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Mini-Workshop: Stochastic Differential Equations: Regularity and Numerical Analysis in Finite and Infinite Dimensions

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Abstract. This Mini-Workshop is devoted to regularity and numerical analysis of stochastic ordinary and partial differential equations (SDEs for both). The standard assumption in the literature on SDEs is global Lipschitz continuity of the coefficient functions. However, many SDEs arising from applications fail to have globally Lipschitz continuous coefficients. Recent years have seen a prosper growth of the literature on regularity and numerical approximations for SDEs with non-globally Lipschitz coefficients. Some surprising results have been obtained – e.g., the Euler–Maruyama method diverges for a large class of SDEs with super-linearly growing coefficients, and the limiting equation of a spatial discretization of the stochastic Burgers equation depends on whether the discretization is symmetric or not. Several positive results have been obtained. However the regularity of numerous important SDEs and the closely related question of convergence and convergence rates of numerical approximations remain open. The aim of this workshop is to bring together the main contributers in this direction and to foster significant progress.

Mathematics Subject Classification (2010): 65C30, 65C35, 60H15, 65N75.

Introduction by the Organisers

The workshop *Mini-Workshop: Stochastic Differential Equations: Regularity and Numerical Analysis in Finite and Infinite Dimensions* was organised by Martin Hutzenthaler (Universität Duisburg-Essen), Annika Lang (Chalmers University, Göteburg), Lukasz Spruch (University of Edinburgh), and Larisa Yaroslavtseva

(University of Passau). It was attended by 16 participants from France, Germany, the Netherlands, Sweden, Switzerland, and the United Kingdom. Most participants were young researchers working on numerical analysis for 'non-standard' SDEs (i.e., SDEs with coefficients that do not satisfy global Lipschitz or monotonicity conditions).

Denis Talay, Stig Larsson, and Arnulf Jentzen agreed to give overview talks (2x45 minutes), in which they also presented some open problems. Arnulf Jentzen gave an overview of positive and negative results regarding both strong and weak convergence (rates) for approximations of non-linear SDEs. He also posed a great number of open problems of varying (presumed) difficulty. Examples include characterizing the strong/weak convergence rates of the Heston model in terms of the model parameters, and determining optimal weak convergence rates for non-linear stochastic partial differential equations (e.g., the non-linear heat equation). One open problem was also mentioned in the talk of Stig Larsson: it concerns obtaining (optimal) strong convergence rates for temporal discretisations of the Cahn–Hilliard–Cook equation. Stig Larsson discussed the Cahn–Hilliard–Cook equation (physical interpretation, well-posedness) and presented some recent results regarding convergence of numerical schemes for this equation. Denis Talay gave a brief introduction to McKean–Vlasov particle interaction systems with smooth kernels, explaining how — when letting the number of particles involved go to infinity — the solution converges weakly to the solution of the McKean–Vlasov SDE. He then continued to discuss the case of singular kernels and to explain how these arise naturally in certain neurological models, and concluded his talk with a discussion of the occurance of blow-ups in such equations.

All other participants also contributed a talk on recent research questions, and some additionally presented open problems. The small scale of the workshop allowed for a very informal atmosphere, leading to numerous discussions during and after the talks and excellent group dynamics.

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Mini-Workshop: Stochastic Differential Equations: Regularity and Numerical Analysis in Finite and Infinite Dimensions

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Abstracts

Approximation and simulation of infinite-dimensional Lévy processes

Andrea Barth
(joint work with Andreas Stein)

Uncertainty quantification plays an increasingly important role in a wide range of problems in the Engineering Sciences and Physics. Examples of sources of uncertainty are imprecise or insufficient measurements and noisy data. In the underlying dynamical system this is modeled via a stochastic operator, stochastic boundary conditions and/or stochastic data. As an example, to model subsurface flow more realistically the coefficients of an (essentially) elliptic equation are assumed to be stochastic. A common approach in the literature is to use correlated random fields that are built from uniform distributions or colored log-normal fields. The resulting marginal distributions of the field are (shifted) normally, resp. log-normally distributed. Neither choice is universal enough to accommodate all possible types of porosity, especially not if fractures are incorporated. In some applications, as for modeling forwards in electricity markets, it might even be necessary that the marginal distribution of the (time-dependent) random field is a pure-jump process (see [3]).

In the case of a (time-dependent) Gaussian random field, the approximation and simulation via its Karhunen-Loève expansion is straightforward. Almost sure and L^p-convergence in terms of the decay of the eigenvalues has been shown for truncated KL-expansions in [4]. For infinite-dimensional Lévy processes, also called *Lévy fields*, the approximation may still be attempted via the KL expansions. In contrast to the Gaussian case, the one-dimensional processes in the spectral representation are not independent but merely uncorrelated. If one were to use independent Lévy processes, the resulting field would not have the desired marginal distributions and would, therefore, not converge to the desired Lévy field. One may circumvent this issue by the use of correlated processes. To this end, one of the main contributions of this work is to introduce a class of Lévy processes that can be expressed in terms of subordinated Brownian motions and derive the corresponding correlation structure. From a simulatory point of view this entails the generation of a certain number of one-dimensional processes with a given set of parameters. To account for the specific structure of the subordinated Brownian motions a second approach is introduced, where one multidimensional process with decorrelated marginals are generated.

In both aforementioned cases it is, however, necessary to simulate one-dimensional Lévy processes. A common way to to do so, is to employ the so called *compound Poisson approximation* (CPA) (see [1],[8] and [9]). Although it is possible to achieve convergence in distribution of the approximation, it is in general not possible to derive mean-square convergence results and obtain error bounds in terms

of the approximation parameters. As a further contribution a novel approximation method for one-dimensional Lévy processes is developed. This new approach, based on Fourier inversion, addresses the abovementioned problems. L^p- and almost sure convergence of the approximation under relatively weak assumptions are proved and precise error bounds are derived. Further, mean-square convergence of the approximation to a given infinite-dimensional Lévy process is shown by combining the Fourier inversion method with an appropriate truncation of the KL expansion.

As a class of subordinated processes, *generalized hyperbolic Lévy processes*, that are based on the generalized hyperbolic distribution and cover for example *normal inverse Gaussian* and *hyperbolic* processes are considered. These processes are widely used in applications such as mathematical Finance, Physics and Biology (see, for instance, [2, 3, 5, 6, 7]). With its fat-tailed distribution a generalized hyperbolic field may also be of value in the modeling of subsurface flows (see [10] for a discussion on fat-tailed distributions).

The corresponding generalized hyperbolic Lévy fields are approximated via truncated Karhunen-Loève expansions with generalized-hyperbolically distributed marginals. Further, it is shown that the approximation converges to an infinite-dimensional generalized hyperbolic process. By the fact that generalized hyperbolic Lévy processes can be represented as subordinated Brownian motions, one is able to simulate generalized hyperbolic fields efficiently using the Fourier inversion method, even if a large number of one-dimensional generalized hyperbolic processes is necessary. Some of the implementational details of the algorithm are highlighted and normal inverse Gaussian and hyperbolic fields as numerical examples conclude.

References

[1] S. Asmussen and J. Rosiński, *Approximations of small jumps of Lévy processes with a view towards simulation*, Journal of Applied Probability, **38** (2001), 482–493.

[2] O. E. Barndorff-Nielsen, *Exponentially decreasing distributions for the logarithm of particle size*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences, **353** (1977), 401–419.

[3] A. Barth and F. E. Benth, *The forward dynamics in energy markets—infinite-dimensional modelling and simulation*, Stochastics An International Journal of Probability and Stochastic Processes, **86** (2014), 932–966.

[4] A. Barth and A. Lang, *Milstein approximation for advection-diffusion equations driven by multiplicative noncontinuous martingale noises*, Appl. Math. Optim., **66** (2012), 387–413.

[5] P. Blæsild, *The two-dimensional hyperbolic distribution and related distributions, with an application to Johannsen's bean data*, Biometrika, **68** (1981), 251–263.

[6] E. Eberlein, *Lévy Processes: Theory and Applications*, Birkhäuser Boston, Boston, MA, 2001, ch. Application of Generalized Hyperbolic Lévy Motions to Finance, 319–336.

[7] E. Eberlein and U. Keller, *Hyperbolic distributions in finance*, Bernoulli, **1** (1995), pp. 281–299.

[8] T. H. Rydberg, *The normal inverse Gaussian Lévy process: simulation and approximation*, Comm. Statist. Stochastic Models, **13** (1997), 887–910.

[9] W. Schoutens, *Lévy Processes in Finance: Pricing Financial Derivatives*, John Wiley and Sons, 2003.
[10] D. Zhang and Q. Kang, *Pore scale simulation of solute transport in fractured porous media*, Geophysical Research Letters, **31** (2004).

Regularization by noise for the stochastic transport equation

Lisa Beck
(joint work with Franco Flandoli, Massimiliano Gubinelli, Mario Maurelli)

We discuss several aspects of regularity and uniqueness for weak (L^∞-) solutions to the (deterministic and) stochastic transport equation

$$du = b \cdot \nabla u dt + \sigma \nabla u \circ dW_t \tag{sTE}$$

on $[0,T] \times \mathbb{R}^d$ with initial values $u_0 \colon \mathbb{R}^d \to \mathbb{R}$ for $t = 0$. Here, $b \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ is a deterministic vector field (the drift), σ a real number, $(W_t)_{t \geq 0}$ a Brownian motion in $\mathbb{R}^d$, $u \colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ the (random) unknown, and the stochastic term is interpreted in the Stratonovich sense.

For the deterministic equation ($\sigma = 0$) the following dichotomy is well-known. If the drift b is sufficiently regular, then the associated equation of characteristics generates a flow $\Phi_t \colon \mathbb{R}^d \to \mathbb{R}^d$ of diffeomorphisms and the initial value problem to (sTE) admits (in suitable function spaces) a unique solution, which preserves C^1-regularity of the initial values and allows for the representation formula $u(t,x) = u_0(\Phi_t^{-1}(x))$, see [2]. In contrast, if b is not regular enough (such as only Hölder continuous in space), then multiple solutions may exist and solutions may blow up from smooth initial data in finite time (which, on the level of the equation of characteristics, means non-uniqueness or coalescence of the trajectories). For the stochastic equation ($\sigma \neq 0$) it turns out that noise can lead to a non-trivial regularization effect, namely that the formation of non-uniqueness and of singularities is prevented.

Similar phenomena of regularization due to noise were observed for different types of partial differential equations, for instance, for reaction diffusion equations in [6], for the transport equation (sTE) in [4, 3, 8] or for stochastic conservation laws in [5]. The main goal in these papers consists in understanding the effect of regularization due to the stochastic term. This requires in particular to find a suitable noise term, as simple as possible, for which the regularization effect takes place, while imposing as little regularity as possible on the deterministic terms.

In the case of the stochastic transport equation (sTE) we work in [1] with multiplicative Stratonovich noise and a mere integrability assumption on the drift (known from fluid dynamics as the Ladyzhenskaya–Prodi–Serrin condition). More precisely, we assume

$$b \in L^q([0,T]; L^p(\mathbb{R}^d, \mathbb{R}^d)) \qquad \text{for } p,q \in (2,\infty) \text{ such that } \frac{d}{p} + \frac{2}{q} \leq 1$$

and in particular, we do not assume any kind of differentiability or Hölder continuity. A scaling argument shows that this integrability condition is subcritical for

strict inequality and critical for equality in $\frac{d}{p} + \frac{2}{q} \leq 1$, meaning that the Gaussian velocity field dominates the drift or that it is comparable to the drift, which suggests its optimality. In this setting, we prove in the purely stochastic case $\sigma \neq 0$ the conservation of Sobolev regularity of the initial values in the sense of

$$u_0 \in W^{1,2m}(\mathbb{R}^d) \quad \Rightarrow \quad \sup_{t\in[0,T]} E\big[\|u(t,\cdot)\|^m_{W^{1,m}(\mathbb{R}^d)}\big] < \infty$$

for $m \in 2\mathbb{N}$ (up to a restriction of the growth at infinity). The techniques needed to reach the critical case are of analytic nature and rely crucially on parabolic equations satisfied by moments of first derivatives of the solution. This is opposite to the previous works [4, 3, 8] based on stochastic flows and their regularity in terms of weak differentiability (which, by means of the result [7], is only known to be true in the subcritical case). Our approach covers in fact stochastic generalized transport equations, containing in particular the stochastic continuity equation

$$dv = \mathrm{div}(bv)dt + \sigma \mathrm{div}(v \circ dW_t)$$

which is in duality correspondence with the stochastic transport equation. By a duality approach in the stochastic setting, this allows to apply our regularity results in order to prove also the restoration of wellposedness for (sTE) provided that both b and $\mathrm{div}\, b$ satisfy the integrability assumption $L^q([0,T]; L^p(\mathbb{R}^d, \mathbb{R}^d))$ (which in fact guarantees weak differentiability of the solutions to the stochastic continuity equation).

References

[1] L. Beck, F. Flandoli, M. Gubinelli, M. Maurelli: *Stochastic ODEs and stochastic linear PDEs with critical drift: regularity, duality and uniqueness*, arXiv:1401.1530 (2014), Preprint.

[2] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. **98** (1989), no. 3, 511–547.

[3] E. Fedrizzi, F. Flandoli, *Noise prevents singularities in linear transport equations*, J. Funct. Anal. **264** (2013), no. 6, 1329–1354.

[4] F. Flandoli, M. Gubinelli, and E. Priola, *Well-posedness of the transport equation by stochastic perturbation*, Invent. Math. **180** (2010), no. 1, 1–53.

[5] B. Gess and P. E. Souganidis, *Long-time behavior, invariant measures and regularizing effects for stochastic scalar conservation laws*, to appear in Commun. Pure Appl. Anal.

[6] I. Gyöngy and É. Pardoux, *On the regularization effect of space-time white noise on quasi-linear parabolic partial differential equations*, Probab. Theory Related Fields **97** (1993), no. 1-2, 211–229.

[7] N. V. Krylov and M. Röckner, *Strong solutions of stochastic equations with singular time dependent drift*, Probab. Theory Related Fields **131** (2005), no. 2, 154–196.

[8] S.-E. A. Mohammed, T. Nilssen, F. Proske, *Sobolev differentiable stochastic flows of SDEs with singular coefficients: Applications to the transport equation*, Ann. Probab. **43** (2015), no. 3, 1535–1576.

Computable a-posteriori bounds for SPDEs

Dirk Blömker

The main question in this work is *"How to numerically compute something that might not exist?"*. Based on a-posteriori error estimates we present a method to obtain existence of globally unique solutions together with error estimates, for equations where this is analytically not known.

Motivation. Our result is motivated by equations where the global existence of solutions is not known, and thus global a-priori estimates are not available for numerical analysis. The final goal of this approach is the three-dimensional Navier–Stokes equation. But for the first deterministic results we focused on a somewhat simpler equation [3] from surface growth with similar properties.

For the latter in [2, 10] a-posteriori analysis was used for the deterministic surface growth PDE to prove numerically the regularity of solutions and thus the global existence and uniqueness. The residual was evaluated using numerical data and analytically an error estimate in terms of the residual was given.

Stochastic Example. In [4] we consider the following simpler SPDE on the Hilbert space $H = L^2([0,\pi])$

$$du = [Au + F(u)]dt + dW \qquad u(0) = u_0 \ . \tag{1}$$

subject to Dirichlet boundary conditions for the Laplacian A. The perturbation W is some cylindrical Q-Wiener process with bounded covariance operator. Finally, F is the locally Lipschitz nonlinearity $F(u) = -u^3$.

For (1) the asymptotic convergence results of numerical schemes are well known, see for example [9, 8, 6] or [1] for a truncated scheme. Moreover, there is no problem with existence and uniqueness of solutions. See for example [5]. Nevertheless, we study it as a starting point for stochastic results.

Discretization. For the spatial discretization we use the spectral Galerkin scheme, where P_N is the projection onto the first N Fourier modes. Moreover, for the time-discretization we use an accelerated exponential scheme introduced in [7, 6]. Unfortunately, no sharp asymptotic rate is known for our scheme.

We use a fixed step-size $h > 0$ and for a fixed realization ω we obtain by a random number generator in principle exact values of $\{P_N Z(hk)\}_{k\in\mathbb{N}}$, defined by

$$Z_0 = 0 \ , \qquad Z_{k+1} = e^{hA} Z_k + X_{k+1} = \sum_{j=1}^{k+1} e^{h(k+1-j)A} P_N X_j$$

with i.i.d. $\mathbb{R}^N$-valued Gaussian random variables

$$X_{k+1} = P_N \int_{hk}^{h(k+1)} e^{(h(k+1)-s)A} dW(s) \sim \mathcal{N}(0, P_N \int_0^h e^{2sA} ds P_N) \ .$$

Given the Z_k, the numerical method provides a realization of the approximation $\{u_k\}_{k=1,\dots,M} \subset P_N H$, which is defined recursively as

$$u_n = e^{Ah} P_N u_{n-1} + \int_0^h e^{A(h-s)} ds F_N(u_{n-1}) + X_n \ .$$

Interpolation linearly in time yields the approximation φ. The main result is a bound on the conditional mean-square error given the numerical data:

$$\mathbb{E}[\|u - \varphi\|^2 \mid \{X_k\}_{k\in\mathbb{N}}] = \mathbb{E}[\|u - \varphi\|^2 \mid \{Z_k\}_{k\in\mathbb{N}}] = \text{ small} \ ,$$

which is not an asymptotic result, but one that holds for the given approximation. The term "small" depends on the the numerical data, and we evaluate this part numerically. The general philosophy is to evaluate as much as possible of the error bounds using the numerical data, and only rely on analytic estimates if no numerical evaluation is possible.

Residual. The residual measures the quality of an arbitrary numerical approximation φ. For $t \in (0, T)$ it is defined as

$$\text{Res}(t) = \varphi(t) - e^{At}\varphi(0) - \int_0^t e^{A(t-s)} F(\varphi(s)) ds - Z(t). \tag{2}$$

This contains terms depending on ...

- ... the data u_k and Z_k, which we evaluate only numerically
- ... infinite dimensional parts from the stochastic convolution, which is independent of the data and expectations are evaluated analytically.
- ... an Ornstein–Uhlenbeck bridge between discretization times, which is also independent of the data and Gaussian.

Approximation. The numerical data mainly comes into play via the residual. We additionally need to quantify the continuous dependence of solutions on additive perturbations like the residual. By putting $d(t) = u(t) - \varphi(t)$ we have

$$d(t) = u(t) - \varphi(t) = e^{At} d(0) + \int_0^t e^{A(t-s)} (F(u(s)) - F(\varphi(s))) ds + \text{Res}(t)$$

with $d(0) = (I - P_N)u_0$. Let $r = d - \text{Res}$ which is the solution of the following differential equation

$$\partial_t r = Ar + F(r + \varphi + \text{Res}) - F(\varphi).$$

Recall that $\text{Res}(0) = 0$, so $r(0) = d(0) = Q_N u_0$. Now we use standard a-priori estimates for the equation for r.

References

[1] S. Becker, A. Jentzen, *Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg–Landau equations.* Preprint, arXiv, 2016.

[2] D. Blömker, C. Nolde, J.C. Robinson, *Rigorous numerical verification of uniqueness and smoothness in a surface growth model*, Journal of Mathematical Analysis and Applications **429**(1):311–325, 2015.

[3] D. Blömker, M. Romito, *Stochastic PDEs and lack of regularity (A surface growth equation with noise: existence, uniqueness, and blow-up)* Jahresbericht der Deutschen Mathematiker-Vereinigung, **117**(4):233–286, 2015.
[4] D. Blömker, M. Kamrani, *Numerically computable a-posteriori bounds for SPDEs*, Preprint, arXiv, 2017
[5] G. Da Prato., J. Zabczyk, *Stochastic Equations in Infinite Dimensions. 2nd ed.*, vol. 152 of Encyclopedia of of Mathematics and its Applications. Cambridge University Press, Cambridge, (2014).
[6] A. Jentzen; P.E. Kloeden, *Overcoming the order barrier in the numerical approximation of stochastic partial differential equations with additive space-time noise.* Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **465**, 649–667, 2009.
[7] A. Jentzen., P. Kloeden, G. Winkel, *Efficient simulation of nonlinear parabolic SPDEs with additive noise*, Annals of Applied Probability. **21**(3), 908–950, 2011.
[8] P.E. Kloeden, G.J. Lord, A. Neuenkirch, T. Shardlow, *The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds.* J. Comput. Appl. Math. **235**(5):1245–1260, 2011.
[9] G.J. Lord, C.E. Powell, T. Shardlow, *An Introduction to Computational Stochastic PDEs.* Cambridge Texts in Applied Mathematics. Cambridge: Cambridge University Press, (2014).
[10] C. Nolde, *Global Regularity and Uniqueness of Solutions in a Surface Growth Model Using Rigorous A-Posteriori Methods.* PhD-thesis, Universität Augsburg, 2017.

Exponential integrators for stochastic Schrödinger equations driven by Itô noise

David Cohen
(joint work with Rikard Anton)

In the first part of the presentation, we consider the numerical discretisation in time of stochastic Schrödinger equations by exponential integrators [1].

We start with the linear case and discuss strong convergence of the exponential integrator as well as its long-time behaviour. In particular, we show that the expected values of the mass, the energy, and the momentum drift linearly with time along the exact solution of the linear stochastic Schrödinger equation as well as along the numerical one. This is not the case for the Euler–Maruyama scheme, the backward Euler–Maruyama scheme or the Crank–Nicolson scheme from [4].

Next, we study the discretisation of linear stochastic Schrödinger equations with a multiplicative potential and driven by multiplicative noise. Especially, we show strong convergence of the exponential method and analyse the behaviour of the numerical discretisation with respect to the expected value of the mass and energy.

Numerical simulations are provided in order to confirm the above theoretical results.

In the second part of the presentation, we discuss some open problems related to long-time behaviour of numerical schemes for Hamiltonian and Poisson systems [3, 8, 7, 5] as well as for stochastic partial differential equations [6, 2, 1].

References

[1] R. Anton, D. Cohen, *Exponential integrators for stochastic Schrödinger equations driven by Itô noise*, To appear in the special issue on SPDEs of J. Comput. Math. (2017).
[2] R. Anton, D. Cohen, S. Larsson, X. Wang, *Full discretisation of semi-linear stochastic wave equations driven by multiplicative noise*, SIAM J. Numer. Anal. **54**(2) (2016), 1093–1119.
[3] J.-M. Bismut, *Mécanique Aléatoire*, Springer-Verlag **866** (1981).
[4] A. de Bouard, A. Debussche, *Weak and strong order of convergence of a semidiscrete scheme for the stochastic nonlinear Schrödinger equation*, Appl. Math. Optim. **54**(3) (2006), 369–399.
[5] D. Cohen, G. Dujardin, *Energy-preserving integrators for stochastic Poisson systems*, Comm. Math. Sci. **12**(8) (2014), 1523–1539.
[6] D. Cohen, S. Larsson, M. Sigg, *A trigonometric method for the linear stochastic wave equation*, SIAM J. Numer. Anal. **51**(1) (2013), 204–222.
[7] J. Hong, S. Zhai, J. Zhang, *Discrete gradient approach to stochastic differential equations with a conserved quantity*, SIAM J. Numer. Anal. **49**(5) (2011), 2017–2038.
[8] G. N. Milstein, Yu. M. Repin, M. V. Tretyakov, *Numerical methods for stochastic systems preserving symplectic structure*, SIAM J. Numer. Anal. **40**(4) (2002), 1583–1604 (electronic).

Convergence in Hölder norms with applications to Galerkin approximations and Monte Carlo methods

Sonja Cox

(joint work with Martin Hutzenthaler, Arnulf Jentzen, Jan van Neerven, Timo Welti)

We demonstrate that if a sequence of piecewise affine linear processes converges in the strong sense with a positive rate to a stochastic process which is strongly Hölder continuous in time, then this sequence converges in the strong sense even with respect to much stronger Hölder norms and the convergence rate is essentially reduced by the Hölder exponent. This principle has a number of applications: it may be used e.g. to derive strong convergence rates of multilevel Monte Carlo approximations of expectations of Banach space valued stochastic processes. Another application is to obtain pathwise convergence rates of spectral Galerkin approximations of non-linear stochastic partial differential equations. This in turn can be used to extend regularity results for SODEs to SPDEs.

1. An estimate for Hölder regularity

Some basic yet subtle manipulations (in combination with the Kolmogorov–Chentsov continuity criterion) allow us to prove the following lemma:

Lemma 1 (A variant of Corollary 2.11 in [1]). *Let $T \in (0,\infty)$, $p \in (1,\infty)$, $\beta \in (1/p, 1)$, $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, E a Banach space, $X \in C^\beta([0,T], \mathcal{L}^p(\mathbb{P}; E))$, $Y^N \colon [0,T]\times\Omega \to E$, $N \in \mathbb{N}$ stochastic processes with continuous sample paths, assume*

$$\sup_{N\in\mathbb{N}} \left(N^\beta \sup_{t\in\{kT/N\colon k=0,1,\ldots,N\}} \|X_t - Y_t^N\|_{\mathcal{L}^p(\mathbb{P};E)} \right) < \infty,$$

and assume at least one of the following two conditions holds:

(1) $\sup_{N\in\mathbb{N}} |Y^N|_{C^\beta([0,T],\mathcal{L}^p(\mathbb{P};E))} < \infty$,

(2) $\forall\, N \in \mathbb{N}\colon Y^N = [Y^N]_N$.

Then for all $\alpha \in [0, \beta - 1/p)$, $\varepsilon \in (0,\infty)$ *it holds that*

$$\sup_{N\in\mathbb{N}} \left[N^{\beta-\alpha-1/p-\varepsilon} \|X - Y^N\|_{\mathcal{L}^p(\mathbb{P};C^\alpha([0,T],E))} \right] < \infty.$$

By considering the Brownian motion, one may check that the convergence rate in the lemma above is essentially optimal (see [1, Lemma 2.14]). Extensions to non-equidistant grids may also be obtained (see [1, Corollary 2.11]), and the proof techniques will also work for processes indexed by e.g. a bounded domain $D \subseteq \mathbb{R}^d$, $d \in \mathbb{N}$, with piecewise affine boundary instead of processes indexed by $[0,T]$.

2. Application 1: Monte Carlo methods

Let $(X, \|\cdot\|_X)$ be a Banach space and let $\xi \in L^2(\mathbb{P}; \|\cdot\|_X)$. When using Monte Carlo approximations to approximate $\mathbb{E}(\xi)$, the convergence rate in $L^2(\mathbb{P}; \|\cdot\|_X)$ depends on the so-called (Rademacher) type $p \in [1,2]$ — indeed, it was demonstrated in [3] (see also [4]) that the the rate is given by $1 - \frac{1}{p}$ (it has also been shown that this is essentially optimal). In particular, if X has only trivial type ($p = 1$), then convergence is not guaranteed. For $T \in (0,\infty)$ the space $(C([0,T],\mathbb{R}), \|\cdot\|_\infty)$ is an example of a Banach space that fails to have type p for any $p \in (1,2]$. This is problematic if one wishes to approximate e.g. $t \mapsto \mathbb{E}(X_t)$ uniformly by means of Monte Carlo approximations, where $(X_t)_{t\in[0,T]}$ is a stochastic process. This problem may be overcome if $(X_t)_{t\in[0,T]}$ allows for additional spatial regularity, e.g., if $(X_t)_{t\in[0,T]} \in L^2(\mathbb{P}, W^{\alpha,p}([0,T],\mathbb{R}))$ for some $\alpha \in (0,1)$ and some $p \in (\frac{1}{\alpha}, \infty)$ (the space $W^{\alpha,p}([0,T],\mathbb{R})$ has type $\min(p,2)$). If moreover $(X_t)_{t\in[0,T]}$ cannot be sampled exactly, then naturally one also needs that $(X_t)_{t\in[0,T]}$ can be approximated in $L^2(\mathbb{P}; W^{\alpha,p}([0,T],\mathbb{R}))$. Approximations in $L^2(\mathbb{P}; C^\alpha([0,T]))$ clearly suffice — which is precisely the topic of Lemma 1. For details see [1, Section 4], in which also the multi level Monte Carlo method is discussed in the Banach space setting.

3. Application 2: pathwise convergence of Galerkin approximations for non-linear SPDEs

Lemma 1 is useful for the study of stochastic partial differential equations (SPDEs). In general, a solution of an SPDE fails to be a semimartingale. As a consequence, Doob's maximal inequality cannot be applied to obtain estimates with respect to the $L^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm. However, it is often feasible to obtain convergence rates with respect to the $C([0,T], \|\cdot\|_{L^2(\mathbb{P};\|\cdot\|_E)})$-norm, after which Lemma 1 can be applied to obtain convergence rates with respect to the $L^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm.

Estimates with respect to the $L^2(\mathbb{P}; \|\cdot\|_{C([0,T],\|\cdot\|_E)})$-norm are essentail for the standard localisation arguments used to extend results for SPDEs with globally Lipschitz continuous nonlinearities to results for SPDEs with nonlinearities that

are only Lipschitz continuous on bounded sets. This is demonstrated in detail in [1, Section 3], where essentially sharp pathwise convergence rates for spatial Galerkin and noise approximations for a large class of SPDEs with non-globally Lipschitz continuous nonlinearities are obtained. Examples include the stochastic Burgers, stochastic Ginzburg–Landau, stochastic Kuramoto–Sivashinsky, and Cahn–Hilliard–Cook equations. The pathwise convergence of the Galerkin approximation is then used in [2] to prove that the aforementioned SPDEs are locally Lipschitz continuous with respect to their initial value.

References

[1] S. Cox, M. Hutzenthaler, A. Jentzen, J.M.A.M. van Neerven, T. Welti, *Convergence in Hölder norms with applications to Monte Carlo methods in infinite dimensions.* Preprint, arXiv:1605.00856v1. To appear in IMA Journal of Numerical Analysis.

[2] S. Cox, M. Hutzenthaler, A. Jentzen *Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations.* Preprint, arXiv:1309.5595

[3] S. Heinrich, *Multilevel Monte Carlo methods*, Large-Scale Scientific Computing, Lect. Notes Comput. Sci., vol. 2179, Springer, Berlin, 2001, pp. 58–67.

[4] M. Ledoux and M. Talagrand, *Probability in Banach spaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 23, Springer-Verlag, Berlin, 1991, Isoperimetry and processes. MR 1102015 (93c:60001)

On arbitrarily slow convergence rates for strong numerical approximations of Cox–Ingersoll–Ross processes and squared Bessel processes

Mario Hefter

(joint work with Arnulf Jentzen)

Stochastic differential equations (SDEs) are a key ingredient in a number of models from economics and the natural sciences. In particular, SDE based models are day after day used in the financial engineering industry to approximately compute prices of financial derivatives. The SDEs appearing in such models are typically highly nonlinear and contain non-Lipschitz nonlinearities in the drift or diffusion coefficient. Such SDEs can in almost all cases not be solved explicitly and it has been and still is a every active topic of research to approximate SDEs with non-Lipschitz nonlinearities; see, e.g., Hu [17], Gyöngy [9], Higham, Mao, & Stuart [14], Hutzenthaler, Jentzen, & Kloeden [20], Sabanis [26], Hutzenthaler & Jentzen [19], Sabanis [27], and the references mentioned therein. In particular, in about the last five years several results have been obtained that demonstrate that approximation schemes may converge arbitrarily slow, see Hairer, Hutzenthaler, & Jentzen [11], Jentzen, Müller-Gronbach, & Yaroslavtseva [21], Müller-Gronbach & Yaroslavtseva [23], Yaroslavtseva [28], and Gerencsér, Jentzen, & Salimova[8]. For example, Theorem 1.2 in [21] demonstrates that there exists an SDE that has solutions with all moments bounded but for which all approximation schemes that use only evaluation points of the driving Brownian motion converge in the strong sense with an arbitrarily slow rate; see also [11, Theorem 1.3], [23, Theorem 3], [28, Theorem 1],

and [8, Theorem 1.2] for related results. All the SDEs in the above examples are purely academic with no connection to applications. The key contribution of this work is to reveal that such slow convergence phenomena also arise in concrete models from applications. To be more specific, in this work we reveal that Cox–Ingersoll–Ross (CIR) processes and squared Bessel processes can in the strong sense in general not be solved approximately in a reasonable computational time by means of schemes using equidistant evaluations of the driving Brownian motion. The precise formulation of our result is the subject of the following theorem.

Theorem 1 (Cox–Ingersoll–Ross processes)**.** *Let $T, a, \sigma \in (0,\infty)$ satisfy $2a < \sigma^2$, let $b, x \in [0,\infty)$, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X\colon [0,T]\times\Omega\to[0,\infty)$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies for all $t\in[0,T]$ $\mathbb{P}$-a.s. that*

$$X_t = x + \int_0^t (a - bX_s)\,\mathrm{d}s + \int_0^t \sigma\sqrt{X_s}\,\mathrm{d}W_s. \tag{1}$$

Then there exists a real number $c\in(0,\infty)$ such that for all $N\in\mathbb{N}$ it holds that

$$\inf_{\substack{\varphi\colon \mathbb{R}^N\to\mathbb{R}\\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_T - \varphi(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T)\big|\Big] \geq c\cdot N^{-(2a)/\sigma^2}. \tag{2}$$

Upper error bounds for strong approximation of CIR processes and squared Bessel processes, i.e., the opposite question of Theorem 1, have been intensively studied in the literature, see, e.g., Delstra & Delbaan [6], Alfonsi [1], Higham & Mao [15], Berkaoui, Bossy, & Diop [3], Gyöngy & Rásonyi [10], Dereich, Neuenkirch, & Szpruch [7], Alfonsi [2], Hutzenthaler, Jentzen, & Noll [18], Neuenkirch & Szpruch [25], Bossy & Olivero Quinteros [4], Hutzenthaler & Jentzen [19], Milstein & Schoenmakers [22], Chassagneux, Jacquier, & Mihaylov [5], Hefter & Herzwurm [12], and Hefter & Herzwurm [13]. Below we relate our result to these results. Using the truncated Milstein scheme with the corresponding error bound from Hefter & Herzwurm [13] we get that the the lower bound obtained in (2) is essentially sharp. The precise formulation of this observation is the subject of the following corollary.

Corollary 2 (Cox–Ingersoll–Ross processes)**.** *Let $T, a, \sigma \in (0,\infty)$ satisfy $4a < \sigma^2$, let $b, x \in [0,\infty)$ let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a normal filtration $(\mathbb{F}_t)_{t\in[0,T]}$, let $W\colon [0,T]\times\Omega \to \mathbb{R}$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-Brownian motion, let $X\colon [0,T]\times\Omega\to[0,\infty)$ be a $(\mathbb{F}_t)_{t\in[0,T]}$-adapted stochastic process with continuous sample paths which satisfies for all $t\in[0,T]$ $\mathbb{P}$-a.s. that*

$$X_t = x + \int_0^t (a - bX_s)\,\mathrm{d}s + \int_0^t \sigma\sqrt{X_s}\,\mathrm{d}W_s. \tag{3}$$

Then there exist real numbers $c, C\in(0,\infty)$ such that for all $N\in\mathbb{N}$ it holds that

$$c\cdot N^{-\frac{2a}{\sigma^2}} \leq \inf_{\substack{\varphi\colon \mathbb{R}^N\to\mathbb{R}\\ \text{Borel-measurable}}} \mathbb{E}\Big[\big|X_T - \varphi(W_{\frac{T}{N}}, W_{\frac{2T}{N}}, \ldots, W_T)\big|\Big] \leq C\cdot N^{-\frac{2a}{\sigma^2}}. \tag{4}$$

The lower bound in (4) is an immediate consequence of Theorem 1 and the upper bound in (4) is an immediate consequence of Hefter & Herzwurm [13, Theorem 2] using the truncated Milstein scheme. We conjecture that in the full parameter range $a, \sigma \in (0, \infty)$ the convergence order in (4) is equal to $\min\{2a/\sigma^2, 1\}$, since for scalar SDEs with coefficients satisfying standard assumptions a convergence order of one is optimal, see, e.g., Hofmann, Müller-Gronbach, & Ritter [16] and Müller-Gronbach [24].

References

[1] Aurélien Alfonsi. On the discretization schemes for the CIR (and Bessel squared) processes. *Monte Carlo Methods Appl.*, 11(4):355–384, 2005.

[2] Aurélien Alfonsi. Strong order one convergence of a drift implicit Euler scheme: application to the CIR process. *Statist. Probab. Lett.*, 83(2):602–607, 2013.

[3] Abdel Berkaoui, Mireille Bossy, and Awa Diop. Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence. *ESAIM Probab. Stat.*, 12:1–11 (electronic), 2008.

[4] M. Bossy and H. Olivero Quinteros. Strong convergence of the symmetrized Milstein scheme for some CEV-like SDEs. *arXiv e-prints*, August 2015.

[5] Jean-François Chassagneux, Antoine Jacquier, and Ivo Mihaylov. An Explicit Euler Scheme with Strong Rate of Convergence for Financial SDEs with Non-Lipschitz Coefficients. *SIAM J. Financial Math.*, 7(1):993–1021, 2016.

[6] G. Deelstra and F. Delbaen. Convergence of discretized stochastic (interest rate) processes with stochastic drift term. *Appl. Stochastic Models Data Anal.*, 14(1):77–84, 1998.

[7] Steffen Dereich, Andreas Neuenkirch, and Lukasz Szpruch. An Euler-type method for the strong approximation of the Cox-Ingersoll-Ross process. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 468(2140):1105–1115, 2012.

[8] M. Gerencsér, A. Jentzen, and D. Salimova. On stochastic differential equations with arbitrarily slow convergence rates for strong approximation in two space dimensions. *arXiv e-prints*, February 2017.

[9] István Gyöngy. A note on Euler's approximations. *Potential Anal.*, 8(3):205–216, 1998.

[10] István Gyöngy and Miklós Rásonyi. A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. *Stochastic Process. Appl.*, 121(10):2189–2200, 2011.

[11] Martin Hairer, Martin Hutzenthaler, and Arnulf Jentzen. Loss of regularity for Kolmogorov equations. *Ann. Probab.*, 43(2):468–527, 2015.

[12] M. Hefter and A. Herzwurm. Optimal Strong Approximation of the One-dimensional Squared Bessel Process. *arXiv e-prints*, January 2016.

[13] M. Hefter and A. Herzwurm. Strong Convergence Rates for Cox–Ingersoll–Ross Processes — Full Parameter Range. *arXiv e-prints*, August 2016.

[14] Desmond J. Higham, Xuerong Mao, and Andrew M. Stuart. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.*, 40(3):1041–1063 (electronic), 2002.

[15] D.J. Higham and X. Mao. Convergence of monte carlo simulations involving the mean-reverting square root process. *Journal of Computational Finance*, 8(3):35–61, 2005.

[16] Norbert Hofmann, Thomas Müller-Gronbach, and Klaus Ritter. The optimal discretization of stochastic differential equations. *J. Complexity*, 17(1):117–153, 2001.

[17] Yaozhong Hu. Semi-implicit Euler-Maruyama scheme for stiff stochastic equations. In *Stochastic analysis and related topics, V (Silivri, 1994)*, volume 38 of *Progr. Probab.*, pages 183–202. Birkhäuser Boston, Boston, MA, 1996.

[18] M. Hutzenthaler, A. Jentzen, and M. Noll. Strong convergence rates and temporal regularity for Cox–Ingersoll–Ross processes and Bessel processes with accessible boundaries. *arXiv e-prints*, March 2014.

[19] Martin Hutzenthaler and Arnulf Jentzen. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc.*, 236(1112):v+99, 2015.
[20] Martin Hutzenthaler, Arnulf Jentzen, and Peter E. Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab.*, 22(4):1611–1641, 2012.
[21] Arnulf Jentzen, Thomas Müller-Gronbach, and Larisa Yaroslavtseva. On stochastic differential equations with arbitrary slow convergence rates for strong approximation. *Commun. Math. Sci.*, 14(6):1477–1500, 2016.
[22] Grigori N. Milstein and John Schoenmakers. Uniform approximation of the Cox–Ingersoll–Ross process. *Adv. in Appl. Probab.*, 47(4):1132–1156, 2015.
[23] T. Müller-Gronbach and L. Yaroslavtseva. On hard quadrature problems for marginal distributions of SDEs with bounded smooth coefficients. *arXiv e-prints*, March 2016.
[24] Thomas Müller-Gronbach. Optimal pointwise approximation of SDEs based on Brownian motion at discrete points. *Ann. Appl. Probab.*, 14(4):1605–1642, 2004.
[25] Andreas Neuenkirch and Lukasz Szpruch. First order strong approximations of scalar SDEs defined in a domain. *Numer. Math.*, 128(1):103–136, 2014.
[26] Sotirios Sabanis. A note on tamed Euler approximations. *Electron. Commun. Probab.*, 18:no. 47, 10, 2013.
[27] Sotirios Sabanis. Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. *Ann. Appl. Probab.*, 26(4):2083–2105, 2016.
[28] L. Yaroslavtseva. On non-polynomial lower error bounds for adaptive strong approximation of SDEs. *arXiv e-prints*, September 2016.

A perturbation theory and exponential moments for SDEs

Martin Hutzenthaler
(joint work with Sonja Cox, Martin Hairer, Arnulf Jentzen, Xiaojie Wang)

If the coefficients of an SDE grow superlinearly and the diffusion coefficient is nontrivial, then the moments of the Euler–Maruyama approximations are unbounded on every finite time interval (see [9, 11]) and the nonadaptive multilevel Monte Carlo Euler approximations are conjectured to diverge almost surely (see [11]). Drift-implicit Euler approximations do not have this divergence property and converge with a positive L^2-rate (see [6, 5]) but are essentially only applicable if the drift coefficient is globally one-sided Lipschitz continuous. Recently, [10] introduced explicit tamed Euler-type approximations and these approximations converge in L^2 essentially if the coefficients satisfy the global monotonicity assumption (see also [19]). Subsequently many tamed Euler-type methods have been proposed for SODEs, see, e.g., [22, 8, 21, 18, 19, 20, 16], and for SPDEs, see, e.g., [3, 17, 15, 1, 13].

Multi-dimensional SDEs from applications, however, rarely satisfy the global monotonicity assumption and it remained an open problem to prove L^2-convergence rates for such SDEs. For solving this problem we estimate the L^2-distance between the solution X of the SDE and an arbitrary Itô process Y with the L^2-distance at time zero and with L^p-distances between the local characteristics of Y and the coefficients evaluated at Y for a suitable $p \in [2, \infty)$, see [7]. The problematic part of this estimate is an exponential moment on the right-hand side in

which the local monotonicity constant appears in the exponent and which require sufficient uniform exponential integrability properties of X and Y. Sufficient exponential moments of the exact solution X are implied by a Lyapunov-type condition on the coefficients which many SODEs from applications satify, see [2]. Sufficient uniform exponential moments of tamed Euler-type approximation processes are implied by essentially the same Lyapunov-type condition, see [14]. We also show an example of an SDE with globally bounded and smooth coefficients which do not satisfy the Lyapunov-type condition from [2] and where the solution process is not locally Hölder continuous in the initial value with respect to the L^2-distance, see [4].

References

[1] Becker, S., and Jentzen, A. Strong convergence rates for nonlinearity-truncated Euler-type approximations of stochastic Ginzburg–Landau equations. *arXiv:1601.05756* (2016), 58 pages.

[2] Cox, S. G., Hutzenthaler, M., and Jentzen, A. Local Lipschitz continuity in the initial value and strong completeness for nonlinear stochastic differential equations. *arXiv:1309.5595* (2013). 84 pages.

[3] Gyöngy, I., Sabanis, S., and Šiška, D. Convergence of tamed Euler schemes for a class of stochastic evolution equations. *arXiv:1308.1796* (2015), 17 pages.

[4] Hairer, M., Hutzenthaler, M., and Jentzen, A. Loss of regularity for Kolmogorov equations. *Ann. Probab. 43*, 2 (2015), 468–527.

[5] Higham, D. J., Mao, X., and Stuart, A. M. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal. 40*, 3 (2002), 1041–1063 (electronic).

[6] Hu, Y. Semi-implicit Euler–Maruyama scheme for stiff stochastic equations. In *Stochastic analysis and related topics*, (1996), 183–202.

[7] Hutzenthaler, M., and Jentzen, A. On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients. *arXiv:1401.0295* (2014), 41 pages.

[8] Hutzenthaler, M., and Jentzen, A. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc. 4* (2015), 1–112.

[9] Hutzenthaler, M., Jentzen, A., and Kloeden, P. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467*, 2130 (2011), 1563–1576.

[10] Hutzenthaler, M., Jentzen, A., and Kloeden, P. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. *Ann. Appl. Probab. 22*, 4 (2012), 1611–1641.

[11] Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations. *Ann. Appl. Probab. 23*, 5 (2013), 1913–1966.

[12] Hutzenthaler, M., Jentzen, A., and Noll, M. Strong convergence rates and temporal regularity for Cox–Ingersoll–Ross processes and bessel processes with accessible boundaries. *arXiv preprint arXiv:1403.6385* (2014).

[13] Hutzenthaler, M., Jentzen, A., and Salimova, D. Strong convergence of full-discrete nonlinearity-truncated accelerated exponential Euler-type approximations for stochastic Kuramoto–Sivashinsky equations. *arXiv preprint arXiv:1604.02053* (2016).

[14] Hutzenthaler, M., Jentzen, A., and Wang, X. Exponential integrability properties of numerical approximation processes for nonlinear stochastic differential equations. *arXiv preprint arXiv:1309.7657* (2013).
[15] Jentzen, A., and Pušnik, P. Strong convergence rates for an explicit numerical approximation method for stochastic evolution equations with non-globally Lipschitz continuous nonlinearities. *arXiv:1504.03523* (2015), 38 pages.
[16] Kumar, C., and Sabanis, S. On tamed Milstein schemes of SDEs driven by Lévy noise. *arXiv:1407.5347* (2015), 45 pages.
[17] Kurniawan, R. Numerical approximations of stochastic partial differential equations with non-globally Lipschitz continuous nonlinearities. *Master thesis, University of Zurich and ETH Zurich* (2014), 74 pages.
[18] Sabanis, S. A note on tamed Euler approximations. *Electron. Commun. Probab. 18* (2013), no. 47, 1–10.
[19] Sabanis, S. Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. *Annals of Applied Probability* (2016), no. 4, 2083–2105.
[20] Szpruch, L. V-stable tamed Euler schemes. *arXiv:1310.0785* (2013), 1–15.
[21] Tretyakov, M. V., and Zhang, Z. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM J. Numer. Anal. 51*, 6 (2013), 3135–3162.
[22] Wang, X., and Gan, S. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *J. Difference Equ. Appl. 19*, 3 (2013), 466–490.

A review on stochastic differential equations with arbitrarily slow convergence rates for strong approximation in two space dimensions

Arnulf Jentzen
(joint work with Máté Gerencsér and Diyora Salimova)

In the recent article [10] it has been established that for every arbitrarily slow convergence speed and every natural number $d \in \{4, 5, \ldots\}$ there exist d-dimensional stochastic differential equations (SDEs) with infinitely often differentiable and globally bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. More specifically, Theorem 1.3 in [10] implies the following theorem.

Theorem 1. *Let $T \in (0, \infty)$, $d \in \{4, 5, \ldots\}$, $\xi \in \mathbb{R}^d$, $m \in \mathbb{N}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0, T]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0, T] \times \Omega \to \mathbb{R}^m$, every continuous $\mathbb{F}$-adapted stochastic process $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ with $\forall\, t \in [0, T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s) = 1$, every $n \in \mathbb{N}$, every measurable function $u\colon (\mathbb{R}^m)^n \times C([\varepsilon_n, T], \mathbb{R}^m) \to \mathbb{R}^d$, and all $t_1, \ldots, t_n \in [0, T]$ it holds that*

$$\mathbb{E}\Big[\big\|X_T - u(W_{t_1}, \ldots, W_{t_n}, (W_s)_{s\in[\varepsilon_n,T]})\big\|_{\mathbb{R}^d}\Big] \geq \delta_n. \tag{1}$$

In the article Gerencsér et al. [1] the above result has been strengthened in the sense that [1, Theorem 1.2] proves that for every arbitrarily slow convergence speed and every natural number $d \in \{2, 3, \ldots\}$ there exist d-dimensional SDEs with infinitely often differentiable and globally bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in absolute mean to the solution faster than the given speed of convergence. More precisely, in Gerencsér et al. [1, Theorem 1.2] the following theorem has been established.

Theorem 2. *Let $T \in (0, \infty)$, $\tau \in (0, T)$, $d \in \{2, 3, \ldots\}$, $\xi \in \mathbb{R}^d$, $m \in \mathbb{N}$, $(\varepsilon_n)_{n\in\mathbb{N}} \subseteq (0, \tau]$, $(\delta_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}$ satisfy $\limsup_{n\to\infty} \delta_n \leq 0$. Then there exist infinitely often differentiable and globally bounded functions $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ such that for every probability space $(\Omega, \mathcal{F}, \mathbb{P})$, every normal filtration $\mathbb{F} = (\mathbb{F}_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, every standard $(\Omega, \mathcal{F}, \mathbb{P}, \mathbb{F})$-Brownian motion $W\colon [0, T] \times \Omega \to \mathbb{R}^m$, every continuous $\mathbb{F}$-adapted stochastic process $X\colon [0, T] \times \Omega \to \mathbb{R}^d$ with $\forall\, t \in [0, T]\colon \mathbb{P}(X_t = \xi + \int_0^t \mu(X_s)\, ds + \int_0^t \sigma(X_s)\, dW_s) = 1$, every $n \in \mathbb{N}$, every measurable function $u\colon (\mathbb{R}^m)^n \times C([\varepsilon_n, T], \mathbb{R}^m) \to \mathbb{R}^d$, and all $t_1, \ldots, t_n \in [0, T]$, $a, b \in [0, \tau]$ with $b - a \geq \varepsilon_n$ it holds that*

$$\mathbb{E}\Big[\big\|X_T - u(W_{t_1}, \ldots, W_{t_n}, (W_s)_{s\in[0,a]\cup[b,T]})\big\|_{\mathbb{R}^d}\Big] \geq \delta_n. \tag{2}$$

Related results concerning lower error bounds for strong and weak numerical approximation schemes for SDEs with non-globally Lipschitz continuous coefficients can be found in [7, 9, 3, 10, 12, 17]. Hairer et al. [3, Theorem 1.3] and Müller-Gronbach & Yaroslavtseva [12, Theorems 1–3] deal with lower bounds for weak approximation errors and Yaroslavtseva [17, Corollary 2] extends [10, Theorem 1.3] (cf. also Theorem 1 above) to numerical approximation schemes where the driving Brownian motion can be evaluated adaptively. Each of the references [3, 10, 12, 17] assumes beside other hypotheses that the dimension d of the considered SDE satisfies $d \geq 4$. The main contribution of Gerencsér et al. [1] is to reveal that a slow convergence phenomena of the form (2) also arises in two ($d = 2$) and three ($d = 3$) space dimensions. Upper error bounds and numerical approximation schemes for SDEs with non-globally Lipschitz continuous coefficients can, e.g., be found in [5, 2, 4, 8, 16, 6, 13, 14, 15] and the references mentioned therein. Lower error bounds for strong approximation schemes for SDEs with globally Lipschitz continuous coefficients can, e.g., found in the overview article Müller-Gronbach & Ritter [11] and the references mentioned therein.

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References

[1] Gerencsér, M., Jentzen, A., and Salimova, D. On stochastic differential equations with arbitrarily slow convergence rates for strong approximation in two space dimensions. *arXiv:1702.03229* (2017), 31 pages.

[2] Gyöngy, I. A note on Euler's approximations. *Potential Anal. 8*, 3 (1998), 205–216.

[3] Hairer, M., Hutzenthaler, M., and Jentzen, A. Loss of regularity for Kolmogorov equations. *Ann. Probab. 43*, 2 (2015), 468–527.
[4] Higham, D. J., Mao, X., and Stuart, A. M. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal. 40*, 3 (2002), 1041–1063 (electronic).
[5] Hu, Y. Semi-implicit Euler-Maruyama scheme for stiff stochastic equations. In *Stochastic analysis and related topics, V (Silivri, 1994)*, vol. 38. Birkhäuser Boston, Boston, MA, 1996, pp. 183–202.
[6] Hutzenthaler, M., and Jentzen, A. Numerical approximations of stochastic differential equations with non-globally Lipschitz continuous coefficients. *Mem. Amer. Math. Soc. 4* (2015), 1–112.
[7] Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 467* (2011), 1563–1576.
[8] Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients. *Ann. Appl. Probab. 22*, 4 (2012), 1611–1641.
[9] Hutzenthaler, M., Jentzen, A., and Kloeden, P. E. Divergence of the multilevel Monte Carlo Euler method for nonlinear stochastic differential equations. *Ann. Appl. Probab. 23*, 5 (2013), 1913–1966.
[10] Jentzen, A., Müller-Gronbach, T., and Yaroslavtseva, L. On stochastic differential equations with arbitrary slow convergence rates for strong approximation. *Commun. Math. Sci. 14* (2016), 1477–1500.
[11] Müller-Gronbach, T., and Ritter, K. Minimal errors for strong and weak approximation of stochastic differential equations. In *Monte Carlo and quasi-Monte Carlo methods 2006*. Springer, Berlin, 2008, pp. 53–82.
[12] Müller-Gronbach, T., and Yaroslavtseva, L. On sub-polynomial lower error bounds for quadrature of SDEs with bounded smooth coefficients. *Stochastic Analysis and Applications 0*, 0 (2017), 1–29.
[13] Sabanis, S. A note on tamed Euler approximations. *Electron. Commun. Probab. 18* (2013), no. 47, 1–10.
[14] Sabanis, S. Euler approximations with varying coefficients: The case of superlinearly growing diffusion coefficients. *Ann. Appl. Probab. 26*, 4 (2016), 2083–2105.
[15] Tretyakov, M. V., and Zhang, Z. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM J. Numer. Anal. 51*, 6 (2013), 3135–3162.
[16] Wang, X., and Gan, S. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *J. Difference Equ. Appl. 19*, 3 (2013), 466–490.
[17] Yaroslavtseva, L. On non-polynomial lower error bounds for adaptive strong approximation of SDEs. *arXiv:1609.08073* (2016), 19 pages.

On a stochastic version of the Prothero–Robinson problem

Raphael Kruse

In 1974 Prothero and Robinson [4] characterized stiffness in some nonlinear ordinary differential equations by studying suitably linearized versions of these systems. Hereby they generalized the notion of *A-stability* and introduced the concept of *stiffly accurate* one-step methods. See also [2, Section IV.15] for more details. In this note we discuss a stochastic version of their problem, illustrating that

their considerations also have significance for the numerical solution of stochastic differential equations (SDEs). In the case of a somewhat extreme choice of the parameter values it will turn out that the standard Euler–Maruyama method amplifies round-off errors exponentially for all practical choices of the step size, while its implicit counterpart is unconditionally stable. This has relevance, for example, in the numerical solution of stochastic partial differential equations or for multi-level Monte Carlo algorithms [1].

1. Derivation of the stochastic Prothero–Robinson problem

Let $T \in (0, \infty)$ and $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space endowed with a right continuous filtration $(\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions. By $Y\colon [0,T]\times\Omega \to \mathbb{R}$ we denote an $(\mathcal{F}_t)_{t\in[0,T]}$-adapted Itō process that is a solution to the SDE

$$\begin{aligned} \mathrm{d}Y(t) &= f(Y(t))\,\mathrm{d}t + g\,\mathrm{d}W(t), \quad t \in [0,T], \\ Y(0) &= Y_0 \in \mathbb{R}. \end{aligned} \tag{1}$$

Here $W\colon [0,T]\times\Omega \to \mathbb{R}$ is a standard $(\mathcal{F}_t)_{t\in[0,T]}$-Wiener process, $g \in \mathbb{R}$, and the drift coefficient function $f\colon \mathbb{R} \to \mathbb{R}$ is assumed to be sufficiently smooth to ensure that Y is uniquely determined with existing exponential moments. For further details on SDEs we refer, for instance, to [3].

Next, we introduce a further Itō process $U\colon [0,T]\times\Omega \to \mathbb{R}$ that also solves the SDE (1) but with a different initial value $U_0 \in \mathbb{R}$. More precisely,

$$U(t) = U_0 + \int_0^t f(U(s))\,\mathrm{d}s + gW(t), \quad t \in [0,T].$$

In order to compare Y and U we apply a first order Taylor expansion on the drift coefficient function along the trajectory of Y and obtain

$$\begin{aligned} U(t) &= U_0 + \int_0^t f(Y(s)) + \big(f(U(s)) - f(Y(s))\big)\,\mathrm{d}s + gW(t) \\ &= Y(t) + \big(U_0 - Y_0\big) + \int_0^t f'(Y(s))\big(U(s) - Y(s)\big) + \xi(s)\,\mathrm{d}s. \end{aligned}$$

If U_0 is close to Y_0 and if $f'(Y(s)) \in (-\infty, 0)$ for all $s \in [0,T]$ it is reasonable to assume that typical trajectories of U and Y will also stay close to each other and that the higher order term ξ can be neglected. Therefore, at least in a close proximity to Y, the dynamics of U should already be captured by the following linear SDE

$$\begin{aligned} \mathrm{d}X(t) &= -\lambda(t)\big(X(t) - Y(t)\big)\,\mathrm{d}t + g\,\mathrm{d}Y(t), \quad t \in [0,T], \\ X(0) &= X_0 := U_0 \in \mathbb{R}, \end{aligned} \tag{2}$$

where $\lambda(t) := -f'(Y(t))$ for all $t \in [0,T]$. Note that (2) is a stochastic generalization of the equation studied by Prothero and Robinson in [4]. In fact, if $g = 0$ we recover their original deterministic problem. In order to simplify the presentation we assume in the following that $\lambda(t) \equiv \lambda \in (0, \infty)$ is some positive given constant.

Before we discuss the numerical approximation of X let us briefly collect a few properties of X. First, due to

$$\mathrm{d}(X-Y)(t) = -\lambda\big(X(t)-Y(t)\big)\,\mathrm{d}t, \quad t\in[0,T], \quad (X-Y)(0) = X_0 - Y_0,$$

we obtain that

$$\big\|X(t)-Y(t)\big\|_{L^2(\Omega;\mathbb{R})} = \mathrm{e}^{-\lambda t}|X_0-Y_0| \tag{3}$$

for all $t\in[0,T]$. In particular, if we have $X_0 = Y_0$, then the Itō process Y is also the uniquely determined solution to (2). Moreover, the relationship (3) stays true on the level of single sample paths. Hence, if Y possesses exponential moments then X will too.

2. Stability of numerical approximations

In this section we investigate the stability of the standard Euler–Maruyama method and its implicit counter-part. To this end, we assume that $|X_0 - Y_0| \le \epsilon \in (0,1)$. Let say $\epsilon \approx 10^{-16}$ is on the level of a typical value for the machine precision. Then, due to (3) we know that the solution X to (2) always stays in an ϵ-neighborhood of Y. This raises the question, if this property is reproduced by a numerical approximation of (2).

We easily find this to be true for the implicit version of the Euler–Maruyama method. For its formulation let $h\in(0,1)$ denote the step size and set $N_h = \lfloor \frac{T}{h} \rfloor \in \mathbb{N}$. Then, the *backward Euler-Maruyama* method for the numerical solution of (2) is determined by $Z^0 = X_0$ and the recursion

$$Z^n = Z^{n-1} - h\lambda\big(Z^n - Y(t_n)\big) + \Delta Y^n, \quad n\in\{1,\dots,N_h\}, \tag{4}$$

with $\Delta Y^n := Y(t_n) - Y(t_{n-1})$. From (4) and $\lambda\in(0,\infty)$ we directly obtain for all values of $h\in(0,1)$ and $n\in\{1,\dots,N_h\}$ the estimate

$$\|Z^n - Y(t_n)\|_{L^2(\Omega;\mathbb{R})} = \frac{1}{1+h\lambda}\|Z^{n-1}-Y(t_{n-1})\|_{L^2(\Omega;\mathbb{R})} = \cdots = \frac{|X_0-Y_0|}{(1+h\lambda)^n} < \epsilon.$$

Moreover, an application of the triangle inequality and (3) also yields

$$\|Z^n - X(t_n)\|_{L^2(\Omega;\mathbb{R})} \le 2\epsilon,$$

for all $n\in\{1,\dots,N_h\}$.

On the other hand, it is not very surprising that the standard (explicit) Euler–Maruyama method fails to have the same unconditional stability property. To be more precise, the explicit version of (4) is given by $X^0 := X_0$ and

$$X^n = X^{n-1} - h\lambda\big(X^{n-1} - Y(t_{n-1})\big) + \Delta Y^n, \quad n\in\{1,\dots,N_h\}. \tag{5}$$

In this case we easily compute

$$\begin{aligned}\|X^n - Y(t_n)\|_{L^2(\Omega;\mathbb{R})} &= |1-h\lambda|\,\|X^{n-1}-Y(t_{n-1})\|_{L^2(\Omega;\mathbb{R})}\\ &= \cdots = |1-h\lambda|^n|X_0-Y_0|.\end{aligned}$$

Then, exactly as in the deterministic situation described in [4], we distinguish between two important cases: If $|1-h\lambda|\le 1$, then the standard Euler–Maruyama method will also stay in an ϵ-neighborhood of Y for all time steps. However,

if $|1 - h\lambda| > 1$ then a rounding error in the initial condition will be amplified exponentially with the number of steps n. Thus, if we want to avoid this very unfavourable behavior we need to choose $h \in (0, \frac{2}{\lambda}]$. Depending on the value of $\lambda \in (0, \infty)$ this can lead to a very restrictive condition on the step size. To make this point clear, set $\lambda = \epsilon^{-1} \approx 10^{16}$. If $T = 2$ then we need at least $N_h > 10^{16}$ steps before the explicit Euler–Maruyama method gives a stable approximation of $X(T)$. Assuming that each step takes about 1 ms on a computer, this sums up to a total computing time of about 316887 years.

3. Conclusions

In this note we have seen that the problem studied by Prothero and Robinson in [4] can easily be generalized to SDEs. If $\lambda \in (0, \infty)$, then the solution Y to (2) is asymptotically stable. In particular, we have seen in (3) that solutions to (2) depend Lipschitz continuously on the initial condition. While this stability property is just more pronounced if the value for λ increases, the effect of large values for λ is sometimes reversed for the numerical solution. As we have seen above, the standard Euler–Maruyama method only gives a stable approximation if $h \in (0, \frac{2}{\lambda})$. Depending on the value of λ this might constitute a very restrictive condition on the choice of the step size h. On the other hand, this kind of step size restriction does not apply to the backward Euler–Maruyama method.

In many applications, this can be an important difference. For instance, for multi-level Monte Carlo algorithms [1] it is decisive to apply one-step methods that give stable approximations already for rather large values of h.

References

[1] M. B. Giles. *Multilevel Monte Carlo path simulation. Oper. Res.*, 56(3):607–617, 2008.
[2] E. Hairer and G. Wanner. *Solving ordinary differential equations. II*, volume 14 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2010. Stiff and differential-algebraic problems, Second revised edition, paperback.
[3] X. Mao. *Stochastic Differential Equations and their Applications*. Horwood Publishing Series in Mathematics & Applications. Horwood Publishing Limited, Chichester, 1997.
[4] A. Prothero and A. Robinson. *On the stability and accuracy of one-step methods for solving stiff systems of ordinary differential equations. Math. Comp.*, 28:145–162, 1974.

Mean-square stability analysis of SPDE approximations

Annika Lang
(joint work with Andreas Petersson, Andreas Thalhammer)

The goal of mean-square stability analysis is to describe the longtime dynamics of a stochastic partial differential equation (SPDE) with initial condition in the neighborhood of an equilibrium, i.e., a constant solution of the SPDE. One can think of this property as continuity in mean-square with respect to the initial condition in the equilibrium. This subject has been studied in the literature, see, e.g., [1]. In what follows we investigate under which conditions approximations of solutions preserve the mean-square stability properties of the SPDE. Therefore, we

first set up the framework and consider the theory from an abstract point of view before applying it to frequently used approximation schemes. For more details, the reader is referred to [2].

Framework. Let H be a real, separable Hilbert space and $(\Omega, \mathcal{A}, (\mathcal{F}_t, t \geq 0), P)$ be a complete filtered probability space satisfying the "usual conditions". In the framework of [3], we consider the linear S(P)DE

$$\mathrm{d}X(t) = (AX(t) + F(X(t)))\,\mathrm{d}t + G(X(t))\,\mathrm{d}M(t) \tag{1}$$

for $t \in \mathbb{R}_+$ with $\mathcal{F}_0$-measurable initial condition $X(0) = X_0 \in L^2(\Omega; H)$. Here, $A : \mathcal{D}(A) \to H$ is the generator of a C_0-semigroup $S = (S(t), t \geq 0)$ on H and $F \in L(H)$. Let us further assume that $M = (M(t), t \geq 0)$ is a U-valued, càdlàg, square-integrable martingale such that

$$\langle\langle M, M\rangle\rangle_t - \langle\langle M, M\rangle\rangle_s \leq (t-s)Q,$$

where $Q \in L(U)$ is a self-adjoint, positive semidefinite trace class operator. Finally, let $G \in L(H; L_{\mathrm{HS}}(Q^{1/2}(U); H))$.

To approximate solutions of (1), let $(V_h, h \in (0,1])$ be a family of finite-dimensional subspaces $V_h \subset H$ with $\dim(V_h) = N_h \in \mathbb{N}$ indexed by a refinement parameter h for the space approximation. In time we choose time steps $t_j = j\Delta t$, $j \in \mathbb{N}_0$, with fixed time step size $\Delta t > 0$. For a given time point t_j, we denote an approximation of $X(t_j)$ in V_h by X_h^j, which is obtained by a numerical approximation scheme given by the recursion

$$X_h^{j+1} = D_{\Delta t,h}^{\mathrm{det}} X_h^j + D_{\Delta t,h}^{\mathrm{stoch},j} X_h^j \tag{2}$$

with initial condition X_h^0 approximating X_0. The linear operator $D_{\Delta t,h}^{\mathrm{det}} \in L(V_h)$ approximates the deterministic linear (partial) differential equation

$$\frac{\partial}{\partial t} u(t) = (A + F)u(t).$$

Furthermore, let $D_{\Delta t,h}^{\mathrm{stoch},j}$ be an $L(V_h)$-valued approximation of the stochastic Itô integral on the time interval $[t_j, t_{j+1})$. We assume that the family of stochastic approximation operators $(D_{\Delta t,h}^{\mathrm{stoch},j}, j \in \mathbb{N}_0)$ is $\mathcal{F}$-compatible.

(Asymptotic) mean-square stability. We are interested in the qualitative behavior of the zero solution of (2), where the zero solution is the solution $X_{h,e}^j = 0$ of (2) that is induced by the zero initial condition $X_h^0 = 0$.

Definition 1. Let $X_h = (X_h^j, j \in \mathbb{N}_0)$ be the numerical approximation given by (2) for fixed time step size Δt and refinement parameter h. Then, the zero solution $(X_{h,e}^j = 0, j \in \mathbb{N}_0)$ of (2) is called *mean-square stable* if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $j \in \mathbb{N}_0$, $\mathbb{E}[\|X_h^j\|_H^2] < \varepsilon$ whenever $\mathbb{E}[\|X_h^0\|_H^2] < \delta$.

It is called *asymptotically mean-square stable* if it is mean-square stable and there exists $\delta > 0$ such that $\mathbb{E}[\|X_h^0\|_H^2] < \delta$ implies $\lim_{j\to\infty} \mathbb{E}[\|X_h^j\|_H^2] = 0$. Furthermore, it is *asymptotically mean-square unstable* if it is not asymptotically mean-square stable.

In the considered abstract framework of approximation schemes, we are able to specify sufficient conditions in terms of tensor products of the approximation operators that ensure asymptotic mean-square stability.

Theorem 2. *Let $X_h = (X_h^j, j \in \mathbb{N}_0)$ be given by (2). Furthermore, let*

$$\mathcal{S}_j = D^{\mathrm{det}}_{\Delta t,h} \otimes D^{\mathrm{det}}_{\Delta t,h} + \mathbb{E}[D^{\mathrm{stoch},j}_{\Delta t,h} \otimes D^{\mathrm{stoch},j}_{\Delta t,h}].$$

Then, the zero solution of (2) is asymptotically mean-square stable if

$$\lim_{j\to\infty} \|\mathcal{S}_j \cdots \mathcal{S}_0\|_{L(V_h^{(2)})} = 0.$$

If we assume additionally that the stochastic operators $(D^{\mathrm{stoch},j}_{\Delta t,h}, j \in \mathbb{N}_0)$ are independent and identically distributed (iid), we obtain an equivalence instead of a sufficient condition.

Corollary 3. *Let $X_h = (X_h^j, j \in \mathbb{N}_0)$ be given by (2) with $(D^{\mathrm{stoch},j}_{\Delta t,h}, j \in \mathbb{N}_0)$ iid. Then, the zero solution of (2) is asymptotically mean-square stable if and only if*

$$\mathcal{S} = D^{\mathrm{det}}_{\Delta t,h} \otimes D^{\mathrm{det}}_{\Delta t,h} + \mathbb{E}[D^{\mathrm{stoch},1}_{\Delta t,h} \otimes D^{\mathrm{stoch},1}_{\Delta t,h}] \in L(V_h^{(2)})$$

satisfies $\rho(\mathcal{S}) < 1$, where $\rho(\mathcal{S})$ is the spectral radius.

Galerkin approximation. Let us consider next the specific case that $-A$ is densely defined, self-adjoint, and positive definite with compact inverse as, e.g., the Laplace operator on a smooth domain. Then, $-A$ has a non-decreasing sequence of positive eigenvalues $(\lambda_i, i \in \mathbb{N})$ for an orthonormal basis of eigenfunctions $(e_i, i \in \mathbb{N})$ in H. Furthermore, we restrict ourselves to a Lévy process L with covariance operator Q as the specific martingale M.

Assume that $(V_h, h \in (0,1])$ is a nested sequence of subspaces of H that satisfies $V_h \subset \mathcal{D}((-A)^{1/2}) \subset H$. Denote by P_h the orthogonal projection onto V_h. We define the discrete operator $-A_h : V_h \to V_h$ by the Galerkin projection of $-A$ onto V_h. This implies that also $-A_h$ has a sequence of orthonormal eigenfunctions $(e_{h,i}, i = 1, \ldots, N_h)$ and positive non-decreasing eigenvalues $(\lambda_{h,i}, i = 1, \ldots, N_h)$. Let $R(\Delta t A_h) = r_d^{-1}(\Delta t A_h) r_n(\Delta t A_h)$ be a rational approximation of the semigroup, where r_n and r_d are polynomials. Then, the *Euler–Maruyama scheme* is given by

$$\begin{aligned} X_h^{j+1} &= \left(R(\Delta t A_h) + r_d^{-1}(\Delta t A_h)(\Delta t P_h F + P_h G(\cdot)\Delta L^j)\right) X_h^j, \\ X_h^0 &= P_h X_0 \end{aligned} \tag{3}$$

for $j \in \mathbb{N}_0$, where the increments of the Lévy process L are denoted by $\Delta L^j = L(t_{j+1}) - L(t_j)$. For this type of numerical approximation, the result from Corollary 3 can be specified.

Theorem 4. *Consider the approximation scheme* (3).

(1) (Backward Euler scheme, $R(z) = (1-z)^{-1}$). *A sufficient condition for the asymptotic mean-square stability of the zero solution is*

$$\frac{(1+\Delta t\|F\|_{L(H)})^2 + \Delta t\,\mathrm{tr}(Q)\|G\|^2_{L(H;L(U;H))}}{(1+\Delta t\lambda_{h,1})^2} < 1.$$

(2) (Crank–Nicolson scheme, $R(z) = (1-z/2)^{-1}(1+z/2)$). *A sufficient condition for the asymptotic mean-square stability of the zero solution is*

$$\left(\max_{k\in\{1,N_h\}}\left|\frac{1-\Delta t\lambda_{h,k}/2}{1+\Delta t\lambda_{h,k}/2}\right| + \Delta t\frac{\|F\|_{L(H)}}{(1+\Delta t\lambda_{h,1}/2)}\right)^2 + \Delta t\frac{\mathrm{tr}(Q)\|G\|^2_{L(H;L(U;H))}}{(1+\Delta t\lambda_{h,1}/2)^2} < 1.$$

(3) (Forward Euler scheme, $R(z) = 1+z$). *A sufficient condition for the asymptotic mean-square stability of the zero solution is*

$$\left(\max_{\ell\in\{1,N_h\}}|1-\Delta t\lambda_{h,\ell}| + \Delta t\|F\|_{L(H)}\right)^2 + \Delta t\,\mathrm{tr}(Q)\|G\|^2_{L(H;L(U;H))} < 1.$$

Similar conditions can be obtained for Milstein schemes. Simulations of the stochastic heat equation with spectral and finite element methods confirm the theoretical findings [2].

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References

[1] K. Liu, *Stability of Infinite Dimensional Stochastic Differential Equations with Applications*, vol. 135 of *Monographs and Surveys in Pure and Applied Mathematics*, Chapman & Hall/CRC, 2006.

[2] A. Lang, A. Petersson, and A. Thalhammer, *Mean-square stability analysis of approximations of stochastic differential equations in infinite dimensions*, arXiv:1702.07700, 2017.

[3] S. Peszat and J. Zabczyk, *Stochastic Partial Differential Equations with Lévy Noise. An Evolution Equation Approach*, vol. 113 of *Encyclopedia of Mathematics and Its Applications*, Cambridge University Press, 2007.

On the convergence of numerical approximations of the stochastic Cahn–Hilliard equation

Stig Larsson
(joint work with Daisuke Furihata, Mihály Kovács, Fredrik Lindgren, and Ali Mesforush)

In two talks I surveyed methods for proving convergence of numerical schemes for the stochastic Cahn–Hilliard equation, also known as the Cahn–Hilliard–Cook equation:

$$\begin{cases} \mathrm{d}u - \Delta v \,\mathrm{d}t = \mathrm{d}W & \text{in } \mathcal{D}\times(0,T]; \\ v + \Delta u - f(u) = 0 & \text{in } \mathcal{D}\times(0,T]; \\ \dfrac{\partial u}{\partial n} = \dfrac{\partial v}{\partial n} = 0 & \text{on } \partial\mathcal{D}\times(0,T]; \\ u(0) = u_0 & \text{in } \mathcal{D}. \end{cases}$$

Here $\mathcal{D}\subset\mathbf{R}^d$, $d\le 3$, is a convex polygonal domain and

$$f(s) = F'(s), \quad F \text{ is a polynomial of degree 4,}$$
$$F(s) \ge c_0 s^4 - c_1,\ c_0>0; \quad F''(s)\ge -\beta^2,$$

Typically: $F(s)=\frac{1}{4}(s^2-\beta^2)^2$, $f(s)=s^3-\beta^2 s$.

Eliminate $v=-\Delta u+f(u)$ and set $X=u\in H=L_2(\mathcal{D})$. Let $A=-\Delta$ be the Neumann Laplacian in H and $W(t)$ – a Q-Wiener process in H with respect to $(\Omega,\mathcal{F},\mathbf{P},\{\mathcal{F}_t\})$. Then we have

$$\begin{cases} \mathrm{d}X + \big(A^2X + Af(X)\big)\,\mathrm{d}t = \mathrm{d}W, \quad t>0; \\ X(0)=X_0. \end{cases}$$

A mild solution satisfies the equation:

$$X(t) = \mathrm{e}^{-tA^2}X_0 - \int_0^t \mathrm{e}^{-(t-s)A^2}Af(X(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s).$$

We first consider spatial discretization by a standard finite element method:

$$\begin{cases} \mathrm{d}X_h + A_h^2X_h\,\mathrm{d}t + A_hP_hf(X_h)\,\mathrm{d}t = P_h\,\mathrm{d}W, \quad t>0; \\ X(0)=P_hX_0. \end{cases}$$

Its mild formulation is:

$$X_h(t) = \mathrm{e}^{-tA_h^2}P_hX_0 - \int_0^t \mathrm{e}^{-(t-s)A_h^2}A_hP_hf(X_h(s))\,\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)A_h^2}P_h\,\mathrm{d}W(s).$$

The implicit Euler method is:

(1)
$$\begin{cases} X_h^j - X_h^{j-1} + kA_h^2X_h^j + kA_hP_hf(X_h^j) = P_h\Delta W^j, \quad t_j=jk,\ j=1,2,\dots,N; \\ X_h^0 = P_hX_0. \end{cases}$$

Let $R^n_{k,h} = (I + k\,A_h^2)^{-n}$. Then the mild formulation is:

$$X_h^n = R^n_{k,h} P_h X_0 - \sum_{j=1}^{n} R^{n-j+1}_{k,h} A_h P_h f(X_h^j) + \sum_{j=1}^{n} R^{n-j+1}_{k,h} P_h \Delta W^j.$$

A first step towards showing convergence is to prove an error estimate for the stochastic convolutions:

$$W_A(t) = \int_0^t \mathrm{e}^{-(t-s)A^2}\,\mathrm{d}W(s), \quad W^n_{A_h} \sum_{j=1}^{n} R^{n-j+1}_{k,h} P_h \Delta W^j.$$

The most advanced result goes as follows: Let $\gamma \in (0, \frac{1}{2}]$, $\beta \in [1,2]$, and $p > \frac{2}{\gamma}$. Then there is $C = C(p,\gamma,T)$ such that

$$\left(\mathbf{E}\Big(\sup_{t_n\in[0,T]} \|W_A(t_n) - W^n_{A_h}\|^p\Big)\right)^{1/p} \le C(h^\beta + k^{\beta/4})\|A^{(\beta-2)/2+\gamma} Q^{1/2}\|_{\mathrm{HS}}.$$

The proof is based on a "factorization argument" from Da Prato–Zabczyk and error estimates for the semigroup.

The challenge in estimating the remaining part of the error $X(t_n) - X_h^n$ is to control the Lipschitz constant of the nonlinear term $G(u) = Af(u)$. It is not globally Lipschitz and does not have a linear growth. Instead we have local Lipschitz conditions, for example,

$$\begin{aligned}\|G(u) - G(v)\|_{H^{-3/2}} &\le C\|f(u) - f(v)\|_{H^{-1}} \le C\|f(u) - f(v)\|_{L_{6/5}} \\ &\le C\|(1 + u^2 + v^2)(u - v)\|_{L_{6/5}} \le C(1 + \|u\|^2_{L_6} + \|v\|^2_{L_6})\|u - v\|_{L_2} \\ &\le C(1 + \|u\|^2_{H^1} + \|v\|^2_{H^1})\|u - v\|_{L_2}.\end{aligned}$$

This indicates the need for moment bounds for $u = X(t_n)$ and $v = X_h^n$.

To obtain such bounds, we introduce the energy functional:

$$J(u) = \tfrac{1}{2}\|\nabla u\|^2 + \int_{\mathcal{D}} F(u)\,\mathrm{d}x = \tfrac{1}{2}\|u\|^2_{H^1} + \mathcal{F}(u).$$

In the deterministic equation ($W = 0$),

$$\dot{u} + Av = 0, \quad v = Au + f(u),$$

we multiply by v:

$$\langle \dot{u}, v\rangle + \langle Av, v\rangle = 0.$$

Here $\langle \dot{u}, v\rangle = D_t J(u)$, so that after integration

$$J(u(t)) + \int_0^t \|v\|^2_{H^1}\,\mathrm{d}s = J(u(0)),$$

that is, we have a Lyapunov functional: $J(u(t)) \le J(u(0))$. Since $\mathcal{F}(u)$ is equivalent to $\|u\|^4_{L_4}$, we now have a bound for $\|u\|^2_{H^1} + \|u\|^4_{L_4}$. This was extended to the

stochastic equation by [4] and to its time-continuous spatial discretization by [2, 3]. The result is: If $\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}}^2 < \infty$, then

$$\mathbf{E}\Big[\sup_{t\in[0,T]}\Big(\|X(t)\|_{H^1}^2 + \|X(t)\|_{L_4}^4 + \|X_h(t)\|_{H^1}^2 + \|X_h(t)\|_{L_4}^4\Big)\Big] \le K_T.$$

The proof is based on Ito's formula for $J(X(t))$ and $J(X_h(t))$. In the time-discrete case we cannot use Ito's formula but instead we follow the deterministic argument above and multiply the equation

$$X_h^j - X_h^{j-1} + kA_hY_h^j = P_h\Delta W^j, \quad Y_h^j = A_hX_h^j + P_hf(X_h^j),$$

by Y_h^j to get

$$\langle X_h^j - X_h^{j-1}, Y_h^j\rangle + k|Y_h^j|_1^2 = \langle Y_h^j, P_h\Delta W^j\rangle$$

Here:

$$\langle X_h^j - X_h^{j-1}, Y_h^j\rangle \ge J(X_h^j) - J(X_h^{j-1}) + \tfrac{1}{2}|X_h^j - X_h^{j-1}|_1^2 - \beta^2\|X_h^j - X_h^{j-1}\|^2.$$

Sum up (with $\Delta X_h^j := X_h^j - X_h^{j-1}$):

$$J(X_h^n) + k\sum_{j=1}^n |Y_h^j|_1^2 + \tfrac{1}{2}\sum_{j=1}^n |\Delta X_h^j|_1^2 \le J(X_h^0) + \sum_{j=1}^n \langle Y_h^j, P_h\Delta W^j\rangle + \beta^2\sum_{j=1}^n \|\Delta X_h^j\|^2.$$

Raise to power p, take $\sup_{1\le n\le N}$ and then $\mathbf{E}$. The most difficult term is:

$$\sum_{j=1}^n \langle \Delta Y_h^j, P_h\Delta W^j\rangle = \sum_{j=1}^n \langle A_h(X_h^j - X_h^{j-1}) + P_h(f(X_h^j) - f(X_h^{j-1})), P_h\Delta W^j\rangle.$$

This leads to a delicate calculation of the Lipschitz constant of f using auxiliary moment bounds. The result is, [1]: Let $p \ge 1$. If $\|A^{1/2}Q^{1/2}\|_{\mathrm{HS}} \le L$ and $|X_h^0|_1 + \mathcal{F}(X_h^0) + |Y_h^0|_1 \le L$, then there exists $C, k_0 > 0$, depending on p, T and L, such that

$$\mathbf{E}\sup_{1\le j\le N}|X_h^j|_1^{2p} + \mathbf{E}\sup_{1\le n\le N}\mathcal{F}(X_h^n)^p + \mathbf{E}\Big(\sum_{j=1}^N k|Y_h^j)|_1^2\Big)^p \le C, \quad k \le k_0.$$

With such bounds we can use Chebyshev's inequality to obtain bounds that are uniform on sets of arbitrarily large probability, $\mathbf{P}(\Omega_{h,k}^\epsilon) > 1-\epsilon$. Then deterministic error analysis applies and we obtain error estimates. However, the information of the rate of convergence is lost when $\epsilon \to 0$.

References

[1] D. Furihata, M. Kovács, S. Larsson, and F. Lindgren, *Strong convergence of a fully discrete finite element approximation of the stochastic Cahn–Hilliard equation*, `arXiv:1612.09459`

[2] M. Kovács, S. Larsson, and A. Mesforush, *Finite element approximation of the Cahn–Hilliard–Cook equation*, SIAM J. Numer. Anal. **49** (2011), 2407–2429.

[3] ———, *Erratum: Finite element approximation of the Cahn–Hilliard–Cook equation*, SIAM J. Numer. Anal. **52** (2014), 2594–2597.

[4] G. Da Prato and A. Debussche, *Stochastic Cahn–Hilliard equation*, Nonlinear Anal. **26** (1996), no. 2, 241–263.

On stochastic Brinkman–Forchheimer anisotropic 3D Navier–Stokes equations

Annie Millet
(joint work with Hakima Bessaih)

The talk is divided in two parts.

We at first recall some results about the stochastic 2D Navier–Stokes equations and more general 2D hydrodynamical models such as the general Bénard problem, the 3D α-model, ... and some known results about their numerical approximations. Well-posedness holds for a general multiplicative noise directed by some infinite dimensional Brownian motion when the bilinear term satisfies some assumptions stated in terms of some Hilbert spaces $V \subset H$ and some interpolation space (see e.g. [5]). Much more is needed for the convergence of numerical schemes. In [4], E. Carelli and A. Prohl study an implicit time discretization of a periodic 2D Navier–Stokes equation on $[0, D]^2$. They impose more restrictive regularity on the initial condition as well as larger moments, and some stronger assumptions on the diffusion coefficient σ. Furthermore, due to the non linearity of the equation, the L^2 convergence is stated localized on large subsets of the probability space. The same is true for the sightly different semi-implicit time scheme coupled with some finite elements discretization in space. In [7] P. Dörsek studies the weak speed of convergence of a spectral Galerkin approximation in space coupled with a Strang time splitting scheme when the stochastic perturbation is driven by a finite dimensional noise. Once more the Navier-Stokes equation is studied on a torus with an H^1-valued initial condition and the function for which the semi-group of the solution and that of the approximation are compared is extremely regular. In [1], we obtain a speed of convergence in probability and a localized speed of convergence in L^2 under conditions similar to those of [4]. One may wonder if some better results could not be obtained for more general 2D hydrodynamical models.

In the setting of "classical" (that is isotropic) Navier–Stokes equations on $\mathbb{R}^3$ or on a bounded domain of $\mathbb{R}^3$ subject to a random perturbation, M. Röckner, T. Zhang and X. Zhang have written several papers (see e.g. [9], [8]) where they study the "tamed" model with the usual incompressibility property $\nabla \cdot u = 0$:

$$\partial_t u + \big[-\nu \Delta u + (u \cdot \nabla)u + g_N(|u|^2)u + \nabla p\big]dt = \sigma(t,u)dW_t.$$

The function g_N satisfies: $g_N(r) = (r - N)/\nu$ for $r > N + 1$, $g_N(r) = 0$ if $r \in [0, N]$ and $0 \le g_N'(r) \le 2/(\nu \wedge 1)$. Under proper growth and Lipschitz conditions on the diffusion coefficient $\sigma(t, u)$ for Hilbert–Schmidt norms in H^0 and H^1, they prove that if $u_0 \in L^2(\Omega, H^1)$ is independent of the driving noise W, the stochastic tamed 3D Navier–Stokes equation has a unique solution in $L^2(\Omega; L^\infty(0, T; H^1)) \cap L^2(\Omega \times [0, T]; H^2)$. In [9] the authors prove that if $\sigma = 1$, that is in case of an additive noise, there exists a unique invariant measure.

In [2] we study an anisotropic Navier–Stokes equation in dimension 3, that is with different viscosity in the horizontal and vertical directions, and subject

to some random forcing with intensity that may depend on the solution. More precisely, we consider the following model of a modified 3D anisotropic Navier–Stokes system on a fixed time interval $[0,T]$ which can be written as follows for $(t,x)\in[0,T]\times\mathbb{R}^3$: $\nabla\cdot u=0$ and

$$\partial_t u+\big[-\nu\,\Delta_h u+\big(u\cdot\nabla\big)u+a\,\big|u\big|^{2\alpha}u+\nabla p\big]dt=\sigma(t,u)\,dW(t), \tag{1}$$

with the initial condition u_0 independent of the driving noise W. Here the viscosity ν and the coefficient a of the nonlinear convective term are strictly positive, $\alpha>1$, ∂_t denotes the time partial derivative, $\Delta_h:=\partial_1^2+\partial_2^2$ and ∂_i denotes the partial derivative in the direction x_i, $i=1,2,3$. As usual the fluid is incompressible, p denotes the pressure. The forcing term $\sigma(t,u)\,\dot{W}$ is a multiplicative noise driven by an infinite dimensional Brownian motion W which is white in time with spatial correlation. The relevance of the anisotropic viscosity is explained through the Ekeman law. The extra term $a|u|^{2\alpha}u$ is of Brinkman–Forchheimer extended Darcy type and has a regularization effect which can balance on one hand the vertical partial derivative of the bilinear term to prove existence, and on the other hand provide some upper bound to obtain uniqueness. Note that the space $L^{2\alpha+2}(\mathbb{R}^3)$ appears naturally in the analysis of the equation; it is equal to $L^4(\mathbb{R}^3)$ if $\alpha=1$. Furthermore, the homogeneous critical Sobolev space $\dot{H}^{1/2}(\mathbb{R}^3)$ for the Navier–Stokes equation is included in $L^4(\mathbb{R}^3)$. Hence it is natural to impose $\alpha>1$.

We set $H=\big\{u\in L^2(\mathbb{R}^3;\mathbb{R}^3)\,;\,\nabla\cdot u=0 \text{ in } \mathbb{R}^3\big\}$ and project the equation on divergence free fields. Due to the anisotropic feature of the model, we use anisotropic Sobolev spaces defined as follows: given $s,s'\in\mathbb{R}$ let $H^{s,s'}$ denote the set of tempered distributions $\psi\in\mathcal{S}'(\mathbb{R}^3)$ such that

$$\|\psi\|_{s,s'}^2:=\int_{\mathbb{R}^3}\big(1+\big|(\xi_1,\xi_2)\big|^{2s}\big)\,\big(1+\big|\xi_3\big|^{2s'}\big)\,\big|\mathcal{F}\psi(\xi)\big|^2\,d\xi<\infty,$$

where $\mathcal{F}$ denotes the Fourier transform. For non negative indices s,s' we set

$$\tilde{H}^{s,s'}:=\big(H^{s,s'}\big)^3\cap H\quad\text{and again}\quad\|\cdot\|_{s,s'}\quad\text{for the corresponding norm.}$$

The deterministic counterpart of (1) (that is the equation with $\sigma=0$), has been studied by H. Bessaih, S. Trabelsi and H. Zorgati in [3]. The authors have proved that if the initial condition $u_0\in\tilde{H}^{0,1}$, for any $T>0$ there exists a unique solution in $L^\infty(0,T;\tilde{H}^{0,1})\cap L^2(0,T;\tilde{H}^{1,1})$ which belongs to $C([0,T];H)$. We generalize this result by allowing the system to be subject to some random external force whose intensity may depend on the solution u and on its horizontal gradient $\nabla_h u$.

Note that in the deterministic setting (that is $\sigma=0$), replacing the Brinkman–Forchheimer term $a|u|^{2\alpha}u$ by $\frac{1}{\epsilon}u\times e_3$ in (1), J.Y. Chemin, B. Desjardin, I. Gallagher and E. Grenier [6] have also studied an anisoropic modified Navier Stokes equation on $\mathbb{R}^3$ with a vertical viscosity $\nu_v\geq 0$, which is allowed to vanish. Using some homogeneous anisotropic spaces, they have proved that if $u_0\in H^{0,s}$ with $s>\frac{3}{2}$,

there exists ϵ_0 depending only on ν and u_0 such that for $\epsilon \in (0, \epsilon_0]$,

$$\partial_t u - \nu\, \Delta_h u + \big(u \cdot \nabla\big)u + \frac{1}{\epsilon} u \times e_3 + \nabla p = 0, \quad \text{for } (t,x) \in [0,T] \times \mathbb{R}^3,$$

with $\nabla \cdot u = 0$, has a unique global solution in $L^\infty(0,T;H^{0,s}) \cap L^2(0,T;H^{1,s})$. The dispersive Brickman–Forchheimer term is "larger" than that used in [6] but the regularity required on the initial condition is weaker and we allow a stochastic forcing term. The power we insert in the Brinkman–Forchheimer term is also larger than that in [8] but our anisotropic setting is not covered by the tamed model. Furthermore, we require more integrability with respect to the probability but for a weaker norm of the initial condition.

We ask the noise intensity σ to satisfy growth conditions for some Hilbert–Schmidt norms from $\tilde{H}^{0,1}$ to H and to $\tilde{H}^{0,1}$ and some Lipschitz condition for a Hilbert–Schmidt norm from $\tilde{H}^{0,1}$ to H. Examples of such coefficients σ of Nemytski type are provided; σ may be a function of u and of a "small multiple" of its horizontal gradient $\nabla_h u$ related to the horizontal viscosity ν. We prove that if $u_0 \in L^4(\Omega, \tilde{H}^{0,1})$ is independent of W, our stochastic anisotropic equation has a unique solution in $L^4(\Omega; L^\infty(0,T;\tilde{H}^{0,1})) \cap L^2(\Omega; L^2(0,T : \tilde{H}^{1,1})) \cap L^{2\alpha+2}(\Omega \times [0,T] \times \mathbb{R}^3)$, which is almost surely continuous from $[0,T]$ to H.

Under stronger assumptions on σ (which may no longer depend on the horizontal gradient $\nabla_h u$), we also prove a large deviations principle in $C([0,T];H) \cap L^2(0,T;\tilde{H}^{1,0})$ when the noise intensity is multiplied by a small parameter $\sqrt{\epsilon}$ converging to 0.

References

[1] H. Bessaih, Z. Brzeźniak & A. Millet, *Splitting up method for the 2D stochastic Navier–Stokes equations, Stochastic Partial Differential Equations*, *Analysis and Computations* **2-4** (2014), p. 433-470.

[2] H. Bessaih & A. Millet, On Stochastic Brinkman–Forchheimer anisotropic 3D Navier–Stokes equations: Well posedness and large deviations, preprint submitted for publication.

[3] H. Bessaih, S. Trabelsi & H. Zorgati, Existence and uniqueness of global solutions for the modified anisotropic ED Navier–Stokes equatiions. *M2AN*, **50** (2016), 1817–1823.

[4] E. Carelli & A. Prohl, *Rates of convergence for discretizations of the stochastic incompressible Navier–Stokes Equations*, SIAM J. Numer. Anal., **50-5** (2012), p. 2467–2496.

[5] I. Chueshov & A. Millet, Stochastic 2D hydrodynamical type systems: Well-posedness and large deviations. *Applied Math. Optim.* **61(3)** (2010) , 379–420.

[6] J.-Y. Chemin, B. Desjardin, I. Gallagher & E. Grenier, Fluids with anisotropic viscosity, *M2AN* **34** (2000), 315-335.

[7] P. Dörsek, *Semigroup splitting and cubature approximations for the stochastic Navier-Stokes Equations*, SIAM J. Numer. Anal., **50-2** (2012), 729–746.

[8] M. Röckner & T. Zhang, Stochastic 3D tamed equations:Existence, uniqueness and small time large deviation principle, *J. Differential Equations* **252** (2012), 717–744.

[9] M. Röckner & X. Zhang, Stochastic tamed 3D Navier-Stokes equations: existence, uniqueness and ergodicity, *Probab. Theory Related Fields* **145 (1-2)** (2009), 211-267.

Simulating rough volatility models

Andreas Neuenkirch
(joint work with Taras Shalaiko)

Let $B = \{B_t, t \in \mathbb{R}\}$ be a fractional Brownian motion (fBm) with Hurst parameter $H \in (0, 1/2)$, i.e. B is a zero mean Gaussian process with continuous sample paths, $B_0 = 0$ and mean square smoothness

$$\mathbf{E}|B_t - B_s|^2 = |t-s|^{2H}, \quad s, t \in \mathbb{R}.$$

Moreover, let $V = \{V_t, t \geq 0\}$, $W = \{W_t, t \geq 0\}$ be two independent Brownian motions (i.e. fBms with $H = 1/2$), $\mu \in \mathbb{R}, \lambda, \theta, s_0 > 0$, $\rho \in (-1, 1)$ and consider

$$\begin{aligned} S_t &= s_0 e^{X_t}, \\ X_t &= -\frac{1}{2}\int_0^t e^{2Y_s} ds + \rho \int_0^t e^{Y_s} dV_s + \sqrt{1-\rho^2} \int_0^t e^{Y_s} dW_s, \\ Y_t &= \mu + \theta e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dB_s. \end{aligned} \tag{1}$$

Here $X = \{X_t, t \geq 0\}$ is the log-price of an asset, whose log-volatility $Y = \{Y_t, t \geq 0\}$ is given by the stationary solution of the Langevin equation

$$dY_t = \lambda(\mu - Y_t)dt + \theta dB_t,$$

i.e. Y is a stationary fractional Ornstein–Uhlenbeck process. The fractional Brownian motion B and the Brownian motion V are correlated, i.e.

$$\mathbf{E} B_t V_s = \gamma(t, s), \qquad t \in \mathbb{R},\ s \geq 0,$$

for some suitable covariance function $\gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}$, while B and W are independent, i.e.

$$\mathbf{E} B_t W_s = 0, \qquad t \in \mathbb{R},\ s \geq 0.$$

Such a model has been proposed by Gatheral, Jaisson and Rosenbaum based on striking empirical evidence that the log-volatility of assets behaves essentially as fBm with $H \approx 0.1$, see [3]. This model has been further analysed in [1].

In this talk, we will give a complexity result for the strong approximation of the log-asset price and we will present a covariance structure γ, which allows the efficient simulation of the corresponding rough volatility model.

1. The complexity Result

[These results can be found in [4].] Here we consider the optimal mean square approximation of X_T based on

$$V_0, V_{T/n}, \ldots, V_T, \ \ W_0, W_{T/n}, \ldots, W_T, \ \ Y_0, Y_{T/n}, \ldots, Y_T. \tag{2}$$

The fractional Ornstein–Uhlenbeck process (fOUp) Y is a Gaussian process with known mean and covariance functions and thus exact joint simulation of V, W, Y

at a finite number of time points is possible. Clearly, the optimal mean square approximation of X_T using (2) is given by

$$X_n^{\mathsf{opt}} = \mathbf{E}\big(X_T \,\big|\, V_{kT/n}, W_{kT/n}, Y_{kT/n},\, k = 0, \dots, n \,\big)$$

and the corresponding minimal errors are

$$e(n) = \big(\mathbf{E}|X_T - X_n^{\mathsf{opt}}|^2\big)^{1/2}.$$

Rough volatility models are numerically challenging in the sense that they admit only low convergence rates for the mean square approximation based on the information given by (2). We will show that

$$\liminf_{n\to\infty} \left(\frac{n}{T}\right)^{2H} e(n)^2 \geq (1-\rho^2)\frac{1}{(2H+1)(2H+2)} T\theta^2 \mathbf{E}|e^{Y_0}|^2.$$

The optimal convergence rate n^{-H} is obtained by the Euler method

$$X_n^{\mathsf{E}} = -\frac{1}{2}\sum_{k=0}^{n-1} e^{2Y_{k\Delta}}\Delta + \rho\sum_{k=0}^{n-1} e^{Y_{k\Delta}}\Delta_k V + \sqrt{1-\rho^2}\sum_{k=0}^{n-1} e^{Y_{k\Delta}}\Delta_k W,$$

where $\Delta = T/n$ and

$$\Delta_k V = V_{(k+1)\Delta} - V_{k\Delta}, \quad \Delta_k W = W_{(k+1)\Delta} - W_{k\Delta}, \qquad k = 0, \dots, n-1,$$

and also if a trapezoidal type discretization (instead of the Euler discretization) is applied to the Riemann integral and the Itō integral with respect to W.

For the Euler scheme we have

$$\lim_{n\to\infty} \left(\frac{n}{T}\right)^{2H} \mathbf{E}|X_T - X_n^{\mathsf{E}}|^2 = \frac{1}{2H+1} T\theta^2 \mathbf{E}|e^{Y_0}|^2,$$

while for the trapezoidal type scheme, we obtain a slightly smaller asymptotic constant. Note that

$$\mathbf{E}|e^{Y_t} - e^{Y_s}|^2 = \theta^2 \mathbf{E}|e^{Y_0}|^2 \cdot |t-s|^{2H} + o(|t-s|^{2H}) \qquad \text{for} \quad |t-s| \to 0,$$

i.e. the limiting constants on the right hand side of the above expression depend on the Hölder constant of the mean square smoothness of the volatility process $\{e^{Y_t},\, t \geq 0\}$.

2. Efficient Simulation

The only known exact method for the simulation of

$$V_0, V_{T/n}, \dots, V_T, \;\; W_0, W_{T/n}, \dots, W_T, \;\; Y_0, Y_{T/n}, \dots, Y_T$$

is the Cholesky method, which has after precomputation of the Cholesky decomposition of the covariance matrix still computational cost $\mathcal{O}(n^2)$, which makes the above complexity result even worse.

Here we discuss the choice of the covariance function γ and how this may simplify the simulation of a rough volatility model. Bayer, Friz and Gatheral [1] use

$$\gamma(t,s) = \frac{G(H)}{H+1/2}\left(t^{H+1/2} - (t - \min\{t,s\})^{H+1/2}\right)\mathbf{1}_{[0,\infty)\times[0,\infty)}(t,s),$$

which arises from the fact that their fBm is given by the Mandelbrot–van Ness representation

$$B_t = G(H) \int_{\mathbb{R}} ((t-s)_+^{H-1/2} - (-s)_+^{H-1/2}) dV_s,$$

where V is a two-sided extension of the original Bm. However, the choice

$$\gamma(t,s) = \min\{t,s\},$$

as in standard (i.e. Brownian) stochastic volatility models, leads to the simple structure

$$\mathbf{E}\Delta_k B \Delta_\ell V = \begin{cases} \Delta & \text{if} \quad k = \ell, \\ 0 & \text{else,} \end{cases}$$

where $\Delta_k B = B_{(k+1)\Delta} - B_{k\Delta}$. So setting

$$\Delta_k B^V = \Delta_k B - \Delta_k V, \qquad k = -m, \dots, n-1,$$

the Gaussian vector

$$\Delta_0 V, \dots, \Delta_{n-1} V, \Delta_0 W, \dots, \Delta_{n-1} W, \Delta_{-m} B^V, \dots, \Delta_{n-1} B^V$$

has independent components, and due to stationarity

$$\Delta_{-m} B^V, \dots, \Delta_{n-1} B^V$$

can be simulated with computational cost $\mathcal{O}((n+m)\log(n+m))$ using the Davies-Harte algorithm. The increments of the fBm can be then reconstructed via $\Delta_k B = \Delta_k B^V + \Delta_k V$.

Using an exponential Euler scheme for the log-volatility, this leads to the following approximation scheme

$$\widehat{X}_n^{\mathsf{E},\mathsf{exp}} = -\frac{1}{2} \sum_{k=0}^{n-1} e^{2\widehat{Y}_k} \Delta + \rho \sum_{k=0}^{n-1} e^{\widehat{Y}_k} \Delta_k V + \sqrt{1-\rho^2} \sum_{k=0}^{n-1} e^{\widehat{Y}_k} \Delta_k W$$

for the log-asset price, where

$$\widehat{Y}_k = \mu + \theta e^{-\lambda k \Delta} \sum_{\ell=-m}^{k-1} e^{\lambda \ell \Delta} \Delta_\ell B, \qquad k = -m+1, \dots, n.$$

If $m = \lceil n \log(n)/\lambda \rceil$, then we have

$$\lim_{n\to\infty} \left(\frac{n}{T}\right)^{2H} \mathbf{E}|X_T - \widehat{X}_n^{\mathsf{E},\mathsf{exp}}|^2 = \frac{1}{2H+1} T\theta^2 \mathbf{E}|e^{Y_0}|^2,$$

and $\widehat{X}_n^{\mathsf{E},\mathsf{exp}}$ has computational cost $\mathcal{O}(n\,\log(n)^2)$.

Thus, in the case of $\gamma(t,s) = \min\{t,s\}$ simulating the fBm and approximating the log-volatility instead of simulating the log-volatility exactly leads to a more efficient method.

References

[1] C. Bayer, P. Friz and J. Gatheral. Pricing under rough volatility, 2015. *Quantitative Finance* **15**, pp 1–18

[2] R.B. Davies and D.S. Harte. Tests for Hurst effect, 1987. *Biometrika* **74**, pp 95–101

[3] J. Gatheral, T. Jaisson and M. Rosenbaum. Volatility is rough, 2014. *arXiv preprint, arXiv:1410.3394*

[4] A. Neuenkirch and T. Shalaiko. The order barrier for strong approximation of rough volatility models, 2016. *arXiv preprint, arXiv:1606.03854*

Multilevel Monte Carlo for McKean–Vlasov SDEs.

Lukasz Szpruch

(joint work with Shuren Tan)

The theory of mean field interacting particle systems was pioneered by the work of H. McKean [1], where he gave a probabilistic interpretations of a class of nonlinear (due to dependence of the coefficients on the solution itself) PDEs arising in fluid dynamics. A general d-dimensional nonlinear McKean–Vlasov PDEs (understood in a weak form) is given by

$$(1)\quad \begin{cases} \frac{\partial}{\partial t}\langle \mathbb{P}_t, f\rangle & = \langle \mathbb{P}_t, \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x,\mathbb{P}_t)\frac{\partial^2 f}{\partial x_i \partial x_j}(x) + \sum_{i=1}^d b_i(x,\mathbb{P}_t)\frac{\partial f}{\partial x_i}(x)\rangle \\ \mathbb{P}_0 & = \mathbb{P}\circ X_0^{-1} = \mathrm{Law}(X_0), \end{cases}$$

where $f(\cdot) \in C_K^\infty(\mathbb{R}^d)$, $a(x,\mathbb{P}_t) = \sigma(X_t,\mathbb{P}_t)^T\sigma(X_t,\mathbb{P}_t)$ and $\mathbb{P}_t$ is a probability measure on $\mathbb{R}^d$ and the solution to the equation. One can derive a probabilistic representation of (1), in the form of the SDEs

$$(2)\quad dX_t = b(X_t,\mathbb{P}_t)dt + \sigma(X_t,\mathbb{P}_t)dW_t, \quad \mathbb{P}_t = \mathbb{P}\circ X_t^{-1} = \mathrm{Law}(X_t), \quad t\in[0,T],$$

where $\{W_t\}_{t\geq 0}$ is k-dimensional Brownian motion and $\mathbb{P}$ is a probability measure on $C([0,T],\mathbb{R}^d)$. We refer to (2) as the McKean–Vlasov SDEs (MV-SDEs). MV-SDEs provide new mathematical tools to tackle key questions such as regularity of (1) and are themselves of independent interest. Crucially, MV-SDEs enable the use of probabilistic numerical methods that alleviate the so-called curse of dimensionality (the exponential increase in computational effort with the dimension). *The theory of the propagation* of chaos, [2], states (under appropriate conditions) that (2) is a limiting equation of the system of *stochastic interacting particles* $X_t^{i,N}$ which are the solution to $(\mathbb{R}^d)^N$ dimensional SDEs

$$(3)\quad \begin{cases} dX_t^{i,N} & = b(X_t^{i,N},\mathbb{P}^N)dt + \sigma(X_t^{i,N},\mathbb{P}_t^N)dW_t^i, \quad i=1,\dots,N, \\ \mathbb{P}_t^N & := \frac{1}{N}\sum_{i=1}^N \delta_{X_t^{i,N}}, \quad t\geq 0, \end{cases}$$

where $\{X_0^{i,N}\}_{i=1,\dots,N}$ are i.i.d samples with law $\mathbb{P}_0$ and $\{W_t^i\}_{i=1,\dots,N}$ are independent Brownian motions. It can be shown, under sufficient regularity conditions on the coefficients, that the convergence of the empirical measures (of $\{X_\cdot^{i,N}\}_i$) on the path space holds in law, i.e $\mathbb{P}^N = \{\mathbb{P}_t^N : t\in[0,T]\} \to \mathbb{P}$, $N\to\infty$. This is a truly spectacular result as the particles are not independent and the standard Law of Large Numbers does not apply. Moreover, (3) can be thought of as a first step

towards numerical schemes for (2) and is also of independent interest (to (2)) as it offers a very rich and versatile modelling framework. On one hand interactions allow us to capture complex dependent structure, on the other provide a great challenge for Monte Carlo simulations. The non-linear dependence of the approximation bias on the statistical error makes classical variance reduction techniques fail in this setting. In this work, we will devise a strategy that will allow us to overcome this difficulty. In particular, we will establish Multilevel Monte Carlo estimator for MV-SDEs and demonstrate it computational superiority over standard Monte Carlo techniques.

References

[1] McKean Jr, H, *A class of Markov processes associated with nonlinear parabolic equations*, Proceedings of the National Academy of Sciences of the United States of America **6** (1966).
[2] Sznitman, Alain-Sol, *Topics in propagation of chaos*, Ecole d'Eté de Probabilités de Saint-Flour XIX—1989 (1991).

On some singular McKean–Vlasov particle systems

Denis Talay

During these three survey lectures I shortly presented standard techniques to analyze McKean–Vlasov particle systems with smooth interaction kernels: Given N independent Brownian motions $(W_t^{(i)})$, multi-dimensional coefficients B and S, and McKean interaction kernels b and σ, the particle dynamics is given by

$$\begin{aligned} X_t^{(i)} = X_0^{(i)} &+ \int_0^t B(s, X_s^{(i)}, \textstyle\int b(X_s^{(i)}, y)\overline{\nu}_s^N(dy))ds \\ &+ \int_0^t S(s, X_s^{(i)}, \textstyle\int \sigma(X_s^{(i)}, y)\overline{\nu}_s^N(dy))dW_s^{(i)}, \end{aligned} \tag{1}$$

where $\overline{\nu}_s^N$ is the marginal distribution at time s of the empirical distribution $\overline{\nu}^N$ of the trajectories of the particles

$$\bar{\nu}^N := \frac{1}{N}\sum_{j=1}^N \delta_{X^{(j)}}.$$

We shortly summarized the seminal techniques developed by Sznitman [5] and the non-linear martingale technique to show that the particle system propagates chaos in the sense that the probability distribution of $\bar{\nu}^N$ converges weakly when N goes to infinity. The limit distribution is concentrated at the probability law of the process (X_t), solution to the McKean–Vlasov SDE

$$\begin{cases} X_t = X_0 + \int_0^t B(s, X_s, \int b(X_s, y)\nu_s(dy))ds \\ \qquad + \int_0^t S(s, X_s, \int \sigma(X_s, y)\nu_s(dy))dW_s, \\ \nu_s(dy) := \text{probability distribution of } X_s. \end{cases} \tag{2}$$

In addition, the flow of the probability distributions ν_t solves the non-linear McKean–Vlasov–Fokker–Planck equation

$$\frac{d}{dt}\nu_t = L^*_{\nu_t}\nu_t, \tag{3}$$

where, A denoting the matrix $S \cdot S^*$, L^*_ν is the formal adjoint of the non-linear differential operator

$$L_\nu := \sum_k B^k(t, x, \int b(x, y)\nu(dy))\partial_k + \frac{1}{2}\sum_{j,k} A^j_k(t, x, \int \sigma(x, y)\nu(dy))\partial_{jk}. \tag{4}$$

We then explained the limitations of these techniques to tackle singular interactions and we discussed some of the difficulties which arise from singularities.

To illustrate our discussion we examined the two following particular situations.

Bossy et al. [1] established existence and uniquess of the solution to the following stochastic differential system

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, \\ U_t = U_0 + \int_0^t B(s, X_s, U_s)ds + \sigma W_t, \\ B(s, x, u) := \mathbb{E}[b(U_s - u) \mid X_s = x], \end{cases} \tag{5}$$

where b is a bounded continuous function. Notice that this equation is very singular because, on the one hand it involves a conditional expectation rather than the integration of a smooth kernel w.r.t. a probability distribution, and, on the other hand the generator of the process (X_t, U_t) is not strongly elliptic. We explained the strategy to establish the propagation of chaos of related particle systems. We also explained how to use estimates on the density of fundamental solutions of ultraparabolic PDEs obtained by Di Francesco and Polidoro [3] in order to get uniqueness of the weak solution of Eq. (5).

We then shifted to a model where interactions do not arise by means of a kernel but arise from a geometric constraint. Consider a finite size network of N-neurons with membrane potentials $X^{(i)}_t$:

$$X^{(i)}_t = X_0 + \int_0^t b(X^{(i)}_s)ds + \frac{\alpha}{N}\sum_{j\neq i} M^{(j)}_t - M^{(i)}_t + W^{(i)}_t,$$

where $M^{(i)}_t$ is the number of times $X^{(i)}_t$ passes the threshold value of 1, i.e. the number of spikes and $\alpha > 0$ is the strength of synaptic connection. After each spike, the membrane potential is reset at 0 when the particle is the only one to spike.

The Fokker–Planck equation for $p(t, y) = \mathbb{P}(X_t \in dy)$ writes

$$\begin{cases} \partial_t p(t, y) + \partial_y[(b(y) + \alpha e'(t))p(t, y)] - \frac{1}{2}\partial^2_{yy} p(t, x) = \delta_0(y)e'(t), \ y < 1, \\ e'(t) = -\frac{1}{2}\partial_y p(t, 1), \end{cases}$$

with boundary conditions $p(t, 1) = p(t, -\infty) = 0$ and initial condition $p(0, y) = p_0(y)$. Carrillo, Perthame and their collaborators have proven that solutions may

blow-up if $\alpha \geq 1$ and for any $\alpha > 0$ there exists an initial density $p(0, y)$ such that blow-up occurs in finite time.

Delarue et al. [2] have established a dual result. Suppose that b is globally Lipschitz. For any $\varepsilon > 0$ there exists an $\alpha_0 > 0$ such that whenever $X_0 = x_0 < 1-\varepsilon$ and $\alpha \in (0, \alpha_0)$, there exists a unique process (X_t, M_t) which is a solution to the mean-field limit equation

$$\begin{cases} X_t = x_0 + \int_0^t b(X_s)ds + \alpha\mathbb{E}(M_t) - M_t + W_t, \\ M_t = \sum_{k\geq 1} \mathbb{I}_{[0,t]}(\tau_k), \\ \tau_k = \inf\{t > \tau_{k-1} : X_{t-} \geq 1\}, \tau_0 = 0, \end{cases}$$

which does not blow-up, that is,

$$e'(t) = \frac{d}{dt}\mathbb{E}(M_t) < \infty, \qquad \forall t > 0.$$

Inglis and Talay [4] have introduced and analyzed the following new model with delayed spike effects and non-constant diffusion and synaptic weigts:

$$\begin{cases} U_t^i = U_0^i + H(t) + \int_0^t b(U_s^i)ds + \sum_{j=1}^N \frac{J_{ij}}{S_i^N} \int_0^t G(t-s)M_s^j ds - M_t^i \\ \qquad + \int_0^t \sigma(U_s^i)dW_s^i, \\ S_i^N := \sum_{j=1}^N J_{ij}, \quad M_t^i := \sum_{k=1}^\infty \mathbb{I}_{[0,t]}(\tau_k^i), \\ \tau_k^i := \inf\{t \geq \tau_{k-1}^i : U_{t-}^i \geq 1\},\ k \in \mathbb{N}\backslash\{0\}, \quad \tau_0^i = 0. \end{cases}$$

Their central goal was to prove, under mild assumptions on the coefficients b and σ and on the weights J_{ij}, the convergence of the particle system towards the limit equation

$$\begin{cases} U_t &= U_0 + H(t) + \int_0^t b(U_s)ds + \int_0^t G(t-s)\mathbb{E}(M_s)ds - M_t \\ &\qquad + \int_0^t \sigma(U_s)dW_s, \\ M_t &:= \sum_{k=1}^\infty \mathbb{I}_{[0,t]}(\tau_k), \\ \tau_k &=: \inf\{t \geq \tau_{k-1} : U_{t-} \geq 1\},\ k \in \mathbb{N}\backslash\{0\}, \quad \tau_0 = 0. \end{cases}$$

In particular, the particle system and the mean-field limit have no blow-up.

References

[1] M. Bossy, J-F. Jabir, and D. Talay, *On conditional McKean Lagrangian stochastic models*, Probab. Theory Related Fields **151** (2011), 319–351.

[2] F. Delarue, J. Inglis, S. Rubenthaler, and E. Tanré, *Global solvability of a networked integrate-and-fire model of McKean–Vlasov type*, Ann. Appl. Probab. **25(4)** (2015), 2096–2133.

[3] M. Di Francesco and S. Polidoro, *Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form*, Adv. Differ. Equations **11(11)** (2006), 1261–1320.

[4] J. Inglis and D. Talay, *Mean-field limit of a stochastic particle system smoothly interacting through threshold hitting-times and applications to neural networks with dendritic component*, SIAM J. Mathematical Analysis **47(5)** (2015), 3884–3916.

[5] A-S. Sznitman. *Topics in propagation of chaos*, in *École d'Été de Probabilités de Saint-Flour XIX-1989*, volume 1464 of Lecture Notes Math. (1991), Springer.

On sub-polynomial lower error bounds for strong approximation of SDEs with smooth coefficients

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(joint work with Arnulf Jentzen, Thomas Müller-Gronbach)

Let $d, m \in \mathbb{N}$, consider a d-dimensional system of autonomous stochastic differential equations (SDEs)

$$\begin{aligned} dX(t) &= \mu(X(t))\,dt + \sigma(X(t))\,dW(t), \quad t \in [0,1], \\ X(0) &= x_0 \end{aligned} \tag{1}$$

with a deterministic initial value $x_0 \in \mathbb{R}^d$, a drift coefficient $\mu\colon \mathbb{R}^d \to \mathbb{R}^d$, a diffusion coefficient $\sigma\colon \mathbb{R}^d \to \mathbb{R}^{d\times m}$ and an m-dimensional driving Brownian motion W, and assume that (1) has a unique strong solution $(X(t))_{t\in[0,1]}$. Our computational task is to approximate $X(1)$ by means of methods that use finitely many evaluations of the driving Brownian motion W. In particular we are interested in the following question: under which assumptions on the coefficients μ and σ does there exist a method of the latter type which converges to $X(1)$ in absolute mean with a polynomial rate?

It is well-known that if the coefficients μ and σ are globally Lipschitz continuous then the classical Euler scheme achieves the rate of convergence $1/2$, see [11]. Moreover, in recent years a number of positive results has been established under substantially weaker assumptions on the coefficients: for such equations new types of algorithms have been constructed that achieve a polynomial rate of convergence, see e.g. [10, 5, 15, 1] and references therein for SDEs with globally monotone coefficients and see e.g. [7, 2, 6, 14, 9, 4] and references therein for SDEs with possibly non-monotone coefficients.

On the other hand, it has recently been shown in [8] that for any sequence $(a_n)_{n\in\mathbb{N}} \subset (0,\infty)$, which may converge to zero arbitrary slowly, there exists an SDE (1) with $d = 4$ and $m = 1$ and with infinitely often differentiable and bounded coefficients μ and σ such that no approximation of $X(1)$ based on finitely many evaluations of the driving Brownian motion W converges in absolute mean faster than the given sequence $(a_n)_{n\in\mathbb{N}}$. More formally,

$$\inf_{s_1,\dots,s_n\in[0,1]} \inf_{\substack{u\colon \mathbb{R}^n\to\mathbb{R}^4 \\ \text{measurable}}} \mathbb{E}\big\|X(1) - u\big(W(s_1),\dots,W(s_n)\big)\big\| \geq a_n. \tag{2}$$

In particular, there exists an SDE (1) with infinitely often differentiable and bounded coefficients μ and σ such that its solution at the final time cannot be approximated with a polynomial rate of convergence based on finitely many evaluations of the driving Brownian motion W.

Note that the time points $s_1, \dots, s_n \in [0,1]$ that are used by an approximation $u(W(s_1), \dots, W(s_n))$ in (2) are fixed, and therefore this negative result does not cover approximations that may choose the number as well as the location of the evaluation sites of the driving Brownian motion W in an adaptive way, e.g. numerical schemes that adjust the actual step size according to a criterion that is based on the values of the driving Brownian motion W observed so far. It

is well-known that for SDEs (1) with (essentially) globally Lipschitz continuous coefficients μ and σ adaptive approximations cannot achieve a better rate of convergence compared to what is best possible for non-adaptive ones, which at the same time coincides with the best possible rate of convergence that can be achieved by any approximation based on $W(\frac{1}{n}), W(\frac{2}{n}), \dots, W(1)$, see [12, 13]. However, as has recently turned out, this is not necessarily the case anymore if the coefficients μ and σ are not both globally Lipschitz continuous. In [3] it has been shown that for the one-dimensional squared Bessel process, which is the solution of the SDE (1) with $d = m = \mu = 1$ and $\sigma(x) = 2\sqrt{|x|}$ for $x \in \mathbb{R}$ the following holds: the best possible rate of convergence that can be achieved by any approximation based on $W(\frac{1}{n}), W(\frac{2}{n}), \dots, W(1)$ equals $1/2$, i.e. there exist $c_1, c_2 > 0$ such that

$$c_1 \cdot n^{-1/2} \leq \inf_{\substack{u\colon \mathbb{R}^n \to \mathbb{R} \\ \text{measurable}}} \mathbb{E}\big|X(1) - u\big(W(\tfrac{1}{n}), W(\tfrac{2}{n}), \dots, W(1)\big)\big| \leq c_2 \cdot n^{-1/2},$$

while the best possible rate of convergence that can be achieved by approximations based on n adaptively chosen evaluations of the driving Brownian motion W equals infinity. More formally, for every $\alpha > 0$ there exists $c > 0$ and a sequence of approximations $\widehat{X}_n$ based on n adaptively chosen evaluations of W such that

$$\mathbb{E}|X(1) - \widehat{X}_n| \leq c \cdot n^{-\alpha}.$$

In view of the latter result one might hope that a sub-polynomial lower error bound a_n in (2) could be overcome by using adaptive approximations. In the first part of the talk we discuss results from [16], where we have shown that the pessimistic alternative is true. More precisely, we have proven in [16] that for any sequence $(a_n)_{n\in\mathbb{N}} \subset (0, \infty)$, which may converge to zero arbitrary slowly, there exists an SDE (1) with $d = 4$ and $m = 1$ and with infinitely often differentiable and bounded coefficients μ and σ such that no approximation based on n adaptively chosen evaluations of the driving Brownian motion W on average can achieve a smaller absolute mean error than the given number a_n, i.e.

$$\mathbb{E}\big\|X(1) - \widehat{X}_n\big\| \geq a_n$$

for any approximation $\widehat{X}_n$ of the latter type. Thus, roughly speaking, these SDEs cannot be solved approximately in the strong sense in a reasonable computational time by means of any kind of adaptive (or nonadaptive) method based on finitely many evaluations of the driving Brownian motion W.

The diffusion coefficients of the pathological SDEs from [8] and [16] are globally Lipschitz continuous, while the first order partial derivatives of the drift coefficients are, essentially, of exponential growth. In the second part of the talk we show that sub-polynomial rates of convergence may happen even when the first order partial derivatives of the drift coefficient have at most polynomial growth, which is one of the typical assumptions in the literature on numerical approximation of SDEs with globally monotone coefficients, see e.g. [10, 5, 15, 1] and references therein. More precisely, we show that for every $p \in \mathbb{N}$ there exists an SDE (1) with $d = 4$ and $m = 1$ and with infinitely often differentiable coefficients μ and σ such that σ

is globally Lipschitz continuous and such that μ satisfies

$$\sum_{i,j=1}^{4} \left| \frac{\partial \mu_i}{\partial x_j} \right| \leq c \cdot (1 + \|x\|^3)$$

and such that the following holds: the p-th moment of the supremum of the solution is finite, i.e. $\mathbb{E} \sup_{t\in[0,1]} |X(t)|^p < \infty$, and no approximation of $X(1)$ based on finitely many evaluations of the driving Brownian motion W converges to the solution $X(1)$ in the p-th mean with a polynomial rate. More formally, for every $\alpha > 0$,

$$\lim_{n\to\infty} \Big(n^\alpha \cdot \inf_{s_1,\dots,s_n\in[0,1]} \inf_{\substack{u\colon \mathbb{R}^n\to\mathbb{R}^4 \\ \text{measurable}}} \mathbb{E} \big\| X(1) - u\big(W(s_1),\dots,W(s_n)\big) \big\|^p \Big) = \infty.$$

We believe that the latter negative statement even holds if adaptive approximations based on finitely many evaluations of the driving Brownian motion W may be used.

References

[1] W.-J. Beyn, E. Isaak, R. Kruse, *Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes*, J. Sci. Comput. **67** (2016), 955–987.

[2] S. Dereich, A. Neuenkirch, L. Szpruch, *An Euler-type method for the strong approximation of the Cox–Ingersoll–Ross process*, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **468** (2012), 1105–1115.

[3] M. Hefter, A. Herzwurm, *Optimal strong approximation of the one-dimensional squared Bessel process*, arXiv:1601.01455 (2016).

[4] M. Hefter, A. Herzwurm, *Strong convergence rates for Cox–Ingersoll–Ross processes — full parameter range*, arXiv:1608.00410 (2016).

[5] M. Hutzenthaler, A. Jentzen, P. Kloeden, *Strong convergence of an explicit numerical method for SDEs with non-globally Lipschitz continuous coefficients*, Ann. Appl. Probab. **22** (2012), 1611–1641.

[6] M. Hutzenthaler, A. Jentzen, *On a perturbation theory and on strong convergence rates for stochastic ordinary and partial differential equations with non-globally monotone coefficients*, arXiv:1401.0295 (2014).

[7] I. Gyongy, M. Rasonyi, *A note on Euler approximations for SDEs with Holder continuous diffusion coefficients*, Stochastic Process. Appl. **121** (2011), 2189–2200.

[8] A. Jentzen, T. Müller-Gronbach, L. Yaroslavtseva, *On stochastic differential equations with arbitrary slow convergence rates for strong approximation*, Commun. Math. Sci. **14** (2016), 1477–1500.

[9] G. Leobacher, M. Szolgyenyi, *A strong srder 1/2 method for multidimensional SDEs with discontinuous drift*, to appear in Ann. Appl. Probab.

[10] X. Mao, L. Szpruch, *Strong convergence rates for backward Euler–Maruyama method for non-linear dissipative-type stochastic differential equations with super-linear diffusion coefficients*, Stochastics **85** (2013), 144–171.

[11] G. Maruyama, *Continuous Markov processes and stochastic equations*, Rend. Circ. Mat. Palermo **4** (1955), 48–90.

[12] T. Müller-Gronbach, *Strong approximation of systems of stochastic differential equations*, Habilitation thesis TU Darmstadt (2002), iv+161.

[13] T. Müller-Gronbach, *Optimal pointwise approximation of SDEs based on Brownian motion at discrete points*, Ann. Appl. Probab. **14** (2004), 1605–1642.

[14] H.-L. Ngo, D. Taguchi, *Strong rate of convergence for the Euler–Maruyama approximation of stochastic differential equations with irregular coefficients*, Math. Comp. **85** (2016), 1793–1819.

[15] S. Sabanis, *Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients*, Ann. Appl. Probab. **26** (2016), 2083–2105.

[16] L. Yaroslavtseva, *On non-polynomial lower error bounds for adaptive strong approximation of SDEs*, arXiv:1609.08073 (2016).

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