# Mathematisches Forschungsinstitut Oberwolfach

# Spectral Structures and Topological Methods in Mathematical Quasicrystals

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ABSTRACT. The mathematical theory of aperiodic order grew out of various predecessors in discrete geometry, harmonic analysis and mathematical physics, and developed rapidly after the discovery of real world quasicrystals in 1982 by Shechtman. Many mathematical disciplines have contributed to the development of this field. In this meeting, the goal was to bring leading researchers from several of them together to exchange the state of affairs, with special focus on spectral aspects, dynamics and topology.

## Introduction by the Organisers

The systematic study of mathematical quasicrystals, otherwise known as the theory of aperiodic order in a wider mathematical setting, has several roots, but was undoubtedly much inspired by the discovery of real world quasicrystals in the 1980s and the need to understand this form of matter from a structural perspective. It became clear rather quickly that any substantial progress needed results from many different mathematical disciplines such as discrete geometry, harmonic analysis and mathematical physics, but also from dynamical systems, topology and number theory.

Within the last 20 years, the field has matured, and has opened into many new directions, thus offering lots of opportunities for fruitful interactions among different mathematical fields. This meeting focused on spectral structures and

topological methods, and their interactions. Here, the word "spectral" refers to three different (but connected) topics, namely to diffraction theory of unbounded measures, to spectral measures of dynamical systems, or to the spectra of (aperiodic) Schrödinger operators. One of the important questions in the field, and one agenda on this workshop, is how strongly these three notions of spectrum are connected. Likewise, "topological" refers to the peculiar topology of aperiodic systems which crosses the topology of the space in which the systems lie with the physical concept of locality, and leads to interesting spaces and dynamical systems.

Since we had participants from different directions, and a large number of young researchers including some newcomers to the field, we opted for a start with some review talks, given by Anton Gorodetski (Schrödinger operators), Nicolae Strungaru (weakly almost periodic measures), John Hunton (topological invariants of tiling spaces) and Alejandro Maass (symbolic dynamical systems), which introduced central "spines" for the rest of the meeting.

Diffraction theory and Fourier analysis form an active part of the field. Beyond Strungaru's introductory lecture, this was demonstrated in talks by Nir Lev on transformable measures with discrete spectrum, by Neil Mañibo on the connection between exact renormalization schemes for substitutions, Fourier matrices and logarithmic Mahler measures, and by the extensions to non-Abelian settings to be mentioned later. Subhro Ghosh gave an introduction to stealthy hyperuniform processes, thus liking diffraction to recent progress on the structure of point processes.

On Schrödinger operators, Siegfried Beckus presented spectral continuity results for studying the structure of the spectrum via suitable approximants. Complementing this, Mark Embree took up the quantitative nature of these results for a plethora of numerical calculations of spectra for approximants to prominent aperiodic tilings, including Penrose's. Jake Fillman explored the more difficult continuum case, where the appearance of pseudo-bands correspond to zeros of the Fricke–Vogt invariant, which result in local thickening of the spectrum. Darren Ong discussed quantum walks, which are an important object in quantum information theory, and how aperiodic order can result in anomalous spreading rates. Yanhui Qu and Qinghui Liu presented their recent work on the Thue–Morse Hamiltonian (non-vanishing dimension of the spectrum at infinite coupling and critical nature of eigenfunctions via trace maps). Qinghui Liu also talked about the structure of the union of Sturmian spectra over all irrational frequencies. This result addresses a seemingly unrelated question from quasi-periodic $SL(2, \mathbb{R})$-cocycle dynamics raised by Bassam Fayad.

With Emil Prodan and Eric Akkermans, it was our intention to include speakers who have contacts to the recent experiments on mesoscopic aperiodic wave guides and even mechanical devices that show the topological phenomena of aperiodic systems. Prodan gave an overview on experiments and the theory of topological boundary resonances in physical systems. Akkermans' talk on the diffraction and the gap labelling for structures built on the Fibonacci chain reported observations of a striking relation between the diffraction spectrum and the spectrum of wave

operators of the same medium. As the very active discussion after this talk showed, there are many open questions that ask for a thorough mathematical investigation in the future.

Topological methods are key to many developments in the field of aperiodic order, let it be for the purpose of classification, for the description of topological effects in physics, or for the understanding of symmetries in tiling and shift systems. How can one understand tilings as limits of growing coronae? (Shigeki Akiyama). What are the possible homeomorphisms between tiling spaces and how many are there? (Antoine Julien). What are the possible (extended) symmetries of subshifts? (Samuel Petite, Reem Yassawi). Topological invariants include cohomology and $K$-theory of tiling spaces. In particular, their ordered versions (dimension groups) have played an important role in the contributions of María Isabel Cortez and Fabien Durand on the dynamical spectrum of symbolic sequences. Recently, more sophisticated invariants like the homology core or representation varieties have been considered (John Hunton) and proven useful to at least partly classify substitutions (Franz Gähler).

Topological dynamics offers many tools to capture order. Most prominent among them are variants of pure point spectrum and zero entropy. Eli Glasner presented a survey of corresponding general notions and results, mostly centred around the Ellis semigroup, culminating in his recent structure theorem for general minimal systems. Felipe García-Ramos discussed characterizations of various strengthenings of pure point spectrum via notions of mean equicontinuity. In terms of specific examples for aperiodic order, the cut and project formalism has been a prominent framework for generating models of aperiodic order. It was originally developed by Yves Meyer in the 1970s in the general context of locally compact Abelian groups, and later (independently) rediscovered by several groups in physics. It is still of central importance today.

A particularly relevant feature in our context is the pure point spectrum with continuous eigenfunctions for model sets with regular windows. Past years have seen developments in various directions: A general introduction into the topic along with recent characterizations of fine combinatorial properties in special Euclidean situations was given by Alan Haynes. Tobias Jäger discussed how model sets with non-regular windows in both Euclidean and non-Euclidean settings give rise to a wealth of (counter)examples. Using special non-regular windows, it is also possible to reformulate questions of number theory and $\mathcal{B}$-free systems via this formalism, as discussed by Mariusz Lemańczyk. A whole new field opens up when setting up this formalism in a non-Abelian context, as shown in a talk given jointly by Tobias Hartnik and Felix Pogorzelski.

On Wednesday evening, an "open session" began with questions and suggestions by the participants. After a short general discussion, they split into three groups of roughly equal size. Under the topic "Eigenfunctions and the Bombieri–Taylor conjecture", various aspects of continuous versus measurable eigenfunctions were discussed, and how eigenfunctions (when viewed as Fourier–Bohr coefficients of

translation bounded measures) are connected with the Bragg peaks from diffraction theory. Short contributions by Lorenzo Sadun, Daniel Lenz and Fabien Durand were the starting points of more detailed discussions.

Another group met to discuss "Cantor spectra of aperiodic Schrödinger operators", where Anton Gorodetski set the scene with a quick summary of what is known about sums of Cantor sets in general, and how this leads to conjectures about the spectra of separable Schrödinger operators in dimension two or higher. This led to a discussion of several multi-dimensional models and the structure of their spectra. Some of these initial thoughts are worthy of further exploration, as they promise to give insight into the spectra associated with products of Toeplitz or quasi-periodic sequences. Also, Mark Embree's numerical results indicate various directions and raise interesting new questions to consider.

Finally, the third group concentrated on "Computational aspects of substitution systems and their topological invariants". Franz Gähler reported on his efforts towards a classification of one-dimensional inflation tilings up to mutual local derivability (MLD), which is analogous to equivalence via sliding block maps in symbolic dynamics. Using invariants like the structure of asymptotic composants and representation varieties, a complete classification of a class of ternary, unimodular Pisot inflations (i.e., substitutions with prototiles of natural length) could be obtained. Dan Rust discussed his work with Timo Spindeler on random substitution systems, in particular its connection to subshifts of finite type.

From the very beginning, the workshop developed the desired interactive flavour, with many questions and intense discussions, all driven by curiosity. The talks were coherent and achieved the goal to unfold the present state of affairs in the field of aperiodic order, as least as far as the topics of the workshop title were concerned. Out of the manifold discussions, various new cooperations were started, and several new results were already obtained during the meeting (some being mentioned in the abstracts) or immediately afterwards. Once again, the magic atmosphere of the MFO added to the success of the workshop and of the stay at the institute for most, if not all, participants, many of whom were MFO newcomers or returning after many years.

## Spectral Structures and Topological Methods in Mathematical Quasicrystals

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# Abstracts

## On corona limits

Shigeki Akiyama

(joint work with Katsunobu Imai, Hajime Kaneko, Jonathan Caalim)

This talk is based on the preprint [1]. Inspired by the growth rate of crystals, we introduce an axiomatic approach to define a corona limit as a limit shape of the successive coronas of a given tiling by suitable normalization.

Assuming its existence, we find that the shape of the corona limit does not depend on the initial patch, and forms a star shape. In the case that directional speeds are uniform, the shape becomes convex and centrally symmetric. Further, if the underlining tiling is periodic, the limit shape is a convex, centrally symmetric polyhedron. (This last result is a rediscovery of a result by V.G. Zhuravlev.)

Many interesting open problems remain around non-periodic tilings. See [2] as well.

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## Topological properties of some quasi-periodic tilings — From structure to spectrum

Eric Akkermans

(joint work with Yaroslav Don, Eli Levy, Dor Gitelman)

The problem addressed is motivated by studies relevant to physical properties of some one-dimensional quasi-periodic tilings and quasicrystals. The meaning of structural and spectral properties is defined below. For the case of periodic tilings (crystals), these two types of properties are related. This constitutes the basis of the Bloch theorem (whose $d = 1$ version is sometimes referred to as Floquet theory). For quasi-periodic tilings, no such relation between structural and spectral data exists as yet. Our purpose is to present some preliminary results which may prove relevant towards such a relation.

We consider a two-letter alphabet $\{a, b\}$. An aperiodic tiling can be obtained from different building rules. The first we consider is the substitution rule defined by its action $\sigma$ on a word $w = l_1 l_2 \ldots l_k$ by the concatenation $\sigma(w) = \sigma(l_1)\sigma(l_2)\ldots\sigma(l_k)$. An occurrence primitive matrix, $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ so that $\sigma(a) = a^\alpha b^\beta$ and $\sigma(b) = a^\gamma b^\delta$, is associated to $\sigma$. It allows to define a sequence of numbers $F_N$ from the recurrence $F_{N+1} = tF_N - pF_{N-1}$ where $t = \text{Tr}M$, $p = \det M$ and

$F_{0,1} = 0, 1$. The largest eigenvalue $\lambda_1$ of $M$ is larger than 1 (Perron–Frobenius) and we consider substitutions whose second eigenvalue $|\lambda_2| < 1$ (Pisot property). For the Fibonacci substitution $M_f = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ which we shall use as a generic example, $\lambda_1 = \tau = \frac{1+\sqrt{5}}{2}$ and $F_N$ are the Fibonacci numbers. Another building rule, the Cut & Project method (hereafter C&P), is equivalent to defining a characteristic function,

$$(1) \qquad \chi(n, \phi) = \text{sign}\left[\cos\left(2\pi\, n\, \lambda_1^{-1} + \phi\right) - \cos\left(\pi\, \lambda_1^{-1}\right)\right].$$

The phason parameter $\phi \in [0, 2\pi]$ is an extra gauge degree of freedom which fixes the origin of a given finite word. We consider first the more restrictive case of quasi-periodic tilings which can be described either by a substitution or by the characteristic function (C&P). Endowed with this description of a quasi-periodic tiling, we consider a distribution of identical atoms placed at the vertices $x_k$ between consecutive letters. This defines the atomic density $\rho(x) = \sum_k \delta(x - x_k)$. The distances $\delta_k = x_{k+1} - x_k$ differ depending on the letters $a$ and $b$. The diffraction spectrum associated to this atomic density is obtained from the structure factor $S(\xi) = |g(\xi)|^2$ where we have defined the Fourier transform $g(\xi) = \sum_k e^{i\xi x_k}$. For C&P tilings, the diffraction spectrum consists of a dense set of Bragg peaks.

We now consider words of finite size $F_N$ for large $N$. Using (1), we obtain an expression of the atomic density Fourier transform (shifted by a non-relevant constant term),

$$(2) \qquad g(\xi, \phi) = \sum_{n=0}^{F_N - 1} \omega^{-\xi n} \chi(n, \phi)$$

with $\omega \equiv e^{\frac{2i\pi}{F_N}}$. For C&P tilings, the corresponding structure factor $S(\xi, \phi) = |g(\xi, \phi)|^2$ is $\phi$-independent. This result expresses that the positions of the discrete diffraction spectrum is independent of the choice of the origin. To prove this result we consider $s_0(n) = \chi(n, 0)$ and apply the translation operator $\mathcal{T}[s_0(n)] = s_0(n+1)$. We then define the $F_N \times F_N$ matrix $\Sigma_0$ whose matrix element $\Sigma_0(n, l) = \mathcal{T}^l[s_0(n)]$ and, more generally, the set of matrices $\Sigma_r(n, l) = \mathcal{T}^{m(l,r)}[s_0(n)]$. We then have $\Sigma_1(n, l) = \chi(n, \phi_l)$ with $m(l, 1) \equiv l F_{N-1}^{-1} \pmod{F_N}$. There, $\phi$ takes the discrete set of values $\phi_l = \frac{2\pi}{F_N} l$. This proves the announced result.[1]

By contrast, the phase $\Theta(\xi, l) \equiv \arg g(\xi, \phi) = \arg \omega^{m(l,1)\xi}$ depends on the phason $\phi$. For each discrete diffraction peak $\xi_q = q F_{N-1}$ obtained from the structure factor $S(\xi_q) = |g(\xi_q, \phi)|^2$, the winding number associated to $\Theta(\xi_q, l)$ is $W_{\xi_q} = q$. This indicates that topological features of C&P quasi-periodic tilings are encoded in the phase of the Fourier transform of the finite size atomic density.

We then consider Schrödinger operators defined on C&P tilings as defined previously. Different approaches have been taken, e.g. tight-binding discrete Hamiltonians [3, 4, 5]. Here, since we are interested in properties of finite size tilings, we propose to obtain spectral properties such as density of states or counting function

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[1]A more thorough study of the group structure of the set of $F_N$ matrices $\Sigma_r$ will be presented in Ref. [8].

(integrated density of states) from the scattering operator. To that purpose, we consider embedding a tiling of finite size $F_N$ between two semi-infinite identical and periodic tilings built out of either the letter $a$ or $b$ with appropriate boundary conditions. These semi-infinite leads support incoming and outgoing plane wave solutions (see [1] for details). In this setup, the unitary scattering operator which relates incoming to outgoing waves is a $2 \times 2$ matrix,

$$(3) \qquad \begin{pmatrix} o_L \\ o_R \end{pmatrix} = \begin{pmatrix} \overrightarrow{r} & t \\ t & \overleftarrow{r} \end{pmatrix} \begin{pmatrix} i_L \\ i_R \end{pmatrix} \equiv \mathcal{S}_{F_N}(\phi) \begin{pmatrix} i_L \\ i_R \end{pmatrix}$$

where the transmission and reflection complex amplitudes are given by $t \equiv |t|e^{i\,\theta_t}$, $\overrightarrow{r} \equiv re^{i\,\overrightarrow{\theta}}$ and $\overleftarrow{r} \equiv re^{i\,\overleftarrow{\theta}}$ with the arrow convention indicating incoming waves from left or right. $\mathcal{S}_{F_N}(\phi)$ is diagonalizable under the form $\mathrm{diag}(e^{i\Phi_1(k,\phi)}, e^{i\Phi_2(k,\phi)})$. We define $\delta(k) \equiv (\Phi_1 + \Phi_2)/2$ known as the total phase shift. It allows to obtain a simple and useful relation for the density of states $\rho(k)$, known as the Krein–Schwinger formula, namely

$$(4) \qquad \rho(k) - \rho_0(k) = \frac{1}{2\pi} \mathrm{Im}\, \frac{\partial}{\partial k} \ln \det \mathcal{S}(k) = \frac{1}{\pi} \frac{d\delta(k)}{dk}\,,$$

where $\rho_0(k)$ is the density of states of the free system, i.e. without the scattering structure [1]. Using the unitarity condition, $\overrightarrow{r}^* t + \overleftarrow{r} t^* = 0$, we obtain the additional expressions $\det \mathcal{S} = e^{2i\delta} = -t/t^* = \overleftarrow{r}/\overrightarrow{r}^*$ and $\delta(k) = \theta_t(k) + \pi/2 = \frac{1}{2}(\overrightarrow{\theta} + \overleftarrow{\theta})$. The notations $\overrightarrow{r}$ and $\overleftarrow{r}$ represent the two possible transmission channels which are identical except for the phases of the reflected amplitudes. Therefore, $\delta(k)$ may be expressed using either the transmitted phase shift or the sum of the two possible reflected phase shifts.

The total phase shift allows to characterize the zero measure Cantor set spectrum of Schrödinger operators defined on C&P quasi-periodic tilings (e.g., gap labelling theorem). In addition to numerous theoretical and numerical studies [3, 4, 5], this Cantor spectrum has also been observed experimentally [6]. It has been shown that $\delta(k)$ is independent of the phason $\phi$ just like the structure factor $S(\xi) = |g(\xi, \phi)|^2$ previously discussed.

A second scattering phase $\Lambda(k, \phi) \equiv (\Phi_2 - \Phi_1)/2$, complementary to $\delta(k)$ is available from the diagonal form of $\mathcal{S}_{F_N}(\phi)$. It carries additional information regarding the structure, unavailable through $\delta(k)$. A useful rewriting of this second phase is $\Theta_{cav}(k, \phi) = 2\delta(k) + \alpha(\phi)$ with $\alpha \equiv \overleftarrow{\theta} - \overrightarrow{\theta}$ which conveniently disentangles the $k$ and $\phi$ dependencies [2, 7]. The winding properties of the phase $\Theta_{cav}(k, \phi)$ for values of $k$ in spectral gaps are identical to those obtained for the phase $\Theta(\xi, l) \equiv \arg g(\xi, \phi)$ in the diffraction spectrum of the finite size atomic density [8]. These identical behaviours have been observed experimentally [9, 10]. This relation between topological features in the diffraction and Schrödinger spectra constitutes a step towards a Bloch theorem for C&P and for certain substitution families of quasi-periodic tilings.

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## Spectral stability of Schrödinger operators in the Hausdorff metric

SIEGFRIED BECKUS

(joint work with Jean Bellissard, Horia Cornean, Giuseppe de Nittis, Felix Pogorzelski)

In this talk, recent developments and results are discussed that are based on various collaborations [1, 2, 3, 4, 5]. Within this research project, we seek to connect dynamical and spectral properties of self-adjoint operators. In the centre of our considerations are Schrödinger operators arising in quantum mechanical models of non-periodic solids. Up to now, the approach by transfer matrices and trace maps led to amazing results showing that new phenomena appear in physical systems. Unfortunately, this techniques do not extend to higher dimensional systems except if the operators decompose to one-dimensional systems [7, 8]. Inspired by the techniques developed so far, an appropriate approximation theory would be helpful for numerical and analytic results. Such a theory is established in [1, 2, 3, 4, 5]. During the talk we discuss this approach while a special focus is put on the quantitative estimates achieved in [3]. For simplicity of the talk, we restrict ourselves to the simpler case of symbolic dynamical systems over $\mathbb{Z}^d$ while most of the results hold in much larger generality.

Given a self-adjoint bounded operator $H$, its spectrum $\sigma(H)$ is a compact subset of $\mathbb{R}$. The space $\mathcal{K}(\mathbb{R})$ of compact subsets of $\mathbb{R}$ is naturally equipped with the Hausdorff metric $d_H$ induced by the Euclidean metric. In [2], a family of self-adjoint bounded operators $(H_t)_{t \in T}$ is studied indexed by a topological (metric) space $T$. There the (Hölder-)continuity of the map $t \mapsto \sigma(H_t)$ is characterized by the (Hölder-)continuity of suitable norms of the operators. This approach is used to show the convergence of the spectra if the underlying dynamical systems converge which is described next.

For a finite set $\mathcal{A}$, we restrict our considerations to the symbolic dynamical system $(\mathcal{A}^{\mathbb{Z}^d}, \mathbb{Z}^d)$. Specifically, the configuration space $\mathcal{A}^{\mathbb{Z}^d} := \prod_{n \in \mathbb{Z}^d} \mathcal{A}$ is equipped with the product topology. Furthermore, $\mathbb{Z}^d$ acts continuously by translation, i.e., $\alpha_n(w) := w(\cdot - n)$ for $n \in \mathbb{Z}^d$ and $w \in \mathcal{A}^{\mathbb{Z}^d}$. A family of Schrödinger operators $H_w \colon \ell^2(\mathbb{Z}^d) \to \ell^2(\mathbb{Z}^d), w \in \mathcal{A}^{\mathbb{Z}^d}$, is defined by

$$H_w := \left( \sum_{h \in \mathcal{R}} M_{q_h, w} \, U_h + U_{-h} \, M_{\overline{q_h}, w} \right) + M_{p, w}$$

where $\mathcal{R} \subseteq \mathbb{Z}^d$ is finite and for $f \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$,

$$U_h u(n) := u(n - h), \quad M_{f, w} u(n) := f\big(\alpha_{-n}(w)\big) u(n), \qquad u \in \ell^2(\mathbb{Z}^d).$$

If $\mathcal{R}$ generates $\mathbb{Z}^d$, the sum over $\mathcal{R}$ is the Laplacian on the Cayley graph of $\mathbb{Z}^d$ with generators $\mathcal{R}$. The corresponding involved multiplication operators are interpreted as weights on the edges. The multiplication operator $M_{p, w}$ by the real valued function $p \circ \alpha_{-\bullet}(w) \colon \mathbb{Z}^d \to \mathbb{R}$ is called the potential term. The multiplication operators by $q_h \circ \alpha_{-\bullet}(w) \colon \mathbb{Z}^d \to \mathbb{C}$ and $p \circ \alpha_{-\bullet}(w) \colon \mathbb{Z}^d \to \mathbb{R}$ should reflect the local structure of the configuration $w$ at the corresponding position in $\mathbb{Z}^d$. In light of this, the functions $q_h \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{C}$ and $p \colon \mathcal{A}^{\mathbb{Z}^d} \to \mathbb{R}$ are assumed to be continuous.

Motivated by the elaborations for one-dimensional systems, periodic approximations are the most promising so far in order to deal with Schrödinger operators $H_w$ where $w$ represents a quasicrystal. This is based on the fact that the spectral properties can be analyzed by the Floquet–Bloch theory and, from experience, periodic approximations admit the best convergence rates. Hence, we address the question which notion of convergence of periodic systems $(w_n)$ to $w$ implies the convergence of suitably many spectral properties of the associated Schrödinger operators. The elaborations [1, 2, 3, 4, 5] show that the Chabauty–Fell topology [6, 9] on dynamical subsystems encodes several spectral properties. More precisely, the compact metrizable space of dynamical subsystems

$$\mathfrak{I} := \big\{ \Omega \subseteq \mathcal{A}^{\mathbb{Z}^d} \text{ closed, invariant} \big\} \subseteq \mathcal{K}\big(\mathcal{A}^{\mathbb{Z}^d}\big)$$

equipped with the Chabauty–Fell topology is studied. A metric on $\mathfrak{I}$ is given by the Hausdorff metric $d_H^{\mathcal{A}}$ induced by the metric

$$d \colon \mathcal{A}^{\mathbb{Z}^d} \times \mathcal{A}^{\mathbb{Z}^d} \to [0, 1], \quad d(w, w') := \frac{1}{\sup \big\{ r \in \mathbb{N}_0 \; : \; w|_{B_r} = w'|_{B_r} \big\} + 1}$$

on $\mathcal{A}^{\mathbb{Z}^d}$ where $B_r \subseteq \mathbb{R}^d$ is the closed ball with radius $r$ centred at 0.

In [1, 4], a special focus is put on the convergence of the spectra of the Schrö-dinger operators. More precisely, the following equivalence is shown

$$\overline{Orb(w_n)} \to \overline{Orb(w)} \text{ in } \mathfrak{I} \quad \Longleftrightarrow \quad \begin{array}{c} \sigma(H_{w_n}) \to \sigma(H_w) \text{ in } \mathcal{K}(\mathbb{R}) \text{ for all } \mathcal{R} \subseteq \mathbb{Z}^d \\ \text{finite and all continuous } p \text{ and } q \end{array}$$

where $\overline{Orb(w)} := \overline{\{\alpha_n(w) : n \in \mathbb{Z}^d\}}$ is the orbit closure in the product topology. The hard direction "$\Rightarrow$" relies on the construction of a continuous field of $C^*$-algebras by gluing together the dynamical systems. The fruitful outcome of this project is based on the embedding of the dynamics in the Chabauty–Fell topology defined on $\mathfrak{I}$. This strategy works out in the generality of topological groupoids as shown in [4]. Hence, an analogous result holds for $\mathcal{R}$ infinite with suitable decay assumptions on the off-diagonal terms, general dynamical systems $(X, G)$ and Schrödinger operators associated with Delone sets. The only restriction is a suitable amenable assumption on the underlying structure. Thus, our approach opens the possibility to handle very interesting examples such as the Penrose tiling.

As discussed before, we focus on the existence and construction of periodic approximations for quasicrystals. The existence and construction of periodic approximations is solved for 1-dimensional systems [1, 4]. In higher dimensions, local symmetries of substitutional systems lead to a specific construction of periodic approximations covering known results such as for the Fibonacci sequence but also for higher dimensional systems like the Table tiling [1]. Furthermore, first elaborations [5] show that cut-and-project sets in the Euclidean setting admit suitable periodic approximations for measured quantities.

In a recent project [5], we analyze the behaviour of measures associated with Delone dynamical systems in the Chabauty–Fell topology within $\mathfrak{I}$. This approach is valid for Delone dynamical systems in general locally compact second countable Hausdorff groups. It turns out that the measures converge if the underlying Delone dynamical systems converge in $\mathfrak{I}$ and the limit object is uniquely ergodic. More precisely, the density of states measure and the autocorrelation measure is considered. As application, cut-and-project sets are studied by approximating the corresponding lattice and the window function by continuous window functions. It is worth mentioning that small changes on the lattice provide periodic approximations in the Euclidean setting which makes this approach very interesting. Continuous window functions need to be considered as the cutting process is highly non-continuous. It is left for further studies if the approximation of the window function also leads to the convergence of the related spectra.

As discussed before, a $C^*$-algebraic approach is used in [1, 4] to show the convergence of the spectra. In the recent work [3], a different proof is provided without this machinery for Schrödinger operators on lattices in $\mathbb{R}^d$ admitting strongly pattern equivariant potentials $q_h$ and $p$. More precisely, the existence of a constant $C > 0$ (depending on the dimension) is proven such that

$$d_H\big(\sigma(H_w), \sigma(H_{w'})\big) \leq C \cdot \|H\|_\alpha \cdot d_H^{\mathcal{A}}\big(\overline{Orb(w)}, \overline{Orb(w')}\big)^\alpha, \qquad w, w' \in \mathcal{A}^{\mathbb{Z}^d}.$$

Here $\|H\|_\alpha$ denotes the Schur $\alpha$-norm where $\alpha \in [0, 1]$ is chosen such that $\|H\|_\alpha$ is finite. The Schur $\alpha$-norm measures the polynomial decay of the off-diagonal terms

of the Schrödinger operator. Thus, $\|H\|_1$ is finite whenever $\mathcal{R}$ is finite implying the Lipschitz continuity of the spectra. The proof is based on a suitable localization of the operator via a quadratic partition of $\mathbb{R}^d$. With this at hand, the main task is to estimate the commutators with such a localization which is discussed during the talk.

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## Orbit equivalence, dimension groups and eigenvalues
María Isabel Cortez, Fabien Durand
(joint work with Samuel Petite)

In a series of papers [3, 1, 2, 6, 5] with X. Bressaud, A. Frank, B. Host, A. Maass and S. Petite we studied continuous and non-continuous eigenvalues of minimal Cantor systems. Among other questions, we were looking for (computable) necessary and sufficient conditions for eigenvalues to be continuous or non-continuous. This was achieved for the continuous eigenvalues in [5].

With S. Petite we investigated the restrictions induced on the groups of eigenvalues that can be realized within a given strong orbit equivalence class. For a minimal Cantor system $(X, T)$, let $E(X, T)$ be the set of real numbers $\alpha$ such that $\lambda = \exp(2i\pi\alpha)$ is a continuous eigenvalue (that is, having a continuous eigenfunction $f$: $f \circ T = \lambda f$).

We know from [9] that strong orbit equivalent minimal Cantor systems share the same subgroup of continuous eigenvalues that are roots of unity. It is no longer true for the orbit equivalence as shown again in [9]. Indeed, Ormes proved that in a prescribed orbit equivalence class it is possible to realize any countable subgroup of the circle as a group of measurable eigenvalues.

It happens that a first restriction has been shown in [8]: the additive group of eigenvalues, $E(X,T)$, of a minimal Cantor system $(X,T)$, is a subgroup of the intersection of all the images of the dimension group by its traces. Dynamically speaking, it is a subgroup of $I(X,T) = \cap_{\mu \in \mathcal{M}(X,T)} \left\{ \int f d\mu | f \in C(X,\mathbb{Z}) \right\}$, where $\mathcal{M}(X,T)$ is the set of $T$-invariant probability measures of $(X,T)$ and $C(X,\mathbb{Z})$ is the set of continuous functions from $X$ to $\mathbb{Z}$. A different proof of this observation can be found in [3] but it was not pointed out.

In [4] we obtained the following additional constraint.

**Theorem.** *Suppose that $(X,T)$ is a minimal Cantor system such that the infinitesimal subgroup of the dimension group $K^0(X,T)$ is trivial. Then the quotient group $I(X,T)/E(X,T)$ is torsion free.*

In [7] the hypotheses were relaxed removing the one on the infinitesimal subgroup.

To illustrate this result, take $K^0(X,T) = \mathbb{Z} + \alpha\mathbb{Z} = I(X,T)$, with $\alpha$ irrational. This is the case for a Sturmian subshift. Then within the strong orbit equivalence class of $(X,T)$ the only groups of continuous eigenvalues that can be realized are $\mathbb{Z}$, which will provide topologically weakly mixing minimal Cantor systems, and $\mathbb{Z} + \alpha\mathbb{Z}$. Moreover, both can be realized, in the first case using results in [9] and in the second case it is realized by a Sturmian subshift.

Apart from such particular examples no general realization results have been obtained so far.

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## Spectral calculations for two-dimensional quasicrystals

Mark Embree

(joint work with David Damanik, Jake Fillman, Anton Gorodetski, May Mei, Charles Puelz)

Mathematical models of one-dimensional quasicrystals, most notably the Fibonacci Hamiltonian, have reached an advanced state of refinement [2]. In contrast, our understanding of two dimensional quasicrystal models remains at a nascent stage. This talk described some analytical and computational results related to two-dimensional models.

### Fast spectral computation for 1d periodic models

The simplest two-dimensional models are constructed by combining one-dimensional quasiperiodic models on a square lattice. The spectrum of such a *square model* then equals the set sum of the corresponding one-dimensional spectra. While these one-dimensional spectra are Cantor sets, the same need not be true for the square model. We presented numerical calculations from [1] that estimate the structure of this spectrum as a function of the *coupling constant* (i.e., the weight of the potential) for the Fibonacci, Thue–Morse, and period doubling models.

In numerous circumstances, periodic approximations of one-dimensional quasi-periodic potentials lead to *covers* (upper bounds) on the spectrum of the quasiperiodic model; long periods yield more accurate estimates. Floquet–Bloch theory shows that the spectrum of a one-dimensional model of period $p$ comprises the union of $p$ real intervals that are traced out by the eigenvalues of a parameterized $p \times p$ matrix $J_p(\theta)$; for $p = 7$,

$$
J_p(\theta) = \begin{bmatrix}
v_1 & 1 & & & & & e^{-i\theta} \\
1 & v_2 & 1 & & & & \\
& 1 & v_3 & 1 & & & \\
& & 1 & v_4 & 1 & & \\
& & & 1 & v_5 & 1 & \\
& & & & 1 & v_6 & 1 \\
e^{i\theta} & & & & & 1 & v_7
\end{bmatrix},
$$

and the spectrum of the one dimensional model is $\cup_{\theta \in [0,\pi]} \sigma(J_p(\theta))$. (The $v_j$ values specify the potential; unspecified entries are zero.) The ends of these intervals are given by $\sigma(J_p(0))$ and $\sigma(J_p(\pi))$, so one can determine the spectrum of a periodic approximation by computing all the eigenvalues of two $p \times p$ symmetric matrices. When $p$ is large, the corner entries $e^{\pm i\theta}$ cause the conventional QR eigenvalue algorithm to use $O(p^2)$ storage and $O(p^3)$ computation time. We describe a simple trick from [6] that relabels the $p$ sites in the periodic potential using a breadth-first ordering, effectively performing an orthogonal similarity transformation (gauge

transformation) to obtain the *pentadiagonal* matrix (illustrated for $p = 7$)

$$PJ_p(\theta)P^* = \begin{bmatrix} v_1 & e^{-i\theta} & 1 & & & & \\ e^{i\theta} & v_7 & 0 & 1 & & & \\ 1 & 0 & v_2 & 0 & 1 & & \\ & 1 & 0 & v_6 & 0 & 1 & \\ & & 1 & 0 & v_3 & 0 & 1 \\ & & & 1 & 0 & v_5 & 0 \\ & & & & 1 & 0 & v_4 \end{bmatrix}.$$

This transformed matrix has fixed bandwidth independent of $p$, so standard algorithms from numerical linear algebra deliver all the eigenvalues of this matrix with $O(p)$ storage and $O(p^2)$ computation. This improvement becomes particularly crucial because large $p$ values often lead to numerically inaccurate eigenvalues (we are seeking good covers of Cantor sets, after all), necessitating the use of expensive extended precision arithmetic. We illustrated this algorithm with results from [6] showing the accuracy of double and quadruple precision computations, along with estimates of the Hausdorff dimension of the Fibonacci model and gap scaling for the Thue–Morse potential.

## Gap openings in 2d periodic models

In the period-$p$ models described above, one can always construct a potential, arbitrarily small in norm, that has a spectrum with $p - 1$ distinct gaps. Is the same true of $(p, q)$-periodic models on a square lattice?

We described a recent result from [3] that shows that this is not the case, extending earlier work of Krüger [5]. Specifically, if both $p$ and $q$ are even, an arbitrarily small $(p, q)$-periodic model can open a gap at $E = 0$; with this exception, arbitrarily small $(p, q)$-periodic potentials cannot open any gaps. The proof of this fact follows from eigenvalue perturbation theory for symmetric matrices.

## Eigenvalues of the graph Laplacian for the Penrose tiling

Arguably the most intriguing two-dimensional quasicrystal model comes from the Laplacian on a graph generated from the Penrose tiling. We closed the talk by presenting numerical results (with Fillman and Mei) based on Robinson's stone inflation rule using triangular tiles. We illustrated modes with local support occurring at $E = 2$ and $E = 4$ (as observed by Kohmoto and Sutherland [4]), which lead to a jump in the integrated density of states at those energies; we showed other intriguing modal structures for this model that merit further investigation.

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## Spectral properties of continuum quasicrystal models

Jake Fillman

(joint work with David Damanik, Mark Embree, Anton Gorodetski, May Mei, Yuki Takahashi, William Yessen)

We consider continuum Schrödinger operators acting in $L^2(\mathbb{R})$ via

$$L_V u = -u'' + Vu,$$

where the potential $V : \mathbb{R} \to \mathbb{R}$ models a quasicrystal; for an archetypal example, let

$$(1) \qquad V_\omega(x) = \sum_{n \in \mathbb{Z}} \big((1 - \omega_n)f_0(x - n) + \omega_n f_1(x - n)\big),$$

where $f_0, f_1 \in L^2[0, 1)$ and $\omega$ denotes the Fibonacci sequence

$$(2) \qquad \omega_n = \chi_{[1-\alpha)}(n\alpha \mod 1), \quad n \in \mathbb{Z},\ \alpha = \frac{\sqrt{5} - 1}{2}.$$

These operators are interesting because their spectra are globally zero-measure Cantor sets [4]; see also [12] for the case of measure-valued potentials. In fact, this holds true for any aperiodic potential of the type (1), as long as the sequence $\omega$ is generated by a minimal subshift satisfying Boshernitzan's criterion (cf. [1, 2, 3]). It is then of interest to assess more delicate fractal properties, such as the Hausdorff dimension. For general potentials, this is currently out of reach, but these questions can be studied in the Fibonacci case using the trace map and tools from hyperbolic dynamics. In particular, for $V_\omega$ and $\omega$ as in Eqs. (1)–(2), one has

$$\lim_{K \to \infty} \inf_{E \in \sigma(L_{V_\omega}) \cap [K, \infty)} \dim_H^{\mathrm{loc}}\big(\sigma(L_{V_\omega}); E\big) = 1;$$

that is, the local Hausdorff dimension of the spectrum tends to one in the high-energy region [4, 7]. This holds for any choice of $f_0$ and $f_1$ and hence this property is independent of the shape of the bump functions one uses to pattern the Fibonacci potential.

Turning to higher dimensions, one may study *separable* Schrödinger operators, i.e., operators of the form

$$\big[L_{V_1, V_2}^{(2)} u\big](x, y) = -\nabla^2 u(x, y) + V_1(x)u(x, y) + V_2(y)u(x, y),$$

where the potential is the sum of two pieces: one piece depends only on $x$, and the other depends only on $y$. The spectra of such operators are amenable to analysis, as they are simply the Minkowski sum of the 1D spectra, that is,

$$\sigma\big(L^{(2)}_{V_1,V_2}\big) = \sigma(L_{V_1}) + \sigma(L_{V_2}) = \big\{a_1 + a_2 : a_j \in \sigma(L_{V_j})\big\}.$$

However, even these ostensibly simple models touch on deep questions in geometric measure theory. In particular, the Minkowski sum of two zero-measure Cantor sets can be a Cantor set, an interval, a finite union of intervals, or something even more exotic. There are parameter ranges for which $\sigma\big(L^{(2)}_{\lambda_1 V_\omega, \lambda_2 V_\omega}\big)$ contains both intervals and Cantor sets [8], where $V_\omega$ is defined by Eqs. (1)–(2). However, there is reason to suspect more is true. Motivated in part by the results of [5] for separable discrete models built around the Fibonacci sequence, we pose the following question:

**Question.** *If $V_\omega$ is defined by Eqs. (1)–(2), is it true that $\sigma\big(L^{(2)}_{\lambda_1 V_\omega, \lambda_2 V_\omega}\big)$ contains a half-line for every choice of $\lambda_1, \lambda_2 > 0$?*

This question is also motivated by and connected with the Bethe–Sommerfeld conjecture for periodic Schrödinger operators, which inspired substantial contributions from many authors, including (but certainly not limited to) [10, 11, 14, 15, 16, 17, 18], and culminating in the paper of Parnovskii [13]. On that note, we conclude with a discussion of the *discrete* Bethe–Sommerfeld conjecture, proved in dimension 2 by Embree–Fillman [6] and in higher dimensions by Han–Jitomirskaya [9]. The spectrum of a 2D periodic discrete Schrödinger operator with a sufficiently small potential consists of either one or two intervals, and is guaranteed to be a single interval as long as at least one period is odd. Using the simple $\ell^\infty$ perturbation theory, this immediately implies that a large class of limit-periodic Schrödinger operators in $\mathbb{Z}^2$ have spectra with one or two connected components.

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# Weak forms of equicontinuity

Felipe García-Ramos

In this report we will discuss the relationship between the regularity of minimal topological dynamical systems, in terms of forms of equicontinuity, and the regularity of topological and measure theoretic factors to equicontinuous sytems. This will provide a hierarchy of non-chaotic dynamical systems.

We say $(X, T)$ is *topological dynamical system* (TDS), if $X$ is a compact metric space (with metric $d$) and $T : X \to X$ is continuous. We say $(X, T, \mu)$ is a *measure preserving topological dynamical system* (MP-TDS), if $(X, T)$ is a TDS and $\mu$ is a $T$-invariant probability measure. We say $(X, T)$ and $(X', T')$ are *conjugate* if there exists a continuous bijective function $f : X \to X'$ such that $T' \circ f = T \circ f$. We say $(X, T, \mu)$ and $(X', T', \mu')$ are *isomorphic* if there exists a bi-measurable invertible function $f : X \to X'$ such that $f$ and $f^{-1}$ push the measures and $T' \circ f = T \circ f$.

There are some non-symmetric notions of similarity that use both the topology and the measure.

We say $(X, T, \mu)$ is an isomorphic extension of $(X', T', \mu')$ if they are isomorphic but we also require $f$ to be continuous and surjective.

We say $(X, T, \mu)$ is a regular isomorphic extension of $(X', T', \mu')$ if there exists a surjective continuous function $f : X \to X'$ such that $T' \circ f = T \circ f$ and $\mu' \{x' \in X' : \{f^{-1}(x')\} \text{ is a singleton}\} = 1$.

Note that for a pair of MP-TDSs, conjugacy $\Rightarrow$ regular isomorphic extension $\Rightarrow$ isomorphic extension $\Rightarrow$ isomorphism.

A TDS is *equicontinuous* if the family $\{T^i\}_{i \in \mathbb{N}}$ is equicontinuous or, equivalently, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every open set with $\operatorname{diam}(U) \le \varepsilon$ then $\operatorname{diam}(T^i U) \le \delta$ for all $i \in \mathbb{N}$ (where $\operatorname{diam}(U)$ denotes the diameter of the set).

Classical equicontinuity is a very strong property and it is not very useful for studying subshifts or Delone systems. Every equicontinuous subshift or Delone system is periodic [2].

A weaker form of equicontinuity was introduced by Fomin [4].

**Definition 1.** *We say a TDS is* mean equicontinuous *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) \leq \delta$ then*

$$\lim \sup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} d(T^i x, T^i y) \leq \varepsilon.$$

Every equicontinuous TDS is mean equicontinuous.

Given the average nature of the definition one might expect that this topological property is strongly related to ergodic properties. Oxtoby showed that every minimal mean equicontinuous system is uniquely ergodic [14, 1]. It was conjectured that this measure must have discrete spectrum [15] (see definition below). This question was answered independently in [13] and in [5].

**Theorem 1** ([3, 13])**.** *A minimal TDS $(X, T)$ is mean equicontinuous if and only if there exists an equicontinuous TDS $(X', T')$ such that $(X, T, \mu)$ is an isomorphic extension of $(X', T', \mu')$ (where $\mu$ and $\mu'$ are the respective invariant measures).*

We say a TDS is *BD-mean equicontinuous* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every open set with $\operatorname{diam}(U) \leq \delta$

$$\lim \sup_{N-M \to \infty} \frac{1}{N-M} \sum_{i=M+1}^{N} \operatorname{diam}(T^i U) \leq \varepsilon.$$

**Theorem 2** ([5, 6])**.** *A minimal TDS $(X, T)$ is BD-mean equicontinuous if and only if there exists an equicontinuous TDS $(X', T')$ such that $(X, T, \mu)$ is a regular isomorphic extension of $(X', T', \mu')$ (where $\mu$ and $\mu'$ are the respective invariant measures).*

It is not hard to show that a TDS is BD-mean equicontinuous if and only if for every $\varepsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that

$$\overline{BD} \left\{ i \in \mathbb{N} : \operatorname{diam}(T^i B_\delta(x)) > \varepsilon \right\} < \varepsilon,$$

where $B_\delta(x)$ is the $\delta$-neighbourhood of $x$, and $\overline{BD}$ denotes the upper Banach density.

Another form of order is zero topological sequence entropy, also known as null systems. For definition and properties see [12, 10].

**Theorem 3** ([5, 6])**.** *Every minimal null TDS is BD-mean equicontinuous.*

**Definition 2.** *Let $(X, T)$ be a TDS and $\mu$ an invariant Borel probability measure. We say $(X, T)$ is $\mu$-mean equicontinuous if for every $\tau > 0$ there exists a compact set $M \subset X$ with $\mu(M) \geq 1 - \tau$, such that for every $\varepsilon > 0$ there exists $\delta > 0$ such*

*that whenever $x, y \in M$ and $d(x, y) \leq \delta$ then*

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} d(T^i x, T^i y) \leq \varepsilon.$$

Halmos and von Neumann showed that an ergodic dynamical system has *discrete spectrum* if and only if it is isomorphic to a minimal equicontinuous TDS (respective to its invariant measure) [11].

**Theorem 4.** *Let $(X, T)$ be a TDS and $\mu$ an invariant ergodic probability measure. Then $(X, T, \mu)$ is isomorphic to a minimal equicontinuous TDS if and only if $(X, T)$ is $\mu$-mean equicontinuous.*

Note that uniquely ergodic topologically weak mixing systems may have discrete spectrum. These systems are never mean equicontinuous [7].

Let $(X, T)$ be a TDS $\mu$ a Borel probability measure and $f \in L^2(X, \mu)$. We say $(X, T)$ is *$\mu$-$f$-mean equicontinuous* if for every $\tau > 0$ there exists a compact set $M \subset X$ with $\mu(M) \geq 1 - \tau$ such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in M$ and $d(x, y) \leq \delta$ then

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} \left| f(T^i x) - f(T^i y) \right| \leq \varepsilon.$$

**Theorem 5** ([8]). *Let $(X, T)$ be a TDS, $\mu$ an invariant ergodic probability measure and $f \in L^2(X, \mu)$. Then $f$ is almost periodic if and only if $(X, T)$ is $\mu$-$f$-mean equicontinuous.*

We have the following hierarchy for minimal uniquely ergodic systems (each implication is strict): equicontinuous $\Rightarrow$ null $\Rightarrow$ BD-mean equicontinuous $\Rightarrow$ mean equicontinuous $\Rightarrow$ $\mu$-mean equicontinuous $\Rightarrow$ $\mu$-$f$-mean equicontinuous (for some but not every $f \in L^2$).

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# Stealthy hyperuniform processes

Subhro Ghosh

(joint work with Joel L. Lebowitz)

In recent years, a special class of hyperuniform particle systems, known as stealthy hyperuniform (henceforth abbreviated as SH) systems, have attracted considerable attention [14, 15, 16, 17, 18]. These systems are characterized by the structure function $S(k)$ vanishing in a neighbourhood of $k = 0$. The quantity $S(k)$ is also referred to as the Bartlett spectrum. A natural generalization of SH point processes is to consider point processes, or random fields, with a gap in the spectrum on an open set $U$ which may not include the origin. We shall denote these processes as Generalized Stealthy (henceforth abbreviated as GS) processes.

The nomenclature "stealthy", as well as the physical interest in SH particle systems, stems from the fact that such systems are optically transparent (invisible) for wave vectors $k$ in the gap $U$. Numerical and experimental investigations have been carried out regarding how to construct SH particle systems. These systems cannot be equilibrium systems, with tempered potentials, at finite temperatures. They may, however, be ground states of such systems, e.g. the periodic (disordered?) zero temperature states of classical systems, or they can be generated as non-equilibrium states. SH systems are an extension of hyperuniform (superhomogeneous) particle systems. Hyperuniform systems, which have been studied extensively both in the physics and the mathematics literature, have reduced fluctuations: the variance of the particle number in a domain $D$ in $\mathbb{R}^d$ or $\mathbb{Z}^d$ grows slower than the volume of $D$. The significance of hyperuniform materials, and in particular SH systems, lies in the fact that they embody properties of both crystalline and disordered or random systems (see [9, 10] and the references therein). For translation invariant systems, an equivalent characterization of hyperuniformity can be obtained by looking at their structure functions. Hyperuniformity then boils down to the vanishing of the structure function $S(k)$ at $k = 0$. SH systems, therefore, involve a specific manner in which this vanishing of the structure function takes place.

In the article [18], Zhang, Stillinger and Torquato provide numerical evidence in support of some remarkable conjectural properties of stealthy hyperuniform processes, in particular that the hole sizes for stealthy hyperuniform processes are uniformly bounded. In [10], we carry out a rigorous mathematical analysis of stealthy hyperuniform processes, and establish the veracity of this conjecture. In particular, we prove that

**Theorem 1.** *Let $\Xi$ be a stealthy hyperuniform point process. Let $B(x; r)$ be the ball with centre $x$ and radius $r$. Then there exists a positive number $r_0$ such that $\mathbb{P}\big[|\Xi \cap B(x; r_0)| = 0\big] = 0$. Further, the quantity $r_0$, can be chosen to be $Cb^{-1}$, where $b$ is the radius of the maximal ball (centred at the origin) that is contained in the gap of the structure function $S$, and $C$ is a universal constant.*

We also establish an anti-concentration property for particle numbers of stealthy hyperuniform processes:

**Theorem 2.** *Let $\Xi$ be a stealthy hyperuniform point process on $\mathbb{R}^d$ or $\mathbb{Z}^d$ with one point intensity $\rho$ and $b$ the radius of the largest ball around the origin (in the wave space) on which the structure function of $\Xi$ vanishes. There exists numbers $C, c > 0$ (independent of all parameters of $\Xi$) such that, the number of points of $\Xi$ in any given $d$-dimensional cube of side-length $Cb^{-1}$ is a.s. bounded above by $c\rho b^{-d}$.*

The fact that holes in SH processes cannot be bigger than a deterministic size is suggestive of a high degree of crystalline behaviour in these processes. In our work, we go further, and establish a remarkable maximal rigidity property of these ensembles. We can, in fact, do this in the setting of GS processes. For a point process (more generally, a random field or a random measure) $\Xi$ on $\mathbb{R}^d$ and a bounded domain $D \subset \mathbb{R}^d$, statistic $\Psi$ defined on $\Xi$ restricted to $D$ is said to be rigid if $\Psi$ is completely determined by (that is, a deterministic function of) the process $\Xi$ restricted to $D^\complement$. To put things in perspective, a point process having rigidity is in notable contrast to the Poisson process, where the process inside and outside of $D$ are statistically independent. Rigidity phenomena for particle systems have been investigated quite intensively in the last few years, and have been shown to appear in many natural models which are, nonetheless, far removed from being crystalline. Key examples include the Ginibre ensemble, Gaussian zeros, the Dyson log gas, Coulomb systems and various determinantal processes related to random matrix theory (see, e.g., [11, 6, 7, 8, 5, 12, 9]).

In [10], we show that GS random measures on $\mathbb{R}^d$ or $\mathbb{Z}^d$ exhibit maximal rigidity: namely, for any domain $D \subset \mathbb{R}^d$, the random measure $[\Xi]_{|D^\complement}$ determines completely the measure $[\Xi]_{|D}$ (that is, the latter is a deterministic measurable function of the former). Stated in formal terms, we prove:

**Theorem 3.** *Let $\Xi$ be a generalized stealthy random measure on $\mathbb{R}^d$ or $\mathbb{Z}^d$. Then for any bounded domain $D$, the random measure $[\Xi]_{|D}$ is almost surely determined by (i.e., is a measurable function of) the random measure $[\Xi]_{|D^\complement}$.*

We further show that, to have maximal rigidity in the sense discussed above, it suffices that the structure function vanishes faster than any polynomial at some point in the wave space. In the 1D discrete setting (i.e. $\mathbb{Z}$-valued processes on $\mathbb{Z}$), this can also be seen as a consequence of a recent theorem of Borichev, Sodin and Weiss [4]; in higher dimensions or in the continuum, such a phenomenon seems novel. The question of inference about a stochastic process from its diffraction spectrum has a long history in diffraction theory, and we believe the results in the present article would be of interest to that body of literature. We refer the reader who is interested in further exploration of this direction to [1, 2, 3]. Note that the central Bragg peak $\rho^2 \delta_{k=0}$ is not included in $S(k)$.

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# The structure of tame minimal dynamical systems for general groups
### Eli Glasner

A dynamical version of the Bourgain–Fremlin–Talagrand dichotomy [1] shows that the enveloping semigroup of a dynamical system is either very large and contains a topological copy of $\beta\mathbb{N}$, or it is a "tame" topological space whose topology is determined by the convergence of sequences. In the latter case the dynamical system is called *tame* [8, 2]. WAP (weakly almost periodic) as well as HNS (hereditarily non-sensitive) systems are tame, and among the typical examples of tame systems one can find many cut and project systems like the classical Sturmian and some Toeplitz flows, [5].

Minimal tame dynamical systems $(X, G)$ with an Abelian acting group $G$ were studied by several authors, and it was shown that such systems are almost automorphic and uniquely ergodic, and that the canonical continuous map from $X$ onto its largest Kronecker factor is a measure theoretical isomorphism, [6, 7, 3].

What happens when the acting group is not assumed to be commutative, or even not amenable? Here, one discover completely new phenomena and a wealth of new examples. The most prominent among them are boundaries of Gromov hyperbolic groups and linear actions on spheres and projective spaces.

In a recent work [4] I use the structure theory of minimal dynamical systems to show that, for a general group $G$, a tame, metric, minimal dynamical system $(X, G)$ has the following structure:
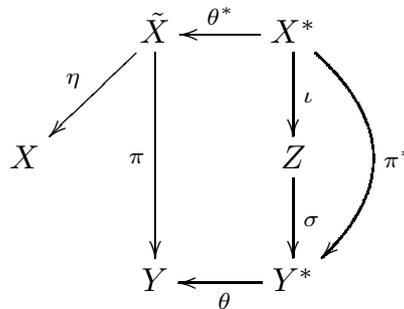


Here (i) $\tilde{X}$ is a metric minimal and tame system (ii) $\eta$ is a strongly proximal extension, (iii) $Y$ is a strongly proximal system, (iv) $\pi$ is a point distal and RIM extension with unique section, (v) $\theta$, $\theta^*$ and $\iota$ are almost one-to-one extensions, and (vi) $\sigma$ is an isometric extension.

When the map $\pi$ is also open this diagram reduces to
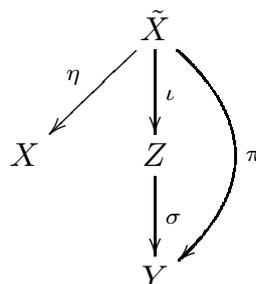
In general the presence of the strongly proximal extension $\eta$ is unavoidable. If the system $(X, G)$ admits an invariant measure $\mu$ then $Y$ is trivial and $X = \tilde{X}$ is an almost automorphic system; i.e. $X \xrightarrow{\iota} Z$, where $\iota$ is an almost one-to-one extension and $Z$ is equicontinuous. Moreover, $\mu$ is unique and $\iota$ is a measure theoretical isomorphism $\iota : (X, \mu, G) \to (Z, \lambda, G)$, with $\lambda$ the Haar measure on $Z$. Thus, this is always the case when $G$ is amenable.

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## Aperiodic Schrödinger operators

### Anton Gorodetski

Most of the questions on spectral properties of higher dimensional aperiodic operators (such as Laplacian on Penrose tilings) are completely open; the only known results are related to existence of a well defined density of states measure and some of its properties [10, 11, 12]. On a one dimensional lattice reasonable models of quasicrystals are substitution sequences (such as Fibonacci, Thue–Morse, period doubling), and Sturmian sequences. In all these cases the spectrum is known to be a Cantor set of zero measure for all non-zero values of the coupling constant [1, 2, 3, 14]. At the same time other characteristics of the spectrum can be very different. For example, gap sizes asymptotics for small coupling is known for Fibonacci Hamiltonian [6], Thue–Morse [1], and period doubling potentials [2], and turn out to be model-dependent. As another example, exact large coupling asymptotics of the Hausdorff dimension of the spectrum of Fibonacci Hamiltonian are known [5] — it tends to zero as an inverse of a logarithm of the coupling; at the same time the Hausdorff dimension of the spectrum in the case of Thue–Morse potential is uniformly bounded away from zero [13]. The Fibonacci Hamiltonian is the most heavily studied since it belongs to both classes — operators with Sturmian potentials, and those with potential given by a substitution sequence. The Trace Map approach allowed to provide a very detailed and almost complete description of the properties of the spectrum, density of states measure, transport properties etc., see [9] and references therein.

One of the ways to use the obtained one-dimensional results to gain some intuition on the spectral properties of the higher dimensional aperiodic operators is via separable potentials. In this case the potential on two (or higher) dimensional lattice is given by the sum of two potentials, each of them depends only on one of the coordinates. The spectrum of the discrete Schrödinger operator in this case turns out to be the Minkowski sum of the spectra of the corresponding one dimensional operators, and the density of states measure is given by convolution of the corresponding density of states measures. In the case of the Fibonacci Hamiltonian the spectrum is a dynamically defined Cantor set [9]. Questions on the structure of sums of dynamically defined Cantor sets appeared before in dynamical systems and number theory. Applying some of the existing methods and using the known results on the spectrum of the Fibonacci Hamiltonian one can show that the spectrum of the square Fibonacci Hamiltonian is an interval for small values of the coupling constant, and is a Cantor set of zero measure for the large coupling [6]. Moreover, typically the density of states is a.c. with respect to the Lebesgue measure for small couplings [8]. Also, interestingly enough, there is a regime (open set in the space of couplings) where typically the spectrum of the square Fibonacci Hamiltonian has positive Lebesgue measure while the density of states measure is singular [7].

For a detailed recent survey of these and many other results on spectral properties of aperiodic Schrödinger operators see [4].

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## Dynamical encodings of patterns in cut and project sets

ALAN HAYNES

(joint work with Antoine Julien, Henna Koivusalo, Jens Marklof, Lorenzo Sadun, James Walton)

The purpose of this talk was to demonstrate how problems about patterns in cut and project sets can be reformulated in terms of questions about higher rank linear actions on tori. One goal of the talk was to emphasize the strong connections between these types of questions and problems in Diophantine approximation.

First we first reviewed classical results of Morse and Hedlund about Sturmian sequences [7, 8], in which patterns of a given size correspond to regions in the circle determined by an irrational rotation $\alpha$ (the slope defining the Sturmian sequence). This point of view leads quickly to detailed knowledge about three quantities associated to patterns in Sturmian sequences: complexity, frequencies, and the repetitivity function. Understanding the complexity is a geometric problem, which corresponds precisely to counting the number of connected components of the circle, with a finite sub-orbit of 0 under the rotation by $\alpha$ removed. Questions about frequencies of patterns are answered by understanding volumes of connected components, which is a simple example of a gaps problem in Diophantine approximation. Questions about repetitivity of patterns are answered by a detailed analysis of the continued fraction expansion of $\alpha$.

Sturmian sequences are examples of one-dimensional cut and project sets obtained by projecting from a two-dimensional total space. For more general cut and project sets, with windows satisfying appropriate regularity conditions (which was a standing assumption in our talk), we can ask analogous questions about complexity, frequencies, and repetitivity of patterns. It is not difficult to see that, as in the case of Sturmian sequences, for $k$ to $d$ cut and project sets there is a correspondence between patterns of a given size and connected components of the window, after a sub-orbit of the boundary under the natural $\mathbb{Z}^d$-action determined by the physical space is removed. This idea was used in [6] to satisfactorily understand complexity of patterns in cut and project sets, as well as how the growth of the complexity function is related to the cohomology of associated topological spaces.

Problems about the number of distinct frequencies of patterns of a given size in cut and project sets are related to higher dimensional gaps problems in Diophantine approximation. Such problems are in general much more difficult than their one-dimensional counterparts. In [2] we proved that, for any $k$ and $d$, there

is a full Hausdorff dimension set of cut and project sets for which, for any $r \geq 1$, the number of distinct frequencies of patterns of size $r$ remains bounded. We also showed that, for almost every $k$ to $d$ cut and project set (with respect to Lebesgue measure), for any $\epsilon > 0$, and for all sufficiently large $r$, the number of distinct frequencies of patterns of size $r$ is bounded above by $(\log r)^{(1+\epsilon)(d+1)(k-d)}$. Furthermore, recent work on higher dimensional Steinhaus problems [5] now also implies a previously elusive result, that for almost every cut and project set, the number of distinct frequencies of patterns of size $r$ is not bounded.

Finally, problems about repetitivity of patterns in cut and project sets involve a careful study of the volumes and shapes of connected components of the regions in the corresponding dynamical encodings. This type of study has recently been undertaken in [3], where we gave an explicit characterization of the collection of all linearly repetitive cut and project sets with cubical windows. Further work on this problem, including a development of the connections with Diophantine approximation, discrepancy theory, and the Littlewood conjecture, can be found in [1] and [4].

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## Quasicrystals beyond amenable groups

Tobias Hartnick, Felix Pogorzelski

(joint work with Michael Björklund)

In this talk, we sketched the diffraction theory of model sets in homogeneous spaces associated with Gelfand pairs (commutative spaces), as developed in [3, 4].

## 1. Model sets in homogeneous spaces

By a *cut-and-project scheme* we shall mean a triple $(G, H, \Gamma)$, where $G$ and $H$ are locally compact second countable (lcsc) groups and $\Gamma < G \times H$ is a lattice which projects injectively to $G$ and densely to $H$. Given such a triple $(G, H, \Gamma)$ and a

compact subset $W \subset H$ with non-empty interior, we define the associated *model set* $P_0 = P_0(G, H, \Gamma, W)$ as

$$P_0 := \mathrm{pr}_G\big((G \times W) \cap \Gamma\big),$$

where $\mathrm{pr}_G$ denotes the projection onto the factor $G$ of $G \times H$. If $K < G$ is a compact subgroup, we also refer to the image $P$ of $P_0$ in the homogeneous space $X = G/K$ as a model set. For $G$ and $H$ Abelian and $K$ trivial, this is the classical cut-and-project construction as used by Meyer and others. In the talk, we discussed non-classical examples in Riemannian symmetric spaces, Bruhat–Tits buildings and nilmanifolds, including both arithmetic and non-arithmetic examples.

In the sequel, we reserve the letter $P_0$ to denote a model set in an lcsc group $G$; we will always assume that the corresponding window $W$ satisfies the following regularity conditions,

$$(1) \qquad W = \overline{W^o}, \quad |\partial W| = 0, \quad \mathrm{Stab}_H(W) = \{e\}, \quad \partial W \cap \mathrm{pr}_H(\Gamma) = \varnothing.$$

On the other hand, we will (at least initially) not assume that $\Gamma$ is uniform. We also fix a compact subgroup $K < G$ and denote by $P \subset X := G/K$ the image of $P_0$ under the canonical projection $p : G \to X$.

Both $P_0$ and $P$ are Delone sets with respect to natural classes of metrics on $G$ and $X$, respectively, and both have $G$-finite local complexity, that is, finitely many patches up to $G$-translation. Moreover, $P_0^{-1}P_0$ is uniformly discrete, which can be seen as a long-range order property.

## 2. The hull of a model set

Given a homogeneous space $Z$ of an lcsc group $G$, we will denote by $\mathcal{C}(Z)$ the collection of all closed subsets of $Z$, considered as a compact metrizable space with respect to the Chabauty–Fell topology, and given a subset $Q \subset Z$, we denote by

$$\Omega_Q := \overline{\big\{g.Q : g \in G\big\}} \subseteq \mathcal{C}(Z)$$

its *hull*. Our model sets $P_0$ and $P$ then give rise to topological dynamical systems $G \curvearrowright \Omega_{P_0} \subset \mathcal{C}(G)$ and $G \curvearrowright \Omega_P \subset \mathcal{C}(X)$, respectively. If the underlying lattice is non-uniform, these hulls contain the empty set as a non-trivial fixpoint. We will thus also consider the *punctured hulls* $\Omega_Q^\times := \Omega_Q \setminus \{\varnothing\}$ for $Q \in \{P_0, P\}$.

A priori, it is not clear whether $\Omega_{P_0}^\times$ or $\Omega_P^\times$ admit any $G$-invariant probability measures. To settle this issue, we establish the following generalization of Schlottmann's torus periodization map [7]; here, $Y := (G \times H)/\Gamma$ is a parameter space, which generalizes the classic torus parametrization [1, 7].

**Theorem 2.1** (Parametrization map, [3]). *Let $P_0 \subset G$ be a model set as above.*

    (1) *There exists a unique surjective Borel $G$-map $\beta : \Omega_{P_0}^\times \to Y$ with closed graph which maps $P_0$ to the basepoint $(e, e)\Gamma$ of $Y$. If $\Gamma$ is uniform, then $\beta$ is continuous.*

    (2) *There exists a subset $Y^{\mathrm{ns}} \subset Y$ of full Haar measure such that $\beta$ is one-to-one over $Y^{\mathrm{ns}}$.*

Explicitly, the set $Y^{\mathrm{ns}}$ of *non-singular parameters* is given by

$$Y^{\mathrm{ns}} := \big\{ (g,h)\Gamma : h^{-1}\partial W \cap \pi_H(\Gamma) = \varnothing \big\}.$$

From Theorem 2.1, one deduces the following consequences.

**Corollary 2.2** (Unique ergodicity and minimality of the hull, [3]). *Let $P_0 \subset G$ and $P \subset X$ be model sets as above.*

(1) *The spaces $\Omega_{P_0}^{\times}$ and $\Omega_P^{\times}$ each admit a unique $G$-invariant probability measure.*

(2) *If $\Gamma$ is uniform, the dynamical systems $G \curvearrowright \Omega_{P_0}$ and $G \curvearrowright \Omega_P$ are minimal.*

In fact, to establish unique ergodicity of $\Omega_P^{\times}$, one needs to establish a stronger property of $\Omega_{P_0}^{\times}$ called unique stationarity. It then follows that also $\Omega_P^{\times}$ is uniquely stationary, hence uniquely ergodic.

## 3. Autocorrelation of model sets

We explained how to associate with our model set $P \subset X$ an autocorrelation measure, which is a Radon measure on the double coset space $K \backslash G / K$, following the general approach of Bartlett from the theory of point processes; compare [5]. The main steps of this construction were as follows:

(1) Construct a periodization map

$$\mathcal{P} : C_c(X) \to C_c(\Omega_P^{\times}), \quad \mathcal{P}f(Q) = \sum_{x \in Q} f(x).$$

(2) Form the second correlation measure $\eta_\nu^{(2)} \in \mathcal{R}(X \times X)^G$ of the $G$-invariant measure $\nu$ on $\Omega_P^{\times}$ by

$$\eta_\nu^{(2)}(f \otimes g) = \int_{\Omega_P^{\times}} \mathcal{P}f(Q)\mathcal{P}g(Q)\,\mathrm{d}\nu(Q) \quad \big(f, g \in C_c(X)\big).$$

(3) Define the autocorrelation measure $\eta_P \in \mathcal{R}(K \backslash G / K)$ as the image of $\eta_\nu^{(2)}$ under the canonical isomorphism

$$\mathcal{R}(X \times X)^G \cong \mathcal{R}\big(G \backslash (G/K \times G/K)\big) \cong \mathcal{R}(K \backslash G / K).$$

For the so-defined autocorrelation measure, we obtain the following formula.

**Theorem 3.1** (Autocorrelation formula, [3]). *Let $P \subset X$ be a model set as above; denote by $p : G \to X$ the canonical projection and by $\mathcal{P}_\Gamma : C_c(G \times H) \to C_c(Y)$ the periodization map along $\Gamma$. Then, $\eta_P$ is uniquely determined by the fact that*

$$\eta_P(f^* * f) = \big\| \mathcal{P}_\Gamma(p^* f \otimes \chi_W) \big\|_{L^2(Y)^K}^2 \quad \big(f \in C_c(K \backslash G / K)\big).$$

Let us compare our definition of the autocorrelation measure to the more classical definition of Hof [6], the latter being a mathematical formulation of the

well-known Patterson function. Given a family of subsets $F_t \subset X$, let us define a family of Radon measures on $K\backslash G/K$ by

$$\sigma_t(f) := \frac{1}{|F_t|} \sum_{x \in P \cap F_t} \sum_{y \in P} f(x^{-1}y) \quad \big(f \in C_c(K\backslash G/K)\big).$$

In the classical Abelian setting, if $(F_t)$ is a van Hove sequence, then it can be shown (see e.g. [2]) that the autocorrelation measure is given by the formula

$$(2) \qquad\qquad \eta_P(f) = \lim_{t \to \infty} \sigma_t(f) \quad \big(f \in C_c(K\backslash G/K)\big),$$

and the classical argument extends to van Hove sequences in arbitrary amenable groups $G$. Remarkably, formula (2) holds also in many non-amenable situations, where Følner sequences, let alone van Hove sequences, do not exist. For example, if $X$ is a Riemannian symmetric space, then (2) holds for Riemannian balls $F_t$. However, the situation is far from simple in general. For example, if $X$ is a tree, then (2) holds along balls of even radius, but in general not along arbitrary balls. Thus, while the dynamical approach to autocorrelation always works in a uniform way, the approach through Hof approximation depends very much on the geometry of the spaces in question.

## 4. Towards diffraction

While autocorrelation can be defined for Delone sets of finite local complexity in arbitrary homogeneous spaces of the form $X = G/K$, the definition of diffraction requires a Fourier transform on the double coset space $K\backslash G/K$, hence we need to make additional assumptions on the pair $(G, K)$ from now on.

**Definition 4.1.** *We say that* $(G, K)$ *is a* Gelfand pair *and that* $X = G/K$ *is a* commutative space *if the* Hecke (convolution) algebra $\mathcal{H}(G, K) = C_c(K\backslash G/K)$ *is commutative.*

This assumption is satisfied in all examples considered above, in particular hyperbolic spaces, Riemannian symmetric spaces, and for nilmanifold pairs. If $(G, K)$ is a Gelfand pair, the Banach algebra $L^1(K\backslash G/K)$ is commutative and its Gelfand spectrum can be identified with the space $\mathcal{S}_b(G, K)$ of all bounded spherical functions; here, a continuous function $\omega : G \to \mathbb{C}$ is called *spherical* if the associated measure $m_\omega \in \mathcal{R}(G)$ as given by

$$m_\omega(f) := \int_G f(x)\omega(x^{-1})\,\mathrm{d}x \quad \big(f \in C_c(G)\big)$$

is bi-$K$-invariant.

The *spherical Fourier transform* of the pair $(G, K)$ is the restriction of the Gelfand transform of $L^1(K\backslash G/K)$ to the subspace $\mathcal{S}_+(G, K) \subseteq \mathcal{S}_b(G, K)$ of positive-definite spherical functions, i.e. for $f \in \mathcal{H}(G, K)$ and $\omega \in \mathcal{S}_+(G, K)$ we define

$$\widehat{f}(\omega) := m_\omega(f).$$

Similarly, if $\eta$ is a positive-definite Radon measure on $K \backslash G / K$, its spherical Fourier transform $\widehat{\eta}$ is determined by the fact that

$$\eta(f^* * f) = \widehat{\eta}(|\widehat{f}|^2), \qquad (f \in \mathcal{H}(G, K)).$$

Since the autocorrelation measure $\eta_P$ is positive-definite, we may thus define the *spherical diffraction measure of $P$* as

$$\widehat{\eta}_P \in \mathcal{R}(\mathcal{S}_+(G, K)).$$

**Theorem 4.2** (Pure point diffraction, [4]). *If the lattice $\Gamma$ is uniform, the spherical diffraction measure $\widehat{\eta}_P$ is pure point, i.e., there is a countable set $S \subset \mathcal{S}_+(G, K)$ and a function $c : S \to \mathbb{R}_{>0}$ such that*

$$\widehat{\eta}_P = \sum_{\omega \in S} c(\omega)\, \delta_\omega.$$

In fact, in the situation of the theorem, we can determine $S$ and the function $c$ explicitly. Indeed, the set $S$ is given by the spherical automorphic spectrum of $\Gamma$, that is, by the collection of all $\omega \in \mathcal{S}_+(G, K)$ for which the eigenspace

$$L^2(Y)_\omega^K := \big\{ g \in L^2(Y)^K : \forall f \in \mathcal{H}(G, K) : f * g = \widehat{f}(\omega)\, g \big\}$$

is non-zero. The coefficient function $c$ can be computed as the squared $L^2$-norm of a certain integral transform of the characteristic function $\chi_W$. In the Abelian case, this transform is simply a normalized Fourier transform on $H$. In the general case, the desired integral transform is obtained as a shadow of the spherical Fourier transform of the pair $(G, K)$ (in the spirit of a Hecke correspondence) and hence is referred to as the *shadow transform*; see [4] for details.

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# Topological invariants for tilings
### John Hunton

This talk gave a brief — and personal — overview of some of the main themes in the recent and current study of aperiodic tilings by methods from topology. It was clearly not possible to cover everything, and similarly it is not possible to give a comprehensive bibliography in the space available here, even for the subjects touched upon. The interested reader should explore the topics further through the selected papers mentioned below and the further work they cite.

We restrict ourselves mainly to tilings of $d$-dimensional Euclidean space which are repetitive, aperiodic and of translationally finite local complexity (FLC). For such a tiling $T \subset \mathbb{R}^d$, the key to the topological approach is the space $\Omega = \Omega_T$, variously known as the *tiling space*, or *continuous hull* of $T$, the completion of the set of translates of $T$ under the *tiling metric*. Under the assumptions above $\Omega$ naturally carries a minimal action of the translation group $\mathbb{R}^d$, and in many of the most popular classes of tilings, a unique ergodic probability measure.

The structure of $\Omega$ is fundamental to this work. Most lines of approach start from one or other of the observations that $\Omega$ can be **(a)** described (up to shape equivalence – see later) as an inverse limit of convenient finite CW complexes (*approximants*), or **(b)** given (up to homeomorphism) the structure of a fibre bundle over a $d$-torus with fibre a Cantor set [24]. The space may also be described as the classifying space of the holonomy groupoid associated with $\Omega$.

For description **(a)**, there are a number of useful models. For primitive substitution tilings, the first constructions were those of [1, 14]. The desire to produce smaller models for the approximants led to a number of developments, including [2, 3] which implicitly involved working in the *shape category*, a notion formally explored in [6]. Recent work has explored further the use of minimal homotopy models for the approximants. For general tilings, inverse limit descriptions exist via various models [1, 3, 10], but without specific structure these are principally of theoretical use. Similarly, the Cantor bundle structure is computationally practical only in the case of a tangible description of the holonomy action of $\mathbb{Z}^d$ on the Cantor fibre; this can be given explicitly in the case of cut and project tilings [9].

Various results have been established exploring the relationships between the spaces $\Omega_T$ and $\Omega_S$ and the possible relationships of the underlying tilings $T$ and $S$. Notable work in this thread includes [8] on deformations of tilings, and most recently [12] characterising homeomorphisms of tiling spaces.

Topological invariants for tilings typically study $\Omega_T$, with or without additional structure, through the application of methods from algebraic topology. Typical applications to date have included characterization results, identification of geometric properties of $T$, issues related to questions about pure point diffraction (for example work related to the Pisot substitution conjecture, see [16, Ch. 2] for an overview), labelling of gaps in the spectrum of the Schrödinger operator associated to $T$ [4, 5, 13], and results on the complexity of $T$, [11].

What algebro-topological tools should be employed? Homotopy groups are rich but hard to compute. For tiling spaces, the relevant variant of these are the *shape groups* $\underline{\pi}_*^{\mathrm{sh}}(-)$. The case of $d = 1$ was studied in [6] where the fundamental shape group $\underline{\pi}_1^{\mathrm{sh}}(\Omega)$ was shown to collect information relevant to embedding one-dimensional tiling spaces in surfaces: the non-Abelian nature of $\pi_1$ registered aspects not picked up by commutative invariants such as cohomology or $K$-theory. This is taken further in recent work of Gähler who uses the representation variety of $\underline{\pi}_1^{\mathrm{sh}}(\Omega)$ (more readily computable than $\underline{\pi}_1^{\mathrm{sh}}(\Omega)$ itself) in his classification of certain classes of one-dimensional substitutions.

Cohomology is a long standing tool used for tiling spaces, but there are several variants in common use; we mention just three. Čech cohomology was the first, and perhaps most natural choice from its behaviour on inverse limits (in which it differs from singular or simplicial cohomology). The models [1, 2] for substitutions mentioned above make this is a computable and well understood invariant for such tilings, at least in low dimensions [23]. Recent work has explored more general situations, such as mixed substitutions [19, 21]. Cohomology gives some clear characterizations: for example, $H^*(\Omega; \mathbb{Q})$ is finite rank for an FLC substitution, but infinite for a generic cut and project tiling; the first cohomology $H^1(\Omega_T, \mathbb{R}^d)$ counts degrees of freedom for deformations of $T$, and so on.

Pattern equivariant cohomology [15, 22] has proved a useful alternative approach, yielding the same algebraic invariant as the Čech theory, but in a way that elements can be realized in terms of geometric patches of $T$. A homological variant [25] shows that tiling spaces satisfy a Poincaré duality property analogous to that of manifolds, and has offered computational advantage, for example in the study of spaces remembering the symmetries of $T$ [26].

The third variant can be thought of as the cohomology of the tiling groupoid, but in the case of an explicit Cantor bundle structure over a $d$-torus $\mathbb{T}^d$, this is equivalent to the group cohomology of $\mathbb{Z}^d = \pi_1(\mathbb{T}^d)$ with coefficients the continuous $\mathbb{Z}$-valued functions on the fibre. This too has its strengths, especially in the case of an explicit description of the bundle, such as for many of the cut and project tilings. See [16, Ch. 4] for a general introduction. Similar methods become natural to apply when studying tilings with rotations, as explored in recent work of the author with Walton.

Cohomology may be enriched with various additional structures, producing finer invariants. Included here are the Ruelle–Sullivan map of [18], the ordered cohomology of [20] and the homology core of [7]. The reader should consult those papers for statements of the advantages gained.

Aperiodic tilings are a fruitful source of examples for non-commutative geometry. Several $C^*$-algebras $A_T$ have been constructed to model $\Omega_T$ and its paraphenalia, and their $K$-groups reflect the space and $\mathbb{R}^d$ action; in the case of a unique ergodic measure, there is also a trace map $K_*(A_T) \to \mathbb{R}$. See [17] for a discussion. Connes' Thom isomorphism identifies $K_*(A_T)$ with the topological $K$-theory $K^*(\Omega)$, and an Atiyah–Hirzebruch spectral sequence gives a method of

calculating $K^*(\Omega)$ from the Čech cohomology $H^*(\Omega)$. Through these the non-commutative invariants can frequently be computed. A key object of study here has been the image of the tracial state, which is related to Bellissard's gap labelling [4, 5, 13].

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# Irregular model sets

Tobias Jäger

(joint work with Michael Baake, Gabriel Fuhrmann, Daniel Lenz, Christian Oertel)

**Cut and project schemes.**  A *cut and project scheme* (CPS) is a triple $(G, H, \mathcal{L})$ consisting of two locally compact Abelian groups $G$ and $H$ and a co-compact discrete subgroup $\mathcal{L} \subseteq G \times H$ (called *lattice*) that projects injectively to $G$ and densely to $H$. Given a compact set $W \subseteq$ that satisfies $\overline{\mathrm{int}(W)} = W$ and is called *window* in this context, a CPS defines a *model set* or *cut and project set* by

$$\Lambda(W) \;=\; \pi_G(\mathcal{L} \cap (G \times W)) \,,$$

where $\pi_G : G \times H \to G$ denotes the projection to the first coordinate. Under the above assumptions, the resulting model set $\Lambda(W)$ is always Delone (relatively dense and uniformly discrete) [10]. CPS were introduced by Meyer in 1972 [9] and have emerged as one of the main constructions to obtain aperiodic structures. In particular, paradigmatic examples such as the Fibonacci quasicrystal or the Penrose tiling can be represented as model sets.

**Hull dynamics and torus parametrization.**  Given a suitable topology on the space of Delone sets, a model set $\Lambda(W)$ defines a topological dynamical system, which is given by the action of $G$ on the dynamical hull

$$\Omega(\Lambda(W)) \;=\; \overline{(\{\Lambda(W) - t \mid t \in \Gamma\})} \,.$$

An important fact for the analysis of this system is the existence of a *torus parametrization* (see [13, 2]), that is, a flow morphism $\beta : \Omega(\Lambda(W)) \to \mathbb{T} = (G \times H)/\mathcal{L}$ from the action on the hull, $(\Omega(\Lambda(W)), G)$, to the canonical $G$-action on the *'torus'* $G$ given by

$$G \times \mathbb{T} \to \mathbb{T} \,, \quad (t, (g, h) + \mathcal{L}) \mapsto (g + t, h) + \mathcal{L} \,.$$

Thereby, the map $\beta$ is uniquely defined by the condition

$$\beta(\Gamma) = (g, h) + \mathcal{L} \quad \Leftrightarrow \quad \Lambda(\mathrm{int}(W) + h) - g \subseteq \Gamma \subseteq \Lambda(W + h) - g \,.$$

**Regular model sets.**  One case which is quite well-understood is that of *regular model sets*, by which we mean model sets $\Lambda(W)$ for which $|\partial W| = 0$, where $|\,.\,|$ denotes the Haar measure on $H$. The reason is the fact that in this situation the flow morphism $\beta$ is almost surely 1-1, that is, $\beta^{-1}((g, h) + \mathcal{L}))$ is a singleton for $\mu$-almost every $(g, h) + \mathcal{L} \in \mathbb{T}$ with respect to the Haar measure $\mu$ on $\mathbb{T}$. This

further entails that the system $(\Omega(\Lambda(W)), G)$ is uniquely ergodic and isomorphic to its factor $(\mathbb{T}, G)$ and has pure point dynamical spectrum and zero topological entropy [13, 2]. Moreover, it can be shown that the diffraction spectrum of $\Lambda(W)$ (which we will not define in detail here) is pure point as well [8]. The latter gives a motivation to consider regular model sets as appropriate models for quasicrystals.

**Irregular model sets.** In contrast to this, a situation that is much less understood is that of *irregular model sets*, that is, of windows with $|\partial W| > 0$. In this case, one expects that 'typically' the dynamics of $\Omega(\Lambda(W))$ should be more complex, and a number of questions have been raised in the literature in this direction. In particular, we want to point out the following two problems, which are attributed to Moody (see [11, 12]) and Schlottmann [13], respectively.

- Does $|\partial W| > 0$ imply positive topological entropy (see [11, 12])?
- Does $|\partial W| > 0$ imply unique ergodicity [13]?

In order to address these questions, we consider two different settings.

**Toeplitz flows.** The first is that of so-called *Toeplitz flows*. A sequence $\xi = (\xi_n)_{n \in \mathbb{Z}} \in \Sigma = \{0, 1\}^{\mathbb{Z}}$ is called a *Toeplitz sequence* if it is aperiodic[1] and for all $n \in \mathbb{Z}$ there exists a *period* $p \in \mathbb{N}$ such that $\xi_{n+kp} = \xi_n$ for all $k \in \mathbb{Z}$. In other words, every symbol in a Toeplitz sequence is repeated periodically, but the period depends on the position $n$ of the symbol. If we let $\mathrm{Per}(\xi, p) = \{n \in \mathbb{Z} \mid \xi_{n+kp} = \xi_n$ for all $k \in \mathbb{Z}\}$, then $\xi$ is Toeplitz if and only if $\bigcup_{p \in \mathbb{N}} \mathrm{Per}(\xi, p) = \mathbb{Z}$.

For any Toeplitz sequence $\xi$, one can choose a *period structure* $(p_\ell)_{\ell \in \mathbb{N}}$ of integers such that $p_\ell$ divides $p_{\ell+1}$ and $\bigcup_{\ell \in \mathbb{N}} \mathrm{Per}(\xi, p_\ell) = \mathbb{Z}$. Let $q_\ell = p_{\ell+1}/p_\ell$ and denote by $\Omega = \prod_{\ell=1}^{\infty} \mathbb{Z}/q_\ell\mathbb{Z}$ the corresponding odometer with minimal group rotation $R$. Then there exists a flow morphism $\pi$ from subshift given by the orbit closure of $\xi$ to the odometer $(\Omega, R)$. Note that different period structures for a given Toeplitz sequence always define the same odometers up to isomorphism (see [3]).

An important distinction between two basic types of Toeplitz flows is the following. For any $p \in \mathbb{N}$, we denote by $D(\xi, p) = \#(\mathrm{Per}(\xi, p) \cap [0, p-1])/p$ the density of the $p$-periodic positions. If $\lim_{\ell \to \infty} D(\xi, p_\ell) = 1$, then $\xi$ is called a *regular Toeplitz sequence*. Otherwise $\lim_{\ell \to \infty} D(\xi, p_\ell) < 1$ and the Toeplitz sequence $\xi$ is called *irregular*. In the regular case, the above flow morphism $\pi$ is $\nu$-almost surely one-to-one, where $\nu$ is the Haar measure on $\Omega$. Hence, similar to the situation for model sets, regular Toeplitz flows are uniquely ergodic and isomorphic to the corresponding odometer and consequently have zero topological entropy and purely discrete dynamical spectrum. These analogies are no coincidence.

**Theorem 1** ([1]). *If $\xi$ is a Toeplitz sequence and $\Omega$ is the corresponding odometer, then the point set $\Lambda_\xi = \{n \in \mathbb{Z} \mid \xi_n = 1\}$ can be represented as a model set with CPS $(\mathbb{Z}, \Omega, \mathcal{L})$, where $\mathcal{L} = \{(n, R^n(0)) \mid n \in \mathbb{Z}\}$ and the window $W$ satisfies $|\partial W| = 1 - \lim_{\ell \to \infty} D(\xi, p_\ell)$.*

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[1]Here $\xi \in \Sigma$ is called *aperiodic* if $\sigma^p(\xi) \neq \xi$ for all $p \in \mathbb{N}$, where $\sigma : \Sigma \to \Sigma$ denotes the left shift map.

Hence, any Toeplitz system can be interpreted and used as an example in the context of model sets, and the notion of regularity coincides in both settings. As there exist examples of irregular Toeplitz flows which are uniquely ergodic and have zero entropy, this allows to give a negative answer to the above questions by Moody and Schlottmann (and a variety of further questions in the same direction).

**Irregular model sets in Euclidean CPS.** The Toeplitz examples allow to answer the above questions in the general setting, where arbitrary locally compact Abelian groups are allowed in the CPS. However, this still leaves the possibility that stronger restrictions exist in the Euclidean setting, where both $G$ and $H$ are Euclidean spaces. In this situation, the following result guarantees that a positive measure of the window boundary "typically" leads to positive entropy (where "typical" is understood in a probabilistic sense).

**Theorem 2** ([7]). *Suppose $(\mathbb{R}^N, \mathbb{R}, \mathcal{L})$ is a Euclidean CPS, $C \subseteq \mathbb{R}$ is a Cantor set of positive measure, $(G_n)_{n \in \mathbb{N}}$ is a numbering of the gaps of $C$, $\omega \in \{0,1\}^{\mathbb{N}}$ and*

$$W(\omega) \ = \ C \cup \bigcup_{n \in \mathbb{N}: \omega_n = 1} G_n \ .$$

*Then for $\mathbb{P}$-almost every $\omega$ we have $\overline{int(W)} = W$, $|\partial W| = C$ and $(\Omega(\Lambda(W)), \mathbb{R}^N)$ has positive entropy, where $\mathbb{P}$ refers to an arbitrary Bernoulli measure on $\{0,1\}^{\mathbb{N}}$.*

However, even in the Euclidean case there exist exceptions.

**Theorem 3** ([4]). *Given any CPS $(\mathbb{R}, \mathbb{R}, \mathcal{L})$, there exists a window $W$ with $|\partial W| > 0$ such that $(\Omega(\Lambda(W)), \mathbb{R})$ is uniquely ergodic and has zero topological entropy.*

**CPS and symbolic dynamics.** Finally, we want to close with an announcement of a result that has been obtained during and shortly after the week in Oberwolfach and was inspired by discussions with Eli Glasner and Felipe García-Ramos during this time. It can therefore be considered a direct outcome of the workshop.

We say a minimal subshift $\Sigma \subseteq \{0,1\}^{\mathbb{Z}}$ is *almost automorphic* if it has a maximal equicontinuous factor $(\Omega, \rho)$ for which the corresponding factor map $\pi$ is almost one-to-one (there exists a point with unique preimage).

**Fact 4.** *Any minimal almost automorphic subshift $(\Sigma, \mathbb{Z})$ is equivalent (up to conjugacy) to the system $(\Omega(\Lambda(W)), (Z))$ obtained from the CPS $(\mathbb{Z}, \Omega, \mathcal{L})$ with lattice $\mathcal{L} = \{(n, \rho^n(\omega_0) \mid n \in \mathbb{Z}\}$, where*

- *$\omega_0 \in \Omega$ has unique preimage under the factor map $\pi$;*
- *$W = \pi([1])$, where $[1] = \{\xi \in \{0,1\}^{\mathbb{Z}} \mid \xi_0 = 1\}$.*

*Moreover, the window $W$ satisfies the topological regularity condition $\overline{int(W)} = W$.*

Analogous to the above situation, $(\Sigma, \mathbb{Z})$ is called *regular* if $\pi$ is $\nu$-almost surely one-to-one, where $\nu$ denotes the unique invariant probability measure on $\Omega$, and *irregular* otherwise. As the examples discussed above already indicate, a subshift may be uniquely ergodic and have zero entropy even if it is regular. This prompts

the obvious question whether irregularity has any dynamical consequences at all. Here, the CPS formalism can be used as a tool to obtain a positive answer. We say *the subshift $(\Sigma, \mathbb{Z})$ has an infinite free set*, if there exists an infinite set $S \subseteq \mathbb{Z}$ such that for any $a \in \{0,1\}^S$ there exists $\xi \in \Sigma$ such that $\xi_s = a_s$ for all $s \in S$.[2]

**Theorem 5.** *If a minimal almost automorphic subshift is irregular, then it has an infinite free set. In particular, it has positive topological sequence entropy.*

The advantage of the CPS formalism in this context is the fact that it translates this dynamical problem into a purely topological questions concerning the structure of the window, which is easier to address. An analogous statement can be obtained for arbitrary irregular model sets with more general groups $G$ and $H$. An important consequence concerns the notion of *tame systems*. (See [5, 6] for a definition and discussion of this notion.) Due to work of Glasner and Megrelishvili [6], it is known that tame subshifts do not allow infinite free sets. Hence, we obtain

**Corollary 6.** *For minimal almost automorphic subshifts, tame implies regular.*

The analogous result for Toeplitz flows is due to Downarowicz. An extension to more general model sets will be the subject of future research.

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[2]Note that if $S$ has positive asymptotic density, then this implies positive entropy for $(\Sigma, \mathbb{Z})$, but this is not required.

# Homeomorphisms between tiling spaces

Antoine Julien

(joint work with Lorenzo Sadun)

A common method for studying aperiodic tilings is to study a *topological space* associated with a given tiling, rather than the tiling itself. This approach is fruitful because properties of the space reflect properties of the tiling.

This talk, based on the results of [3], addresses the following two questions:

- Whenever two spaces are equivalent what can be said about the tilings?
- What remains of a tiling space when one forgets about the tiling?

The answer to the first question depends of course of what is meant by *equivalent*. Two tiling spaces are equivalent whenever there is a map between them which preserves some *structure*. How much structure is given to a space by the virtue of being a tiling space is the answer to the second question.

Given a tiling $T$ of $\mathbb{R}^d$, its space is $\Omega := \overline{\{T - x \; : \; x \in \mathbb{R}^d\}}$, where the closure in the set of all tilings is taken for an appropriate topology. It is an $\mathbb{R}^d$-dynamical system. We assume that $T$ is aperiodic, repetitive, with finite local complexity — FLC (meaning $T$ has finitely many patterns of size $r$, for all $r$). This implies that $\Omega$ is compact, minimal and has no periodic orbit.

The tiling space in itself can be given several structures:

(1) it is a topological space;
(2) with an orbit structure (equivalence relation inherited by the $\mathbb{R}^d$-action);
(3) it is a dynamical system (a specific parametrization of the orbits by $\mathbb{R}^d$);
(4) with a certain transverse "rigid" structure.

Let us specify point (4). It is known that a tiling space with finite local complexity can be given an atlas of charts in which neighbourhoods are all homeomorphic to $B(0, r) \times X$ where $B(0, r)$ is an Euclidean ball of $\mathbb{R}^d$ and $X$ is a Cantor set. The image in $\Omega$ of a set of the form $\{0\} \times X$ is called a *vertical transversal*. The translate of a vertical transversal is still a vertical transversal (or a finite union of such sets), and the property of being a (finite union of) vertical transversals does not depend on the chart. Spaces having such a transverse structure are sometimes called *tileable laminations* in the literature.

We can now categorize maps between tiling spaces according to which structure they preserve. We will always require our maps to be continuous (or homeomorphisms if invertible). Because of the local structure of FLC tiling spaces, such a map always sends path-connected component to path-connected component hence orbit to orbit. The weakest notion of equivalence we therefore consider is *orbit-equivalence*, i.e., a homeomorphism sending orbit to orbit. Additional structure can be preserved:

- maps preserving (3) are the *topological conjugacies*;
- maps preserving (3) and (4) are the *mutual local derivations* (MLD), as introduced in [1];

- maps preserving (4) are called *local* maps. The archetypes of local maps between tiling spaces are given by *shape changes* (or *tiling deformations*), see [2].

For example: consider the Fibonacci sequence given by the substitution $a \mapsto ab; b \mapsto a$. We can consider two tiling spaces based on it: one in which the $a$ and $b$-tiles have respective length $\phi$ and 1 (with $\phi$ the golden ratio); one with constant tile length $c$. It is known that these two spaces are conjugate for a good choice of $c$: they are the same dynamical system. However, their canonical transverse structures are different, and it is not preserved by the conjugacy.

In essence, our work establishes that any homeomorphism between FLC tiling spaces is within bounded distance of (and isotopic to) a homeomorphism which preserves the transverse structure. In the results below, all spaces are aperiodic, minimal, FLC tiling spaces.

**Theorem 1.** *Let $h : \Omega \to \Omega'$ be a homeomorphism between tiling spaces. Assume $\Omega$ is uniquely ergodic. Then there exists $\alpha : \Omega \to \Omega'$ a homeomorphism, which is* local *in the sense above, such that $h = \alpha \circ \tau_s$, where $\tau_s(T) = T - s(T)$ for some continuous (hence bounded) function $s : \Omega \to \mathbb{R}^d$.*

While any homeomorphism between two tiling spaces is isotopic to a homeomorphism which preserves (4), there is also a topological invariant measuring how this map changes the parametrization of the orbits by the action.

**Theorem 2.** *A continuous map $h : \Omega \to \Omega'$ defines a cohomology class $[h] \in \check{H}^1(\Omega; \mathbb{R}^d)$. Furthermore, whenever $h_1, h_2$ are homeomorphisms $h_i : \Omega \to \Omega_i$ such that $[h_1] = [h_2]$, then there exists an MLD map between $\Omega_1$ and $\Omega_2$.*

If $\Omega$ is uniquely ergodic, one defines $C_\mu : \check{H}^1(\Omega; \mathbb{R}^d) \to M_d(\mathbb{R})$, a matrix-valued map (see [4]). A non-singular class in $\check{H}^1$ is a class having a non-singular image under $C_\mu$. Given a homeomorphism $h$, the matrix $C_\mu[h]$ describes how $h$ maps orbits to orbits at large scales.

**Theorem 3.** *Given a homeomorphism $h : \Omega \to \Omega'$ (with $\Omega$ uniquely ergodic), $C_\mu[h]$ is a non-singular matrix. Conversely, for any non-singular $[\alpha] \in \check{H}^1(\Omega; \mathbb{R}^d)$, there exists an FLC tiling space $\Omega_\alpha$ and a homeomorphism $h_\alpha : \Omega \to \Omega_\alpha$ such that $[h_\alpha] = [\alpha]$.*

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# Dynamics of $B$-free sets

MARIUSZ LEMAŃCZYK

Given an infinite $B \subset \mathbb{N} \setminus \{1\}$, the set

$$\mathcal{F}_B := \{n \in \mathbb{Z} : \text{no } b \in B \text{ divides } n\}$$

is called the set of $B$-free numbers. Although such sets need not possess asymptotic density, they always have logarithmic density (Davenport–Erdös theorem from 1936). Prominent examples of $B$-free sets are: the set of square-free numbers, the set of deficient numbers or even the set of prime numbers itself (consider $B = \{pq : p, q \text{ are primes}\}$). It is not hard to see that $A \subset \mathbb{Z}$ is $B$-free, i.e. $A = \mathcal{F}_B$ (for some $B$) if and only if $A$ is closed under taking divisors. By setting $\eta$ to be the characteristic function of $\mathcal{F}_B$ and treating it as a point in the shift space $\{0,1\}^{\mathbb{Z}}$, we obtain the subshift $(X_\eta, S)$, where $S$ stands for the left shift. $B$-free subshifts constitute an important class of examples of dynamical systems arising in the cut-and-project scheme. Indeed, set $G = \mathbb{Z}$ for the physical space, $H = \prod_{b \in B} \mathbb{Z}/b\mathbb{Z}$ for the external space (we assume for simplicity that $B$ is coprime but this is not essential), $\mathcal{L} = \{(n, \underline{n}) : n \in \mathbb{Z}\}$ for the lattice in $G \times H$ (here, $\underline{n} = (n, n, \ldots)$) and $W = \{h = (h_b) \in H : h_b \neq 0 \text{ for all } b \in B\}$ for the window. However, some other natural subshifts appear in this context, for example, $(X_B, S)$ the subshift of $B$-admissible sequences (those sequence whose support taken mod an arbitrary $b \in B$ misses at least one residue class mod $b$). It is not hard to see that $X_\eta \subset X_B$ and the latter set is hereditary. This yields

$$X_\eta \subset \widetilde{X}_\eta \subset X_B,$$

where $\widetilde{X}_\eta$ stands for the hereditary closure of $X_\eta$.

The talk, based mainly on two recent papers [2] and [4], focuses on dynamical properties of subshifts given by $B$-free sets. Several theorems classifying characteristic properties of such subshifts are usually given as an equivalence between dynamical, topological and arithmetical viewpoints. For example, the proximality (dynamics) of $(X_\eta, S)$ is equivalent to the fact that $B$ contains an infinite coprime subset (arithmetic) which in turn is equivalent to $\text{Int}(W) = \varnothing$ (topology). In this spirit we go through proximality, minimality (which turns out to be closely related to the theory of Toeplitz systems), Behrend property and, the most surprising part concerning the tautness property of $B$. Tautness is a classical, purely arithmetical property telling us that, for each $b \in B$, the logarithmic density of $\mathcal{F}_{B \setminus \{b\}}$ is strictly larger than the logarithmic density of the original $B$-free set. Surprisingly, this condition is equivalent to the fact that the restriction of Haar measure to the window has full (topological) support. Some of these developments have profited from the recent progress on weak model sets [5, 1], which give an alternative description of such systems and various generalizations to higher dimensions.

In turn, from the dynamical point of view, taut $B$-free systems have the property that their Mirsky measure is supported by the set with the maximal number of residue classes. Moreover, from the ergodic theory point of view, only taut systems are interesting. Indeed, for each $B$-free system there is a unique taut $B'$-free system

such that the locus of all invariant measures of $(\widetilde{X}_\eta, S)$ is given by $\widetilde{X}_{\eta'}$. We will also discuss the entropy problems (the entropies of $B$-admissible and the hereditary closure of $X_\eta$ turn out to be the same). Finally, as an application, we show a role of $B$-free sets in the multiple recurrence problems in dynamics discussed in [3], which in particular yields a reinforcement of the famous Szemerédi theorem on arithmetic progressions precising that the difference of such progressions can be taken from a self-shift of a $B$-free set whenever $B$ is taut.

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## Fourier quasicrystals and Poisson summation formulas

### Nir Lev

### (joint work with Alexander Olevskii)

By a Fourier quasicrystal one often means an (infinite) pure point measure $\mu$ on $\mathbb{R}^d$ whose Fourier transform is also a pure point measure. The classical example of such a measure is the sum of unit masses over a lattice, and the spectrum is the dual lattice.

The subject has received a new peak of interest after the experimental discovery in the middle of the 80's of non-periodic atomic structures with diffraction patterns consisting of spots. The "cut-and-project" construction, introduced by Y. Meyer in the beginning of the 70's, may serve as a good model for this phenomenon. It provides many examples of measures with uniformly discrete support and dense countable spectrum.

On the other hand, we proved with A. Olevskii [1, 2] that if both the support and the spectrum of a measure on $\mathbb{R}$ are uniformly discrete sets, then the measure has a periodic structure. A similar result was proved for positive measures on $\mathbb{R}^d$. In our paper [3] with A. Olevskii we establish in a strong sense the sharpness of the uniform discreteness requirement in this result. Namely, we proved there the existence of a measure $\mu$ on $\mathbb{R}$ whose support and spectrum are both discrete closed sets, but such that the support contains only finitely many elements of any arithmetic progression. The latter result thus reveals the existence of "non-classical" Poisson summation formulas.

In the crystallography community, it seems to be commonly agreed that the support of the measure $\mu$ should be a uniformly discrete set. So it is a natural problem, to what extent can the spectrum of a non-periodic quasicrystal be discrete, assuming that the support is uniformly discrete? In our paper [4] with A. Olevskii we address this problem, and consider quasicrystals with non-symmetric discreteness assumptions on the support and the spectrum. We obtain several results which show that, under various conditions, if the spectrum is a discrete closed set, then in fact it must be uniformly discrete. These results thus reduce the situation to the setting in [2], which in turn allows us to conclude that the measure has a periodic structure. On the other hand, we present an example of a non-periodic quasicrystal such that the spectrum $S$ is a nowhere dense countable set. Finally, we extend our results to the more general situation, where the Fourier transform of the measure $\mu$ has both a pure point component and a continuous one.

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## On the union of spectra for all Sturm potentials

QINGHUI LIU

(joint work with Bassam Fayad, Yanhui Qu)

### 1. INTRODUCTION

Taking $V > 0$, irrational $\alpha \in (0,1)$ and $\theta \in [0,1)$, the Schrödinger operator with Sturm potential $H_{V,\alpha,\theta}$ acting on $l^2(\mathbb{Z})$ is defined by, for any $(\phi(n))_{n \in \mathbb{Z}} \in l^2(\mathbb{Z})$,

$$\left(H_{V,\alpha,\theta}\phi\right)(n) = \phi(n+1) + \phi(n-1) + v_n\phi(n),$$

where $v_n = V\chi_{[1-\alpha,1[}(\{n\alpha + \theta\})$ and $\chi_{[1-\alpha,1[}$ is the characteristic function, and $V$ is called coupling, $\alpha$ is called frequency, $\theta$ is called phase. Since the spectrum $\sigma(H_{V,\alpha,\theta})$ is independent of $\theta$, we take $\theta = 0$ and denote the operator by $H_{V,\alpha}$.

We study the union of spectra of the Schrödinger operator with Sturm potential of fixed coupling and all frequencies, i.e, for any $V > 0$, the set

$$S_V = \bigcup_{\alpha \in \mathbb{Q}^c \cap (0,1)} \sigma(H_{V,\alpha}).$$

**Theorem 1.** [1] $\mathfrak{L}(\sigma(H_{V,\alpha})) = 0$.

In this paper, they show that $\sigma(H_{V,\alpha}) \subset [-2,2] \cup [-2+V, 2+V] = [-2,2] + \{0,V\} := \Gamma$. Notice that $[-2,2] = \{2\cos t\pi \ : \ 0 \leq t \leq 1\}$, we have

**Theorem 2** (Fayad, Liu, Qu, preprint). *For $V > 0$, there exists $\Theta \subset \Gamma$ at most countable such that*

$$S_V = \Gamma \backslash \Theta,$$

*where*

$$\Theta \subset \{2\cos t\pi \ : \ 0 \leq t \leq 1, rational\} + \{0,V\}$$
$$\Theta \supset \{\pm 2, 0, \pm 2 + V, V\} = \{2\cos t\pi \ : \ t = 0, 1/2, 1\} + \{0,V\}, V > 4$$
$$\Theta \cap \left(\{2\cos t\pi \ : \ \varepsilon < t < 1 - \varepsilon\} + \{0,V\}\right) \ is \ a \ finite \ set.$$

## 2. Transfer matrix and trace polynomial

Define the transfer matrices by

$$T_n(E) = \begin{bmatrix} E - v_n & -1 \\ 1 & 0 \end{bmatrix}$$

and by $T_{1\to n}(E) = T_n(E)T_{n-1}(E)\cdots T_1(E)$.

For $\alpha \in [0,1]\backslash\mathbb{Q}$, let $\alpha = [0; a_1, a_2, \cdots]$ be the continued fraction expansion. For any $k \geq 0$, let $p_k/q_k = [0; a_1, a_2, \cdots, a_k]$, which satisfies,

$$p_{-1} = 1, \qquad p_0 = 0, \qquad p_{k+1} = a_{k+1}p_k + p_{k-1}, \ k \geq 0,$$
$$q_{-1} = 0, \qquad q_0 = 1, \qquad q_{k+1} = a_{k+1}q_k + q_{k-1}, \ k \geq 0.$$

For $k \geq 0$, define

$$M_k(E) := T_{1\to q_k}(E)$$
$$x_k(E) := \operatorname{tr} M_k(E)$$
$$\sigma_k := \{E \in \mathbb{R} \ : \ |x_k(E)| \leq 2\}.$$

Note that $x_k(E)$ is a polynomial with degree $q_k$.

**Theorem 3.** [1] *One has*

$$M_{k+1}(E) = M_{k-1}(E)M_k^{a_{k+1}}(E), \ \ \forall k \geq 0,$$
$$\sigma(H_{V,\alpha}) = \bigcap_{k\geq 0}(\sigma_{k-1} \cup \sigma_k),$$

*where*

$$M_{-1}(E) \equiv \begin{bmatrix} 1 & -V \\ 0 & 1 \end{bmatrix}, \quad M_0(E) = \begin{bmatrix} E & -1 \\ 1 & 0 \end{bmatrix},$$

*and $\sigma_k := \{E \in \mathbb{R} \ : \ |\operatorname{tr} M_k(E)| \leq 2\}$ for $k = 0, -1$.*

Note that $x_{-1}(E) := \operatorname{tr} M_{-1}(E) \equiv 2$, and $x_0(E) := \operatorname{tr} M_0(E) = E$.

## 3. Sketch of $\big\{2\cos t\pi : t \in [0,1]\backslash\mathbb{Q}\big\} \subset S_V$

We choose $a_k$ step by step. The idea in the proof comes from [4, 3].

**Lemma 1.** *For $E \in \mathbb{R}$, $k \geq 0$, if $\big|x_{k-1}(E)\big| < 2$, $\big|x_k(E)\big| < 2$, then there exists $a_{k+1}$ such that $\big|x_{k+1}(E)\big| < 2$.*

**Corollary 1.** *For $E \in \mathbb{R}$, if there exists $k \geq 0$ so that $|x_{k-1}(E)| < 2$, $|x_k(E)| < 2$, then $E \in S_V$.*

**Proposition 1.** *If $t \in [0,1]$ be irrational, then there exists $a_1 > 0$ so that*

$$\big|x_0(2\cos t\pi)\big| < 2, \quad \big|x_1(2\cos t\pi)\big| < 2,$$

*i.e., $2\cos t\pi \in S_V$.*

## 4. If $V > 4$, then $2 \notin S_V$

**Lemma 2.** [1] *For any $V > 0$, $\alpha$ irrational and $E \in \mathbb{R}$, $(x_k(E))_{k\geq -1}$ grow exponentially if and only if there exists $k \geq 0$ such that*

$$\big|x_{k-1}(E)\big| \leq 2, \quad \big|x_k(E)\big| > 2, \quad \big|x_{k+1}(E)\big| > 2.$$

**Lemma 3.** [2] *For any $V > 0$, $\alpha$ irrational, $\delta \geq 0$ and $E \in \mathbb{C}$, $(x_k(E))_{k\geq -1}$ grow exponentially if and only if there exists $k \geq 0$ such that*

$$\big|x_{k-1}(E)\big| \leq 2 + \delta, \quad \big|x_k(E)\big| > 2 + \delta, \quad \big|\operatorname{tr} M_{k-1}M_k(E)\big| > 2 + \delta.$$

We can modify these results by

**Lemma 4.** *Take any $V > 0$, $\alpha$ irrational, and $E \in \mathbb{C}$. If there exists $k \geq 0$ such that*

$$\big|x_{k-1}(E)\big| \leq 2, \quad \big|x_k(E)\big| \geq 2, \quad \big|\operatorname{tr} M_{k-1}M_k(E)\big| > 2,$$

*then $\big(x_k(E)\big)_{k\geq -1}$ grow exponentially.*

Since $x_{-1}(2) = 2$, $x_0(2) = 2$, $\operatorname{tr} M_{-1}M_0(2) = 2 - V$, we have $2 \notin S_V$.

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# On continuous and measure-theoretical eigenvalues of minimal Cantor systems and applications

Alejandro Maass

(joint work with Fabien Durand, Alexander Frank)

The study of eigenvalues of topological dynamical systems, either from a measure-theoretical or a topological perspective, is a fundamental topic in ergodic theory. Particularly interesting and rich has been the study of eigenvalues and weakly mixing properties of classical systems like interval exchange transformations or other systems arising from translations on surfaces. From the symbolic dynamics point of view most of these systems have representations as minimal Cantor systems of finite topological rank, i.e., there is a symbolic extension that can be represented by a Bratteli–Vershik system such that the number of Kakutani–Rohlin towers per level is globally bounded. To characterize eigenvalues of the original systems it is enough to consider this class of Cantor systems.

With these examples in mind and extensions to the study of tiling systems, our main motivation is to provide general necessary and sufficient conditions for a complex number to be the eigenvalue, either continuous or measure-theoretical, of a minimal Cantor system of finite topological rank and when possible to get the same kind of results for any minimal Cantor system.

Some results for different subclasses of minimal Cantor systems of finite topological rank have been produced since the pioneering work of Dekking [5] and Host [11]. There, it was stated that measurable eigenvalues of primitive substitution dynamical systems are always associated to continuous eigenfunctions. Later, necessary and sufficient conditions to characterize continuous and measurable eigenvalues of linearly recurrent minimal Cantor systems were provided in [3] and [1]. These conditions are very effective and rely on the combinatorial data carried by their Bratteli–Vershik representations. Even if linearly recurrent systems are natural from the symbolic dynamics point of view (see [6, 7]), this class is "small", meaning that in many classical cases, like interval exchange transformations, only a few maps have a symbolic representation of this kind. In fact, most of them are of finite topological rank and not linearly recurrent. There are few general results concerning eigenvalues of minimal Cantor systems of finite topological rank. Some preliminary results are given in [2] and a detailed study of eigenvalues of Toeplitz systems of finite topological rank is given in [8].

After reviewing the results described above we provide novel necessary and sufficient conditions that a complex number should satisfy to be a measurable eigenvalue of a minimal Cantor system of finite topological rank (we follow [9]). In addition, we give a necessary and sufficient condition for a complex number to be a continuous eigenvalue of a minimal Cantor system, that is, we succeeded in dropping the finite rank hypothesis. In its conception, the conditions are very similar to those proposed for linearly recurrent systems. They are given in the form of the convergence of some series or special sequences and only depend on the combinatorial data provided by the Bratteli–Vershik representations. The main

difference here is that we need to include in an algebraic way the information of the local orders carried by these representations.

We illustrate the use of the conditions giving examples and applications. First we prove that our conditions extend the results in [8] to characterize eigenvalues of finite rank Toeplitz minimal systems. Then, a first application relates the notions of continuous eigenvalues and strong orbit equivalence. We use our necessary and sufficient condition in the continuous case to prove that, by doing controlled modifications of the local orders of a Bratteli–Vershik system, one can alter the group of continuous eigenvalues. In particular, starting from a minimal Cantor system without roots of unity as continuous eigenvalues we produce a strong orbit equivalent system that is topologically weakly mixing and which shares the Kronecker factor with the original system for any ergodic measure. In [12] a similar example is developed in the context of tiling systems. In a second example, the conditions to be measurable eigenvalues and previous application are used to construct a topologically weakly mixing minimal Cantor system of rank two admitting all rational numbers as measure theoretical eigenvalues, showing that topological rank is not an obstruction to have non continuous rational eigenvalues as in the Toeplitz case. Finally, inspired by questions in [4] and [10], we use our main theorems to produce an expansive minimal Cantor system whose group of continuous eigenvalues coincides with the intersection of the images of the so-called group of traces.

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# Spectral analysis of primitive inflation rules

Neil Mañibo

(joint work with Michael Baake, Michael Coons, Nathalie P. Frank, Franz
Gähler, Uwe Grimm and E. Arthur Robinson Jr.)

The spectral analysis of inflation rules is explained for a characteristic class of
examples in one dimension. We determine the spectral type of the diffraction
measure $\widehat{\gamma}$ for the one-parameter family of binary substitutions given by

$$(1) \qquad\qquad \varrho_m : \quad 0 \mapsto 01^m, \quad 1 \mapsto 0$$

with corresponding inflation factor $\lambda = \lambda_m$, where $m \geqslant 2$ (we exclude $m = 1$ be-
cause this is the well-known Fibonacci inflation). We achieve this by using the geo-
metric realization as an inflation tiling with prototiles (intervals) of natural length
[5] and by examining exact renormalization equations for the corresponding pair
correlations $\nu_{ij}(z)$; see [3], as well as Eq. (6) below for a generalization to arbitrary
dimension. These relations extend to a (measure-valued) renormalization equation
for the measure vector $\Upsilon$, whose components are given by $\Upsilon_{ij} = \sum_{z \in \Lambda} \nu_{ij}(z)\, \delta_z$.
These components determine the autocorrelation $\gamma$ for general weights $u_i \in \mathbb{C}$ via
a simple quadratic form [2]. All measures $\Upsilon_{ij}$ are Fourier transformable.

Via Fourier transform, one obtains a measure-valued renormalization equation
for $\widehat{\Upsilon}$; compare [3, 2, 6]. This new equation, which holds for each of the three
spectral types (pp, sc and ac) separately, involves the Fourier matrices $B(k)$,
where $\overline{B_{ij}(k)} = \widehat{\delta_{T_{ij}}}(k)$ with $T_{ij}$ being the set of positions of tiles of type $i$ in level-1
supertiles of type $j$. In particular, for the absolutely continuous components, when
described by a vector $h$ of Radon–Nikodym densities, this implies an iterative
equation for a.e. $k \in \mathbb{R}$,

$$(2) \qquad\qquad h(\lambda k) = \lambda \left( B^{-1}(k) \otimes \overline{B^{-1}(k)} \right) h(k).$$

This iteration can be reduced to an equation of lower dimension [2, 4], namely

$$(3) \qquad\qquad v(\lambda k) = \sqrt{\lambda}\, B^{-1}(k)\, v(k),$$

where $h_{ij}(k) = v_i(k)\, \overline{v_j(k)}$. Exponential growth of $\|v(k)\|$ implies an exponential
growth of the norm of $h$ and hence contradicts the translation-boundedness of the
corresponding measure $\widehat{\Upsilon}_{\text{ac}}$ if $v(k) \neq 0$ for a subset of positive measure [2]. To
rule out the existence of a non-trivial component $\widehat{\Upsilon}_{\text{ac}}$, it thus suffices to show that,
for any chosen $0 \neq v(k) \in \mathbb{C}^2$ and a.e. $k \in \mathbb{R}$, $\|v(k)\|$ grows exponentially under
the iteration (3). One way to analyse this is to obtain bounds for the Lyapunov
exponents of the associated cocycle $B^{(n)}(k) = B(k)B(\lambda k)\cdots B(\lambda^{n-1}k)$; see [14]
for background. This was done rigorously for $m = 3$ in [2] and extended to the
entire family in [6].

These substitutions give rise to 2-dimensional cocyles, which ensures that there
can be at most two distinct exponents, denoted by $\chi_{\min}$ and $\chi_{\max}$. Whenever $\lambda$ is
not an integer, i.e., for cases other than $m = \ell(\ell + 1)$ with $\ell \in \mathbb{N}$, the existence of
these exponents and Lyapunov regularity for a.e. $k$ are not guaranteed (Oseledec's

theorem does not necessarily apply here [2]). Nevertheless, it can be shown that the exponents (then defined via a lim sup) add up to $\log(\lambda_m)$ for all $m$ via some extension of Sobol's theorem to almost periodic functions [7]. A useful sufficient criterion for the positivity of all exponents, i.e., the positivity of the smallest exponent, is given by

$$(4) \qquad\qquad \log(\lambda_m) \; > \; \mathbb{M}\left(\log \|B(k)\|_{\mathrm{F}}^2\right),$$

where $\mathbb{M}$ denotes the mean of a function and $\|.\|_{\mathrm{F}}$ is the Frobenius norm. It can be shown [4] that this mean is recoverable as the logarithmic Mahler measure $\mathfrak{m}(q_m)$ of a polynomial $q_m \in \mathbb{Z}[x]$ given by

$$(5) \qquad\qquad q_m(x) \; = \; 2x^{m-1} + (1 + x + x^2 + \ldots + x^{m-1})^2.$$

Furthermore, one finds that this family of Mahler measures is bounded. In particular, it is dominated by $\log(\lambda_m)$ for all $m \geqslant 18$; see [6] for a proof. For $m < 18$, the smallest exponent has a bound that depends on the mean of a quasiperiodic function $\frac{1}{N}\mathbb{M}\left(\log \|B^{(N)}(k)\|_{\mathrm{F}}^2\right)$ with two incommensurate frequencies, which cannot be expressed as a one-dimensional Mahler measure. However, this mean is computable as a finite integral over $\mathbb{T}^2$, and hence an appropriate $N$ can be chosen so that this quantity is surpassed by $\log(\lambda)$. From this, we conclude the desired positivity by invoking a one-sided inequality due to some version of the subadditive ergodic theorem; see [2, 12]. This confirms the absence of $\widehat{\Upsilon}_{\mathsf{ac}}$ for all $m$, of which $\widehat{\gamma}_{\mathsf{ac}} = 0$ is an immediate consequence.

For all $\varrho_m$ with non-Pisot inflation multiplier, this means that the diffraction is singularly continuous (except for the Bragg peak at $k = 0$, which corresponds to the constant eigenfunction of the inflation dynamical system). In contrast, the systems with an integer inflation multiplier ($m = \ell(\ell + 1)$ and $\lambda_m = \ell + 1$ for some $\ell \in \mathbb{N}$) are MLD to constant-length substitutions with a coincidence at the first column, and hence are automatically pure point due to Dekking's classic result [11]. Oseledec's multiplicative ergodic theorem can be applied to this class, from which one can obtain a closed form of $\chi_{\min}$ that is related to the (logarithmic) Mahler measure of a $\{-1, 0, 1\}$-polynomial in one variable [6, 13, 1]. This minimal exponent can be shown to be strictly positive, which provides an independent argument of why the diffraction is singular.

Some general results on the absence of $\widehat{\gamma}_{\mathsf{ac}}$ in the one-dimensional case via this method have already been written down; in particular, it has been shown for all binary aperiodic constant-length substitution in [13], for which the exponents are bounded appropriately by considering relevant polynomials of height 1; compare [8, 9]. General constant-length substitutions on $n$ letters follow a similar scheme, and positivity can be proved for some general families (bijective Abelian, some families with coincidences [4]). We comment briefly that the renormalization scheme in [3] also holds for higher-dimensional analogues (primitive stone inflations of finite local complexity with a suitably chosen reference point in each prototile [5]).

The renormalization equations then read

$$(6) \qquad \nu_{ij}(\boldsymbol{z}) \, = \, \frac{1}{|\det(A_s)|} \sum_{k,\ell} \sum_{\boldsymbol{u} \in T_{ik}} \sum_{\boldsymbol{v} \in T_{j\ell}} \nu_{k\ell} \left( A_s^{-1}(\boldsymbol{z} + \boldsymbol{u} - \boldsymbol{v}) \right),$$

where $\boldsymbol{z}, \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^d$, while $A_s$ is the linear map that expands the system to one that is MLD with the original one via the stone inflation in question [5, 4]. The indices run over a set of labels for the finite prototile set. An application to the higher-dimensional case of block substitutions boils down to finding appropriate bounds for Mahler measures of polynomials in more than one variable [10, 1].

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### Quasicrystalline structures and quantum walks

DARREN C. ONG

(joint work with David Damanik, Jake Fillman)

Consider first the following classical random walk problem. A walker is travelling on the integers, and flips a coin. If the coin lands heads, the walker moves to the left, and if the coin lands tail, the walker moves right. Furthermore, imagine that that weighted coins are distributed at each integer, and whenever the walker is at that integer we use the coin placed there.

We consider now a "quantum mechanical" version of this problem. In this model, the walker (which we imagine as a quantum particle) possesses a spin (either $\uparrow$ or $\downarrow$) as well as an integer location; moreover, the walker may be in a superposition of pure states, rather than being purely localized at a particular site with a definite spin. Instead of a weighted coin at each location, we have a unitary operator (which we call the quantum coin) at each location that interacts with the particle differently depending on its spin. This model has attracted a lot of interest in mathematics, computer science, and physics. Please refer to [6, 7] for some recent surveys on the subject.

We now introduce a second object, the CMV operator. These are operators on $\ell^2(\mathbb{N})$ or $\ell^2(\mathbb{Z})$ that can be viewed as a unitary analogue to the Jacobi operator. See [4, 5] for a survey on the spectral theory of the CMV operator. The CMV operator on $\ell^2(\mathbb{Z})$ looks like

$$
\mathcal{E} = \begin{bmatrix}
\ddots & \ddots & \ddots & \ddots & & & \\
& \overline{\alpha_0}\rho_{-1} & -\overline{\alpha_0}\alpha_{-1} & \overline{\alpha_1}\rho_0 & \rho_1\rho_0 & & \\
& \rho_0\rho_{-1} & -\rho_0\alpha_{-1} & -\overline{\alpha_1}\alpha_0 & -\rho_1\alpha_0 & & \\
& & & \overline{\alpha_2}\rho_1 & -\overline{\alpha_2}\alpha_1 & \overline{\alpha_3}\rho_2 & \rho_3\rho_2 \\
& & & \rho_2\rho_1 & -\rho_2\alpha_1 & -\overline{\alpha_3}\alpha_2 & -\rho_3\alpha_2 \\
& & & & \ddots & \ddots & \ddots & \ddots
\end{bmatrix}.
$$

Here, the $\alpha_n$ are a sequence of complex numbers in the open unit disk, and $\rho_n = \sqrt{1 - |\alpha_n|^2}$.

Cantero, Grünbaum, Moral and Velazquez discovered in [1] that CMV operators can be used to understand quantum walks. Furthermore, in [3] the authors discover a connection between the spectral properties of a CMV operator and the spreading rate of a walker in a quantum walk.

In our paper [2], we discover upper and lower bounds on quantum walk spreading that depend on the growth rates of the transfer matrices of the corresponding CMV operator. As an application, this enables us to understand quantum walk problems where the coin distributions are given by an aperiodically ordered sequence.

To be more precise, consider two types of coins, each weighted differently. We arrange these coins on the integers using a Fibonacci binary string, such that the

0's correspond to the first type of coin and the 1's correspond to the second. Using the upper and lower bounds we developed, we can calculate that it is possible to obtain anomalous transport this way: that is, the quantum walker leaves the origin, but at sub-ballistic speeds. To our knowledge, this was the first example of anomalous transport in quantum walks when the coins do not vary in time.

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## Automorphism and extended symmetry groups of shifts
### Samuel Petite, Reem Yassawi

A $d$-dimensional *shift* is a closed set $X$ of sequences on a finite alphabet, indexed by $\mathbb{Z}^d$, and invariant by the $d$ shift maps $\sigma_1, \ldots \sigma_d$ defined by the canonical basis for $\mathbb{Z}^d$. Shifts form a rich class of dynamical systems. An *automorphism* (or *symmetry*) of a shift is a homeomorphism $\Phi : X \to X$ that commutes with the shift maps $\sigma_1, \ldots \sigma_d$. The Curtis–Hedlund–Lyndon theorem tells us that such an automorphism is defined by a sliding block map; i.e., there exists a local rule, defined on a set of finite configurations, such that the image of a point $x$ at index $\boldsymbol{m} = m_1, \ldots m_d$ is defined by applying the local rule table to a neighbourhood of $\sigma_1^{m_1} \ldots \sigma_d^{m_d} x$ . The set of automorphisms $\mathrm{Aut}(X)$ of a shift, endowed with the composition operation, form a group, which, other than being countable, is in general hard to describe. We will present in this talk a survey of recent results, where we give a finer description of the automorphism groups of certain small one-dimensional shifts. In particular, if the complexity function of a one-dimensional minimal shift is linear, we show that $\mathrm{Aut}(X)/\langle \sigma \rangle$ is finite. We also give conditions on an automorphism $\Phi$, in terms of its radius, so that it has finite order, and deduce that if such an automorphism $\Phi$ exists, then the automorphism group does not contain a group with an exponentially distorted element. Staying in the zero-complexity case, we show that for a minimal shift whose complexity function is $\mathrm{o}(n^5)$, any finitely generated, torsion-free subgroup of $\mathrm{Aut}(X)$ is virtually Abelian.

We also consider *extended symmetry groups* of shifts. While the automorphism group can be recast as the centralizer of the group $\langle \boldsymbol{\sigma} \rangle$ generated by the shift action in the group $\mathcal{H}(X)$ of homeomorphisms on $X$, the extended symmetry group $\mathcal{R}(X)$ is defined to be the normalizer of $\langle \boldsymbol{\sigma} \rangle$ in $\mathcal{H}(X)$. If $(X, \sigma)$ is a minimal one-dimensional shift, then work of Putnam, Giordano and Skau tells us that $\mathcal{R}(X)/\langle \sigma \rangle$ is the group of *outer autormphisms* of the topological full group of a shift, whose commutator subgroup has recently been shown by Juschenko and Monod to have interesting algebraic properties. We also define and discuss extended symmetry groups in higher dimension. We illuminate these concepts by computing the extended symmetry groups of two celebrated and qualitatively different shifts: the chair shift and the Ledrappier shift.

This is based on recent and active works [1, 2, 3, 4, 5, 6, 7, 8].

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## Topological boundary spectrum in physical systems

### Emil Prodan

Let $\mathcal{P}$ be a Delone set in $\mathbb{R}^d$, $\Omega$ be its discrete hull (transversal) and $H_\Omega = \{H_\omega\}_{\omega \in \Omega}$ a family of covariant Hamiltonians on $\mathbb{C}^N \otimes \ell^2(\mathcal{P})$. We denote by $\mathrm{spec}(H_\Omega) = \cup_{\omega \in \Omega} \mathrm{spec}(H_\omega)$. A spectral gap $G$ is defined as a connected component of $\mathbb{R} \setminus \mathrm{spec}(H_\Omega)$. A mobility gap is a connected region $\Delta$ of the real axis where the direct conductivity tensor $d^{-1} \sum_{i=1}^{d} \sigma_{ii}(E_F, T = 0)$ vanishes, whenever $E_F \in \Delta$. We use the terminology (mobility) gapped Hamiltonians for a pair $(H_\Omega, G)$ with $G$ a non-empty spectral (mobility) gap.

Given a covariant bulk family $H_\Omega$, one can define a family of Hamiltonians $\widehat{H}_\Omega = \{\hat{H}_\omega\}_{\omega \in \Omega}$ with a boundary, defined on $\mathbb{C}^N \otimes \ell^2(\mathcal{P} \cap \mathbb{R}_+^d)$, $\mathbb{R}_+^d = \{x = (x_1, \ldots, x_d) \in \mathbb{R}^d, \ x_d \geq 0\}$. More precisely, $\widehat{H}_\Omega = \widehat{H}_\Omega^{(D)} + \widetilde{H}_\Omega$, where $\widehat{H}_\Omega^{(D)}$ is obtained from $H_\Omega$ by imposing the Dirichlet boundary condition and $\widetilde{H}_\Omega$ is a family of covariant Hamiltonians w.r.t. translations parallel to the boundary that

are localized near the boundary (hence $\widetilde{H}_\Omega$ can be seen as defining the boundary condition). As it turns out, $\mathrm{spec}(H_\Omega) \subseteq \mathrm{spec}(\widehat{H}_\Omega)$ and one defines the boundary spectrum as $\mathrm{spec}_b(\widehat{H}_\Omega) = \mathrm{spec}(\widehat{H}_\Omega) \setminus \mathrm{spec}(H_\Omega)$.

For such data, the physics community propose the following programs.

**Bulk.**

(1) Classify the gapped Hamiltonians w.r.t. the equivalence relation defined by $(H_\Omega, G) \sim (H'_\Omega, G')$ iff there exists a homotopy of gapped Hamiltonians $(H_\Omega(t), G(t))_{t \in [0,1]}$ such that $(H_\Omega(0), G(0)) = (H_\Omega, G)$ and $(H_\Omega(1), G(1)) = (H'_\Omega, G')$.

(2) Same as above but with spectral gap replaced by mobility gap.

**Boundary.**

(1) Find all $(H_\Omega, G)$ such that

$$\mathrm{spec}_b(\widehat{H}_\Omega) \cap G = G,$$

regardless of the boundary condition. This type of boundary spectrum is called topological.

(2) Same as above but with additional requirement that $\mathrm{spec}_b(\widehat{H}_\Omega)$ is not Anderson localized.

**Bulk+Boundary.**

(1) Find a relations between the bulk and boundary programs. Whenever such relations exist, they go by the name of bulk-boundary principle.

The most ambitious programs are Bulk #2, Boundary #2 and, of course, establishing the bulk-boundary principle which relates the two. In the context of the electronic degrees of freedom in disordered crystals, the physics community put forward a conjecture which comes in the form of a classification table of all possible topological phases displaying delocalized topological boundary spectrum [13, 7, 12]. The conjecture survived a large number of numerical tests (too many to be mentioned here). For the phases classified by the $\mathbb{Z}$ group, a proof of the conjecture can be found in the monograph [11]. It is based on the index theorems discovered in [9, 10], which extended the previous pioneering works [1, 6] on the integer quantum Hall effect. For the topological phases classified by $\mathbb{Z}_2$, progress with program Bulk #2 has been made in [4], while the program Bulk #1 has been carried out in [14, 2, 5]. A unifying bulk-boundary principle for the programs Bulk #1 and Boundary #1 has been carried in [3]. Work on the conjecture contained in the classification table continues these days.

Recently, the interest of the physics community is rapidly shifting from crystals to meta-materials and from the electronic degrees of freedom to the electromagnetic and acoustic degrees of freedom. The search for topological photonic and acoustic crystals is vigorously underway, on both theoretical and experimental fronts. Since meta-materials give great control over the structure of the materials, there is a strong movement among the physics community to move away from the

simple disordered crystalline patterns and experiment with more interesting patterns. Following this effort, myself and collaborators [8] introduced the concept of dynamically generated patterns, defined as below.

**Proposition.** *Let* $(\Omega, \mathbb{Z}^d, \tau)$ *be a topological dynamical system. Denote the set of generators of* $\mathbb{Z}^d$*, taken with both signs, by* $\mathcal{G}_d$*. Assume the existence of the continuous maps:*

$$(1) \qquad\qquad F_{\boldsymbol{e}} : \Omega \to \mathbb{R}^{\alpha}, \quad \boldsymbol{e} \in \mathcal{G}_d, \quad \alpha \in \mathbb{N}_+,$$

*obeying the consistency relations:*

$$(2) \qquad\quad F_{-\boldsymbol{e}} = -F_{\boldsymbol{e}} \circ \tau_{\boldsymbol{e}}, \quad F_{\boldsymbol{e}'} - F_{\boldsymbol{e}'} \circ \tau_{-\boldsymbol{e}} = F_{\boldsymbol{e}} - F_{\boldsymbol{e}} \circ \tau_{-\boldsymbol{e}'}.$$

*Then, for each* $\omega \in \Omega$*, the algorithm:*

$$(3) \qquad \boldsymbol{p}_0 = 0, \quad \boldsymbol{p}_{\boldsymbol{n}+\boldsymbol{e}} = \boldsymbol{p}_{\boldsymbol{n}} + (F_{\boldsymbol{e}} \circ \tau_{\boldsymbol{n}+\boldsymbol{e}})(\omega), \quad \boldsymbol{n} \in \mathbb{Z}^d, \quad \boldsymbol{e} \in \mathcal{G}_d,$$

*generates a point pattern* $\mathcal{P} = \{\boldsymbol{p}_{\boldsymbol{n}}\}_{\boldsymbol{n} \in \mathbb{Z}^d}$ *whose points are indexed by* $\mathbb{Z}^d$*.*

Under precise conditions, the discrete hull of these patterns coincides with $\Omega$. While these patterns are algorithmically simple, the patterns themselves can be very complex and, through various limits, one can explore patterns that are not from this category. Hence, we think they are very interesting. For many examples, the bulk-boundary principle for programs Bulk #1 and Boundary #1 can be carried out completely using tools from $K$-theory. As a result, we obtained a large number of new classes of systems displaying topological boundary spectrum.

The goals of my talk given for the workshop "Spectral Structures and Topological Methods in Mathematical Quasicrystals", organized at Oberwolfach Institute, were: 1) communicate the research programs described in the first part of this note; 2) exemplify the success stories with physical examples; 3) communicate the interest of the physics community in patterns that go beyond disordered crystals; 4) introduce the dynamically-generated patterns; 5) establish the bulk-boundary principle for equivariant Hamiltonians defined over such patterns; 6) exemplify the physical consequences using laboratory results and numerical simulations.

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## The "mixed" spectral nature of the Thue–Morse Hamiltonian

Yanhui Qu

(joint work with Qinghui Liu, Xiao Yao)

We find a subset $\Sigma$ of the spectrum of Thue–Morse Hamiltonian $H_\lambda$, such that for any $E \in \Sigma$, the following properties hold:

(i) The related trace orbit $\{t_n(E) : n \geq 1\}$ is unbounded;

(ii) The norms of the transfer matrices grow as
$$e^{c_1\gamma\sqrt{n}} \leq \left\|T_n(E)\right\| \leq e^{c_2\gamma\sqrt{n}},$$
where $0 < c_1 < c_2$ are two absolute constants, $\gamma > 0$ is a constant only depending on $E$;

(iii) There exists a subordinate solution $\psi$ of $H_\lambda\psi = E\psi$, such that $|\psi_n|$ is polynomially bounded; $|\psi_{\pm 2^n}|$ decreases as $e^{-2^n\gamma}$; $(\psi_{2^{2n+1}+1}, \psi_{2^{2n+1}})$ tends to $(\pm 1, 1)$. We call such a $\psi$ a pseudo-localized state.

It is known that there exists a dense subset $\tilde{\Sigma}$ of the spectrum such that, for any $E \in \tilde{\Sigma}$ and any solution $\phi$ of $H_\lambda\phi = E\phi$, $\phi$ is an extended state [1, 2]. Since the extended states and pseudo-localized states co-exist, we may say that the Thue–Morse Hamiltonian exhibits "mixed" spectral nature [3].

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# Almost periodic measures and diffraction

Nicolae Strungaru

(joint work with Robert V. Moody)

Almost periodicity plays an important role in the study of mathematical diffraction. Given a point set $\Lambda \subset \mathbb{R}^d$, representing the positions of atoms in an idealized solid, Hof [7] defined the diffraction $\widehat{\gamma}$ of $\Lambda$ as the Fourier transform of the positive and positive definite measure $\gamma$, the autocorrelation (or 2-point correlation) measure of $\Lambda$; see [2] for a general exposition.

Each positive definite measure $\gamma$ (or, more generally, each weakly almost periodic measure $\gamma$) admits a (unique) Eberlein decomposition

$$\gamma = \gamma_{\mathrm{s}} + \gamma_0$$

into a strongly almost periodic measure $\gamma_{\mathrm{s}}$ and a null-weakly almost periodic measure $\gamma_0$ (see the review [8] or [4, 5, 6] for details).

As proved by Eberlein for finite measures [5] and by Gil de Lamadrid–Argabright for twice Fourier transformable measures [6], the Eberlein decomposition of the autocorrelation $\gamma$ is Fourier dual to the Lebesgue decomposition of the diffraction measure $\widehat{\gamma}$. Recently, we proved that this result more generally holds for translation bounded, Fourier transformable measures as follows.

**Theorem 1** ([8]). *Let $\gamma$ be a translation bounded, Fourier transformable measure. Then, $\gamma$ is weakly almost periodic, $\gamma_{\mathrm{s}}$ and $\gamma_0$ are Fourier transformable, and*

$$\widehat{\gamma_{\mathrm{s}}} = (\widehat{\gamma})_{\mathrm{pp}} , \qquad \widehat{\gamma_0} = (\widehat{\gamma})_{\mathrm{c}} .$$

An immediate consequence of this result is that strong almost periodicity and pure point diffraction are Fourier dual concepts:

- a Fourier transformable measure $\mu$ is strongly almost periodic if and only if $\widehat{\mu}$ is pure point [8];
- a Fourier transformable measure $\mu$ is pure point if and only if $\widehat{\mu}$ is strongly almost periodic [6].

These results allow us to study the pure point spectrum $(\widehat{\gamma})_{\mathrm{pp}}$ and the continuous spectrum $(\widehat{\gamma})_{\mathrm{c}}$ of $\Lambda$, which are measures in the Fourier dual space $\widehat{\mathbb{R}^d} \simeq \mathbb{R}^d$, by studying instead the measures $\gamma_{\mathrm{s}}$ and $\gamma_0$, respectively, in the real space $\mathbb{R}^d$. This approach has led to many general results about the diffraction of Meyer sets (see [9, 10] for example).

To gain further insight into the absolutely continuous and the singular continuous spectrum, we would like to extend the Eberlein decomposition to another decomposition step, $\gamma_0 = \gamma_{0a} + \gamma_{0s}$, which is Fourier dual to the spectral decomposition $(\widehat{\gamma})_{\mathrm{c}} = (\widehat{\gamma})_{\mathrm{ac}} + (\widehat{\gamma})_{\mathrm{sc}}$.

While the general question about the existence of this decomposition is still open, recent progress has been made in the case of positive definite measures with Meyer set support as follows.

**Theorem 2.** [11] *Let $\gamma$ be a translation bounded, positive definite measure that is supported inside a Meyer set. Then, there exist three positive definite measures $\gamma_{\mathrm{s}}$, $\gamma_{\mathrm{0s}}$, $\gamma_{\mathrm{0a}}$ supported inside Meyer sets such that $\gamma = \gamma_{\mathrm{s}} + \gamma_{\mathrm{0s}} + \gamma_{\mathrm{0a}}$ and*

$$\widehat{\gamma_{\mathrm{s}}} = (\widehat{\gamma})_{\mathrm{pp}} \,, \qquad \widehat{\gamma_{\mathrm{0s}}} = (\widehat{\gamma})_{\mathrm{sc}} \,, \qquad \widehat{\gamma_{\mathrm{0a}}} = (\widehat{\gamma})_{\mathrm{ac}} \,.$$

If $\Lambda$ is a Meyer set, and $\gamma$ its autocorrelation, it follows that each of the measures $(\widehat{\gamma})_{\mathrm{pp}}$, $(\widehat{\gamma})_{\mathrm{ac}}$, $(\widehat{\gamma})_{\mathrm{sc}}$ is strongly almost periodic. In particular, each of these measures is either trivial or has a relatively dense support.

Theorem 2 holds if $\mathbb{R}^d$ is replaced by an arbitrary metrizable locally compact Abelian group $G$. It follows more generally that, if $\omega$ is any translation bounded measure with Meyer set support in $G$, and $\gamma$ is any autocorrelation of $\omega$, then each of the measures $(\widehat{\gamma})_{\mathrm{pp}}$, $(\widehat{\gamma})_{\mathrm{ac}}$, $(\widehat{\gamma})_{\mathrm{sc}}$ is either trivial or has a relatively dense support.

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