann
Institut f. Diskr. Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10/104
1040 Wien
AUSTRIA

Dr. Ronen Eldan
Department of Mathematics
The Weizmann Institute of Science
P. O. Box 26
Rehovot 76100
ISRAEL

Dmitry Faifman
Department of Mathematics
University of Toronto
Bahen Centre, Rm. 6290
40 St. George Street
Toronto, Ontario M5S 2E4
CANADA

Dr. Matthieu Fradelizi
Laboratoire d'Analyse et Math. Appliq.
Université de Paris-Est Marne-la-Vallée
Cité Descartes, 5 Bd. Descartes
Champs-sur-Marne
77454 Marne-la-Vallée Cedex 2
FRANCE

Dr. Omer Friedland
Institut de Mathématiques
Projet Analyse Fonctionnelle
Boite 186, Université Paris 6
4, place Jussieu
75005 Paris Cedex
FRANCE

Prof. Dr. Richard J. Gardner
Department of Mathematics
Western Washington University
Bellingham, WA 98225-9063
UNITED STATES

Prof. Dr. Apostolos A. Giannopoulos
Department of Mathematics
University of Athens
Panepistemiopolis
157 84 Athens
GREECE

Julian Grote
Fakultät für Mathematik
Ruhr-Universität Bochum
44780 Bochum
GERMANY

Prof. Dr. Olivier Guédon
Laboratoire d'Analyse et Math. Appliq.
Université de Paris-Est Marne-la-Vallée
Cité Descartes, 5 Bd. Descartes
Champs-sur-Marne
77454 Marne-la-Vallée Cedex 2
FRANCE

Prof. Dr. Martin Henk
Institut für Mathematik
Technische Universität Berlin
Sekr. MA 4-1
Strasse des 17. Juni 136
10623 Berlin
GERMANY

Prof. Dr. Maria A. Hernandez Cifre
Departamento de Matematicas
Facultad de Matematicas
Universidad de Murcia
30100 Espinardo, Murcia
SPAIN

Prof. Dr. Daniel Hug
Institut für Stochastik
Karlsruher Institut f. Technologie (KIT)
Englerstrasse 2
76131 Karlsruhe
GERMANY

Prof. Dr. Alexander Koldobsky
Department of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia, MO 65211-4100
UNITED STATES

Dr. Joseph Lehec
Centre de Recherche de
Mathématiques de la Decision
CEREMADE
Université Paris IX
75016 Paris Cedex
FRANCE

Prof. Dr. Alexander Litvak
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1
CANADA

Galyna Livshyts
School of Mathematics
Georgia Institute of Technology
Skiles 228
686 Cherry Street
Atlanta, GA 30332-0160
UNITED STATES

Prof. Dr. Monika Ludwig
Institut f. Diskr. Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstraße 8-10/104
1040 Wien
AUSTRIA

Prof. Dr. Erwin Lutwak
Department of Mathematics
Polytechnic Institute of NYU
Six Metro Tech Center
Brooklyn, NY 11201
UNITED STATES

Prof. Dr. Mathieu Meyer
Equipe d'Analyse et de
Mathématiques Appliquées
Université de Paris Est Marne-la-Vallée
5, bd. Descartes
77454 Marne-la-Vallée Cedex 2
FRANCE

Prof. Dr. Vitali D. Milman
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. Alain Pajor
Equipe d'Analyse et de
Mathématiques Appliquées
Université de Paris Est Marne-la-Vallée
5, bd. Descartes
77454 Marne-la-Vallée Cedex 2
FRANCE

Prof. Dr. Grigoris Paouris
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Dr. Lukas Parapatits
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Peter Pivovarov
Department of Mathematics
University of Missouri-Columbia
202 Mathematical Science Bldg.
Columbia, MO 65211-4100
UNITED STATES

Prof. Dr. Luis Rademacher
School of Computer Science &
Engineering
Ohio State University
Dreese Labs 495
2015 Neil Avenue
Columbus, OH 43210
UNITED STATES

Liran Rotem
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Prof. Dr. Mark Rudelson
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Dmitry Ryabogin
Department of Mathematical Sciences
Kent State University
P.O. Box 5190
Kent OH 44242-0001
UNITED STATES

Dr. Eugenia Saorín Gómez
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg
GERMANY

Prof. Dr. Christos Saroglou
Department of Mathematics
Texas A & M University
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Rolf Schneider
Mathematisches Institut
Universität Freiburg
Eckerstrasse 1
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Franz Schuster
Institut für Diskr. Mathematik &
Geometrie
Technische Universität Wien
Wiedner Hauptstrasse 8 - 10
1040 Wien
AUSTRIA

Prof. Dr. Carsten Schütt
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
24098 Kiel
GERMANY

Prof. Dr. Alina Stancu
Department of Mathematics & Statistics
Concordia University
1455 De Maisonneuve Blvd. West
Montreal, QC H3G 1M8
CANADA

Dr. Beatrice-Helen Vritsiou
Department of Mathematics
University of Michigan
East Hall, 525 E. University
Ann Arbor, MI 48109-1109
UNITED STATES

Prof. Dr. Elisabeth Werner
Department of Mathematics
Case Western Reserve University
Cleveland OH 44106
UNITED STATES

Prof. Dr. Vladyslav Yaskin
Department of Mathematics and
Statistical Sciences
University of Alberta
632 Central Academic Building
Edmonton, Alberta T6G 2G1
CANADA

Dr. Jesús Yepes Nicolás
Instituto de Sciencias Matemáticas
(ICMAT)
Campus de Cantoblanco, UAM
C/Nicolás Cabrera, no. 13-15
28049 Madrid
SPAIN

Prof. Dr. Gaoyong Zhang
Department of Mathematics
Polytechnic Institute of NYU
Six Metro Tech Center
Brooklyn, NY 11201
UNITED STATES

Prof. Dr. Artem Zvavitch
Department of Mathematical Sciences
Kent State University
P.O. Box 5190
Kent OH 44242-0001
UNITED STATES