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Topology of Arrangements and Representation Stability

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Abstract. The workshop "Topology of arrangements and representation stability" brought together two directions of research: the topology and geometry of hyperplane, toric and elliptic arrangements, and the homological and representation stability of configuration spaces and related families of spaces and discrete groups. The participants were mathematicians working at the interface between several very active areas of research in topology, geometry, algebra, representation theory, and combinatorics. The workshop provided a thorough overview of current developments, highlighted significant progress in the field, and fostered an increasing amount of interaction between specialists in areas of research.

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Introduction by the Organisers

The workshop "Topology of arrangements and representation stability" brought together over 50 mathematicians from Austria, Australia, Canada, Denmark, France, Germany, Italy, Japan, Mexico, Sweden, Switzerland, the UK, and the USA. The participants were from all career stages, ranging from graduate students to senior faculty. The aim of the workshop was to bring together two directions of current research: the topology and geometry of hyperplane, toric and elliptic arrangements, and the homological and representation stability of configuration spaces and related families of spaces and discrete groups.

Since the participants came from more than one mathematical community, the speakers on the first day each gave one hour talks which were mandated to be partially expository and to summarize some aspect of the current state of research. Nathalie Wahl gave an introductory talk on homological stability, Toshitake Kohno presented some deep connections between three approaches to study linear representations of braid groups, Jennifer Wilson gave an overview on representation stability and FI-modules, while Mike Falk presented an expository talk on the theory of arrangements and Artin groups. Later in the week, Andrew Snowden gave an overview of some of his work with Steven Sam on twisted commutative algebras, which provides methods to establish finite generation of families of group representations.

The rest of the presentations were 40-minute talks on recent advances on some important topics related to the main themes of the workshop:

- Polynomial functors and the way they relate to representation and homological stability (Djament, Soulié, Vespa).
- Cohomology of braid groups with local coefficients, Artin groups, and configurations spaces (Callegaro, Liu, Knudsen, J. Miller, Ramos, Wiltshire-Gordon)
- The Milnor fiber of reflection arrangements (Dimca), and the Milnor fiber complex associated to a finite Coxeter or Shepard group (A. R. Miller).
- Semimatroids as a tool for understanding the combinatorics of abelian arrangements (Delucchi), as well as Kazdhan–Lusztig polynomials for matroids (Wakefield).
- Logarithmic derivations and free (multi)arrangements (Abe, Röhrle).
- Johnson homomorphisms of automorphism groups of free groups (Satoh).
- Topology of enumerative problems on cubic curves (Chen), resolvent degree problems connected with Hilbert's 13th problem, as well as Hilbert's Sextic and Octic conjectures (Wolfson).

In addition to the regular talks, there were also two problem sessions. Both sessions were very well-attended and led to some animated discussions. The problems proposed have been included at the end of this report, since we think that some of them will be of interest to a wider audience.

Several participants commented favorably on the overview of current developments that was presented at the start of the meeting, bridging some of the gaps between experts on the various different aspects of the subject. The schedule allowed for time for informal discussions among the participants. As a result, several collaborations started or grew at the meeting, involving various groups and projects at various stages of development.

We wish to thank the Oberwolfach Mathematics Institute and its staff for creating a stimulating atmosphere and making the workshop possible.

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Workshop: Topology of Arrangements and Representation Stability

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Abstracts

Introduction to homological stability

Nathalie Wahl

A sequence of spaces $X_1 \to X_2 \to \cdots$ or groups $G_1 \to G_2 \to \cdots$ satisfies *homological stability* if $H_i(X_n) \to H_i(X_{n+1})$ (resp. $H_i(G_n) \to H_i(G_{n+1})$) is an isomorphism whenever n is large enough (larger than a function of i). Replacing G_n by its classifying space $X_n = BG_n$, we see that groups are a special case of spaces. We note also that homological stability for a sequence $X_1 \to X_2 \to \cdots$ is equivalent to the statement that $H_i(X_n) \cong H_i(X_\infty)$ when n is large enough, where $X_\infty = \bigcup_i X_i$ is the colimit of the sequence. The homology $H_*(X_\infty)$ is called the *stable homology*, and is thus what the homology of the spaces X_n stabilizes to.

Examples of sequences of spaces that stabilize are the unordered configuration spaces $X_n = \mathrm{Conf}(n, \mathbb{R}^k)$ whenever $k \geq 2$, or configuration spaces in more general manifolds, the moduli space $X_g = \mathcal{M}_{g,k}$ of Riemann surfaces of genus g with $k \geq 1$ boundary components, or moduli spaces of higher dimensional manifolds, see eg. [17, 8, 7]. For groups, examples include the symmetric groups, braid groups, the mapping class groups $G_n = \pi_0 \,\mathrm{Diff}(M \#_n N)$ of connected sums of 3-manifolds, the automorphisms of free groups $G_n = \mathrm{Aut}(F_n)$, the general linear groups $G_n = GL_n(R)$ for R a ring satisfying a mild condition, or $G_r = V_{n,r}$ the Higman–Thompson groups, see eg. [15, 1, 10, 9, 21, 20].

Empirical observation 1. The stable part of the homology is often easier to compute. This is how homological stability many times has turned out to be a powerful tool for computations. Examples of such computations are

$$\begin{aligned} H_i(\mathrm{Conf}(n, \mathbb{R}^k)) &\cong H_i(\Omega_0^k S^k) && \text{whenever } n \geq 2i+1 \\ H_i(\mathcal{M}_{g,k}) &\cong H_i(\Omega_0^\infty \mathbb{C}P_{-1}^\infty) && \text{whenever } g \geq \frac{3i+2}{2} \\ H_i(V_{n,r}) &\cong H_i(\Omega_0^\infty M(\mathbb{Z}/(n-1))) && \text{all } r, \end{aligned}$$

where the last computation in particular shows that Thompson's group V is acyclic, a fact that is so far only known using stability methods. (See [2, 12, 20].)

Empirical observation 2. When homological stability holds with constant coefficients, it usually also holds more generally with certain types of twisted coefficients. There are though fewer computations of stable homology with twisted coefficients (and most of the authors of such computations were present at the workshop, see eg. [3, 4, 6]). Another observation is that stability with twisted coefficients is related to *representation stability* (as in Jenny Wilson's talk at the present workshop), but this relationship is not yet fully understood. See for instance [16] for work in this direction.

1. E_2-algebras and stability

Roughly speaking, a space X is called an E_2-algebra if it possesses a multiplication $X \times X \to X$ which is unital, associative and commutative up to homotopy; an E_k-algebra, for $k > 2$, is "more commutative" and an E_∞-algebra is commutative up to all higher homotopies. For example, $X = \coprod_{n\geq 0} \mathrm{Conf}(n, \mathbb{R}^k)$ is an E_k-algebra. (See eg. [13] for the definition of E_k-algebra in terms of the little cube operads.)

If X is an E_k-algebra, we can construct from X a sequence of spaces as above by picking an element $x \in X$ and multiplying with it:

$$X_1 \xrightarrow{+x} X_2 \xrightarrow{+x} X_3 \xrightarrow{+x} \cdots$$

where we define X_1 to be the component of x, X_2 that of $x + x$, and so on. In the case $X = \coprod_{n\geq 0} X_n$ with $k \geq 2$, the classical *recognition principle* and *group completion theorem* [5, 13, 14] assemble to show that

$$H_*(\mathbb{Z} \times X_\infty) \cong H_*(\Omega^k Y),$$

for some explicitly constructed space Y, where X_∞ is the colimit of the sequence as above. So the fact that the sequence $X_1 \to X_2 \to \cdots$ comes from an E_k-structure (for $k \geq 2$), implies that the stable homology, if stability happens, will be that of a k-fold loop space.

It has recently been understood that E_k-structures for $k \geq 2$ also play a role in the question of whether a sequence stabilizes or not. To make this precise, we first recall Quillen's recipe for proving homological stability in the case of groups: To prove homological stability for a sequence of groups $G_1 \to G_2 \to \cdots$, one can look for a collection of G_n*-simplicial objects* W_n (simplicial or semi-simplicial sets, or simplicial complexes), one for each n, such that the action of G_n on W_n is as transitive as possible and the stabilizer of a p-simplex is the group G_{n-p-1}. Using the action of G_n on W_n and the skeletal filtration of W_n, one can construct a spectral sequence which, if the W_n's are highly connected, proves homological stability for the groups G_n, see eg. [10, Sec 5]. The following result says that such spaces W_n can be canonically constructed from an E_2-structure on $X = \coprod_n BG_n$.

Theorem 1 ([18] for groups and [11] for general E_2-algebras)**.** *If $X = \coprod_{n\geq 0} X_n$ is an E_2-algebra as above, then there is a canonical sequence of semi-simplicial spaces W_n such that the vanishing $\widetilde{H}_i(W_n) = 0$ for $i \leq \frac{n-2}{m}$, for some $m \geq 2$, implies that*

$$H_i(X_n) \xrightarrow{\cong} H_i(X_{n+1}) \quad \text{for } i \leq \frac{n-1}{m}.$$

Moreover,

$$H_i(X_n, M_n) \xrightarrow{\cong} H_i(X_{n+1}, M_{n+1}) \quad \text{for } n \gg i$$

for $\{M_n\}_{n\geq 0}$ any polynomial or abelian coefficient sequence; see [18, 11] for the definition of such coefficient systems and the precise stability range in this case .

In each of the stability examples mentioned above, there is an underlying E_k-algebra which makes it fit into the above theorem (or a slight generalization of it, using a module structure over an E_2-algebra, see [11]). In particular, stability is known to hold with polynomial and twisted coefficients in all these cases. Typically, an E_1-algebra which is not E_2 (so that no homotopy commutativity for the multiplication is assumed) will not yield a homologically stable sequence when multiplying with a fixed element x as above. Although we know that not every sequence obtained this way from an E_2-algebra is homologically stable, we observe that stability most often happens when the algebra is at least E_2.

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Configuration spaces, KZ connections and conformal blocks

Toshitake Kohno

The purpose of this report is to clarify a relation among the following three approaches for linear representations of braid groups.

(1) Homological representations (Burau and Lawrence–Krammer–Bigelow (LKB) representations).
(2) Monodromy representations of Kniznhnik–Zamolodchikov (KZ) connections.
(3) Generalized Jones representations arising from R matrices in the theory of quantum groups.

We first recall basic notions concerning hyperplane arrangements. Let $\mathcal{A} = \{H_1, \ldots, H_\ell\}$ be an arrangement of affine hyperplanes in the complex vector space $\mathbf{C}^n$. We consider the complement $M(\mathcal{A}) = \mathbf{C}^n \setminus \bigcup_{H\in\mathcal{A}} H$. Let $\mathcal{L}$ be a complex rank one local system over $M(\mathcal{A})$ associated with a representation of the fundamental group $r\colon \pi_1(M(\mathcal{A}), x_0) \longrightarrow \mathbf{C}^*$. We denote by f_j be a linear form defining the hyperplane H_j, $1 \leq j \leq \ell$. We associate a complex number $a_j = a(H_j)$ called an exponent to each hyperplane and consider a multivalued function $\Phi = f_1^{a_1} \cdots f_\ell^{a_\ell}$. The associated local system is denoted by $\mathcal{L}_\Phi$. We choose a smooth compactification $i\colon M(\mathcal{A}) \longrightarrow X$. We shall say that the local system $\mathcal{L}$ is generic if and only if there is an isomorphism $i_*\mathcal{L} \cong i_!\mathcal{L}$ where i_* is the direct image and $i_!$ is the extension by 0. If the local system $\mathcal{L}$ is generic in the above sense, then there is an isomorphism $H_*(M(\mathcal{A}), \mathcal{L}) \cong H_*^{lf}(M(\mathcal{A}), \mathcal{L})$ and we have $H_k(M(\mathcal{A}), \mathcal{L}) = 0$ for any $k \neq n$. Here H_*^{lf} stands for the homology with locally finite chains.

Let D_n be a disk with n-punctured points and consider the configuration space of unordered distinct m points in D_n, which is denoted by $\mathrm{Conf}(m, D_n)$. We have $H_1(\mathrm{Conf}(m, D_n); \mathbf{Z}) \cong \mathbf{Z}^{\oplus n} \oplus \mathbf{Z}$. Consider the homomorphism

$$\alpha\colon H_1(\mathrm{Conf}(m, D_n); \mathbf{Z}) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$$

defined by $\alpha(x_1, \ldots, x_n, y) = (x_1 + \cdots + x_n, y)$. Composing with the abelianization map, we obtain the homomorphism $\beta\colon \pi_1(\mathrm{Conf}(m, D_n), x_0) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$. We denote by $\widetilde{\mathcal{C}}_{n,m}$ the abelian covering of $\mathrm{Conf}(m, D_n)$ corresponding to $\mathrm{Ker}\,\beta$. The homology group $H_*(\widetilde{\mathcal{C}}_{n,m}; \mathbf{Z})$ is considered to be a $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$-module by deck transformations. We express $\mathbf{Z}[\mathbf{Z} \oplus \mathbf{Z}]$ as the ring of Laurent polynomials $R = \mathbf{Z}[q^{\pm 1}, t^{\pm 1}]$. We put $H_{n,m} = H_m(\widetilde{\mathcal{C}}_{n,m}; \mathbf{Z})$, which is a free R-module. There is a homomorphism

$$\rho : B_n \longrightarrow \mathrm{Aut}_R\, H_{n,m}$$

called the homological (LKB) representation of the braid group. The case $m = 1$ corresponds to the Burau representation.

Let $\mathfrak{g}$ be a complex semi-simple Lie algebra and $\{I_\mu\}$ be an orthonormal basis of $\mathfrak{g}$ with respect to the Cartan–Killing form. Let $r_i\colon \mathfrak{g} \to \mathrm{End}(V_i)$, $1 \leq i \leq n$, be representations of $\mathfrak{g}$. We consider the Casimir element $\Omega = \sum_\mu I_\mu \otimes I_\mu$ and denote by Ω_{ij} the action of Ω on the i-th and j-th components of $V_1 \otimes \cdots \otimes V_n$. We set

$$\omega = \frac{1}{\kappa} \sum_{i,j} \Omega_{ij} d\log(z_i - z_j), \quad \kappa \in \mathbf{C} \setminus \{0\}.$$

The 1-form ω defines a flat connection for a trivial vector bundle over X_n, the configuration space of ordered distinct n points in $\mathbf{C}$ with fiber $V_1 \otimes \cdots \otimes V_n$. As the holonomy we have representations of pure braid groups

$$\theta_\kappa \colon P_n \longrightarrow \mathrm{Aut}(V_1 \otimes \cdots \otimes V_n),$$

which are called the monodromy representations of KZ connections.

In the following, we consider the case $\mathfrak{g} = sl_2(\mathbf{C})$ with the standard basis H, E, F. For a complex number λ we denote by M_λ the Verma module of $\mathfrak{g}$ with highest weight vector v such that $Hv = \lambda v$ and $Ev = 0$. For an n-tuple $\Lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ we set $|\Lambda| = \lambda_1 + \cdots + \lambda_n$. We consider the tensor product $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$. For a non-negative integer m we set

$$W[|\Lambda| - 2m] = \{x \in M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n} \ ; \ Hx = (|\Lambda| - 2m)x\}$$

and define the space of null vectors by

$$N[|\Lambda| - 2m] = \{x \in W[|\Lambda| - 2m] \ ; \ Ex = 0\}.$$

The KZ connection ω commutes with the diagonal action of $\mathfrak{g}$ on $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_n}$ and acts on the space of null vectors $N[|\Lambda| - 2m]$. For parameters κ and λ we consider the multi-valued function

$$\Phi_{n,m} = \prod_{1 \le i < j \le n} (z_i - z_j)^{\frac{\lambda_i \lambda_j}{2\kappa}} \prod_{1 \le i \le m, 1 \le \ell \le n} (t_i - z_\ell)^{-\frac{\lambda_\ell}{\kappa}} \prod_{1 \le i < j \le m} (t_i - t_j)^{\frac{2}{\kappa}}$$

defined over X_{n+m}. Let $\mathcal{L}$ be the local system over X_{n+m} associated to the multi-valued function $\Phi_{n,m}$. We denote by $\pi : X_{m+n} \to X_n$ the projection defined by $\pi(x_1, \cdots, x_n, t_1, \cdots, t_m) = (x_1, \cdots, x_n)$. Let $X_{n,m}$ denote a fiber of π and put $Y_{n,m} = X_{n,m}/\mathfrak{S}_m$, where $\mathfrak{S}_m$ acts as the permutation of coordinates. Let us notice that $Y_{n,m}$ is homotopy equivalent to $\mathrm{Conf}(m, D_n)$. We denote by $\overline{\mathcal{L}}$ the induced local system on $Y_{n,m}$. The symbol $\mathcal{L}^*$ stands for the dual local system of $\mathcal{L}$.

We have the monodromy of the KZ connection

$$\theta_{\kappa,\lambda} : P_n \to \mathrm{Aut}\, N[|\Lambda| - 2m].$$

On the other hand, we have a homological representation

$$\rho_{n,m} : P_n \to \mathrm{Aut}\, H_m(Y_{n,m}, \mathcal{L}^*).$$

By using a construction of horizontal sections of KZ connections by hypergeometric integrals using $\Phi_{n,m}$ due to Schechtman and Varchenko [6], we can construct a period map

$$\phi : H_m(Y_{n,m}, \mathcal{L}^*) \longrightarrow N[|\Lambda| - 2m]^*.$$

It turns out that ϕ is an isomorphism for generic parameters λ, κ and is equivariant with respect to the action of the pure braid group P_n. We fix a complex number λ and consider the case $\lambda_1 = \cdots = \lambda_n = \lambda$. Then the above representation is the specialization of the LKB representation with $q = e^{-2\pi\sqrt{-1}\lambda/\kappa}, \quad t = e^{2\pi\sqrt{-1}/\kappa}$ (see [4]). A relation between the monodromy representations of KZ connections and R matrices in the theory of quantum groups $U_h(\mathfrak{g})$ was originally found in [2] and [1]. By defining the action of $U_h(\mathfrak{g})$ on chains with local system coefficients and identifying the action of $E \in U_h(\mathfrak{g})$ with the twisted boundary operator, we can recover the quantum group symmetry in homological representations.

Finally, we briefly discuss a relation to conformal field theory (see [3]). We consider the affine Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbf{C}((\xi)) \oplus \mathbf{C}c$ with the commutation relation

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg + \mathrm{Res}_{\xi=0}\, df\, g\, \langle X, Y \rangle c.$$

We fix a positive integer K called a level. For an integer λ with $0 \leq \lambda \leq K$ we can associate the integrable highest weight module $\mathcal{M}_\lambda$, which is an irreducible $\widehat{\mathfrak{g}}$ module containing V_λ and c acts as $K \cdot \mathrm{id}$. We call such λ a level K highest weight. We consider the Riemann sphere $\mathbf{C}P^1$ with $n{+}1$ marked points $p_1, \cdots, p_n, p_{n+1}$, where $p_{n+1} = \infty$. We assign level K highest weights $\lambda_1, \cdots, \lambda_n, \lambda_{n+1}$ to $p_1, \cdots, p_n, p_{n+1}$. We denote by $\mathcal{M}_p$ the set of meromorphic functions on $\mathbf{C}P^1$ with poles at most at $p_1, \cdots, p_{n+1}$. The space of conformal blocks is defined as the space of coinvariants

$$\mathcal{H}_\Sigma(p, \lambda) = \mathcal{H}_{\lambda_1} \otimes \cdots \otimes \mathcal{H}_{\lambda_{n+1}} / (\mathfrak{g} \otimes \mathcal{M}_p)$$

where $\mathfrak{g} \otimes \mathcal{M}_p$ acts diagonally via Laurent expansions at $p_1, \ldots, p_{n+1}$. The space of conformal blocks forms a vector bundle over X_n with the KZ connection such that $\kappa = K + 2$. By means of horizontal sections of the KZ connection using hypergeometric integrals we can construct a period map

$$\phi : H_m(Y_{n,m}, \overline{\mathcal{L}}^*) \to \mathcal{H}(p, \lambda)^*.$$

with $m = \frac{1}{2}(\lambda_1 + \cdots + \lambda_n - \lambda_{n+1})$. This might not be a generic case and the period map is not an isomorphism in general. There is a subtle point concerning fusion rules and resonance at infinity. We refer the reader to [5] for recent progress on this aspect.

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A brief introduction to representation stability

Jenny Wilson

This was an expository talk in two parts. The first part gave a brief history of the field of representation stability from the perspective developed in work of Church, Ellenberg, Farb, and Napgal [14, 12, 13]. The second part was an illustration of a proof technique—adapting Quillen's methods in homological stability—for proving representation stability in certain applications. The full text of the talk is available on the author's webpage under "Notes".

Over the past five years, the field of representation stability has taken several directions. One objective has been to exhibit representation stability phenomena in particular families of groups or spaces. Applications include congruence subgroups of linear groups [72, 13, 10, 73, 38, 62, 15, 64], complements of arrangements [14, 12, 98, 99, 2, 4, 29, 82], configuration spaces [9, 12, 13, 53, 25, 45, 76, 95, 63, 78, 61, 15, 1], mapping class groups and moduli space [50, 48, 49, 96], Torelli groups [3, 18, 69, 16, 62], variations on the pure braid groups and related automorphisms groups, objects in graph theory, etc [97, 54, 83, 77, 81].

Authors have constructed categories for actions by families of groups other than the symmetric group, or for sequences of symmetric group representations with additional structure [98, 99, 86, 71, 40, 41, 73, 28, 74], and studied their algebraic structure [14, 12, 13, 32, 100, 33, 34, 35, 55, 75, 31, 36, 101, 57, 89, 10, 58, 37, 39, 56, 59, 60, 79, 80, 68, 70, 15, 64]. These results are closely related to the theory of polynomial functors [22, 19, 23, 17, 24, 44, 21, 20, 94] and the theory of twisted commutative algebras [85, 84, 88, 90, 91, 67, 66].

Other goals have been to explore connections between representation stability results and objects in number theory [11, 26, 7, 46, 47, 8, 51, 27, 52, 5, 30, 6] or algebraic combinatorics, or the modular representation theory of the symmetric groups [42, 65, 87, 43, 92].

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Arrangements and Artin groups

Michael J. Falk

We give an introduction to the general theory of complex hyperplane arrangements and its origins in the study of finite reflection groups and their discriminants. We state some solved and still open problems, in particular involving the Milnor fibration. We present recent constructions of the author with E. Delucchi [4], and from work in progress with D. Ernst and S. Riedel, of finite combinatorial models for the complements of general complexified real arrangements, and of complements of discriminants of finite real reflection groups, respectively. Natural generalizations to Artin groups and pure Artin groups on one hand, and to finite complex reflection groups and complex braid groups on the other, should be of interest in the study of representation and homological stability.

Complex hyperplane arrangements. A *complex hyperplane arrangement* is a finite set $\mathcal{A}$ of linear hyperplanes in $V = \mathbb{C}^\ell$. For $H \in \mathcal{A}$ choose $\alpha_H \colon V \to \mathbb{C}$, a nonzero linear form satisfying $H = \ker(\alpha_H)$. The product $Q := \prod_{H\in\mathcal{A}} \alpha_H$ is called the *defining polynomial* of $\mathcal{A}$. The *union* of $\mathcal{A}$, $D := \{Q = 0\} = \bigcup_{H\in\mathcal{A}} H$ is an affine algebraic variety with a (homogeneous) singularity at 0 for $\ell \geq 2$, isolated only if $\ell = 2$. The *complement* $M := V \setminus D$ of $\mathcal{A}$ is a non-compact 2ℓ-manifold, connected but not simply-connected. In case all α_H can be chosen to have real coefficients, we say $\mathcal{A}$ is a *complexified real* arrangement. Then one has real hyperplanes $H_\mathbb{R} = H \cap \mathbb{R}^\ell$ for $H \in \mathcal{A}$ comprising the associated real arrangement $\mathcal{A}_\mathbb{R}$ in $\mathbb{R}^\ell$.

As an example, consider the arrangement $\mathcal{A}$ in $V = \mathbb{C}^3$ with defining polynomial $Q = (x - y)(x - z)(y - z)$. The complement of $\mathcal{A}$ consists of ordered triples of distinct points in the plane; that is, M is the ordered configuration space $\mathrm{Conf}(\mathbb{R}^2, 3)$. All three hyperplanes contain the line L given by $x = y = z$, so $\mathcal{A}$ determines an arrangement of hyperplanes in the quotient vector space V/L. This is a complexified real arrangement, whose real part is pictured in Figure 1.

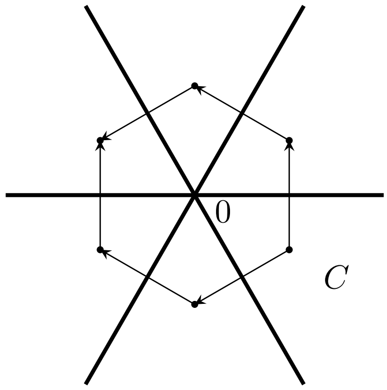

Figure 1. The Coxeter arrangement of type A_2.

A quick sketch of the general theory. Since the α_H are homogeneous, they define hyperplanes $\overline{H}$ in complex projective space $\mathbb{P}^{\ell-1}$, comprising the associated projective arrangement $\overline{\mathcal{A}}$, with complement $\overline{M}$. The projectivization map $M \to \overline{M}$ is a trivial $\mathbb{C}^\times$-bundle, and $\overline{M}$ is diffeomorphic to the complement of an arrangement of $|\mathcal{A}| - 1$ affine hyperplanes in $\mathbb{C}^{\ell-1}$. This "deconing" process is useful for induction arguments [2].

One motivation for research in the field is the theorem of Orlik and Solomon [5]: the cohomology ring $H^*(M, \mathbb{C})$ has a presentation that depends only on the combinatorics of $\mathcal{A}$. Here the "combinatorics of $\mathcal{A}$" means the function $d_{\mathcal{A}}$ given by $d_{\mathcal{A}}(S) = \dim_{\mathbb{C}}(\bigcap_{H\in S} H)$, for $S \subseteq \mathcal{A}$. There are arrangements with different combinatorics but isomorphic cohomology rings; a complete classification of these rings has not yet been accomplished.

There are known presentations for the *arrangement group* $\pi_1(M)$, all depending on a choice of coordinates. Rybnikov [6] showed that $\pi_1(M)$ is not determined by $d_{\mathcal{A}}$; the counterexample is a pair of thirteen-line (projective) arrangements with no real form. For complexified real arrangements one has, for instance, the *Salvetti complex* $\mathcal{S}$ of $\mathcal{A}$, a finite regular cell complex of dimension ℓ with the homotopy type of M, determined by the stratification of $\mathbb{R}^\ell$ coming from $\mathcal{A}_{\mathbb{R}}$. $\mathcal{S}$ is the nerve of a partially-ordered set on the set of pairs (C, F), where C is a chamber of $\mathcal{A}_{\mathbb{R}}$ and F is a face of C. The boundary of a typical top-dimensional cell, corresponding to the pair $(C, 0)$, is illustrated in Figure 1. Recently a pair of complexified real arrangements with the same combinatorics but different arrangement groups has been found [1], resolving a long-standing open problem. These arrangements also have 13 lines.

Milnor fibers. The restriction $Q\colon M \to \mathbb{C}^\times$ of the defining polynomial Q is a locally trivial fibration, known as the Milnor fibration; the fiber $F = \{Q = 1\}$ is the *Milnor fiber* of $\mathcal{A}$. It is an open question whether the betti numbers $b_i(F)$

are determined by $d_{\mathcal{A}}$, although the author has a conjectural solution for the case $i = 1$. Since $\mathbb{C}^\times$ is aspherical, F has the homotopy type of the $\mathbb{Z}$-cover of M classified by $Q_*\colon \pi_1(M) \to \pi_1(\mathbb{C}^\times) = \mathbb{Z}$. The cyclic group of order n acts freely on F, by scalar multiplication, with orbit space $\overline{M}$. It is also an open problem whether the monodromy homomorphism $H^1(F) \to H^1(F)$ is determined by $d_{\mathcal{A}}$.

Coxeter arrangements and discriminants. Suppose $\mathcal{A}$ is a complexified real arrangement. For $H \in \mathcal{A}$, let $s_H\colon \mathbb{R}^\ell \to \mathbb{R}^\ell$ denote the orthogonal reflection across the hyperplane $H_{\mathbb{R}}$. If $s_H(K_{\mathbb{R}}) \in \mathcal{A}_{\mathbb{R}}$ for every $H, K \in \mathcal{A}$, $\mathcal{A}$ is a *Coxeter arrangement.* This is the case for the example in Figure 1. The finite real reflection group W generated by $\{s_H \mid H \in \mathcal{A}\}$ is a Coxeter group. It acts on V and the space of orbits is isomorphic to $\mathbb{C}^\ell$; coordinates on the orbit space are given by a set of homogeneous invariant polynomials for W. The image Δ_W of D under the orbit map $\mathbb{C}^\ell \to \mathbb{C}^\ell$ is the *discriminant* associated with W. The fundamental group of $\mathbb{C}^\ell \setminus \Delta_W$ is the Artin group associated with the Coxeter presentation of W—see [2]. These spaces are aspherical [3]. In the example, W is the symmetric group S_3 and the associated Artin group is the full braid group on three strands. The Salvetti complex $\mathcal{S}$ can be constructed equivariantly, so as to yield a finite cell complex with the homotopy type of $\mathbb{C}^\ell \setminus \Delta_W$, and hence a finite model for the associated Artin group of finite type.

Some new models. In [4] we defined a variation on the Salvetti complex for complexified real arrangements. One defines a partial ordering on the set Q of ordered pairs of chambers of $\mathcal{A}_{\mathbb{R}}$: $(R, S) \leq (U, V)$ if there is a minimal gallery from R to S that can be extended to a minimal gallery from U to V. The nerve of this poset is homotopy equivalent to M. If $\mathcal{A}$ is a Coxeter arrangement, the poset Q carries an action of W, and yields a model for the complement $\mathbb{C}^\ell \setminus \Delta_W$ of the discriminant, and hence for the associated finite-type Artin group. This model is the nerve of the acyclic category with set of objects W and morphisms $u \to v$ labelled by pairs (x, y) of group elements giving a reduced factorization $v = xuy$ relative to the Coxeter generators. The composite of morphisms $(x, y)\colon u \to v$ and $(s, t)\colon w \to u$ is $(xs, ty)\colon w \to y$. In Figure 2 we illustrate the category for W of

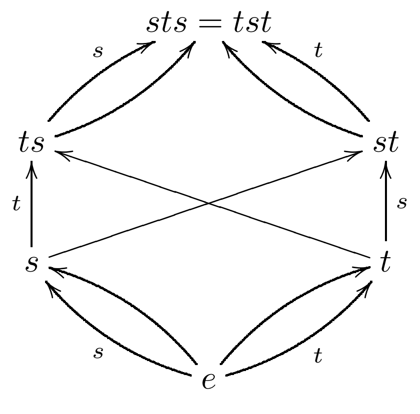

FIGURE 2. A model for the braid group on three strands.

type A_2, the example in Figure 1. Only non-identity indecomposable morphisms are pictured; the morphism (x, y) is labeled x or is unlabelled if $x = e$.

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Polynomial functors and homological stability

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(joint work with Aurélien Djament)

The definition of polynomial functors on a category of modules over a ring R has been introduced by Eilenberg and Mac Lane [5] using the notion of cross-effects. A typical example of a polynomial functor of degree n is the n-th tensor power $T^n\colon R\text{-}Mod \to R\text{-}Mod$ defined by $T^n(V) = V^{\otimes n}$. The definition of Eilenberg and Mac Lane can easily be extended to functors on a monoidal category whose unit is a null object. Several natural functors having polynomial properties are defined only on monoidal categories $(\mathcal{M}, \oplus, 0)$ whose unit 0 is an initial object but is not a terminal object. Examples of such categories are

- the category $(FI, \amalg, \emptyset)$ of finite sets and injections;
- the category $(S(\mathbb{Z}), \oplus, 0)$ having as objects the finitely generated free abelian groups and as morphisms

$$S(\mathbb{Z})(\mathbb{Z}^{\oplus n}, \mathbb{Z}^{\oplus m}) = \big\{(u, v) \in Ab(\mathbb{Z}^{\oplus n}, \mathbb{Z}^{\oplus m}) \times Ab(\mathbb{Z}^{\oplus m}, \mathbb{Z}^{\oplus n}) / v \circ u = Id\big\}$$

  where Ab is the category of abelian groups,
- the homogeneous category associated to braid groups $(U\beta, \amalg, \emptyset)$. (See [7] for the definition of homogeneous category. For examples of polynomial functors on this category see [9] or the extended abstract of Soulié in this report).

In [4] we introduce two notions of polynomial functors on a symmetric monoidal category whose unit is an initial object, extending the original definition of Eilenberg and Mac Lane: the strong polynomial and the weak polynomial functors. This talk is an overview of [4].
The strong polynomial functors are related to representation stability by the following proposition.

Proposition 1. [4] *Let F be a functor from FI to Ab. The functor F is strong polynomial with finitely generated values iff it is finitely generated.*

To describe stable phenomena the weak polynomial degree is more suitable than the strong polynomial degree. For example, the functor $T^n_{\geq i} : FI \to Ab$ defined by $T^n_{\geq i}(k) = T^n(\mathbb{Z}^k)$ for $k \geq i$ and 0 otherwise, is strong polynomial of degree $n + i$ and weak polynomial of degree n. Stably this functor behaves as T^n.

Let $(\mathcal{M}, \oplus, 0)$ be a small symmetric monoidal category where 0 is an initial object and generated by an object x (i.e. for each object $m \in \mathcal{M}$ there exists $k \in \mathbb{N}$ such that $m \simeq x^{\oplus k}$). For example FI is generated by 1 the set having one element.

1. Strong polynomial functors

1.1. **Definition.** Let $Func(\mathcal{M}, Ab)$ be the category of functors from $\mathcal{M}$ to Ab. The shift functor $\tau_x : Func(\mathcal{M}, Ab) \to Func(\mathcal{M}, Ab)$ is defined by $\tau_x(F) = F(x \oplus -)$. Since 0 is initial, there is a unique map $0 \to x$ inducing a natural transformation $i_x : Id \to \tau_x$. The cokernel of this transformation is the difference functor denoted by δ_x and the kernel is the evanescence functor denoted by κ_x.

Definition 2. *A functor $F : \mathcal{M} \to Ab$ is strong polynomial of degree $\leq d$ if $\delta_x^{d+1} F = 0$.*

If the unit 0 is also terminal, i_x splits so $\kappa_x = 0$. We recover the definition of usual polynomial functors using the difference functor (see for example [6]). This definition is equivalent to the original definition of Eilenberg and Mac Lane.

1.2. **Examples.**

- The constant functor $\mathbb{Z} : FI \to Ab$ defined by $\mathbb{Z}(k) = \mathbb{Z}\ \forall k \in \mathbb{N}$ is strong polynomial of degree 0.
- The atomic functor $\mathbb{Z}_i : FI \to Ab$ defined by $\mathbb{Z}_i(k) = \mathbb{Z}$ for $k = i$ and 0 otherwise, is strong polynomial of degree i.
- The functor $\mathbb{Z}_{\geq i} : FI \to Ab$ defined by $\mathbb{Z}_{\geq i}(k) = \mathbb{Z}$ for $k \geq i$ and 0 otherwise, is strong polynomial of degree i.

Since $\mathbb{Z}_{\geq i}(k)$ is a subfunctor of $\mathbb{Z}$ we deduce that the category of strong polynomial functors of degree $\leq d$ is not closed under subobjects.
By Proposition 1, the examples of finitely generated FI-modules given in [1] are examples of strong polynomial functors.

2. Weak polynomial functors

Stably the functors $\mathbb{Z}$ and $\mathbb{Z}_{\geq i}$ are equal. We will introduce a quotient of the category $Func(\mathcal{M}, Ab)$, named the stable category, in which these two functors are equal and we will define polynomial functors in this quotient category.

2.1. **The stable category** $St(\mathcal{M}, Ab)$**.** A functor $F\colon \mathcal{M} \to Ab$ is stably zero if $\underset{n\in\mathbb{N}}{\operatorname{colim}} F(x^{\oplus n}) = 0$. For example, $\mathbb{Z}_i$ is stably zero. We denote by $\mathcal{SN}(\mathcal{M}, Ab)$ the full subcategory of $Func(\mathcal{M}, Ab)$ of stably zero functors. The category $\mathcal{SN}(\mathcal{M}, Ab)$ is a thick subcategory of $Func(\mathcal{M}, Ab)$ so we can give the following definition.

Definition 3. *The stable category $St(\mathcal{M}, Ab)$ is the quotient category*

$$Func(\mathcal{M}, Ab)/\mathcal{SN}(\mathcal{M}, Ab).$$

We denote by $\pi_{\mathcal{M}}$ the functor $Func(\mathcal{M}, Ab) \to Func(\mathcal{M}, Ab)/\mathcal{SN}(\mathcal{M}, Ab)$. The functor κ_x takes its values in $\mathcal{SN}(\mathcal{M}, Ab)$.

Definition 4.
(1) A functor $F \in St(\mathcal{M}, Ab)$ is polynomial of degree $\leq d$ if $\delta_x^{d+1} F = 0$.
(2) A functor $F \in Func(\mathcal{M}, Ab)$ is weak polynomial of degree $\leq d$ if $\pi_{\mathcal{M}}(F)$ is polynomial of degree $\leq d$.

A strong polynomial functor of degree d is weak polynomial of degree $\leq d$. For example, the functor $\mathbb{Z}_{\geq i}$ is strong polynomial of degree i and weak polynomial of degree 0. The converse of the previous statement is not true. For example, the functor $\bigoplus_{i\in\mathbb{N}} \mathbb{Z}_{\geq i}$ is weak polynomial of degree 0 but is not strong polynomial.

If the unit 0 is also terminal $St(\mathcal{M}, Ab) = Func(\mathcal{M}, Ab)$. In this case the notions of strong polynomiality, weak polynomiality, polynomiality in $St(\mathcal{M}, Ab)$ and polynomiality in the sense of Eilenberg and Mac Lane are equivalent.

The category of polynomial functors of degree $\leq d$ in $St(\mathcal{M}, Ab)$ (denoted by $Pol_d(\mathcal{M}, Ab)$) is thick. In [4] we study the quotient categories

$$Pol_d(\mathcal{M}, Ab)/Pol_{d-1}(\mathcal{M}, Ab).$$

Note that in [2] the authors call "stable degree" the weak polynomial degree.

2.2. **Examples.**

(1) In [3] (see also the extended abstract of Djament in this report) Djament computes the weak polynomial degree of the homology of congruence subgroups.
(2) Let $\phi\colon Aut(F_n) \to GL_n(\mathbb{Z})$ be the map induced by the abelianisation and $IA_n = ker(\phi)$, Djament gives the following conjecture.

    Conjecture 5. *The functor $H_k(IA_\bullet)\colon S(\mathbb{Z}) \to Ab$ is weak polynomial of degree $3k$.*

(3) Let γ_{k+1} be the lower central series, $\psi\colon Aut(F_n) \to Aut(F_n/\gamma_{k+1}(F_n))$ and $\mathcal{A}_k(F_n) = ker(\psi)$. We have a functor $\mathcal{A}_k/\mathcal{A}_{k+1} : S(\mathbb{Z}) \to Ab$.

    Proposition 6. [4, Proposition 6.3] *The functor $\mathcal{A}_k/\mathcal{A}_{k+1}\colon S(\mathbb{Z}) \to Ab$ is weak polynomial of degree $k+2$.*

    The keystone of the proof of this proposition is the description of the cokernel of the Johnson homomorphism for $Aut(F_n)$ given by Satoh in [8].

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Homology of braid group with coefficients in symplectic representations

Filippo Callegaro
(joint work with Mario Salvetti)

We consider the family of hyperelliptic curves

$$\mathrm{E}_n^d := \{(\mathrm{P}, z, y) \in \mathrm{C}_n \times \mathrm{D} \times \mathbb{C} | y^d = (z - x_1) \cdots (z - x_n)\}.$$

where D is the unit open disk in $\mathbb{C}$, C_n is the configuration space of n distinct unordered points in D and $\mathrm{P} = \{x_1, \ldots, x_n\} \in \mathrm{C}_n$. For each configuration $\mathrm{P} \in \mathrm{C}_n$ the equation

$$y^d = (z - x_1) \cdots (z - x_n)$$

defines a curve that we call Σ_n^d. Each curve Σ_n^d in the family is a d-fold covering of the disk D ramified along the set P and there is a fibration $\pi : \mathrm{E}_n^d \to \mathrm{C}_n$ which takes Σ_n^d onto its set of ramification points.

The bundle $\pi : \mathrm{E}_n^d \to \mathrm{C}_n$ has a global section, so $H_*(\mathrm{E}_n^d)$ splits as a direct sum $H_*(\mathrm{C}_n) \oplus H_*(\mathrm{E}_n^d, \mathrm{C}_n)$ and by the Serre spectral sequence $H_*(\mathrm{E}_n^d, \mathrm{C}_n) = H_{*-1}(\mathrm{Br}_n; H_1(\Sigma_n^d))$, where Br_n is the classical Artin braid group on n strands.

The surface Σ_n^d has Euler characteristic $\chi = d - n(d-1)$ and the number of connected component of the boundary is $\gcd(n, d)$. In particular when $d = 2$ the surface Σ_n^d has genus $\frac{n-1}{2}$ if n is odd and $\frac{n-2}{2}$ if n is even. The representation of the group Br_n on the fundamental group of the surface Σ_n^2 is described in [5] (see also [4]).

Following some ideas in [1] we can project the space E_n^d to the product $\mathrm{C}_n \times \mathrm{D}$, that decomposes as a union of two open sets: the first one is homotopy equivalent to the configuration space $\mathrm{C}_{1,\mathrm{n}}$ of n distinct marked points and one additional distinguished point in D, the second one is homotopy equivalent to the configuration space $\mathrm{C}_{1,\mathrm{n\text{-}1}}$ and their intersection is homotopy equivalent to the product

$\mathrm{C}_{1,\text{n-1}} \times S^1$. This induces a decomposition of E_n^d. The associated Mayer-Vietoris long exact sequence can be used to compute the homology of E_n^d.

The rational homology of the space E_n^d has been computed in [3].

Our main results give a complete description of the homology of E_n^2 for n odd:

Theorem 1. *For odd n :*

(1) the integral homology $H_i(\mathrm{Br}_n; H_1(\Sigma_n^2;\mathbb{Z}))$ has only 2-torsion.

(2) the rank of $H_i(\mathrm{Br}_n; H_1(\Sigma_n^2;\mathbb{Z}))$ as a $\mathbb{Z}_2$-module is the coefficient of $q^i t^n$ in the series

$$\widetilde{P}_2(q,t) = \frac{qt^3}{(1-t^2q^2)} \prod_{i\geq 0} \frac{1}{1-q^{2^i-1}t^{2^i}}.$$

In particular the series $\widetilde{P}_2(q,t)$ is the Poincaré series of the homology group

$$\bigoplus_{n\,odd} H_*(\mathrm{Br}_n; H_1(\Sigma_n^2;\mathbb{Z}))$$

as a $\mathbb{Z}_2$-module.

The homology group $H_1(\Sigma_n^d)$ can be seen as a polynomial coefficient system for Br_n (see [6] for a definition of polynomial coefficient system). Hence the homology computed in the previous theorem stabilizes. For $d = 2$ the stable homology is described in the following result.

Theorem 2. *Let us consider homology with integer coefficients.*

(1) The homomorphism

$$H_i(\mathrm{Br}_n; H_1(\Sigma_n^2)) \to H_i(\mathrm{Br}_{n+1}; H_1(\Sigma_{n+1}^2))$$

is an epimorphism for $i \leq \frac{n}{2} - 1$ and an isomorphism for $i < \frac{n}{2} - 1$.

(2) For n even $H_i(\mathrm{Br}_n; H_1(\Sigma_n^2))$ has no p torsion (for $p > 2$) when $\frac{pi}{p-1} + 3 \leq n$ and no free part for $i + 3 \leq n$. In particular for n even, when $\frac{3i}{2} + 3 \leq n$ the group $H_i(\mathrm{Br}_n; H_1(\Sigma_n^2))$ has only 2-torsion.

(3) The Poincaré polynomial of the stable homology $H_i(\mathrm{Br}_n; H_1(\Sigma_n^2;\mathbb{Z}))$ as a $\mathbb{Z}_2$-module is the following:

$$P_2(\mathrm{Br}; H_1(\Sigma^2))(q) = \frac{q}{1-q^2} \prod_{j\geq 1} \frac{1}{1-q^{2^j-1}}.$$

When d is greater than 2 the same argument gives a partial description of the homology of E_n^d and of its stabilization.

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Homology of surface and graph braid groups

Ben Knudsen
(joint work with Byung Hee An and Gabriel C. Drummond-Cole)

For a topological space X, we consider the unordered *configuration space* of k points in X, which is the quotient

$$B_k(X) = \left\{(x_1, \ldots, x_k) \in X^k : x_i \neq x_j \text{ if } i \neq j\right\}_{/\Sigma_k}.$$

It is convenient to consider the graded space $B(X) = \coprod_{k\geq 0} B_k(X)$.

Example. If X is an aspherical surface or a graph, then $B_k(X)$ is a classifying space for its fundamental group, the kth surface or graph *braid group*, respectively.

When the background space X is a manifold, we have the following calculation [7], which unifies and extends partial results of [3, 6].

Theorem (K). *Let M be an n-manifold. There is an isomorphism of bigraded Abelian groups*

$$H_*(B(M); \mathbb{Q}) \cong H^{\mathcal{L}}(\mathfrak{g}_M),$$

where $\mathfrak{g}_M$ is the graded Lie algebra defined by

$$\mathfrak{g}_M = \begin{cases} H_c^{-*}(M; \mathbb{Q}^w) \otimes v & n \text{ odd} \\ H_c^{-*}(M; \mathbb{Q}^w) \otimes v \oplus H_c^{-*}(M; \mathbb{Q}) \otimes [v, v] & n \text{ even.} \end{cases}$$

Here,

- $H^{\mathcal{L}}$ *denotes Lie algebra homology,*
- H_c *denotes compactly supported cohomology,*
- $\mathbb{Q}^w$ *denotes the orientation sheaf of M, and*
- v *and* $[v, v]$ *are formal parameters in bigrading* $(n-1, 1)$ *and* $(2n-2, 2)$*, respectively.*

The Lie algebra homology in question may be computed by means of the classical Chevalley–Eilenberg complex. This complex is very amenable to computation; for example, we are able to determine explicit formulas for $\dim H_i(B_k(\Sigma); \mathbb{Q})$ for every $i, k \geq 0$ and surface Σ [5].

Remark. The dual of the Chevalley–Eilenberg complex for $\mathfrak{g}_M$ coincides with the direct sum over k of the Σ_k-invariant part of the E_2 page of the spectral sequence considered in [11]. Thus, our result may be interpreted as asserting the vanishing of all higher differentials in these spectral sequences of Σ_k-invariants. In contrast, the full spectral sequence is known not to collapse in general.

The Lie algebra homology of any Lie algebra is naturally a cocommutative coalgebra. Consideration of this structure leads to an alternate proof of homological stability for configuration spaces [4].

Corollary. *Suppose that M is connected and $n > 1$. The cap product with $1 \in H^0(M;\mathbb{Q}) \subseteq H_*(B(M);\mathbb{Q})^\vee$ induces an isomorphism*

$$H_i(B_{k+1}(M);\mathbb{Q}) \xrightarrow{\simeq} H_i(B_k(M);\mathbb{Q})$$

for $i \leq k$ and a surjection in the next degree.

The proof is based on elementary combinatorial facts about the Chevalley–Eilenberg complex. If M is not an orientable surface, a slightly better stable range obtains.

We turn now to the case of a graph Γ with set of vertices V, set of edges E, and set of half-edges H. For $v \in V$, we write

$$S(v) = \mathbb{Z}\langle \varnothing, v, h \in H(v)\rangle,$$

where $H(v)$ denotes the set of half-edges incident on v. The *Swiatkowski complex* of Γ is the Abelian group

$$S(\Gamma) = \mathbb{Z}[E] \otimes \bigotimes_{v \in V} S(v),$$

endowed with the differential determined by the equation $\partial(h) = e(v) - v(h)$ and the bigrading determined by declaring that $|\varnothing| = (0,0)$, $|v| = |e| = (0,1)$, and $|h| = (1,1)$. We prove the following [1].

Theorem (A–D-C–K)**.** *There is a natural isomorphism of bigraded $\mathbb{Z}[E]$-modules*

$$H_*(B(\Gamma);\mathbb{Z}) \cong H_*(S(\Gamma)).$$

The complex $S(\Gamma)$ is a functorial and algebraic enhancement of the cellular chains on the cubical model considered in [10] (see also [8]). The $\mathbb{Z}[E]$-action arises geometrically from the process of *edge stabilization*, which replaces a subconfiguration on the edge e with the collection of pairwise averages of the points in this subconfiguration and the endpoints (a similar stabilization mechanism for trees is considered in [9]).

This structure provides a natural setting in which to study analogues of classical homological stability phenomena.

Corollary. *The bigraded $\mathbb{Z}[E]$-module $H_*(B(\Gamma);\mathbb{Z})$ is finitely presented.*

This algebraic structure is usually rather complicated; indeed, $S(\Gamma)$ is formal as a $\mathbb{Z}[E]$-module if and only if each component of Γ is homeomorphic to a graph in which each vertex has at most two edges [2].

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Topology of Enumerative Problems: Inflection Points on Cubic Curves

Weiyan Chen

A plane cubic curve is given by the vanishing locus of a complex homogeneous polynomial $F(x, y, z)$ of degree 3. It is a classical result that every smooth cubic plane curve has exactly 9 inflection points (also called "flexes"), *i.e.* points where the Hessian vanishes. In other words, every smooth cubic plane curve naturally comes with 9 marked points. Motivated by this classical result, Benson Farb asked the following question:

Question 1 (Farb)**.** What are all the possible ways to continuously choose n distinct unlabeled points on any smooth cubic plane curve, as the curve varies in family?

To make this question precise, we define the following space parameterizing smooth cubic curves:

$$\mathcal{X} := \{F(x, y, z) : F \text{ is a homogeneous polynomial of degree 3 and is smooth}\}/\sim$$

where $F \sim cF$ for any $c \in \mathbb{C}^\times$. Each $F \in \mathcal{X}$ gives a well-defined smooth cubic curve C_F in $\mathbb{C}\mathrm{P}^2$. This construction gives the following fiber bundle over $\mathcal{X}$:

$$\begin{tikzcd} C_F \arrow[r] & E \arrow[d, "\xi"] \\ & \mathcal{X} \end{tikzcd}$$

where the total space $E := \left\{(F, p) \in \mathcal{X} \times \mathbb{C}\mathrm{P}^2 : p \in C_F\right\}$ can be viewed as the universal cubic curve. Each fiber C_F is a Riemann surface of genus 1.

Definition 1. A *multisection* for ξ of degree n is a triple $(\widetilde{\mathcal{X}}, p, i)$ where $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ is a cover of degree n, and $i : \widetilde{\mathcal{X}} \to E$ is a continuous injection such that $p = \xi \circ i$, making the following diagram commute:

$$\begin{tikzcd} \{n \text{ points}\} \arrow[r] \arrow[d, hook] & \widetilde{\mathcal{X}} \arrow[r, "p"] \arrow[d, hook, "i"] & \mathcal{X} \arrow[d, "="] \\ C_F \arrow[r] & E \arrow[r, "\xi"'] & \mathcal{X} \end{tikzcd} \tag{1}$$

Thus, a multisection is a continuous choice of n distinct points on the cubic curve C_F as F varies in the family $\mathcal{X}$. Sometimes we will just call $\widetilde{\mathcal{X}}$ a multisection when p and i are clear from the context.

Example 1 (Inflection points)**.** Define $\widetilde{\mathcal{X}}^{\text{flex}}$ to be

$$\widetilde{\mathcal{X}}^{\text{flex}} := \{(F, q) \in \mathcal{X} \times \mathbb{C}\mathrm{P}^2 : q \text{ is an inflection point on } C_F\}.$$

Let $p_{\text{flex}} : \widetilde{\mathcal{X}}^{\text{flex}} \to \mathcal{X}$ be the projection onto the first factor. Then $\widetilde{\mathcal{X}}^{\text{flex}}$ defines a multisection for ξ of degree 9.

Question 1 asks for a classification of multisections of ξ. To aim for a partial answer, Farb made the following conjecture:

Conjecture 1 (Farb)**.** There is no multisection for ξ of degree $n < 9$.

It turns out that ξ does admit multisections of degree $n > 9$:

Example 2 (A multisection of degree 36)**.** Define $\widetilde{\mathcal{X}}^{36}$ to be

$$\widetilde{\mathcal{X}}^{36} := \Big\{(F, q) \in \mathcal{X} \times \mathbb{C}\mathrm{P}^2 : \\ q \text{ is a point on } C_F \text{ whose tangent line passes through a flex}\Big\}$$

Let $p : \widetilde{\mathcal{X}}^{36} \to \mathcal{X}$ be the projection onto the first factor. Then $\widetilde{\mathcal{X}}^{36}$ defines a multisection for ξ of degree 36.

Let us make the following two observations from Example 2. First, $\widetilde{\mathcal{X}}^{36}$ has an intermediate cover which is exactly $\widetilde{\mathcal{X}}^{\text{flex}}$. Thus, the multisection $\widetilde{\mathcal{X}}^{36}$ "factors through" the multisection of 9 flexes. Second, if we choose a flex p on C_F to be the identity for the elliptic curve, then the multisection $\widetilde{\mathcal{X}}^{\text{flex}}$ picks the 3-torsions of (C_F, p), while $\widetilde{\mathcal{X}}^{36}$ picks the 6-torsions of (C_F, p). Thus, $\widetilde{\mathcal{X}}^{36}$ comes from an algebraic construction.

Therefore, what is behind Conjecture 1 is the following metaconjecture:

Metaconjecture 2 (Farb)**.** There is no multisection for ξ unless there is an algebraic reason for it to exist.

To state the main theorem, we need to first introduce a cover $\widetilde{\mathcal{X}}^{\text{ncf}}$ of $\mathcal{X}$:

$$\widetilde{\mathcal{X}}^{\text{ncf}} := \Big\{ \Big(F, \{p,q,r\}\Big) \in \mathcal{X} \times \operatorname{Sym}^3(\mathbb{C}\mathrm{P}^2) : \ \{p,q,r\} \text{ is a triple of} \\ \text{three non-collinear infection points on } C_F \Big\}.$$

$\widetilde{\mathcal{X}}^{\text{ncf}}/\mathcal{X}$ is a cover of degree 72, since there are $\binom{9}{3} = 84$ unordered triples of flexes, 12 of which are collinear.

We will say a cover $\widetilde{\mathcal{X}}_1/\mathcal{X}$ *factors through* another cover $\widetilde{\mathcal{X}}_2/\mathcal{X}$ if the later is an intermediate cover of the former.

Theorem 1. *If $\widetilde{\mathcal{X}}/\mathcal{X}$ gives a multisection of ξ, then each connected component of $\widetilde{\mathcal{X}}$ must factor through either $\widetilde{\mathcal{X}}^{\text{flex}}/\mathcal{X}$ or $\widetilde{\mathcal{X}}^{\text{ncf}}/\mathcal{X}$.*

We already knew that $\widetilde{\mathcal{X}}^{\text{flex}}$ is a multisection of degree 9 (Example 1). However, currently it is not known whether $\widetilde{\mathcal{X}}^{\text{ncf}}$ can be made into a multisection or not. What is missing is the injection i in the commutative diagram (1).

Question 2. Does there exist a continuous injective map $i : \widetilde{\mathcal{X}}^{\text{ncf}} \to E$ making the diagram (1) commute? Equivalently, is it possible to associate 72 distinct points $x_{\{p,q,r\}}$ to the 72 triples $\{p,q,r\}$ such that the choice varies continuously with $(F, \{p,q,r\}) \in \widetilde{\mathcal{X}}^{\text{ncf}}$?

Either a positive or a negative answer to Question 2 will be very interesting because:

- If $\widetilde{\mathcal{X}}^{\text{ncf}}/\mathcal{X}$ is not a multisection, then Theorem 1 implies that every multisection must factor through $\widetilde{\mathcal{X}}^{\text{flex}}$, and therefore $\widetilde{\mathcal{X}}^{\text{flex}}$ is the universal multisection.
- If $\widetilde{\mathcal{X}}^{\text{ncf}}/\mathcal{X}$ is a multisection given by certain algebraic construction, then every smooth cubic curve naturally has 72 special points on it. These 72 special points are perhaps as interesting as the 9 flexes.
- If $\widetilde{\mathcal{X}}^{\text{ncf}}/\mathcal{X}$ is a multisection that is continuous but is not from any algebraic construction, then it is a counter-example to Farb's Metaconjecture 2.

Theorem 1 implies that Farb's Conjecture 1 is true. In fact, it implies the following corollary which is stronger than Conjecture 1:

Corollary 2. *The bundle ξ admits no multisection of degree n if n is not a multiple of 9.*

Proof. The degree of a cover is multiplicative when two covers are composed. Either $\widetilde{\mathcal{X}}^{\text{flex}}/\mathcal{X}$ or $\widetilde{\mathcal{X}}^{\text{ncf}}/\mathcal{X}$ is of degree a multiple of 9. Thus, Theorem 1 implies that degree of any multisection $\widetilde{\mathcal{X}}/\mathcal{X}$ must be a multiple of 9. □

Every smooth cubic plane curve is a Riemann surface of genus 1, and thus is an elliptic curve. However, Corollary 2 implies that the bundle ξ does not admit a multisection of degree 1, or equivalently, that it is not possible to continuously choose one point on every smooth cubic curve to serve as the identity. Therefore, we conclude:

Corollary 3. *It is not possible to continuously choose an elliptic curve structure for all smooth cubic plane curves.*

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Milnor monodromy of plane curves, space surfaces and hyperplane arrangements

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(joint work with Gabriel Sticlaru)

Let $C : f(x, y, z) = 0$ be a reduced plane curve in the complex projective plane $\mathbb{P}^2$, defined by a degree d homogeneous polynomial f in the graded polynomial ring $S = \mathbb{C}[x, y, z]$. The smooth affine surface $F : f(x, y, z) = 1$ in $\mathbb{C}^3$ is called the Milnor fiber of f. The mapping $h : F \to F$ given by $(x, y, z) \mapsto (\theta x, \theta y, \theta z)$ for $\theta = \exp(2\pi i/d)$ is called the monodromy of f. There are induced monodromy operators $h^j : H^j(F, \mathbb{C}) \to H^j(F, \mathbb{C})$, $h^j(\omega) = (h^{-1})^*(\omega)$, for $j = 0, 1, 2$, and we can look at the corresponding characteristic polynomials

$$\Delta^j_C(t) = \det(t \cdot Id - h^j | H^j(F, \mathbb{C})).$$

Given the degree d reduced plane curve $C : f = 0$ and a d-th root of unity $\lambda \neq 1$, how to determine the multiplicity $m(\lambda)$ of λ as a root of the Alexander polynomial $\Delta^1_C(t)$? This question, or higher dimensional versions of it, has a very long tradition, see for instance [1, 2, 3, 4, 6, 7, 8, 10, 11, 12, 13, 16]. A general answer is given by the following result, see [5, 6, 8, 10, 15]. Let $K^*_f = (\Omega^*, df\wedge)$ be the Koszul complex of the partial derivatives f_x, f_y, f_z of f in S.

Theorem 1. *For any integer k with $1 \leq k \leq d$, there is an E_1- spectral sequence $E_*(f)_k$ such that*

$$E_1^{s,t}(f)_k = H^{s+t+1}(K^*_f)_{td+k}$$

and converging to

$$E_\infty^{s,t}(f)_k = Gr^s_P H^{s+t}(F, \mathbb{C})_\lambda,$$

where P^* is a decreasing filtration on the Milnor fiber cohomology, called the pole order filtration. Moreover, $E_2(f)_k = E_\infty(f)_k$ if and only if C has only weighted homogeneous singularities.

In fact this result is valid for projective hypersurfaces of any dimension. In the case of plane curves this gives the following, see [8, 10]. This results says that for a plane curve the computations of a limited number of the terms in the second page of the spectral sequence is enough to determine all the Alexander polynomials $\Delta^j_C(t)$.

Theorem 2. *Let $C : f = 0$ be a reduced degree d curve , and let $\lambda = \exp(-2\pi ik/d)$, with $k \in (0, d)$ an integer. Then λ is a root of the Alexander polynomial $\Delta_C^1(t)$ of multiplicity $m(\lambda)$ given by*

$$m(\lambda) = \dim E_2^{1,0}(f)_k + \dim E_2^{1,0}(f)_{k'}.$$

In the case of hyperplane arrangements in $\mathbb{P}^n$, one has the following **Conjecture**. For any hyperplane arrangement $V : f = 0$ in $\mathbb{P}^n$ and any integer k with $1 \leq k \leq d$, d being the number of hyperplanes in V, the E_1- spectral sequence $E_*(f)_k$ degenerates at the second page, i.e.

$$E_2(f)_k = E_\infty(f)_k.$$

Moreover, one has $E_2^{s,t}(f)_k = 0$ for $t > 1$, see [10].

A lot of examples suggest that this conjecture holds, and this is very important for doing computations in terms of computing time. The corresponding SINGULAR codes are available at `http://math1.unice.fr/~dimca/singular.html`. These codes are very effective for plane curves, especially for the free and nearly free curves, as explained in [8, 10]. Indeed, for plane curves, the graded cohomology group $H^2(K_f^*)$ is determine by the graded S-module of Jacobian syzygies

$$AR(f) = \{(a, b, c) \in S^3 \ \mid \ af_x + f_y + cf_z = 0\},$$

which is free for a free curve, and has a very precise resolution in the case of a nearly free curve, see [9].

There are many interesting relations of the above results with the roots of the Bernstein–Sato polynomials for projective hypersurfaces, which can be found in [10, 15].

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Long-Moody constructions and generalizations

Arthur Soulié

In [4], Long and Moody gave a construction on representations of braid groups which associates a representation of $\mathbf{B}_n$ with a representation of $\mathbf{B}_{n+1}$. This construction complexifies in a sense the initial representation: for instance, starting from a dimension one representation, one obtains the unreduced Burau representation. This construction inspires endofunctors, called Long–Moody functors, on a suitable category of functors (see [7]). This construction also generalizes to other families of groups such as automorphism groups of free groups, mapping class groups of orientable and non-orientable surfaces or mapping class groups of 3-manifolds (see [8]). Moreover adapting notions of strong polynomial functors in this context, the Long–Moody functors increase by one the degree of polynomiality (see [7, 8]).

1. Case of braid groups

1.1. **Categories.** The braid groupoid $\boldsymbol{\beta}$ has the natural numbers $n \in \mathbb{N}$ as objects and the braid groups B_n as automorphisms. A monoidal product $\natural : \boldsymbol{\beta} \times \boldsymbol{\beta} \to \boldsymbol{\beta}$ is defined assigning the usual addition for the objects and connecting two braids side by side for the morphisms (see [5]). The object 0 is the unit of this monoidal product. The strict monoidal groupoid $(\boldsymbol{\beta}, \natural, 0)$ is braided, its braiding is denoted by $b_{-,-}$.

Remark that a family of representations of braid groups is this way equivalent to define a functor $\boldsymbol{\beta} \to \mathbb{K}\text{-}\mathfrak{Mod}$ where $\mathbb{K}\text{-}\mathfrak{Mod}$ is the category of $\mathbb{K}$-modules for $\mathbb{K}$ a commutative ring. Note that all the classical representations of braid groups, such as Burau representations (see [2]), Tong–Yang–Ma representations (see [9]) or Lawrence–Krammer representations (see [3]) satisfy compatibility relations when one passes from $\mathbf{B}_n$ to $\mathbf{B}_{n+1}$. This motivates the use of Quillen's bracket construction over $\boldsymbol{\beta}$.

Definition 1.1. [6] *The category $\mathfrak{U}\boldsymbol{\beta}$ is defined by:*

- $Obj\,(\mathfrak{U}\boldsymbol{\beta}) = Obj\,(\boldsymbol{\beta}) = \mathbb{N}$;
- $Hom_{\mathfrak{U}\boldsymbol{\beta}}\,(n, n') = \underset{\boldsymbol{\beta}}{colim}\,[Hom_{\boldsymbol{\beta}}\,(-\natural n, n')]\,.$

Proposition 1.2. [6] *The category $\mathfrak{U}\boldsymbol{\beta}$ satisfies the following properties.*

- *The unit* 0 *is initial object. We denote by* $\iota_n : 0 \to n$ *the unique morphism from* 0 *to* N.
- $\natural$ *extends to give a monoidal structure* $(\mathfrak{U}\boldsymbol{\beta}, \natural, 0)$. *This category is not braided but pre-braided (see* [6]*).*

In [6, 7], it is proven that the families of Burau, Tong-Yang-Ma and Lawrence-Krammer representations define functors over the category $\mathfrak{U}\boldsymbol{\beta}$.

1.2. **Definition.** Consider $a_n : \mathbf{B}_n \to Aut\,(\mathbf{F}_n)$ a Wada representation (see [10]), where $\mathbf{F}_n$ is the free group on n generators. Let $\varsigma_n : \mathbf{F}_n \to \mathbf{F}_n \underset{a_n}{\rtimes} \mathbf{B}_n \to \mathbf{B}_{n+1}$ be a group morphism. Denote by $\mathcal{I}_{\mathbb{K}[\mathbf{F}_n]}$ the augmentation ideal of the free group $\mathbf{F}_n$. Long–Moody functors are defined by:

Theorem 1.3. [4, 7] *Let* $G \in Obj\,(\mathbf{Fct}\,(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}))$. *Assign:*

- $\forall n \in \mathbb{N}$, $\mathbf{LM}\,(G)\,(n) = \mathcal{I}_{\mathbb{K}[\mathbf{F}_n]} \underset{\mathbb{K}[\mathbf{F}_n]}{\otimes} G\,(n+1)$.
- *For* $[n'-n, \sigma] \in Hom_{\mathfrak{U}\boldsymbol{\beta}}\,(n, n')$*:*

$$\mathbf{LM}\,(G)\,([n'-n,\sigma]) = a_{n'}\,(\sigma) \underset{\mathbb{K}[\mathbf{F}_{n'}]}{\otimes} G\,(id_1 \natural\, [n'-n,\sigma])\,.$$

Then we define this way an object of $\mathbf{Fct}\,(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})$ *and this naturally extends to give an exact functor:* $\mathbf{LM}\colon \mathbf{Fct}\,(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}\,(\mathfrak{U}\boldsymbol{\beta}, \mathbb{K}\text{-}\mathfrak{Mod})\,.$

Using the Artin representation as a_n, the induced Long–Moody functor recovers the unreduced Burau functor, and its iteration allows to obtain Lawrence–Krammer as subfunctor. Another choice of a_n defines a Long–Moody functor which recovers the Tong–Yang–Ma functor.

2. Generalizations

We can generalize the principle of Long–Moody functors to other families of groups. Consider $(\mathcal{G}, \natural, 0)$ a braided monoidal groupoid, such that $Obj\,(\mathcal{G}) = \mathbb{N}$. We denote the automorphism groups of $\mathcal{G}$ by G_n. Quillen's construction $\mathfrak{U}$ applies in the same way as before and defines a pre-braided homogenous category $(\mathfrak{U}\mathcal{G}, \natural, 0)$. Let $\{H_m\}_{m\in\mathbb{N}}$ be a family of free groups with injections $H_m \hookrightarrow H_{m+1}$. Let us make the following analogy:

- $\mathbf{B}_n \longleftrightarrow G_n$
- $\mathbf{F}_n \longleftrightarrow H_n$
- $(\mathbf{B}_n \to Aut\,(\mathbf{F}_n)) \longleftrightarrow (G_n \to Aut\,(H_n))$
- $(\mathbf{F}_n \to \mathbf{F}_n \rtimes \mathbf{B}_n \to \mathbf{B}_{n+1}) \longleftrightarrow (G_n \to H_n \rtimes G_n \to H_{n+1})$

Theorem 2.1. [8] *Repeating mutatis mutandis the assignments of Theorem* 1.3*, we define an exact functor:*

$$\mathbf{LM} : \mathbf{Fct}\,(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod}) \to \mathbf{Fct}\,(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod})\,.$$

Applications 2.2. [8] *The following families of groups fit into this framework.*

- *The automorphism groups of free groups* $Aut(\mathbf{F}_n)$*. We can obtain the abelianization functor thanks to a Long–Moody functor.*
- *The mapping class groups of compact orientable connected surfaces with genus* g *and one boundary component* $\{\mathbf{\Gamma}_{g,1}\}_{g\in\mathbb{N}}$*. For example, a Long–Moody functor recovers the family of symplectic representations.*
- *The mapping class groups of compact, connected, oriented* 3*-manifold with boundary. This includes handlebody mapping class groups* $\mathcal{H}_{n,1}$ *or symmetric automorphisms of free groups* $\Sigma Aut(\mathbf{F}_n)$*.*

3. Polynomial behaviour

3.1. **Polynomial functors.** The notions of strong and weak polynomial functors for symmetric monoidal categories are introduced by Djament and Vespa in [1]. This is extended to the case of a pre-braided monoidal category in [7]. We take up the framework, definitions and terminology of the extended abstract of Vespa in this report. Recall that $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod}))$, the shift, difference and evanescence functors define a short exact sequence:

$$0 \to \kappa_1 F \to F \to \tau_1 F \to \delta_1 F \to 0.$$

Definition 3.1. *F is very strong polynomial of degree 0 if it is constant. For $d \geq 1$, F is very strong polynomial of degree $\leq d$ if it is strong polynomial of degree $\leq d$, $\kappa_1 F = 0$ and $\delta_1 F$ is very strong polynomial of degree $\leq d-1$.*

The concept of very strong polynomial functor corresponds to the one of coefficient system of finite degree at 0 in the terminology of [6].

3.2. **Effect of Long–Moody functors.**

Theorem 3.2. [8] *Let* $\mathbf{LM}$ *be any generalized Long–Moody functor. It induces a functor:*

$$\mathbf{LM} : \mathcal{P}ol_d^{strong}(\mathfrak{U}\mathcal{G}) \to \mathcal{P}ol_{d+1}^{strong}(\mathfrak{U}\mathcal{G}).$$

If $F \in Obj(\mathbf{Fct}(\mathfrak{U}\mathcal{G}, \mathbb{K}\text{-}\mathfrak{Mod}))$ is very strong (resp. weak) polynomial of degree $\leq d$, then $\mathbf{LM}(F)$ is very strong (resp. weak) polynomial of degree $\leq d+1$.

Thus, the Long–Moody constructions will provide new examples of twisted coefficients fitting into the framework developed by Randal-Williams and Wahl in [6] where prove homological stability for different families of groups, in particular for braid groups, mapping class groups of surfaces and 3-manifolds.

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Introduction to twisted commutative algebras

Andrew Snowden

1. Overview

In representation stability, one often has a sequence of representations connected by some kind of transition maps. Such a structure can often be conveniently encoded as a representation of a category. Recall that a representation of a category $\mathcal{C}$ is a functor from $\mathcal{C}$ to the category of vector spaces. Here are some examples of the kinds of categories that come up, and some sample applications of them.

Name	Definition	Application
$\mathbf{FI}$	finite sets / injections	Cohomology of configuration spaces [1]
$\mathbf{FI}_d$	finite sets / injections with a d-coloring on the complement of the image	Configuration spaces of graphs [6] Syzygies of Segre embeddings [10]
$\mathbf{FIM}$	finite sets / injections with a perfect matching on the complement of the image	Secondary stability [4] Representations of $\mathbf{O}_\infty$ [7]
$\mathbf{OI}$	finite totally ordered sets / order preserving injections	Homology of unipotent groups [5]
$\mathbf{FS}^{\mathrm{op}}$	finite sets / surjections (opposite category)	Homology of $\overline{\mathcal{M}}_{g,n}$ [11]
$\mathbf{VA}(\mathbf{F})$	finite dimensional vector spaces over $\mathbf{F}$ / linear maps	Steenrod algebra [3]

Representations of the first three categories are equivalent to modules over three specific twisted commutative algebras (tca's). The second three categories are not directly related to tca's. We hope this gives the reader some insight into the place that tca's occupy within representation stability.

2. Twisted commutative algebras: three definitions

We now give three equivalent ways to define tca's. Fix a commutative ring k.

Definition 1. A twisted commutative algebra is a graded associative unital k-algebra $A = \bigoplus_{n=0}^{\infty} A_n$ equipped with an action of the symmetric group S_n on A_n such that the following two conditions hold:

(1) The multiplication map $A_n \times A_m \to A_{n+m}$ is $S_n \times S_m \subset S_{n+m}$ equivariant.
(2) (Twisted commutativity.) Given $x \in A_n$ and $y \in A_m$, we have $yx = \tau(xy)$ where $\tau \in S_{n+m}$ is the element given by $\tau(i) = i + m$ for $1 \leq i \leq n$ and $\tau(i) = i - n$ for $n + 1 \leq i \leq n + m$.

Definition 2. Let **FB** be the groupoid of finite sets and bijections. A tca is then a lax symmetric monoidal functor $A\colon \mathbf{FB} \to \mathrm{Mod}_k$, where the monoidal structure on the source is disjoint union and on the target is tensor product. Precisely, this means A is a functor assigning to every finite set S a k-module A_S together with maps $A_S \otimes A_T \to A_{S \amalg T}$ (this is the "lax monoidal" part) such that the diagram

$$\begin{tikzcd} A_S \otimes A_T \arrow[r] \arrow[d] & A_{S\amalg T} \arrow[d] \\ A_T \otimes A_S \arrow[r] & A_{T\amalg S} \end{tikzcd}$$

commutes, where the vertical maps are the given isomorphisms (this is the "symmetric" part). The commutativity of this diagram corresponds to the twisted commutativity axiom in Definition 1.

Definition 3. A **representation of** S_* is a sequence $M = (M_n)_{n\geq 0}$ where M_n is a representation of S_n over k. If M and N are two representations of S_*, we define their tensor product to be the representation of S_* given by

$$(M \otimes N)_n = \bigoplus_{i+j=n} \mathrm{Ind}_{S_i \times S_j}^{S_n}(M_i \otimes_k N_j).$$

There is a natural isomorphism $M \otimes N \to N \otimes M$; this makes use of the element τ in Definition 1. In this way, the category $\mathrm{Rep}_k(S_*)$ of representations of S_* has a symmetric monoidal structure. A tca is just a commutative algebra object in this tensor category; that is, it is an object A of $\mathrm{Rep}_k(S_*)$ equipped with a multiplication map $A \otimes A \to A$ and a unit map $k \to A$ satisfying the usual axioms.

We note that there is a notion of module over a tca. From the perspective of Definition 3, a module is just a module object in the general sense of tensor categories.

Examples. We now give some examples of the definitions:

(1) Let A be the graded k-algebra $k[t]$, where t has degree 1. We regard this as a tca by letting S_n act trivially on each graded piece. An A-module is a representation M of S_* equipped with maps $M_n \to M_{n+1}$ (multiplication

by t). This looks a lot like what one gets from an **FI**-module, and, in fact, one can show that the category of A-modules is equivalent to the category of **FI**-modules.

(2) Let V be a k-module and let A be the tensor algebra on V, equipped with its usual grading. We let S_n act on $A_n = V^{\otimes n}$ by permuting tensor factors. Then A is a tca. If we regard V as a representation of S_* concentrated in degree 1 then A is in fact the symmetric algebra on V (in the tensor category $\mathrm{Rep}_k(S_*)$). If V is a free module of rank d then A-modules are equivalent to $\mathbf{FI}_d$-modules.

(3) Taking the perspective of Definition 2, let A_S be the free k-module on the set of matchings on S. (A matching is an undirected graph in which each vertex belongs to precisely one edge.) The multiplication map $A_S \otimes A_T \to A_{S \amalg T}$ is induced by taking the disjonit union of matchings. This is a tca. In fact, it is the symmetric algebra on the trivial representation of S_2, regarded as an object of $\mathrm{Rep}_k(S_*)$ concentrated in degree 2. Modules for this tca are equivalent to **FIM**-modules.

3. Structure theory

One of the main problems in the subject of tca's is to understand the structure of module categories. Some examples of the kinds of problems and results that have been studied:

(1) One of the main problems is noetherianity: if A is a finitely generated tca over a noetherian coefficient ring, is the module category Mod_A locally noetherian? This is known for the three examples given above (although only in characteristic 0 for the third example), and this nearly exhausts the list of known results. Draisma [2] has proven a topological version of noetherianity in general, which strongly suggests that Mod_A is locally noetherian in general.

(2) If M is a module over a tca (and k is a field), its Hilbert series is defined to be $\mathrm{H}_M(t) = \sum_{n \geq 0} \dim(M_n) \frac{t^n}{n!}$. One would like to know the form of this series. Much is known for modules over the three example tca's: for example, for Example 2 the Hilbert series is a polynomial in t and e^t. For more general tca's, not much is yet known.

(3) There are a whole manner of finer structural results for modules over the three example tca's, many of which are analogous to classical results in commutative algebra. For example, there is a theory of local cohomology and depth. See [8, 9].

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Configuration space in a product

John D. Wiltshire-Gordon

This talk explains how to compute the homotopy type of $\mathrm{Conf}(n, X \times Y)$ using the homotopy types of certain configuration spaces in X and Y separately.

First, note that the homotopy types of $\mathrm{Conf}(n, X)$ and $\mathrm{Conf}(n, Y)$ alone will not be enough to recover the homotopy type of $\mathrm{Conf}(n, X \times Y)$. For example, when $X = \{0, 1\}$ and $Y = \mathbb{R}$, the inclusion $X \subset Y$ induces an equivalence

$$\mathrm{Conf}(2, \{0, 1\}) \simeq \mathrm{Conf}(2, \mathbb{R})$$

but $\mathrm{Conf}(2, \mathbb{R} \times \mathbb{R}) \not\simeq \mathrm{Conf}(2, \{0, 1\} \times \{0, 1\})$. To avoid this pitfall, our theorem relies on a richer kind of configuration space.

If X is a space and Γ is a graph, define the **graphical configuration space**

$$\mathrm{Conf}(\Gamma, X) = \{f \colon V(\Gamma) \to X \ \text{ so that } a \sim_\Gamma b \implies f(a) \neq f(b)\ \}.$$

Writing GI for the category of finite graphs with injections, we have a functor

$$\mathrm{Conf}(-, X) \colon \mathrm{GI}^{op} \to \mathrm{Top}$$

given by relabeling vertices and forgetting both vertices and edges. The restriction of this functor along the complete graph functor $K \colon \mathrm{FI} \to \mathrm{GI}$ recovers the usual FI^{op}-structure on configuration space.

We introduce a new category called GI_2 to help with configuration space in a product. It is defined as the full subcategory of $\mathrm{GI} \times \mathrm{GI}$ spanned by pairs of graphs (Γ', Γ'') for which $V(\Gamma') = V(\Gamma'')$. In other words, an object of GI_2 is a pair of graph structures on the same underlying set of nodes. The union functor

$$U \colon \mathrm{GI}_2 \to \mathrm{GI}$$

sends a pair (Γ', Γ'') to the graph on the same vertex set as Γ' and Γ'' that includes every edge that appears in either graph.

Theorem 1. *The natural map from the homotopy left Kan extension along U^{op}*

$$\mathbb{L}(U^{op})_! \big[\mathrm{Conf}(-, X) \times \mathrm{Conf}(-, Y)\big] \to \mathrm{Conf}(-, X \times Y)$$

to the configuration space in the product is a pointwise weak equivalence.

Since the left hand side of the map in Theorem 1 only depends on homotopy-theoretic information, we have found the desired description of $\mathrm{Conf}(n, X \times Y)$, and could even iterate to find $\mathrm{Conf}(n, X \times Y \times Z)$ for example.

The rest of the talk switches to the following reformulation of Theorem 1.

Theorem 2. *If $\mathcal{P}(\Gamma)$ is the poset of pairs of subgraphs $\Gamma', \Gamma'' \subseteq \Gamma$ so that $\Gamma = U(\Gamma', \Gamma'')$, then the natural map*

$$\operatorname*{hocolim}_{(\Gamma', \Gamma'') \in \mathcal{P}(\Gamma)^{op}} \big[\mathrm{Conf}(\Gamma', X) \times \mathrm{Conf}(\Gamma'', Y)\big] \to \mathrm{Conf}(\Gamma, X \times Y)$$

is a weak equivalence.

Braid matroid Kazhdan–Lusztig polynomials

Max Wakefield

Kazhdan–Lusztig polynomials for matroids mimic the classical Kazhdan–Lusztig polynomials (originally defined in [4]) in many ways. Both these polynomials fit into the wider combinatorial theory of Kazhdan-Lusztig-Stanley polynomials defined in [7] and refined in [1]. In the case of matroids there is a significant amount of combinatorial machinery one can use to interpret these polynomials. From many perspectives braid matroids are the most important family of matroids. It is an open problem to find a simple closed formula for the braid matroid Kazhdan–Lusztig polynomials. In this note we will briefly survey matroid Kazhdan–Lusztig polynomials and discuss some recent results on computing them for braid matroids. We will focus on the combinatorial perspective, however a rich algebraic view was recently taken by Proudfoot and Young in [6] which yielded some crucial information about the generating functions of the coefficients for braid matroid Kazhdan–Lusztig polynomials.

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an arrangement of hyperplanes in $\mathbb{C}^\ell$ with linear forms $\ker(\alpha_i) = H_i$. The complex complement $U_{\mathcal{A}} = \mathbb{C}^\ell \setminus \bigcup H_i$ is an important manifold. In the case where $\mathcal{A}$ is the braid arrangement (i.e. all the hyperplanes of the form $\{x_i - x_j = 0\}$) the manifold $U_{\mathcal{A}}$ is the configuration space of ℓ-points in $\mathbb{C}$. The Zariski closure in $\mathbb{C}^n$ of the compositions of the maps $f : U_{\mathcal{A}} \to (\mathbb{C}^*)^n$ and $g : (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$ defined by $f(\vec{x}) = (\alpha_i(\vec{x}))$ and $g(x_i) = (x_i^{-1})$ respectively is called the reciprocal plane which we denote by $X_{\mathcal{A}} = \overline{g \circ f(U_{\mathcal{A}})}$. The coordinate ring of $X_{\mathcal{A}}$ is the Orlik-Terao algebra $OT(\mathcal{A}) = \mathbb{C}[\alpha_1^{-1}, \ldots, \alpha_n^{-1}]$. Let

$$\overline{P}(X_{\mathcal{A}}, t) = \sum \dim(IH^{2i}(X_{\mathcal{A}}))t^i$$

where $IH(X_{\mathcal{A}})$ is the topological intersection cohomology over $\mathbb{C}$. This Poincaré polynomial was the motivation for the study matroid Kazhdan–Lusztig polynomials.

Let M be a matroid with lattice of flats $L(M)$. If M is realizable with arrangement $\mathcal{A}$ then we write $M(\mathcal{A})$ for the matroid and $L(\mathcal{A})$ for its lattice of flats. For $F \in L(M)$ we call $L(M)_F = \{X \in L(M) | X \leq F\}$ the localization of $L(M)$ at F and $L(M)^F = \{X \in L(M) | X \geq F\}$ the restriction of $L(M)$ at F. Then we will denote M_F and M^F the matroids associated to the lattices $L(M)_F$ and $L(M)^F$ respectively. Now we can define the matroid Kazhdan–Lusztig polynomials.

Definition 1 (Theorem 2.2 in [2])**.** *There is a unique way to assign to each matroid M a polynomial $P(M,t) \in \mathbb{Z}[t]$ such that the following conditions are satisfied:*

(1) If $\mathrm{rk}M = 0$, then $P(M,t) = 1$.
(2) If $\mathrm{rk}M > 0$, then $\deg P(M,t) < \frac{1}{2}\mathrm{rk}M$.
(3) For every M, $t^{\mathrm{rk}M} P(M, t^{-1}) = \sum_F \chi(M_F, t) P(M^F, t)$.

In this definition $\chi(M,t)$ is the characteristic polynomial of the matroid. This combinatorial definition turns out to give the Poincaré polynomial of the intersection cohomology.

Theorem 1 (Theorem 3.10 in [2])**.** $\overline{P}(X_{\mathcal{A}}, t) = P(M(\mathcal{A}), t)$

Theorem 1 gives a few nice corollaries. First it shows that the Betti numbers of the intersection cohomology are combinatorial. Second it shows that for representable matroids the polynomials $P(M,t)$ have non-negative coefficients. So, one can naturally conjecture that these polynomials have non-negative coefficients for all matroids.

A question remains: does this combinatorial definition help us compute these polynomials for infinite families of matroids? The most important family of matroids is the so called braid matroids, denoted here as B_n, associated to the configuration spaces, type A Coxeter groups, and complete graphs. Conveniently, the lattice of flats of the braid matroid is the set partition lattice. Unfortunately, there are no known closed formulas for any coefficients for these polynomials and we do not even have a conjectural formula except for the top coefficient. Fortunately, using Stirling numbers we can obtain a few different formulas for the braid matroid Kazhdan–Lusztig polynomials.

Definition 2. *Suppose $\mathrm{rk}(L(M)) = N$ and $I = (i_1, i_2, \ldots, i_r)$ has $0 \leq i_1 \leq i_2 \leq \cdots \leq i_r \leq N$. The set of partial flags of $L(M)$ associated to I is*

$$L(M)_I = \{(X_1, X_2, \ldots, X_r) \in L(M)^r \mid \mathrm{rk}(X_j) = i_j, \text{ and } X_1 \leq X_2 \leq \cdots \leq X_r\}.$$

The partial flag (also called multi-indexed) Whitney numbers of the second kind are $W_I = |L(M)_I|$.

Using these partial flag Whitney numbers once can define an index set S_i and two functions $s_i \colon S_i \to \mathbb{Z}$ and $t \colon S_i \to 2^{\mathbb{Z}[N]}$ constructed in [8].

Theorem 2 (Theorem 11 in [8]). *For any finite, ranked lattice P such that* $\mathrm{rk}(P) = N$*, the degree i coefficient of the Kazhdan–Lusztig polynomial of P with $1 \leq i < N/2$ is*

$$\sum_{I \in S_i} (-1)^{s_i(I)} (W_{t(I)}(P) - W_I(P)).$$

A very similar result was given in [5].

Theorem 3 (Theorem 3.3 in [5]). *For all $i > 0$, the degree i coefficient of the matroid Kazhdan–Lusztig Polynomial, C_i for a matroid of rank N is*

$$\sum_{r=1}^{i} \sum_{D \subset [r]} (-1)^{|D|} \sum_{(a_m)} W_{(N-(a_{t_r(S)}+a_{r+1}),\ldots,N-(a_{t_1(S)}+a_0))}$$

where W is the multi-indexed Whitney number of the sequence of integers (a_m) such that $a_0 = 0$, $a_r = i$, $a_{r+1} = \mathrm{rk}(M) - i$, $a_0 < a_1 < \cdots < a_r < a_{r+1}$, and

$$t_j(S) = \min\{k \mid k \geq j \text{ and } k \notin S\} \in [r+1].$$

Now both of these theorems use flag Whitney numbers of the second kind. For braid matroids these flag Whitney numbers are products of Stirling numbers of the second kind $S(n,k) =$ the number of set partitions of a set with size n with k blocks. For $I = (i_1, i_2, \ldots, i_k)$,

$$W_I(B_n) = \prod_{j=0}^{k-1} S(n - i_j, n - i_{j+1}).$$

Since both Theorem 2 and Theorem 3 use Stirling numbers of the second kind for braid matroids wouldn't it be nice to have a Stirling number of the first kind formula. This was the subject of recent work by Trevor Karn and the author in [3]. To state the theorem, we need a little notation.

Definition 3 (Theorem 3.3 in [3]). *Let $n \geq 2$ and $i < \frac{n-1}{2}$. Define $\mathcal{K}_{n,i}$ to be the set of all triples (Λ, A, Ξ) where $\Lambda = [\lambda_1, \ldots, \lambda_q]$ is a sequence of number partitions, and $A = [\alpha_1, \ldots, \alpha_q]$ and $\Xi = [\xi_1, \ldots, \xi_q]$ are sequences of integers which satisfy:*

(1) $\lambda_1 \vdash n$
(2) $\lambda_j \vdash \ell(\lambda_{j-1})$ *for all* $1 < j \leq q$
(3) $\alpha_1 + \xi_1 = n - 1 - i$
(4) $\alpha_j + \xi_j = \ell(\lambda_{j-1}) - 1 - \xi_{j-1}$ *for* $j > 1$
(5) $0 \leq \alpha_j \leq |\lambda_j| - \ell(\lambda_j)$ *for all* j
(6) $\xi_j = 0$ *when* $\ell(\lambda_j) = 1$
(7) $0 \leq \xi_j < \frac{\ell(\lambda_j)-1}{2}$ *when* $\ell(\lambda_j) \geq 2$
(8) $\xi_j = 0$ *if and only if* $q = j$.

Using $\mathcal{K}_{n,i}$ as an index set, we can state a Stirling number of the first kind formula for the braid matroid Kazhdan–Lusztig polynomials.

Theorem 4. *For $n \geq 2$ and $i < \frac{n-1}{2}$,*

$$P(B_n, t)_i = \sum_{(\Lambda, A, \Xi)} \left[\prod_{j=1}^{q} m(\lambda_j) \sum_{(d_k^j)} \prod_{k=1}^{\ell(\lambda_j)} s(b_k^j, d_k^j) \right],$$

where $(\Lambda, A, \Xi) = ([\lambda_1, \dots, \lambda_q], [\alpha_1, \dots, \alpha_q], [\xi_1, \dots, \xi_q]) \in \mathcal{K}_{n,i}$, b_k^j is the k^{th} block of λ_j, and the last sum is over all sequences $(d_k^j) = (d_1^j, \dots, d_{\ell(\lambda_j)}^j)$ satisfying $\sum_{k=1}^{\ell(\lambda_j)} d_k^j = \alpha_j + \ell(\lambda_j)$ and $1 \leq d_k^j \leq b_k^j$.

Many questions remain open for braid matroid Kazhdan–Lusztig polynomials. (1) Do Theorems 2, 3, and 4 imply anything about the complexity of these coefficients? (2) Could Theorems 2, 3, and 4 help in computing the top coefficient of $P(B_n, t)$, conjectured in [2]?

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Resolvent Degree, Hilbert's 13th Problem and Geometry

Jesse Wolfson

(joint work with Benson Farb)

We start with a problem central to classical (and modern) mathematics.

Problem 1. Find and understand formulas for the roots of a polynomial

$$P(z) = z^n + a_1 z^{n-1} + \cdots + a_n$$

in terms of the coefficients $a_1, \dots, a_n$.

It is well known that if $n \geq 5$ then no formula exists using only radicals and arithmetic operations in the coefficients a_i.[1] Less known is Bring's 1786 theorem [Bri] that any quintic can be reduced via radicals to a quintic of the form $Q(z) = z^5 + az + 1$ (see [CHM] for a contemporary translation). In 1836, Hamilton [Ham]

[1] This was claimed by Ruffini in 1799; a complete proof was given by Abel in 1824.

extended Bring's results to higher degrees, showing, for example, that any sextic can be reduced via radicals to $Q(z) = z^6+az^2+bz+1$, that any degree 7 polynomial can be reduced via radicals to one of the form

$$Q(z) = z^7 + az^3 + bz^2 + cz + 1. \tag{1}$$

and that any degree 8 polynomial can be reduced via radicals to one of the form $Q(z) = z^8 + az^4 + bz^3 + cz^2 + d + 1$. Hilbert conjectured explicitly that one cannot do better: solving a sextic (resp. septic, resp. octic) is fundamentally a 2-parameter (resp. 3-parameter, resp. 4-parameter) problem. More precisely, we have the following invariant, first introduced by Brauer [Bra].

Definition 2 (Resolvent degree). Fix a field k. Let $\widetilde{X} \to X$ be a generically finite dominant map of k-varieties. The *resolvent degree* $\mathrm{RD}(\widetilde{X} \to X)$ is the smallest $d \geq 0$ with the following property: there is a chain of generically finite dominant maps

$$X_r \to X_{r-1} \to \cdots \to X_0 = X$$

such that $X_r \to X$ factors through a dominant map to $\widetilde{X}$ and such that for each i there is a variety Y with $\dim(Y) \leq d$ so that $X_{i+1} \to X_i$ is a pullback

$$\begin{array}{ccc} \widetilde{X}_{i+1} & \to & \widetilde{Y} \\ \downarrow & & \downarrow \\ X_i & \to & Y \end{array}$$

Example 3. Let $\mathcal{P}_n$ denote the space of monic degree n polynomials (i.e. $\mathbb{A}^n_k/S_n$), let $\widetilde{\mathcal{P}_n}$ denote the space of monic degree n polynomials with a choice of root (i.e. $\mathbb{A}^n_k/S_{n-1}$), and let $\widetilde{\mathcal{P}_n} \to \mathcal{P}_n$ be the finite map obtained by forgetting a root. In this language, the classical results on reduction of parameters can be restated succinctly as:

$$\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n) = 1 \;\; \forall n \leq 5, \quad \text{and} \quad \mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n) \leq n-4 \;\; \forall n > 5.$$

Buhler-Reichstein, Merkujev and others have developed a beautiful and widely applicable theory of *essential dimension* $\mathrm{ed}(\widetilde{X} \to X)$, where one forces $r = 1$ in Definition 2; see Reichstein's 2010 ICM paper [Re] for a survey. This disallowing of so-called "accessory irrationalities" captures more of the arithmetic of the ground field $k(X)$, whereas RD captures more of the intrinsic complexity of the branched cover.

Hilbert's problems. As already noted by Brauer [Bra], Hilbert's conjecture (explicitly asked by Hilbert in [Hi1, p.424] and [Hi2, p.247]) that Hamilton's reduction of parameters for the general polynomial of degree 6, 7, or 8 is optimal, can now be stated precisely, as can the problem for all degrees. Both Klein and Hilbert worked on this general problem for decades (see [Kl1, Hi1, Hi2]).

Problem 4 (Klein, Hilbert, Brauer). Compute $\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n)$. In particular:

Hilbert's Sextic Conjecture ([Hi2], p.247): $\mathrm{RD}(\widetilde{\mathcal{P}}_6 \to \mathcal{P}_6) = 2$.
Hilbert's 13th Problem ([Hi1],p.424): $\mathrm{RD}(\widetilde{\mathcal{P}}_7 \to \mathcal{P}_7) = 3$.

Hilbert's Octic Conjecture ([Hi2]*, p.247):* $\mathrm{RD}(\widetilde{\mathcal{P}}_8 \to \mathcal{P}_8) = 4$.

Beyond these, we have the following, which are implicitly due to Hilbert, and probably Brauer.

Conjecture 5. *There exists an example with* $\mathrm{RD}(\widetilde{X} \to X) > 1$.

Conjecture 6. $\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n) \to \infty$ *as* $n \to \infty$.

Along with the Hilbert Sextic Conjecture and Hilbert 13, these are clearly among the most important conjectures in this area. Amazingly, no progress has been made on these conjectures or on either of the three special cases above since Hilbert stated them. In 1957, Arnol'd and Kolmogorov proved (see [Ar]) that there is no local topological obstruction to reducing the number of variables; however, as Arnol'd and many others have noted, the global problem remains open. While these conjectures provide the primary challenges for the field, the explicit study of resolvent degree is already yielding improvements on old theorems, and striking relationships between seemingly different problems. Two sample theorems provide an indication of what to expect.

First, we expect rapid improvement should be possible on existing upper bounds on resolvent degree. Brauer in 1975 proved that for $n \geq 4$, $\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n) \leq n - r$ once $n > B(r) := (r-1)!$. In [FW2], we prove the following. Let $\mathrm{RD}(r, N)$ denote the resolvent degree of finding an r-dimensional linear subspace on a cubic hypersurface in $\mathbb{P}^N$. A dimension count shows that the function $\mathrm{RD}(r, N)$ grows at most polynomially in r and N.

Theorem 7 (Farb-W). *There exist a pair of polynomial functions* $f, g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that, for* $n \geq \frac{(d+k)!}{d!}$,

$$\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n) \leq \max\{n - (d+k+1), \mathrm{RD}(f(d,k), g(d,k))\}.$$

Corollary 8. *There exist monotone increasing functions* $FW, \varphi\colon \mathbb{N} \to \mathbb{N}$ *s.t.*

- *For* $n > FW(r)$, $\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n) \leq n - r$,
- *For all* $d \geq 0$, $r \geq \varphi(d)$, *then* $B(r)/FW(r) \geq d!$.

This improvement over Brauer uses ideas of Hilbert [Hi2], who used lines on cubic surfaces to simplify the degree 9 polynomial. We are confident that further improvements will follow from incorporating ideas of Hamilton and Sylvester. While such theorems do not address the fundamental questions of lower bounds above, they provide a testing ground for new methods and give us evidence as to the eventual shape of the function $\mathrm{RD}(\widetilde{\mathcal{P}}_n \to \mathcal{P}_n)$.

For a second example of the type of theorems we expect to follow from renewed interest in resolvent degree, we prove the following in [FW1].

Theorem 9 (Farb-W.)**.** *The following statements are equivalent:*

(1) *Hilbert's Sextic Conjecture is true:* $\mathrm{RD}(\widetilde{\mathcal{P}}_6 \to \mathcal{P}_6) = 2$.
(2) $\mathrm{RD} = 2$ *for the problem of finding the* 27 *lines on a cubic, given a "double six" set of lines.*

(3) RD $= 2$ *for the problem of finding a fixed point for the hyperelliptic involution on a genus* 2 *curve.*

In fact, the resolvent degrees of all of the above problems coincide.

We also prove similar reformulations for Hilbert's 13th Problem and Hilbert's Octic Conjecture. While we make no definite progress toward proving nontrivial lower bounds for RD, we hope that with renewed attention to Hilbert's conjectures and to resolvent degree, future progress may be more forthcoming.

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On the Johnson homomorphisms of the automorphism groups of free groups

Takao Satoh

In the 1980s, Dennis Johnson established a remarkable method to investigate the group structure of the mapping class groups of surfaces in a series of his works. In particular, he constructed a certain homomorphism τ to determine the abelianization of the Torelli group. Today, his homomorphism τ is called the first Johnson homomorphism. Over the last three decades, good progress was made in the study of the Johnson homomorphisms of mapping class groups through the works of a large number of authors, including Morita [13], Hain [9], Cohen-Pakianathan [3, 4] and Farb [8] as pioneer works. In addition to this, we have a lot of interesting and remarkable works given by participants of this workshop, including Brendle [2], Papadima–Suciu [15], Patzt [16], Djament–Vespa [7].

The definition of the Johnson homomorphisms can be naturally generalized to the automorphism groups of free groups. Let F_n be a free group of rank n, H the abelianization of F_n, and $\mathrm{Aut}\, F_n$ the automorphism group of F_n. The kernel of the homomorphism $\mathrm{Aut}\, F_n \to \mathrm{GL}(n, \mathbb{Z})$ induced from the action of $\mathrm{Aut}\, F_n$ on H, is called the IA-automorphism group of F_n, and is denoted by IA_n. In 1965, Andreadakis [1] introduced a descending central filtration

$$\mathrm{IA}_n = \mathcal{A}_n(1) \supset \mathcal{A}_n(2) \supset \cdots$$

of IA_n, and showed that each graded quotient $\mathrm{gr}^k(\mathcal{A}_n) := \mathcal{A}_n(k)/\mathcal{A}_n(k+1)$ is a free abelian group of finite rank. We call the above filtration the Andreadakis–Johnson filtration of $\mathrm{Aut}\, F_n$. Johnson studied this kind of filtration for the mapping class groups in the 1980s. In order to investigate the structure of $\mathrm{gr}^k(\mathcal{A}_n)$, we consider the k-th Johnson homomorphism

$$\tau_k : \mathrm{gr}^k(\mathcal{A}_n) \to H^* \otimes_{\mathbb{Z}} \mathcal{L}_n(k+1).$$

Each of τ_k is $\mathrm{GL}(n, \mathbb{Z})$-equivariant and injective. Based on our previous work and a recent remarkable work by Darné [5], the stable cokernel of τ_k has been determined as

$$\mathrm{Coker}(\tau_k) \cong H^{\otimes k}/(\text{Cyclic Permutation}) \quad (n \geq k+2).$$

On the other hand, the mapping class group case is much more difficult and mysterious, and the image of the Johnson homomorphisms are not determined yet.

One of the motivations to study of the Johnson homomorphisms is to consider applications to twisted cohomology groups. Kawazumi [12] extended the first Johnson homomorphism to $\mathrm{Aut}\, F_n$ as a crossed homomorphism. We [19] computed $H^1(\mathrm{Aut}\, F_n, (H^* \otimes \Lambda^2 H)_{\mathbb{Z}} \otimes \mathbb{Q}) = \mathbb{Q}^{\otimes 2}$, and described generators with the extension of τ_1. So far, there are only a few computations of stable twisted cohomology groups, including those by Hatcher–Wahl [10], Djament–Vespa [6] and Randal-Williams–Wahl [18]. Pettet [17] determined the $\mathrm{GL}(n, \mathbb{Q})$-decomposition of the image of the cup product $\cup \colon \Lambda^2 H^1(\mathrm{IA}_n, \mathbb{Q}) \to H^2(\mathrm{IA}_n, \mathbb{Q})$. Based on her results, recently we showed that $H^2(\mathrm{Aut}\, F_n, (\mathrm{Im}(\cup))^*) \supset \mathbb{Q}^{\oplus d_n}$ where d_n is the number of the irreducible components of $\mathrm{Im}(\cup)$.

Recently, we discovered that the framework of the theory of the Johnson homomorphisms can be applied to the ring of complex functions on $\mathrm{SL}(2, \mathbb{C})$-representations of F_n. Let $R(F_n)$ be the set of all $\mathrm{SL}(2, \mathbb{C})$-representations of F_n, and $\mathcal{F}(F_n)$ the set of all complex-valued functions on $R(F_n)$. Then $\mathcal{F}(F_n)$ naturally has the $\mathbb{C}$-algebra structure by the pointwise sum and product. Furthermore, $\mathrm{Aut}\, F_n$ naturally acts on $\mathcal{F}(F_n)$ from the right. For any $x \in F_n$ and any $1 \leq i, j \leq 2$, define $a_{ij}(x)$ of $\mathcal{F}(F_n)$ to be

$$(a_{ij}(x))(\rho) := (i, j)\text{-component of } \rho(x)$$

for any $\rho \in R(F_n)$. Let $\mathfrak{R}_{\mathbb{Q}}(F_n)$ be the $\mathbb{Q}$-subalgebra of $\mathcal{F}(F_n)$ generated by all $a_{ij}(x)$ for $x \in F_n$ and $1 \leq i, j \leq 2$. Let J be the ideal of $\mathfrak{R}_{\mathbb{Q}}(F_n)$ defined by

$$J := (a_{ij}(x) - \delta_{ij} \mid x \in F_n,\ 1 \leq i, j \leq 2) \subset \mathfrak{R}_{\mathbb{Q}}(F_n)$$

where δ_{ij} is Kronecker's delta. Then, we have the descending filtration $J \supset J^2 \supset J^3 \supset \cdots$, and each graded quotient $\mathrm{gr}^k(J) := J^k/J^{k+1}$ is an $\mathrm{Aut}\,F_n$-invariant finite dimensional $\mathbb{Q}$-vector space. Set $H_{\mathbb{Q}} := H \otimes_{\mathbb{Z}} \mathbb{Q}$. For $n \geq 3$, we [20] obtained the following.

(1) $\bigcap_{k\geq 1} J^k = \{0\}$.
(2) For any $k \geq 1$, $\mathrm{gr}^k(J) \cong \displaystyle\bigoplus_{e_{11}+e_{12}+e_{21}=k} S^{e_{11}}H_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{e_{12}}H_{\mathbb{Q}} \otimes_{\mathbb{Q}} S^{e_{21}}H_{\mathbb{Q}}$.
(3) $\mathfrak{R}_{\mathbb{Q}}(F_n)$ is an integral domain, and is isomorphic to the universal SL_2-representation ring of F_n.

Now, for any $k \geq 1$, let $\mathcal{D}_n(k)$ be the kernel of the homomorphism $\mathrm{Aut}\,F_n \to \mathrm{Aut}(J/J^{k+1})$ induced from the action of $\mathrm{Aut}\,F_n$ on J/J^{k+1}. Then the groups $\mathcal{D}_n(k)$ define a descending filtration $\mathcal{D}_n(1) \supset \mathcal{D}_n(2) \supset \cdots$ of $\mathrm{Aut}\,F_n$. This is an $\mathrm{SL}(2,\mathbb{C})$-representation analogue of the Andreadakis-Johnson filtration. For $n \geq 3$, we [20] showed

(1) $[\mathcal{D}_n(k), \mathcal{D}_n(l)] \subset \mathcal{D}_n(k+l)$ for any $k, l \geq 1$.
(2) $\mathcal{A}_n(k) \subset \mathcal{D}_n(k)$ for any $k \geq 1$. Furthermore, this is equal for $1 \leq k \leq 4$.

From Part (1), we see that the graded quotients $\mathrm{gr}^k(\mathcal{D}_n) := \mathcal{D}_n(k)/\mathcal{D}_n(k+1)$ are abelian groups for any $k \geq 1$. In order to study the structure of $\mathrm{gr}^k(\mathcal{D}_n)$, we have introduced the homomorphisms $\eta_k : \mathrm{gr}^k(\mathcal{D}_n) \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{gr}^1(J), \mathrm{gr}^{k+1}(J))$ defined by

$$\sigma \pmod{\mathcal{D}_n(k+1)} \mapsto \Big(f \pmod{J^2} \mapsto f^\sigma - f \pmod{J^{k+1}}\Big).$$

The homomorphisms η_k is $\mathrm{SL}(2,\mathbb{C})$-representation analogues of the Johnson homomorphisms. In [20], we showed that each η_k is $\mathrm{Aut}\,F_n/\mathcal{D}_n(1)$-equivariant injective homomorphism. This implies that each of $\mathrm{gr}^k(\mathcal{D}_n)$ is torsion-free, and that $\dim_{\mathbb{Q}}(\mathrm{gr}^k(\mathcal{D}_n) \otimes_{\mathbb{Z}} \mathbb{Q}) < \infty$. Now, we conjecture that $\mathcal{A}_n(k) = \mathcal{D}_n(k)$ for any $k \geq 1$.

Finally we remark that in [21], we consider the above framework for $\mathrm{SL}(m,\mathbb{C})$-representations of F_n for any $m \geq 2$, and obtained similar results as a part of the above.

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Polynomial behaviour for stable homology of congruence groups

Aurélien Djament

An ideal in a (unital) ring is the same as a ring without unit: such a (non-unital) ring I can be seen as the two-sided ideal given by the kernel of the augmentation $\mathbb{Z} \ltimes I \twoheadrightarrow \mathbb{Z}$, where $\mathbb{Z} \ltimes I$ is the unital ring obtained by formally adding a unit to I. The framework in the preprint [6], on which this talk is reporting, is more general, but most of the ideas and applications are already in this classical setting.

The *congruence groups* associated to I are defined by

$$\Gamma_n(I) := \operatorname{Ker}\left(GL_n(\mathbb{Z} \ltimes I) \twoheadrightarrow GL_n(\mathbb{Z})\right).$$

We look for qualitative properties of the homology of these groups. As in the case of usual linear groups, we have obvious stabilisation maps $H_*(\Gamma_n(I);\mathbb{Z}) \to H_*(\Gamma_{n+1}(I);\mathbb{Z})$: we will deal only with *stable* properties (as in algebraic K-theory), that is, properties of the colimit of this sequence of graded abelian groups. But we have also a richer structure: $H_*(\Gamma_n(I);\mathbb{Z})$ in endowed with a natural action of $GL_n(\mathbb{Z})$ (induced by the conjugation action) which is generally not trivial (even stably). We will later express these structures (and their compatibility properties) in a functorial setting.

Earlier known results

Suslin [12] proved the following striking Theorem (which improves the rational result that he got with Wodzicki in [13], with a different method).

Theorem 1 (Suslin 1995). *Let $d > 0$ be an integer.*

(1) The following statements are equivalent.
- *(a) Stably in n, the action of $GL_n(\mathbb{Z})$ on $H_i(\Gamma_n(I);\mathbb{Z})$ is trivial for $i < d$;*
- *(b) I is excisive for algebraic K-theory in homological degree $< d$;*
- *(c) $\operatorname{Tor}_i^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z}) = 0$ for $0 < i < d$.*

(2) There is a natural map $H_d(\Gamma_n(I);\mathbb{Z}) \to \mathfrak{gl}_n\big(\operatorname{Tor}_d^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})\big)$ (where $\mathfrak{gl}_n(M)$ denotes the $n\times n$ matrices with entries in M) which is $GL_n(\mathbb{Z})$-equivariant, compatible with stabilisation in n, and whose kernel and cokernel bear a trivial $GL_n(\mathbb{Z})$-action stably in n if the previous conditions are fulfilled.

(Note that $\operatorname{Tor}_1^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z}) \simeq I/I^2$, so the conditions are only seldom fulfiled for $d > 1$; for $d = 1$, the last statement is classical and not hard.)

Other known results give informations on $H_*(\Gamma_n(I))$ for each homological degree, but only for particular non-unital rings I.

In [1], Calegari proved the following asymptotic polynomial behaviour for homology of classical congruence groups.

Theorem 2 (Calegari 2015). *Let p be a prime number and k, i be non-negative integers. Then*

$$\dim H_k(\Gamma_n(p^i\mathbb{Z});\mathbb{F}_p) = \frac{n^{2k}}{k!} + O(n^{2k-2}).$$

Another important recent result (whose methods are completely independent from the ones used to prove both previous Theorems) is due to Putman [9], with an input given by an older work by Charney [4]. This result was quickly improved by the systematic use of functorial methods that we will remind now.

Statements in terms of polynomial functors

Let $(\mathcal{C},+,0)$ be a small symmetric monoidal category whose unit 0 is an initial object. For convenience we will assume that the objects of $\mathcal{C}$ are the natural integers and that $+$ is the usual sum on objects. The precomposition by $-+1$ is an exact endofunctor, denoted by τ, of the category $\mathcal{C}$-**Mod** of functors from $\mathcal{C}$ to abelian groups; with Vespa we studied in [7] the quotient category $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$ of $\mathcal{C}$-**Mod** obtained by killing the functors which are stably zero, that is, by quotienting out the localising subcategory of $\mathcal{C}$-**Mod** generated by functors F such that the canonical map $F \to \tau(F)$ is zero (equivalently, a functor F is stably zero if and only if $\operatorname{colim}_{n\in\mathbb{N}} F(n) = 0$).

We introduced two notions of polynomial functor of degree d: a strong one, which captures also unstable phenomena, and a weak one, which depends only of the isomorphism class of the functor in $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$. For example, a functor in $\mathcal{C}$-**Mod** is weakly polynomial of degree ≤ 0 if and only if it is isomorphic in $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$ to a constant functor. For the definition of strongly and weakly polynomial functors and properties, we refer to [7] or to the talk by Vespa in this meeting. Weakly polynomial functors of (weak) degree $\leq d$ (or more precisely, their images in $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$) form a localising subcategory of $\mathbf{St}(\mathcal{C}\text{-}\mathbf{Mod})$ denoted by $\mathcal{P}ol_d(\mathcal{C}\text{-}\mathbf{Mod})$. For example, $\mathfrak{gl}_\bullet(M)$ is a strongly polynomial functor of degree 2 in $\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}$ (where $\mathbf{S}(\mathbb{Z})$ is defined just below), for any abelian group M.

We are interested here in the following monoidal categories $\mathcal{C}$ with the previous properties: the category $\mathbf{FI}$ for which $\mathbf{FI}(n,m)$ is the set of injections from $\mathbf{n} := \{1,\dots,n\}$ to $\mathbf{m}$ (the monoidal structure being given by disjoint union) and the category $\mathbf{S}(R)$, where R is a unital ring, for which

$$\mathbf{S}(R)(n,m) := \{(f,g) \in \mathrm{Hom}_R(R^n,R^m) \times \mathrm{Hom}_R(R^m,R^n) \,|\, g \circ f = \mathrm{Id}\}$$

(the monoidal structure being given by direct sum). These categories are also *homogeneous categories* in the sense of Randal-Williams and Wahl [10] (a very general framework which is related to the one used at the beginning of [8]).

For any unital ring R, $n \mapsto GL_n(R)$ defines a functor $GL_\bullet(R)$ from $\mathbf{S}(R)$ to the category of groups. If I is a non-unital ring, $n \mapsto \Gamma_n(I)$ is a subfunctor of $GL_\bullet(\mathbb{Z} \ltimes I)$. By taking the homology, we get a functor $H_d(\Gamma_\bullet(I))$ in $\mathbf{St}(\mathbf{S}(\mathbb{Z} \ltimes I)\text{-}\mathbf{Mod})$ for each d, which lives indeed in $\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}$ (because inner automorphisms act trivially in homology). By restricting it along the canonical monoidal functor $\mathbf{FI} \to \mathbf{S}(\mathbb{Z})$, several authors, improving Putman [9], showed that $H_d(\Gamma_\bullet(I))$ is strongly polynomial for each d if the ring I is nice enough—see Church–Ellenberg–Farb–Nagpal [3] of Church–Ellenberg [2]. Recently, Church–Miller–Nagpal–Reinhold [5] obtained the following result, always by using $\mathbf{FI}$-modules.

Theorem 3 (Church–Miller–Nagpal–Reinhold, preprint 2017). *If I is an ideal in a unital ring R satisfying Bass condition (SR_{r+2}), then for each non-negative integer d, $H_d(\Gamma_\bullet(I);\mathbb{Z})$ is a weakly polynomial functor of (weak) degree $\leq 2d+r$.*

In [6], the following stronger result is proven.

Theorem 4. *Let I be a ring without unit and $e > 0$ an integer such that $\mathrm{Tor}_i^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z}) = 0$ for $0 < i < e$ (for example, $e = 1$).*

1. *(1) For each integers $r, d \geq 0$ and each object F in $\mathcal{P}ol_r(\mathbf{S}(\mathbb{Z} \ltimes I)\text{-}\mathbf{Mod})$, the functor $H_d(\Gamma_\bullet(I);F)$ belongs to $\mathcal{P}ol_{2[d/e]+r}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})$ (where the brackets denote the floor function).*
2. *(2) If e is odd (respectively even), then for each integer $n \geq 0$, $H_{ne}(\Gamma_\bullet(I);F)$ is isomorphic in the quotient category $\mathcal{P}ol_{2n}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})/\mathcal{P}ol_{2n-2}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})$ to $\Lambda^n(\mathfrak{gl}_\bullet(\mathrm{Tor}_e^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})))$ (resp. $S^n(\mathfrak{gl}_\bullet(\mathrm{Tor}_e^{\mathbb{Z}\ltimes I}(\mathbb{Z},\mathbb{Z})))$), where Λ^n (resp. S^n) denotes the n-th exterior (resp. symmetric) power (over the integers).*

For $n = 1$, the second part of this theorem is equivalent to Suslin's Theorem 1.

Ingredients of the proof

The input of the proof of Theorem 4 is a version in degree 0 with *twisted* coefficients: one has an (easy) stable natural isomorphism $H_0(\Gamma_\bullet(I);F) \simeq \Phi_*(F)$ in

$\mathbf{St}(\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod})$ for any functor F in $\mathbf{S}(\mathbb{Z} \ltimes I)\text{-}\mathbf{Mod}$, where $\Phi : \mathbf{S}(\mathbb{Z} \ltimes I) \to \mathbf{S}(\mathbb{Z})$ denotes the reduction modulo the ideal I and Φ_* the left Kan extension along Φ.

One can then derive this isomorphism (even in a quite more general framework) to get a stable spectral sequence

$$E^2_{i,j} = \mathbf{L}_i\big((- \underset{\oplus}{\otimes} H_j(\Gamma_\bullet(I);\mathbb{Z})) \circ \Phi_*\big)(F) \Rightarrow H_{i+j}(\Gamma_\bullet(I);F)$$

where $\underset{\oplus}{\otimes} : (\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}) \times (\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}) \to \mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}$ is the composition of the external tensor product with the left Kan extension along the direct sum functor $\mathbf{S}(\mathbb{Z}) \times \mathbf{S}(\mathbb{Z}) \to \mathbf{S}(\mathbb{Z})$.

When F factorises through $\Phi : \mathbf{S}(\mathbb{Z} \ltimes I) \to \mathbf{S}(\mathbb{Z})$, the abutment $H_*(\Gamma_\bullet(I);F)$ of the spectral sequence can be expressed simply from F and $H_*(\Gamma_\bullet(I);\mathbb{Z})$, thanks to the universal coefficients exact sequence for group homology. So the spectral sequence gives informations on $H_*(\Gamma_\bullet(I);\mathbb{Z})$. One needs several steps to show the wished result with this program, especially:

- a comparison theorem of stable (in the sense of categories **St** introduced above!) derived categories of $\mathbf{S}(\mathbb{Z})\text{-}\mathbf{Mod}$ and $\mathbf{F}(\mathbb{Z})\text{-}\mathbf{Mod}$, where $\mathbf{F}(\mathbb{Z})$ denotes Quillen's category of factorizations of free abelian groups of finite rank, on (weakly) polynomial functors. This is inspired by Scorichenko's thesis [11];
- A study of the left derived functors of Φ_* on polynomial functors (using the first step);
- a study of the tensor product $\underset{\oplus}{\otimes}$ and its left derivatives on polynomial functors (also using the first step);
- a concrete argument of triangular groups inspired by Suslin–Wodzicki [13];
- some functorialities of the above spectral sequence.

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Secondary representation stability for configuration spaces

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(joint work with Jennifer Wilson)

1. Indecomposables of an FI-module

Throughout, we work over a commutative unital base ring R. All homology groups will be understood to have coefficients in R, all tensor products will be over R, and the term FI-module will mean a functor from the category of finite sets and injections to the category of R-modules.

One of the most popular notions of representation stability is the notion of *finite generation*. The following is a measure of the generators of an FI-module. Following the notation of [2], we make the following definition.

Definition 1.1. Let V be an FI-module and S be a set. Let

$$H_0^{\mathrm{FI}}(V)_S = \operatorname{coker}\left(\bigoplus_{T\subset S, T\neq S} V_T \to V_S\right).$$

We say that V has *generation degree* $\leq d$ if $V_S = 0$ for all sets S of cardinality strictly larger than d.

Note that if the modules V_S are finitely generated R-modules for all S, then an FI-module is finitely generated as an FI-module if and only if it has finite generation degree.

In general, the groups $H_0^{\mathrm{FI}}(V)$ should not be thought of as the generators of V as an FI-module but only a measure of how large a minimal generating set must be. Generators are naturally subobjects while the groups $H_0^{\mathrm{FI}}(V)$ are a quotient. Although not standard terminology, it would be reasonable to call the groups $H_0^{\mathrm{FI}}(V)$ *indecomposables* as they are analogous to the indecomposables of a graded algebra.

The subscript 0 in $H_0^{\mathrm{FI}}(V)$ is used because often one considers higher left derived functors of H_0^{FI} which are denoted by H_i^{FI}. Note that if V is an FI$\sharp$-module in the sense of [3], then

$$H_i^{\mathrm{FI}}(V) \cong 0$$

for all $i > 0$. Moreover, we can recover V from $H_0^{\mathrm{FI}}(V)$ by an explicit formula given in [3]. For this and other reasons, FI$\sharp$-modules are often called *induced* or *free*. In this case, $H_0^{\mathrm{FI}}(V)_S$ naturally sit as a submodule of V_S and can be more reasonably thought of as generators.

2. Configuration spaces

Definition 2.1. For S a set and M a space, let $\mathrm{Conf}_S(M)$ denote the space of injections of the set S into the space M. Topologize $\mathrm{Conf}_S(M)$ with the subspace topology inside the space of all maps from S to M equipped with the compact open topology and with S equipped with the discrete topology. Let $[k] = \{1, \ldots, k\}$ and denote $\mathrm{Conf}_{[k]}(M)$ by $\mathrm{Conf}_k(M)$.

We call $\mathrm{Conf}_k(M)$ the *configuration space of k ordered points in M*. One of the most intensely studied families of FI-modules are the cohomology of ordered configuration spaces of points in a manifold. See [1] [3], [4], [5]. Here, we will instead study the homology of configuration spaces. For this to make sense, we must restrict to a certain class of manifolds.

We will assume that M is a connected, non-compact n-dimensional manifold with $n > 1$. In this case, there exists an embedding

$$e : \mathbb{R}^n \sqcup M \hookrightarrow M$$

which we will fix once and for all. If M has multiple ends, then the isotopy class of e will not be unique. This embedding induces a map

$$\mathrm{Conf}_S(\mathbb{R}^n) \times \mathrm{Conf}_T(M) \to \mathrm{Conf}_{S \sqcup T}(M)$$

which in turn induces a map on homology

$$H_i(\mathrm{Conf}_S(\mathbb{R}^n)) \otimes H_j(\mathrm{Conf}_T(M)) \to H_{i+j}(\mathrm{Conf}_{S \sqcup T}(M)).$$

For fixed $\alpha \in H_i(\mathrm{Conf}_S(\mathbb{R}^n))$, we denote the induced map by

$$t_\alpha : H_j(\mathrm{Conf}_T(M)) \to H_{i+j}(\mathrm{Conf}_{S \sqcup T}(M))$$

and call it the *stabilization map* associated to the homology class α.

Let $\mathfrak{p}$ denote the class of a point in $H_0(\mathrm{Conf}_1(\mathbb{R}^n))$. Implicit in [3] is the fact that the maps

$$t_{\mathfrak{p}} : H_i(\mathrm{Conf}_k(M)) \to H_i(\mathrm{Conf}_{k+1}(M))$$

induce an FI-module structure on the functor

$$S \mapsto H_i(\mathrm{Conf}_S(M)).$$

We denote this FI-module by $H_i(\mathrm{Conf}(M))$. In [3], Church–Ellenberg–Farb proved the following (also see [8]).

Theorem 2.2 (Church–Ellenberg–Farb)**.** *The* FI*-modules* $H_i(\mathrm{Conf}(M))$ *have a natural* FI$\sharp$*-module structure and have generation degree* $\leq 2i$*.*

3. Secondary stability

From now on, we assume M is a surface. Let $\mathfrak{l}$ denote a generator of $H_1(\mathrm{Conf}_2(\mathbb{R}^2)) \cong R$. This class induces a map

$$t_{\mathfrak{l}} : H_i(\mathrm{Conf}_k(M)) \to H_{i+1}(\mathrm{Conf}_{k+2}(M)).$$

One can check that this induces a map on indecomposables

$$t_{\mathfrak{l}} : H_0^{\mathrm{FI}}(H_i(\mathrm{Conf}_k(M))) \to H_0^{\mathrm{FI}}(H_{i+1}(\mathrm{Conf}_{k+2}(M))).$$

Use the convention that fractional dimensional homology groups are zero and define

$$W(M,i)_k = H_0^{\mathrm{FI}}(H_{i+k/2}(\mathrm{Conf}_k(M))).$$

We get a sequence of symmetric group representations and equivariant maps

$$W(M,i)_0 \xrightarrow{t_{\mathfrak{l}}} W(M,i)_2 \xrightarrow{t_{\mathfrak{l}}} W(M,i)_4 \xrightarrow{t_{\mathfrak{l}}} W(M,i)_6 \xrightarrow{t_{\mathfrak{l}}} \dots$$

and

$$W(M,i)_1 \xrightarrow{t_{\mathfrak{l}}} W(M,i)_3 \xrightarrow{t_{\mathfrak{l}}} W(M,i)_5 \xrightarrow{t_{\mathfrak{l}}} W(M,i)_7 \xrightarrow{t_{\mathfrak{l}}} \dots$$

Note that one of these sequences will be zero depending on the parity of i. Denote the other sequence by $W(M,i)$.

The sequence $W(M,i)$ consists of the indecomposables of the FI-module

$$H_*(\mathrm{Conf}(M))$$

which lie a distance i above the stable range. There does not seem to be an interesting FI-module structure on $W(M,i)$. However, the maps $t_{\mathfrak{l}}$ do give the sequences $W(M,i)$ the structure of a $\bigwedge \mathrm{Sym}^2 R$-module. Here $\bigwedge \mathrm{Sym}^2 R$ is the free twisted skew commutative algebra on the trivial one dimensional representation in degree two. See [9] for a definition of $\bigwedge \mathrm{Sym}^2 R$. The main theorem of [8] is the following.

Theorem 3.1 (M.–Wilson)**.** *If R has characteristic zero and M is finite type, then the sequences $W(M,i)$ are finitely generated for all i as modules over $\bigwedge \mathrm{Sym}^2 R$.*

We call this phenomenon *secondary representation stability* as it involves a stability pattern outside the classical stable range which only manifests itself after one appropriately accounts for the primary stability pattern. This was inspired by *secondary homological stability* which is a similar pattern discovered by Galatius–Kupers–Randal-Williams [6] and is also present in the work of Hepworth [7]. The need to work in characteristic zero stems from the fact that currently, the category of $\bigwedge \mathrm{Sym}^2 R$ is only known to be locally Noetherian if R is a ring of characteristic zero, a result of Nagpal–Sam–Snowden [9].

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Combinatorics of Abelian arrangements

Emanuele Delucchi

1. Abelian arrangements

Let $\mathbb{G}$ be one of the groups $\mathbb{C}$, $\mathbb{C}^*$ or $\mathbb{E}$ (an elliptic curve), and let Λ be a free abelian group of rank rank d. Any choice of elements $\alpha_1, \ldots, \alpha_n \in \Lambda$ determines a family of homomorphisms $\alpha_i : \mathrm{Hom}(\Lambda, \mathbb{G}) \to \mathbb{G}$ and thus an *abelian arrangement* in $\mathbb{G}^d \simeq \mathrm{Hom}(\Lambda, \mathbb{G})$:

$$\mathscr{A} = \{H_1, \ldots, H_n\}, \text{ where } H_i := \ker \alpha_i.$$

From a topological point of view, the object of interest is the *complement*

$$M(\mathscr{A}) := \mathbb{G}^d \setminus \bigcup \mathscr{A}.$$

As the combinatorial data associated to an abelian arrangement we consider

- the *dimension function* $\delta_{\mathscr{A}} : 2^{[n]} \to \mathbb{N}$, $I \mapsto \dim(\cap_{i\in I} H_i) = \mathrm{corank}\langle \alpha_i : i \in I\rangle$;
- the *multiplicity function* $m_{\mathscr{A}} : 2^{[n]} \to \mathbb{N}$, $I \mapsto \beta_0(\cap_{i\in I} H_i)$;
- the *poset of layers* $\mathcal{C}(\mathscr{A})$, the set of all connected components of intersections of the H_i, partially ordered by reverse inclusion.

Algebraic models for the cohomology of $M(\mathscr{A})$ have been given by Bibby [3] and by Dupont [16], who also addressed formality questions [17]. Local systems cohomology has been studied by Levin and Varchenko [18] and Denham, Suciu and Yuzvinsky [15]. We aim at a structural understanding of the combinatorial structures, and motivate our approach with a review of some special cases.

1.1. **Weyl arrangements.** Let Φ_l be a rank l root system of type $ABCD$. Taking $\{\alpha_i\} := \Phi_l$ as a subset of the associated coroot lattics $\Lambda := \langle \Phi_l^\vee \rangle$, we obtain the *Weyl arrangements* $\mathscr{A}_{\Phi_l}$. Bibby [3] proved representation stability for the action of the Weyl groups $W(\Phi_l)$ on $M(\mathscr{A}_{\Phi_l})$ by way of explicit descriptions of the posets of layers. Using this we can prove that all posets $\mathcal{C}(\mathscr{A}_{\Phi_l})$ are EL-shellable [12].

1.2. **Linear arrangements:** $\mathbb{G} = \mathbb{C}$**.** We refer to the contribution of Michael Falk in this volume for a detailed overview of the theory in this case. Here we stress that, from a combinatorial point of view, the function $\delta_{\mathscr{A}}$ and the poset $\mathcal{C}(\mathscr{A})$ encode equivalent combinatorial data: by knowing one of them it is possible to reconstruct the other. (Notice that, in this case, $m_{\mathscr{A}}$ is constant equal to 1, hence it does not add any information.) As an abstract poset, $\mathcal{C}(\mathscr{A})$ has the structure of a *geometric lattice*. The class of geometric lattices is larger than that of intersection posets of arrangements, but still corresponds to a class of functions defined by some of the combinatorial properties of $\delta_{\mathscr{A}}$. Every such abstract function – respectively, every geometric lattice – defines a *matroid* [20].

1.3. **Toric arrangements:** $\mathbb{G} = \mathbb{C}^*$**.** As an update to the overview given in [7, Introduction] we mention the study of the ring $H^*(M(\mathscr{A}), \mathbb{Z})$ in [6] and [21] and De Concini and Gaiffi's work constructing projective wonderful models for $M(\mathscr{A})$ and computing their cohomology [10, 11]. Combinatorial aspects of toric arrangements appeared in many contexts - see the introduction to [13]. In particular, d'Adderio and Moci [9] and Brändén and Moci [5], devised a theory of *arithmetic matroids* designed to underpin some properties of Moci's *arithmetic Tutte polynomial* [19]

$$T_{\mathscr{A}}(x,y) := \sum_{S \subseteq \mathscr{A}} m_{\mathscr{A}}(S)(x-1)^{\delta(S)}(y-1)^{|S|+\delta(S)-\delta(S)}. \tag{1}$$

2. Group actions on semimatroids

We propose a combinatorial theory based on the observation that abelian arrangements are quotients of periodic arrangements of affine hyperplanes. In fact, to any locally finite[1] set of affine hyperplanes $\widetilde{\mathscr{A}}$ in a vectorspace we can associate the poset $\mathcal{L}$ of all intersection subspaces ordered by reverse inclusion and a function $\delta : 2^{\widetilde{\mathscr{A}}} \to \mathbb{Z}$, $\delta(X) := \dim(\cap X)$, where we set $\dim(\emptyset) := -1$. Posets of the form $\mathcal{L}$ are *geometric semilattices*, and the function δ satisfies the axioms for a *semimatroid* with $\widetilde{\mathscr{A}}$ as its ground set. Semimatroids and geometric semilattices are abstractly equivalent in the sense explained in §1.2, see [1, 13].

Definition 1. Let G be a group. A G-semimatroid $\mathfrak{S}$ is given by a δ-preserving action of G on the ground set E of a semimatroid or, equivalently, an action of G by poset automorphisms on the associated geometric semilattice $\mathcal{L}$. We can define

– the poset $\mathcal{P}_{\mathfrak{S}} := \mathcal{L}/G$ of orbits;
– a function $\delta_{\mathfrak{S}} : 2^{E/G} \to \mathbb{Z}$ induced by δ (see [13, Definition 3.2]);
– an "orbit-counting" function $m_{\mathfrak{S}} : 2^{E/G} \to \mathbb{N}$,

$$m_{\mathfrak{S}}(X) = |\{p \in \mathcal{P}_{\mathfrak{S}} \mid p \text{ is a supremum of } X\}/G|$$

– a polynomial $T_{\mathfrak{S}}(x,y)$ defined from $\delta_{\mathfrak{S}}$ and $m_{\mathfrak{S}}$ as in Equation (1).

Definition 2. Call the G-semimatroid $\mathfrak{S}$

[1] I.e., every point of the space has a neighbourhood that meets only finitely many hyperplanes.

– *translative* if $\mathcal{L}/G$ is a finite poset and, for all $x \in \mathcal{L}$ and every $g \in G$, the existence of any $y \in \mathcal{L}$ with $y \geq x$ and $y \geq gx$ implies $x = gx$.
– *refined* if it is translative, the group G is finitely generated free abelian and, for all $x \in \mathcal{L}$, $\mathrm{stab}(x)$ is a free direct summand of G of rank $\delta(x)$.

Every abelian arrangement $\mathscr{A}$ gives rise to a (refined) G-semimatroid $\mathfrak{S}$, with $\mathcal{P}_{\mathfrak{S}} \simeq \mathcal{C}(\mathscr{A})$, $m_{\mathscr{A}} = m_{\mathfrak{S}}$, $T_{\mathscr{A}}(x, y) = T_{\mathfrak{S}}(x, y)$. Many well-known properties of matroids can be generalized, as the following sample of [8, 13] shows.

Theorem 1. *If $\mathfrak{S}$ is translative, then $T_{\mathfrak{S}}(x, y)$ satisfies deletion-contraction and*

$$\chi_{\mathcal{P}_{\mathfrak{S}}}(t) = (-1)^{\delta(\emptyset)} T_{\mathfrak{S}}(1 - t, 0).$$

Moreover, if $\mathfrak{S}$ is refined, then $\widetilde{H}^i(\widehat{\Delta}(\mathcal{P}_{\mathfrak{S}}), \mathbb{Z}) = 0$ for $i < \dim(\widehat{\Delta}(\mathcal{P}_{\mathfrak{S}}))$.

Notice that there are translative, nonrefined, representable actions for which the topological claim of the theorem fails.

See [13] for a discussion of representability and of conditions under which $m_{\mathfrak{S}}$ satisfies the axioms of arithmetic matroids. However, we do not know whether every arithmetic matroid arises from a G-semimatroid.

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Solomon-Terao algebra of hyperplane arrangements

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(joint work with Toshiaki Maeno, Satoshi Murai and Yasuhide Numata)

This is a short report on the forthcoming paper [2]. Let $V = \mathbb{C}^\ell$ and $\mathcal{A}$ a hyperplane arrangement in V, i.e., a finite set of linear hyperplanes in V. For each $H \in \mathcal{A}$ fix a linear form $\alpha_H \in V^*$ such that $\ker(\alpha_H) = H$. Let $S = \mathbb{C}[x_1, \ldots, x_\ell]$ the coordinate ring of V and $\mathrm{Der}S := \bigoplus_{i=1}^{\ell} S\partial_{x_i}$ the free S-module of S-derivations. The most important algebra of the hyperplane arrangement is the logarithmic derivation module defined as follows:

$$D(\mathcal{A}) := \{\theta \in \mathrm{Der}S \mid \theta(\alpha_H) \in S\alpha_H \ (\forall H \in \mathcal{A})\}.$$

$\mathcal{A}$ is free with exponents $\exp(\mathcal{A}) = (d_1, \ldots, d_\ell)$ if $D(\mathcal{A})$ is a free S-module with homogeneous basis $\theta_1, \ldots, \theta_\ell \in D(\mathcal{A})$ with $\deg(\theta_i) = d_i$ $(i = 1, \ldots, \ell)$. The most important consequence of the freeness is the following factorization theorem due to Terao in [5]:

$$\pi(\mathcal{A}; t) := \mathrm{Poin}(\mathbb{C}^\ell \setminus \bigcup_{H \in \mathcal{A}} H; t) = \prod_{i=1}^{\ell} (1 + d_i t).$$

Based on logarithmic derivation modules, Solomon and Terao introduced a polynomial $\Psi(\mathcal{A}; x, t) \in \mathbb{Q}[x, t]$ and proved that $\Psi(\mathcal{A}; 1, t) = \pi(\mathcal{A}; t)$. For details, see [4] and [2]. Our purpose is to consider the other specialization $\Psi(\mathcal{A}; x, 1)$. First let us define the new algebra $ST(\mathcal{A}, \eta)$ of $\mathcal{A}$ and a homogeneous polynomial η of degree $d > 0$ as follows:

$$ST(\mathcal{A}, \eta) := S/\mathfrak{a}(\mathcal{A}, \eta),$$

here $\mathfrak{a}(\mathcal{A}, \eta) := \{\theta(\eta) \mid \theta \in D(\mathcal{A})\}$ is the **Solomon-Terao ideal**, and $ST(\mathcal{A}, \eta)$ the **Solomon-Terao algebra** with respect to $\mathcal{A}$ and η. Let us introduce an example on the above objects.

Let $\mathcal{A}$ be the arrangement defined by $x_1 x_2 (x_1^2 - x_1^2) = 0$ in $\mathbb{C}^2$. Then we can compute

$$D(\mathcal{A}) = \langle x_1 \partial_{x_1} + x_2 \partial_{x_2}, x_1^3 \partial_{x_1} + x_2^3 \partial_{x_2} \rangle_S,$$

thus $\mathcal{A}$ is free with exponents $(1, 3)$. When $\eta = x_1^2 + x_2^2$, the Solomon-Terao algebra is $\mathfrak{a}(\mathcal{A}, \eta) = (x_1^2 + x_2^2, x_1^4 + x_2^4)$. Thus $ST(\mathcal{A}, \eta)$ coincides with the coinvariant algebra of the type B_2. This observation is true for all the other Weyl arrangements, which justifies our definition.

To justify this definition more, let us recall two results. The first one is by Solomon and Terao in [4], asserting that for each $d > 0$, there exists a non-empty Zariski open set $U_d(\mathcal{A})$ of the homogeneous polynomial of degree d such that $\dim_{\mathbb{C}} ST(\mathcal{A}, \eta) < \infty$ for all $\eta \in U_d(\mathcal{A})$. Hence for a generic η, $ST(\mathcal{A}, \eta)$ is Artinian. Second one is due to the first author, Horiguchi, Masuda, the third author and Sato in [1]. To state it, let us introduce a notation. Let W be the Weyl group acting on V, Φ the corresponding root system and Φ^+ a fixed positive system. Let $\alpha_1, \ldots, \alpha_\ell$ be the simple system. A subset $I \subset \Phi^+$ is a lower ideal if $\alpha \in I,\ \beta \in \Phi^+$ satisfy $\alpha - \beta \in \sum_{i=1}^{\ell} \mathbb{Z}_{\geq 0}\alpha_i$, then $\beta \in I$. For a lower ideal I, we can defie the ideal arrangement $\mathcal{A}_I$ as the set of all reflecting hyperplanes corresponding to the roots in I. Also, for I, we can define the regular nilpotent Hessenberg varierty $X(I)$, see [1] for details. Then

$$ST(\mathcal{A}_I, P_1) \simeq H^*(X(I), \mathbb{C}),$$

where P_1 is the lowest degree W-invariant polynomial. Hence the Solomon–Terao algebra could be a cohomology ring of certain algebraic varieties.

Also, when $\mathcal{A}$ satisfies a generic condition called the tameness, it is essentially shown in [4] (see also [2]) that

$$\mathrm{Hilb}(ST(\mathcal{A}, \eta); x) = \Psi(\mathcal{A}; x, 1)$$

for $\eta \in U_2(\mathcal{A})$. Thus for an ideal arrangement $\mathcal{A}_I$, it holds that

$$\Psi(\mathcal{A}; x, 1) = \mathrm{Hilb}(ST(\mathcal{A}, P_1); x) = \mathrm{Poin}(X; \sqrt{x}).$$

Thus the Solomon–Terao algebra gives an algebraic counter part of $\Psi(\mathcal{A}; x, 1)$ with possible nice geometric interpretations. Moreover, as an algebra itself, we can show the following main result about the Solomon–Terao algebra in [2].

Theorem ([3], [2]). $ST(\mathcal{A}, \eta)$ is a complete intersection ring if and only if $\mathcal{A}$ is free for all $d > 0$ and all $\eta \in U_d(\mathcal{A})$.

Note that the above theorem is shown by Epure and Schulze independently in [3] when $d = 2$ with more general setup, i.e., for hypersurface singularities. Hence we can give another characterization of the freeness in terms of the complete intersections of the Solomon–Terao algebra.

Since Solomon–Terao algebras provide a way to construct Artinian algebras from hyperplane arrangements, we can consider several problems related to Artinian rings. For example, the following questions are important.

Problem ([2]). Let $\eta \in S_2$. Then when is $ST(\mathcal{A}, \eta)$ Gorenstein?

If we want $ST(\mathcal{A}, \eta)$ to be a cohomology ring of some varieties, then it is necessary for $ST(\mathcal{A}.\eta)$ to be Gorenstein. However, we have no example of a non-free arrangement $\mathcal{A}$ such that $ST(\mathcal{A}, \eta)$ is Gorenstein.

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Configuration spaces of graphs

Eric Ramos

Definition 1. *A* ***graph*** *is a connected and compact 1-dimensional CW complex. We call the 0-cells of a graph G the* ***vertices*** *of G, while the 1-cells are referred to as the* ***edges*** *of G. The number of edges adjacent to a vertex v will be called the* ***degree*** *of v, and will be denoted $\mu(v)$.*

Given a graph G, we write $\mathrm{Conf}_n(G)$ to denote the ***n-stranded configuration space of*** *G,*

$$\mathrm{Conf}_n(G) = \{(x_1, \ldots, x_n) \in G^n \mid x_i \neq x_j\}.$$

We will write $\mathrm{UConf}_n(G)$ to denote the ***unordered n-stranded configuration space of*** *G*

$$\mathrm{UConf}_n(G) = \mathrm{Conf}_n(G)/\mathfrak{S}_n$$

In the first part of this survey, we will briefly recount the main structural theorems about the spaces $\mathrm{Conf}_n(G)$ and $\mathrm{UConf}_n(G)$.

One of the primary techniques in the study of graph configuration spaces is the use of certain cellular models for $\mathrm{Conf}_n(G)$ and $\mathrm{UConf}_n(G)$. The first such model was proposed by Abrams in [Ab]. Using this model, he proved the following.

Theorem 1 (Abrams [Ab], Theorem 3.10)**.** *Let G be a graph. Then $\mathrm{Conf}_n(G)$ and $\mathrm{UConf}_n(G)$ are aspherical.*

Other cellular models have been constructed by Ghrist [Gh], Swiatkowski [Sw], Farley and Sabalka [FS], and Lütgehetmann [Lu]. Each of these models has proven to be useful in different circumstances. For instance, the models of Ghrist and Swiatkowski each showed the following.

Theorem 2 (Ghrist [Gh], Theorem 3.3; Swiatkowski [Sw], Theorem 0.1)**.** *Let G be a graph which is not homeomorphic to a circle. Then $\mathrm{Conf}_n(G)$ is homotopy equivalent to a CW complex whose dimension is at most the number of vertices of G of degree ≥ 3. The same statement is true for $\mathrm{UConf}_n(G)$.*

As a consequence of the above, we immediately obtain that the homological dimensions of $\mathrm{Conf}_n(G)$ and $\mathrm{UConf}_n(G)$ are bounded independently of n. This behavior seems to be largely unique in the study of configuration spaces. We will also see that, in combination with the following theorem of Gal, it leads to problems when trying to compute the relevant homology groups.

Theorem 3 (Gal [Ga], Theorem 2)**.** *Let G be a graph with e edges, and let $\mathfrak{e}(t)$ denote the power series*

$$\mathfrak{e}(t) = \sum_{n \geq 0} \chi(\mathrm{Conf}_n(G)) \frac{t^n}{n!}.$$

Then,

$$\mathfrak{e}(t) = \frac{1}{(1-t)^e} \prod (1 + (1 - \mu(v))t))$$

where the product is over the vertices of G.

The following theorem was proven using the Farley-Sabalka model for $\mathrm{UConf}_n(G)$.

Theorem 4 (Ko and Park [KP], Theorem 3.5)**.** *Let G be a graph. Then $H_1(\mathrm{UConf}_n(G))$ is torsion-free if and only if G is planar. If $H_1(\mathrm{UConf}_n(G))$ has torsion, then it must be 2-torsion.*

In contrast to the above, it is conjectured that $H_i(\mathrm{Conf}_n(G))$ is always torsion-free [CL]. In fact, this has been proven for trees.

Theorem 5 (Chettih and Lütgehetmann [CL], Theorem A)**.** *If G is a tree, then $H_q(\mathrm{Conf}_n(G))$ is torsion-free for all $q \geq 0$.*

To conclude, we outline the work that has been done towards applying techniques from representation stability theory to understand the homology groups of $\mathrm{Conf}_n(G)$ and $\mathrm{UConf}_n(G)$. One should observe that Theorems 2 and 3 imply that at least one of the homology groups $H_q(\mathrm{Conf}_n(G))$ has Betti numbers which grow at least factorially in n. While this would seem to preclude the usual FI-module techniques, there are still some conclusions one can draw.

Theorem 6. *Let G be a graph, and let S_G denote the integral polynomial ring whose variables are labeled by the edges of G. Then:*

(1) [*An, Drummond-Cole, Knudsen* [ADK], *Theorem 4.5*] *For all $q \geq 0$, the abelian group $\bigoplus_n H_q(\mathrm{UConf}_n(G))$ can be equipped with an action of S_G, turning it into a finitely generated graded S_G-module.*

(2) [*Ramos* [Ra], *Theorem D*] *If G is a tree, then the S_G-module $\bigoplus_n H_q(\mathrm{UConf}_n(G))$ decomposes as a direct sum of graded twists of square-free monomial ideals. Moreover, this decomposition only depends on q and the degree sequence of G.*

Note that Maciazek and Sawicki independently proved the statement about the homology groups only depending on the degree sequence of the tree independently of the author [MS, Theorem V.3].

Theorem 7 (Lütgehetmann [Lu2], Theorem I)**.** *Let G be a 3-connected graph with at least 4 vertices of degree ≥ 3. Then the* FI*-module $H^1(\mathrm{Conf}_n(G); \mathbb{Q})$ is finitely generated.*

Fundamentally, one of the main difficulties with configuration spaces of graphs is that it is difficult for points to move around one another. Therefore, allowing the number of points being configured to grow leads to very unstable behaviors. One way to combat this is to fix the number of points being configured and instead allow the graph itself to vary. Let $\mathcal{T}$ denote the category of trees and injective maps, and let $\mathcal{G}$ denote the category of graphs and injective maps. The following theorem is to appear in future work.

Theorem 8 (Lütgehetmann and Ramos)**.** *For all $k, q \geq 0$, The functor $T \mapsto H_q(\mathrm{Conf}_k(T))$ from $\mathcal{T}$ to the category of abelian groups is finitely generated. The same is true of the functor $G \mapsto H_q(\mathrm{Conf}_k(G))$ from $\mathcal{G}$ to the category of abelian groups, so long as $q \leq 1$.*

It is unknown whether the second half of the above theorem can be expanded to include all $q \geq 0$.

Another theorem in the same vein is the above is the following. Note that for any injection of sets $f\colon \{1, \ldots, n\} \hookrightarrow \{1, \ldots, m\}$, one obtains a map of graphs $K_n \to K_m$ between complete graphs. This induces an FI-module structure on the homology groups $H_q(\mathrm{Conf}_k(K_n))$ and $H_q(\mathrm{UConf}_k(K_n))$, for each fixed $k, q \geq 0$, where we allow n to vary.

Theorem 9 (Ramos and White [RW], Theorem G)**.** *The* FI*-modules $H_q(\mathrm{Conf}_k(K_n))$ and $H_q(\mathrm{UConf}_k(K_n))$ are finitely generated for all choices of q and k.*

In fact, the above theorem will hold whenever K_n is replaced by any vertex-stable FI-graph (see [RW] for definitions).

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Freeness of multi-reflection arrangements for complex reflection groups

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(joint work with Torsten Hoge, Toshiyuki Mano, Christian Stump)

In his seminal work [13], Ziegler introduced the concept of multi-arrangements generalizing the notion of hyperplane arrangements. In [10], Terao showed that every reflection multi-arrangement of a real reflection group with a constant multiplicity is free. The aim of the joint work [4], reported on in this talk, is to generalize this result from real reflection groups to unitary reflection groups.

Before reporting on our generalizations of [10, Thm. 1.1] in [4], I want to briefly recall the background and motivation for Terao's work. In [2, Conj. 3.3], Edelman and Reiner conjectured that the cones over the extended Shi arrangements and the extended Catalan arrangements are free with prescribed exponents. Edelman and Reiner were able to prove their conjecture in case of the extended Catalan arrangement for the underlying root system of type A in *loc. cit.*

If the above conjecture is true, then Ziegler's theorem [13, Thm. 11] implies the freeness of the multi-arrangements of the underlying Weyl arrangements with constant multiplicity at every hyperplane (with exponents derived from the conjecture). Terao's theorem [10, Thm. 1.1] confirms this consequence of the conjecture. Ultimately, the conjecture of Edelman and Reiner was proved by Yoshinaga in [12] by combining [10, Thm. 1.1] with a local criterion for freeness, [12, Thm. 2.5].

From now on suppose that W is an irreducible unitary reflection group with reflection representation $V \cong \mathbb{C}^\ell$. Denote the set of reflections of W by $\mathcal{R} = \mathcal{R}(W)$, and the associated reflection arrangement in V by $\mathscr{A} = \mathscr{A}(W)$. Following [3], the *Coxeter number* of W is given by

$$h := \frac{1}{\ell} \sum_{H \in \mathscr{A}} e_H = \frac{1}{\ell}\Big(|\mathcal{R}| + |\mathscr{A}|\Big),$$

generalizing the usual Coxeter number of a real reflection group to irreducible unitary reflection groups.

Let $\mathrm{Irr}(W)$ denote the irreducible complex representations of W up to isomorphism. For U in $\mathrm{Irr}(W)$ of dimension d, denote by

$$\exp_U(W) := \big\{n_1(U) \leq \ldots \leq n_d(U)\big\}$$

the U*-exponents* of W given by the d homogeneous degrees in the coinvariant algebra of W in which U appears. In particular, the *exponents* of W are

$$\exp(W) := \exp_V(W) = \{n_1(V) \le \ldots \le n_\ell(V)\}$$

and the *coexponents* of W are

$$\operatorname{coexp}(W) := \exp_{V^*}(W) = \{n_1(V^*) \le \ldots \le n_\ell(V^*)\}.$$

The group W is *well-generated* if $n_i(V) + n_{\ell+1-i}(V^*) = h$, e.g., see [7, 6, 1].

For $H \in \mathscr{A}$, let e_H denote the order of the point-wise stabilizer of H in W. Consider the *order multiplicity* function

$$\omega : \mathscr{A} \to \mathbb{N}, \quad \omega(H) = e_H$$

for each hyperplane $H \in \mathscr{A}$. For $m \in \mathbb{N}$ let $m\omega$ and $m\omega+1$ denote the multiplicities defined by $m\omega(H) = me_H$ and $m\omega(H) + 1 = me_H + 1$ for $H \in \mathscr{A}$, respectively.

The following is [4, Thm. 1.1], generalizing [10, Thm. 1.1] to the case of well-generated finite unitary reflection groups.

Theorem 1. *Let W be an irreducible, well-generated unitary reflection group with reflection arrangement $\mathscr{A}$. Let $\omega : \mathscr{A} \to \mathbb{N}$ given by $\omega(H) = e_H$, and let $m \in \mathbb{N}$. Then*

(i) the reflection multi-arrangement $(\mathscr{A}, m\omega)$ is free with exponents

$$\exp(\mathscr{A}, m\omega) = \{mh, \ldots, mh\},$$

(ii) the reflection multi-arrangement $(\mathscr{A}, m\omega + 1)$ is free with exponents

$$\exp(\mathscr{A}, m\omega + 1) = \{mh + n_1(V^*), \ldots, mh + n_\ell(V^*)\}.$$

Note from above that $\operatorname{coexp}(W) = \exp_{V^*}(W) = \{n_1(V^*), \ldots, n_\ell(V^*)\}$.

In the special case when W is a Coxeter group, Theorem 1 recovers [10, Thm. 1.1], as then $\omega \equiv 2$ and $\operatorname{coexp}(W) = \exp(W)$.

In [4, Thm. 4.20], we prove a more general version of Theorem 1 based on a generalization of Yoshinaga's approach [11] to [10, Thm. 1.1]. More precisely, we first extend Yoshinaga's construction of a basis of the module of derivations and of Saito's Hodge filtration to well-generated unitary reflection groups by using recent developments of flat systems of invariants in the context of isomonodromic deformations and differential equations of Okubo type due to Kato, Mano and Sekiguchi [5].

Our second main result [4, Thm. 1.2] extends Theorem 1 further to the infinite three-parameter family $W = G(r, p, \ell)$ of imprimitive reflection groups. It turns out that the corresponding multi-arrangements are also free. However, the description of the exponents is considerably more involved and depends on the representation theory of the Hecke algebra associated to the group W. To this end, let Ψ denote the permutation on $\operatorname{Irr}(W)$ introduced by Malle in [6, Sec. 6C], having the semi-palindromic property on the fake degrees of W. This is, for any U in $\mathrm{Irr}(W)$ of dimension d, we have

$$n_i(U) + n_{d+1-i}(\Psi(U^*)) = h_U,$$

where $h_U = |\mathscr{A}| - \sum_{r\in\mathcal{R}} \chi(r)/\chi(1)$, where χ is the character of U. A direct calculation shows that $h_V = h$ is the Coxeter number of W.

Theorem 2. *Let $W = G(r,p,\ell)$ with reflection arrangement $\mathscr{A}$. Let $\omega : \mathscr{A} \to \mathbb{N}$ given by $\omega(H) = e_H$, and let $m \in \mathbb{N}$. Then*

(i) the reflection multi-arrangement $(\mathscr{A}, m\omega)$ is free with exponents

$$\exp(\mathscr{A}, m\omega) = \{mh, \dots, mh\},$$

(ii) the reflection multi-arrangement $(\mathscr{A}, m\omega + 1)$ is free with exponents

$$\exp(\mathscr{A}, m\omega + 1) = \{mh + n_1(\Psi^{-m}(V^*)), \dots, mh + n_\ell(\Psi^{-m}(V^*))\}.$$

Note this time that $\exp_{\Psi^{-m}(V^*)}(W) = \{n_1(\Psi^{-m}(V^*)), \dots, n_\ell(\Psi^{-m}(V^*))\}$.

Remarks 3. (i). Observe that the group $G(r,p,\ell)$ is well-generated if and only if $p \in \{1, r\}$. Moreover, $\Psi(V^*) = V^*$ if and only if W is well-generated [6, Cor. 4.9]. Thus, Theorem 2 extends Theorem 1 to the class of imprimitive reflection groups that are not well-generated.

(ii). While the reflection arrangements of the reflection groups $G(r,1,\ell)$ and $G(r,p,\ell)$ for $1 < p < r$ coincide, the multi-arrangements in Theorem 2 depend on the structure of the underlying group.

(iii). Computational evidence for small values for m suggest that Theorem 2 extends to the remaining eight irreducible complex reflection groups of exceptional type that are not well-generated.

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Homology of Artin groups: A combinatorial group theoretic approach

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(joint work with Toshiyuki Akita)

Given an arbitrary Coxeter system (W, S), or equivalently a Coxeter graph Γ, the Artin group $A = A(\Gamma)$ associated to Γ is obtained from the standard Coxeter presentation of $W = W(\Gamma)$ by dropping the relations $s^2 = 1$ for $s \in S$. The celebrated $K(\pi, 1)$ conjecture asserts that A admits a nice $K(\pi, 1)$ space by realizing W as a reflection group acting on a Tits cone [6]. Homology of Artin group $A(\Gamma)$ can be computed from the conjectural space if the $K(\pi, 1)$ conjecture is proved.

However the $K(\pi, 1)$ conjecture has only been proved for certain classes of Artin groups (see [1] or [6]) and remained open in general. In this talk, we start a combinatorial group theoretic approach to the computation of homology of Artin groups, without assuming that the $K(\pi, 1)$ conjecture holds.

The first step is the following easy observation.

Theorem 1. *For an arbitrary Coxeter graph* Γ*, we have*

$$H_1(A(\Gamma); \mathbb{Z}) \cong \mathbb{Z}^{c(\Gamma)},$$

where $c(\Gamma)$ *is the number of connected components of* Γ_{odd}*, the induced subgraph of* Γ *obtained by deleting edges with even number label and edges with* ∞ *label.*

This result follows from the fact that $H_1(A; \mathbb{Z})$ is the abelianization of A, and the latter is obtained by imposing commutating relations for each pair of standard generators.

Our next result is less trivial. Let us define the following numbers associated to a Coxeter graph Γ. Denote by $P(\Gamma)$ the set of pairs of non-adjacent vertices of Γ. We say that $\{s, t\} \equiv \{s, t'\}$ in $P(\Gamma)$ if $t, t' \in S$ are joined by an edge with odd number label (i.e. $m(t, t')$ is odd). Let $\sim$ be the equivalence relation in $P(\Gamma)$ generated by $\equiv$. Now define $n_1(\Gamma) = \#\left(P(\Gamma)/\sim\right)$, $n_2(\Gamma) = \#\{\{s, t\} \subset S \mid m(s, t) \geq 4, m(s, t) \text{ is even}\}$ and $n_3(\Gamma) = \operatorname{rank} H_1(\Gamma_{odd}; \mathbb{Z})$.

Theorem 2 ([1])**.** *For an arbitrary Coxeter graph* Γ*, we have*

$$H_2(A(\Gamma); \mathbb{Z}_2) \cong \mathbb{Z}_2^{n(\Gamma)},$$

where $n(\Gamma) = n_1(\Gamma) + n_2(\Gamma) + n_3(\Gamma)$*.*

The idea of proof is to use the Hopf's formula.

Theorem 3 (Hopf's formula). *If a group G has a presentation $\langle S \mid R\rangle$, then*

$$H_2(G;\mathbb{Z}) \cong \frac{N\cap[F,F]}{[F,N]},$$

where $F = F(S)$ is the free group generated by S and $N = N(R)$ is the normal closure of R.

Applying Hopf's formula to Coxeter groups and Artin groups with their standard presentations, we may construct explicitly second homology classes as cosets $x[F,N]$ with $x \in N\cap[F,F]$ as above. We manage to find a set $\Omega(W)$ of generators of $H_2(W;\mathbb{Z})$ and a set $\Omega(A)$ of generators of $H_2(A;\mathbb{Z})$ such that the homomorphism $p_* : H_2(A;\mathbb{Z}) \to H_2(W;\mathbb{Z})$ induced by the natural map $p : A \to W$ maps $\Omega(A)$ onto $\Omega(W)$. Moreover we have by construction $\#\Omega(W) = n(\Gamma)$. On the other hand, Howlett proved the following.

Theorem 4 ([5]). *For an arbitrary Coxeter graph Γ, we have*

$$H_2(W(\Gamma);\mathbb{Z}) \cong \mathbb{Z}_2^{n(\Gamma)}.$$

Hence we know that $\Omega(W)$ is a basis of $H_2(W;\mathbb{Z})$, and we have proved that p_* is surjective. Theorem 2 follows without difficulties.

We expect that the above computation extends to higher homology of Artin groups. In fact, we have the similar ingredients: $H_3(W;\mathbb{Z})$ has been computed in [4] and [2], the higher Hopf formulae have been studied in [3].

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Milnor fiber complexes and some representations

Alexander R. Miller

H. O. Foulkes discovered some amazing characters for the symmetric group S_n by summing Specht modules of certain ribbon shapes according to height [7]. These characters have some remarkable properties and have been the subject of many investigations, most recently because of connections with adding random numbers, shuffling cards, the Veronese embedding, and combinatorial Hopf algebras, see [2, 5, 6, 9, 17]. We give a new approach to these characters which works for a wide variety of reflection groups. The approach is geometric and based on an object called the Milnor fiber complex. It gives new results and it unifies, explains, and extends previously known (type A) ones. This work appears in [12, 13, 14, 15].

Coxeter and Shephard groups. Let V be an ℓ-dimensional vector space over $\mathbf{C}$, and let G be a finite group with presentation

$$\langle\, r_1, r_2, \dots, r_\ell \mid r_i^{p_i} = 1,\ \underbrace{r_i r_j r_i \dots}_{m_{ij}\text{ terms}} = \underbrace{r_j r_i r_j \dots}_{m_{ji}\text{ terms}}\ \ i \neq j \,\rangle \tag{1}$$

where $p_i \geq 2$, $m_{ij} = m_{ji} \geq 2$, and $p_i = p_j$ when m_{ij} is odd. Write $R = \{r_1, \dots, r_\ell\}$. Finite Coxeter groups are the ones where each p_i is 2. In general G has a Coxeter-like diagram Γ and a canonical faithful representation $G \subset \mathrm{GL}(V)$ as a *(complex) reflection group* in which the generators r_i act on V as *reflections* in the sense that they have finite order and the fixed spaces $\ker(1 - r_i)$ are hyperplanes [10]. The group is identified with its canonical representation as a reflection group and called irreducible if it acts irreducibly on V. Being irreducible is equivalent to the diagram having exactly one connected component. Finite groups with presentation (1) were classified in [10]. The irreducible ones are precisely the finite irreducible Coxeter groups and the groups known as Shephard groups (symmetry groups of objects called *regular complex polytopes* [3] studied by Shephard and Coxeter).

Milnor fiber complex. Associated to G is an abstract simplicial complex Δ with simplices (labeled by) cosets $g\langle I\rangle$ of standard parabolic subgroups $\langle I\rangle$ $(I \subset R)$ with face relation "$g\langle I\rangle$ is a face of $h\langle J\rangle$" $\Leftrightarrow$ $g\langle I\rangle \supset h\langle J\rangle$, and with G acting by left translation. If G is a Coxeter group, then this is the classical abstract description of the Coxeter complex [26]. See [22, 19, 12, 15] for details, geometry, and history.

Foulkes characters. Each type-selected subcomplex Δ_S $(S \subset R)$ is a bouquet of spheres, and we call the $\mathbf{C}G$-module on the top reduced homology group $H_{|S|-1}(\Delta_S)$ a *ribbon representation*, see [12]. Its character ρ_S is an alternating sum of characters induced by principal characters of parabolic subgroups [12]. The *(generalized) Foulkes characters* defined in [13] are

$$\phi_k = \sum_{\substack{S \subset R \\ |S| = k}} \rho_S \qquad (k = 0, 1, \dots, \ell). \tag{2}$$

An immediate benefit of this approach is the following formula [13, Theorem 1]

$$\phi_k(g) = \sum_{i=0}^{\ell} (-1)^{k-i} \binom{\ell - i}{k - i} f_{i-1}(\Delta^g) \tag{3}$$

where $\Delta^g = \{\sigma \in \Delta : g\sigma = \sigma\}$ and $f_k(\Sigma)$ is the number of k-simplices in Σ. The face numbers $f_k(\Sigma)$ can be computed with a formula of Orlik and Solomon. Assume G irreducible. Let L be the set of all intersections of reflecting hyperplanes ordered by reverse inclusion, and let μ be the Möbius function. For $X \in L$ define $B_X(t) = (-1)^{\dim X} \sum_{Y \geq X} \mu(X, Y)(-t)^{\dim Y}$. Let $d_1 \leq d_2 \leq \ldots \leq d_\ell$ be the basic degrees of G. Then Orlik [22] (after Orlik–Solomon in the Coxeter case) proved

$$f_{i-1}(\Delta^g) = \textstyle\sum_Y B_Y(d_1 - 1) \tag{4}$$

where the sum is over all i-dimensional subspaces Y above $V^g = \ker(1-g)$ in L.

Elucidating and generalizing classical (type A) results. Our approach elucidates and extends the type A theory (due to Foulkes, Kerber–Thürlings, Diaconis–Fulman, and Isaacs), which previously rested on ad hoc proofs by induction. See [13]. For example, if G is the wreath product $Z_r \wr S_n$ (Z_r cyclic of order r), then L is a Dowling lattice and the restrictions L^X depend only on the dimension of $X \in L$, so that by (3) and (4) the ϕ_i's *depend only on fixed-space dimension* in the sense that $\phi_i(g) = \phi_i(h)$ whenever $\dim V^g = \dim V^h$. The $r = 1$ case of this is the classical fact that the Foulkes characters $\phi_i(g)$ of S_n depend only on the number of cycles of g. The only previous proof of this for S_n is the original one due to Foulkes [7] which uses the Murnaghan–Nakayama rule and induction.

Adding random numbers. Interestingly, these generalized Foulkes characters have recently been connected to adding random numbers in other number systems. Persi Diaconis and Jason Fulman [6] connected the hyperoctahedral ones (type B) to adding random numbers in balanced ternary and other number systems that minimize carries, and Nakano–Sadahiro [16] connected the Foulkes characters for $Z_r \wr S_n$ to a generalized carries process and riffle shuffles.

New phenomena. If G is the wreath product $Z_r \wr S_n$, then the Foulkes characters form a basis for the space of class functions $\chi(g)$ of G that depend only on length $\ell(g) = \min\{k : g = t_1 t_2 \dots t_k,\ t_i \text{ a reflection}\}$, see [13, 14]. Danny Goldstein, Robert M. Guralnick, and Eric M. Rains together made the remarkable experimental observation [18] that in fact the hyperoctahedral Foulkes characters play the role of irreducibles among the hyperoctahedral characters that depend only on length, in the sense that the characters of the hyperoctahedral group B_n that depend only on length are precisely the $\mathbf{N}$-linear combinations of the hyperoctahedral Foulkes characters. We prove this conjecture in [14]. In fact we prove that the same is true for all wreath products $Z_r \wr S_n$ with $r > 1$, not just $r = 2$.

It is an open problem to give a nice description of the characters $\chi(g)$ for S_n ($r = 1$) that depend only on $\ell(g)$, or in other words, that depend only on the number of cycles of g. Kerber [9, p. 306] noticed that already for S_5 the $\mathbf{N}$-linear combinations of Foulkes character do not account for all the characters of S_5 that depend only on length. In [14] we prove that this is always the case for symmetric groups S_n with $n \geq 3$. NOTE: This line of investigation makes sense for any finite group with given set of generators closed under conjugation.

Curious classification. In [13] we determined all the irreducible cases of G where the ϕ_i's depend only on fixed-space dimension. This led to a curious classification with 11 equivalent conditions [13, Thm. 14]. For example, we find that the ϕ_i's depend only on fixed-space dimension if and only if the sequence of basic degrees $d_1, d_2, \dots, d_\ell$ is arithmetic. Another equivalent condition is that the diagram of G contains no subdiagram of type D_4, F_4, or H_4. We recently found this condition in [1] Abramenko's answer to a geometric problem: In which Coxeter complexes Δ are all walls Δ^r (r a reflection) Coxeter complexes? In [15] we extend Abramenko's result to Milnor fiber complexes in two ways and find another equivalent condition for the Foulkes characters to depend only on fixed-space dimension. In the course of that work we also discovered a beautiful enumerative condition [15, Thm. 11]: if G is irreducible, then the diagram contains no subdiagram of type D_4, F_4, or H_4 if and only if for each $g \in G$ the number of top cells in Δ^g is given by

$$f_{p-1}(\Delta^g) = d_1 d_2 \cdots d_p, \quad p = \dim V^g. \tag{5}$$

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Problem Session

Nate Harman, Aurélien Djament, Roberto Pagaria, Jeremy Miller, Weiyan Chen, Jesse Wolfson, Masahiko Yoshinaga, Alexander R. Miller, Graham Denham, Dan Petersen, Michael Falk

We hosted two evening problem sessions during the workshop. Various participants from very different backgrounds proposed open questions to the audience. The sessions stimulated active discussions among the participants. The problems are collected below in the order that they were proposed.

1. **Nate Harman (University of Chicago).** The following standard theorem from representation theory of the symmetric groups roughly says that "FI-modules see all representations of polynomial growth":

Theorem 1. *Suppose V_n is a sequence of irreducible representations of S_n. If there exists a constant d such that $\dim V_n < n^d$ for all $n \gg 0$, then either V_n or $V_n \otimes \mathrm{sign}$ is a factor of an FI-module generated in degree at most d.*

With the motivation to understand low dimensional representations of the braid group, we ask the following question:

Question 1. *Is there an analog if we replace S_n by the braid group B_n?*

Conjecture 1. *All representations of B_n with slow growth come from finitely generated modules over certain category.*

Ivan Marin remarked that the conjecture is known for linear growth, *e.g.*, in the case when the dimension is $n-1$. (see [5])

2. **Aurélien Djament (CNRS, Nantes).** Let k be a maximal ordered field (*e.g.*, $k = \mathbb{R}$). Let's consider the following monomorphisms between orthogonal groups, for all n and i:

$$O_n(k) \times O_i(k) \hookrightarrow O_{n,i}(k).$$

Question 2. *Does this map induces an isomorphism of $H_d(-,\mathbb{Z})$ for $n \gg d, i$?*

The homology here is understood as the group homology of discrete groups.

For $i = 1$, a theorem of Bökstedt, Brun, and Dupont [2] shows that the answer is yes for $d < n$.

3. **Roberto Pagaria (SNS, Pisa).** Let $\mathcal{A}$ be a central toric arrangement in the torus T, let $\mathcal{L}$ be its poset of layers, and let $M(\mathcal{A})$ be the complement in T. Consider the cohomology algebra with rational coefficients $H^\bullet(M(\mathcal{A}))$ and its associated graded algebra with respect to the Leray filtration, $\operatorname{gr} H^\bullet(M(\mathcal{A}))$.

Theorem 2 ([9, Theorem 4.6])**.** *The poset of layers $\mathcal{L}$ determines the algebra $\operatorname{gr} H^\bullet(M(\mathcal{A}))$.*

Moreover, a stronger statement holds: the poset $\mathcal{L}$ describes the cohomology algebra $H^\bullet(M(\mathcal{A}))$. We ask whether the converse holds:

Question 3. *Does the cohomology algebra $H^\bullet(M(\mathcal{A}))$ determine the poset of layers $\mathcal{L}$?*

The analogous statement in the setting of hyperplane arrangements has a negative answer (see [4] or [3]), but the combinatorics of toric arrangements is richer than combinatorics of hyperplane arrangements. In order to solve this problem, a deep study of characteristic varieties of toric arrangements could be useful.

4. **Jeremy Miller (Purdue University).** It was previously known that (by Proposition A.1. of [10])

$$H^2(\mathrm{Out}(F_n), \mathbb{Z}^n) = \mathbb{Z}/(n-1)\mathbb{Z} \quad \text{for } n \geq 9.$$

Question 4. *Is $H^i(\mathrm{Out}(F_n), (\mathbb{Z}/p\mathbb{Z})^n)$ eventually periodic as n increases? If yes, we ask the same question replacing the twisted coefficient $(\mathbb{Z}/p)^n$ by a polynomial functor, or a VIC-module.*

Andrew Snowden remarked that this question is motivated by Rohit Nagpal's work (Theorem C of [8]), and the subsequent works by various authors.

5. **Weiyan Chen (University of Minnesota, Twin Cities).**

Definition 1. *The Schwarz genus of a cover $f : Y \to X$ is the minimum number k such that there exists a cover of X by connected open subsets $U_1, U_2, ..., U_k$ where the restriction of f to each open subset is trivial.*

Alex Suciu remarked that Schwarz genus is a special case of a more general concept called "sectional category" which can be defined for any fiberation.

Consider the following spaces.

$$X := \{\text{Smooth complex homogeneous polynomials } F(x,y,z) \text{ of degree } 3\}/\mathbb{C}^\times$$
$$Y := \{(F,p) \in X \times \mathbb{C}P^2 : p \text{ is a flex on the smooth cubic curve } F = 0\}.$$

Then Y is a covering space of X of degree 9 (since there are 9 flexes on every smooth cubic curve).

Question 5. *What is the Schwarz genus of the flex cover $Y \to X$ as defined above?*

Weiyan Chen remarked that he was able to bound the number to be no smaller than 3 and no larger than 9. This question is motivated by the work of Smale [12], who first interpreted the Schwarz genus as a lower bound for the topological complexity of any algorithm solving certain problems. In a similar way, an answer to the question above gives a lower bound for any algorithm that finds a flex for any given smooth cubic curve. One can ask the similar question for many enumerate problems.

6. **Jesse Wolfson (University of California, Irvine).**

Theorem 3 (Jacobi, 1850). *Every smooth quartic plane curve has 28 bitangent lines.*

Let $H_{4,2}$ denote the space of smooth quartic curves in $\mathbb{C}P^2$. Precisely, let

$$H_{4,2} = (P^{\binom{4+2}{2}} \setminus \Sigma)/\mathrm{PGL}_3(\mathbb{C})$$

where Σ denote the discriminant locus containing singular homogeneous quartic polynomials that give singular quartic curves. Let $H_{4,2}(1)$ denote the space of smooth quartic curves equipped with a bitangent line. Jacobi's theorem tells us that $H_{4,2}(1)$ is a degree 28 cover of $H_{4,2}$. Other classical covers of interest are $H_{4,2}(S)$, $H_{4,2}(A)$, and $H_{4,2}(C)$, the moduli of quartics equipped with a Steiner complex, Aronhold set, and Cayley octad respectively. These give degree 63, 288, and 36 covers of $H_{4,2}$.

Question 6. *What is $H^*(H_{4,2}(1), \mathbb{Q})$? Similarly, what is $H^*(H_{4,2}(X), \mathbb{Q})$ for $X = S, A$ or C?*

Weiyan Chen commented that the computation of $H^*(H_{4,2}(1), \mathbb{Q})$ can be found in a paper by Orsola Tommasi [13]. Dan Petersen commented that work of Olof Bergvall [1] is also relevant.

7. **Masahiko Yoshinaga (Hokkaido University).**

Definition 2. *$f : \mathbb{Z} \to \mathbb{C}$ is quasi-polynomial if there exists $\rho > 0$ and $g_1(t), ..., g_\rho(t) \in \mathbb{C}[t]$ such that*

$$f(n) = \begin{cases} g_1(n), & \text{if } n = 1 \mod \rho \\ g_2(n), & \text{if } n = 2 \mod \rho \\ \cdots & \\ g_\rho(n), & \text{if } n = \rho \mod \rho. \end{cases}$$

Furthermore, f has the GCD-property if

$$g_i(t) = g_j(t) \quad \text{if } (i, \rho) = (j, \rho).$$

In other words, f has the GCD-property if the constituent depends only on the GCD with the period ρ.

Question 7. *Which rational polytope has Ehrhart quasi-polynomial with GCD-property?*

Example 1. *Let* $P_1 = [0, 1/3]$*.*

$$L_{P_1}(n) = \begin{cases} \frac{n+3}{3}, & \textit{if } n = 0 \mod 3 \\ \frac{n+2}{3}, & \textit{if } n = 1 \mod 3 \\ \frac{n+1}{3}, & \textit{if } n = 2 \mod 3 \end{cases}$$

Hence the Ehrhart quasi-polynomial of P_1 *does not have GCD-property.*

Example 2. *Let* $P_2 = [1/3, 4/3]$*.*

$$L_{P_2}(n) = \begin{cases} n+1, & \textit{if } n = 0 \mod 3 \\ n, & \textit{if } n = 1, 2 \mod 3 \end{cases}$$

Hence the Ehrhart quasi-polynomial of P_2 *has GCD-property.*

Example 3. *The Ehrhart quasi-polynomial of the fundamental alcove of a root system has GCD-property.* [14]

Question 8. *Let* P *be an integral zonotope in* $\mathbb{Z}^\ell$*. Let* $a \in \mathbb{Q}^\ell$*. Does the translated zonotope* $P' = a + P$ *have the Ehrhart quasi-polynomial with GCD-property?*

8. **Alexander R. Miller (Universität Wien).** For λ, μ partitions of n, let χ_λ denote the character of the irreducible S_n-representation corresponding to λ. Here is an observation: as $n \to \infty$,

$$\text{Prob}(\chi_\lambda(g) = 0 \ \text{ for } \lambda \vdash n \text{ and } g \in S_n) \longrightarrow 1. \tag{1}$$

Conjecture 2. *As* $n \to \infty$*,*

$$\text{Prob}(\chi_\lambda(\mu) = 0 \mod 2 \ \textit{ for } \lambda, \mu \vdash n) \longrightarrow 1. \tag{2}$$

In other words, the conjecture says that an entry chosen uniformly at random from the character table of S_n *is even with probability* $\to 1$ *as* $n \to \infty$*.*

The probability measures in (1) and in (2) are different: the former is uniform over elements in S_n, while the latter is uniform over conjugacy classes of S_n.

Alexander Miller remarked that he has done some computer experiments which suggest that the probability in (2) converges to 1 following the graph of

$$2\pi^{-1} \arctan(\sqrt{n/2} - 1).$$

It was also remarked that similar unexpected behavior occurs for other primes.

9. **Graham Denham (University of Western Ontario).** If we set

$$Q(x_1, \dots, x_n) := \prod_{1 \le i < j \le n} (x_i - x_j),$$

then the ordered configuration space becomes

$$\text{Conf}(n, \mathbb{C}) = \{(x_1, \dots, x_n) : Q \neq 0\}.$$

Consider the Milnor fiber

$$F_n := \{(x_1, \dots, x_n) : Q^2 = 1\}.$$

F_n carries commuting actions of two groups: an action of S_n by permuting the coordinates, and an action of the group $\mu_{n(n-1)}$ of $n(n-1)$-th roots of unity by the diagonal multiplication.

Question 9. *What is $H^*(F_n, \mathbb{Q})$ as a $(S_n \times \mu_{n(n-1)})$-module?*

A stability phenomenon was discovered by Simona Settepanella: the action of $\mu_{n(n-1)}$ on $H^i(F_n, \mathbb{Q})$ is trivial when $n \gg i$.

Question 10. *Is there a category $\mathcal{C}$ similar to FI such that $H^i(F_n, \mathbb{Q})$ becomes a finitely generated $\mathcal{C}$-module?*

Notice that there is no obvious way to make $H^i(F_n, \mathbb{Q})$ an FI-module, since there is no natural map between F_n and F_{n+1}.

Kevin Casto remarked that it may be fruitful to study the cohomology of the maximal abelian cover of $\mathrm{Conf}(n, \mathbb{C})$, which is also the classifying space of the commutator subgroup of the pure braid group, and whose cohomology is naturally an FI-module.

10. **Dan Petersen (Stockholm University).**

Theorem 4 (Randal-Williams–Wahl, [11])**.** *If V_n is a sequence of polynomial coefficient system for the braid group B_n, then $H_*(B_n, V_n)$ stabilizes as $n \to \infty$.*

Question 11. *Let P_n be the pure braid groups. The collection $H_*(P_n, V_n)$ is an FI-module. Is it finitely generated?*

Notice that the theorem of Randal-Williams–Wahl above implies that $H_*(P_n, V_n)$ satisfies multiplicity stability. One can also prove that the homology grows polynomially in n.

Since the (co)homology of pure braid groups was the original motivating example for the theory of representation stability, it is surprising that this is not known.

11. **Michael Falk (Northern Arizona University).** Let $M_{d,n}$ denote the space of unlabeled affine arrangements of n hyperplanes in general position in $\mathbb{C}^d$.

Problem 1. *Find a presentation for $\pi_1(M_{d,n})$ that specializes to Artin's presentation of the full braid group for $d = 1$.*

There has been some work on the space of labeled affine arrangements in general position [6, 7], but one would expect a nicer presentation for the unlabeled version, since that is the case for $d = 1$: there are fewer generators and more symmetric relations in Artin's presentation of the full braid group, than in the standard presentation of the pure braid group.

After a brief dicussion, Alex Suciu asked the following question:

Question 12. *Does $M_{d,n}$ have a nice compactification?*

12. **Dan Petersen (Stockholm University).** Suppose E and V are finite dimensional vector spaces over $\mathbb{C}$. Define a commutative algebra $A := \mathbb{C} \oplus E$ where elements in E multiply to 0 (a square zero extension). Let $\mathfrak{g} := \mathrm{Lie}(V)$, a free Lie algebra.

Question 13. *What is the Lie algebra homology $H_*(\mathfrak{g} \otimes A)$?*

The answer should be expressed as a sum of polynomial functors in E and V.

When E is 1-dimensional, an answer can be calculated by hand (even this case is not obvious). A complete answer to the question in its general form will help us understand the cohomology of the "link" of the tropical moduli space of curves $\mathcal{M}_{2,n}$.

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