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Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstract. The general topic of the 2016 workshop *Heat kernels, stochastic processes and functional inequalities* was the study of linear and non-linear diffusions in geometric environments including smooth manifolds, fractals and graphs, metric spaces and in random environments. The workshop brought together leading researchers from analysis, geometry and probability, and provided an excellent opportunity for interactions between scientists from these areas at different stages of their career.

The unifying themes were heat kernel analysis, mass transportation problems and functional inequalities while the program straddled across a great variety of subjects and across the divide that exists between discrete and continuous mathematics. Other unifying concepts such as the notions of metric measure space, Otto Calculus and Lott-Sturm-Villani synthetic Ricci curvature bounds played an important part in the discussions. Novel directions including the study of Liouville quantum gravity were included. The workshop provided participants with an opportunity to discuss how these ideas and techniques can be used to approach problems regarding optimal transport, Riemannian and sub-Riemannian geometry, and analysis and stochastic processes in random media.

Mathematics Subject Classification (2010): 31, 60, 35, 58, 46, 58J65, 53C23, 60F17, 60J45, 35B27.

Introduction by the Organisers

The workshop *Heat kernels, stochastic processes and functional inequalities*, organized by Masha Gordina (University of Connecticut), Takashi Kumagai (RIMS, Kyoto University), Laurent Saloff-Coste (Cornell University), and Karl-Theodor Sturm (University of Bonn) was attended by over 50 participants from Austria, Belgium, Canada, China, France, Germany, Italy, Japan, Luxembourg, Poland, Portugal, Spain, Switzerland, United Kingdom, and USA. The program consisted of 27 talks and 5 short contributions, leaving sufficient time for informal discussions. The general topic of the workshop was the study of linear and non-linear diffusions in geometric environments: metric measure spaces, Riemannian and sub-Riemannian manifolds, fractals and graphs, and in random environments. The workshop brought together leading experts in three different major fields of mathematics: analysis, stochastics and geometry. It also provided a unique opportunity for interactions between established and young scientists from these different areas. One after-dinner session was devoted to short communications by junior participants of the workshop.

The list of the talks provided below illustrates the wide variety of the topics treated during the workshop. Even so no particular pressure was put on the speakers to stress connections across fields, such connections were overwhelmingly present, loud and clear. The questions during and following the talks demonstrated both the high interest of the problems and results that were presented from the point of view of the experts in the field and the curiosity of many participants for concepts and ideas that were unfamiliar to them.

The notions of metric measure spaces, curvature-dimension bounds and related problems regarding optimal transport and functional inequalities provided one area of focus and several novel developments and techniques were discussed including recent progress on time-dependent metric measure spaces (Eva Kopfer), new insights in monotonicity formulas a la Pereleman (Kazumaza Kuwada) and sharp functional inequalities via an innovative powerful 1-D localization method (Andrea Mondino). Optimal transport techniques were successfully extended to degenerate situations (Chen Li, Giuseppe Savaré) as well as to genuine probabilistic problems like matching problems (Luigi Ambrosio) or the Skorokhod embedding problem (Martin Huesmann).

Graphs and metric graphs present a particular challenge as some of the curvature techniques are not easily applicable there. Still, recent efforts show one can prove a weak form of the Bakry-Émery estimate for some metric graphs (Fabrice Baudoin), or following the work of S.-T. Yau and co-authors, a version of the Li-Yau gradient estimates for the heat kernel on graphs (Moritz Kassmann). The Gromov-Hausdorff-vague topology for graph-like metric spaces was used to prove an invariance principle for variable speed random walks on trees (Anita Winter). Embedding of Cayley graphs in Hilbert spaces based on spectral profile was used to provide a sharp sufficient condition for the Liouville property (Tianyi Zheng) and variational methods was used to prove a limit shape theorem for certain domino tilings (Georg Menz).

Dirichlet forms and heat kernel estimates play key roles in a number of topics discussed during the workshop. In particular, they provide a useful tool to prove various functional inequalities in absence of an immediately available well-defined geometry. Analysis on fractals, non-local operators, and random media are some examples of applications. Talks on such topics included: non-local Dirichlet forms (Zhen-Qing Chen), analysis on fractals (Naotaka Kajino), analysis on metric measure spaces (Mathav Murugan), as well as two talks by junior participants (Melchior Wirth and Alberto Chiarini). In particular, some recent progresses were presented on stability of Harnack inequalities under quasi-isometries for local and non-local Dirichlet forms on various metric measure spaces.

Random media and random environment provided another major area of focus for the workshop. The flexibility of functional inequality techniques was demonstrated by new developments regarding homogenization theory (Antoine Gloria, Felix Otto), random conductance models (Jean–Dominique Deuschel, Pierre Mathieu, Tuan Anh Nguyen), and other discrete random models (Nathanael Berestycki, Perla Sousi, Alain-Sol Sznitman).

A new direction that has seen significant progress recently is the Liouville quantum gravity. While some participants (Sebastian Andres, Nathanael Berestycki, Christophe Garban and Naotaka Kajino) have constructed the Liouville Brownian motion and/or provided detailed heat kernel estimates for the process previously, the workshop included a review talk on the Liouville quantum gravity (Christophe Garban), which gave the participants an opportunity to see the larger picture. This includes concepts from random planar geometry, Gaussian free fields and more generally certain probability measures on Riemannian surfaces, sometimes referred to as the two-dimensional quantum gravity. In particular, the Liouville quantum gravity measure can be viewed as a probabilistic formulation of the Knizhnik, Polyakov, Zamolodchikov relation in conformal field theory (Bertand Duplantier).

Several talks were centered on stochastic differential geometry when the underlying manifold is equipped with a Riemannian or sub-Riemannian metric. Some of these results involved geometric methods used to study the fundamental solution to the parabolic problem for weighted Schrödinger operators (Xue-Mei Li), to describe small-time heat kernel asymptotics at cut points on sub-Riemannian manifolds (Robert Neel), to construct a reflected Brownian motion in a Riemannian manifold with boundary (Marc Arnaudon), or used probabilistic methods to study sub-Riemannian manifolds via Hamiltonian random walks (Thomas Laetsch). One of the talks described the Dirac operator on a compact globally hyperbolic Lorentzian spacetime with a spacelike Cauchy boundary, and gave an index formula for this operator connecting it to the Atiyah-Patodi-Singer index formula for Riemannian manifolds with boundary (Christian Bär).

This diversity of topics and mix of participants stimulated many extensive and fruitful discussions. It also helped initiate new collaborations, in particular for the junior researchers, and strengthen existing ties between researchers in different fields of mathematics.

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Workshop: Heat Kernels, Stochastic Processes and Functional Inequalities

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Abstracts

Stability of elliptic Harnack inequality

Mathav Murugan
(joint work with Martin T. Barlow)

A well known theorem of Moser [4] is that an elliptic Harnack inequality (EHI) holds for solutions associated with uniformly elliptic divergence form PDE. Let $\mathcal{A}$ be given by

$$\mathcal{A}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\Big(a_{ij}(x)\frac{\partial f}{\partial x_j}\Big),$$

where $(a_{ij}(x), x \in \mathbb{R}^d)$ is bounded, measurable and uniformly elliptic. Let h be a non-negative $\mathcal{A}$-harmonic function in a domain $B(x, 2R)$, and let $B = B(x, R) \subset B(x, 2R)$. Moser's theorem states that there exists a constant C_H, depending only on d and the ellipticity constant of $a_{..}(\cdot)$, such that

$$\text{esssup}_{B(x,R)}\, h \le C_H \,\text{essinf}_{B(x,R)}\, h. \tag{1}$$

A few years later Moser [5] extended this to obtain a parabolic Harnack inequality (PHI) for solutions $u = u(t, x)$ to the heat equation associated with $\mathcal{A}$:

$$\frac{\partial u}{\partial t} = \mathcal{A}u. \tag{2}$$

This states that if u is a non-negative solution to (2) in a space-time cylinder $Q = (0, T) \times B(x, 2R)$, where $R = T^2$, then writing $Q_- = (T/4, T/2) \times B(x, R)$, $Q_+ = (3T/4, T) \times B(x, R)$,

$$\text{esssup}_{Q_-}\, u \le C_P \,\text{essinf}_{Q_+}\, u. \tag{3}$$

If h is harmonic then $u(t, x) = h(x)$ is a solution to (2), so the PHI implies the EHI.

A major advance in understanding the PHI was made in 1992 by Grigoryan and Saloff-Coste [2, 6], who proved that the PHI is equivalent to two conditions: volume doubling (VD) and a family of Poincaré inequalities (PI). The context of [2, 6] is the Laplace-Beltrami operator on Riemannian manifolds, but the basic equivalence VD+PI $\Leftrightarrow$ PHI also holds for graphs and metric measure spaces with a Dirichlet form. This characterisation of the PHI implies that it is stable with respect to rough isometries. One consequence of the EHI is the Liouville property – that all bounded harmonic functions are constant. However, the Liouville property is not stable under rough isometries – see [3].

These papers left open the following questions:

(1) Is EHI stable under perturbations?
(2) If so, find a characterization of EHI by properties that are easily seen to be stable under perturbations.

Our main results in [1] answers these questions.

Theorem 1. [1] *Under mild regularity hypothesis on the underlying metric measure Dirichlet space, the elliptic Harnack inequality is stable under bounded perturbations of the Dirichlet form and under rough isometries.*

Our proof of the above result also gives a characterization of EHI by properties that are stable under perturbations.

A main difficulty in proving the above result stems from the following fact. Every 'robust' method to prove EHI relies on the volume doubling property in an essential way. However there are several examples of spaces satisfying EHI that do not satisfy volume doubling.

Our starting point to overcome this difficultly is the following simple observation. Time change of the process (or equivalently change of the reference measure of the Dirichlet form) does not affect the sheaf of harmonic functions. We show that any space satifying EHI admits a doubling measure satisfying Poincaré and Sobolev type inequalities with respect to the corresponding time-changed Dirichlet form. The existence of doubling measure along with suitable function inequalities provide a robust characterization of EHI.

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The geometry of multi-marginal Skorokhod embedding

Martin Huesmann
(joint work with Mathias Beiglboeck, Alexander Cox)

This talk is based on [2].

Given $\mu_1, \dots, \mu_n \in \mathcal{P}(\mathbb{R})$ centered and increasing in convex order the *multi-marginal Skorokhod embedding problem* is to find an increasing sequence $\tau_1 \leq \dots \leq \tau_n$ of stopping times of Brownian motion B minimising

$$\text{(MSEP)} \qquad \mathbb{E}[\gamma((B_s)_{s\leq\tau_n}, \tau_1, \dots, \tau_n)]$$

among all stopping times satisfying $B_{\tau_i} \sim \mu_i$ and $B_{\cdot\wedge\tau_n}$ is uniformly integrable. Here, γ is some functional depending on the path of the Brownian motion up to time τ_n as well as the n stopping times $\tau_1, \dots, \tau_n$. Typical examples are $h(\tau_n)$ for some convex/concave function h, the runnning maximum $\max_{s\leq\tau_n} B_s$, or some functional of local time $\phi(L_{\tau_n})$. The uniform integrability condition ensures some minimality of the solutions.

The case $n = 1$ is the well known classical Skorokhod embedding problem with 20+ solutions by various authors, e.g. Azèma, Yor, Root, Rost, Perkins, Hobson,.... We refer to [8] for a nice survey of the existing solutions (up to 2004). The one-marginal solutions found various applications in probability, cf. [8]. In fact, many of these applications have natural multi-marginal counterparts which caused increased interest in the optimization problem (MSEP) in the recent years, e.g. [4, 5, 6]

Let us mention one particular application the *martingale optimal transport problem:* Given two probability measure ν_1, ν_2 on $\mathbb{R}^d$ increasing in convex order and a cost function $c : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ the martingale optimal transport problem is to find a minimizer of

$$\int c(x, y)\ Q(dx, dy) \tag{1}$$

among all martingale couplings Q of ν_1 and ν_2, i.e. all couplings Q of ν_1 and ν_2 such that the coordinate process (x, y) is a martingale under Q. This problem is well understood in dimension one. Essentially as a consequence of the Dambis-Dubins-Schwarz Theorem (1) can be recast as a Skorokhod embedding problem. Vice versa any solution to the Skorokhod embedding problem induces a solution to (1). Hence, the Skorokhod embedding problem can be seen as a continuous time version of the martingale optimal transport problem in dimension one, cf. [7]. Moreover, (1) has a clear multi-marginal counterpart corresponding to n-step martingales instead of 1-step martingales.

The key idea to study (MSEP), similar to [1], is to interprete a stopping time τ as a way to transport mass attached to a given path ω to the position $\omega(\tau(\omega))$. Correspondingly we associate to each tuple $\tau_1 \leq \ldots \leq \tau_n$ of stopping times on the Wiener space the random measure

$$\bar{\tau}(d\omega, ds_1, \ldots, ds_n) = \delta_{\tau_1(\omega)}(ds_1) \cdots \delta_{\tau_n(\omega)}(ds_n)\ \mathbb{W}(d\omega)\ ,$$

where $\mathbb{W}$ denotes the Wiener measure. In the language of optimal transport this corresponds to a Monge-type solution since to any tracjectory we associate exactly one n-tuple of positions where to stop. Following a key lesson from optimal transport we relax this to the class of randomised multi-stopping times satisfying

$$\bar{\tau}(d\omega, ds_1, \ldots, ds_n) = \bar{\tau}_\omega(ds_1, \ldots, ds_n)\ \mathbb{W}(d\omega)\ ,$$

with $\bar{\tau}_\omega \in \mathcal{P}(\{0 \leq t_1 \leq \ldots \leq t_n\})$ plus some linear constraints ensuring enough adaptedness to keep the stopping time properties. Using these notions the minimization problem (MSEP) turns into

$$\inf_{\bar{\tau} \in \mathsf{RMST}(\mu_1, \ldots, \mu_n)} \int \gamma(\omega_{s \leq t_n}, t_1, \leq, t_n)\ \bar{\tau}(d\omega, dt_1, \ldots, dt_n)\ , \tag{2}$$

where $\mathsf{RMST}(\mu_1, \ldots, \mu_n)$ is the set of all randomised stopping times embedding the measures $\mu_1, \ldots, \mu_n$ together with a suitable variant of the uniform integrability condition. The formulation (2) is very useful since it turns our optimization problem into an optimization problem over a *convex and compact set*. Hence, we directly get

Theorem 1. *Let γ be lower semi continuous and bounded from below. There exists a minimizer to* (MSEP).

Theorem 2 (cf. [3])**.** *Let γ be lower semi continuous and bounded from below. There exists a dual theory to* (MSEP).

Most importantly we can prove a *geometric characterization* of minimizers to (MSEP) allowing us to establish multi-marginal extensions of all known solutions to the classical Skorokhod embedding problem and many more. Moreover, all of these solutions, including the ones corresponding to martingale optimal transport problems, share a common geometric structure which we exemplify by stating the n-Root solution to (MSEP). A barrier $\mathcal{R}$ is a set $\mathcal{R} \subset \mathbb{R}_+ \times \mathbb{R}$ such that $(s,x) \in \mathcal{R}$ and $t \geq s$ implies $(t,x) \in \mathcal{R}$.

Theorem 3 (n-marginal Root embedding)**.** *Put $\gamma_i((\omega_s)_{s \leq s_n}, s_1, \dots, s_n) = h(s_i)$ for some strictly convex function $h : \mathbb{R}_+ \to \mathbb{R}$ and assume that* (MSEP) *is well defined for all $\bar{\tau} \in \mathsf{RMST}(\mu_1, \dots, \mu_n)$ and finite for one such $\bar{\tau}$. Then there exist n barriers $(\mathcal{R}^i)_{i=1}^n$ such that defining*

$$\tau_1^{Root}(\omega) = \inf\{t \geq 0 : (t, B_t(\omega)) \in \mathcal{R}^1\}$$

and for $1 < i \leq n$

$$\tau_i^{Root}(\omega) = \inf\{t \geq \tau_{i-1}^{Root}(\omega) : (t, B_t(\omega)) \in \mathcal{R}^i\}$$

the multi stopping time $(\tau_1^{Root}, \dots, \tau_n^{Root})$ minimises

$$\mathbb{E}[h(\tilde{\tau}_i)]$$

simultaneously for all $1 \leq i \leq n$ among all increasing families of stopping times $(\tilde{\tau}_1, \dots, \tilde{\tau}_n)$ such that $B_{\tilde{\tau}_j} \sim \mu_j$ for all $1 \leq j \leq n$. This solution is unique in the sense that for any solution $\tilde{\tau}_1, \dots, \tilde{\tau}_n$ of such a barrier-type we have $\tau_i^{Root} = \tilde{\tau}_i$ a.s.

Finally we remark that this approach is not limited to one-dimensional Brownian motion and readily extends to sufficiently regular Markov process such as $3-d$ Bessel processes, Ornstein Uhlenbeck processes, or geometric Brownian motion.

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Stability of parabolic Harnack inequalities for symmetric non-local Dirichlet forms

Zhen-Qing Chen
(joint work with Takashi Kumagai and Jian Wang)

Harnack inequalities are inequalities that control the growth of non-negative harmonic functions and caloric functions (solutions of heat equations) on domains. The inequalities were first proved for harmonic functions for Laplacian in the plane by Carl Gustav Axel von Harnack, and later became fundamental in the theory of harmonic analysis, partial differential equations and probability. One of the most significant implications of the inequalities is that (at least for the cases of local operators/diffusions) they imply Hölder continuity of harmonic/caloric functions.

Because of their fundamental importance, there has been a long history of research on Harnack inequalities. A lot is known now for parabolic Harnack inequalities (PHI) for diffusions and for elliptic differential operators on Euclidean spaces, manifolds, graphs and on general metric spaces. In particular, stable charachterizations of parabolic Harnack inequalities have been obtained. See the introduction part of [2] for a brief history. However, little is known about the stable characterization of PHI for discontinuous Markov processes.

In this talk, we report recent advances in the study of stable characterizations of PHI for symmetric purely discontinuous Markov processes, or equivalently, for symmetric pure jump Dirichlet forms on general metric measure spaces.

Let X be a strong Markov process on a locally compact separable metric space M. Denote by $Z_t = (V_0 - t, X_t)$ the corresponding space-time process. We say that a nearly Borel measurable function $u(t,x)$ on $[0,\infty) \times M$ is *parabolic* (or *caloric*) on $Q = (a,b) \times B(x_0, r)$ for X if for every relatively compact open subset U of Q, $s \mapsto u(Z_{s\wedge\tau_U})$ is a $\mathbb{P}^{(t,x)}$ uniformly integrable martingale for every $(t,x) \in U$, where τ_U is the first exit time of Z from U. If we denote the generator of X by $\mathcal{L}$, then intuitively, $u(t,x)$ is parabolic in $Q = (a,b) \times B(x_0, r)$ if and only if $\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x)$ in Q.

Let ϕ be an increasing function on $[0,\infty)$ with $\phi(0) = 0$. We say parabolic Harnack inequality with scale function ϕ (PHI(ϕ)) holds, if there exist constants $C_1 > 0$, $0 < C_2 < 1$ and $C_3 > 0$ such that for any non-negative $u = u(t,x)$ caloric in the cylinder $Q(t_0, x_0, 4C_1\phi(R), R) := (t_0, t_0 + 4C_1\phi(R)) \times B(x_0, R)$,

$$\sup_{Q_-} u \le C_3 \inf_{Q_+} u,$$

where $Q_- := (t_0 + C_1\phi(R), t_0 + 2C_1\phi(R)) \times B(x_0, C_2R)$ and $Q_+ := (t_0 + 3C_1\phi(R), t_0 + 4C_1\phi(R)) \times B(x_0, C_2R)$.

Now consider a metric measure space (M, ρ, μ) that satisfies volume doubling and reversed volume doubling property in the sense that there are constants $c_2 > c_1 > 0$ and $L > 1$ so that

$$c_1\mu(B(x,r)) \leq \mu(B(x,Lr)) \leq c_2\mu(B(x,r)) \quad \text{for every } x \in M \text{ and } r > 0.$$

Here $B(x,r)$ denotes the open ball centered at x with radius r. Let ϕ be a continuous increasing function on $[0,\infty)$ with $\phi(0) = 0$ and $\phi(1)$ that has the doubling and reversed doubling property.

Suppose that X is a pure jump μ-symmetric Markov process on M with jumping kernel $J(x,y)$ with respect to $\mu(dx)\mu(dy)$ associated with a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E;\mu)$, where

$$\mathcal{E}(u,v) = \int_{M\times M} (u(x)-u(y))(v(x)-v(y))J(x,y)\mu(dx)\mu(dy), \quad u,v \in \mathcal{F}.$$

Suppose that $\widetilde{X}$ is another purely jump $\widetilde{\mu}$-symmetric Markov process with jumping kernel $\widetilde{J}(x,y)$ with respect to $\widetilde{\mu}(dx)\widetilde{\mu}(dy)$ associated with a regular Dirichlet form $(\widetilde{\mathcal{E}}, \mathcal{F})$ on $L^2(E;\widetilde{\mu})$, where

$$c_3\widetilde{\mu} \leq \mu \leq c_4\widetilde{\mu} \quad \text{and} \quad c_3\widetilde{J}(x,y) \leq J(x,y) \leq c_4\widetilde{J}(x,y)$$

for some constants $c_4 \geq c_3 > 0$.

We show that PHI(ϕ) holds for X if and only if it holds for $\widetilde{X}$. This stability result result is a direct consequence of a more precise characterization of PHI(ϕ) obtained in [2] in terms of a Sobolev inequality, a cutoff energy inequality, an upper bound and an averaging property for the jumping kernel $J(x,y)$.

Other equivalent characterizations for PHI(ϕ) in terms of heat kernel estimates, mean exit time bounds, Hölder regularity of parabolic functions and harmonic functions, as well as its relation to elliptic Harnack inequalites are also given in the talk.

The talk is based on the recent joint work [1], [2] and [3].

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On groups slow decay of heat kernel implies Liouville property

Tianyi Zheng
(joint work with Yuval Peres, Laurent Saloff-Coste)

Let G be a finitely generated infinite group equipped with a generating set S, and let μ be a probability measure on G. Let $X_1, X_2, \dots$ be a sequence of i.i.d. random variables with distribution μ, so $W_n = X_1 \cdots X_n$ is the random walk on G with step distribution μ. The law of W_n is the n-fold convolution power $\mu^{(n)}$. The return probability of the μ-random walk to the identity e after $2n$ steps is

$$\mathbf{P}\left(W_{2n} = e\right) = \mu^{(2n)}(e).$$

The Shannon entropy of W_n is

$$H_\mu(n) = H(W_n) = -\sum_{x \in G} \mu^{(n)}(x) \log \mu^{(n)}(x).$$

The pair (G, μ) has the *Liouville property* if all bounded μ-harmonic functions on G are constant. By classical work of Avez [1], Derrienic [2] and Kaimanovich-Vershik [5], for μ with finite entropy $H_\mu(1) < \infty$, the pair (G, μ) has the Liouville property if and only if the Avez *asymptotic entropy* $h_\mu = \lim_{n\to\infty} \frac{H_\mu(n)}{n}$ is 0. We say a probability measure μ on G is *symmetric* if $\mu(g) = \mu(g^{-1})$ for all $g \in G$.

We show a link between the decay of the return probability and the growth of entropy of a symmetric random walk on G. Namely, we derive an upper bound on $H_\mu(n)$ from a lower bound on $\mu^{(2n)}(e)$, provided that $\mu^{(2n)}(e)$ decays sufficiently slowly. More precisely, in joint work with Peres [7], we show the following. Suppose μ is a symmetric probability measure of finite entropy on G such that

$$\mu^{(2n)}(e) \geq \exp\left(-\gamma(n)\right)$$

where $\gamma : [1, \infty) \to \mathbb{R}_+$ is a function such that both $\gamma(n)$ and $n^{\frac{1}{2}}/\gamma(n)$ are increasing, that satisfies

$$\lim_{n\to\infty} \frac{\gamma(n)}{n^{\frac{1}{2}}} = 0.$$

Then (G, μ) has the Liouville property.

Kotowski and Virág [6] analyzed a group G on which simple random walk satisfies $\mu^{(2n)}(e) \geq \exp(-cn^{1/2+o(1)})$ and the entropy $H_\mu(n)$ has linear growth. The Kotowski-Virg example shows that the exponent 1/2 is the critical value in the setting of the result above. It is an interesting open problem whether $\mu^{(2n)}(e) \geq \exp\left(-cn^{\frac{1}{2}}\right)$ for some constant $c > 0$ implies that (G, μ) has the Liouville property. Simple random walk on the lamplighter group over the two-dimensional lattice $G = \mathbb{Z}_2 \wr \mathbb{Z}^2$ satisfies $\mu^{(2n)}(e) \simeq \exp(-n^{1/2})$ and $H_\mu(n) \simeq n/\log n$, see [3, 8]. This example is just beyond the limit of application of our result.

The decay of the return probability enjoys good stability properties, see Pittet and Saloff-Coste [9]. However, it remains a major open problem whether the Liouville property is stable under changing the generating set of the group. We deduce the following corollary regarding stability of the Liouville property provided that $\mu^{(2n)}(e)$ decays slower than $\exp\left(-n^{1/2}\right)$. Suppose G is a finitely generated group such that for some symmetric probability measure μ with finite generating support on G,

$$\lim_{n\to\infty} \frac{-\log \mu^{(2n)}(e)}{n^{\frac{1}{2}}} = 0.$$

Let Γ be a finitely generated group that is quasi-isometric to G. Then (Γ, η) has the Liouville property for any symmetric probability measure η of finite second moment on Γ. Here we say a probability measure μ on G has finite second moment if $\sum_{g\in G} |g|^2\mu(g) < \infty$ where $|\cdot|$ is the word distance on the Cayley graph (G, S).

When the decay of the return probability $\mu^{(2n)}(e)$ is much slower than $\exp\left(-n^{\frac{1}{2}}\right)$, in [7] we have the following explicit entropy upper bound. This improves earlier results of Gournay [4] and Saloff-Coste with the author [10]. Let μ be a symmetric probability measure of finite entropy on G. Suppose there exists constants $C > 0$, $\beta \in \left(0, \frac{1}{2}\right)$ such that

$$\mu^{(2n)}(e) \geq \exp\left(-Cn^{\beta}\right).$$

Then there exists a constant $C_1 = C_1(\beta, C)$ such that

$$H_\mu(n) \leq C_1 n^{\frac{\beta}{1-\beta}}.$$

This bound is sharp on a family of groups which are extensions of the bubble groups considered in [6], see Section 5 in [7].

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Quenched invariance principle for the dynamic random conductance model

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(joint work with Sebastian Andres, Alberto Chiarini and Martin Slowik)

We are interested in establishing a quenched invariance principle for the *dynamic random conductance model* on the d-dimensional Euclidean lattice $(\mathbb{Z}^d, E^d)$. The dynamic conductance model is an time-inhomogenous Markov process $\{X_t : t \geq 0\}$ on $\mathbb{Z}^d$ in continuous time with generator, $\mathcal{L}^\omega$, which acts on bounded functions $f \colon \mathbb{Z}^d \to \mathbb{R}$ as

$$\big(\mathcal{L}_t^\omega f\big)(x) \;=\; \sum_{y\sim x} \omega_t(\{x,y\})\,\big(f(y)-f(x)\big),$$

where $\omega = \{\omega_t(e) \in (0,\infty) : e \in E^d, t \in \mathbb{R}\}$ is a family of non-negative weights. Our main objective is to study this model under the following assumption on the law of the conductances.

Assumption 1. *Assume that the law $\mathbb{P}$ of the conductances satisfies:*

(1) $\mathbb{E}\big[\omega_t(e)\big] < \infty$ *and* $\mathbb{E}\big[\omega_t(e)^{-1}\big] < \infty$ *for all* $e \in E_d$ *and* $t \in \mathbb{R}$.
(2) $\mathbb{P}$ *is ergodic and stationary with respect to space-time shifts* $\tau_{t,x}$.
(3) *For every* $A \in \mathcal{F}$ *the mapping* $(\omega, t, x) \mapsto 1_A(\tau_{t,x}\omega)$ *is jointly measurable with respect to the* σ*-algebra* $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{Z}^d)$.

Theorem 1. *Suppose that* $d \geq 2$ *and Assumption 1 holds. Futher, assume that there exists* $p, q \in (1,\infty]$ *satisfying*

$$\frac{1}{p-1} + \frac{1}{(p-1)q} + \frac{1}{q} \;<\; \frac{2}{d}$$

such that

$$\mathbb{E}\big[\omega_t(e)^p\big] \;<\; \infty \qquad \text{and} \qquad \mathbb{E}\big[\omega_t(e)^{-q}\big] \;<\; \infty$$

for all $e \in E_d$ *and* $t \in \mathbb{R}$. *Then, the QFCLT holds for* X *with a deterministic non-degenerate covariance matrix* Σ^2.

Quenched invariance principles have been shown for various models for random walks evolving in dynamic random environments (see [1, 5, 4, 8, 9, 14, 13]). Here analytic, probabilistic and ergodic techniques were invoked, but assumptions on the ellipticity and the mixing behaviour of the environment remained a pivotal requirement. For instance, the QFCLT for the time-dynamic RCM in [1] required strict ellipticity, i.e. the conductances are almost surely uniformly bounded and bounded away from zero, as well as polynomial mixing, i.e. the polynomial decay of the correlations of the conductances in space and time.

One motivation to study the dynamic RCM is to consider random walks in an environment generated by some interacting particle systems like zero-range or exclusion processes (cf. [7, 12]). Recently, some on-diagonal upper bounds for the transition kernel of a degenerate time-dependent conductances model are obtained in [12], where the conductances are uniformly bounded from above but they are allowed to be zero at a a given time satisfying a lower moment condition. In [11] it is shown that for uniformly elliptic dynamic RCM in discrete time – in contrast to the time-static case – two-sided Gaussian heat kernel estimates are not stable under perturbations. In a time dynamic balanced environment a QFCLT under moment conditions has been recently shown in [7].

An annealed FCLT has been obtained for strictly elliptic conductances in [1], for non-elliptic conductances generated by an exclusion process in [2] and for a similar one-dimensional model allowing some local drift in [3] and recently for environments generated by random walks in [10].

Finally, let us remark that there is a link between the time dynamic RCM and Ginzburg-Landau interface models as such random walks appear in the so-called Helffer-Sjöstrand representation of the space-time covariance in these models (cf. [6, 1]). However, in this context the annealed FCLT is relevant.

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New estimates on the matching problem

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(joint work with Federico Stra, Dario Trevisan)

Optimal matching problems are random variational problems widely investigated in the mathematical and physical literature, having many variants (monopartite, bipartite, matching to the reference measure, grid matching...). See the monographs [4] and [5] for many more informations on this subject. We provide new results on the optimal matching of the empirical measure $\sum_i \frac{1}{n}\delta_{X_i}$ built from an i.i.d. sequence (X_i) of points in a d-dimensional domain with law μ, and on the bipartite problem, where μ is replaced by another empirical measure $\sum_i \frac{1}{n}\delta_{Y_i}$.

Denoting by W_p the Wasserstein distance induced by the transport cost $c = d^p$, $1 \le p < \infty$, the problem is to estimate the rate of convergence to 0 of

$$\mathbb{E}\left[W_p^p\left(\sum_i \frac{1}{n}\delta_{X_i}, \mu\right)\right], \qquad \mathbb{E}\left[W_p^p\left(\sum_i \frac{1}{n}\delta_{X_i}, \sum_i \frac{1}{n}\delta_{Y_i}\right)\right]. \tag{1}$$

If μ is uniformly distributed, the typical distance between points is expected to be of order $n^{-1/d}$, and therefore it is natural to guess that the quantities in (1) behave as $n^{-p/d}$ (the lower bound can be achieved using duality and the random test function $\min_i |\cdot - x_i|$). However, it is by now well known that this expectation is true for $d \ge 3$, while it is false for $d = 1$ and $d = 2$. Leaving aside the 1-dimensional case, for which many explicit computations are possible (since optimal allocations are monotone rearrangement), the most striking result has been obtained in [1], where it has been proved that a logarithmic correction appears: $\mathbb{E}\left[W_p^p\left(\sum_i \frac{1}{n}\delta_{X_i}, \mu\right)\right] \sim (\log n)^{p/2}$. In [2] we obtain a new proof of this result based on semigroup tools and spectral analysis; in addition, for $p = 2$, we are able to show that

$$\lim_{n\to\infty} \frac{n}{\log n}\mathbb{E}\left[W_2^2(\sum_i \frac{1}{n}\delta_{X_i}, \mu_D)\right] = \frac{\mu(D)}{4\pi}$$

whenever D is a compact 2-dimensional Riemannian manifold without boundary. Here μ is the Riemannian volume measure and $\mu_D = \mu/\mu(D)$ is its normalization; our result covers also the classical case of the unit square, by a comparison argument. In the bipartite case we also prove that the limit is $\mu(D)/2\pi$, using independence of the two empirical measures.

In our proof the geometry of the domain D enters only through the (asymptotic) properties of the spectrum of the Laplacian with Neumann boundary conditions; for this reason we are able to cover also abstract manifolds.

The idea of the proof comes from a recent work [3], where scaling and expansion hypotheses are made, in the bipartite case on the 2-dimensional torus. The main idea in [3] is to linearize the Monge-Ampère equation, formally treating the empirical measures as absolutely continuous measure. We can confirm part of the predictions of [3] using a delicate smoothing technique, together with Dacorogna-Moser interpolation and estimates on the Hopf-Lax semigroup.

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Monotonicity and rigidity of the $\mathcal{W}$-entropy on $\mathsf{RCD}^*(0,N)$ spaces

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(joint work with Xiang-Dong Li)

Perelman's $\mathcal{W}$-entropy plays a crucial role in his seminal work on Ricci flow [9]. It is well-known by Perelman's entropy formula that the $\mathcal{W}$-entropy is nonincreasing along the heat distribution in (reversed) time and a time derivative vanishes if and only if the space is isomorphic to a gradient shrinking Ricci soliton. L. Ni [7] brought the notion of $\mathcal{W}$-entropy to time-homogeneous Riemannian manifolds, and the corresponding results has been studied in the literature under nonnegative Ricci curvature in an appropriate sense (see [6, 7, 8] for instance).

In this talk, we consider the corresponding problem on $\mathsf{RCD}^*(0,N)$ metric measure spaces. They are "Riemannian" spaces with nonnegative Ricci curvature ($\mathrm{Ric} \geq 0$) and an upper bound of dimension by N ($\dim \leq N$), defined in terms of optimal transport (see [1] and references therein). It includes all (weighted) Riemannian manifold with nonnegative N-Bakry-Émery Ricci tensor, and Ricci limit spaces with an appropriate curvature-dimension bound.

Let (X,d) be a Polish geodesic metric space and $\mathfrak{m}$ is a Borel measure on X. We suppose $\mathfrak{m}(B_r(x)) \in (0,\infty)$ for any open metric ball $B_r(x)$ centered at $x \in X$ of radius $r > 0$. On metric measure space $(X,d,\mathfrak{m})$, we can define Cheeger's L^2-energy functional Ch. When X is a complete Riemannian manifold, d is the associated Riemannian distance and $\mathfrak{m}$ is the Riemannian volume measure, Ch is identified with the Dirichlet energy functional:

$$\mathsf{Ch}(f) = \frac{1}{2}\int_X |Df|^2 d\mathfrak{m}.$$

As a gradient flow of Ch on $L^2(\mathfrak{m})$, we can define the heat flow $(f_t)_{t\geq 0}$, the associated heat semigroup P_t satisfying $f_t = P_t f_0$ and its generator Δ. Up to regularity assumptions, we say that $(X,d,\mathfrak{m})$ is $\mathsf{RCD}^*(0,N)$ space ($N \geq 2$ for simplicity), if P_t is linear operator, $\|P_t f\|_{L^1(\mathfrak{m})} = \|f\|_{L^1(\mathfrak{m})}$ for $f \in L^1(\mathfrak{m}) \cap L^2(\mathfrak{m})$ with $f \geq 0$ and the following space-time W_2-control of heat flows

$$W_2(P_t f\mathfrak{m}, P_t g\mathfrak{m})^2 \leq W_2(f\mathfrak{m}, g\mathfrak{m})^2 + 2N(\sqrt{t}-\sqrt{s})^2$$

holds for any probability densities f and g, where W_2 is L^2-Wasserstein distance. We can define $\mathsf{RCD}^*(K,N)$ spaces (spaces with $\mathrm{Ric} \geq K$ and $\dim \leq N$) in a similar manner. See [1] for the precise definition and other equivalent formulations. In this framework, we can extend P_t to a (continuous) map from the space of probability measures $\mathcal{P}_2(X)$ with a finite finite moment to $\mathcal{P}_2(X)$ itself.

According to [7] (with a different expression and up to additive constants although), we define the $\mathcal{W}$-entropy $\mathcal{W} : \mathcal{P}_2(X) \times (0,\infty) \to \mathbb{R}$ as follows:

$$\mathcal{W}(\mu,t) := t\mathrm{I}_{\mathfrak{m}}(\mu) - \mathrm{Ent}_{\mathfrak{m}}(\mu) - \frac{N}{2}\log t$$

for $\mu \in \mathcal{P}_2(X)$ and $t > 0$, where $\mathrm{Ent}_{\mathfrak{m}}$ is the relative entropy functional with respect to $\mathfrak{m}$ and $\mathrm{I}_{\mathfrak{m}}$ is the Fisher information. Note that $\mathrm{Ent}_{\mathfrak{m}}(P_t\mu)' = -\mathrm{I}_{\mathfrak{m}}(P_t\mu)$ holds for a.e. $t > 0$ and $\mu \in \mathcal{P}_2(X)$.

Our first result asserts that, for $\mu_t = P_t\mu$, $\mathcal{W}(\mu_t,t)$ is nonincreasing in t on $\mathsf{RCD}^*(0,N)$ space $(X,d,\mathfrak{m})$. By following Topping's approach [10] to this problem on a (backward) Ricci flow by means of optimal transport, we can show this monotonicity from our space-time W_2-control of heat flows. Indeed, the identification of W_2-metric speed with $\sqrt{\mathrm{I}_{\mathfrak{m}}}$

$$\limsup_{\delta\downarrow 0} \frac{W_2(\mu_t,\mu_{t+\delta})}{\delta} = \sqrt{\mathrm{I}_{\mathfrak{m}}(\mu_t)} \quad \text{a.e. } t$$

plays an essential role. Unlike previous results [6, 7, 8, 9], our proof goes without deriving the entropy formula which describes the derivative of $\mathcal{W}(\mu_t,t)$ explicitly. As a by-product, even when X is a smooth (weighted) noncompact Riemannian manifold, the monotonicity holds without additional technical assumptions in [6, 8] (see [5] also).

Moreover, we also show the rigidity of this monotonicity. Here the rigidity means that $(X,d,\mathfrak{m})$ is a $(0,N)$-cone of an $\mathsf{RCD}^*(N-2,N-1)$ space if the upper right derivative of $\mathcal{W}(\mu_t,t)$ vanishes (see [2] for instance for the notion of $(0,N)$-cone). Among smooth Riemannian manifolds, $\mathbb{R}^N$ is the only possible choice of such spaces and it recovers previous rigidity results [6, 7, 8] as a special case. It also means that, some other singular spaces than Euclidean spaces admit a vanishing time derivative of the $\mathcal{W}(\mu_t,t)$. For the rigidity, we reduce the problem to verifying the assumption of the main result of [2]. For this, the Li-Yau inequality [3] and the Varadhan-type short time asymptotic for the heat kernel [4] will be used.

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Hessian Estimates

Xue-Mei Li

Gradient and hessian estimates for heat kernel are useful in the study of loops spaces over a complete Riemannian manifold. Specifically they are needed for obtaining an integration by parts for the Brownian bridge measure, and estimating the tails of the measure in terms of exponential integrability of Lipschitz continuous functions, and for proving the existence of a spectral gap for the Laplacian on loop space.

The following type of estimates for the heat kernel $p(t,x,y)$

$$|\nabla \log p(t,x,y)| \le C(\frac{1}{\sqrt{t}} + \frac{d}{t}), \qquad |\nabla^2 \log p(t,x,y)| \le C(\frac{1}{t} + \frac{d^2}{t^2})$$

are standard assumptions in the study of Brownian bridges measures on loop spaces. These are proved by Sheu for Euclidean spaces, by Malliavin-Stroock for compact manifolds and extended by Stroock-Turetsky to a class of manifolds with bounded curvature and with the gradient of Ricci growing at most linearly.

We generalise this result to include more general manifolds and obtain a similar estimate for the weighted Schrödinger operator with a potential,

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u + Du(\nabla h) - Vu$$

where h and V are real valued functions on the manifold. Our main tool is a Hessian formula for which we introduce the doubly damped stochastic parallel translation equation and study its exponential integrability. We also give several criteria for non-explosion of Brownian motion with a gradient drift and the strong 1-completeness for gradient SDEs, which are needed for the above mentioned estimates.

Using the Hessian formula mentioned earlier, the semi-classical bridge introduced by Elworthy-Truman, and a triple Girsanov transform, we obtain exact formulas for the Hessian of the weighted heat kernel on manifolds with a pole (for example for simply connected manifolds of negative curvature). The semi-classical bridge is Brownian motion with drift $\nabla \log k_{1-t}$. Its radial part is the Bessel bridge and hence we expect its probability distribution on the pinned path space on $[0,1]$ or on the loop space to have nicer properties than the Brownian bridge measure.

These formulas are expressed in terms of the product of an exact Gaussian kernel and an explicit correction term E_2,

$$\nabla dp_t(x_0, y_0) = k_t(x_0, y_0)E_2, \quad k_t(x_0, y_0) = (2\pi t)^{-\frac{n}{2}} e^{-d^2(x_0,y_0)/2t} J^{-\frac{1}{2}}(x_0, y_0).$$

Here J is the Jacobian determinant of the exponential map at y_0. The correction term E_2 can easily be bounded by $C(\frac{1}{t} + \frac{d^2}{t^2})$, but since it is explicit (in terms of the semi-classical bridge), more precise asymptotics can also be obtained. Combining this with the elementary formula of Elworthy-Truman for heat kernels of the same form, $p_t(x_0, y_0) = k_t(x_0, y_0)E_0$, and an estimate of the form $\nabla p_t(x_0, y_0) = k_t(x_0, y_0)E_1$ obtained in an earlier work in collaboration with my Ph.D. student Thompson, we obtain estimates for the Hessian of the logarithm of the heat kernel, which also hold for the weighted Laplacian.

This study raises a number of questions including whether there exists a curvature comparison theorem for the Ruse invariant appearing in the Girsanov term and and for the gradient of the Jacobian determinant J. These estimates should also extend to include less smooth data. In a future study we also hope to use these formulas and estimates to examine the validity of Poincaré and Logarithmic Sobolev inequalities on the loop space.

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Molchanov's technique for small-time heat kernel asymptotics at cut points

Robert Neel

(joint work with Ugo Boscain, Davide Barilari, and Grégoire Charlot)

In [14], Molchanov described a method to compute the small-time heat kernel asymptotics of the heat kernel at the cut locus of a Riemannian manifold. This method turns out to be fairly broadly applicable, and in [4], it was extended to sub-Riemannian geometry (where Riemannian geometry can be seen as a special case), with further development in [3] and [2].

We equip our (complete) n-dimensional sub-Riemannian manifold with a sub-Laplacian Δ, which gives rise to a hypoelliptic diffusion, and a smooth volume, which serves as a reference measure for the associated heat kernel $p_t(x, y)$ (the fundamental solution to $\partial_t u_t(x) = \Delta u_t(x)$ in the correct variable).

This method is based on three ingredients: the Chapman-Kolmogorov equation, a "global" coarse estimate, which in the sub-Riemannian context is given by $-2t \log p_t(x, y) \to \frac{1}{2} d^2(x, y)$ as $t \to 0$ uniformly on compacts (due to Leandre [10, 11]), and a finer estimate off of the cut locus, which in the sub-Riemannian context is provided by $p_t(x, y) \sim \left(\frac{1}{4\pi t}\right)^{n/2} e^{-d^2(x,y)/4t} \sum_{i=0}^{\infty} H_i(x, y) t^i$ on M minus $\mathrm{Cut}(x)$, x itself, and any abnormals (due to Ben Arous [6]). Concretely, take x and y to be distinct points in M, let Γ be the set of midpoints of minimal geodesics from x to y and let Γ_ϵ be an ϵ-neighborhood. For example, if M is the standard sphere and x and y the north and south poles, Γ is the equator. The idea is to glue two copies of the expansion at Γ. Let $h_{x,y}(z) = \frac{1}{2} d^2(x, z) + \frac{1}{2} d^2(z, y)$ be the hinged energy function. Then we derive

$$p_t(x, y) = \left(\frac{1}{2\pi t}\right)^n \int_{\Gamma_\epsilon} (H_0(x, z) H_0(z, y) + O(t)) \, e^{-h_{x,y}(z)/t} \, dz,$$

giving the small-time asymptotics of p_t as a Laplace integral where $h_{x,y}$ is the phase function. Note that $h_{x,y}(z)$ achieves its minimum (of $d^2(x, y)/4$) exactly on the set Γ. Also, for $z \in \Gamma$, $\operatorname{Hess} h_{x,y}(z)$ is non-degenerate if and only if the geodesic from x to y through z is non-conjugate.

For some broader context, we note that integral representations of hypoelliptic heat kernels for left-invariant structures on Lie groups (or other spaces with a lot of symmetry) have been studied algebraically going back to classical work of Gaveau and Hulanicki on the Heisenberg group and remain active, as seen in recent work by Bonnefont [7], Baudoin-Wang [5], and Asaad-Gordina [1]. In a different vein, within the past year, Inahama-Taniguchi [8] used Watanabe's distributional Malliavin calculus to give a general approach to sub-Riemannian heat kernel asymptotics, in a related direction to earlier work of Kusuoka-Stroock [9], and Ludewig [13, 12] gave similar asymptotics for Riemannian vector bundles via a path-integral-type approach.

Returning to Molchanov's method, we note that the integral formula above is well suited to studying broad classes of examples of Riemannian and sub-Riemannian structures. We illustrate this in the next two theorems.

In the most commonly studied cases in the literature, M possesses some rotational symmetry and $h_{x,y}$ is a Morse-Bott function (the Hessian is not degenerate on the normal bundle to $T\Gamma$). Then we have the following.

Theorem. *Let M be an n-dimensional Riemannian or sub-Riemannian manifold with an associated heat kernel as above, and let x and y be distinct with every optimal geodesic joining x to y strongly normal. Define*

$$\mathcal{O} := \{p \in T_x^* M \mid \mathrm{Exp}_x(p, d(x, y)) = y\}$$

Assume that $\mathcal{O}$ is a submanifold of T_x^*M of dimension r and that for every $p \in \mathcal{O}$ we have $\dim\ker D_{p,d(x,y)}\mathrm{Exp}_x = r$. Then there exists a positive constant C such that

$$p_t(x,y) = \frac{C+O(t)}{t^{\frac{n+r}{2}}} e^{-d^2(x,y)/4t} \quad \textit{for small } t.$$

In a different direction, recall the ADE classification of generic singularities of maps of the Arnold school, and say that a geodesic γ is, for example, A_3-conjugate if the exponential map has an A_3-singularity at γ. If γ is A_m-conjugate, then near the midpoint of γ, $h_{x,y}$ has the form

$$h_{x,y}(z) = \frac{1}{4} d^2(x,y) + z_1^2 + \ldots + z_{n-1}^2 + z_n^{m+1}.$$

Note this implies a minimizing geodesic can't be A_{2k}-conjugate. Suppose that, for some $\ell \in \{3,5,7,\ldots\}$ every minimizing geodesic from x to y is non-conjugate or A_m-conjugate for some $3 \le m \le \ell$, and at least one is A_ℓ . Then there exists $C > 0$ such that

$$p_t(x,y) = \frac{C + O\left(t^{\frac{2}{\ell+1}}\right)}{t^{\frac{n+1}{2} - \frac{1}{\ell+1}}} e^{-d^2(x,y)/4t}.$$

Moreover, we have the following.

Theorem. *Let M be a smooth manifold, $\dim M = n \le 5$, and $x \in M$. For a generic Riemannian metric on M and any minimizing geodesic γ from x to some y, γ is either non-conjugate, A_3-conjugate, or A_5-conjugate. Then the only possible heat kernel asymptotics are (here $C > 0$ is some constant which can differ from line to line):*

- *If no minimizing geodesic from x to y is conjugate, then* $p_t(x,y) = \frac{C+O(t)}{t^{\frac{n}{2}}} e^{-d^2(x,y)/4t}$,
- *If at least one minimizing geodesic from p to q is A_3-conjugate but none is A_5-conjugate,* $p_t(x,y) = \frac{C+O(t^{1/2})}{t^{\frac{n}{2}+\frac{1}{4}}} e^{-d^2(x,y)/4t}$,
- *If at least one minimizing geodesic from p to q is A_5-conjugate,* $p_t(x,y) = \frac{C+O(t^{1/3})}{t^{\frac{n}{2}+\frac{1}{6}}} e^{-d^2(x,y)/4t}$.

We discuss further examples, including generic three-dimensional contact sub-Riemannian structures and non-generic examples that show that the expansion need not proceed in integer or half-integer powers of t. In addition, if the exact structure of the exponential map near all minimizing geodesics is not known, one can still give estimates on the heat kernel for small times.

Finally, we briefly describe how this method can be extended to study logarithmic derivatives of the heat kernel, at least on a Riemannian manifold, as discussed in [15].

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An index theorem for hyperbolic operators

Christian Bär
(joint work with Alexander Strohmaier)

Let M be a Lorentzian manifold with boundary; the boundary is assumed to consist of two smooth and spacelike Cauchy hypersurfaces, one lying in the past of the other. We assume that M carries a spin structure so that the spinor bundle $SM \to M$ is defined. Moreover, let the dimension of M be even; then the spinor bundle splits into the two subbundles of left-handed and right-handed spinors, $SM = S_L M \oplus S_R M$. Finally, let $E \to M$ be a Hermitian vector bundle, equipped with a compatible connection. Then we have the bundles of spinors with coefficients in E, $V_{L/R} = S_{L/R}M \otimes E$.

The boundary is a Riemannian manifold and the induced operator on the boundary is a self-adjoint elliptic differential operator. Therefore the Atiyah-Patodi-Singer boundary conditions make sense in this Lorentzian setting; they say

$$P_+(u|_{\partial M}) = 0$$

where P_+ denotes the spectral projector onto the subspace of L^2-spinors over ∂M spanned by the eigenspinors to the non-negative eigenvalues of the boundary Dirac operator.

The twisted Dirac operator $D : C^\infty(M, V_R) \to C^\infty(M, V_L)$ on M is a *hyperbolic* linear differential operator of first order. Usually, index theory is closely tied to ellipticity of the operator and hyperbolic operators are not Fredholm. Moreover, solutions of $Du = 0$ need not be smooth; they can be very irregular.

In this particular setting however, we have a complete analog to the Atiyah-Patodi-Singer index theorem [1]:

Theorem 1 (Bär-Strohmaier [2]). *Under Atiyah-Patodi-Singer boundary conditions, D is a Fredholm operator. The kernel consists of smooth spinor fields and the index is given by*

$$\mathsf{ind}(D_{\mathsf{APS}}) = \int_M \widehat{\mathsf{A}}(M) \wedge \mathsf{ch}(E) + \int_{\partial M} T(\widehat{\mathsf{A}}(M) \wedge \mathsf{ch}(E)) - \frac{h+\eta}{2}.$$

Here $\widehat{\mathsf{A}}(M)$ is the $\widehat{\mathsf{A}}$-form computable in terms of the curvature of M and ch is the Chern character form, an expression in the curvature of E. By $T(\widehat{\mathsf{A}}(M)\wedge\mathsf{ch}(E))$ we denote the corresponding transgression form and h and η denote the dimension of the kernel and the η-invariant of the boundary operator, respectively.

There are also important differences to the Riemannian case. First of all, it is possible to replace the Atiyah-Patodi-Singer boundary conditions by the complementary *anti-Atiyah-Patodi-Singer boundary conditions*

$$P_-(u|_{\partial M}) = 0.$$

In the Riemannian case this would not yield a Fredholm operator. In the Lorentzian setting the operator turns out to be Fredholm and the same index formula as in Theorem 1 holds, except for a global sign. Moreover, the index can be written as

$$\begin{aligned}\mathsf{ind}(D_{\mathsf{APS}}) = \mathsf{dim\,ker}[D &: C^\infty_{\mathsf{APS}}(M, V_R) \to C^\infty(M, V_L)] \\ - \mathsf{dim\,ker}[D &: C^\infty_{\mathsf{aAPS}}(M, V_R) \to C^\infty(M, V_L)]\end{aligned}$$

where the subscripts APS and aAPS indicate that (anti-)Atiyah-Patodi-Singer boundary conditions are imposed. In the corresponding Riemannian formula the negative term would have to be replaced by $-\mathsf{dim\,ker}[D : C^\infty_{\mathsf{APS}}(M, V_L) \to C^\infty(M, V_R)]$ (up to a subtlety if $h \neq 0$).

In the Lorentzian setup the APS-boundary conditions have a natural physical interpretation in terms of a particle-antiparticle splitting. This allows to use Theorem 1 to directly derive a geometric formula for the *chiral anomaly* in quantum field theory on curved spacetimes without the need to resort to mathematically fishy arguments such as a Wick rotation. See [3] for details and computed examples.

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Geometric properties of Dirichlet forms under order isomorphisms

Melchior Wirth

(joint work with Matthias Keller, Daniel Lenz, Marcel Schmidt)

Kac' famous question "Can one hear the shape of a drum?" can mathematically be formulated as follows: Given Laplacians with Dirichlet boundary on Euclidean domains Ω_1, Ω_2 and given a unitary operator U that intertwines the Laplacians,

$$U\Delta_{\Omega_1} = \Delta_{\Omega_2} U,$$

are the domains Ω_1, Ω_2 necessarily isometric?

As the answer is no in general, it is a natural follow-up to ask for conditions on U that do enforce congruence of the domains. This was accomplished by Arendt [1], who showed that the answer to Kac' question is positive if one assumes U to be an order isomorphism instead of a unitary operator. An order isomorphism U between L^p-spaces is an invertible linear operator such that U and U^{-1} map non-negative functions to non-negative functions.

In this talk we present the results of [4, 5], where we address this kind of question in the general setting of Dirichlet forms: Let $\mathcal{E}_1$, $\mathcal{E}_2$ be Dirichlet forms on $L^2(X_1, m_2)$, $L^2(X_2, m_2)$ with generators L_1, L_2, and assume there is an order isomorphism U such that $UL_1 = L_2 U$. We investigate which geometric properties of the Dirichlet forms are preserved in this situation.

This setting allows us to treat a variety of new classes of examples including graphs, metric graphs, fractals, and metric measure spaces.

As a first result we establish that the assumption of U being an order isomorphism is indeed stronger than the one in Kac' original question. This is new even in the Euclidean case.

Proposition 1. *The Dirichlet form $\mathcal{E}_1$ is irreducible if and only if $\mathcal{E}_2$ is irreducible. In this case, $\|U\|^{-1}U$ is unitary.*

It is a classical result that every order isomorphism has the form of a weighted composition operator:

$$Uf = h \cdot f \circ \tau$$

with $h\colon X_2 \longrightarrow (0,\infty)$ measurable and $\tau\colon X_2 \longrightarrow X_1$ measurable with measurable inverse.

For regular Dirichlet forms we can prove stronger regularity properties of τ.

Proposition 2. *Assume that $\mathcal{E}_1$, $\mathcal{E}_2$ are irreducible and regular. Under some additional regularity assumption, there exist polar sets $N_1 \subset X_1$, $N_2 \subset X_2$ and an m_2-version $\tilde{\tau}$ of τ such that $\tilde{\tau}\colon X_2 \setminus N_2 \longrightarrow X_1 \setminus N_1$ is a homeomorphism.*

Roughly speaking, the additional regularity assumption ensures that there are sufficiently many bump functions in $D(\mathcal{E}_1)$ and $D(\mathcal{E}_2)$ that are mapped to continuous functions by U and U^{-1} respectively. This condition is satisfied for example for Dirichlet forms where points have positive capacity (graphs, metric graphs,and more generally Dirichlet forms induced by resistance forms), complete Riemannian manifolds [2], and $\mathsf{RCD}^*(K,N)$ spaces with finite measure.

In many situations, $\tilde{\tau}$ can be extended to a homeomorphism on the entire space and even an isometry with respect to suitable distance functions.

If $\mathcal{E}$ is a strongly local Dirichlet form on $L^2(X,m)$ with energy measure Γ, one defines the associated intrinsic metric by

$$d(x,y) = \sup\{|u(x) - u(y)|\colon u \in D(\mathcal{E})_{\mathrm{loc}} \cap C(X),\, \Gamma(u) \leq m\}.$$

In the case of a jump-type form $\mathcal{E}$, there is no distinguished intrinsic metric, but instead a family of intrinsic metrics d characterized by

$$d(\,\cdot\,,A) \wedge T \in D(\mathcal{E})_{\mathrm{loc}} \cap C(X),\, \Gamma(d(\,\cdot\,,A) \wedge T) \leq m$$

for all $A \subset X$, $T > 0$ [3]. Note that we allow in both cases metrics that attain the value 0 off the diagonal or are infinite at some points.

We call a map $\phi\colon Y_1 \longrightarrow Y_2$ between sets endowed with families of distance functions $\mathfrak{D}(Y_1)$, $\mathfrak{D}(Y_2)$ an isometry if

$$\rho \mapsto \rho(\phi(\cdot),\phi(\cdot))$$

is a bijection between $\mathfrak{D}(Y_1)$ and $\mathfrak{D}(Y_2)$.

Theorem 1. *Make the same regularity assumptions on $\mathcal{E}_1$, $\mathcal{E}_2$ as before. If $\mathcal{E}_1$, $\mathcal{E}_2$ are strongly local, then $\tilde{\tau}\colon X_2 \setminus N_2 \longrightarrow X_1 \setminus N_1$ is an isometry with respect to d_1, d_2.*

If $\mathcal{E}_1$, $\mathcal{E}_2$ are Dirichlet forms of jump-type, the same assertion is true with respect to the sets of intrinsic metrics for $\mathcal{E}_1$, $\mathcal{E}_2$ under the additional assumption that $\mathcal{E}_1$, $\mathcal{E}_2$ are recurrent.

If intrinsic metrics d_1, d_2 induce the original topology on X_1, X_2 and are complete, then $\tilde{\tau}$ extends to an isometry $X_2 \longrightarrow X_1$ with respect to d_1, d_2.

In some special situations, there are other metrics that are better adapted to the Dirichlet form (e.g. the resistance metric or the combinatorial graph metric for Dirichlet forms on discrete spaces [4]). We show that $\tilde{\tau}$ is also an isometry with respect to these metrics under suitable conditions.

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Conical square functions for degenerate elliptic operators

Li Chen

(joint work with José María Martell, Cruz Prisuelos-Arribas)

The second order divergence form degenerate elliptic operators with degeneracy in form of Muckenhoupt weights arise naturally from boundary value problems for elliptic equations. In this talk, we consider the conical square functions that one can construct using the heat or Poisson semigroup associated with degenerate elliptic operators, which is a natural generalization of Littlewood-Paley functions. We study their weighted and unweighted L^p boundedness. As a consequence of our methods, we find a class of degeneracy weights w for which L^2-estimates for these conical square functions hold. This opens the door to the study of weighted and unweighted Hardy spaces and of boundary value problems associated with degenerate elliptic operators.

Non-regular weighted Sobolev spaces and Dirichlet forms

Alberto Chiarini

(joint work with Pierre Mathieu)

Dirichlet form theory has seen a wide spread appreciation in both the analytic and probabilistic community. This success is due to the rich interplay between the theory of strongly continuous semigroups and stochastic processes.

A very well studied model in Dirichlet forms theory is the so called *distorted Brownian motion*. Let $d \geq 2$, we consider a weight $\rho : \Omega \to [0, +\infty)$ measurable and such that $\rho, \rho^{-1} \in L^1_{loc}(\mathbb{R}^d)$. Then a natural way to build a regular Dirichlet form is to look at the formal generator

$$Lu(x) := \frac{1}{\rho(x)} \nabla \cdot (\rho(x) \nabla u(x))$$

which is not well defined being ρ only measurable. Formally integrating by parts $\int_{\mathbb{R}^d} v \cdot (-Lu)\, \rho dx$ in $L^2(\mathbb{R}^d, \rho dx)$ we obtain the bilinear form

$$\mathcal{E}(u, v) := \int_{\mathbb{R}^d} \nabla u \cdot \nabla v \, \rho dx, \quad u, v \in H$$

where H is the completion of $C_0^\infty(\mathbb{R}^d)$ with respect to the energy norm $\mathcal{E}_1 = \mathcal{E} + (\cdot,\cdot)_{L^2(\mathbb{R}^d,\rho dx)}$. Since $(\mathcal{E}, H)$ on $L^2(\mathbb{R}^d, \rho dx)$ is a strongly local regular Dirichlet form, there exists an Hunt process starting from quasi-every point associated to it.

The choice of the domain H is very important to characterize the process. Very naturally, one could also take as domain the set

$$W := \left\{ u \in W^{1,1}_{loc}(\mathbb{R}^d) \ : \ \mathcal{E}_1(u,u) := \int_{\mathbb{R}^d} \big(|u|^2| + \nabla u|^2\big)\rho dx < \infty \right\}$$

where $W^{1,1}_{loc}(\mathbb{R}^d)$ is the classical space of locally integrable functions in $\mathbb{R}^d$, whose gradient in the sense of distributions ∇u belongs to $L^1_{loc}(\mathbb{R}^d)^d$. Observe that W is complete with respect to the energy norm and in particular $(\mathcal{E}, W)$ is a strongly local Dirichlet form on $\mathbb{R}^d$. Moreover the closure of $C_0^\infty(\mathbb{R}^d)$ in $(W, \mathcal{E}_1)$ is exactly H. The classical Sobolev space corresponds to $\rho = 1$ and in that case it is well known that $H = W$. However we learn from [2] that H = W is not true for a general weight ρ, i.e. smooth functions are not dense in W in general. Sufficient conditions for which this holds true are given in [2].

Since any Dirichlet form admits a regular representation, a natural question is to describe the stochastic process associated to the regular representation of $(\mathcal{E}, W)$ on $L^2(\mathbb{R}^d, \rho dx)$ and understand how it relates to the one associated to $(\mathcal{E}, H)$ on $L^2(\mathbb{R}^d, \rho dx)$.

In the talk we provide a very concrete example and a precise description of the two processes. Related research on this subject can be found in [1] and references therein. There, the authors deal with the problem of one-point symmetric extension of a Dirichlet form.

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Exploring Manifolds via Hamiltonian Dynamics

Thomas Laetsch

(joint work with Maria Gordina)

Let M be a connected, smooth, d-dimensional manifold without boundary. Let $\pi : T^*M \to M$ be the standard projection from the cotangent bundle onto the manifold $(\mathbf{x}, \mathbf{p}) \mapsto \pi(\mathbf{x}, \mathbf{p}) := \mathbf{x}$. Coordinates $\mathbf{x} = (x^1, ..., x^d) : U \to \mathbb{R}^d$ on some open subset $U \subset M$ induce coordinates $(\mathbf{x}, \mathbf{p}) = (x^1, ..., x^d, p_1, ..., p_d)$ on $\pi^{-1}(U)$ by expanding cotangent vectors within the $dx^1, ..., dx^d$ basis and prescribing $p_1, ..., p_d$ as the corresponding coefficients: $p_1 dx^1 + \cdots + p_d dx^d$. Hence T^*M is endowed with a canonical symplectic stucture with symplectic form $\omega = \sum_{k=1}^{d} dx^k \wedge dp_k$. The symplectic structure induces a bundle isomorphism $\underline{\omega} : T(T^*M) \to T^*(T^*M)$ by

$X \mapsto \underline{\omega}(X) := \omega(X, \cdot)$, and in turn, this map defines the *Hamiltonian vector field* X_h of a smooth map (Hamiltonian) $h : T^*M \to \mathbb{R}$ via $X_h := \underline{\omega}^{-1}(dh)$. Finally, an integral curve $\gamma(t) := (\mathbf{x}(t), \mathbf{p}(t))$ of X_h can be defined using *Hamilton's Equations*:

$$\begin{aligned} \dot{x}^i(t) &= \frac{\partial h}{\partial p_i}(\gamma(t)) \\ \dot{p}_i(t) &= -\frac{\partial h}{\partial x^i}(\gamma(t)) \end{aligned} \tag{1}$$

Smooth, symmetric bundle homomorphisms $\beta : T^*M \to TM$ with image $\mathcal{H} \subset TM$ are in one-to-one correspondence with smooth, symmetric maps $g : \bigcup_{\mathbf{x} \in M} \mathcal{H}_{\mathbf{x}} \times \mathcal{H}_{\mathbf{x}} \to \mathbb{R}$ where $g(X, Y) := \phi(\beta(\varphi))$ with $\beta(\phi) = X$ and $\beta(\varphi) = Y$. In the case that β is positive definite, g is a standard Riemannian metric, and β is represented as a matrix via the familiar index-raising maps g^{ij}. When β is positive semi-definite, then, depending on certain further desired restrictions on $\mathcal{H}$, g corresponds to a sub-Riemannian metric with $\mathcal{H}$ the horizontal bundle. Given a bundle homomorphism β, the canonical Hamiltonian is defined locally as

$$H(\mathbf{x}, \mathbf{p}) = \frac{1}{2} p_i \beta^{ij}_{\mathbf{x}} p_j \tag{2}$$

with $\beta^{ij}_{\mathbf{x}} = dx^i(\beta(dx^j))|_{\mathbf{x}}$. We recognize (2) as the familiar kinetic energy, one-half the momentum squared, and hence integral curves along the Hamiltonian vector field X_H correspond to paths of conserved energy.

For the Hamiltonian H in (2), let $(t, \mathbf{x}, \mathbf{p}) \mapsto \Phi_t(\mathbf{x}, \mathbf{p})$ be the *Hamiltonian flow*, where $t \mapsto \Phi_t(\mathbf{x}, \mathbf{p})$ is an integral curve along X_H starting at $(\mathbf{x}, \mathbf{p}) \in T^*M$. Restricting to the case where β is positive semi-definite and (M, g) is a Riemannian or sub-Riemannian manifold, we let $\mathcal{V} \subset TM$ be a choice of smooth *vertical bundle* so that $TM = \mathcal{H} \oplus \mathcal{V}$, and define $g^{\mathcal{V}}$ as a smooth, positive-definite extension of g to TM such that $\mathcal{V}$ is the perpendicular space to $\mathcal{H}$ under $g^{\mathcal{V}}$. This added structure induces the isomorphism $TM \to T^*M$ via $X \mapsto g^{\mathcal{V}}(X, \cdot)$, where we realize that $\beta(g^{\mathcal{V}}(X, \cdot))$ is the $g^{\mathcal{V}}$-orthogonal projection of X onto $\mathcal{H}$. For each $\mathbf{x} \in M$, let $\nu_{\mathbf{x}}$ be the rotationally invariant probability measure, supported on the g-unit sphere in $\mathcal{H}_{\mathbf{x}}$. We define the second order differential operator $\mathcal{L}^{\mathcal{V}}$ on bounded, smooth functions $f : T^*M \to \mathbb{R}$ as

$$\mathcal{L}^{\mathcal{V}} f(\mathbf{x}, \mathbf{p}) = \int_{\mathcal{H}_{\mathbf{x}}} \frac{d^2}{dt^2}\Big|_{t=0} f(\Phi_t(\mathbf{x}, g^{\mathcal{V}}(\mathbf{v}, \cdot)))\, \nu_{\mathbf{x}}(d\mathbf{v}). \tag{3}$$

It's notable that (3) becomes a scaled version of the Laplace-Beltrami operator on $C_b^\infty(M)$, the bounded and smooth functions on M, when M is a Riemannian manifold and we identify $f \in C_b^\infty(M)$ with its lift $f \circ \pi \in C_b^\infty(T^*M)$.

Relegating details to the referenced literature, we here describe an intuitive construction of a $\mathcal{V}$-dependent random walk on M using our above buildup of Hamiltonian dynamics. The culmination of this will be to advertise that under appropriate scaling, the semigroup of this random walk limits to the semigroup with generator $\mathcal{L}^{\mathcal{V}}$, and hence generalizes the case when M is Riemannian, $\mathcal{L}^{\mathcal{V}}$ is the Laplace-Beltrami operator, and thusly the limiting process is Brownian motion.

The development of the walk on M proceeds as follows: start at a point $\mathbf{x} \in M$; randomly and uniformly choose a unit-direction $\mathbf{v} \in \mathcal{H}_{\mathbf{x}}$; walk the process along the flow $t \mapsto \Phi_t(\mathbf{x}, g^{\mathcal{V}}(\mathbf{v}, \cdot))$ until some random exponential clock $\xi \sim \mathrm{Exp}(\epsilon)$ goes off; repeat this with $\mathbf{x} \leftarrow \pi(\Phi_\xi(\mathbf{x}, \mathbf{v}))$ and using a new, independent exponential clock at each iteration. Letting the semigroup of this process be denoted by T_t^ϵ, the following limit theorem holds on $C_b^\infty(M)$:

$$\lim_{\epsilon \to 0} T^\epsilon_{t/\epsilon^2} f = e^{t\mathcal{L}^{\mathcal{V}}} f \tag{4}$$

where the limit is under the sup-norm on $C_b^\infty(M)$.

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Looking for a metric in Liouville Quantum Gravity: overview and tools to address this problem

Christophe Garban

The following formal Riemannian metric tensor arises naturally in the context of Liouville Quantum Gravity:

$$e^{\gamma X}(dx^2 + dy^2)$$

where X is a well known and highly oscillating distribution on the plane called the "Gaussian Free Field". On the one hand, giving a proper mathematical sense to its associated volume form $e^{\gamma X} dxdy$ is now well understood since the work of Kahane in the 80's. On the other hand, extracting a Riemannian metric out of the above formal tensor for general values of γ remains a major challenge in this area. The main goal of this talk was to discuss in details this open problem in front of an audience which is likely to have the appropriate tools for this question. I started by introducing and motivating the problem. I then gave an overview of some tools that have been designed recently in order to analyse this question. The plan of the talk was as follows.

(1) Context: Liouville quantum gravity, KPZ formula etc.
(2) Gaussian Free Field
(3) Liouville measures $M_\gamma, \gamma \leq \gamma_c = 2$
  (a) Construction/existence
  (b) Regularity properties / thick points
(4) Liouville Brownian motion
  (a) Construction
  (b) Liouville heat kernel
(5) Liouville metric

(a) Miller-Sheffield breakthrough theorem in the case $\gamma = \sqrt{\frac{8}{3}}$
(b) Discussion when $\gamma \neq \sqrt{8/3}$

I ended the talk with the following two important warnings:

A. Lack of doubling property

Many other talks in the same workshop assumed a **volume doubling property** for the measures considered. It is important to point out here that it is a.s. not the case for the Liouville measures M_γ. This was quantified by the following result.

Theorem. (Berestycki, Garban , Rhodes, Vargas, 2014). *The following essentially tight upper bound holds*

$$M_\gamma(B(x,2r)) \leq C\, M_\gamma(B(x,r))^{1-\frac{\gamma^2}{4+\gamma^2}}$$

B. Questioning the form of the metric tensor

By analogy with the classical smooth Riemannian setting, all the works so far in this field have assumed that the following correspondance between metric tensor and volume form holds.

$$\boxed{e^{\gamma h} dxdy \quad \leftrightarrow e^{\gamma h}(dx^2+dy^2)}$$

I pointed out at the end of my presentation that in the fractal setting of M_γ, there is no reason to believe that this correspondance should still hold. I stated the following conjecture which predicts a different multiplicative exponent in front of the field X.

Conjecture. *The metric tensor corresponding to γ-Liouville Quantum Gravity corresponds to*

$$\boxed{e^{\frac{2\gamma}{\beta}h}(dx^2+dy^2)},$$

where $\beta = \beta(\gamma)$ stands for the Hausdorff dimension of the Liouville metric d_γ. (For example $\beta(\sqrt{8/3}) = 4$).

This conjecture can be heuristically justified as follows. Assume the appropriate metric tensor is given by $e^{\alpha h}(dx^2+dy^2)$ for some $\alpha > 0$ possibly $\neq \gamma$. It is then natural to expect that

$$\begin{aligned} M_\gamma(B(x,r)) &\asymp r^{2+\gamma^2/2} e^{\gamma h_r(x)} \\ &\asymp (e^{\frac{\alpha}{2} h_r(x)} r)^\beta \end{aligned}$$

which suggests $\beta \frac{\alpha}{2} = \gamma$

Remark. In particular, we conjecture that the metric introduced by Miller-Sheffield (built out of their QLE processes) does not correspond to the regularization of $e^{\sqrt{8/3}X_\epsilon}(dx^2+dy^2)$ as $\epsilon \to 0$, but rather to (since $\beta(\sqrt{8/3}) = 4$)

$$\epsilon^\alpha e^{\frac{1}{2}\sqrt{8/3}X_\epsilon}(dx^2+dy^2)$$

as the regularisation $\epsilon \to 0$ and where α is some unknown renormalization exponent which is chosen so that the "diameter" of $\epsilon^\alpha\, e^{\frac{1}{2}\sqrt{8/3}X_\epsilon}(dx^2 + dy^2)$ is tight when $\epsilon \to 0$.

On disconnection and level sets

Alain-Sol Sznitman

In this talk I reviewed several large deviation estimates obtained in part in collaboration with Xinyi Li (now at the University of Chicago) concerning the probability that in $\mathbb{Z}^d$, $d \geq 3$, a simple random walk starting form the origin, or random interlacements in the regime where the vacant set is percolative, or the sub-level set $\{\varphi < h\}$ of the Gaussian free field, in the regime where the super-level set $\{\varphi \geq h\}$ is percolative, disconnect a large box $B_N = \{x \in \mathbb{Z}^d,\ |x|_\infty \leq N\}$ from the boudnary of a larger concentric box ∂B_{MN}, $M > 1$. I discussed some of the links between these various questions.

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The cutoff phenomenon on random graphs

Nathanael Berestycki
(joint work with Eyal Lubetzky, Yuval Peres and Allan Sly)

The mixing time of a graph is a fundamental quantity which measures the time it takes for a random walk to reach its equilibrium distribution. More precisely, we set $d(t) = \sup_x \|P^t(x,\cdot) - \pi(\cdot)\|$ where $P^t(x,\cdot)$ denotes the heat kernel of random walk (in discrete time, say); $\|\cdot\|$ denotes total variation; and the sup is taken over all vertices x. The mixing time is the first time t such that $d(t)$ drops below level α, where $\alpha \in (0,1)$ is a predetermined threshold (typically one takes $\alpha = 1/4$ for convenience). A remarkable phenomenon conjectured to occur in a wide variety of examples is the *cutoff phenomenon*, where $d(t)$ drops abruptly from one to zero asymptotically as some parameter n tends to infinity.

In this work we study random walks on the giant component of the Erdős–Renyi random graph model $G(n,p)$ where $p = c/n$ for $c > 1$ fixed (so each potential edge in the complete graph is present with probability $p = c/n$, independently of all the other edges). In this regime it is known that there is a unique giant component whose size is a positive fraction of the total number of vertices, n.

The mixing time from the worst starting point on the giant component was shown by Fountoulakis and Reed, and independently by Benjamini, Kozma and Wormald, to have order $\log^2 n$, with convergence to equilibrium taking place gradually (no cutoff phenomenon). As shown in the second of these papers, this has to do with the geometry of the giant component, which consists of an expander (where mixing occurs quickly) to which decorations of logarithmic depth have been added. These spoil the mixing when started near the end of such decorations.

We prove that by contrast, when started from a uniformly chosen vertex in the giant component (equivalently, from a fixed vertex conditioned to belong to the giant component), then mixing occurs on a much faster scale, and moreover the cutoff phenomenon occurs: that is, convergence to equlibrium in the sense of total variation takes place abruptly. The mixing time is

$$t_{\text{mix}} = (1/vd)(\log n) \pm (\log n)^{1/2+o(1)},$$

where the constants v and d are respectively the speed of random walk and dimension of harmonic measure on a Poisson(c)-Galton-Watson tree (shown to exist by fundamental work of Lyons, Pemantle and Peres). Since d is known to be strictly smaller than the exponential growth rate of the tree, this implies that there is a range of times for which the random walk has reached its equilibrium distance from its starting point, but its distribution is far from equilibrium. This is in sharp contrast to the case of random regular graphs, where the cutoff was proved by Lubetzky and Sly (following a conjecture by Berestycki and Durrett) at a time where the random walk reaches its equilibrium distance.

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Contraction and regularization properties of the heat flow with respect to Hellinger, Kantorovich, and Hellinger-Kantorovich distances

Giuseppe Savaré
(joint work with G. Luise)

Since the pioneering contribution by F. Otto [15], contraction properties of second order diffusion equations with respect to the Kantorovich-Rubinstein-Wasserstein distance W_2 played an important role, from both the geometric and the analytic point of view. These aspects have been deeply studied by a series of contributions of Otto [15] (nonlinear diffusion), Otto-Villani [16] (heat flow and Ricci curvature), Carrillo-McCann-Villani [6] (contraction of a general class of evolution equations combining diffusion, interaction and drift), Ambrosio-Gigli-S. [1] (gradient flows and geodesic convexity in $\mathbb{R}^d$), Otto-Westdickenberg [17] (the Eulerian approach), Sturm-Von Renesse [18] (equivalence of contraction with lower Ricci bounds), Daneri-S. [8] (the Eulerian approach to contraction and geodesic convexity), Erbar '10 [9], Villani [19] (geodesic convexity in Riemannian manifold), Ambrosio-S.-Zambotti [4] (Hilbert spaces), Kuwada [12] (duality with gradient estimates), Gigli-Kuwada-Ohta [11] (Alexandrov spaces), Ambrosio-Gigli-S. [2] (RCD metric measure spaces and Bakry-Émery condition), Bakry-Gentil-Ledoux [5], Erbar-Kuwada-Sturm [10] Ambrosio-Mondino-S. [3] (refined contraction for finite dimensional RCD spaces).

Perhaps one of the most general formulation concerns the mass preserving Markov semigroup $(\mathsf{P}_t)_{t\geq 0}$ associated to a strongly local symmetric Dirichlet form $\mathcal{E}$ on $L^2(X,\mathfrak{m})$ admitting a Carré du Champ Γ: for every $u_0 \in L^2(X,\mathfrak{m})$ the curve $u_t := \mathsf{P}_t u_0$ is a solution of the differential equation

$$\partial_t u_t = L u_t$$

where $L : D(L) \subset L^2(X,\mathfrak{m}) \to L^2(X,\mathfrak{m})$ is the selfadjoint operator induced by $\mathcal{E}$. For the sake of simplicity, we assume here that $\mathfrak{m}$ is a finite Borel measure on the complete and separable metric space (X,d) and Γ is compatible with d, in the sense that every function $u \in D(\mathcal{E})$ with $\Gamma(u) \leq 1$ $\mathfrak{m}$-a.e. admits a d-continuous representative (still denoted by u) and

$$\mathsf{d}(x,y) := \sup\Big\{u(x) - u(y) : u \in D(\mathcal{E}),\ \Gamma(u) \leq 1\Big\}.$$

In this case, L satisfies (a suitable weak formulation of) the Bakry-Émery condition $\mathrm{BE}(K,\infty)$, $K \in \mathbb{R}$,

$$\Gamma_2(u) = \frac{1}{2} L\Gamma(u) - \Gamma(u, Lu) \geq K\,\Gamma(u) \tag{1}$$

if and only if $(\mathsf{P}_t)_{t\geq 0}$ admits a (unique) extension to the space of finite Borel measures $\mathcal{M}(X)$ satisfying the contraction property

$$\mathrm{W}_2(\mathsf{P}_t\mu_0, \mathsf{P}_t\mu_1) \leq \mathrm{e}^{-Kt}\mathrm{W}_2(\mu_0,\mu_1) \quad \text{for every } \mu_0,\mu_1 \in \mathcal{M}(X),\ \mu_0(X) = \nu_0(X).$$

Such an equivalence is strictly related to a few basic facts: the identification of $\Gamma(u)$ with the squared weak gradient $|\mathrm{D}u|_w^2$ of the metric-Sobolev space $W^{1,2}(X,\mathsf{d},\mathfrak{m})$, the duality formula expressing the distance W_2 in terms of regular subsolutions $\zeta \in \mathrm{C}^1([0,1];\mathrm{Lip}_b(X))$ to the Hamilton-Jacobi equation

$$\frac{1}{2}\mathrm{W}_2^2(\mu_0,\mu_1) = \sup\Big\{ \int \zeta_1 \,\mathrm{d}\mu_1 - \int \zeta_0 \,\mathrm{d}\mu_0 : \partial_t\zeta_t + \frac{1}{2}|\mathrm{D}\zeta_t|^2 \le 0\Big\},$$

and the pointwise gradient estimate (in fact equivalent to (1))

$$\Gamma(\mathsf{P}_t u) \le \mathrm{e}^{-2Kt}\mathsf{P}_t\Gamma(u).$$

A similar approach can be used to obtain new contraction and regularization estimates involving other interesting distances. A first example is provided by the Hellinger-Kakutani distance

$$\mathrm{H}^2(\mu_0,\mu_1) := \int \Big(\sqrt{\varrho_1} - \sqrt{\varrho_0}\Big)^2 \,\mathrm{d}\mu, \quad \mu_i = \varrho_i\mu,$$

which can also be characterized by the dynamic duality formula

$$\mathrm{H}^2(\mu_0,\mu_1) = \sup\Big\{ \int \zeta_1 \,\mathrm{d}\mu_1 - \int \zeta_0 \,\mathrm{d}\mu_0 : \partial_t\zeta_t + \zeta_t^2 \le 0\Big\}.$$

It is then possible to show (Luise-S., in preparation) that for every couple of finite Borel measures $\mu_0,\mu_1 \in \mathcal{M}(X)$ absolutely continuous w.r.t. $\mathfrak{m}$ and for every Markov semigroup $(\mathsf{P}_t)_{t\ge 0}$ (without any curvature assumption)

$$\mathrm{H}(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1) \le \mathrm{H}(\mu_0,\mu_1).$$

A more refined estimate involves the recently introduced Hellinger-Kantorovich distance HK [7, 14, 13], which can be defined in terms of an Optimal Entropy–Transport problem [13]

$$\mathrm{HK}^2(\mu_0,\mu_1) := \min_{\boldsymbol{\gamma}\in\mathcal{M}(X\times X)} \mathrm{Ent}(\gamma_0|\mu_0) + \mathrm{Ent}(\gamma_1|\mu_1) + \int_{X\times X} \ell(x_0,x_1)\,\mathrm{d}\boldsymbol{\gamma},$$

where γ_0,γ_1 are the marginals of $\boldsymbol{\gamma}$, Ent is the logarithmic entropy functional

$$\mathrm{Ent}(\gamma|\mu) := \int_X \Big(\varrho\log\varrho - \varrho + 1\Big)\,\mathrm{d}\mu, \quad \gamma = \varrho\mu \ll \mu,$$

and ℓ is the cost function

$$\ell(x_0,x_1) := \begin{cases} \log\big(1+\tan^2(\mathsf{d}(x_0,x_1))\big) & \text{if } \mathsf{d}(x_0,x_1) < \pi/2,\\ +\infty & \text{otherwise.}\end{cases}$$

It turns out that HK admits a dual dynamic representation formula [13]

$$\frac{1}{2}\mathrm{HK}^2(\mu_0,\mu_1) = \sup\Big\{ \int \zeta_1 \,\mathrm{d}\mu_1 - \int \zeta_0 \,\mathrm{d}\mu_0 : \partial_t\zeta_t + \frac{1}{2}|\mathrm{D}\zeta_t|^2 + 2\zeta_t^2 \le 0\Big\},$$

so that when the Bakry-Émery condition $\mathrm{BE}(0,\infty)$ holds one has [13]

$$\mathrm{HK}(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1) \le \mathrm{HK}(\mu_0,\mu_1) \quad \text{for every } \mu_0,\mu_1 \in \mathcal{M}(X).$$

Actually, the stronger Hellinger distance at time $t > 0$ can be estimated in terms of the weaker Hellinger-Kantorovich one: for every $t > 0$ (Luise-S.)

$$\mathrm{H}(\mathsf{P}_t\mu_0,\mathsf{P}_t\mu_1) \le \frac{c}{\sqrt{t}}\mathrm{HK}(\mu_0,\mu_1) \quad \text{for every } \mu_0,\mu_1 \in \mathcal{M}(X). \tag{2}$$

Differently from other well known properties, estimate (2) cannot be deduced by a regularization effect on a single initial datum, since H and HK are not translation invariant. In this respect, the dual dynamic approach plays a crucial role.

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Heat equation on time-dependent metric measure spaces

Eva Kopfer
(joint work with Karl-Theodor Sturm)

Let (X, d_t, m_t) be a family of metric measure space, where $t \in [0, T]$, X is a Polish space and

(1) d_t is a geodesic metric such that

$$|\log(d_t(x, y)/d_s(x, y))| \leq L|t - s|,$$

(2) $m_t = e^{-f_t}m$, $m \in \mathcal{P}(X)$, $f \in \mathrm{Lip}([0, T] \times X)$.

For every $t \in [0, T]$ we define the Cheeger's energy $\mathrm{Ch}_t \colon L^2(X, m_t) \to [0, +\infty]$ by

$$\mathrm{Ch}_t(u) := \frac{1}{2} \inf \left\{ \liminf \int (\mathrm{lip}_t u_n)^2 \, dm_t | u_n \in \mathrm{Lip}(X), u_n \to u \text{ in } L^2(X) \right\}.$$

Under the assumption that each (X, d_t, m_t) satisfies $\mathrm{RCD}(K, N)$ we have $2\mathrm{Ch}_t(u) = \mathcal{E}_t(u)$, where $\mathcal{E}_t(u) = \int \Gamma_t(u) \, dm_t$ is the strongly local Dirichlet form with self-adjoint operator $\Delta_t \colon Dom(\Delta_t) \to L^2(X)$

$$-\int \Delta_t uv \, dm_t = \mathcal{E}_t(u, v) \qquad \forall u \in Dom(\Delta_t), v \in Dom(\mathcal{E}_t).$$

Theorem 1 ([2])**.** *There exists a kernel $p_{t,s}(x, y)$ such that*

(1) *given $\bar{u} \in L^2(X)$*

$$(t, x) \mapsto P_{t,s}\bar{u}(x) := \int p_{t,s}(x, y)\bar{u}(y) \, dm_s(y)$$

is the unique solution to the heat equation

$$\partial_t u_t = \Delta_t u_t \qquad \text{on } (s, T) \times X$$

with $u_s = \bar{u}$;

(2) *given $\bar{v} \in L^2(X)$*

$$(s, y) \mapsto P_{t,s}\bar{v}(y) := \int p_{t,s}(x, y)\bar{v}(x) \, dm_t(x)$$

is the unique solution to the adjoint heat equation

$$\partial_s v_s = -\Delta_s v_s + (\partial_s f_s)v_s \qquad \text{on } (0, t) \times X$$

with $v_t = \bar{v}$.

Both flows admit a gradient flow interpretation. The heat equation can be interpreted as the gradient flow of the time-dependent Cheeger's energy, and the adjoint heat equation as the "upward" gradient flow of the time-dependent Boltzmann entropy $S_t(\rho m_t) = \int \rho \log \rho \, dm_t$, see [3].

We define the dual heat flow $\hat{P}_{t,s}\mu \colon \mathcal{P}(X) \to \mathcal{P}(X)$ by

$$\int u \, d\hat{P}_{t,s}\mu = \int P_{t,s}u \, d\mu.$$

EVI-formulation of the dual heat flow. We introduce a kind of dynamic version of L^2-Kantorovich distance by

$$W_{s,t}^2(\mu_0,\mu_1) := 2\sup\left\{\int \varphi_1\, d\mu_1 - \int \varphi_0\, d\mu_0\right\},$$

where the supremum runs over all subsolutions $\varphi \in \mathrm{Lip}_b([0,1]\times X)$ to the "Hamilton-Jacobi equation" $\partial_a\varphi \le -\frac{1}{2}\Gamma_{s+a(t-s)}\varphi$. It is important to note that this is not a distance. Let W_s denote the L^2-Kantorovich distance with respect to d_t. Then $W_{s,t} = W_s$ if $s = t$ and $W_{s,t}(\mu,\mu) = 0$. The next theorem can be thought of as a time-dependent variant of "Ric ≥ 0".

Theorem 2 ([2]). *The following are equivalent.*

(1) $\forall t \in (0,T)$ *and every* W_t*-geodesic* $(\mu^a)_{a\in[0,1]}$ *with* $\mu^0,\mu^1 \in Dom(S)$

$$\partial_a^+ S_t(\mu^a)\Big|_{a=1-} + \partial_a^- S_t(\mu^a)\Big|_{a=0+} \ge -\frac{1}{2}\partial_t^- W_{t-}^2(\mu^0,\mu^1)$$

(2) $\forall\, 0 \le s \le t \le T$, $\mu,\nu \in \mathcal{P}(X)$

$$W_s(\hat{P}_{t,s}\mu, \hat{P}_{t,s}\nu) \le W_t(\mu,\nu)$$

(3) $\forall\, 0 \le s \le t \le T$, $\forall\, u \in Dom(\mathcal{E})$

$$\Gamma_t(P_{t,s}u) \le P_{t,s}\Gamma_s(u)$$

(4) $\forall\, 0 \le s \le t \le T$ *we have for all* u *"*$\Gamma_{2,t}(u) \ge \partial_t\Gamma_t(u)$*"*

Theorem 3 ([2]). *If one of the assertions of the previous theorem holds, we have* $\forall \sigma \in Dom(S)$ *and* $t \le \tau$

$$\frac{1}{2}\partial_s^- W_{s,t}^2(\mu_s,\sigma)\Big|_{s=t-} \ge S_t(\mu_t) - S_t(\sigma),$$

where $\tau < T$ *and* $\mu_t = \hat{P}_{\tau,t}\mu$.

Sketch of Proof. The proof basically follows the idea by Ambrosio, Gigli and Savaré in [1]. We suppose that (3) holds. Let $(\rho_a)_{a\in[0,1]}$ be the W_t-geodesic connecting μ_t with σ. Define the linear interpolation $\vartheta(a) = s + a(t-s)$. Consider $\rho_{a,\vartheta} = \hat{P}_{t,\vartheta(a)}(\rho_a)$. Then $\rho_{a,\vartheta}$ connects μ_s with σ. We obtain then using (3) after various cancellations

$$\begin{aligned}
&\frac{1}{2}W_{s,t}^2(\mu_s,\sigma) - (t-s)(S_t(\sigma) - S_s(\mu_s))\\
&\overset{(3)}{\le}\frac{1}{2}\int |\dot\rho_a|_t^2\, da - (t-s)^2\int_0^1\int \dot f_{\vartheta(a)}\, d\rho_{a,\vartheta}\, da\\
&=\frac{1}{2}W_t^2(\mu_t,\sigma) - (t-s)^2\int_0^1\int \dot f_{\vartheta(a)}\, d\rho_{a,\vartheta}\, da.
\end{aligned}$$

Then dividing by $(t-s)$ and letting $s \to t$, we obtain the result. $\square$

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Variational principles for discrete random maps

Georg Menz

(joint work with Martin Tassy)

In the talk we present the results of the preprint [MT16] where a new robust technique is developed to deduce variance principles for non-integrable discrete systems. To illustrate this technique we show the existence of a variational principle for graph homomorphisms from $\mathbb{Z}^m$ to a d-regular tree. This seems to be the first non-trivial example of a variational principle in a non-integrable model. Instead of relying on integrability the technique is based on a discrete Kirszbraun theorem and a concentration inequality obtained through the dynamic of the model.

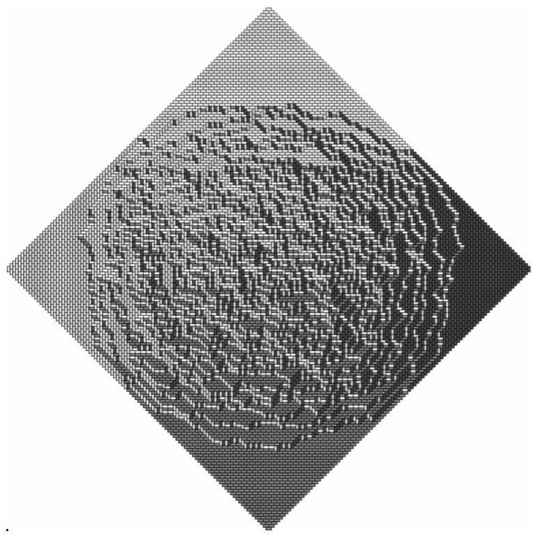

Figure 1. An Aztec diamond for domino tilings. The combinatorics of the model is similar to Lipschitz functions from $\mathbb{Z}^2$ to $\mathbb{Z}$. (see [CKP01])

The appearance of limit shapes as a limiting behavior of discrete systems is a well-known and studied phenomenon in statistical physics and combinatorics (e.g. [Geo88]). Among others, models that exhibits limits shapes are domino tilings and dimer models (e.g. [Kas63, CEP96, CKP01] and see Figure 1), polymer models, lozenge and ribbon tilings (e.g. [LRS01, Wil04], also see Figure 2), Gibbs models (e.g. [She05]), the Ising model (e.g. [DKS92, Cer06]), asymmetric exclusion processes (e.g. [FS06]), sandpile models (e.g.[LP08]), the Young tableaux (e.g. [LS77, VK77, PR07]) and many more.

Limit shapes appear whenever fixed boundary conditions force a certain response of the system. The main tool to explain those shapes is a variational principle. The variational principle asymptotically characterizes the number of microscopic states, i.e. the microscopic entropy Ent_n, via a variational problem. This means that for large system sizes n, the entropy of the system is given by maximizing a macroscopic entropy $\mathrm{Ent}(f)$ over all admissible limiting profile $f \in$ '$\mathcal{A}$. The boundary conditions are usually incorporated in the admissibility condition. In formulas, the variational principle can be expressed as (see for example Theorem 2.7 in [MT16])

$$\mathrm{Ent}_n \approx \inf_{f\in\mathcal{A}} \mathrm{Ent}(f), \tag{1}$$

where the macroscopic entropy

$$E(f) = \int \mathrm{ent}(\nabla f(x))dx$$

can be calculated via a local quantity $\mathrm{ent}(\nabla f(x))$. This local quantity is called local surface tension in this article.

Often, a simple consequence of those variational principles is that the uniform measure on the microscopic configurations, concentrates around configurations that are close to the minimizer of the variational problem. This explains the appearance of limit shapes on large scales. In analogy to classical probability theory, one can understand the variational principle as an elaborated version of the law of large numbers. On large scales, the behavior of the system is determined by a deterministic quantity, namely the minimizer f of the macroscopic entropy. Hence, deriving a variational principle is often the first step in analyzing discrete models, before studying other questions like the fluctuations of the model.

As a motivating example serves the variational principle of domino tilings [CKP01] (see Figure 1). It was the first variational principle for two-dimensional random maps. It is one of the fundamental results for studying domino tilings and the other integrable discrete models. A detail analysis of the limit shapes for domino tilings was given in [KOS06]. So far, all the tools that were developed to study variational principles of discrete models rely on the integrability of the model. Up to the knowledge of the authors, there is no non-trivial example of a variational principle for which the underlying model is not integrable. However, simulations as the ones in Figure 2 and Figure 3 show that those limit shapes still appear for a large class of non-integrable models. Limit shapes appear to be a universal phenomenon. The purpose of our study is to go beyond integrability and to find out what properties of a discrete system lead to variational principles and limit shapes.

We deduce the variational principle for the non-integrable model of graph homomorphisms form $\mathbb{Z}^m$ to a d-regular tree (see Figure 3). We choose this model because of its central role among graph homomorphisms, which stems from the

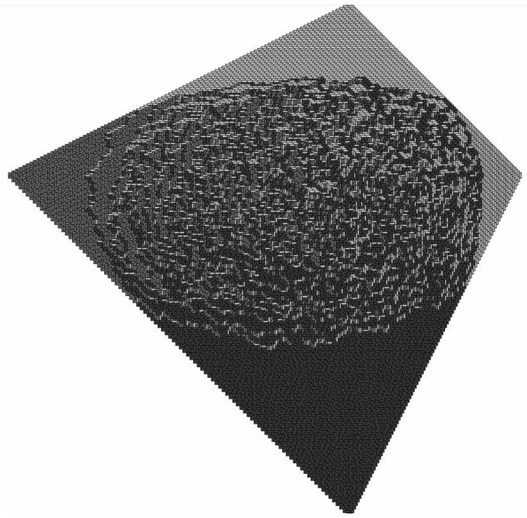

FIGURE 2. An Aztec diamond for ribbon tilings. The combinatorics of the model is similar to Lipschitz functions from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ (see [She02]).

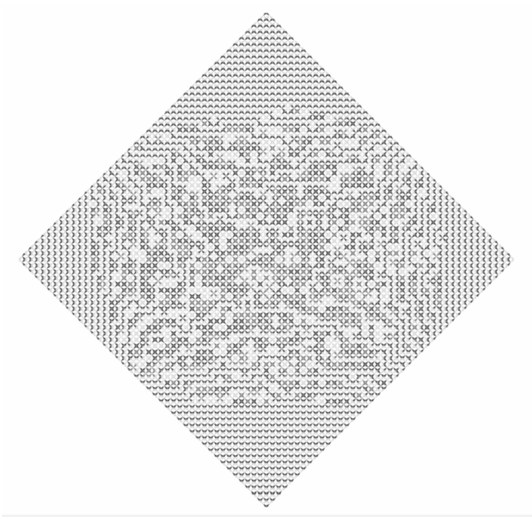

FIGURE 3. An Aztec diamond for Graph homomorphisms in a 3-regular tree. Each color represent one of the α_i's introduced in Section 3.

fact that regular trees are the universal cover of d-regular graph with no four cycle. Hence, it provides valuable information for those systems. One should also note that the underlying lattice can have arbitrary dimension $m \geq 1$. We identified two properties that a model of discrete maps needs to have in order to have a variational principle: The first one is a stability property i.e. the Kirszbraun theorem. It means that changes of the boundary condition on a microscopic scale do not change the macroscopic properties of the model. The second one is a concentration property which is natural because a variational principle is a type of law of large numbers.

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Random walk on dynamical percolation

Perla Sousi

(joint work with Yuval Peres, Jeff Steif)

We study random walk on dynamical percolation on $\mathbb{Z}_n^d$. The edges refresh at rate $\mu \leq 1$ and switch to open with probability p and closed with probability $1-p$ where $p > p_c(\mathbb{Z}^d)$ with $p_c(\mathbb{Z}^d)$ the critical probability for bond percolation on $\mathbb{Z}^d$. The random walk X moves at rate 1. When his exponential clock rings, the walk chooses one of the $2d$ adjacent edges with equal probability. If the bond is open, then it makes the jump, otherwise it stays in place. We call η_t the configuration of the edges at time t, i.e. $\eta_t \in \{0,1\}^{E(Z_n^d)}$.

We study the mixing time for the Markov chain described. We will be concerned with the quenched mixing time. We start by defining the different notions of mixing that we will be using. First of all we write $\mathbb{P}_{x,\eta}(\cdot)$ for the probability measure of the walk, when the environment process is conditioned to be $\eta = (\eta_t)_{t\geq 0}$ and the walk starts from x. We write $\mathcal{P}$ for the distribution of the environment which is dynamical percolation on the torus, a measure on càdlàg paths $[0,\infty) \mapsto \{0,1\}^{E(\mathbb{Z}_n^d)}$, where $E(\mathbb{Z}_n^d)$ stands for the edges of the torus. We write $\mathcal{P}_{\eta_0}$ to denote the measure $\mathcal{P}$ when the starting environment is η_0.

This process was introduced by Peres, Stauffer and Steif. They focused on the subcritical regime $p < p_c$ of the dynamical percolation. They proved

Theorem 1 (Peres, Stauffer and Steif)**.** *For all $p < p_c$ the mixing time of the process (X,η) satisfies*

$$t_{\text{mix}} \asymp \frac{n^2}{\mu}.$$

The upper bound of the above result was established using coupling. A crucial ingredient of the proof was to define the so-called regeneration times, which relied heavily on the fact that the process was subcritical. For the lower bound, the proof used the so-called Markov type property of metric spaces.

For the supercritical regime, using Markov type they established a lower bound of order $n^2 + 1/\mu$. The question that remained open is to find a matching upper bound. In our work, we do this in both the quenched and the annealed setting.

For $\epsilon \in (0,1)$, $x \in \mathbb{Z}_n^d$ and a fixed environment $\eta = (\eta_t)_{t\geq 0}$ we write $t_{\mathrm{mix}}(\epsilon, x, \eta)$ to denote

$$t_{\mathrm{mix}}(\epsilon, x, \eta) = \min\left\{t \geq 0 : \|\mathbb{P}_{x,\eta}(X_t = \cdot) - \pi\|_{\mathrm{TV}} \leq \epsilon\right\},$$

where π is the uniform measure on $\mathbb{Z}_n^d$. We also write

$$t_{\mathrm{mix}}(\epsilon, \eta) = \max_x t_{\mathrm{mix}}(\epsilon, x, \eta).$$

Theorem 2. *Let $p > p_c(\mathbb{Z}^d)$ with $\theta(p) > 1/2$. Then there exists $a > 0$ so that for all $\epsilon > 0$ for all n sufficiently large and all starting environments η_0 we have as $n \to \infty$*

$$\mathcal{P}_{\eta_0}\left(\eta = (\eta_t)_{t\geq 0} : t_{\mathrm{mix}}(\epsilon, \eta) \leq (\log n)^a \left(n^2 + \frac{1}{\mu}\right)\right) = 1 - o(1).$$

Our second main result concerns Cesaro mixing in the quenched regime for all values of $p > p_c(\mathbb{Z}^d)$. First we recall the definition of Cesaro mixing. For every t let U_t be a uniform random variable on $\{1, \ldots, t\}$ independent of the chain. Then we define

$$t_{\mathrm{Ces}}(\epsilon, \eta) = \min\left\{t \geq 0 : \max_x \|\mathbb{P}_{x,\eta}(X_{U_t} = \cdot) - \pi\|_{\mathrm{TV}} \leq \epsilon\right\}.$$

Theorem 3. *Let $p > p_c(\mathbb{Z}^d)$. Then there exists $a > 0$ so that for all $\epsilon > 0$ and all n sufficiently large and all starting environments η_0 we have*

$$\mathcal{P}_{\eta_0}\left(\eta = (\eta_t)_{t\geq 0} : t_{\mathrm{Ces}}(\epsilon, \eta) \leq (\log n)^a \left(n^2 + \frac{1}{\mu}\right)\right) \geq 1 - \epsilon.$$

As a corollary of the Theorem above we get a bound on the mixing time for the chain (X, η). We write t_{mix} for its mixing time.

Corollary 4. *Let $p > p_c(\mathbb{Z}^d)$. Then there exists $a > 0$ so that for all $\epsilon > 0$ and all n sufficiently large*

$$t_{\mathrm{mix}}(\epsilon) \leq (\log n)^a \left(n^2 + \frac{1}{\mu}\right).$$

We now explain the main ideas behind the proofs. First we note that when we fix the environment to be η, we obtain a time inhomogeneous Markov chain. To study its mixing time, we use the theory of evolving sets developed by Morris and Peres [1] adapted to the inhomogeneous setting. In particular, a beautiful coupling due to Diaconis and Fill transfers to this setting. This coupling is going to be crucial for us in the proofs of Theorems 2 and 3. What it says is that conditional on the Doob transform of the evolving set up to time t, the random walk at time t is uniform on it.

The Doob transform of the evolving set in the inhomogeneous setting is again a submartingale, just like in the homogeneous one. The crucial quantity we want to control is by how much its size increases. This increase will be large only at *good times*, i.e. when the intersection of the Doob transform of the evolving set with the giant cluster is a substantial proportion of the evolving set. Hence we want to ensure that there are enough *good times*. We achieve this using the coupling of the walk with the evolving set together with a result of Gabor Pete [3] who established that the isoperimetric profile of the giant cluster in supercritical percolation coincides with its lattice profile.

We conclude by showing that there exists a stopping time bounded by the mixing time with high probability so that at this time the Doob transform of the evolving set has size at least $(1-\delta)(\theta(p)-\delta)n^d$. In the case when $\theta(p) > 1/2$ we can take $\delta > 0$ sufficiently small so that $(1-\delta)(\theta(p)-\delta) > 1/2$. Using the uniformity of the walk on the Doob transform of the evolving set again, we deduce that at this stopping time the walk is close to the uniform distribution in total variation with high probability.

To finish the proof of Theorem 3 the idea is to repeat the above procedure to obtain k sets whose union covers at least $1-\delta$ of the whole space. Then define τ by choosing one of these times uniformly at random. At time τ the random walk will be uniform on a set with measure at least $1-\delta$, and hence this means that the total variation from the uniform distribution at this time is going to be small. Since this time is with high probability smaller than k times the mixing time, this implies the Cesaro mixing time bound.

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On Li-Yau inequalities on graphs

Moritz Kassmann

(joint work with Dominik Dier, Rico Zacher)

The aim of the talk was to present a new approach to differential Harnack inequalities on graphs. The presentation was based on [3].

1. Introduction

The classical gradient estimate given by Li-Yau [4] holds true for positive solutions $u : [0,\infty) \times M \to (0,\infty)$ of the heat equation $\partial_t u - \Delta u = 0$ on a complete d-dimensional Riemannian manifold M with $Ric(M) \geq 0$: For every $t \in (0,\infty)$ and $x \in M$

$$\frac{|\nabla u(t,x)|^2}{u^2(t,x)} - \frac{\partial_t u(t,x)}{u(t,x)} \leq \frac{d}{2t} \tag{1}$$

or, equivalently,

$$|\nabla \log u(t,x)|^2 - \partial_t(\log u)(t,x) \leq \frac{d}{2t}. \tag{2}$$

An important consequence of this estimate is a pointwise bound on the solution itself, which can be obtained from integration over a path that connects two given points (t_1, x_1) and (t_2, x_2) with $t_2 > t_1$:

$$u(t_1, x_1) \leq \left(\frac{t_2}{t_1}\right)^{d/2} u(t_2, x_2) \exp\left(\frac{r^2(x_1, x_2)}{4(t_2 - t_1)}\right). \tag{3}$$

Note that estimates (1), (2), and (3) are sharp in the sense that corresponding equalities hold true for the fundamental solution to the heat equation on $\mathbb{R}^d$, i.e., if $u(t,x)$ equals $(4\pi t)^{-d/2} \exp\left(\frac{-|x|^2}{4t}\right)$.

The aim of the current project is to study estimates of the type of (1), (2), and (3) for positive solutions to the heat equation on graphs. In order to establish a corresponding theory, we establish new computation rules for functions defined on discrete spaces. Furthermore, we provide a condition for graphs, which serves as a substitute for the assumption that the Ricci-curvature is nonnegative.

Let $G = (V, E)$ be a locally finite graph with weights $\omega_{xy} > 0$. Let $\mu : V \to (0, \infty)$. The Laplace operator on G maps functions $v \in \mathbb{R}^V$ to $\Delta v \in \mathbb{R}^V$ as follows:

$$\Delta v(x) = \frac{1}{\mu(x)} \sum_{y: y \sim x} (v(y) - v(x))\, \omega_{xy}$$

As it is usual, we define the carré du champ-operator $\Gamma : \mathbb{R}^V \times \mathbb{R}^V \to \mathbb{R}^V$ as follows: $2\Gamma(v, w) = \Delta(uv) - v\Delta w - w\Delta v$.

One approach to Li-Yau type estimates on graphs is given in [1] and related subsequent works. The authors establish the following estimate

$$\frac{\Gamma(\sqrt{u})(t,x)}{u(t,x)} - \frac{\partial_t(\sqrt{u})(t,x)}{u(t,x)} \leq \frac{n}{2t} \quad (t > 0, x \in V) \tag{4}$$

for positive solutions u to the heat equation on G. The graph G is assumed to satisfy a so-called exponential curvature dimension inequality $CDE(n, 0)$. The significance of this assumption and alternative conditions are investigated in [1] and other works. As a consequence of (4), the authors obtain a Harnack inequality

$$u(t_1, x_1) \leq u(t_2, x_2) \left(\frac{t_2}{t_1}\right)^n \exp\left(\frac{4Dr^2(x_1, y_1)}{t_2 - t_1}\right) \quad (0 < t_1 < t_2, x_i \in V), \tag{5}$$

where D equals the maximal degree of a vertex in G. Note that $\mathbb{Z}^d$ satisfies $CDE(n, 0)$ with $n = 2d$. We prove an estimate like (5) with a more general expression in place of $(t_2/t_1)^n$. The result is optimal for some graphs. Note that [5] contains a different approach to Li-Yau estimates on (finite) graphs with advancements over [1] in some cases.

2. Results

Assume that $G = (V, E)$ is a locally finite graph and Δ is the Laplace operator as explained above. In order to work with the logarithm of solutions to the heat equation, we need a new computation rule.

Definition 1:

(i) Given $H : \mathbb{R} \to \mathbb{R}$, we define $\Psi_H : \mathbb{R}^V \to \mathbb{R}^V$ by

$$\Psi_H(u)(x) = \frac{1}{\mu(x)} \sum_{y:y\sim x} H(u(y) - u(x))\omega_{xy}$$

(ii) We will need the function $\Upsilon : \mathbb{R} \to \mathbb{R}$,

$$\Upsilon(z) = \exp(z) - z - 1.$$

Lemma 2: *Let u be a positive solution of $\partial_t u - \Delta u = 0$ on G and set $v = \log u$. Then v satisfies $\partial_t v - \Delta v = \Psi_\Upsilon(v)$.*

Proof: The assertion follows from the chain rule:

$$\Delta(\log u) = \frac{1}{u}\Delta u - \Psi_\Upsilon(\log u).$$

Working with the logarithm of positive solutions rather than with the square root as in [1] is one achievement of our approach. Another one concerns the functions $t \to \frac{d}{2t}$ resp. $t \to \frac{n}{2t}$ in (1), (2) resp. in (4). We choose a function for each class of graphs separately. As we discuss below, this choice is optimal for some graphs. Note that, for finite graphs, it is desirable to consider a function which is integrable at $t = 0$.

We call a continuous function $F : [0, \infty) \to [0, \infty)$ a CD-function if $F(0) = 0, F(x)/x$ is strictly increasing on $(0, \infty)$, and if $1/F$ is integrable at ∞. A model case is given by $F(x) = x^2$.

Lemma 3: *If F is a CD-function, then there is a unique positive solution φ of*

$$\varphi'(t) + F(\varphi(t)) = 0 \quad (0 < t < \infty) \tag{6}$$

with $\varphi(0+) = \infty$. The function φ is strictly decreasing and log-*convex. Moreover, it satisfies $\varphi(\infty) = 0$. A model case is $\varphi(t) = \frac{1}{t}$.*

Now we can formulate our main assumption and a first result.

Definition 4: The graph G satisfies the condition $CD(x, F, 0)$ at vertex $x \in V$, if for every function $v \in \mathbb{R}^V$ with

$$-\Delta v(x) > 0, \text{ and } -\Delta v(x) \geq -\Delta v(y) \text{ for every } y \sim x,$$

the following estimate holds true:

$$\Delta\Psi_{\Upsilon'}(v)(x) \geq F(-\Delta v(x)).$$

We say that the graph G satisfies $CD(F, 0)$ if it satisfies $CD(x, F, 0)$ for every $x \in V$.

In the definition above, we use the notation "CD" because the condition has a relation to curvature dimension inequalities. On the first hand, it does not look easy to check $CD(F, 0)$ for a given graph. In fact, this is possible in many cases.

The unweighted two-point graph satisfies $CD(F,0)$ with $F(a) = 2\sinh(a)$. The function φ from Lemma 3 then is $\varphi(t) = -\log(\tanh t)$. We are able to show that every regular unweighted Ricci-flat graph in the sense of [2] satisfies $CD(F,0)$ and F can be computed explicitly. Here is our main result in the case of finite graphs:

Theorem 5: *Let G satisfy $CD(F,0)$ and $\varphi : (0,\infty) \to (0,\infty)$ be associated with F via (6). Suppose $u : [0,\infty) \times V \to (0,\infty)$ solves $\partial_t u - \Delta u = 0$ on G. Then*

$$\Psi_\Upsilon(\log u)(t,x) - \partial_t(\log u)(t,x) \leq \varphi(t)$$

for $t > 0, x \in V$.

In the forthcoming article [3] we explain implications and extensions of this result. Most important, the pathwise integration from [1] can be applied to obtain a Harnack estimate. We provide examples, for which our approach leads to sharp results. Theorem 5 can be extended to infinite graphs. Though, the localization procedure does not seem very simple. We discuss the condition $CD(F,0)$ in detail.

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Brownian motion on graph-like metric spaces and the cover time bound

Anita Winter

In this talk we consider a class of Feller processes with compact metric state space (X, r), which are symmetric with respect to a finite measure ν of full support, and which are assumed to admit local times. We interested in a condition under which these processes have finite cover time.

To answer this question we merge two complementary type of results. First, in [5] it is shown for simple random walk on a finite, connected graph that the cover time equals (up to a constant which does not depend on the size of the graph) the square of the mean maximum of the associated discrete Gaussian free field. Secondly, in [2] and [4] it has been established for tree- and graph-like metric spaces (resistance networks) that stochastic processes converge weakly in path space if and only if the associated metric measure spaces converge Gromov-Hausdorff-weakly. Moreover, in [1] and [6], Feller processes were associated with a given compact metric measure space. These processes satisfy the above property and can be considered as the extension of simple random walk on finite graphs to Brownian motion on graph-like metric spaces.

Our main result shows that the techniques presented in [5] can be extended to our class of Feller processes. In particular, it states that the cover time is finite if the majorizing measure yields a finite entropy integral:

Theorem. Let $(V, \mathcal{E}, \mathcal{F}, \nu)$ be a compact measured resistance network, and $X = (X_t)_{t\geq 0}$ the associated ν-symmetric V-valued Feller process. Assume that there exists a probability measure m on $(V, R_{(V,\mathcal{E},\mathcal{F})})$ such that

$$\sup_{z\in V}\int_0^\infty \mathrm{d}\varepsilon\sqrt{-\log\left(m(B_{R_{(V,\mathcal{E},\mathcal{F})}}(z,\varepsilon))\right)} < \infty.$$

Then the cover time is finite.

(based on joint work with Siva Athreya and Wolfgang Löhr and on ongoing discussions with Omer Angel, Siva Athreya and Manjunath Krishnapur)

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Functional inequalities via a 1-dimensional localization method

Andrea Mondino

(joint work with Fabio Cavalletti)

The 1-dimensional localization method roughly consists in reducing an N-dim problem to a (often easier) 1-dim statement to be proved. It has its roots in a work of Payne-Weinberger [10] of 1960 about the sharp Poincaré inequality in a bounded convex subset of $\mathbb{R}^n$; apart from the interesting result by itself, the novelty of that paper was the proof based on an iterative bisection argument which finally reduced the problem to a one dimensional statement. Such a procedure, then called *1-dimensional localization*, was then formalized by Gromov-V. Milman [4] and by Kannan-Lovats-Simonovits [5], still by using iterative bisections. Since such an approach relies on the high symmetry of the space, the localization method was for a long time confined to euclidean space (or very symmetric spaces like spheres, hyperbolic spaces, Hilbert spaces).

A new approach via L^1-optimal transportation was proposed in a ground-breaking work of Klartag [7], who extended the localization technique to smooth Riemannian manifolds endowed with weighted measures. Such an approach has the tremendous advantage of dropping any symmetry assumption, on the other hand it heavily relies on the smoothness of the space.

In a recent joint work with Cavalletti [1], we extended even further the localization method to include the class of non smooth metric measure spaces satisfying Ricci curvature lower bounds in the synthetic sense of Lott-Villani [8] and Sturm [11, 12]. More precisely we proved that the 1-dimensional localization method extends to essentially non-branching $\mathrm{CD}_{loc}(K,N)$ spaces. The statement is the following.

Theorem [1]. Let $(X, \mathsf{d}, \mathsf{m})$ be an essentially non-branching metric measure space with $\mathsf{m}(X) = 1$, verifying $\mathrm{CD}_{loc}(K,N)$, with $1 < N < \infty$. Let $f : X \to \mathbb{R}$ with $\int f\, \mathsf{m} = 0$ and $\int |f(x)| \mathsf{d}(x, x_0) \mathsf{m}(dx) < \infty$.

Then $X = Z \cup \mathcal{T}$, $Z \cap \mathcal{T} = \emptyset$, with $f = 0$ m-a.e. over Z and
(1) there exists a partition $\{X_q\}_{q \in Q}$ of $\mathcal{T}$;
(2) such a partition induces a disintegration $\mathsf{m} = \int_Q \mathsf{m}_q\, \alpha(dq)$, with $\alpha(Q) = 1$ and $\mathsf{m}_q(X_q) = \mathsf{m}_q(X) = 1$ for α-a.e. $q \in Q$;
(3) X_q is a geodesic in X and $(X_q, |\cdot|, \mathsf{m}_q)$ is a $\mathrm{CD}(K,N)$ space;
(4) for α-a.e. $q \in Q$ it holds $|X_q| > 0$ and $\int f\, \mathsf{m}_q = 0$.

Having the above localization theorem at hand we [1] could extend the Levy-Gromov isoperimetric inequality [3] (and as well the generalization by E. Milman [9] to general lower Ricci curvature bounds) to essentially non-branching metric measure spaces verifying $\mathrm{CD}_{loc}(K,N)$. In case the isoperimetric lower bound is achieved and the space is $\mathrm{RCD}^*(K,N)$ for some $K > 0$, by using the maximal diameter Theorem (originally proved by Cheng for smooth manifolds and extended by Ketterer [6] to $\mathrm{RCD}^*(K,N)$-spaces), we showed rigidity: namely the space must be a spherical suspension. Moreover, also the almost rigidity holds: if the isoperimetric lower bound is almost achieved, then the space is close to a spherical suspension in measured Gromov-Hausdorff sense.

With similar methods [2] we have been able to prove a number of inequalities in sharp form (in class of spaces with Ricci curvature bounded below and dimension bounded above), answering some open problems proposed in the celebrated optimal transport book of Villani [13]: sharp p-spectral gap, Sobolev, log-Sobolev, Brunn-Minkowsky. A remarkable feature of such results is that they ensure inequalities with *sharp constants* under the *local* curvature condition $\mathrm{CD}_{loc}(K,N)$ (instead of the a-priori stronger global $\mathrm{CD}(K,N)$).

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Weyl's eigenvalue asymptotics for the Laplacian on circle packing limit sets of certain Kleinian groups

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1. Introduction: circle packing limit sets of Kleinian groups

The purpose of this talk[1] was to present the author's recent results on the construction of a "canonical" Laplacian on circle packing fractals invariant under the action of certain Kleinian groups and on the asymptotic behavior of its eigenvalues.

Recall that each $\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{PSL}_2(\mathbb{C})$ acts on the Riemann sphere $\widehat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ as a Möbius (linear fractional) transformation $z \mapsto (\alpha z + \beta)/(\gamma z + \delta)$, which is a biholomorphism of $\widehat{\mathbb{C}}$ and *maps circles to circles* (with each straight line considered as a circle containing ∞). A discrete subgroup G of $\mathrm{PSL}_2(\mathbb{C})$ is called a *Kleinian group*, and the smallest closed subset $\Lambda(G)$ of $\hat{\mathbb{C}}$ invariant under the action of G is called the *limit set* of G. It is known in the theory of Kleinian groups (see, e.g., [1, 5]) that the limit sets of certain classes of Kleinian groups are circle packing fractals, and typical examples of such circle packing fractals are provided in the book [3] with a number of beautiful pictures of them.

Aiming at developing a rich theory of analysis *on* circle packing fractals, the author has recently identified a candidate for the "canonical" Laplacian on them and proved Weyl's eigenvalue asymptotics for this Laplacian in important special cases. This talk presented these results, which are summarized in this abstract. The identification of the Laplacian is based on the preceding studies on the Apollonian gasket (Fig. 1) and is explained in Section 2. Then Section 3 gives an extension

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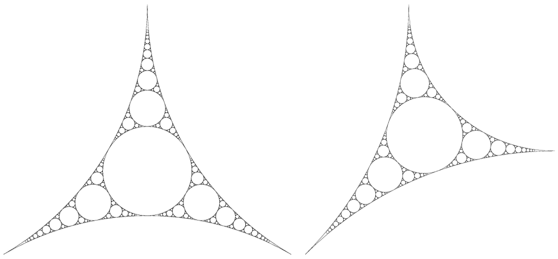

Fig. 1. Some Apollonian gaskets

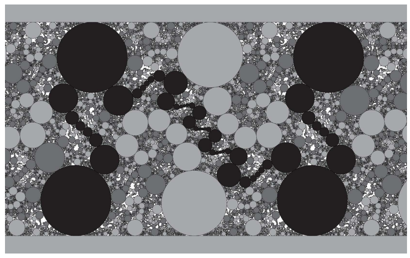

Fig. 2. Limit set of $\frac{7}{43}$ double cusp group

of the construction of the Laplacian to a certain important class of circle packing fractals and states the author's result on Weyl's eigenvalue asymptotics.

2. Preceding results for the Apollonian gasket

The *Apollonian gasket* $K_{\alpha,\beta,\gamma}$ associated with an ideal triangle (the closed subset of $\mathbb{R}^2$ enclosed by mutually tangent three circles) formed by three circles of radii $\alpha,\beta,\gamma \in (0,\infty)$ (Fig. 1) is the compact fractal subset of $\mathbb{R}^2$ obtained from the given ideal triangle by repeating indefinitely the process of removing the interior of the inner tangent circles of the ideal triangles. $K_{\alpha,\beta,\gamma}$ is homeomorphic to the (usual) Sierpiński gasket K as can be easily seen from its construction.

An essential idea for constructing a "canonical Laplacian" on the Apollonian gasket was proposed by Teplyaev [4]. His idea was to try to *make the given geometry of $K_{\alpha,\beta,\gamma}$ harmonic* by equipping $K_{\alpha,\beta,\gamma}$ with a suitable energy functional (Dirichlet form). Specifically, Teplyaev [4] proved the following proposition. For each $m \in \mathbb{N}\cup\{0\}$, let V_m denote the set of all the vertices of the 3^m ideal triangles $\{\Delta_{m,k}\}_{k=1}^{3^m}$ obtained after the mth step of the construction of $K_{\alpha,\beta,\gamma}$, and equip V_m with the natural graph (edge) structure B_m given by $B_m := \bigl\{\{x,y\} \bigm| x,y \in \Delta_{m,k} \text{ for some } k \in \{1,\dots,3^m\}, \text{ and } x \neq y\bigr\}$.

Proposition 1 ([4]). *There exists a unique (up to constant multiples) sequence $\{c_m^{\alpha,\beta,\gamma}\}_{m=0}^{\infty}$ where $c_m^{\alpha,\beta,\gamma} = (c_{m,x,y}^{\alpha,\beta,\gamma})_{\{x,y\}\in B_m} \in (0,\infty)^{B_m}$ for each $m \in \mathbb{N}\cup\{0\}$, such that the bilinear forms $\mathcal{E}_m^{\alpha,\beta,\gamma} : \mathbb{R}^{V_m}\times\mathbb{R}^{V_m} \to \mathbb{R}$ defined by $\mathcal{E}_m^{\alpha,\beta,\gamma}(u,v) := \sum_{\{x,y\}\in B_m} c_{m,x,y}^{\alpha,\beta,\gamma}(u(x)-u(y))(v(x)-v(y))$ satisfy the following: for $m \in \mathbb{N}\cup\{0\}$,*

(2.1) $\mathcal{E}_m^{\alpha,\beta,\gamma}(u,u) = \min\{\mathcal{E}_{m+1}^{\alpha,\beta,\gamma}(v,v) \mid v \in \mathbb{R}^{m+1},\, v|_{V_m} = u\}$ *for any* $u \in \mathbb{R}^{V_m}$,

(2.2) $\mathcal{E}_m^{\alpha,\beta,\gamma}(h|_{V_m}, h|_{V_m}) = \mathcal{E}_0^{\alpha,\beta,\gamma}(h|_{V_0}, h|_{V_0})$ *for any affine function* $h : \mathbb{R}^2 \to \mathbb{R}$.

(2.2) means that, on the complement of V_0, the inclusion map $K_{\alpha,\beta,\gamma} \hookrightarrow \mathbb{R}^2$ is harmonic with respect to the sequence $\{\mathcal{E}_m^{\alpha,\beta,\gamma}\}_{m=0}^{\infty}$ of forms. By virtue of the compatibility condition (2.1), we can further take the natural limit as $m \to \infty$, as in the following definition. Set $C(K_{\alpha,\beta,\gamma}) := \{u \mid u : K_{\alpha,\beta,\gamma} \to \mathbb{R},\, u \text{ is continuous}\}$.

Definition 2. *We define $\mathcal{C}_{\alpha,\beta,\gamma} \subset C(K_{\alpha,\beta,\gamma})$ and $\mathcal{E}^{\alpha,\beta,\gamma} : \mathcal{C}_{\alpha,\beta,\gamma}\times\mathcal{C}_{\alpha,\beta,\gamma} \to \mathbb{R}$ by $\mathcal{C}_{\alpha,\beta,\gamma} := \bigl\{u \in C(K_{\alpha,\beta,\gamma}) \bigm| \lim_{m\to\infty}\mathcal{E}_m^{\alpha,\beta,\gamma}(u|_{V_m}, u|_{V_m}) < \infty\bigr\}$ and*

(2.3) $$\mathcal{E}^{\alpha,\beta,\gamma}(u,v) := \lim_{m\to\infty}\mathcal{E}_m^{\alpha,\beta,\gamma}(u|_{V_m}, v|_{V_m}) \in \mathbb{R}, \qquad u,v \in \mathcal{C}_{\alpha,\beta,\gamma}.$$

While Proposition 1 proved just the unique existence of the sequence of forms $\{\mathcal{E}_m^{\alpha,\beta,\gamma}\}_{m=0}^{\infty}$ satisfying (2.1) and (2.2) *without* giving their weights $(c_{m,x,y}^{\alpha,\beta,\gamma})_{\{x,y\}\in B_m}$

explicitly, the author has recently determined the values of $(c^{\alpha,\beta,\gamma}_{m,x,y})_{\{x,y\}\in B_m}$ as concrete rational functions in α,β,γ and $\sigma := (\alpha^{-1}\beta^{-1}+\beta^{-1}\gamma^{-1}+\gamma^{-1}\alpha^{-1})^{-1/2}$, from which the following expression of $\mathcal{E}^{\alpha,\beta,\gamma}$ can be deduced:

Theorem 3 (K.)**.** *$\mathcal{C}^{\mathrm{LIP}}_{\alpha,\beta,\gamma} := \{u|_{K_{\alpha,\beta,\gamma}} \mid u : \mathbb{R}^2 \to \mathbb{R},\ u$ is Lipschitz continuous$\} \subset \mathcal{C}_{\alpha,\beta,\gamma}$, and (after specifying a constant multiple of $\mathcal{E}^{\alpha,\beta,\gamma}$) for any $u \in \mathcal{C}^{\mathrm{LIP}}_{\alpha,\beta,\gamma}$,*

$$\mathcal{E}^{\alpha,\beta,\gamma}(u,u) = \sum\nolimits_{C\in\mathcal{A}_{\alpha,\beta,\gamma}} \mathrm{rad}(C) \int_C |\nabla_C u|^2 d\mathrm{vol}_C, \tag{2.4}$$

where $\mathcal{A}_{\alpha,\beta,\gamma}$ denotes the set of all the arcs appearing in the construction of $K_{\alpha,\beta,\gamma}$, $\mathrm{rad}(C)$ the radius of C, ∇_C the gradient on C and vol_C the length measure on C.

3. The limit sets of the double cusp groups on Maskit's boundary

Let $p/q \in \mathbb{Q}\cap(0,1)$, $\mu\in\mathbb{C}$ and define $a,b\in \mathrm{PSL}_2(\mathbb{C})$ by $a := a_\mu := \left(\begin{smallmatrix} i\mu & i \\ i & 0\end{smallmatrix}\right)$ and $b := \left(\begin{smallmatrix} 1 & 2 \\ 0 & 1\end{smallmatrix}\right)$. Set $\varepsilon_0 := \{z\in\mathbb{C} \mid \mathrm{Im}\, z < 0\}$ and $\varepsilon_1 := \varepsilon_{\mu,1} := \{z\in\mathbb{C} \mid \mathrm{Im}(z-\mu) > 0\}$. The framework of this section is summarized in the following theorem.

Theorem 4 ([2, 1, 5])**.** *There exists a unique $\mu = \mu(p/q)\in\mathbb{C}$ with $\mathrm{Im}\,\mu \geq 1$ such that for some disjoint open discs $\{\delta_k\}_{k=0}^{p+q}$ contained in $\mathbb{C}\setminus(\varepsilon_0\cup\varepsilon_1)$, $a(\delta_k) = \delta_{k+p}$ for any $k\in\{0,\dots,q\}$, $b(\delta_k) = \delta_{k+q}$ for any $k\in\{0,\dots,p\}$, $(\varepsilon_0,\varepsilon_1)$, (ε_0,δ_0), (ε_0,δ_q), (ε_1,δ_p), $(\varepsilon_1,\delta_{p+q})$ and (δ_k,δ_{k+1}) for $\{0,\dots,p+q-1\}$ are mutually tangent, and no other two distinct discs from $\{\varepsilon_0,\varepsilon_1,\delta_1,\dots,\delta_{p+q}\}$ are mutually tangent.*

We set $\mu := \mu(p/q)$ throughout the rest of this section. It then follows from Theorem 4 that the subgroup $G := G_{\mu(p/q)} := \langle a,b\rangle$ of $\mathrm{PSL}_2(\mathbb{C})$ generated by $a = a_{\mu(p/q)}, b$, called the *p/q double cusp group*, is a free group in the two alphabets a,b and is a Kleinian group, and that the complement of its limit set $\Lambda(G)$ is given by $\widehat{\mathbb{C}}\setminus\Lambda(G) = \bigcup_{g\in G}\bigl(g(\varepsilon_0)\cup g(\delta_0)\bigr)$ and is a disjoint union of open discs in $\widehat{\mathbb{C}}$.

Let D, D' be the two connected components of $\mathbb{C}\setminus\overline{\varepsilon_0\cup\varepsilon_1\cup\delta_0\cup\cdots\cup\delta_p}^{\mathbb{C}}$ where $-t+\mu/2\in D$ and $t+\mu/2\in D'$ for sufficiently large $t\in(0,\infty)$, and set $F := \Lambda(G)\cap\overline{D'}^{\widehat{\mathbb{C}}}$ and $\Gamma := \{w\in \mathrm{PSL}_2(\mathbb{C}) \mid w^{-1}(\infty)\in D\}$. For $w\in\Gamma$, we also set $F_w := w(F)$ and $\mathcal{A}_w := \bigl\{F_w\cap\partial_{\mathbb{C}}(wg(\delta)) \bigm| g\in G,\ \delta\in\{\varepsilon_0,\delta_0\}\bigr\}$, so that $\mathcal{A}_w$ is a family of arcs in $\mathbb{C}$ with $F_w = \overline{\bigcup_{C\in\mathcal{A}_w} C}^{\mathbb{C}}$.

Now we adopt (2.4) as the *definition* of the Dirichlet form on our fractal F_w.

Definition 5. Let $w\in\Gamma$ and $\mathcal{C}_w := \{u|_{F_w} \mid u:\mathbb{C}\to\mathbb{R},\ u$ is Lipschitz continuous$\}$. We define a Borel measure ν^w on F_w and a bilinear form $\mathcal{E}^w : \mathcal{C}_w\times\mathcal{C}_w\to\mathbb{R}$ by

$$\nu^w := \sum_{C\in\mathcal{A}_w} \mathrm{rad}(C)\cdot\mathrm{vol}_C, \quad \mathcal{E}^w(u,v) := \sum_{C\in\mathcal{A}_w}\mathrm{rad}(C)\int_C \langle\nabla_C u,\nabla_C v\rangle d\mathrm{vol}_C. \tag{3.1}$$

Proposition 6 (K.)**.** *On $L^2(F_w,\nu^w)$, $(\mathcal{E}^w,\mathcal{C}_w)$ is closable and its closure $(\mathcal{E}^w,\mathcal{F}_w)$ is a strongly local regular Dirichlet form. Moreover, with $\mathcal{F}_w$ equipped with the inner product $\mathcal{E}^w(u,v)+\int_{F_w} uv\, d\nu^w$, the inclusion $\mathcal{F}_w\hookrightarrow L^2(F_w,\nu^w)$ is compact.*

Let d be the Hausdorff dimension of F_w with respect to the Euclidean metric and let $\mathcal{H}^d$ be the d-dimensional Hausdorff measure on $\mathbb{C}$ with respect to the

Euclidean metric. It is easy to see that d is independent of w, and it is known that $d \in (1,2)$ and $\mathcal{H}^d(F_w) \in (0,\infty)$. The following is our main theorem.

Theorem 7 (K.). *There exists $c_1 \in (0,\infty)$ such that for any $w \in \Gamma$, the eigenvalues $\{\lambda_n^w\}_{n\in\mathbb{N}}$ (with each eigenvalue repeated according to its multiplicity) of the non-negative self-adjoint operator on $L^2(F_w,\nu^w)$ associated with $(\mathcal{E}^w,\mathcal{F}_w)$ satisfies*

$$\lim_{\lambda\to\infty} \lambda^{-d/2}\#\{n \in \mathbb{N} \mid \lambda_n^w \le \lambda\} = c_1\mathcal{H}^d(F_w). \tag{3.2}$$

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Reflected Brownian Motion: selection, approximation and Linearization

Marc Arnaudon

(joint work with Xue-Mei Li)

Let M be a Riemannian manifold with smooth boundary and (X_t) a Brownian motion in M with normal reflection at boundary. We construct a family (W_t) of damped transports along (X_t): $W_t(\omega)$ is a linear map $T_{X_0(\omega)}M \to T_{X_t(\omega)}M$, which solves the heat equation for differential 1-forms with absolute boundary conditions

$$\frac{\partial\phi}{\partial t} = \frac{1}{2}\Delta^1\phi \quad \text{in } M^o, \quad \phi(t,\nu) = 0 \;\text{ on } \partial M, \quad \phi(0,\cdot) = \phi_0$$

where $\nu(x)$ is the inward normal vector at boundary. The damped transport process evolves pathwise by the Ricci curvature $\mathrm{Ric}^\sharp$ in the interior, by the shape operator $\mathcal{S}$ on the boundary driven by the boundary local time L_t, and has its normal part erased on the boundary: its Itô covariant differential DW_t, which measures difference with parallel transport along (X_t), satisfies

$$DW_t = -\frac{1}{2}\mathrm{Ric}^\sharp(W_t)\,dt - \mathcal{S}(W_t)\,dL_t - 1_{\{t\in\mathcal{R}(\omega)\}}\langle W_t, \nu(X_t)\rangle\nu(X_t)$$

where $\mathcal{R}(\omega)$ is the set of end times of excursions outside boundary.

A representation of the solution to the heat equation for 1-forms is

$$\phi_t(v) = \mathbf{E}[\phi_0(W_t(v))],$$

valid when Ric and $\mathcal{S}$ are bounded from below.

From this we can prove a Bismut type formula: if $F : [0,T] \times M \to \mathbf{R}$ is $C^{1,2}$ and satisfies $(\partial_t + \frac{1}{2}\Delta)F = 0$ on M^o and $\langle \nabla F, \nu \rangle = 0$ on ∂M then $\langle dF_t, W_t \rangle$ is a local martingale, so

$$\langle dF_0, v \rangle = -\mathbf{E}\left[F(\tau, X_\tau) \int_0^\tau \langle W_s \dot{h}_s, dX_s \rangle\right]$$

with $h_0 = v$, $h_\tau = 0$, $\tau = T \wedge \tau_{\partial M}$, $\tau_{\partial M}$ is the hitting time of ∂M by (X_t), $\dot{h} \in L^{1+\varepsilon}([0,T] \times \Omega)$.

When M is compact, we prove that, taking $a > 0$ suitably small, we can approximate $(X_t, W_t)_{t\in[0,T]}$ as close as we want in the S^p topology for any $p \geq 1$, by $(X_t^a, W_t^a)_{t\in[0,T]}$, where X_t^a is Brownian motion with drift $\nabla \ln \tanh\left(\frac{\mathrm{dist}(\cdot, \partial M)}{a}\right)$ and W_t^a is the damped parallel translation along X_t^a, i.e.

$$DW_t^a = \left(-\frac{1}{2}\mathrm{Ric}^\sharp(W_t^a) + \nabla_{W_t^a} \nabla \ln \tanh\left(\frac{\mathrm{dist}(\cdot, \partial M)}{a}\right)\right) dt.$$

But W_t^a is known to be a derivative flow for X_t^a for some special construction of Brownian motion with drift. Taking a weak limit as $a \to 0$, we prove that W_t is a derivative flow of X_t.

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