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Nonlinear Partial Differential Equations on Graphs

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Abstract. One-dimensional metric graphs in two and three-dimensional spaces play an important role in emerging areas of modern science such as nano-technology, quantum physics, and biological networks. The workshop focused on the analysis of nonlinear partial differential equations on metric graphs, especially on the bifurcation and stability of nonlinear waves on complex graphs, on the justification of Kirchhoff boundary conditions, on spectral properties and the validity of amplitude equations for periodic graphs, and the existence of ground states for the NLS equation with and without potential.

Mathematics Subject Classification (2010): 35R02, 35Q55, 81Q37.

Introduction by the Organisers

In many applications, a specific waveguide geometry of the spatial domain suggests the use of metric graphs as suitable way to approximate dynamics of nonlinear PDEs on such spatial domains. At the junction between different pieces of metric graphs, suitable boundary conditions are given to define the coupling between the edges. These boundary conditions ensure continuity of the wave functions and conservation of the current flow through the network junction. Mathematical studies of nonlinear PDEs on graphs are developed by using different approaches, like the calculus of variation, bifurcation theory, dynamical systems methods, applied harmonic analysis, perturbation theory, spectral theory, and numerical simulations.

This workshop brought together 24 participants from 11 different countries from all over the world, including well-known international experts as well as promising young postdocs and PhD students.

The central topics of the workshop were the existence of ground states, bifurcations of solutions of the NLS equation on complex graphs, and their stability. The existence and non-existence of ground states and bifurcations from ground states has been discussed by Gregory Berkolaiko, Claudio Cacciapuoti, Simone Dovetta, Domenico Finco, Diego Noja, Jeremy L. Marzuola, and Enrico Serra. The instability of stationary states has been discussed by Riccardo Adami, Dmitry Pelinovsky, and Lorenzo Tentarelli.

A number of lectures were about infinite periodic graphs. Pavel Exner talked about an unusual spectrum for such graphs, Hiroaki Niikuni on the effects of a broken periodicity on the spectrum, Martina Chirilus-Bruckner on the construction of breathers, and Guido Schneider on the validity of the NLS approximation.

Another topic of the workshop was an approximation of thin domains with metric graphs. An overview of the approximation technique was given by Olaf Post. Stefan Teufel considered the NLS limit for bosons in a quantum waveguide.

Dispersive estimates on trees and star-graphs were discussed by Valeria Banica and Andreea Grecu. Dynamics of other nonlinear PDEs, on or related to metric graphs, was the subject of the lectures of Andrew Comech, Reika Fukuizumi, and Liviu Ignat. Evgeny Korotyaev presented estimates for the Schrödinger operator on the lattice and Davron Matrasulov discussed relativistic solitons on graphs. Finally Braxton Osting presented Dirichlet partitions by using geometric graphs.

The lectures stimulated many discussions and inspired the participants to new projects. New research fields have emerged in the discussions such as the analysis of nonlinear PDEs with vertex conditions different from Kirchhoff boundary conditions or the justification of the Kirchhoff boundary conditions for time-dependent solutions. Altogether, we think that the workshop was a great success.

The organisers thank the Oberwolfach staff which helped a lot to create the unique atmosphere during the workshop.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1049268, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Davron Matrasulov in the "Simons Visiting Professors" program at the MFO.

Workshop: Nonlinear Partial Differential Equations on Graphs

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Abstracts

Nonlinear Instability of Half-Solitons on Star Graphs

Riccardo Adami
(joint work with Enrico Serra, Paolo Tilli)

We review some basic results on the existence of ground states for the NLS on metric graphs in the subcritical case. More specifically, we look for global minimizers of the energy functional

$$E(u,\mathcal{G}) = \frac{1}{2}\|u'\|^2_{L^2(\mathcal{G})} - \frac{1}{p}\|u\|^p_{L^p(\mathcal{G})} = \frac{1}{2}\int_{\mathcal{G}} |u'|^2 dx - \frac{1}{p}\int_{\mathcal{G}} |u|^p dx$$

where $\mathcal{G}$ is a connected, noncompact metric graph $\mathcal{G}$, under the mass constraint

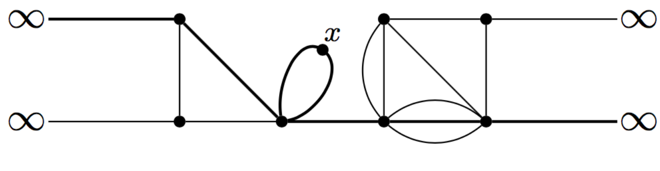
$$\|u\|^2_{L^2(\mathcal{G})} \;=\; \mu.$$

As for the ordinary NLS on the line, subcriticality here means $2 < p < 6$. The problem has been investigated in several papers during the last years [1, 2, 3, 4]. Non-compact means that at least one edge in the graph is a half-line, i.e. there is at least one vertex at infinity.

Preliminarily, we recall that the problem is solved for $\mathcal{G} = \mathcal{R}$, where for every $\mu > 0$ a ground state exists and is given by the NLS-soliton ϕ_μ at mass μ (together with all its translated and multiplications by a phase factor), and $\mathcal{G} = \mathcal{R}^+$, where the ground states are given by a half-soliton with the correct mass (again, modulo a global phase). Furthermore, a direct computation shows that

$$\inf_{\int_{\mathcal{R}^+} |u|^2=\mu} E(u,\mathcal{R}^+) \;<\; \inf_{\int_{\mathcal{R}} |u|^2=\mu} E(u,\mathcal{R}) = E(\phi_\mu,\mathcal{R}) < 0.$$

The first result holding for general graphs is a topological obstruction to the existence of ground states, called hypothesis (H), stating that *every edge of $\mathcal{G}$ belongs to a trail connecting two distinct point at infinity* (we recall that a trail is a path made of adjacent edges in which edges cannot be repeated, but vertices can). Here is an example of a graph that satisfies hypothesis (H):

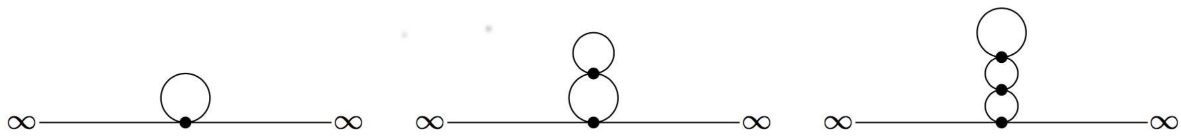


We have the following

Theorem. Let $\mathcal{G}$ be a non-compact graph satisfying hypothesis (H). Then, for every $\mu > 0$

$$\inf_{\int_{\mathcal{G}} |u|^2=\mu} E(u,\mathcal{G}) = E(\phi_\mu,\mathcal{R})$$

and there is no ground state at mass μ, except if $\mathcal{G}$ is one of the bubble towers displayed as follows:



(the number of the so-called bubbles range on natural numbers, including zero).

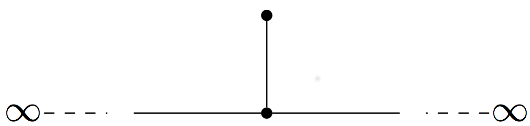
Roughly speaking, the theorem states that if $\mathcal{G}$ is more intricated than a line, then, as regards the issue of minimizing energy with mass constraint, it makes worse than the line. In order to make better than the line, hypothesis (H) must be broken. Furthermore, it turns out that the following operative criterion holds:

Theorem. Given $\mathcal{G}$, if there exists $u \in \mathcal{G}$ with mass equal to one, such that

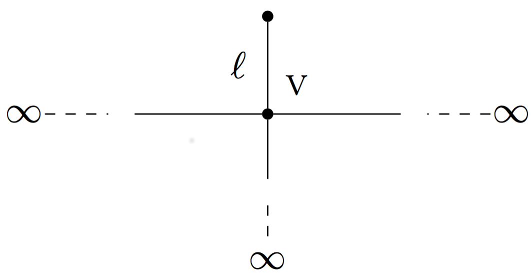
$$E(u, \mathcal{G}) \leq E(\phi_\mu, \mathcal{R}),$$

then there exists a ground states for $E(\cdot, \mathcal{G})$ at mass μ.

Thus, one can exhibit examples of graphs in which it is possible to sit a soliton, after a suitable sequence of the following operations: cutting, pasting, and monotonically rearranging. The two first operations leave the energy untouched, while the third one makes it decrease. So, we finally get a function that makes better than the soliton, so that a ground state exists. The simplest example is given by the line with a pendant:



In all cases where this strategy works, the existence of a ground state is a topological fact, in the sense that the metric plays no role. On the other hand, for the graph made of three half-lines and a pendant



we prove that a ground state exists if and only if the length ℓ of the pendant is beyond a certain threshold value ℓ^*. In this case topology is not sufficient to determine existence, so metric plays a crucial roles. Many other examples like this can be constructed.

The main message we would like to convey is that existence of ground states is the result of a competition between half-line and compact core of the graph. If the compact core is simple enough (in the sense that it violates hypothesis (H)), then it wins the competition and gives rise to a ground state.

The techniques mix standard variational results with a thorough insight in the rearrangement theory, that is used not only in order to restrict the set possible minimizers, but also in a constructive way.

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Dispersive properties for the linear Schrödinger equation on trees and consequences

Valeria Banica

(joint work with Liviu Ignat)

The Schrödinger equation is classified among the PDEs as a dispersive equation. This is due to the following property of the linear equation, valid in the classical case when for the space variable is $\mathbb{R}^n$:

$$\|e^{it\Delta}u_0\|_{L^\infty(\mathbb{R}^n)} \leq \frac{C}{\sqrt{|t|}}\|u_o\|_{L^1(\mathbb{R}^n)}, \quad \forall\, t \in \mathbb{R}^*.$$

This property together with the conservation of the L^2 norm implies other dispersives properties as the Strichartz estimates. All these dispersive estimates have an important role in the analysis of the nonlinear equation. For instance the local existence of classes of general nonlinear solutions is proved using the dispersive estimates. They are also used at the global in time level for proving scattering.

The linear Schrödinger equation on a metric graph has been extensively studied since the 90's from the point of view of the spectral theory. However, the theory for time-dependent NLS on graphs started only very recently. Mainly there are two directions since 2010's: the study of the existence and properties of stationary states, and the study of the dispersive estimates and their consequences. For the first one we can cite as authors Adami, Cacciapuoti, Finco, Kairzhan, Marzuola, Matrasulov, Nakamura, Noja, Pelinovsky, Sabirov, Sawada, Schneider, Serra, Shaikhova, Sobirov Tilli, ... (see for instance the survey [15] and also several others abstracts of this workshop). We shall focus now on the dispersive properties.

We first recall that the dispersion estimate is easily obtained on $\mathbb{R}^n$ by using Fourier analysis. The issue on a general metric graph is the absence of this tool, explaining the lack of results in this direction.

The first dispersive results on metric graphs were obtained in particular cases of graphs by doing a reduction to a setting where the dispersive estimate was already known. For instance Ignat showed in [12] the dispersion estimate on regular trees with Kirchhoff conditions (all the vertices of the same generation have the same number of descendants and all the edges of the same generation are of the same length) by a very nice reduction to the laminar Schrödinger equation on $\mathbb{R}$, enjoying dispersion ([5]). Cascaval and Hunter gave in [9] a detailed analysis for the linear equation for particular type of data and particular shape of trees with Kirchhoff conditions allowing for a reduction to the Schrödinger equation on half-line with particular boundary conditions. Also, Ali Mehmeti, Ammari and Nicaise proved in [3] the dispersion for the Laplacian with a potential on star-shaped trees with Kirchhoff conditions by obtaining a reduction to a system on $\mathbb{R}$.

For proving the dispersion directly at the level of the graph the natural way is to compute the resolvent of the Laplacian and to use a spectral formula for the Schrödinger evolution in time. This can be done easily for simple graphs, as for instance star-shaped graphs, since the resolvent can be easily compluted. For general graphs the situation gets very intricate. In [6] and [7] we have obtained the dispersion for the case of general trees Γ having a finite number of vertices with infinite last edges and Kirchhoff and respectively delta coupling conditions. More precisely, we have the following results.

1) The solution of the linear Schrödinger equation on Γ with Kirchhoff coupling conditions is a summable superposition of classical linear evolutions:

$$e^{it\Delta_\Gamma}u_0(x) = \sum_{\lambda\in\mathbb{R}} a_\lambda \int_{I_\lambda} u_0(y) \int_{-\infty}^{\infty} e^{it\tau^2} e^{i\tau\phi_\lambda(x,y)}\ d\tau\, dy,$$

with $\phi_\lambda(x,y) \in \mathbb{R}$, I_λ among the parametrizations of the edges, $\sum_{\lambda\in\mathbb{R}} |a_\lambda| < \infty$, and it satisfies the dispersion inequality

$$\|e^{it\Delta_\Gamma}u_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\Gamma)}, \quad \forall\, t \in \mathbb{R}^*. \tag{1}$$

2) The solution of the linear Schrödinger equation on Γ with δ coupling conditions of positive strengths satisfies the dispersion inequality (1). For generic strengths of the δ conditions and lengths of the edges[1], the solution of the linear Schrödinger equation on a tree satisfies the dispersion inequality

$$\|e^{it\Delta_\Gamma^\delta}Pu_0\|_{L^\infty(\Gamma)} \leq \frac{C}{\sqrt{|t|}}\|u_0\|_{L^1(\Gamma)}, \quad \forall\, t \in \mathbb{R}^*,$$

where P is the projection outside the discrete spectrum.

To end this extended abstract we shall make a series of remarks.

The proof is based on a recursive way to compute the resolvent for a tree, that eventually allows for obtaining the dispersion. The nature of this recursive way imposes the shape of a tree for the considered graph, which is a technical

[1]typically, the situation to be avoided for 2 vertices is that the length L of the internal edge is equal to $-\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}$, that is precisely the situation of the presence of zero resonance.

constraint. Indeed, the simplest case of graph containing a cycle, the tadpole graph (only one vertex from which emerge a cycle and an infinite edge) enjoys also the dispersion, as it was proved by computing explicitly the resolvent by Ali Mehmeti, Ammari and Nicaise in [4].

In [7] we also give conditions for the dispersion to hold in the case of trees with general coupling conditions. However, these conditions look too intricate. On the other hand, a resolvent formula for the Laplacian on general metric graphs for general coupling conditions was given in [14] by Kostrykin, Potthoff and Schrader. The issue is that this formula is not easy to handle in order to decide dispersion. Only the case of the star-shaped tree has been settled via this formula by Grecu and Ignat in [11].

As a consequence of our second result, the dispersion is proved for the Schrödinger equation with several Dirac potentials on $\mathbb{R}$. This equation was extensively studied, at least from the spectral point of view (see for instance [2]), and for its stationary states properties. Previous studies considered dispersion only the case of one or two potentials, with one exception by Duchêne, Marzuola and Weinstein that proved a weaker dispersion in [10] in the case of several Dirac potentials.

As mentioned before the dispersive estimates imply as in the $\mathbb{R}^n$ case results for the nonlinear equation, as local existence results and small data scattering results. They are an important piece in the proofs of other types of results for the nonlinear equation on $\mathbb{R}^n$, proofs that contain also other ingredients which are not obvious to transfer to general graphs. In this direction, the propagation of fast soliton was considered on star-shaped trees with $\delta-$type coupling conditions by Adami, Cacciapuoti, Finco and Noja in [1]. Also, large data scattering on the line with one repulsive Dirac potential was proved in [8] in the defocusing case, and by Ikeda and Inui [13] in focusing case.

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Nonlinear resonances: an exploration of NLS on a simple open metric graph

Gregory Berkolaiko
(joint work with Diego Noja)

When studying families of stationary solutions of nonlinear Schroedinger equation on open metric graphs, one sometimes observes that a whole solution branch may disappear under a small perturbation of the graph's geometry. To investigate this phenomenon we propose a simple model and an adaptation to NLS of the concept of linear resonances. We discuss how the disappearing branch can then be viewed as a branch of complex solutions moving into the "unphysical" half of the complex plane.

Existence of the ground state for the NLS with potential on graphs

Claudio Cacciapuoti
(joint work with Riccardo Adami, Domenico Finco, Diego Noja)

The problem we are interested in is the minimization of the nonlinear Schrödinger (NLS) energy functional

$$(1) \quad E[\Psi] := \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{v\in\mathcal{V}} \alpha_v |\Psi(v)|^2 - \frac{1}{\mu+1}\|\Psi\|_{2\mu+2}^{2\mu+2} \qquad 0 < \mu \leq 2$$

defined on a metric graph $\mathcal{G}$, where W is a potential on the graph, $\mathcal{V}$ is the set of vertices of the graph, and α_v are some real constants that take into account possible delta-interactions in the vertices. As for the standard NLS with power-type nonlinearity on the real-line, the case $0 < \mu < 2$ is called *subcritical* while the case $\mu = 2$ is called *critical.*

We shall focus attention on the constrained minimization problem

$$(2) \qquad -\nu_\mu(m) := \inf\{E[\Psi] \mid \Psi \in H^1(\mathcal{G}),\ \|\Psi\|^2 = m\}.$$

The parameter m in the constraint is usually referred to as *mass.*

We call *ground state* (of mass m), a minimizer of problem (2), i.e., a function $\hat{\Psi} \in H^1(\mathcal{G})$, such that $\|\hat{\Psi}\|^2 = m$, and $E[\hat{\Psi}] = -\nu_\mu(m)$. We want to identify general conditions which guarantee the existence of the ground state for small mass.

We make the following assumptions:

Assumption 1. $\mathcal{G}$ is a finite, connected graph, with at least one external edge.

We call *finite* a graph that has a finite number of edges and vertices; a graph $\mathcal{G}$ is *connected* if given any two points of the graph there is always a path in $\mathcal{G}$ connecting them; and we call *external* an edge of infinite length.

Assumption 2. $W = W_+ - W_-$ with $W_\pm \geq 0$, $W_+ \in L^1(\mathcal{G}) + L^\infty(\mathcal{G})$, and $W_- \in L^r(\mathcal{G})$ for some $r \in [1, 1 + 1/\mu]$.

Denote by $E^{lin}[\Psi]$ the quadratic part of energy functional $E[\Psi]$,

$$E^{lin}[\Psi] := \|\Psi'\|^2 + (\Psi, W\Psi) + \sum_{v \in \mathcal{V}} \alpha_v |\Psi(v)|^2, \tag{3}$$

and by $-E_0$ the infimum

$$-E_0 := \inf\left\{E^{lin}[\Psi] \mid \Psi \in H^1(\mathcal{G}),\ \|\Psi\|^2 = 1\right\}. \tag{4}$$

Assumption 3. $E_0 > 0$.

In the statements of our main results, some constraints on μ and m, with an interplay between the two parameters, are needed. Two quantities will enter the constraints. For the critical case it will be relevant the best constant $K_{6,2}(\mathcal{G})$ satisfying the Gagliardo-Nirenberg inequality

$$\|\Psi\|_6 \leq K_{6,2}(\mathcal{G}) \|\Psi'\|^{1/3} \|\Psi\|^{2/3}. \tag{5}$$

In the subcritical case, instead, the constraint will depend on a positive constant, denoted by γ_μ, which is related to the infimum of the "free" nonlinear NLS energy functional on the real-line

$$-t_\mu(m) := \inf\left\{E_{\mathbb{R}}[\psi] \,\middle|\, \psi \in H^1(\mathbb{R}),\ \|\psi\|^2_{L^2(\mathbb{R})} = m\right\} \qquad 0 < \mu \leq 2,$$

with

$$E_{\mathbb{R}}[\psi] := \|\psi'\|^2_{L^2(\mathbb{R})} - \frac{1}{\mu+1}\|\psi\|^{2\mu+2}_{L^{2\mu+2}(\mathbb{R})}.$$

$t_\mu(m)$ can be explicitly computed, and γ_μ is related to $t_\mu(m)$ by

$$t_\mu(m) = \gamma_\mu m^{1+\frac{2\mu}{2-\mu}} \qquad \text{for} \quad 0 < \mu < 2. \tag{6}$$

Our first result concerns the existence of a lower bound for the infimum in Eq. (2).

Theorem 1. Let Assumption 1 hold true and assume $W \in L^1(\mathcal{G}) + L^\infty(\mathcal{G})$. If $0 < \mu < 2$ then $\nu_\mu(m) < +\infty$ for any $m > 0$. If $\mu = 2$ then $\nu_\mu(m) < +\infty$ for any $0 < m < \sqrt{3}/K_{6,2}^3(\mathcal{G})$.

The second result concerns the existence of the ground state.

Theorem 2. Let Assumptions 1, 2, and 3 hold true, then $-\nu_\mu(m) \leq -mE_0$. Moreover, let

$$m_\mu^* := \begin{cases} (E_0/\gamma_\mu)^{\frac{1}{\mu}-\frac{1}{2}} & \text{if} \quad 0 < \mu < 2 \\ \sqrt{3}/K_{6,2}^3(\mathcal{G}) & \text{if} \quad \mu = 2 \end{cases} \tag{7}$$

Then the ground state $\hat{\Psi}$ exists for all $0 < m < m_\mu^*$.

The subcritical case was discussed in [3], while the analysis of the critical case is in [2]. When $\mathcal{G}$ is a star-graph, $W = 0$, and $\alpha_v < 0$, both the subcritical an the critical case were discussed in [1].

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On the construction of breathers by tailoring periodicity

Martina Chirilus-Bruckner

Consider the nonlinear Klein-Gordon equation

$$s(x)\partial_t^2 u = \partial_x^2 u + q(x)u + r(x)u^3$$

for $x, t, u = u(x,t) \in \mathbb{R}$ and with spatially periodic coefficients s, q, r. In [2] this equation was demonstrated to support breathers – that is, time-periodic, spatially localized solutions – if the coefficient s is a specific step-function (with q and r then chosen accordingly). This result came as a surprise since breathers were considered a rare phenomenon for nonlinear wave equations. In fact, the main novelty was to first carefully tailor the periodic coefficients s and q of the linear part such that the construction of the breather could be carried out using a spatial dynamics formulation and a blend of center manifold reduction and bifurcation theory.

The unexpected success of this method led to the quest of finding a whole class of coefficients s and q for which the above method could be employed. The core difficulty of the construction is to find coefficients for which the band structure of the linear part is such that every other gap is uniformly open around some multiple of a squared integer. This, in turn, can be reformulated as an inverse spectral problem for Dirichlet and Neumann eigenvalues of a weighted Sturm-Liouville operator. In [3] this question has been addressed showing that one can indeed find coefficients $s \in H^1$ (close to $s = 1$) that give rise to the desired band gap structure. In order to further extend this result for the purposes of the construction of breather solutions it is necessary to address this problem for $s \in L^2 \setminus H^{1/2}$ which would need a whole set of new tools (since this problem is directly related to inverse spectral theory for Schrödinger operators with potentials in $H^s, s < -1$).

A new natural direction which might lead to a similar breather construction is to replace the spatially periodic coefficients by periodic point interactions or to consider nonlinear Klein-Gordon equations on graphs (see, for instance, [1] or [4]). In both cases, the tailoring of the band gap structure seems much more accessible with the main difficulty now being the development of invariant manifold theory for these new settings.

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Global attraction to solitary waves

Andrew Comech

(joint work with Alexander Komech)

Bohr's second postulate states that the electrons can jump from one quantum stationary state (Bohr's *stationary orbit*) to another. This postulate suggests the dynamical interpretation of Bohr's transitions as long-time attraction

$$\Psi(t) \longrightarrow |E_\pm\rangle, \qquad t \to \pm\infty \tag{1}$$

for any trajectory $\Psi(t)$ of the corresponding dynamical system, where the limiting states $|E_\pm\rangle$ depend on the trajectory. Then the *quantum stationary states*, denote them $\mathcal{S}$, should be viewed as points of the *global attractor*, which we denote by $\mathfrak{A}$.

The attraction (1) takes the form of the long-time asymptotics

$$\psi(x,t) \sim \phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, \qquad t \to \pm\infty, \tag{2}$$

which holds for each finite energy solution. However, because of the superposition principle, the asymptotics of type (2) are generally impossible for the linear autonomous equation, be it the Schrödinger equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\Delta\psi - \frac{\mathrm{e}^2}{|\mathbf{x}|}\psi \tag{3}$$

or relativistic Schrödinger or Dirac equation in the Coulomb field. An adequate description of this process requires to consider the equation for the electron wave function (Schrödinger or Dirac equation) coupled to the Maxwell system which governs the time evolution of the four-potential $A(x,t) = (\varphi(x,t), \mathbf{A}(x,t))$:

$$\begin{cases} (i\hbar\partial_t - \mathrm{e}\varphi)^2\psi = (c\frac{\hbar}{i}\nabla - \mathrm{e}\mathbf{A})^2\psi + m^2c^4\psi, \\ \Box\varphi = 4\pi\mathrm{e}(\bar\psi\psi - \delta(\mathbf{x})), \qquad \Box\mathbf{A} = 4\pi\mathrm{e}\frac{\bar\psi\cdot\nabla\psi - \nabla\bar\psi\cdot\psi}{2i}. \end{cases} \tag{4}$$

Consideration of such a system seems inevitable, because, again by Bohr's postulates, the transitions from one orbit to another are followed by electromagnetic radiation responsible for the atomic spectra which we observe in the experiment. Moreover, the Lamb shift (a relatively small difference between $2S_{1/2}$ and $2P_{1/2}$ energy levels) can not be explained in terms of the linear Dirac equation in the external Coulomb field. Its theoretical explanation within the Quantum Electrodynamics is based on taking into account the higher order interaction of the electron wave function with the electromagnetic field.

One might expect the following generalization of asymptotics (2) for solutions to the coupled Maxwell–Schrödinger (or Maxwell–Dirac) equations:

$$(\psi(x,t), A(x,t)) \sim \left(\phi_{\omega_\pm}(x)e^{-i\omega_\pm t}, A_{\omega_\pm}(x)\right), \qquad t\to\pm\infty. \tag{5}$$

The asymptotics (5) would mean that the set of all solitary waves

$$\{\left(\phi_\omega e^{-i\omega t}, A_\omega\right) : \omega\in\mathbb{R}\}$$

forms a global attractor for the coupled system. The asymptotics of this form are not available yet in the context of coupled systems. Let us mention that the existence of the solitary waves for the coupled Maxwell–Dirac equations (without external potential) was established in [1].

We mention that convergence to a global attractor is well known for dissipative systems, like Navier–Stokes equations. For such systems, the global attractor is formed by the *static, stationary states*, and the convergence to the attractor only holds for $t\to+\infty$.

We would like to know whether dispersive Hamiltonian systems could, in the same spirit, possess finite dimensional global attractors, and whether such attractors are formed by the solitary waves. Although there is no dissipation per se, we expect that the attraction is caused by certain friction via the dispersion mechanism (local energy decay). Because of the difficulties posed by the system of interacting Maxwell and Dirac (or Schrödinger) fields (and, in particular, absence of the a priori estimates for such systems), we will work with simpler models which share certain key properties of the coupled Maxwell–Dirac or Maxwell–Schrödinger systems. Let us try to single out these key features:

(1) *The system is* $\mathbf{U}(1)$*-invariant.*
This invariance leads to the existence of solitary wave solutions $\phi_\omega(x)e^{-i\omega t}$.
(2) *The linear part of the system has a dispersive character.*
This property provides certain dissipative features in a Hamiltonian system, due to local energy decay via the dispersion mechanism.
(3) *The system is nonlinear.*
The nonlinearity is needed for the convergence to a single state of the form $\phi_\omega(x)e^{-i\omega t}$. Bohr type transitions to pure eigenstates of the energy operator are impossible in a linear system because of the superposition principle.

We suggest that these are the very features responsible for the global attraction to "quantum stationary states".

Let us briefly list our results. In [2] we prove global attraction to solitary waves for the Klein–Gordon field coupled to a nonlinear oscillator at one point. Under the assumption that the potential energy of the oscillator depends polynomially of its amplitude, the convergence to the attractor is proved in the local energy norm, for an arbitrary initial data of finite energy. The main ingredient of the proof is the Titchmarsh convolution theorem, which ensures that the spectrum of the "omega-limit" trajectory consists of (at most) one point.

In [3], this result is extended to the Klein–Gordon field coupled to finitely many oscillators.

In [4], we prove the global attraction to solitary waves for the Klein–Gordon field in the discrete time-space, coupled to a nonlinear oscillator at one point. The peculiarity of this model is that the solitary manifold is formed by solutions with one, two, and four frequencies, of the following form:

$$\phi(X)e^{-i\omega T},$$
$$\phi_1(X)e^{-i\omega T} + \phi_2(X)e^{-i(\omega+\pi)T},$$
$$\varphi_1(X)e^{-i\omega T} + \varphi_2(X)e^{-i(\omega+\pi)T} + \varphi_3(X)e^{-i\omega' T} + \varphi_4(X)e^{-i(\omega'+\pi)T}.$$

Above, $(X,T) \in \mathbb{Z}^d \times \mathbb{Z}$ and $\omega \in \mathbb{R} \mod 2\pi$ corresponds to the Fourier transform of $T \in \mathbb{Z}$; the amplitudes $\phi(X)$, $\phi_j(X)$, and $\varphi_j(X)$ belong to $l^2(\mathbb{Z}^n)$.

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NLS ground states on the two-dimensional grid: dimensional crossover and a continuum of critical exponents

Simone Dovetta

(joint work with Riccardo Adami, Enrico Serra, Paolo Tilli)

We consider the nonlinear Schrödinger (NLS) energy functional

$$E(u,\mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}_\ell} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{G}_\ell} |u|^p\,dx \tag{1}$$

on the two-dimensional grid of edge-length $\ell > 0$, $\mathcal{G}_\ell$.

We are interested in ground states of fixed mass $\mu > 0$. These are the absolute minimizers of E with mass constraint

$$\int_{\mathcal{G}_\ell} |u|^2\,dx = \mu,$$

and solve the NLS equation

$$u'' + |u|^{p-1}u = \omega u \qquad \text{on } \mathcal{G}_\ell,$$

with Kirchhoff boundary conditions at the vertices of $\mathcal{G}_\ell$.

The purpose of this talk is to analyse whether ground states exist or not in both the subcritical and the critical regime, i.e. when $p \in (2,6)$ and $p = 6$ respectively.

Such a problem has already been addressed on metric graphs with a finite number of nodes and with at least one half-line ([1],[2] for the subcritical case, [3] for the critical one), where the authors revealed how topological and metric features of the domain may play a significant role. In shorts, for graphs like these, it turned out that ground states on the real line (the so-called solitons) provide a natural energy threshold, and possibility to beat it implies existence of global minimizers.

Due to the absence of half-lines and to the periodicity of the grid we are dealing with, however, it appears that solitons on the line play no longer such a crucial role in our setting. Indeed, we derive a first existence criterion that ensures compactness of minimizing sequences on $\mathcal{G}_\ell$, provided that the infimum of the energy functional (1) is strictly negative.

Theorem 1. *If*

$$-\infty < \inf_{u\in H^1_\mu(\mathcal{G}_\ell)} E(u,\mathcal{G}_\ell) < 0$$

then the infimum is attained, i.e. there exists a function $u \in H^1_\mu(\mathcal{G}_\ell)$ *such that*

$$E(u,\mathcal{G}_\ell) = \inf_{v\in H^1_\mu(\mathcal{G}_\ell)} E(v,\mathcal{G}_\ell)$$

If, on the contrary, $E(u,\mathcal{G}_\ell) > 0$ *for every* $u \in H^1_\mu(\mathcal{G}_\ell)$*, then*

$$\inf_{u\in H^1_\mu(\mathcal{G}_\ell)} E(u,\mathcal{G}_\ell) = 0$$

and it is never attained.

The periodical structure of the grid deeply helps in gaining compactness, and the natural threshold for the energy is enhanced to 0, and this highlight a first significant difference with graphs sharing half-lines.

An even more important difference is rooted in dimension. Indeed, defining what is the proper dimension of $\mathcal{G}_\ell$ reveals to be not as easy as one may expect, since two different scales coexist: a one-dimensional scale, proper of every edge building up the graph, and a two-dimensional one, that enter the game when we are looking at the grid from far beyond th edge-length scale.

Dimension of the space determines critical exponent for the NLS equation, i.e. the exponent p_c for which, as μ increases, the infimum of the energy passes from 0 to a negative (possibly unbounded) value. It is well-known that

· for $\mathbb{R}$ and for graphs with half-lines and compact core: $p_c = 6$;

· for $\mathbb{R}^2$: $p_c = 4$.

Critical exponents are rooted in the so-called Gagliardo-Nirenberg inequalities, that are cornerstones of investigations like the one we are performing.

On a general non-compact graphs, and thus on $\mathcal{G}_\ell$ too, the following Gagliardo-Nirenberg inequality always holds:

$$\|u\|^p_{L^p(\mathbb{R})} \leq K_1 \|u\|^{\frac{p}{2}+1}_{L^2(\mathbb{R})} \|u'\|^{\frac{p}{2}-1}_{L^2(\mathbb{R})} \tag{2}$$

for every $p \in [2, \infty)$ and every $u \in H^1(\mathbb{R})$. Here K_1 denotes the optimal constant in the inequality.

Moreover, on $\mathbb{R}^2$, a similar inequality holds:

$$\|u\|^p_{L^p(\mathbb{R}^2)} \leq K_2 \|u\|^2_{L^2(\mathbb{R}^2)} \|\nabla u\|^{p-2}_{L^2(\mathbb{R}^2)} \tag{3}$$

for every $p \in [2, \infty)$ and $u \in H^1(\mathbb{R}^2)$.

Since these inequalities provide lower bounds for the energy functional, plugging them into (1) shows why $p = 6$ and $p = 4$ are the critical exponents in the one-dimensional and the two-dimensional problem respectively (it is easy to see that coercivity of (1) fails in both cases).

On $\mathcal{G}_\ell$, we prove that both (2) and (3) hold, and by interpolation, that an entire new family of such inequalities arises. In particular, the following interpolated version

$$\|u\|^p_{L^p(\mathcal{G}_\ell)} \leq K_p \|u\|^{p-2}_{L^2(\mathcal{G}_\ell)} \|u'\|^2_{L^2(\mathcal{G}_\ell)} \tag{4}$$

holds for every $4 \leq p \leq 6$.

Inequality (4) reveals the existence of a critical behaviour on $\mathcal{G}_\ell$ for a continuum of exponents, i.e. exponents varying between 4 and 6. In some sense, the grid performs a crossover between the purely one-dimensional and two-dimensional cases, and existence of ground states reflects this feature. Indeed, if for $p \in (2, 4)$, ground states exist for every value of the mass $\mu > 0$, as it happens on the plane $\mathbb{R}^2$, critical masses appear at every $p \in [4, 6]$ (depending on the exponent), and global minimizers exist only for masses greater than these thresholds. The following theorems thus summarize our main results.

Theorem 2 (Subcritical case)**.** *Let the functional* $E(\cdot, \mathcal{G}_\ell)$ *be defined as in* (1) *and* $2 < p < 4$. *Then, for every* $\mu > 0$, *there exists a ground state at mass* μ.

Theorem 3 (Dimensional Crossover)**.** *Let the functional* $E(\cdot, \mathcal{G}_\ell)$ *be defined as in* (1) *and* $4 \le p \le 6$. *Then, for every* p *there exists a threshold value* $\mu_p > 0$ *such that*

(i) *if* $p = 4$ *then ground states exist if* $\mu > \mu_4$ *and do not exist if* $\mu < \mu_4$.
(ii) *if* $4 < p < 6$ *then ground states exist if and only if* $\mu \ge \mu_p$
(iii) *if* $p = 6$ *then ground states never exist, regardless of the value of* μ.
Furthermore,

$$\inf_{u \in H^1_\mu(\mathcal{G}_\ell)} E(u, \mathcal{G}_\ell) = \begin{cases} 0 & \text{if } \mu \le \mu_6 \\ -\infty & \text{if } \mu > \mu_6 \end{cases} \tag{5}$$

We note that the case whit $p = 4$ and $\mu = \mu_4$ remains uncovered, and existence or non-existence of ground states will be object of further investigations.

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Unusual spectra of periodic graphs

Pavel Exner

(joint work with Stepan Manko, Daniel Vašata, and Ondřej Turek)

It is a standard part of the quantum lore that spectrum of periodic system has a number of familiar properties:

- it is absolutely continuous
- it has a band-and-gap structure
- in the one-dimensional case the number of open gaps is infinite except for a particular class of potentials
- on the contrary, in higher dimensions the *Bethe-Sommerfeld conjecture*, nowadays verified for a wide class of interactions, says that the number of open gaps is *finite*

The aim of the talk is to show that if the system in question is a quantum graph, nothing of that needs to be true. To demonstrate this, we discuss two simple example, a loop chain exposed to a magnetic field and a rectangular lattice.

The former one was discussed in [1, 2]. We consider an infinite array Γ of identical loops of circumference 2π on which we have the magnetic Laplacian,

$\psi_j \mapsto -\mathcal{D}^2\psi_j$ with $\mathcal{D} := -i\nabla - \mathbf{A}$ and a δ-coupling at the vertices connecting the rings, i.e. with the domain consisting of functions from $H^2_{\text{loc}}(\Gamma)$ satisfying

$$\psi_i(0) = \psi_j(0) =: \psi(0)\,, \quad i,j \in \mathfrak{n}\,, \quad \sum_{i=1}^{n} \mathcal{D}\psi_i(0) = \gamma\,\psi(0)\,,$$

where $\mathfrak{n} = \{1, 2, \dots, n\}$ is the index set numbering the edges and $\gamma \in \mathbb{R}$ is the coupling constant. If the array has a mirror symmetry, the system exhibits *Dirichlet eigenvalues* with eigenfunctions vanishing at the vertices. If the field is homogeneous, the latter are interlaced with absolutely continuous bands, however, if $A - \frac{1}{2} \in \mathbb{Z}$, where A is the tangent component of the vector potential, those shrink to points and the spectrum consists of infinitely degenerate eigenvalues only.

It is also expected that local perturbations give rise to discrete eigenvalues in the spectral gaps. We analyze several examples using the well-known duality technique which allows us to rephrase the spectral problem as a difference equation. We show, in particular, that no such eigenvalues need to appear; this happens, for instance, when the magnetic field is changed on a single ring, $A = \{\dots, A, A_1, A \dots\}$, and the inequality $|\cos A_1\pi| \leq |\cos A\pi|$ is satisfied [2].

If the perturbation is nonlocal the spectral behavior may be very different. We focus on the example where it changes linearly along the chain, $A_j = \alpha j + \theta$ for some $\alpha, \theta \in \mathbb{R}$ and every $j \in \mathbb{Z}$. Using deep results of Jitomirskaya, Avila and Krikorian about the *almost Mathieu equation* we prove that the spectrum of $\sigma(-\Delta_{\gamma,A})$ has the following properties [3]:

(a) If $\alpha, \theta \in \mathbb{Z}$ and $\gamma = 0$, then $\sigma(-\Delta_{\gamma,A}) = \sigma_{ac}(-\Delta_{\gamma,A}) \cup \sigma_{pp}(-\Delta_{\gamma,A})$ where $\sigma_{ac}(-\Delta_{\gamma,A}) = [0, \infty)$ and $\sigma_{pp}(-\Delta_{\gamma,A}) = \{n^2 |\, n \in \mathbb{N}\}$.

(b) If $\alpha = p/q$ with p and q relatively prime, $\alpha j + \theta + \frac{1}{2} \notin \mathbb{Z}$ for all $j = 0, \dots, q-1$ and assumptions of (a) do not hold, then $-\Delta_{\gamma,A}$ has *infinitely degenerate eigenvalues* at the points of $\{n^2 |\, n \in \mathbb{N}\}$ and *an* ac *part* of the spectrum in each interval $(-\infty, 1)$ and $\big(n^2, (n+1)^2\big)$, $n \in \mathbb{N}$, consisting of q closed intervals possibly touching at the endpoints.

(c) If $\alpha = p/q$, where p and q are relatively prime, and $\alpha j + \theta + \frac{1}{2} \in \mathbb{Z}$ for some $j = 0, \dots, q-1$, then the spectrum $-\Delta_{\gamma,A}$ is of *pure point type* and such that in each interval $(-\infty, 1)$ and $\big(n^2, (n+1)^2\big)$, $n \in \mathbb{N}$, there are exactly q distinct eigenvalues and the remaining eigenvalues form the set $\{n^2 |\, n \in \mathbb{N}\}$. All the eigenvalues are infinitely degenerate.

(d) Finally, if $\alpha \notin \mathbb{Q}$, then $\sigma(-\Delta_{\gamma,A})$ does not depend on θ and it is a disjoint union of the isolated-point family $\{n^2 |\, n \in \mathbb{N}\}$ and *Cantor sets*, one inside each interval $(-\infty, 1)$ and $\big(n^2, (n+1)^2\big)$, $n \in \mathbb{N}$. Moreover, *overall Lebesgue measure* of $\sigma(-\Delta_{\gamma,A})$ *is zero*.

Moreover, using a fresh result of Last and Shamis we can also show that there exists a dense G_δ set of the slopes α for which, and all θ, the Haussdorff dimension of the spectrum vanishes, $\dim_H \sigma(-\Delta_{\gamma,A}) = 0$.

The graphs in this example had ‘many’ gaps indeed. Next we ask whether periodic graphs can have ‘just a few’ gaps. In their book, Berkolaiko and Kuchment recall the reasoning behind the Bethe–Sommerfeld conjecture and say that the situation with graphs is similar, however, they add immediately that *this is not a strict law* recalling resonant gaps created by a graph 'decoration'. One may ask whether *it is a 'law' at all*, that is, whether infinite periodic graphs with a *finite nonzero* number of open gaps exist. From obvious reasons we would call them *Bethe–Sommerfeld graphs*. The answer depends on the vertex coupling; it is not difficult to show that such a situation is excluded, if the coupling is *scale invariant* or it belongs to a wider class associated with a scale invariant coupling [4, 5].

On the other hand, we are going to show that *Bethe–Sommerfeld graphs do exist*. To this aim we analyze the example of an *infinite rectangular lattice* of the call sides a, b and a δ coupling with the parameter γ at the vertices. It is known that a number $k^2 > 0$ belongs to a gap if and only if $k > 0$ satisfies

$$\tan\left(\frac{ka}{2} - \frac{\pi}{2}\left\lfloor\frac{ka}{\pi}\right\rfloor\right) + \tan\left(\frac{kb}{2} - \frac{\pi}{2}\left\lfloor\frac{kb}{\pi}\right\rfloor\right) < \frac{\gamma}{2k} \quad \text{for } \gamma > 0$$

and

$$\cot\left(\frac{ka}{2} - \frac{\pi}{2}\left\lfloor\frac{ka}{\pi}\right\rfloor\right) + \cot\left(\frac{kb}{2} - \frac{\pi}{2}\left\lfloor\frac{kb}{\pi}\right\rfloor\right) < \frac{|\gamma|}{2k} \quad \text{for } \gamma < 0\,;$$

we neglect the Kirchhoff case, $\gamma = 0$, where the spectrum is $[0, \infty)$.

The spectrum depends on the ratio $\theta = \frac{a}{b}$. If θ is rational, $\sigma(H)$ has infinitely many gaps unless $\gamma = 0$, the same is true if θ is is *an irrational well approximable by rationals*, which means equivalently that in the continuous fraction representation $\theta = [a_0; a_1, a_2, \dots]$ the sequence $\{a_j\}$ is unbounded. On the other hand, $\theta \in \mathbb{R}$ is *badly approximable* if there is a $c > 0$ such that

$$\left|\theta - \frac{p}{q}\right| > \frac{c}{q^2}$$

for all $p, q \in \mathbb{Z}$ with $q \neq 0$. We analyze in detail the case of the *golden mean lattice*, $\theta = \frac{\sqrt{5}+1}{2} = [1; 1, 1, \dots]$, which can be regarded as corresponding to the 'worst' irrational, and prove that [4, 5]:

(i) If $\gamma > \frac{\pi^2}{\sqrt{5}a}$ or $\gamma \le -\frac{\pi^2}{\sqrt{5}a}$, there are *infinitely many spectral gaps*.

(ii) If

$$-\frac{2\pi}{a}\tan\left(\frac{3-\sqrt{5}}{4}\pi\right) \le \gamma \le \frac{\pi^2}{\sqrt{5}a}\,,$$

there are *no gaps* in the positive spectrum.

(iii) If

$$-\frac{\pi^2}{\sqrt{5}a} < \gamma < -\frac{2\pi}{a}\tan\left(\frac{3-\sqrt{5}}{4}\pi\right),$$

there is *a nonzero and finite number of gaps* in the positive spectrum.

Furthermore, we present a finer classification of the case (iii) indicating the intervals of γ at which the spectrum has exactly N gaps. We also show examples of ratios θ for which a lattice can belong to the Bethe–Sommerfeld class for both signs of the coupling constant γ.

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Stationary states for NLS on a graph

Domenico Finco

(joint work with Riccardo Adami, Claudio Cacciapuoti, Diego Noja)

Very interesting bifurcation phenomena appears when looking for stationary states of NLS on graphs. Here we consider the case of focusing power nonlinearities for sake of simplicity and we consider two graphs where one can gives explicit formulas: a star graph and a tadpole. We report some results from [1, 2, 3].
A star graph $\mathcal{G}$ is a graph with one vertex v and N infinite edges e_i. We place a δ-interaction of strength $\alpha < 0$ in the vertex. A function $\Psi : \mathcal{G} \to \mathcal{C}$ is a collection of functions $\psi_i : e_i \to \mathcal{C}$. Stationary solutions of the NLS

$$i\frac{d}{dt}\Psi = H\Psi - |\Psi|^{2\mu}\Psi. \tag{1}$$

have the from $\Psi(t) = e^{-i\omega t}\Phi$ and Φ satisfies the following equation.

$$H\Phi_\omega - |\Phi_\omega|^{2\mu}\Phi_\omega = \omega\Phi_\omega \qquad \Phi \in \mathcal{D}(H),\ \omega < 0, \tag{2}$$

The domain of the operator H is defined in the following way:

$$\mathcal{D}(H) := \left\{ \Psi \in H^2 \text{ s.t. } \psi_1(0) = \ldots = \psi_n(0) \ \ \sum_i \psi_i'(0) = \alpha\psi_1(0) \right\}.$$

The action of H is defined by $(H\Psi)_i = -\psi_i''$ Notice that $\sigma_p(H) = \{-\alpha^2/N^2\}$ and $\sigma_c(H) = [0, \infty)$. A stationary state must coincide on each edge, up to phase multiplication and translations, with ϕ where ϕ is given by :

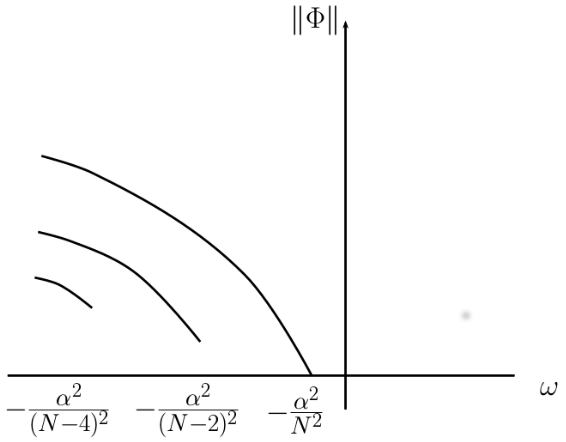
$$\phi(x) = [(\mu+1)|\omega|]^{\frac{1}{2\mu}} \operatorname{sech}^{\frac{1}{\mu}}(\mu\sqrt{|\omega|}x)$$

Matching the free parameters in order satisfy the boundary conditions in $\mathcal{D}(H)$, we obtain that there are up to$[(N-1)/2]+1$ stationary solutions Φ_ω^j, with $j = 0, \cdots, [\frac{N-1}{2}]$, given, up to permutations of the edges, by:

$$\left(\Phi_\omega^j\right)_i(x) = \begin{cases} \phi(x - a_j) & i = 1, \ldots, j \\ \phi(x + a_j) & i = j+1, \ldots, N \end{cases}$$

$$a_j = \frac{1}{\mu\sqrt{|\omega|}} \operatorname{arctanh}\left(\frac{\alpha}{(2j-N)\sqrt{|\omega|}}\right) .$$

Due to bell shape of ϕ we say that in a edge there is a bump if the stationary state reads $\phi(x-a_j)$ while on the contrary we say there is a tail. For $N=3$ there just two stationary states (up to edge permutations), the three tail state and the one bump state, see the picture. Notice that the index j counts the number of bumps. The stationary state Φ_ω^j exists for $\omega \leqslant \omega_*^j = -\frac{\alpha^2}{(N-2j)^2}$ and for $\omega \to \omega_*^j$ we have $a_j \to \infty$. In this limit we notice that Φ_ω^0 vanishes and all its $L^p(\mathcal{G})$ norms go to zero while Φ_ω^j for $j \geqslant 1$ escapes at infinity all its $L^p(\mathcal{G})$ norms stay away from zero. For $\omega \to -\infty$ we have $a_j \to 0$ and all the stationary state converge to the unique stationary states of the Kirchhoff case.
We summarize the situation with the following power diagram (here $1 < \mu < 2$).



From the point of view of bifurcation theory, the branch of Φ_ω^0 corresponds to a bifurcation from the simple eigenvalue of H. The bifurcation mechanism of the higher branches is different and not yet well understood and it does not fit into a general perturbative scheme at the bifurcation point. A tadpole $\mathcal{G}$ is graph with a closed loop and one infinite edge. We use as coordinates on the graph: $x \in [-L, L]$, $y \in [0, \infty)$ while functions on the graph will be written in the following way:

$$\Psi = (u, \eta) \qquad \text{with} \quad u : [-L, L] \to \mathbb{C}\,;\ \eta : [0, \infty) \to \mathbb{C}$$

The operator H is the Laplacian with Kirchhoff b.c., $H\Psi = (-u'', -\eta'')$.

$$\mathcal{D}(H) = \Big\{\Psi \in H^2\ :\ u(L) = u(-L) = \eta(0)\,;\ -u'(L) + u'(-L) + \eta'(0) = 0\Big\}$$

The Spectrum of H is $\sigma(H) \equiv \sigma_{ess}(H) = [0, \infty)$, moreover we have embedded eigenvalues: $\{\lambda_n\} = \{(\frac{n\pi}{L})^2, n \in\}$ with corresponding (normalized) eigenfunctions given by: $\Upsilon_n = L^{-1/2}(\sin(n\pi x/L), 0)$. Notice also that H has a zero energy resonance, $\Upsilon_{res} = (1, 1)$.

We limit the analysis to the cubic case $\mu = 1$. The situation is more complicate compared to a star graph since besides ϕ there two type of oscillating solutions, u_{cn} and u_{dn}, which can be placed on the compact part of the graph and we have many different ways to match Kirchhoff boundary conditions.

$$\begin{aligned} u_{cn}(x;k) &= \sqrt{\tfrac{2\omega k^2}{1-2k^2}}\,\mathrm{cn}\left(\sqrt{\tfrac{\omega}{1-2k^2}}x;k\right),\ \omega\in\mathbb{R} \\ u_{dn}(x;\kappa) &= \sqrt{\tfrac{2|\omega|}{2-\kappa^2}}\,\mathrm{dn}\left(\sqrt{\tfrac{|\omega|}{2-\kappa^2}}x;\kappa\right),\ \omega<0 \end{aligned}$$

For $\omega > 0$ we have to take $\eta \equiv 0$. The match with u_{cn} fixes the period to be a integer multiple of $2L$; the condition on the period has infinite solutions $k_n(\omega)$.

$$\Phi_{\omega,n}(x,y) = (u_{cn}(x - nL/2, k_n), 0) \quad n = 1, 2, \ldots$$

Stationary solutions $\Phi_{\omega,n}$ exist for $\omega \in \mathbb{R}$. For $\omega < 0$ we can match $u_{cn}(\cdot - a, k_n)$ with ϕ if we choose the right translation. This is possible in two different ways and for each n we obtain two states $\Phi^{\pm}_{\omega,n}$.
If we try to match u_{dn} and ϕ we find two families $\{\Xi_{\omega,0,n}\}$ and $\{\Xi_{\omega,1,n}\}$ with

$$\Xi_{\omega,0,n}(x,y) = \big(u_{dn}(x;\kappa_{0,n}), \phi(y - b_{0,n})\big) \qquad \omega < 0$$

The two parameters $\kappa_{0,n}$ and $b_{0,n}$ are determined by the equations:

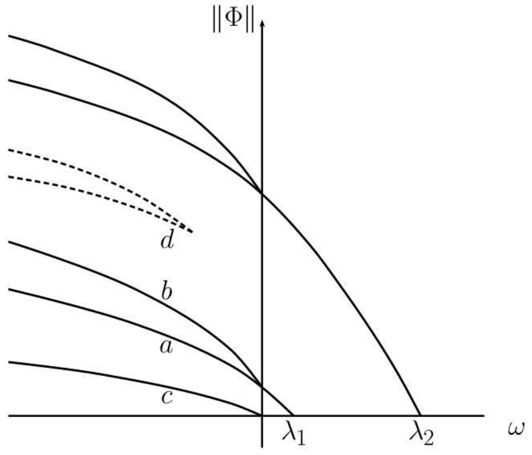
$$\begin{aligned} \frac{3\kappa^4}{1-\kappa^2}\,\mathrm{cn}^2\left(\frac{L\sqrt{|\omega|}}{\sqrt{2-\kappa^2}};\kappa\right)\left(1-\mathrm{cn}^2\left(\frac{L\sqrt{|\omega|}}{\sqrt{2-\kappa^2}};\kappa\right)\right) &= 1, \\ \cosh^{-2}(\sqrt{|\omega|}b_{0,n}) &= u^2_{dn}(L;\kappa_{0,n})/(2|\omega|) \end{aligned}$$

There is a finite number of solutions for any $L\sqrt{|\omega|}$ and exactly one solution $\Xi_{\omega,0,1}$ for $L\sqrt{|\omega|}$ small. Further solutions appear in couples as $L\sqrt{|\omega|}$ increases and the number of solutions diverges as $L\sqrt{|\omega|} \to \infty$. The second family is defined by:

$$\Xi_{\omega,1,n}(x,y)\big(u_{dn}(x - T_{dn}(\kappa_{1,n})/2;\kappa_{1,n}), \phi(y - b_{1,n})\big)$$

$$\begin{aligned} \frac{3\kappa^4\,\mathrm{cn}^2\left(\frac{L\sqrt{|\omega|}}{\sqrt{2-\kappa^2}};\kappa\right)\left(1-\mathrm{cn}^2\left(\frac{L\sqrt{|\omega|}}{\sqrt{2-\kappa^2}};\kappa\right)\right)}{\mathrm{dn}^4\left(\frac{L\sqrt{|\omega|}}{\sqrt{2-\kappa^2}};\kappa\right)} &= 1 \\ \cosh^{-2}(\sqrt{|\omega|}b_{1,n}) &= \tfrac{u^2_{dn}(L-T_{dn}(\kappa_{1,n})/2;\kappa_{1,n})}{2|\omega|} \end{aligned}$$

There is a finite number of solutions of any $L\sqrt{|\omega|}$ and no solutions for $L\sqrt{|\omega|}$ small. After $\Xi_{\omega,1,n}$ further solutions appear in couples as $L\sqrt{|\omega|}$ increases and the number of solutions diverges as $L\sqrt{|\omega|} \to \infty$.
We summarize the behavior of stationary states of a tadpole in the following diagram.



The branches $\Phi_{\omega,n}$ bifurcate from the vanishing state in correspondence of the embedded eigenvalues of the linear since $\Phi_{\lambda_n-\varepsilon,n} = c\sqrt{\varepsilon}\Upsilon_n + O(\varepsilon^{3/2})$.
The branch $\Xi_{\omega,0,1}$ bifurcates from the vanishing state in correspondence of the zero energy resonance at the threshold of the continuum spectrum of the linear Hamiltonian since $\Xi_{\omega,0,1} = \sqrt{2|\omega|}(1,1) + O(|\omega|^{3/2})$. At the secondary bifurcations where $\Phi^{\pm}_{\omega,1}$ are born, the linearized operator has a zero energy resonance. The formation mechanism of higher dn-states bears some resemblance to the case of excited states of the star graph.

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Schrödinger equation with point nonlinearity

Reika Fukuizumi

(joint work with Riccardo Adami, Justin Holmer)

In this talk we consider the following nonlinear Schrödinger equation with point nonlinearity in 1d:

$$i\partial_t\psi + \partial_x^2\psi + K(x)|\psi|^{p-1}\psi = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R} \tag{1}$$

where $p > 1$, and $K = \delta$, δ is the Dirac mass at $x = 0$. The singularity in the nonlinearity is interpreted as the linear Schrödinger equation:

$$i\partial_t\psi + \partial_x^2\psi = 0, \qquad t \in \mathbb{R}, \quad x \neq 0$$

together with the jump condition at $x = 0$

$$\psi(0,t) := \psi(0-,t) = \psi(0+,t)$$
$$\partial_x\psi(0+,t) - \partial_x\psi(0-,t) = -|\psi(0,t)|^{p-1}\psi(0,t).$$

Remark that the equation (1) appears as a limiting case of nonlinear Schrödinger equation with a concentrated nonlinearity (see [2]).

This equation (1) obeys the scaling law, i.e. if $\psi(x,t)$ solves (1), then $\psi_\lambda(x,t) = \lambda^{1/(p-1)}\psi(\lambda x, \lambda^2 t)$ solves (1). Thus the regularity of the scale invariant Sobolev norm $\dot{H}^{\sigma_c}$ satisfying $\|\psi\|_{\dot{H}^{\sigma_c}} = \|\psi_\lambda\|_{\dot{H}^{\sigma_c}}$ is $\sigma_c = \frac{1}{2} - \frac{1}{p-1}$. We can therefore observe that the nonlinear power $p = 3$ (cubic) is the L^2 critical setting for $\sigma_c = 0$, and $3 < p < \infty$ is the L^2 super critical setting for $0 < \sigma_c < \frac{1}{2}$. We remark that in contrast, $p = 5$ is the L^2 critical case for 1D standard NLS (i.e. the case $K \equiv 1$ in (1)).

An H^1 local well-posedness theory is available in [1] (see also [3]). More precisely,

Proposition. Let $\psi_0 \in H^1$, there exists $T = T(\|\psi_0\|_{H^1}) > 0$ and a unique solution $\psi(x,t)$ to (1) satisfying $\psi(0) = \psi_0$ and $\psi \in C([0,T]; H^1_x(\mathbb{R})) \cap C(x \in \mathbb{R}; H^{3/4}_{[0,T]})$ and $\partial_x \psi \in C(x \in \mathbb{R} \setminus \{0\}; H^{1/4}_{[0,T]})$.

From this proposition, it follows that if the maximal forward time $T^* > 0$ of existence is finite (i.e. $T^* < \infty$), then necessarily

$$\lim_{t \nearrow T^*} \|\partial_x \psi(t)\|_{L^2} = +\infty.$$

We say this that the solution $\psi(t)$ blows up at time $t = T^* > 0$.

The conservation law for (1) takes the form

$$\begin{aligned} M(\psi) &= \|\psi\|^2_{L^2}, \\ E(\psi) &= \frac{1}{2}\|\partial_x \psi\|^2_{L^2} - \frac{1}{p+1}|\psi(0,\cdot)|^{p+1}. \end{aligned}$$

In the Duhamel form of (1)

$$\begin{aligned} \psi(x,t) &= e^{it\partial_x^2}\psi_0 + i\int_0^t e^{i(t-s)\partial_x^2}\delta(x)|\psi(x,s)|^{p-1}\psi(x,s)ds \\ &= e^{it\partial_x^2}\psi_0 + i\int_0^t \frac{e^{\frac{ix^2}{4(t-s)}}}{\sqrt{4\pi i(t-s)}}|\psi(0,s)|^{p-1}\psi(0,s)ds, \end{aligned}$$

if we specialize to the value $x = 0$, we obtain an equation for $\psi(0,\cdot)$:

$$\psi(0,t) = e^{it\partial_x^2}\psi_0(0) + i\int_0^t \frac{1}{\sqrt{4\pi i(t-s)}}|\psi(0,s)|^{p-1}\psi(0,s)ds. \tag{2}$$

Namely the point is that the equation to $\psi(x,t)$ is completely solved once the one-variable complex function $\psi(0,\cdot)$ is known.

The solitary waves are the solutions to (1) of the form

$$\psi(x,t) = e^{it}\varphi_0(x).$$

Here, φ_0 satisfies the stationary equation

$$0 = \varphi_0 - \partial_x^2\varphi_0 - \delta|\varphi_0|^{p-1}\varphi_0.$$

It is straightforward that the unique solution is $\varphi_0(x) = 2^{1/(p-1)}e^{-|x|}$, and the rescalings of this solution are the only solitary waves for (1).

The authors in [3] established L^2 supercritical global existence, and blow-up dichotomy as follows.

Proposition. Suppose that $\psi(t)$ is an H^1 solution to (1) for $p > 3$ satisfying

$$M(\psi)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi) < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0).$$

Let

$$\eta(t) = \frac{\|\psi\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} \|\partial_x \psi(t)\|_{L_x^2}}{\|\varphi_0\|_{L^2}^{\frac{1-\sigma_c}{\sigma_c}} \|\partial_x \varphi_0\|_{L^2}}$$

Then

- If $\eta(0) < 1$, then the solution $\psi(t)$ is global in both time directions and $\eta(t) < 1$ for all $t \in \mathbb{R}$.
- If $\eta(0) > 1$, then the solution $\psi(t)$ blows up in the negative time direction at some $T_- < 0$, blows-up in the positive time direction at some $T_+ > 0$, and $\eta(t) > 1$ for all $t \in (T_-, T_+)$.

As far as we know the problem of scattering for (1) has not yet been studied. In the case of global existence for standard NLS (i.e. $K \equiv 1$), the question of scattering has been addressed by, among many others, [4, 5, 6].

We address in this talk a L^2 supercritical scattering result in H^1 to (1).

Theorem. Let $p > 3$. Let $\psi_0 \in H^1$ and let $\psi(t)$ be a H^1 solution to (1) with $\psi(0) = \psi_0$ satisfying

$$M(\psi)^{\frac{1-\sigma_c}{\sigma_c}} E(\psi) < M(\varphi_0)^{\frac{1-\sigma_c}{\sigma_c}} E(\varphi_0)$$

and $\eta(0) < 1$. Then, there exist $\psi^+, \psi^- \in H^1$ such that

$$\lim_{t \nearrow \pm\infty} \|e^{-it\partial_x^2}\psi(t) - \psi^{\pm}\|_{H^1} = 0.$$

Remark that the defocusing case is similarly proved (without mass-energy condition). To show this theorem we use the Kenig-Merle method [5] which is these days becoming standard, but it is required to use an appropriate function space according to the smoothing properties of the integral equation (2).

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Dispersive properties for the linear Schrödinger equation on star-graphs. Application to NLS

Andreea Grecu

We consider the linear Schrödinger equation:

$$(1) \qquad \begin{cases} iu_t(t,x) + \Delta_x u(t,x) = 0, & t \neq 0, \quad x \in \mathcal{G} \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathcal{G} \end{cases},$$

where $\mathcal{G}$ is a metric graph given by a finite number $n \in \mathbb{N}^*$ of infinite length edges attached to a common vertex (so-called *star-graph*), having each edge identified with the positive real axis; u_t represents the time derivative of u, and the Laplace operator $\Delta_x =: \Delta(A, B)$ with domain

$$D(\Delta(A,B)) = \{u \in H^2(\mathcal{G}) : A\underline{u} + B\underline{u'} = 0\},$$

acts as the second derivative along the edges. The $n \times n$ real matrices A and B which express the coupling conditions at the vertex are assumed to satisfy:

$(H1)$ The horizontally concatenated matrix (A, B) has maximal rank;

$(H2)$ $AB^\dagger$ is self-adjoint;

ensuring the self-adjointness of the corresponding laplacian (according to [2, 3]).

For $q \in [2, \infty]$, $\frac{1}{p} + \frac{1}{q} = 1$, we obtain $L^p(\mathcal{G}) - L^q(\mathcal{G})$ dispersive and space-time Strichartz estimates for the solution of (1), and well-posedness for the nonlinear Schrödinger equation (NLS) for a class of power nonlinearities.

The proof of the dispersive properties is based on an explicit form of the solution obtained via spectral theory, which is further estimated using classical results for oscillatory integrals. We point out that the explicit form of the resolvent's integral kernel (given in [2]) and its properties (delivered in [3]) are of great use. Once the dispersive properties are obtained, the Strichartz estimates follow by the result of Keel and Tao in [4]. We make use of these estimates to prove well-posedness of the NLS

$$\begin{cases} iu_t(t,x) + \Delta_x u(t,x) + \lambda |u|^{p-1} u = 0, & t \neq 0, \quad x \in \mathcal{G} \\ u(0,x) = u_0(x), & \qquad\quad\; x \in \mathcal{G} \end{cases},$$

where $p \in (1, 5)$, $\lambda \in \mathbb{R}$. The latter is based on a fixed point argument and follows in the spirit of [1].

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Kuramoto-Sivashinsky equation on star-shaped trees. A controllability result

Liviu Ignat

(joint work with C.M. Cazacu and A.F. Pazoto)

In this talk we present some controllability properties for the linear Kuramoto-Sivashinsky equation on a network with two types of boundary conditions. More precisely, the equation is considered on a star-shaped tree with Dirichlet and Neumann boundary conditions. By using the moment theory we can derive null-controllability properties with boundary controls acting on the external vertices of the tree. In particular, the controllability holds if the *anti-diffusion* parameter of the involved equation does not belong to a critical countable set of real numbers. We point out that the critical set for which the null-controllability fails differs from the first model to the second one.

Our main goal is to study boundary null-controllability properties for the Kuramoto-Sivashinsky (KS) equation

$$y_t + \lambda y_{xx} + y_{xxxx} = 0, \tag{1}$$

on a star-shaped tree denoted Γ. More precisely, Γ is a simplified topological graph with $N \geq 2$ edges of the same given length $L > 0$ and $N + 1$ vertices. Besides, all edges intersect at a unique endpoint which is the interior vertex of the graph. The mathematical formulation of the control problems that we address on Γ stands for a system of N-KS equations on the interval $(0, L)$ coupled through the left endpoint $x = 0$ as follows

$$(2) \quad \begin{cases} y^k_t + \lambda y^k_{xx} + y^k_{xxxx} = 0, & (t,x) \in (0,T) \times (0,L) \\ y^i(t,0) = y^j(t,0), & t \in (0,T), \quad i,j \in \{1,\dots,N\} \\ \sum_{k=1}^N y^k_x(t,0) = 0, & t \in (0,T) \\ y^i_{xx}(t,0) = y^j_{xx}(t,0), & t \in (0,T), \quad i,j \in \{1,\dots,N\} \\ \sum_{k=1}^N y^k_{xxx}(t,0) = 0, & t \in (0,T) \\ y^k(0,x) = y^k_0(x), & x \in (0,L). \end{cases}$$

For system (2) we study two types of boundary control conditions:

$$(\mathrm{I}) : \quad \begin{cases} y^k(t,L) = 0, \\ y^k_x(t,L) = u^k(t), \qquad k \in \{1,\dots,N\}, \end{cases}$$

respectively

$$(\mathrm{II}) : \quad \begin{cases} y^k_x(t,L) = a^k(t), \\ y^k_{xxx}(t,L) = b^k(t), \qquad k \in \{1,\dots,N\}. \end{cases}$$

The main problem we analyse here is on the following null-controllability issue:

Given any finite time $T > 0$ and any initial state $y_0 = (y^k_0)_{k=1,N}$, can we find proper control inputs in (I) or (II) ($u = (u^k)_{k=1,N}$ and $a = (a^k)_{k=1,N}$, $b = (b^k)_{k=1,N}$, respectively) to lead the solution of system (2) *to the zero state, i.e.,*

$$(3) \qquad y^k(T,x) = 0, \quad \text{for any } x \in (0,L), \quad k \in \{1,\dots,N\}?$$

Our main results are as given in the following theorems. For simplicity we will not make explicit the exceptional sets $\mathcal{N}_{odd}$, $\mathcal{N}_{even}$, $\mathcal{N}_{mixt}$. The interested reader can consult [1].

Theorem 1 (Null-controllability for model (2)-(I))**.** *Let $T > 0$ be fixed. For any $\lambda \notin \mathcal{N}_{even} \cup \mathcal{N}_{odd}$ and any initial state $y_0 = (y^k_0)_{k=1,N} \in L^2(\Gamma)$ there exists a control $u = (u^k)_{k=1,N} \in H^1(\Gamma)$ having at most $N-1$ non-identically vanishing components such that the solution of system* (2)*-(I) satisfies*

$$(4) \qquad y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}.$$

Theorem 2 (Null-controllability for model (2)-(II))**.** *Let $T > 0$ be fixed. For any $\lambda \notin \mathcal{N}_{even} \cup \mathcal{N}_{mixt}$ and any initial state $y_0 = (y^k_0)_{k=1,N} \in L^2(\Gamma)$ there exist the controls $a = (a^k)_{k=1,N}, b = (b^k)_{k=1,N} \in H^1(\Gamma)$ having at most $2N-1$ non-identically vanishing components such that the solution of system* (2)*-(II) satisfies*

$$(5) \qquad y^k(T,x) = 0, \text{ for any } x \in (0,L), \quad k \in \{1,\dots,N\}.$$

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Estimates for Laplacians and Schrödinger operators on the lattice

Evgeny Korotyaev

We consider Schrödinger operators with complex decaying potentials on the lattice. We determine trace formulae and estimate of eigenvalues and singular measure in terms of potentials.

We consider Schrödinger operators $H = \Delta + V$ on the lattice $\mathbb{Z}^d, d \geq 3$, where Δ is the discrete Laplacian on $\ell^2(\mathbb{Z}^d)$ given by

$$\big(\Delta f\big)(n) = \frac{1}{2} \sum_{|n-m|=1} f_m, \qquad n = (n_j)_1^d \in \mathbb{Z}^d,$$

where $f = (f_n)_{n\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$. We assume that the potential V satisfies

$$(1) \qquad (Vf)(n) = V_n f_n, \qquad V \in \ell^p(\mathbb{Z}^d), \qquad \begin{cases} 1 \leq p < \frac{6}{5} & if \quad d = 3 \\ 1 \leq p < \frac{4}{3} & if \quad d \geq 4 \end{cases}.$$

Here $\ell^p(\mathbb{Z}^d), p > 0$ is the space of sequences $f = (f_n)_{n\in\mathbb{Z}^d}$ such that $\|f\|_p < \infty$, where

$$\|f\|_p = \|f\|_{\ell^p(\mathbb{Z}^d)} = \begin{cases} \sup_{n\in\mathbb{Z}^d} |f_n|, & p = \infty, \\ \big(\sum_{n\in\mathbb{Z}^d} |f_n|^p\big)^{\frac{1}{p}}, & p \in [1,\infty). \end{cases}$$

It is well-known that the spectrum of the Laplacian Δ is absolutely continuous and $\sigma(\Delta) = \sigma_{\rm ac}(\Delta) = [-d, d]$ and the essential spectrum of H is given by $\sigma_{\rm ess}(H) = [-d, d]$. The operator H has $N \leq \infty$ eigenvalues $\{\lambda_n, n = 1, \ldots, N\}$ outside the interval $[-d, d]$. Introduce the operator-valued function $Y_0(\lambda)$ by

$$Y_0(\lambda) = |V|^{\frac{1}{2}}(\Delta - \lambda)^{-1}|V|^{\frac{1}{2}} e^{i \arg V}, \quad \lambda \in \mathbb{C} \setminus [-d, d].$$

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the trace and the Hilbert-Schmidt classes equipped with the norm $\|\cdot\|_{\mathcal{B}_1}$ and $\|\cdot\|_{\mathcal{B}_2}$ correspondingly. In order to study Schrödinger operators with complex potentials we need the following results from [6].

Theorem 1. *Let $u, v \in \ell^q(\mathbb{Z}^d), q \geq 2$. Then for all $t \in \mathbb{R} \setminus [-1, 1]$, we have*

$$(2) \qquad \|ue^{it\Delta}v\| \leq C_o^{\frac{2d}{q}} |t|^{-\frac{2d}{3q}} \|u\|_q \|v\|_q,$$

where the constant $C_o < \frac{4}{5}$. If V satisfy (1), then the function $Y_0 : \mathbb{C} \setminus [-d, d] \to \mathcal{B}_2$ is analytic and Hölder continuous up to the boundary and satisfies

$$(3) \qquad \|Y_0(\lambda)\|_{\mathcal{B}_2} \leq C_* \|V\|_p$$

for some constant C_* depending from p, d only. If in addition V is real, then the wave operators

$$(4) \qquad W_\pm = s - \lim e^{itH} e^{-it\Delta} \qquad as \qquad t \to \pm\infty$$

exist and are complete, i.e., $\mathcal{H}_{\rm ac}(H) = \operatorname{Ran} W_{\pm}$.

We define the disc $\mathbb{D}_r = \{z \in \mathbb{C} : |z| < r\}, r > 0$. Define **the new spectral variable** $z \in \mathbb{D}$ by

$$\lambda = \lambda(z) = \frac{d}{2}\Big(z + \frac{1}{z}\Big) \in \Lambda = \mathbb{C} \setminus [-d, d], \qquad z \in \mathbb{D} := \mathbb{D}_1.$$

Here $\lambda(z)$ is a conformal mapping from $\mathbb{D}$ onto the spectral domain Λ such that

- *The function* $\lambda(z)$ *maps the point* $z = 0$ *to the point* $\lambda = \infty$.
- *The inverse mapping* $z(\cdot) : \Lambda \to \mathbb{D}$ *is given by* $z = \frac{1}{d}(\lambda - \sqrt{\lambda^2 - d^2}\,), \quad \lambda \in \Lambda$

defined by asymptotics $z = \frac{d}{2\lambda} + \frac{O(1)}{\lambda^3}$ *as* $|\lambda| \to \infty$.

Define the Hardy space $\mathcal{H}_\infty = \mathcal{H}_\infty(\mathbb{D})$. Let F be analytic in $\mathbb{D}$. We say F belongs the Hardy space $\mathcal{H}_\infty$ if F satisfies $\|F\|_{\mathcal{H}_\infty} = \sup_{z \in \mathbb{D}} |F(z)| < \infty$.

We define the regularized determinant $\mathcal{D}(\lambda)$ in the cut domain Λ and the modified determinant D in the disc $\mathbb{D}$ by

$$(5) \qquad \mathcal{D}(\lambda) = \det\big[(I + Y_0(\lambda))e^{-Y_0(\lambda)}\big], \qquad \lambda \in \Lambda, \quad D(z) = \mathcal{D}(\lambda(z)), \quad z \in \mathbb{D}.$$

The function $\mathcal{D}$ is analytic in Λ and the function D is analytic in the disc $\mathbb{D}$. It has $N \le \infty$ zeros (counted with multiplicity) $z_1, z_2, \ldots$ in the disc $\mathbb{D}$. Note that $\lambda_j = \lambda(z_j)$ is an eigenvalue of H (counted with multiplicity).

In this paper we combine classical results about Hardy spaces and the free resolvent estimates from [6]. This gives us new trace formulae for discrete Scrödinger operators $H = \Delta + V$ on the lattice $\mathbb{Z}^d$, where the potential V is complex and satisfies the condition (1). We improve results from [5], where potentials are considered under the weaker condition $|V|^{\frac{2}{3}} \in \ell^1(\mathbb{Z}^d)$.

Introduce another conformal mapping $\varkappa : \Lambda \to \mathbb{C} \setminus [id, -id]$ by $\varkappa = \sqrt{\lambda^2 - d^2}, \lambda \in \Lambda$, where the branch is defined by $\varkappa = \lambda - \frac{d^2}{2\lambda} + \frac{O(1)}{\lambda^3}$ as $|\lambda| \to \infty$.

Theorem 2. *Let a potential* V *satisfy (1). Then the modified determinant* D *is analytic in the disc* $\mathbb{D}$ *and is Hölder up to the boundary and satisfies*

$$(6) \qquad \|D\|_{\mathcal{H}_\infty(\mathbb{D})} \le e^{C_*^2 \|V\|_p^2/2},$$

where the constant C_* *is from (3). It has* $N \le \infty$ *zeros* $\{z_j\}_{j=1}^N$ *in the disc* $\mathbb{D}$, *such that*

$$(7) \qquad 0 < r_0 = \inf |z_j| = |z_1| \le |z_2| \le \ldots \le |z_j| \le |z_{j+1}| \le |z_{j+2}| \le \ldots,$$

Moreover, $\psi(z) = \log D(z)$ *is analytic in* $\mathbb{D}_{r_0}$ *and has the Taylor series*

$$(8) \qquad \psi(z) = -\psi_2 z^2 - \psi_3 z^3 - \psi_4 z^4 + \ldots, \qquad as \quad |z| < r_0,$$

where $\psi_2 = \frac{a^2}{2} \operatorname{Tr} V^2, \quad \psi_3 = \frac{a^3}{3} \operatorname{Tr} V^3, \ldots$ *and* $a = \frac{2}{d}$.

For the function D we define the Blaschke product $B(z) \prod_{j=1}^N \frac{|z_j|}{z_j} \cdot \frac{(z_j - z)}{(1 - \overline{z}_j z)}, z \in \mathbb{D}$ if $N \ge 1$ and $B = 1$ if $N = 0$. It is well known that the Blaschke product $B(z)$, $z \in \mathbb{D}$ converges absolutely for $\{|z| < 1\}$ and satisfies $B \in \mathcal{H}_\infty$ with $\|B\|_{\mathcal{H}_\infty} \le 1$, since $D \in \mathcal{H}_\infty$. The Blaschke product B has the Taylor series at $z = 0$:

$$\log B(z) = B_0 - B_1 z - B_2 z^2 - \ldots \qquad as \qquad z \to 0,$$

where $B_0 = \log B(0) < 0$, $\quad B_1 = \sum_{j=1}^N \left(\frac{1}{z_j} - \overline{z}_j\right)$, $\quad B_2 = \frac{1}{2}\sum_{j=1}^N \left(\frac{1}{z_j^2} - \overline{z}_j^2\right), ...$

We describe the canonical representation of $D(z), z \in \mathbb{D}$.

Corollary 3. *Let a potential V satisfy (1). Then there exists a singular measure $\nu \geq 0$ on $[-\pi, \pi]$, such that $D(z) = B(z)e^{-K_\nu(z)}e^{K(z)}$ for all $|z| < 1$, where*

$$K_\nu(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} d\nu(t), \quad K(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log|D(e^{it})|dt,$$

where $\log|D(e^{it})| \in L^1(-\pi, \pi)$ *and* $\operatorname{supp}\nu \subset \{t \in [-\pi, \pi] : D(e^{it}) = 0\}$.

Theorem 4. *Let V satisfy (1) and let $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. Then*

$$\frac{\nu(\mathbb{T})}{2\pi} - B_0 = \frac{1}{2\pi}\int_{\mathbb{T}} \log|D(e^{it})|dt \geq 0,$$

$$\sum_{j=1}^N \big(\operatorname{Re}\varkappa(\lambda_j) + i\operatorname{Im}\lambda_j\big) = \frac{d}{2\pi}\int_{\mathbb{T}} e^{-it}d\mu(t) = \frac{d}{2}\sum_{j=1}^N \Big(\frac{1}{z_j} - \overline{z}_j\Big),$$

$$\sum(1 - |z_j|) \leq -B_0 \leq \frac{C_*^2}{2}\|V\|_p^2 - \frac{\nu(\mathbb{T})}{2\pi},$$

$$\Big|\sum_{j=1}^N \big(\operatorname{Re}\varkappa_j + i\operatorname{Im}\lambda_j\big)\Big| \leq dC_*^2\|V\|_p^2,$$

There are a lot of papers about eigenvalues of Schrödinger operators in $\mathbb{R}^d$ with complex-valued potentials decaying at infinity, see [2, 7] and references therein.

Schrödinger operators with decreasing potentials on the lattice $\mathbb{Z}^d, d > 1$ have been considered by Boutet de Monvel-Sahbani [1], Isozaki-Korotyaev [3], Isozaki-Morioka [4], Korotyaev-Moller [6], Shaban-Vainberg [8] and see references therein.

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Ground state on the dumbbell graph

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(joint work with Dmitry Pelinovsky)

In [1], we consider a cubic nonlinear Schr odinger equation on a dumbbell graph consisting of two rings attached to a central line segment under the normal Kirchhoff boundary conditions. At small mass, we construct the branches bifurcating off of the linear spectrum of the graph using Lyapunov-Schmidt expansions. In particular, we are able to observe symmetry breaking bifurcations into odd and even solutions off of a constrant branch. At large mass, we construct bound states that live both in the ring and in the central link by modifying the bound state that occurs on the real line in an appropriate manner. The method of construction at large mass uses careful analysis of Jacobi elliptic functions and should have implications for other solutions on graphs, and should be able to be modified to other nonlinearities as well through more general special function analysis.

To clarify things, let the central line segment be placed on $I_0 := [-L, L]$, whereas the end rings are placed on $I_- := [-L - 2\pi, -L]$ and $I_+ := [L, L + 2\pi]$. The Laplacian operator is defined piecewise by

$$\Delta\Psi = \begin{bmatrix} u_-''(x), \ x \in I_-, \\ u_0''(x), \ x \in I_0, \\ u_+''(x), \ x \in I_+, \end{bmatrix}, \quad \text{acting on} \quad \Psi = \begin{bmatrix} u_-(x), \ x \in I_-, \\ u_0(x), \ x \in I_0, \\ u_+(x), \ x \in I_+, \end{bmatrix},$$

subject to the Kirchhoff boundary conditions at the two junctions:

$$\text{(1)} \qquad \begin{cases} u_-(-L-2\pi) = u_-(-L) = u_0(-L), \\ u_-'(-L) - u_-'(-L-2\pi) = u_0'(-L), \end{cases}$$

and

$$\text{(2)} \qquad \begin{cases} u_+(L+2\pi) = u_+(L) = u_0(L), \\ u_+'(L) - u_+'(L+2\pi) = u_0'(L). \end{cases}$$

The Laplacian operator Δ is equipped with the domain $\mathcal{D}(\Delta)$ given by a subspace of $H^2(I_- \cup I_0 \cup I_+)$ closed with the boundary conditions (1) and (2).

The cubic NLS equation on the dumbbell graph is given by

$$\text{(3)} \qquad i\frac{\partial}{\partial t}\Psi = \Delta\Psi + 2|\Psi|^2\Psi \ , \quad \Psi \in \mathcal{D}(\Delta),$$

where the nonlinear term $|\Psi|^2\Psi$ is also defined piecewise on $I_- \cup I_0 \cup I_+$. The energy of the cubic NLS equation (3) is given by

$$\text{(4)} \qquad E(\Psi) = \int_{I_- \cup I_0 \cup I_+} \left(|\partial_x \Psi|^2 - |\Psi|^4\right) dx,$$

and it is conserved in the time evolution of the NLS equation (3). The energy is defined in the energy space $\mathcal{E}(\Delta)$ given by

$$\mathcal{E}(\Delta) := \left\{ \Psi \in H^1(I_- \cup I_0 \cup I_+) : \quad \begin{array}{l} u_-(-L-2\pi) = u_-(-L) = u_0(-L) \\ u_+(L+2\pi) = u_+(L) = u_0(L) \end{array} \right\}.$$

Ground states can be defined as solutions of the constrained minimization problem

$$E_0 = \inf\{E(\Psi): \quad \Psi \in \mathcal{E}(\Delta), \quad Q(\Psi) = Q_0\}. \tag{5}$$

Standing waves of the focusing NLS equation (3) are given by the solutions of the form $\Psi(t,x) = e^{i\Lambda t}\Phi(x)$, where Λ and $\Phi \in \mathcal{D}(\Delta)$ are considered to be real. This pair satisfies the stationary NLS equation

$$-\Delta\Phi - 2|\Phi|^2\Phi = \Lambda\Phi \qquad \Lambda \in \mathbb{R}, \ \Phi \in \mathcal{D}(\Delta). \tag{6}$$

The stationary NLS equation (6) is the Euler–Lagrange equation of the energy functional $H_\Lambda := E - \Lambda Q$, where the charge

$$Q(\Psi) = \int_{I_- \cup I_0 \cup I_+} |\Psi|^2 dx \tag{7}$$

is another conserved quantity in the time evolution of the NLS equation (3).

The two main theorems can be stated as follows.

Theorem 1. *There exist Q_0^* and Q_0^{**} ordered as $0 < Q_0^* < Q_0^{**} < \infty$ such that the ground state of the constrained minimization problem (5) for $Q_0 \in (0, Q_0^*)$ is given (up to an arbitrary rotation of phase) by the constant solution of the stationary NLS equation (6):*

$$\Phi(x) = p, \quad \Lambda = -2p^2, \quad Q_0 = 2(L + 2\pi)p^2. \tag{8}$$

The constant solution undertakes the symmetry breaking bifurcation at Q_0^* and the symmetry preserving bifurcation at Q_0^{**}, which result in the appearance of new positive non-constant solutions. The asymmetric standing wave is a ground state of (5) for $Q \gtrsim Q_0^*$ but the symmetric standing wave is not a ground state of (5) for $Q \gtrsim Q_0^{**}$.

Theorem 2. *In the limit of large negative Λ, there exist two standing wave solutions of the stationary NLS equation (6). One solution is a positive asymmetric wave localized in the ring:*

$$\Phi(x) = |\Lambda|^{1/2}\text{sech}(|\Lambda|^{1/2}(x - L - \pi)) + \tilde{\Phi}(x), \quad Q_0 = 2|\Lambda|^{1/2} + \tilde{Q}_0, \tag{9}$$

and the other solution is a positive symmetric wave localized in the central line segment:

$$\Phi(x) = |\Lambda|^{1/2}\text{sech}(|\Lambda|^{1/2}x) + \tilde{\Phi}(x), \quad Q_0 = 2|\Lambda|^{1/2} + \tilde{Q}_0, \tag{10}$$

where $\|\tilde{\Phi}\|_{H^2(I_- \cup I_0 \cup I_+)} \to 0$ and $|\tilde{Q}_0| \to 0$ as $\Lambda \to -\infty$ in both cases. The positive symmetric wave satisfying (10) is a ground state of the constrained minimization problem (5) for Q_0 sufficiently large.

The paper also includes careful numerical verification of these theorems using a Petviashvilli type iteration argument as well as Newton solvers on graphs.

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Dynamics of relativistic solitons in networks: Metric graph based approach

Davron Matrasulov

(joint work with K.Sabirov, D. Babajanov and P.Kevrekidis)

In this research we study dynamics of Dirac solitons in networks. The dynamics of such solitons are modeled in terms of nonlinear Dirac equations on metric graphs. Explicit soliton solutions of the problem are derived. Conditions(constraints) for integrability are obtained.

It is shown that under such constraints transmission of relativistic solitons through the network branching points (vertices) are reflectionless. Such a feature can be useful from the viewpoint of ballistic transport in branched structures. ...

Nonlinear evolution equations on networks have attracted much attention recently [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Such interest is caused by a broad variety of potential applications of the nonlinear wave dynamics and soliton transport in networks, such as Bose-Einstein condensates (BECs) in branched traps, Josephson junction networks, DNA double helix, polymer chains, etc.

Despite the rapidly growing interest in wave dynamics on networks, most of the studies are mainly focused on nonrelativistic wave equations such as the nonlinear Schrödinger equation [1, 2, 3, 4, 5, 6, 7, 10]. Nevertheless, there is a number of works on the sine-Gordon equation in branched systems [8, 9, 11]. However, relativistic wave equations such as the nonlinear Klein-Gordon and Dirac equations are important in field theory and condensed matter physics and hence exploring them on metric graphs is of interest in its own right. we address the problem of In this work we address the sine-Gordon equation and nonlinear Dirac equation on simple metric graphs by focusing on conservation laws and soliton transmission at the graph vertices. Our prototypical example will be the Y-junction. Such problem can be effective model for describing BEC dynamics graphene nanoribbon Y-junctions.

For a metric star graph consisting of three semi-infinite bonds , the nonlinear Dirac equation for each bond can be written as L_j where j parametrizes the bond and L for each is of the form [12]

$$(i\gamma^{\mu}\partial_{\mu} - m)\,\Psi + g^2\left(\bar{\Psi}\Psi\right)\Psi = 0, \tag{1}$$

where

$$\Psi(x,t) = \begin{pmatrix} \phi(x,t) \\ \chi(x,t) \end{pmatrix}; \quad \bar{\Psi} = \Psi^{\dagger}\gamma^0, \tag{2}$$

and $\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Complete formulation of the problem requires imposing f vertex boundary conditions which can be obtained from fundamental conservation laws. Here we use conservations of energy and current. The current conservation, $\dot{J} = 0$ yields the following vertex boundary condition:

$$\mathrm{Re}\,[\phi_1\chi_1^*]|_{x=0} = \mathrm{Re}\,[\phi_2\chi_2^*]|_{x=0} + \mathrm{Re}\,[\phi_3\chi_3^*]|_{x=0}\,. \tag{3}$$

Here we used the asymptotic conditions

$$\Psi_1 \to 0 \text{ at } x \to -\infty \text{ and } \Psi_{2,3} \to 0 \text{ at } x \to \infty. \tag{4}$$

The energy conservation, $\dot{E} = 0$ gives rise to

$$\begin{aligned} &\mathrm{Im}\,[\phi_1\partial_t\chi_1^* + \chi_1\partial_t\phi_1^*]|_{x=0} \\ &= \mathrm{Im}\,[\phi_2\partial_t\chi_2^* + \chi_2\partial_t\phi_2^*]|_{x=0} + \mathrm{Im}\,[\phi_3\partial_t\chi_3^* + \chi_3\partial_t\phi_3^*]|_{x=0}\,. \end{aligned} \tag{5}$$

Representing soliton solution of NLDE on a metric star graph in the form

$$\Psi_j(x,t) = e^{-i\omega t}\begin{pmatrix} A_j(x) \\ iB_j(x) \end{pmatrix}. \tag{6}$$

Then, from Eqs. (1) and (6) we have

$$\frac{dA_j}{dx} + (m+\omega)B_j - g_j^2(A_j^2 - B_j^2)B_j = 0, \tag{7}$$

$$\frac{dB_j}{dx} + (m-\omega)A_j - g_j^2(A_j^2 - B_j^2)A_j = 0. \tag{8}$$

The vertex boundary conditions for the functions A_j and B_j can be written as

$$\begin{aligned} &\alpha_1 A_1|_{x=0} = \alpha_2 A_2|_{x=0} + \alpha_3 A_3|_{x=0}, \\ &\frac{1}{\alpha_1}B_1|_{x=0} = \frac{1}{\alpha_2}B_2|_{x=0} = \frac{1}{\alpha_3}B_3|_{x=0}. \end{aligned} \tag{9}$$

A static solution of the system (7), (8) obeying the asymptotic conditions (4) can be written as [12]

$$\begin{aligned} A_j(x) = &\sqrt{\frac{(m+\omega)\cosh^2(\beta x)}{m+\omega\cosh(2\beta x)}}\,\cdot \\ &\times\sqrt{\frac{2\beta^2}{g_j^2(m+\omega\cosh(2\beta x))}}, \end{aligned} \tag{10}$$

$$\begin{aligned} B_j(x) = &\sqrt{\frac{(m-\omega)\sinh^2(\beta x)}{m+\omega\cosh(2\beta x)}}\,\cdot \\ &\times\sqrt{\frac{2\beta^2}{g_j^2(m+\omega\cosh(2\beta x))}}, \end{aligned} \tag{11}$$

where $\beta = \sqrt{m^2 - \omega^2}$. In order for these solutions of Eqs. (7) and (8) to solve the problem on the metric graph, they need to also satisfy the vertex boundary conditions (9). This can be achieved if the constants α_j and coupling parameters g_j fulfill the following conditions:

$$\frac{1}{g_1^2} = \frac{1}{g_2^2} + \frac{1}{g_3^2}. \tag{12}$$

Having found the these traveling wave solutions one can analyze vertex transmission of Dirac solitons through the graph vertex [12].

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Spectral analysis of periodic Schrödinger operators on a broken carbon nanotube

Hiroaki Niikuni

Carbon nanotubes (see Fig. 1), which were discovered in 1991, have been playing important roles as materials due to its outstanding mechanical properties such as thermal conduction, electrical conduction, hardness and tribology. Its electrical conduction is related to the structure of spectrum of Schrödinger operators on carbon nanotubes. In the representative papers [2, 4], spectral properties of Schrödinger operators on carbon nanotubes were studied from the view point of quantum graphs. In general, a quantum graph is defined as a triplet of a metric graph, a differential (Schrödinger) operator and a suitable boundary vertex condition. In [4], Kuchment and Post studied spectral properties of periodic Schrödinger operators on all class of carbon nanotubes, namely, zigzag, armchair, and chiral carbon nanotubes. In [2], Korotyaev and Lobanov dealt with direct and inverse spectral problems for periodic Schrödinger operators on a class of zigzag carbon nanotubes. They gave an unitary equivalence between their quantum graph and the direct sum of its corresponding periodic Schrödinger operator on a one-dimensional periodic metric graph with a necklace structure (see also

[1]). Moreover, they showed the existence of eigenvalues with infinite multiplicities and the band-gap structure of the absolutely continuous spectrum by utilizing one-dimensional tools for the corresponding Hill operators such as the monodromy matrix and the Lyapunov function. They also give the results for resonances, the asymptotic formulas of the spectral band edges. In [3], a scattering theory has been discussed by Korotyaev and Saburova. They proved the existence and completeness of the wave operators for a class of perturbations to quantum graphs.

In the talk, the results discussed in [5] was introduced. In [5], we considered periodic Schrödinger operators on a broken carbon nanotube. In a process to refine single wall carbon nanotubes, we need metals such as Ni, Co, Y and Fe. So, carbon nanotubes are strained with tiny particles of metals. In order to get rid of these metals, we need to clean by acids. Carbon nanotubes are broken in this process. Furthermore, carbon nanotubes can also be broken by abrasion. Thus, it is important to study spectral properties of periodic Schödinger operators on broken carbon nanotubes. Although there are a lot of patterns of broken carbon nanotubes, we deal with a model of broken carbon nanotubes as in the right hand side of Fig. 1.

Let us define a broken zigzag carbon nanotube and a periodic Schrödinger operator on it. We fix $N \in \mathbb{N}$, denote the number of zigzags in one round by $2N$. Let Γ^N be a metric graph like the left hand side of Fig.1(We describe the precise definition of Γ^N in the talk.). Since there are 8 zigzags in the case of the right hand side of Fig. 1, we consider that it is Γ^4. Furthermore, we put $\mathcal{Z} := \mathbb{Z} \times \mathbb{J} \times \mathbb{Z}_N$, where $\mathbb{J} = \{1, 2, 3, 4, 5\}$ and $\mathbb{Z}_N := \mathbb{Z}/(N\mathbb{Z})$. As in Fig. 2, we give a triplet of numbers $(n, j, k) \in \mathcal{Z}$ for each edge of Γ^N without any reduplication, and call it $\Gamma_{n,j,k}$. We assume that the length of each edge is equal to 1. Then, $\Gamma_{n,j,k}$ can be identified with the closed interval $[0, 1]$. Moreover, we define the Hilbert space $L^2(\Gamma^N) = \oplus_{\omega \in \mathcal{Z}} L^2(\Gamma_\omega)$ as $\oplus_{\omega \in \mathcal{Z}} L^2(0, 1)$. For a real-valued function $q \in L^2(0, 1)$, we define

$$(Hf_\omega)(x) = -f''_\omega(x) + q(x)f_\omega(x), \quad x \in (0, 1) \simeq \Gamma^\circ_\omega, \quad \omega \in \mathcal{Z}.$$

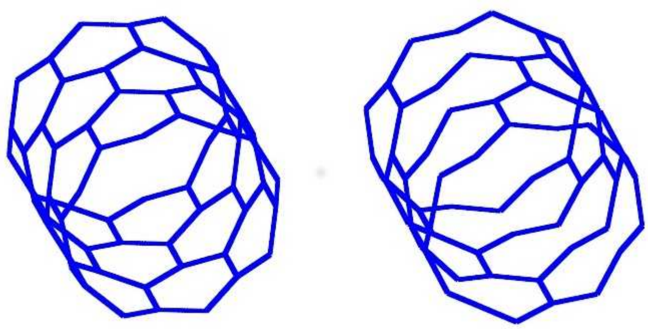

FIG. 1. The left one is a standard zigzag carbon nanotube. A broken zigzag carbon nanotube which we deal with throughout this talk is in the right picture.

Assume that $f \in \mathrm{Dom}(H)$ satisfies the Kirchhoff vertex condition. Namely, f is continuous on each vertex and satisfies no flux condition. Then, H is a self-adjoint operator. In order to study the spectrum of H, we utilize the same method as in [2]. In short, we put $\Gamma_{n,j} = \Gamma_{n,j,1}$ for $n \in \mathbb{Z}$ and $j \in \mathbb{J}$. Let us consider the following N operators on Γ^1 (see Fig. 3), putting $s = e^{i\frac{2\pi}{N}}$. For $k = 1, 2, \ldots, N$, we define N operators H_k in $L^2(\Gamma^1)$ as follows:

$$(H_k f_{n,j})(x) = -u''_{n,j}(x) + q(x)u_{n,j}(x), \quad x \in (0,1) \simeq \Gamma^{\circ}_{n,j}, \quad (n,j) \in \mathcal{Z}_1.$$

Let $f = (f_{n,j}) \in \mathrm{Dom}(H_k)$ satisfies the Kirchhoff condition except for

$$-u'_{n,1}(1) + u'_{n,2}(0) - s^k u'_{n,5}(1) = 0, \quad u_{n,1}(1) = u_{n,2}(0) = s^k u_{n,5}(1) \text{ for } (n,j) \in \mathcal{Z}_1.$$

Then, we obtain $\sigma(H) = \bigcup_{k=1}^{N} \sigma(H_k)$. Thus, it is sufficient to examine $\sigma(H_k)$.

In order to describe our main results, we give notations. Let $\theta(x,\lambda)$ and $\varphi(x,\lambda)$ be the solutions to the Schrödiger equation corresponding to the operator $L := -\frac{d^2}{dx^2} + q(x)$ in $L^2(\mathbb{R})$ as well as the initial conditions $\theta(0,\lambda) = 1$, $\theta'(0,\lambda) = 0$ and $\varphi(0,\lambda) = 0$, $\varphi'(0,\lambda) = 1$, respectively. The function $\Delta(\lambda) = \frac{\theta(1,\lambda)+\varphi'(1,\lambda)}{2}$ is called the Lyapunov function corresponding to L. Moreover, let $\sigma_D(L)$ be the set of the Dirichlet eigenvalues corresponding to L. For $k = 1, 2, \ldots, N$, we put $s_k = \sin\frac{\pi k}{N}$ and define

$$D(k,\lambda) = \frac{2\Delta^2(\lambda)D(\lambda) + s_k^2}{\sqrt{4\Delta^4(\lambda) - 4\Delta^2(\lambda)s_k^2 + s_k^2}} \quad \text{on} \quad \mathbb{C} \setminus \mathcal{P}_k, \tag{1}$$

where $\theta_1(\lambda) = \theta(1,\lambda)$, $\varphi_1'(\lambda) = \varphi'(1,\lambda)$, $\mathcal{P}_k = \{\lambda \in \mathbb{C} | \quad 4\Delta^4(\lambda) - 4\Delta^2(\lambda)s_k^2 + s_k^2 = 0\}$ and

$$\Delta^2(\lambda)D(\lambda) = 4\Delta^4(\lambda) + \frac{\theta_1(\lambda)\varphi_1'(\lambda) - 7}{2}\Delta^2(\lambda) + \frac{1 - \theta_1(\lambda)\varphi_1'(\lambda)}{8}. \tag{2}$$

Let $\sigma_\infty(H_k)$ be the set of eigenvalues with infinite multiplicities of H_k. Then, we have the following:

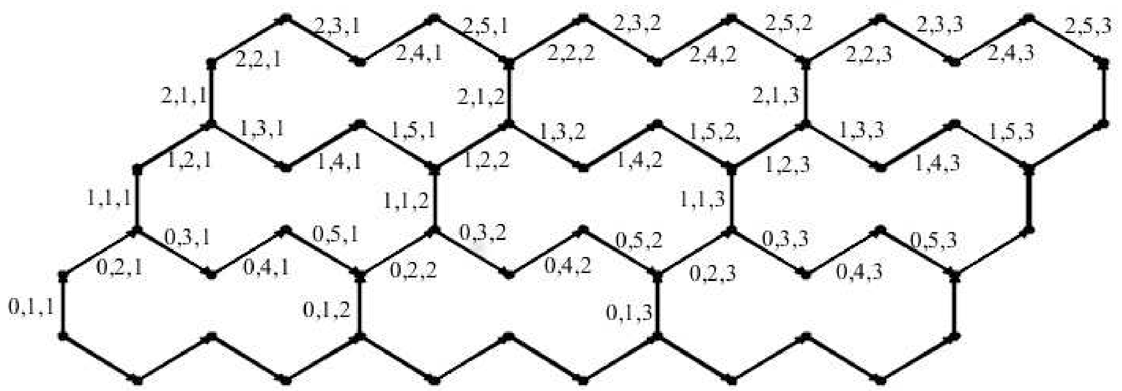

FIG. 2. Before we rolled up Γ^3, we obtain the sheet seen in the picture . The indexes in this picture imply the ones of $\Gamma_{n,j,k}$.

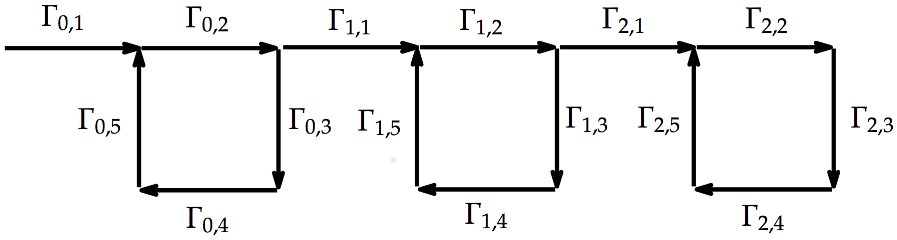

Fig. 3. The degenerate broken zigzag nanotube

Theorem 1. *For $k = 1, 2, \ldots, N$, we have $\sigma(H_k) = \sigma_\infty(H_k) \cup \sigma_{ac}(H_k)$, where*

$$\sigma_\infty(H_k) = \sigma_D(L) \quad \text{and} \quad \sigma_{ac}(H_k) = \{\lambda \in \mathbb{R} | \quad D(k,\lambda) \in [-1,1]\}. \tag{3}$$

Thus, $D(k,\lambda)$ plays a role of the spectral discriminant of H_k. The properties of $D(k,\lambda)$ will be studied by utilizing the Rouché's theorem of the version for meromorphic functions. In the talk, the following results were introduced.

Theorem 2. *(i) We have $\sigma_{ac}(H) = \bigcup_{k=1}^N \sigma_{ac}(H_k)$.*

(ii) For $k = 1, 2, \ldots, N$, we have $\sigma_{ac}(H_k) = \sigma_{ac}(H_{N-k})$.

(iii) For $k = 1, 2, \ldots, N$, there exists real sequence

$$\lambda_{k,0}^+ < \lambda_{k,1}^- \le \lambda_{k,1}^+ < \lambda_{k,2}^- \le \lambda_{k,2}^+ < \cdots < \lambda_{k,n}^- \le \lambda_{k,n}^+ < \cdots$$

such that $\sigma_{ac}(H_k) = \bigcup_{j=1}^\infty [\lambda_{k,j-1}^+, \lambda_{k,j}^-]$.

(iv) We have the following inequality:

$$\begin{aligned} \lambda_{0,0}^+ \quad &< \quad \lambda_{0,1}^- < \lambda_{0,1}^+ < \lambda_{0,2}^- < \lambda_{0,2}^+ < \lambda_{0,3}^- < \lambda_{0,3}^+ < \lambda_{0,4}^- \le \lambda_{0,4}^+ \\ &< \quad \lambda_{0,5}^- < \lambda_{0,5}^+ < \lambda_{0,6}^- < \lambda_{0,6}^+ < \lambda_{0,7}^- < \lambda_{0,7}^+ < \lambda_{0,8}^- \le \lambda_{0,8}^+ < \cdots . \end{aligned}$$

(v) Assume that $N = 2\ell$. Then, we have $\lambda_{\ell,n}^- < \lambda_{\ell,n}^+$ for all $n \in \mathbb{N}$.

(vi) For $k = 1, 2, \ldots, \ell_N$, we have

$$\begin{aligned} \lambda_{k,0}^+ \quad &< \quad \lambda_{k,1}^- \le \lambda_{k,1}^+ < \lambda_{k,2}^- \le \lambda_{k,2}^+ < \lambda_{k,3}^- \le \lambda_{k,3}^+ < \lambda_{k,4}^- < \lambda_{k,4}^+ \\ &< \quad \lambda_{k,5}^- \le \lambda_{k,5}^+ < \lambda_{k,6}^- \le \lambda_{k,6}^+ < \lambda_{k,7}^- \le \lambda_{k,7}^+ < \lambda_{k,8}^- < \lambda_{k,8}^+ < \cdots . \end{aligned}$$

Furthermore, we see the followings on the equality of the above inequality:

(a) *If $s_k \neq \sqrt{\frac{7}{8}}$, then we have $\lambda_{k,2n-1}^- \neq \lambda_{k,2n-1}^+$ for $n \in \mathbb{N}$ and $k = 1, 2, \ldots, \ell_N$. If $q \equiv 0$ and $s_k = \sqrt{\frac{7}{8}}$, then we have $\lambda_{k,2n-1}^- = \lambda_{k,2n-1}^+$ for $n \in \mathbb{N}$ and $k = 1, 2, \ldots, \ell_N$.*

(b) *If $k \neq \frac{N}{6}$, then we have $\lambda_{k,4n-2}^- \neq \lambda_{k,4n-2}^+$ for any $k = 1, 2, \ldots, \ell_N$. If $q \equiv 0$ and $k = \frac{N}{6}$, then we have $\lambda_{k,4n-2}^- = \lambda_{k,4n-2}^+$ for any $n \in \mathbb{N}$ and $k = 1, 2, \ldots, \ell_N$.*

(vii) Let $\{\eta_n\}_{n=1}^\infty = \{\lambda \in \mathbb{R} | \quad \Delta(\lambda) = 0\}$, $\{\mu_n\}_{n=1}^\infty = \sigma_D(L)$ and $\{\xi_n\}_{n=1}^\infty = \{\lambda \in \mathbb{R} | \quad \Delta^2(\lambda) = \frac{5}{12}\}$ be labelled in the increasing order each other. Then, we have $\lambda_{k,4n-2}^- \le \eta_n \le \lambda_{k,4n-2}^+$, $\lambda_{k,4n}^- \le \mu_n \le \lambda_{k,4n}^+$ for any $n \in \mathbb{N}$ and $k =$

$0, 1, 2, \ldots, N$. *If* $k = 0, 1, 2, \ldots, \ell_N$, *we have* $\lambda^-_{k,4n-3} \leq \xi_{2n-1} \leq \lambda^+_{k,4n-3}$ *and* $\lambda^-_{k,4n-1} \leq \xi_{2n} \leq \lambda^+_{k,4n-1}$ *for any* $n \in \mathbb{N}$.

(viii) For $n \in \mathbb{N}$, *we put* $\lambda^-_n = \max_{0 \leq k \leq \ell_N} \lambda^-_{k,n}$ *and* $\lambda^+_n = \min_{0 \leq k \leq \ell_N} \lambda^+_{k,n}$. *Then, we have*

$$\bigcup_{k=0}^{\ell_N} \sigma_{ac}(H_k) = \bigcup_{n=1}^{\infty} [\lambda^+_{n-1}, \lambda^-_n].$$

Especially, we have

$$\sigma_{ac}(H) = \begin{cases} \bigcup_{n=1}^{\infty} [\lambda^+_{n-1}, \lambda^-_n] & \text{if } N = 2\ell - 1, \\ \left(\bigcup_{n=1}^{\infty} [\lambda^+_{n-1}, \lambda^-_n]\right) \cup \sigma_{ac}(H_\ell) & \text{if } N = 2\ell. \end{cases}$$

(ix) For $n \in \mathbb{N}$, *we put* $\gamma_n := (\lambda^-_n, \lambda^+_n)$. *Then, we have the followings:*

(a) *For* $n \in \mathbb{N}$, *we see that* $\lambda^-_{0,4n} \neq \lambda^+_{0,4n}$ *if and only if* $\gamma_{4n} \neq \emptyset$.

(b) *For* $n \not\equiv 0 \pmod 4$, *we see that* $\gamma_n \neq \emptyset$ *if and only if there does not exist* $k \in \{1, 2, \ldots, \ell_N\}$ *satisfying* $\lambda^-_{k,n} = \lambda^+_{k,n}$.

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Standing waves for the NLS on the double-bridge graph and a rational-irrational dichotomy

Diego Noja

(joint work with Sergio Rolando and Simone Secchi)

We study a boundary value problem related to the search of standing waves for the cubic focusing nonlinear Schrödinger equation (NLS) on graphs (see for general introduction and more recent results [1, 2] and references therein). Precisely we are interested in characterizing a class of standing waves of NLS posed on the *double-bridge graph* $\mathcal{G}$, in which two semi-infinite half-lines are attached at a circle at different vertices.

- $e_1 = (0, L_1)$, $e_2 = (L_1, L_1 + L_2)$, $\quad L_1 + L_2 = L$
- $e_3 = (0, +\infty) = e_4$

At vertices Kirchhoff or "free" boundary conditions are imposed, assuring self-adjointness of the corresponding laplacian $H_\mathcal{G}$. The NLS equation is

$$i\frac{d\psi_t}{dt} = H_\mathcal{G}\psi_t - |\psi_t|^2 \psi_t$$

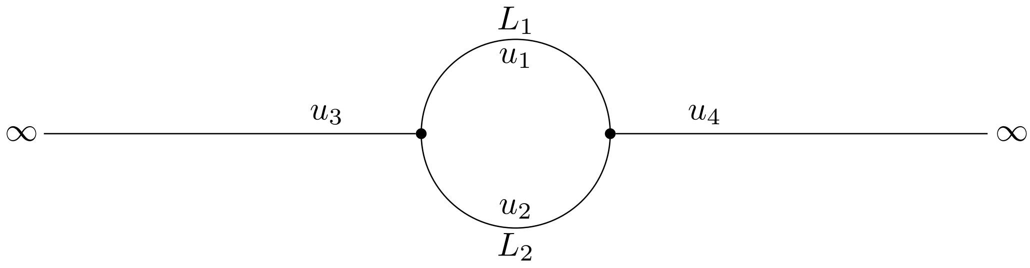


FIG. 1. The double bridge graph

and the standing waves are solutions of the form $\psi_t = e^{-i\omega t}u(x)$ where $\omega \in \mathbb{R}$ and u is a real function on the double bridge $\mathcal{G}$ with components u_j, see Figure 1. So the boundary value problem for standing waves profiles is

$$(1) \qquad \begin{cases} -u_j'' - u_j^3 = \omega u_j, & u_j \in H^2(I_j) \\ u_1(0) = u_2(L) = u_3(0), & u_1(L_1) = u_2(L_1) = u_4(0) \\ u_1'(0) - u_2'(L) + u_3'(0) = 0, & u_1'(L_1) - u_2'(L_1) - u_4'(0) = 0. \end{cases}$$

It is easy to show that for $\omega > 0$ only solution compactly supported on the ring exist and only for a rational value of L_1/L_2. Their profiles are given by Jacobi cnoidal functions and constitute continuous branches of standing waves bifurcating from the linear eigenstates of the problem (cubic term suppressed). They can be continued in all the range $(-\infty, \lambda_n)$ where λ_n is any eigenvalue of the linear Schrödinger equation on the double bridge graph; see figure (A). All the compactly supported standing waves belong to these branches.
For $\omega < 0$ one can have nontrivial solutions on the half-line and any solution on the double bridge graph is composed by an elliptic function (cnoidal or dnoidal, and not necessarily the same) on every bounded edge and a piece of soliton on the half-lines.
It is known that no ground state exists for the NLS on the double bridge graph ([3]), at variance with the simpler structure of the tadpole graph ([4, 5]).
However, the complete classification of standing waves for this BVP is open for $\omega < 0$, and so we simplify the analysis imposing the following requests

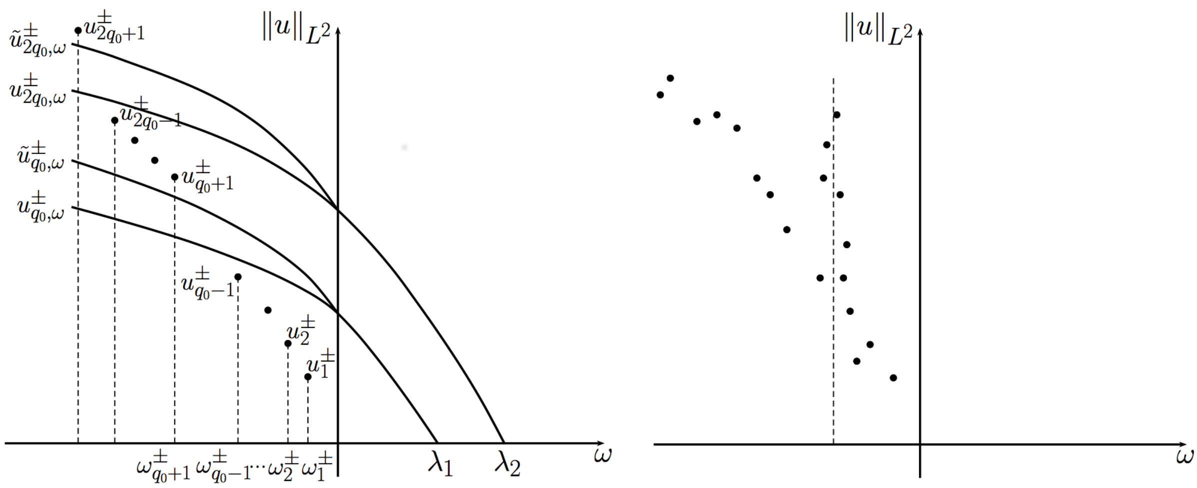
- u_3, u_4 are nontrivial
- u_1, u_2 are the restriction to e_1, e_2 of some $u \in H^2_{per}([0, L])$

The corresponding reduced BVP becomes

$$(P_\pm) \qquad \begin{cases} -u'' - u^3 = \omega u, & u \in H^2_{per}([0, L]), \quad \omega < 0 \\ u(0) = \pm u(L_1) = \sqrt{2|\omega|} \end{cases}$$

where the sign $\pm$ distinguishes the cases of u_3 and u_4 with the same sign or with different signs. $(P_\pm)$ is a nonlinear boundary value problem in which the spectral parameter ω appears explicitly in the boundary conditions (a so called "energy dependent" boundary condition).

Ultimately, our results refer to this system and are described in [6].
Two main phenomena occur, depending on the quantity L_1/L_2 being rational or not.
If $L_1/L_2 \in \mathbb{Q}$ there are two families of new solutions. Firstly, from every solution branch coming from a linear eigenvalue, a secondary global branch bifurcates at $\omega = 0$ (a pitchfork bifurcation). A second independent family is given by a denumerable sequence of solutions $u_n^\pm$ to both problems $(P_\pm)$. The corresponding sequence ω_n diverges to $-\infty$ (see Figure A).
If $L_1/L_2 \in \mathbb{R} \setminus \mathbb{Q}$ a denumerable sequence of solutions for both $(P_\pm)$ again exists, the corresponding frequency sequence ω_n is unbounded from below (a subsequence diverges to $-\infty$), but other limit points of the sequence ω_n exist. Using basic tools from diophantine analysis, it can be shown that whatever be $L_1/L_2 \in \mathbb{R} \setminus \mathbb{Q}$, a limit point of frequencies always exist in a certain interval $[a_\pm, 0]$, with $a_\pm < 0$ (see Figure B). Moreover numerical evidence shows that other limit points are possible, depending on the special choice of the ratio $L_1/L_2 \in \mathbb{R} \setminus \mathbb{Q}$.
Finally, any non positive real number (so, including 0) can be a limit point of a set of admitted frequencies up to the choice of a suitable irrational geometry L_1/L_2.
We end with the following remarks.
As stressed, the above sketched results refer to the systems $(P_\pm)$. In this sense they describe properties of the "nonlinear spectrum" of the corresponding eigenparameter dependent boundary value problem. A non trivial structure of the discrete spectrum appears, and in the irrational case a limit point of frequencies always exists.
Are these properties general features of a nonlinear BVP with an energy dependent boundary condition?



(A) $L_1/L_2 \in \mathbb{Q}$ (B) $L_1/L_2 \in \mathbb{R} \setminus \mathbb{Q}$

As regards the original boundary value problem (1) describing the totality of standing waves on a double bridge graph, its complete set of solutions is not yet known, and neither is its relation with the reduced problem ($P_{\pm}$). The structure for $\omega > 0$ is the same as in the reduced case: branches bifurcating from linear eigenvalues for $L_1/L_2 \in \mathbb{Q}$ and their disappearance in the case $L_1/L_2 \in \mathbb{R} \setminus \mathbb{Q}$. For $\omega < 0$ a reasonable conjecture is that new continuous branches arise, both in the rational and the irrational geometry. These branches, not related to any evident linear background, should contain the discrete families displayed in the reduced case and previously discussed. Work is in progress by these same authors.

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Approximating Dirichlet partitions using geometric graphs

Braxton Osting

(joint work with Todd H. Reeb)

Let $U \subseteq \mathbb{R}^d$ with $d \geq 2$ be an open bounded domain with Lipschitz boundary. A *k-partition* of U is a collection of k disjoint open sets $\{U_\ell\}_{\ell \in [k]}$, such that $\overline{U} = \cup_{\ell \in [k]} \overline{U_i}$. A *Dirichlet k-partition* of U is a k-partition, $\{U_\ell\}_{\ell \in [k]}$, that attains

$$\min_{\{U_\ell\}_{\ell \in [k]}} \sum_{\ell=1}^{k} \lambda_1(U_\ell), \qquad \text{where} \quad \lambda_1(U) := \min_{\substack{u \in H_0^1(U) \\ \|u\|_{L^2(U)}=1}} E(u). \tag{1}$$

Here, $E(u) := \int_U |\nabla u|^2 \, dx$ is the Dirichlet energy so that $\lambda_1(U)$ is the first Laplace-Dirichlet eigenvalue of U. In the upper right panel of Figure 1, a partition of a "clover" domain is plotted together with the first Laplace-Dirichlet eigenfunction of each component. *In this talk, we consider the problem of approximating a Dirichlet partition of U via a discrete problem solved on a weighted geometric graph, constructed by uniformly sampling points from U.*

We denote by $B \subset \mathbb{R}^d$, a Euclidean ball compactly containing U. We use this *auxiliary domain* to emulate Dirichlet boundary conditions on ∂U in the discrete problem. Similar to [2, 3], we construct a sequence of weighted geometric graphs $G^{(n)} = (V^{(n)}, W^{(n)})$ from the first n points $V^{(n)} = \{x_i\}_{i \in [n]}$ of a sequence of random points $\{x_n\}_{n\in\mathbb{N}}$ of B sampled uniformly and independently. The edge incident to vertices x_i and x_j ($i = j$ possibly) has weight

$$W_{ij}^{(n)} = \frac{1}{\varepsilon_n^d}\eta\left(\frac{x_i - x_j}{\varepsilon_n}\right),$$

where $\eta\colon \mathbb{R}^d \to [0,\infty)$ is a *similarity kernel* and $\varepsilon_n > 0$ is an *admissible sequence* tending to zero as $n \to \infty$; see [4] for the precise assumptions on η and $\{\varepsilon_n\}$. In Figure 1, a geometric graph is illustrated in the lower left panel.

We say that a vertex k partition, $\{V_\ell\}_{\ell\in[k]} \subset V^{(n)}$ is a *discrete Dirichlet k-partition of the geometric graph $G^{(n)} = (V^{(n)}, W^{(n)})$* if it attains

$$(2) \qquad \min_{\{V_\ell^{(n)}\}_{\ell\in[k]}} \sum_{\ell=1}^{k} \lambda_1(V_\ell^{(n)}), \qquad \text{where} \quad \lambda_1(S) := \min_{\substack{u\in L_U^2(V^{(n)}) \\ u|_{S^c}=0 \\ \|u\|_{\nu_n}=1}} E_{n,\varepsilon}(u).$$

Here $E_{n,\varepsilon}(u) := \frac{1}{n^2\varepsilon^2}\sum_{i,j=1}^{n} W_{ij}^{(n)}(u(x_i) - u(x_j))^2$ is a *weighted discrete Dirichlet energy*, $L_U^2(V^{(n)}) := \{u\colon V^{(n)} \to \mathbb{R}\colon u(x_i) = 0 \text{ if } x_i \in B\setminus U\}$, is the class of L^2 vertex functions which vanish on $B \setminus U$, and $\|u\|_{\nu_n} = \left(\frac{1}{n}\sum_{i=1}^{n} u(x_i)^2\right)^{\frac{1}{2}}$ is a weighted variant of the L^2-norm. In Figure 1, a k-Dirichlet graph partition is illustrated in the lower right panel. Computational methods for computing Dirichlet graph partitions and many examples can be found in [5, 6, 7, 8].

Theorem 1 (Convergence of Dirichlet partitions [4, Corollary 1]). *In the above setting, let $\{V_\ell^{(n)}\}_{\ell\in[k]}$ and $\{U_\ell\}_{\ell\in[k]}$ denote Dirichlet k-partitions of $G^{(n)}$ and U, respectively. Then, with probability one with respect to the sampled points, $V_\ell^{(n)}$ converges along a subsequence to $\overline{U_\ell}$ in the Hausdorff distance as $n \to \infty$ for all $\ell \in [k]$.*

Theorem 1 is illustrated in Figure 2 and can be proven as follows. The continuum (respectively graph) Dirichlet partitioning problem admits a variational formulation: solutions are characterized by ground states of a Dirichlet energy $\mathbf{E}$ (resp. $\{\mathbf{E}_{n,\varepsilon_n}\}_{n\in\mathbb{N}}$) for mappings from U (resp. $V^{(n)}$) into a singular space, $\Sigma_k := \{x \in \mathbb{R}^k\colon \sum_{i\neq j}^{k} x_i^2 x_j^2 = 0\}$, given by the coordinate axes [1]. Using methods developed in [2, 3], we show that with probability one with respect to the sampled points, as $n \to \infty$, the discrete Dirichlet energies, $\mathbf{E}_{n,\varepsilon_n}$, Γ-converge to (a scalar multiple of) the continuum Dirichlet energy, $\mathbf{E}$, with respect to the TL^2 metric, coming from the theory of optimal transport. This, along with a compactness property for the aforementioned energies that we prove, implies the convergence, along a subsequence, of the discrete ground states to a continuum ground state in the TL^2 sense. This, in turn, can be used to show the Hausdorff convergence of the associated Dirichlet k-partitions along this subsequence.

Many *open questions* remain for both continuum and graph Dirichlet partitions. One direction is to generalize Theorem 1 to non-uniformly sampled points, which would require showing that a generalized Dirichlet energy admits a continuous minimizer on $H_0^1(U;\Sigma_k)$. Another direction would be a convergence result for the

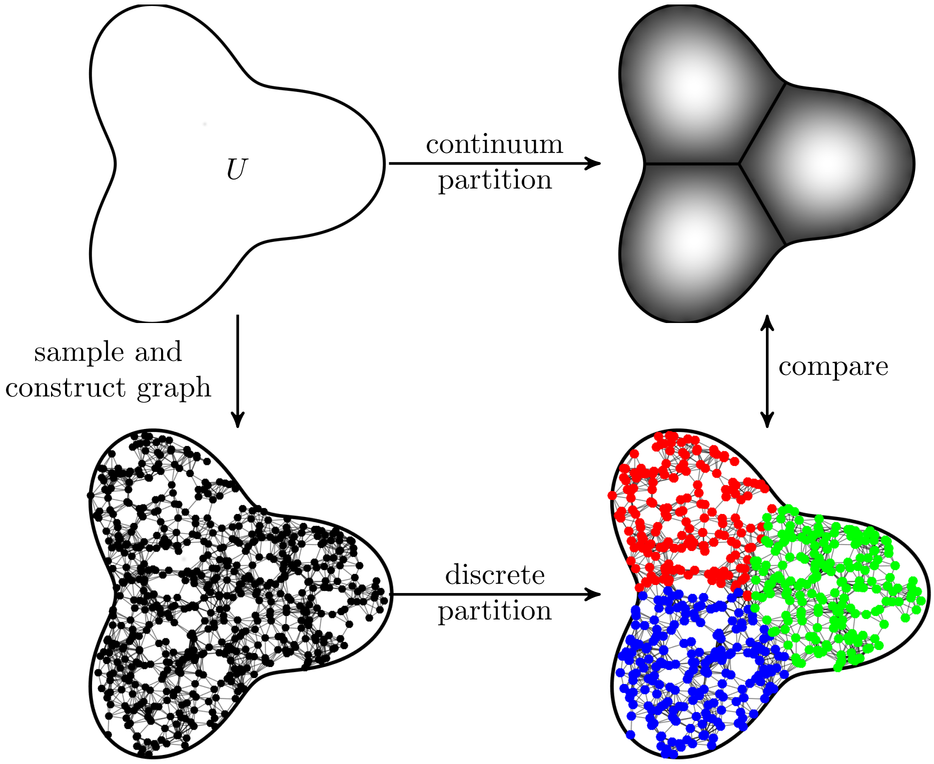

Fig. 1. Illustration for the problem of approximating a Dirichlet partition of U via a discrete problem on a geometric graph.

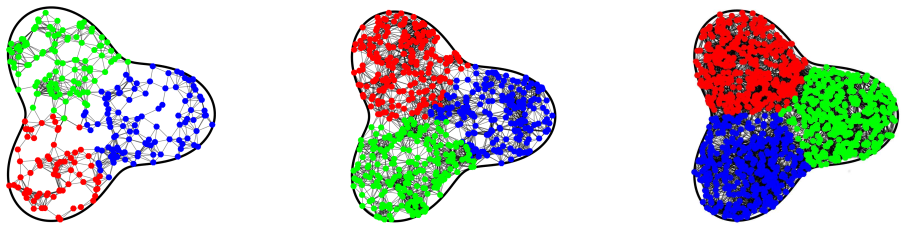

Fig. 2. An illustration of convergence for the clover domain in Figure 1. As $n \to \infty$, Theorem 1 shows that the graph Dirichlet partitions, $\{V_\ell^{(n)}\}_{\ell \in [k]}$, converge to a continuum Dirichlet partition, $\{U_\ell\}_{\ell \in [k]}$, in the Hausdorff sense.

more general objective, $\sum_{\ell=1}^{k} \lambda_1^p(U_\ell)$ with $p \in [1, \infty]$. In terms of computation, there is a stochastic process associated with Dirichlet partitions, which could lead to novel and improved numerical methods. Finally, Dirichlet graph partitions can be used for the clustering task arising in machine learning. Here, for a very large dataset, it is common to subsample the edges and/or vertices of the graph, which is sometimes referred to as *graph sparsification*. The *convergence* result stated here, or more appropriately referred to as a *consistency* result in the context of statistical learning, supports this practice, but quantifying the error incurred would require a convergence rate of the Dirichlet partitions.

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Nonlinear Instability of Half-Solitons on Star Graphs

Dmitry Pelinovsky

(joint work with Adilbek Kairzhan)

In a series of papers [1, 2, 3, 4, 5], Adami, Cacciapuoti, Finco, and Noja analyzed variational properties of stationary states on a star graph, which is the union of N half-lines (edges) connected at a single vertex. For the standard Kirchhoff boundary conditions at the vertex and for odd N, there is only one stationary state of the NLS on the star graph. This state is represented by the half-solitons along each edge glued by their unique maxima at the vertex. By using a one-parameter deformation of the NLS energy constrained by the fixed mass, it was shown that the half-soliton state is a saddle point of the constrained NLS energy [2]. On the other hand, by adding a focusing delta impurity to the vertex, it was proven that there exists a global minimizer of the constrained NLS energy for a sufficiently small mass below the critical mass [1, 3, 4]. This minimizer coincides with the N-tail state symmetric under exchange of edges, which has monotonically decaying tails and which becomes the half-soliton state if the delta impurity vanishes. In the concluding paper [5], it was proven that although the constrained minimization problem admits no global minimizers for a sufficiently large mass above the critical mass, the N-tail state symmetric under exchange of edges is still a local minimizer of the constrained NLS energy when a focusing delta impurity is added to the vertex.

Due to local minimization property, the N-tail state symmetric under exchange of edges is orbitally stable in the time evolution of the NLS in the presence of the focusing delta impurity. Although the second variation of the constrained energy was mentioned in the first work [1], the authors obtained all the variational results in [3, 4, 5] from the energy formulation avoiding the linearization procedure. In the same way, the saddle point geometry of energy at the half-soliton state in the case of vanishing delta impurity was not related in [2] to the instability of the half-soliton state in the time evolution of the NLS. It is quite well known that the saddle point geometry does not necessarily imply instability of stationary states in Hamiltonian systems because of the presence of neutrally stable modes of negative energy [8].

The recent works of Adami, Serra, and Tilli [6, 7] were devoted to the existence of ground states on the unbounded graphs that are connected to infinity after removal of any edge. It was proven that the infimum of the constrained NLS energy on the unbounded graph coincides with the infimum of the constrained NLS energy on the infinite line and it is not achieved (that is, no ground state exists) for every such a graph with the exception of graphs isometric to the real line [6]. The reason why the infimum is not achieved is a possibility to minimize the constrained NLS energy by a family of NLS solitary waves escaping to infinity along one edge of the graph. The star graph with vanishing delta impurity is an example of the unbounded graphs with no ground states, moreover, the constrained NLS energy of the half-soliton state is strictly greater than its infimum. Thus, the study in [6] provides a general argument of the computations in [2], where it is shown that the one-parameter deformation of the half-soliton state with the fixed mass reduces the NLS energy and connects the half-soliton state with the solitary wave escaping along one edge of the star graph.

In the present work, we provide a dynamical characterization of the result in [2] for the NLS with the power nonlinearity and in the case of an arbitrary star graph. By using dynamical system methods (in particular, normal forms), we verify that the half-soliton state is the saddle point of the constrained NLS energy on the star graph and moreover it is dynamically unstable due to the slow growth of perturbations. This nonlinear instability is likely to result in the destruction of the half-soliton state pinned to the vertex and the formation of a solitary wave escaping to infinity along one edge of the star graph.

Since the nonlinear saddle points are rarely met in applications of the NLS equations, it is the first time to the best of our knowledge when the energy method is adopted to the proof of the nonlinear instability of the stationary states.

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Limits of Laplacians on thin network-like domains

Olaf Post
(joint work with Pavel Exner)

In this talk, we presented results from [P05, EP09, P12, EP13] about domains in $\mathbb{R}^d$ ($d \geq 2$) shrinking to a metric graph. In particular, we show that the (linear) Neumann Laplacian converges in a generalised norm resolvent sense to the Kirchhoff Laplacian on the metric graph.

Metric graphs, thin network-like domains and their Laplacians. Let X_0 be a metric graph with underlying discrete graph having V and E as vertex and edge set, respectively. To each edge $e \in E$ we associate a length $\ell_e > 0$, and identify an edge in X_0 with an interval $I_e := [0, \ell_e]$. A function on X_0 can hence be seen as a family $f = (f_e)_{e \in E}$ with $f_e \colon I_e \longrightarrow \mathbb{C}$, $f \in \mathsf{L}_2(X_0) := \bigoplus_{e \in E} \mathsf{L}_2(I_e)$. A natural self-adjoint Laplacian A_0 on X_0 is now the so-called *Kirchhoff* or *standard* Laplacian acting on each edge as $(A_0 f)_e = -f_e''$ for weakly differentiable functions f_e being continuous at each vertex and fulfilling $\sum_{e \in E_v} f_e'(v) = 0$, where $f_e'(v)$ is the inwards derivative of f_e at a vertex $v \in V$.

An example of a thin network-like domain X_ε based on a metric graph X_0 can be defined as follows: assume that X_ε is embedded in $\mathbb{R}^d$ ($d \geq 2$) — for simplicity with straight edges — and let X_ε be the (open) ε-neighbourhood of X_0 in $\mathbb{R}^d$. As operator A_ε, we consider (minus) the Neumann Laplacian on X_ε.

We can decompose X_ε into open edge and vertex neighbourhoods $X_{\varepsilon,e}$ and $X_{\varepsilon,v}$, respectively (up to subsets of measure 0), where each $X_{\varepsilon,v}$ is ε-homothetic and where $X_{\varepsilon,e}$ is isometric to a product $(0, \ell_e - \varepsilon a_e) \times Y_{\varepsilon,e}$ with $Y_{\varepsilon,e}$ being a ball of radius ε. Here, $a_e > 0$ is due to the fact that the vertex neighbourhoods need some space. To make the presentation technically simpler, it is conveniant to assume that $a_e = 0$ (the embedded case $a_e > 0$ is then a perturbation, see e.g.[P12, Sec. 5.3.2]). Moreover, we assume that the measure on Y_e (the unscaled space $Y_{1,e}$) is 1, by a suitable scaling of the measure. If we allow different radii, we will end up with $p_e = \mathrm{vol}_{d-1}(Y_{1,e})$. leading to a Laplacian with so-called *weighted* or *generalised* Kirchhoff conditions $\sum_{e \sim v} p_e f_e'(v) = 0$ (see e.g [EP09]).

Convergence of the Laplacian on thin network-like domains. As the spaces $\mathcal{H}_0 := \mathsf{L}_2(X_0)$ and $\mathcal{H}_\varepsilon := \mathsf{L}_2(X_\varepsilon)$ are a priori unrelated, we need a suitable identification operator, namely a bounded operator $J\colon \mathsf{L}_2(X_0) \longrightarrow \mathsf{L}_2(X_\varepsilon)$ with

$$(Jf)_e = f_e \otimes \mathbb{1}_{\varepsilon,e} \qquad \text{and} \qquad (Jf)_v = 0$$

where $(Jf)_e$ is the contribution on the edge neighbourhoood $X_{\varepsilon,e}$ and $(Jf)_v$ is the contribution on the vertex neighbourhood, according to the decomposition into edge and vertex neighbourhoods. Moreover, $\mathbb{1}_{\varepsilon,e}$ is the constant function on $Y_{\varepsilon,e}$ with value $\varepsilon^{-m/2}$ (the first normalised eigenfunction of $Y_{\varepsilon,e}$).

The setting $(Jf)_v = 0$ seems at first sight a bit rough, but we cannot set something like $(Jf)_v = \varepsilon^{-m/2} f(v)$, since on $\mathsf{L}_2(X_0)$, the value of f at v is not defined. There is a finer version of identification operators on the level of the quadratic form domains, see [P12, Ch. 4] for details or [P11] for a review article.

Let us now calculate the sandwiched resolvent difference $R_\varepsilon J - JR_0$, where $R_\varepsilon = (A_\varepsilon + 1)^{-1}$ for $\varepsilon \geq 0$: Let $g \in \mathsf{L}_2(X_0)$ and $w \in \mathsf{L}_2(X_\varepsilon)$, then we have

$$\begin{aligned} \langle R_\varepsilon J - JR_0)g, w\rangle &= \langle Jg, R_\varepsilon w\rangle_{\mathsf{L}_2(X_\varepsilon)} - \langle JR_0 g, w\rangle_{\mathsf{L}_2(X_\varepsilon)} \\ &= \sum_{e\in E} \Big(\langle (-f_e'' \otimes \mathbb{1}_{\varepsilon,e}, u_e\rangle_{\mathsf{L}_2(X_{\varepsilon,e})} - \langle f_e \otimes \mathbb{1}_{\varepsilon,e}, -u_e''\rangle_{\mathsf{L}_2(X_{\varepsilon,e})} \Big) \\ &= \sum_{e\in E} \varepsilon^{m/2} \Big[\int_{Y_e} (-f_e' \overline{u}_e + f_e \overline{u}_e') \,\mathrm{d}Y_e \Big]_{\partial I_e} \\ &= \sum_{v\in V} \sum_{e\sim v} \varepsilon^{m/2} \int_{Y_e} \big(-f_e'(v)\overline{u}_e(v) + f_e(v)\overline{u}_e'(v)\big) \,\mathrm{d}Y_e, \end{aligned}$$

where $u = R_\varepsilon w \in \operatorname{dom} A_\varepsilon$, $f = R_0 g \in \operatorname{dom} A_0$ and where u_e is the restriction of u onto $X_{\varepsilon,e}$. For the second equality, we used the self-adjointness of the transversal (Neumann) Laplacian on $Y_{\varepsilon,e}$. For the third equality, we use Green's first formula, and then some reordering. Consider now the averages

$$⨍_v u_v := \frac{1}{\operatorname{vol} X_v} \int_{X_v} u_v \,\mathrm{d}X_v \qquad \text{and} \qquad ⨍_e u_e(v) := \frac{1}{\operatorname{vol} Y_e} \int_{Y_e} u_e(v) \,\mathrm{d}Y_e,$$

then we express the sum over the first summands as

$$\sum_{e\sim v} \varepsilon^{m/2} \int_{Y_e} f_e'(v)\overline{u}_e(v) = \sum_{e\sim v} \varepsilon^{m/2} f_e'(v) \big(⨍_e \overline{u}_e(v) - ⨍_v \overline{u}_v \big). \tag{1a}$$

The equality holds as $f \in \operatorname{dom} A_0$ fulfils the Kirchhoff condition. For the sum over the second summands, we use the fact that $f_e(v) = f(v)$ is independent of $e \sim v$ and obtain

$$\sum_{e\sim v} \varepsilon^{m/2} \int_{Y_e} f_e(v)\overline{u}_e'(v)\big) \,\mathrm{d}Y_e = \varepsilon^{m/2} f(v) \int_{X_v} \Delta_{X_v} \overline{u}_v \,\mathrm{d}X_v, \tag{1b}$$

performing again a partial integration (Green's first formula, writing u_v as $1 \cdot u_v$). Summing up these contributions, we can express $R_\varepsilon J - JR_0$ as a sum of two operators $S_\varepsilon, T_\varepsilon\colon \mathsf{L}_2(X_0) \longrightarrow \mathsf{L}_2(X_\varepsilon)$ defined via (1a) resp. (1b), respectively.

Moreover, it can be shown that the operator norm of the first is of order $\varepsilon^{1/2}$, while the second is of order $\varepsilon^{3/2}$. In particular, we can show:

Theorem 1. *The sandwiched resolvent difference $R_\varepsilon J - JR_0$ of the Neumann Laplacian A_ε on the thin neighbhourhood X_ε and the Kirchhoff Laplacian A_0 on the metric graph X_0 can be estimated in operator norm by a term of order $\varepsilon^{1/2}$. Moreover, A_ε converges in generalised norm resolvent convergence to A_0 (notion explained below).*

Convergence of operators acting in different Hilbert spaces. We have already stated a relation of the two resolvents using an identification operator $J = J_\varepsilon \colon \mathcal{H}_0 := \mathsf{L}_2(X_0) \longrightarrow \mathcal{H}_\varepsilon := \mathsf{L}_2(X_\varepsilon)$. We also want that J is close to some unitary operator, although J need not to be injective nor surjective. We state the following concept for abstract Hilbert spaces $\mathcal{H}$, $\widetilde{\mathcal{H}}$ and self-adjoint, non-negative operators A, $\widetilde{A}$ with resolvents $R := (A+1)^{-1}$, $\widetilde{R} := (\widetilde{A}+1)^{-1}$, respectively. The following generalises classical notions:

Definition 2. Let $\delta \geq 0$.

(1) A bounded operator $J \colon \mathcal{H}_0 \longrightarrow \mathcal{H}_\varepsilon$ is called *δ-quasi unitary* if
$$\|(\mathrm{id}_{\mathcal{H}} - J^*J)R\|_{\mathcal{H}\to\mathcal{H}} \leq \delta \qquad \text{and} \qquad \|(\mathrm{id}_{\widetilde{\mathcal{H}}} - JJ^*)\widetilde{R}\|_{\widetilde{\mathcal{H}}\to\widetilde{\mathcal{H}}} \leq \delta.$$
(2) We say that A and $\widetilde{A}$ are *δ-quasi unitarily equivalent* if there is a δ-quasi unitary operator J such that $\|JR - \widetilde{R}J\|_{\mathcal{H}\to\widetilde{\mathcal{H}}} \leq \delta$.
(3) We say that A_ε (acting in $\mathcal{H}_\varepsilon$) converges to A_0 (acting in $\mathcal{H}_0$) in the *generalised norm resolvent sense* if A_0 and A_ε are δ_ε-quasi unitarily equivalent for some $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$.

Remark 3. If $\delta = 0$ in the first to cases, then 0-quasi-unitarity is just unitarity, and similarly for 0-quasi-unitary equivalence. If $\mathcal{H}_\varepsilon = \mathcal{H}_0$ and $J = \mathrm{id}_{\mathcal{H}_0}$ in (3), then A_0 and A_ε are δ_ε-quasi unitarily equivalent if and only if $A_\varepsilon \geq 0$ converges to $A_0 \geq 0$ in *norm resolvent sense*. Hence, generalised norm resolvent convergence is a generalisation of classical norm resolvent convergence and unitary equivalence.

From the generalised norm resolvent convergence, we can conclude basically the same statements as for the classical notion (such as norm convergence of functions of A_ε and A_0, appropriately sandwiched etc.) and also uniform convergence of the spectrum on compact subsets, see e.g. [P12, Ch. 4] and [P06, EP09, P11, EP13].

Outlook. To what extend can we adopt this general scheme of convergence of operators acting in varying Hilbert spaces also to (mildly) *non-linear* operators? The closest result here is probably [Kos00, Kos02]: Kosugi shows uniform convergence of a family of semi-linear (Laplace-like) operators on thin tubular neighbourhoods to a non-linear (Kirchhoff-like) operator on the metric graph.

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The NLS approximation for dispersive systems on graphs

Guido Schneider

For the nonlinear wave equation

$$\partial_t^2 u = \partial_x^2 u - u - u^3, \tag{1}$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, $u(x,t) \in \mathbb{R}$ the NLS equation

$$2i\omega_0 \partial_T A = (1 - c_g^2)\partial_X^2 A - 3A\,|A|^2\,, \tag{2}$$

with $T \in \mathbb{R}$, $X \in \mathbb{R}$, and $A(X,T) \in \mathbb{C}$ occurs as a universal amplitude equation which can be derived via the multiple scaling ansatz

$$\varepsilon\psi_{\text{NLS}} = \varepsilon A\left(\varepsilon(x - c_g t), \varepsilon^2 t\right) e^{i(k_0 x + \omega_0 t)} + \text{c.c.} \tag{3}$$

in order to describe slow modulations in time and space of the envelope of the spatially and temporarily oscillating wave packet. Herein, $0 < \varepsilon \ll 1$ is a small perturbation parameter, c_g is the group velocity, and the basic temporal and spatial wave number ω_0 and k_0 are related by the linear dispersion relation $\omega_0^2 = k_0^2 + 1$. In [3] the following approximation result has been shown.

Theorem. *For all s there exists an s_A sufficiently large such that the following holds: Let $A \in C([0,T_0], H^{s_A})$ be a solution of the NLS equation* (2)*. Then there exists an $\varepsilon_0 > 0$ and a $C > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ there are solutions u of* (1) *which can be approximated by $\varepsilon\psi_{NLS}$ defined in* (3) *such that*

$$\sup_{t\in[0,T_0/\varepsilon^2]} \|u(t) - \varepsilon\psi_{NLS}(t)\|_{H^s} < C\varepsilon^2.$$

In case of a cubic nonlinearity as for (1) the proof follows by a simple application of Gronwall's inequality. In case of quadratic terms in the original system the proof is more involved and is based on normal form transformations and the validity of non-resonance conditions. The NLS equation is a universal amplitude equation which can be derived and justified for many spatially homogeneous dispersive systems. By applying the Fourier transform all such systems can be brought into the form

$$\partial_t \widehat{u}_j(k,t)) = i\omega_j(k)\widehat{u}_j(k,t)) + \text{nonlinear terms in convolution form} \tag{4}$$

for j in some index set. The NLS ansatz in Fourier space is strongly concentrated at the wave number k_0 and the NLS equation describes solutions of (4) whose initial conditions are strongly concentrated in Fourier space at the wave number k_0.

In [1] by applying Bloch transform the nonlinear wave equation

$$\partial_t^2 u(x,t) = \chi_1(x)\partial_x^2 u(x,t) - \chi_2(x)u(x,t) - \chi_3(x)u^3(x,t), \tag{5}$$

with $x \in \mathbb{R}$, $t \in \mathbb{R}$, $u = u(x,t) \in \mathbb{R}$, and smooth 2π-spatially periodic coefficient functions $\chi_j = \chi_j(x)$, has been brought into the form of (4). The linearized problem

$$\partial_t^2 v(x,t) = \chi_1(x)\partial_x^2 v(x,t) - \chi_2(x)v(x,t)$$

is solved by the Bloch waves

$$v(x,t) = f_n(\ell,x)e^{i\ell x}e^{\pm i\omega_n(\ell)t}$$

where $n \in \mathbb{N}$, $\ell \in (-1/2, 1/2]$, with $\omega_{n+1}(\ell) \geq \omega_n(\ell)$, and $f_n(x,\ell)$ satisfying

$$f_n(\ell,x) = f_n(\ell,x+2\pi) \quad \text{and} \quad f_n(\ell,x) = f_n(\ell+1,x)e^{ix}.$$

By the ansatz

$$u(x,t) = \varepsilon A(\varepsilon(x+c_{\mathrm{g}}t), \varepsilon^2 t) f_{n_0}(\ell_0,x)e^{i\ell_0 x}e^{i\omega_{n_0}(\ell_0)t} + c.c., \tag{6}$$

again a NLS equation can be derived and justified in the sense of the above theorem. Hence, the NLS equation is a universal amplitude equation which can be derived and justified for many spatially periodic dispersive systems. In case of quadratic terms in the nonlinearity the normal form transforms are more involved since no convolution structure w.r.t. the Bloch index n exists.

Finally in [2] the validity of the NLS approximation for one-dimensional infinite periodic quantum graphs such as drawn in Figure 1 has been established by transferring the scalar problem into a vector valued problem of the form (4) on the real line.

…-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-O-…

Fig. 1. The periodic graph with Kirchhoff boundary conditions at the vertices.

The justification of the NLS approximation for original systems on periodic graphs with quadratic nonlinearities is an open problem. Since Kirchhoff boundary conditions correspond to very irregular coefficients in (5) the regularity conditions on the periodic coefficients in [1] are not satisfied.

As in the spatially homogeneous and spatially periodic situation the theory also transforms to higher dimensional periodic graphs such as rectangular or hexagonal graphs. One-dimensional periodic graphs are good candidates for the construction of breather solutions.

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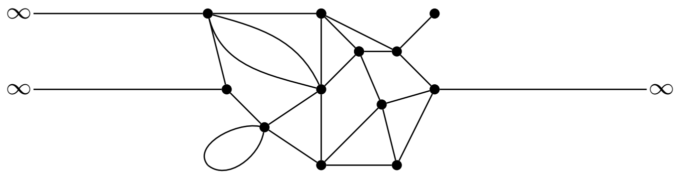
Nonlinear Schrödinger ground states on metric graphs: the L^2–critical case

Enrico Serra

(joint work with Riccardo Adami, Paolo Tilli)

On a noncompact metric graph $\mathcal{G}$:



we consider the nonlinear Schrödinger (NLS) energy functional

$$E(u, \mathcal{G}) = \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{6}\int_{\mathcal{G}} |u|^6\,dx \tag{1}$$

We are interested in ground states of fixed mass $\mu > 0$. These are the absolute minimizers of E with mass constraint

$$\int_{\mathcal{G}} |u|^2\,dx = \mu,$$

and solve the quintic NLS equation

$$u'' + |u|^4 u = \omega u \qquad \text{on } \mathcal{G},$$

with Kirchhoff boundary conditions at the vertices of $\mathcal{G}$.

When the exponent of the potential term in (1) is strictly less than 6, the problem is called L^2–subcritical, and the existence or non existence of ground states depends

- on the topology of $\mathcal{G}$
- on the interplay between μ and the metric properties of $\mathcal{G}$

This has been analyzed in the papers [1], [2].

When the exponent is 6, the problem is called L^2–critical and it turns out that it is much more delicate than the subcritical one. One of the reasons is that under the formal mass-preserving transformation

$$u(x) \mapsto u_\lambda(x) = \sqrt{\lambda}u(\lambda x),$$

the kinetic and the potential terms in E scale in the same way:

$$E(u_\lambda, \lambda^{-1}\mathcal{G}) = \lambda^2 E(u, \mathcal{G}),$$

which is typical of problems with serious loss of compactness.

In the critical case the problem depends very strongly on μ and on the ground state energy function

$$\mathcal{E}_{\mathcal{G}}(\mu) = \inf_{u \in H^1_\mu(\mathcal{G})} E(u, \mathcal{G}),$$

which plays a central role in all of our results.

When $\mathcal{G}$ is the real line $\mathbb{R}$, it is known that there exists a number $\mu_{\mathbb{R}} > 0$, the critical mass, such that

$$\mathcal{E}_{\mathbb{R}}(\mu) = \begin{cases} 0 & \text{if } \mu \leq \mu_{\mathbb{R}} \\ -\infty & \text{if } \mu > \mu_{\mathbb{R}} \end{cases} \qquad \left(\mu_{\mathbb{R}} = \pi\sqrt{3}/2\right).$$

Moreover $\mathcal{E}_{\mathbb{R}}(\mu)$ is achieved if and only if $\mu = \mu_{\mathbb{R}}$ and consequently all ground states have zero energy. The same behavior takes place on the half-line $\mathbb{R}^+$, with the appropriate value for the critical mass: $\mu_{\mathbb{R}^+} = \mu_{\mathbb{R}}/2$.

Thus on the standard domains $\mathbb{R}$ and $\mathbb{R}^+$ the minimization process is extremely unstable, with solutions existing for a single value of the mass. This behavior is due to the same homogeneity of the kinetic and potential terms under mass-preserving scalings and the invariance of $\mathbb{R}$ and $\mathbb{R}^+$ under dilations.

On a generic noncompact graph $\mathcal{G}$ however, the problem can be highly nontrivial and entirely new phenomena may arise, depending on the topology of the graph.

The purpose of this talk is to describe these new phenomena, essentially by classifying all graphs from the point of view of existence of ground states. The results we present are contained in the paper [3].

To describe our results we first define the best constant in the Gaglardo-Nirenberg inequality on $\mathcal{G}$

$$\|u\|_6^6 \leq K_{\mathcal{G}} \|u\|_2^4 \cdot \|u'\|_2^2 \qquad \forall u \in H^1(\mathcal{G})$$

as

$$K_{\mathcal{G}} = \sup_{\substack{u \in H^1(\mathcal{G}) \\ u \not\equiv 0}} \frac{\|u\|_6^6}{\|u\|_2^4 \cdot \|u'\|_2^2}$$

and then the critical mass for a generic noncompact graph $\mathcal{G}$ as

$$\mu_{\mathcal{G}} = \sqrt{\frac{3}{K_{\mathcal{G}}}}.$$

It turns out that for every noncompact graph $\mathcal{G}$,

$$\mu_{\mathbb{R}^+} \leq \mu_{\mathcal{G}} \leq \mu_{\mathbb{R}}.$$

The existence of ground states depends mainly on the topology of the graph $\mathcal{G}$, according to four mutually exclusive cases:

1. $\mathcal{G}$ has a terminal point
2. $\mathcal{G}$ admits a cycle covering
3. $\mathcal{G}$ has exactly one half-line and no terminal point
4. $\mathcal{G}$ has none of the above properties

Precisely, the following results hold.

Theorem 1. *Assume that $\mathcal{G}$ has at least one terminal point. Then*

- $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$
- *when* $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$, $\mathcal{E}_{\mathcal{G}}(\mu) = -\infty$
- *when* $\mu = \mu_{\mathbb{R}^+}$, $\mathcal{E}_{\mathcal{G}}(\mu) = 0$ *but is achieved if and only if* $\mathcal{G}$ *is a half-line.*

Theorem 2. *Assume that $\mathcal{G}$ has a cycle covering. Then*

- $\mu_{\mathcal{G}} = \mu_{\mathbb{R}}$
- $\mathcal{E}_{\mathcal{G}}(\mu_{\mathbb{R}}) = 0$ *and is achieved if and only if* $\mathcal{G}$ *is* $\mathbb{R}$ *or a tower of bubbles.*

In these first two cases, as a rule, ground states do not exist. In Theorem 1, the terminal edge behaves like $\mathbb{R}^+$, almost supporting a half-soliton. In Theorem 2, the graph behaves like $\mathbb{R}$, almost supporting a soliton. The "almost" however cannot be eliminated, resulting in nonexistence of ground states.

Theorem 3. *Assume that $\mathcal{G}$ has exactly one half-line and no terminal point. Then*

- $\mu_{\mathcal{G}} = \mu_{\mathbb{R}^+}$
- $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ *(and finite) for every* $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$
- $\mathcal{E}_{\mathcal{G}}(\mu)$ *is achieved if and only if* $\mu \in (\mu_{\mathbb{R}^+}, \mu_{\mathbb{R}}]$.

Theorem 4. *Assume that $\mathcal{G}$ has no terminal point, no cycle covering and more than one half-line. If, in addition,*

$$\mu_{\mathcal{G}} < \mu_{\mathbb{R}},$$

then

- $\mathcal{E}_{\mathcal{G}}(\mu) < 0$ *(and finite) for every* $\mu \in (\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$
- $\mathcal{E}_{\mathcal{G}}(\mu)$ *is achieved if and only if* $\mu \in [\mu_{\mathcal{G}}, \mu_{\mathbb{R}}]$.

The last two theorems result unveil completely new phenomena:

- ground states exist for a whole interval of masses
- ground states have negative energy

The proof of these two results is quite involved. It is easy to show that when μ is larger than $\mu_{\mathcal{G}}$, $\mathcal{E}_{\mathcal{G}}(\mu) < 0$. The value of $\mathcal{E}_{\mathcal{G}}(\mu)$ is then stabilized by the presence of the compact part of $\mathcal{G}$, preventing it to collapse to $-\infty$. However, due to the criticality of the exponent 6, much effort is required to prove that minimizing sequences converge. The key point is the establishment of a modified Gagliardo-Nirenberg inequality especially suited to deal with the critical case.

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NLS equation on metric graphs with localized nonlinearity

Lorenzo Tentarelli

(joint work with Enrico Serra)

We present some recent results on the bound states of the *focusing* NLS equation on noncompact *metric graphs*. In particular, our work focuses on the case (introduced in [9, 10]) where the nonlinearity is localized only on the "compact part" of the graph.

Precisely, let $\mathcal{G} = (\mathrm{V}, \mathrm{E})$ be a connected *noncompact* metric graph (for details see [6, 8, 12]) with a *finite* number of edges (and vertices), and denote by $\mathcal{K} = (\mathrm{V}_{\mathcal{K}}, \mathrm{E}_{\mathcal{K}})$ the *compact core* of $\mathcal{G}$, that is, the metric subgraph of $\mathcal{G}$ consisting of all its bounded edges.

Then, for fixed $p > 2$, we recall that a function $u = (u_e)_{e\in\mathrm{E}} : \mathcal{G} \to \mathbb{C}$ is said to be a *bound state* of the NLS equation on $\mathcal{G}$ with homogeneous *Kirchhoff boundary conditions* and *nonlinearity localized on* $\mathcal{K}$ if and only if $u \in H^1(\mathcal{G})$ (see again [6, 8, 12]) and satisfies:

(1) $$u_e'' + \chi_{\mathcal{K}}|u_e|^{p-2}u_e = \lambda u_e, \qquad \forall e \in \mathrm{E},$$

where $\lambda \in \mathbb{R}$ and $\chi_{\mathcal{K}}$ is the characteristic function of $\mathcal{K}$, and

(2) $$\sum_{e\succ \mathrm{v}} \frac{du_e}{dx_e}(\mathrm{v}) = 0, \qquad \forall \mathrm{v} \in \mathrm{V}_{\mathcal{K}},$$

where "$e \succ \mathrm{v}$" means that the edge e is incident at the vertex v and $\frac{du_e}{dx_e}(\mathrm{v})$ stands for $u_e'(\mathrm{v})$ or $-u_e'(\mathrm{v})$ depending on the "orientation" of the parametrization of e (i.e., according as v is the starting or the endpoint of e). As a consequence, the function $\psi : \mathbb{R}^+ \times \mathcal{G} \to \mathbb{C}$, defined by $\psi(t,x) = e^{i\lambda t}u(x)$ is a *stationary solution* of the NLS equation (on $\mathcal{G}$ with homogeneous Kirchhoff boundary conditions and nonlinearity localized on $\mathcal{K}$), namely, for all $t \geq 0$, it satisfies (2) and

$$i\dot{\psi}_e = -\psi_e'' - \chi_{\mathcal{K}}|\psi_e|^{p-2}\psi_e, \qquad \forall e \in \mathrm{E}$$

($\dot{\psi}_e$ representing the derivative with respect to time).

We also stress that our investigation focuses only on the case $p \in (2, 6)$, which is usually called the L^2–*subcritical* case.

Now, it is well known that, if u is a *ground state of mass* μ, namely a minimizer of the energy functional

$$E(u) := \frac{1}{2}\int_{\mathcal{G}} |u'|^2\,dx - \frac{1}{p}\int_{\mathcal{K}} |u|^p\,dx$$

constrained on the manifold

$$H^1_\mu(\mathcal{G}) := \{u \in H^1(\mathcal{G}) : \|u\|^2_{L^2(\mathcal{G})} = \mu\}, \qquad \mu > 0,$$

then it fulfills (1)&(2), and hence it is a bound state of mass μ.

On the *existence/nonexistence* of the ground states we obtained two results.

Theorem 1 ([15]). *Let $p \in (2,6)$ and $\mu > 0$. Then $\inf_{u\in H^1_\mu(\mathcal{G})} E(u) \le 0$. In addition,*

$$\text{if} \quad \inf_{u\in H^1_\mu(\mathcal{G})} E(u) < 0 \quad \Rightarrow \quad \text{the infimum is attained,}$$

namely, there exists a ground state of mass μ.

Theorem 2 ([14, 15]). *If $p \in (2,4)$, then for every $\mu > 0$ there exists at least a ground state of mass μ. On the other hand, if $p \in [4,6)$, then there exist two constants $\mu_1 > \mu_2 > 0$ such that:*

(i) *for every $\mu > \mu_1$, there exists at least a ground state of mass μ;*
(ii) *for every $\mu < \mu_2$, there cannot exist any ground state of mass μ.*

It is also interesting to discuss existence and nonexistence of those bound states of mass μ that are *not necessarily* ground states. They may arise as constrained critical points of $E_{|_{H^1_\mu(\mathcal{G})}}$ at higher energies.

Concerning existence and *multiplicity* of these bound states (with increasing mass), we proved the following

Theorem 3 ([13]). *For every $k \in \mathbb{N}$, there exists $\widetilde{\mu}_k > 0$ such that for all $\mu \ge \widetilde{\mu}_k$ there exist at least k distinct pairs $(\pm u_j)$ of (real–valued) bound states of mass μ. Moreover, for every $j = 1, \dots, k$*

$$E(\pm u_j) \le -\mathcal{C}_p \frac{\mu^{\frac{p+2}{6-p}}}{j^{\frac{2p-4}{6-p}}} + \sigma_k(\mu) < 0,$$

where $\sigma_k(\mu) \downarrow 0$ (exponentially fast), as $\mu \to \infty$, and $\mathcal{C}_p$ is a positive constant depending only on p. Finally, for each j, the Lagrange multiplier λ_j related to u_j is positive.

On the other hand, we also established two nonexistence results, which depend on the sign of the *Lagrange multiplier* λ.

Theorem 4 ([14]). *Let $p \in [4,6)$. There exists $\mu_3 > 0$ such that, if $\mu < \mu_3$, then there is no bound state of mass μ with $\lambda \ge 0$.*

Theorem 5 ([14]). *Let $\mathcal{G}$ be a tree with at most one pendant (i.e., an edge incident at a vertex of degree one). Then, for every $p > 2$ and $\mu > 0$, there is no bound state of mass μ with $\lambda \le 0$.*

Remark 6. Easy computations show that Theorem 4 entails that, if $p \in [4, 6)$ and $\mu < \mu_3$, then there does not exist any bound state *supported on* $\mathcal{G}$ (i.e., that do not vanish on at least one half–line).

Remark 7. Combining Theorems 4&5 one obtains a *sufficient condition* for the nonexistence of bound states.

Finally, we showed an existence result for those particular bound states that are *supported on* $\mathcal{K}$ (i.e., that do vanish on all the half–lines).

Theorem 8 ([14])**.** *Let* $p > 2$ *and* $\lambda \in \mathbb{R}$*. Assume that* $\mathcal{G}$ *contains a cycle, whose edges have pairwise commensurable lengths. Then, there exists* $\mu > 0$ *such that there is at least a bound state of mass* μ *with Lagrange multiplier* λ *and supported on* $\mathcal{K}$*.*

Remark 9. Note that, in contrast to Theorems 4&5, Theorem 8 does not present any restriction on the sign of λ. In addition, an analogous result holds even is $\mathcal{G}$ has no cycles, provided that it contains *two or more* pendants.

Remark 10. Theorems 5&8 are valid even if the nonlinearity affects the whole graph, that is, when there is no characteristic function in front of the nonlinear term of (1). However, for a complete discussion of the "everywhere nonlinear" problem (ground/bound states existence/nonexistence) we refer the reader to [1, 2, 3, 4, 5, 7, 11], and the references therein.

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The NLS limit for bosons in a quantum waveguide

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(joint work with Johannes von Keler)

We consider a system of N identical weakly interacting bosons confined to a thin waveguide, i.e. to a region $\mathcal{T}_\varepsilon \subset \mathbb{R}^3$ contained in an ε-neighborhood of a curve $c : \mathbb{R} \to \mathbb{R}^3$. The Hamiltonian of the system is

$$H_{\mathcal{T}_\varepsilon}(t) = \sum_{i=1}^{N} \left(-\mathrm{d}elta_{z_i} + V(t, z_i)\right) + \sum_{i \leq j} \frac{a}{\mu^3}\, w\left(\frac{z_i - z_j}{\mu}\right), \tag{1}$$

where $z_j \in \mathbb{R}^3$ is the coordinate of the jth particle, $\mathrm{d}elta_{z_j}$ the Laplacian on $\mathcal{T}_\varepsilon$ with Dirichlet boundary conditions, V a possibly time-dependent external potential and w a positive pair interaction potential. The coupling $a := \varepsilon^2/N$ is chosen such that for N-particle states supported along a fixed part of the curve the interaction energy per particle remains of order one for all $N \in \mathbb{N}$ and $\varepsilon > 0$. For $\beta > 0$ the effective range of the interaction $\mu := \left(\varepsilon^2/N\right)^\beta$ goes to zero for $N \to \infty$ and $\varepsilon \to 0$ and $\mu^{-3} w(\cdot/\mu)$ converges to a point interaction. We consider in the following only $\beta \in (0, 1/3)$, the so called mean-field regime where a/μ^3 still goes to zero.

We show in [2] that when taking simultaneously the NLS limit $N \to \infty$ and the limit of strong confinement $\varepsilon \to 0$, the time-evolution of such a system starting in a state close to a Bose-Einstein condensate is approximately captured by a non-linear Schrödinger equation in one dimension. More precisely, we show that all M-particle density matrices $\gamma_M(t)$ of the solution $\psi^{N,\varepsilon}(t)$ of the Schrödinger equation

$$\mathrm{i}\tfrac{\mathrm{d}}{\mathrm{d}t}\psi^{N,\varepsilon}(t) = H_{\mathcal{T}_\varepsilon}(t)\,\psi^{N,\varepsilon}(t)$$

are asymptotically close to $|\varphi(t)\rangle\langle\varphi(t)|^{\otimes M}$, where $\varphi(t) = \Phi(t)\chi$ with $\Phi(t)$ the solution of the one-dimensional non-linear Schrödinger equation

$$\mathrm{i}\partial_t \Phi(t,x) = \left(-\tfrac{\partial^2}{\partial x^2} + V_{\mathrm{geom}}(x) + V(t,x,0) + b|\Phi(t,x)|^2\right) \Phi(t,x) \text{ with } \Phi(0) = \Phi_0\,.$$

Here χ is the ground state in the confined direction. The strength b of the nonlinearity depends on the details of the asymptotic limit. We distinguish two regimes: In the case of moderate confinement the width ε of the waveguide shrinks slower than the range μ of the interaction and $b = \int_{\Omega_\mathrm{f}} |\chi(y)|^4\,\mathrm{d}^2y \cdot \int_{\mathbb{R}^3} w(r)\,\mathrm{d}^3r$, where Ω_f is the cross section of the waveguide and χ the ground state of the $2d$-Dirichlet

Laplacian on Ω_{f}. In the case of strong confinement the width ε of the waveguide shrinks faster than the range μ of the interaction and $b = 0$. The geometric potential $V_{\mathrm{geom}}(x)$ depends on the geometry of the waveguide and is the sum of two parts. The curvature $\kappa(x)$ of the curve contributes a negative potential $-\kappa(x)^2/4$, while the twisting of the cross-section relative to the curve contributes a positive potential. Our analysis is based on an approach to mean-field limits developed by Pickl [1].

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