Mathematisches Forschungsinstitut Oberwolfach

Mini-Workshop: Gibbs Measures for Nonlinear Dispersive Equations

Organised by
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Benjamin Schlein, Zürich
Vedran Sohinger, Coventry

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ABSTRACT. In this mini-workshop we brought together leading experts working on the application of Gibbs measures to the study of nonlinear PDEs. This framework is a powerful tool in the probabilistic study of solutions to nonlinear dispersive PDEs, in many ways alternative or complementary to deterministic methods. Among the special topics discussed were the construction of the measures, applications to dynamics, as well as the microscopic derivation of Gibbs measures from many-body quantum mechanics.

Introduction by the Organisers

The mini-workshop *Gibbs Measures for Nonlinear Dispersive Equations*, organised by Giuseppe Genovese (Zürich), Benjamin Schlein (Zürich) and Vedran Sohinger (Coventry) brought together 16 participants with broad geographic representation.

The topic of the meeting, i.e. the notion of Gibbs measure, occurs naturally in the context of nonlinear dispersive PDEs, probability theory, as well as in many-body quantum mechanics. Therefore one of the main scientific aims was to strengthen the bridge between mathematicians of these different communities.

The workshop unfolded around three main themes involving Gibbs measures, namely nonlinear dispersive equations, stochastic differential equations and many-body quantum mechanics. A majority of the contributions were devoted to the first theme. They included analytical and deterministic aspects, as discussed in the talks of N. Visciglia and F. Cacciafesta, along with the construction of invariant or quasi-invariant measures for different models, as in the talks by A. B. Cruzeiro,

R. Lucá, L. Thomann, A. Nahmod, J. Lukkarinen and G. Genovese. As for the theme of stochastic differential equations, we had the talk of H. Weber in the parabolic setting and the talk of L. Tolomeo in the dispersive one. Finally the theme of many-body quantum mechanics was concentrated on recent advances on the derivation of the Gibbs measure starting from a quantum mechanical ensemble in a semiclassical limit. P. T. Nam and N. Rougerie presented a derivation for the Bose gas in dimensions 1 and 2 and A. Knowles discussed a different approach in dimensions 1,2 and 3. V. Sohinger finally explained the derivation of time-dependent correlation functions in the 1-dimensional case.

The format of the workshop consisted of several lectures per day (from two to four). Much time was reserved to discussions, deepening the arguments of the lectures and developing scientific interaction.

Mini-Workshop: Gibbs Measures for Nonlinear Dispersive Equations

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Abstracts

Conservation laws and almost conservation laws for dispersive PDEs with applications

Nicola Visciglia

Several dispersive equations are completely integrable, we mention in particular cubic NLS, dNLS (derivative NLS), KdV, BO (Benjamin-Ono equation). As a consequence there exist infinitely many energies (**conservation laws**) preserved along the corresponding flows and having the following structure:

$$E_k(u) = \frac{1}{2}\|u\|_{\dot{H}^k}^2 + R_k(u),$$

for k integer (in the case of BO k can be fractional as well) and $R_k(u)$ involving lower order terms. Two important consequences of this fact are in order:

(1) **boundedness of Sobolev norms**, namely $\sup_t \|u(t,x)\|_{H^k} < \infty$ where $u(t,x)$ is any solution of the dispersive model (with a further L^2-smallness assumption in the case of dNLS);

(2) existence of **invariant measures** for the corresponding flow, of the type $f(u)d\mu_k$ where $d\mu_k = $ ” $\exp^{-\|u\|_{H^k}^2} du$” and $f(u)$ is a suitable density.

Concerning the issue of invariant measure there is a huge literature, we only quote a few of them: [8] for cubic NLS and KdV, [3] and [4] for dNLS, [6] and [7] for BO.

The connection between **exact conservation laws and invariant measures** comes from the fact that one (roughly speaking) can exponential the conservation law and hence the measure

$$\exp(-E_k(u))du = \underbrace{\exp(-R_k(u))}_{f(u)} d\mu_k$$

is expected to be **invariant** along the flow. At least formally, this fact follows by the Hamiltonian structure of the PDEs and Liouville theorem.

It is worth mentioning that, for dispersive equations which are no more completely integrable, the situation is more involved and for instance it is not clear whether or not the high order Sobolev norms can grow when time goes over, and eventually to understand how fast they can grow (see [1] and [2]). Let's focus for simplicity on the case of NLS

$$\begin{cases} i\partial_t u + \Delta_g u = u|u|^{2m}, & (t,x) \in \mathbb{R} \times M^d \\ u(0,x) = \varphi(x) \in H^s \end{cases}$$

where (M^d, g) is a compact d-dimensional Riemannian manifold, Δ_g is the Laplace Beltrami operator, s the regularity of the initial data. Following pioneering paper by Bourgain (and further subsequent generalizations) one can show that for $d = 1, 2$ the growth of high Sobolev norms is at most polynomial. The approach by Bourgain relies heavily on the celebrated spaces $X^{s,b}$.

An alternative approach has been developed in the paper [5] where, beside results on the polynomial growth in dimensione $d = 1, 2$, it has been established exponential growth for solutions to cubic NLS posed on a generic compact manifold of dimension $d = 3$. The key point in [5] is the introduction of suitable **almost invariant conservation laws** $\widetilde{E}_k(u)$, namely energies whose time derivative along solutions involve lower order terms that can be estimated by quite elementary arguments, hence getting **polynomial growth of the high Sobolev norms along the evolution**.

Another important question for dispersive models which are not completely integrable, concerns the **quasi-invariance** of the measure $d\mu_k$ along the flow, namely to understand whether or not $\Phi_t^*(d\mu_k)$ is absolutely continuous w.r.t to $d\mu_k$, where Φ_t is the corresponding flow. We believe that the analysis of the measures $\exp(-\widetilde{E}_k(u))du$, where $\widetilde{E}_k(u)$ are the aforementioned **almost invariant conservation laws**, can shed some light on this problem at least for NLS in dimension $d = 1$.

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The Dirac equation and invariant measures

Federico Cacciafesta

The Dirac equation is among the fundamental equations in relativistic quantum mechanics, and it is widely used in physics to describe relativistic particles of spin $1/2$. We recall that the 3D Dirac equation can be written as

$$(1) \qquad i\partial_t u + \mathcal{D}u + m\beta u = 0, \quad u(t,x) : \mathbb{R}_t \times \mathbb{R}_x^3 \to \mathbb{C}^4$$

where $\mathcal{D} = -i \sum_{k=1}^{3} \alpha_k \partial_k = -i(\alpha \cdot \nabla)$ and α_j and β are the standard Dirac matrices. From a dynamical point of view, the Dirac equation can be related to the Klein-Gordon or the wave ones (depending whether the mass $m \geq 0$), as indeed the operator $\mathcal{D}$ is constructed such that $\mathcal{D}^2 = -I_4 \Delta$. Therefore, the flow $e^{it(\mathcal{D}+\beta m)} f$ satisfies the same apriori estimates as the corresponding one for the Klein-Gordon flow (Strichartz, time-decay, local smoothing...). The situation becomes definitely more involved when perturbations come into play, either of potential type (electric or magnetic), nonlinear interaction or even geometric ones. Indeed, in this kind of problems and especially in a low regularity setting or close-to-the scaling critical regime, the algebraic structure of the operator typically needs to be exploited in order to obtain precise results.

In this talk we will describe how to prove dispersive estimates for the Dirac equation on some curved spacetimes. In particular, we will discuss how to obtain:

- *Local smoothing estimates* for asymptotically flat or some warped-products manifolds by a direct application of the classical multiplier method;
- *Strichartz estimates* in the spherically symmetric case in various settings (which include the hyperbolic space), by relying on the so called "partial wave decomposition", which is roughly speaking a radial decomposition of the Dirac operator in 2-dimensional spaces. These results, in particular, allow applications to the well-posedness for some nonlinear models.

Both these results are ongoing joint works with Anne-Sophie de Suzzoni.

Despite its major impact in physical applications and quantum chemestry, the understanding of several aspects related to the Dirac equation are still far to be satisfactory, the main reason for that being the rich and complicated algebraic structure of the operator that make mosts of the tools that give strong results for the Schrödinger equation (the non-relativistic counterpart) typically difficult to be applied. Among the many open problems related to dynamical aspects of the Dirac equations we mention the following:

- *Dispersive estimates for the Dirac-Coulomb equation.* The Coulomb operator is critical with respect to the natural scaling of the (massless) Dirac operator and, as it is known, the study of dispersive dynamics of flows perturbed with scaling-critical potentials is typically difficult as in fact all the perturbative strategies are ruled out. Differently from the Schrödinger and wave equations perturbed with inverse square potentials, for which Strichartz estimates are well known, only a weak local smoothing effect has been proved in the present contest (see [2]).
- *The Dirac equation on curved spacetime.* Much still needs to be done on the topic: e.g. nothing is known on the dynamics of the Dirac equation on compact manifolds.
- *Randomized initial data/ existence of an invariant measure.* At the moment, it has not been possible to exploit any probabilistic technique to improve the deterministic results for the nonlinear Dirac equation. The

main obstruction in this direction is the indefinite sign of the energy, which makes it impossibile to obtain uniform bounds in the standard construction of the Gibbs measure. Nevertheless, one could (maybe) start from the 1D cubic equation (Thirring model) and rely on the invariance of the L^2 norm. Also, one could hope to obtain the existence of a "weak" invariant measure, i.e. of a random variable distributional valued that is a weak solution of the nonlinear equation and such that its law is a measure ρ independent on time, to this case by adapting the strategy developed in [1] which relies, essentially, on the Prokhorov-Skorokhod's method combined with Feynman-Kac theory for oscillatory processes.

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The interacting 2D Bose gas and nonlinear Gibbs measures

Nicolas Rougerie, Phan Thành Nam
(joint work with Mathieu Lewin)

During the MFO workshop "Gibbs measures for nonlinear dispersive equations", we have announced a new theorem bearing on high-temperature 2D Bose gases. The purpose of this note is to state the result in a concise manner. Background, details, generalizations, discussion, references and proofs will appear elsewhere shortly.

Hilbert space and state space. We consider the grand-canonical picture of the homogeneous 2D Bose gas. We assume periodic boundary conditions and thus particles live in the 2D unit flat torus $\mathbb{T}$. The particle number is not fixed: we work in the bosonic Fock space

$$(1) \qquad \mathfrak{F} = \mathbb{C} \oplus L^2(\mathbb{T}^2) \oplus \ldots \oplus L^2_{\mathrm{sym}}\left(\left(\mathbb{T}^2\right)^n\right) \oplus \ldots$$

where $L^2_{\mathrm{sym}}\left(\left(\mathbb{T}^2\right)^n\right)$ is the usual n-particle bosonic space of symmetric square-integrable wave-functions (identified with the n-fold symmetric tensor product of $L^2(\mathbb{T}^2)$ with itself).

We denote

$$(2) \qquad \mathcal{S}\left(\mathfrak{F}\right) := \{\Gamma \text{ self-adjoint operator on } \mathfrak{F}, \Gamma \geq 0, \mathrm{Tr}_{\mathfrak{F}}[\Gamma] = 1\}$$

the set of all (mixed) quantum states on the bosonic Fock space $\mathfrak{F}$. For any state $\Gamma \in \mathcal{S}(\mathfrak{F})$ of the form

$$\Gamma = \Gamma_0 \oplus \Gamma_1 \oplus \ldots \oplus \Gamma_n \oplus \ldots$$

we define its reduced k-body density matrix, a positive trace-class operator on $L^2_{\text{sym}}\left(\left(\mathbb{T}^2\right)^k\right)$, by the formula

$$\Gamma^{(k)} := \sum_{n \geq k} \binom{n}{k} \text{Tr}_{n+1 \to k}\left[\Gamma_n\right].$$

The partial trace $\text{Tr}_{n+1 \to k}$ is taken on $n - k$ variables, no matter which by symmetry.

Hamiltonian. We are interested in the equilibrium states of

(3)
$$\mathbb{H}_\lambda = \mathbb{H}_0 + \lambda \mathbb{W}, \quad \mathbb{H}_0 = \bigoplus_{n \geq 1} \sum_{j=1}^{n} -\Delta_j$$

with $\lambda > 0$ a coupling constant and

$$\mathbb{W} = \bigoplus_{n \geq 2} \sum_{1 \leq i < j \leq n} w(x_i - x_j), \quad \widehat{w} \geq 0$$

where $w : \mathbb{T}^2 \mapsto \mathbb{R}$ is even and $\widehat{w}$ is its Fourier transform (sequence of its Fourier coefficients). Equivalently,

$$\mathbb{H}_\lambda = \sum_{\mathbf{k}} |\mathbf{k}|^2 a_{\mathbf{k}}^\dagger a_{\mathbf{k}} + \frac{\lambda}{2} \sum_{\mathbf{k}, \mathbf{p}, \mathbf{q}} \widehat{w}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^\dagger a_{\mathbf{q}-\mathbf{k}}^\dagger a_{\mathbf{p}} a_{\mathbf{q}}$$

with annihilation $a_{\mathbf{k}}$ and creation $a_{\mathbf{k}}^\dagger$ operators associated to the Fourier modes $e^{i\mathbf{k}\cdot x}$, annihilating/creating a particle with momentum $\mathbf{k} \in (2\pi\mathbb{Z})^2$.

Quantum Gibbs state. We investigate the minimizer, amongst states $\Gamma \in \mathcal{S}(\mathfrak{F})$, of the free-energy functional at temperature T and chemical potential ν, setting an energy reference E_0:

(4)
$$\mathcal{F}_{\lambda,T}[\Gamma] := \text{Tr}_{\mathfrak{F}}\left[\left(\mathbb{H}_\lambda - \nu \mathcal{N}\right)\Gamma\right] + T \, \text{Tr}_{\mathfrak{F}}\left[\Gamma \log \Gamma\right] + E_0.$$

Here $\mathcal{N} = \bigoplus_{n \geq 0} n = \sum_{\mathbf{k}} a_{\mathbf{k}}^\dagger a_{\mathbf{k}}$ is the particle number operator. The minimum free-energy is achieved by the Gibbs state

(5)
$$\Gamma_{\lambda,T} := \frac{1}{\mathcal{Z}_{\lambda,T}} \exp\left(-\frac{1}{T}\left(\mathbb{H}_\lambda - \nu \mathcal{N}\right)\right)$$

where the partition function $\mathcal{Z}_{\lambda,T}$ fixes the trace equal to 1. The minimum free-energy is then

$$F_{\lambda,T} = -T \log \mathcal{Z}_{\lambda,T} + E_0.$$

Nonlinear Gibbs measure. Let $\kappa > 0$ and μ_0 be the gaussian measure with covariance $(-\Delta + \kappa)^{-1}$. This is a probability measure supported on the negative Sobolev spaces $\bigcap_{s<0} H^s(\mathbb{T}^2)$. Let P_K be the orthogonal projector on the span of

the Fourier modes with $|\mathbf{k}| \leq K$. Consider then an interaction energy with local mass renormalization

(6)
$$
\begin{aligned}
\mathcal{E}_K^{\text{int}}[u] = \frac{1}{2} \iint_{\mathbb{T}^2 \times \mathbb{T}^2} &\left(|P_K u(x)|^2 - \left\langle |P_K u(x)|^2 \right\rangle_{\mu_0} \right) \\
&\times w(x-y) \left(|P_K u(y)|^2 - \left\langle |P_K u(y)|^2 \right\rangle_{\mu_0} \right) dx dy.
\end{aligned}
$$

Here $\langle \, . \, \rangle_{\mu_0}$ denotes expectation in the measure μ_0. One can show that the sequence $\mathcal{E}_K^{\text{int}}[u]$ converges to a limit $\mathcal{E}^{\text{int}}[u]$ in $L^1(d\mu_0)$ and that

(7)
$$
d\mu(u) := \frac{1}{z} \exp\left(-\mathcal{E}^{\text{int}}[u] \right) d\mu_0(u),
$$

with $0 < z < \infty$ a normalization constant, makes sense as a probability measure.

Result: the high-temperature/mean-field limit. Let $\kappa > 0$ and denote

$$
N_0(T) := \sum_{\mathbf{k} \in (2\pi\mathbb{Z})^2} \frac{1}{e^{\frac{|k|^2 + \kappa}{T}} - 1}
$$

the expected particle number of the non-interacting quantum Gibbs state ($\lambda = 0$) at temperature T and chemical potential $-\kappa$. This number is easily seen to be of order $T \log T$ for large T and fixed κ. Assume that

$$
\widehat{w}(\mathbf{k}) \geq 0 \text{ for all } \mathbf{k} \in (2\pi\mathbb{Z})^2 \text{ and } \sum_{\mathbf{k}} \left(1 + |\mathbf{k}|^2 \right)^{1/2} \widehat{w}(\mathbf{k}) < \infty.
$$

Then, we have the following

Theorem (High-temperature/mean-field limit of the 2D Bose gas).
Set, for some $\kappa > 0$,

(8)
$$
\nu = \widehat{w}(0) \lambda N_0(T) - \kappa \quad and \quad E_0 = \frac{1}{2} \lambda \widehat{w}(0) N_0(T)^2.
$$

Then, in the limit $T \to \infty, \lambda T \to 1$ we have

(9)
$$
\frac{F_{\lambda,T} - F_{0,T}}{T} \to -\log z.
$$

Moreover, for every $k \geq 1$ and $p > 1$

(10)
$$
\operatorname{Tr} \left| \frac{k!}{T^k} \Gamma_{\lambda,T}^{(k)} - \int |u^{\otimes k}\rangle\langle u^{\otimes k}| d\mu(u) \right|^p \to 0.
$$

Finally

(11)
$$
\operatorname{Tr} \left| \frac{1}{T} \left(\Gamma_{\lambda,T}^{(1)} - \Gamma_{0,T}^{(1)} \right) - \int |u\rangle\langle u| \left(d\mu(u) - d\mu_0(u) \right) \right| \to 0.
$$

Comments. A detailed discussion is postponed to a future paper, that will also contain the proof of the theorem. The following remarks are thus intentionally kept to a bare minimum.

1. The 1D analogue of this theorem was proved first in [14], see also [15] and [9, 10]. No renormalization is necessary to define the limit object in this case. The 2D and

3D cases are investigated in [9] where the analogue of the above result is proved for some modified Gibbs state instead of the minimizer of the free-energy functional.

2. The construction of the nonlinear Gibbs measure μ requires renormalization because the natural interaction

$$\frac{1}{2} \iint_{\mathbb{T}^2 \times \mathbb{T}^2} |u(x)|^2 w(x - y)|u(y)|^2 dxdy$$

does not make sense on the support of the gaussian measure. The renormalized version (6) is relatively simple to control because $\widehat{w} \geq 0$. Positivity of the interaction is then preserved: $\mathcal{E}_K^{\text{int}}[u] \geq 0$ for all u. In more involved cases one can rely on tools from constructive quantum field theory, see [7, 11, 22, 24] for reviews.

3. Gibbs measures related to μ are known [5, 6, 17] to be invariant under suitably renormalized nonlinear Schrödinger flows. They also appear as long-time asymptotes for stochastic nonlinear heat equations, see [16, 18, 23] and references therein for recent results.

4. The above theorem is part of the more general enterprise of gaining mathematical understanding on positive-temperature equilibria of the interacting Bose gas. The ground state and mean-field dynamics of this system are now well-understood, but rigorous works showing the effect of temperature seem rather rare [4, 8, 19, 20, 21, 25].

5. In the physics literature, classical field theories [26] of the type we rigorously derive are used as effective descriptions at criticality, i.e. aroung the BEC phase transition, to obtain the leading order corrections due to interaction effects [1, 2, 3, 12, 13]. Results of these papers are not easy to relate to our theorem, in particular because we work in 2D where there is no phase transition in the strict sense of the word. However (11) is reminiscent of methods for calculating the critical density/critical temperature of the Bose gas in presence of interactions.

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Solutions and invariant measures for Euler equations on the plane
Ana Bela Cruzeiro

We present the author's recent results in collaboration with Alexandra Symeonides ([3],[4]) about the construction of invariant (and quasi-invariant) probability measures for Euler equations and their importance for the definition of solutions.

We show how invariant Gibbs-type Gaussian measures are helpful to prove existence of solutions for Euler equations starting (almost everywhere) from their support. The corresponding functional spaces, endowed with probability measures that we know a priori to be invariant or quasi-invariant for the equations (more precisely, that we know to be infinitesimally invariant) are abstract Wiener spaces (similar to path the space of the Brownian motion). One can consider on an abstract Wiener space the Malliavin calculus ([5]) a kind of differential calculus adapted to deal with non regular functionals. Once formulated in functional spaces, some partial differential equations such as the ones we consider here become ordinary differential equations. Flows for ordinary differential equations in context of Wiener spaces were first studied in [2]. These methods can be applied to construct flows for Euler equations.

The Euler equation on the plane. Let us consider the incompressible Euler equation

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \qquad \text{div } u = 0$$

on the 2-d torus $\mathbb{T} \simeq [0, L] \times [0, L]$ (space periodic boundary conditions). Writing $u = \nabla^{\perp}\varphi = (-\partial_2\varphi, \partial_1\varphi)$, the corresponding vorticity equation is the following

$$\frac{\partial \Delta\varphi}{\partial t} = -(\nabla^{\perp}\varphi.\nabla)\Delta\varphi.$$

Among the conserved quantities of the Euler equation we have the energy and the enstrophy, namely

$$E = -\frac{1}{2}\int_{\mathbb{T}} \varphi\Delta\varphi dx, \quad S = \frac{1}{2}\int_{\mathbb{T}} |\Delta\varphi|^2 dx$$

In [3] we considered the invariant Gibbs measure, formally given by

$$d\mu_L(\varphi) = \frac{1}{Z} e^{-S(\varphi)} \mathcal{D}\varphi$$

that we have used to construct irregular solutions. More precisely, set $e_k^L(x) = \frac{1}{L}e^{\frac{2\pi i}{L}k.x}$, $k \in \mathbb{Z}^2$, a basis of $L^2(\mathbb{T})$, decompose $\varphi = \sum_{k>0} \varphi_k^L(t)e_k^L(x)$ and define the random variables

$$\Phi_L(\omega, x) = \sum_k a_k^L(\omega)e_k^L(x),$$

where

$$a_k^L(\omega) = \sqrt{2} \, (\frac{L}{2\pi k})^2 \, [W_{k^2+1}(\omega) - W_{k^2}(\omega)],$$

and W_k are i.i.d. complex-valued Brownian motions. Then μ_L is its law, the Gaussian measure with mean zero and covariance

$$< a, C^{-1}a > = \frac{1}{2}\sum_k (\frac{2\pi k}{L})^4 \, |a_k^L|^2$$

The support of μ_L is the Sobolev space $H^\beta(\mathbb{T})$, $\beta < 1$.

One can prove that Φ_L is a Cauchy sequence in $L^2(\Omega; H_{loc}^\beta(\mathbb{R}^2))$ for $\beta < 1$; if Φ denotes its limit and μ its law, then μ_L converges weakly to μ in H_{loc}^β.

The vorticity equation (periodic case), written in the functional spaces, is, explicitly,

$$\frac{d}{dt}\varphi^L = B(\varphi^L)$$

where

$$B_L(\varphi^L) = \sum_{k>0} B_k^L(\varphi^L)e_k^L(x)$$

and

$$B_k^L(\varphi^L) = \frac{1}{L}\left(\frac{2\pi}{L}\right)^2 \sum_{\substack{h>0 \\ h\neq k}}\left[\frac{(h^\perp \cdot k)(k \cdot h)}{k^2} - \frac{h^\perp \cdot k}{2}\right]\varphi_h^L\varphi_{k-h}^L,$$

$h^\perp = (-h_2, h_1)$.

Using the L_μ^p regularity of the vector field B as well as the invariance of μ for B (the fact that the divergence, i.e. the dual of the gradient applied to B is zero), we can prove the following:

Theorem. Let $\beta < -1$. There exists a unique flow $U(t, \varphi)$ defined for $t \in \mathbb{R}$, $\varphi \in H_{loc}^\beta$ $\mu - a.e.$ such that

$$U(t, \varphi) = \varphi + \int_0^t B(U(s, \varphi))ds \qquad \mu - a.e.$$

The flow is continuous in H_{loc}^β and the measure μ is invariant for the flow, namely

$$\int f(U(t, \varphi))d\mu(\varphi) = \int f d\mu \qquad \forall t \quad \forall f.$$

A modified Euler equation. Let $\sigma(x) = \frac{\sqrt{\varepsilon}}{2\pi}e^{-\frac{\varepsilon |x|^2}{2}}$ the Gaussian probability density in $\mathbb{R}^2$, $\rho = \sigma^{-1}$. We consider the equation

$$\frac{\partial \widetilde{u}}{\partial t} + (\widetilde{u} \cdot \nabla)\widetilde{u} = -\nabla p + \varepsilon x p, \qquad \text{div } \widetilde{u} = 0$$

Notice that $\widetilde{u}, p$ depend on ε.

The corresponding vorticity equation for this modified Euler equation reads

$$\frac{\partial}{\partial t}L\varphi = -(\nabla^\perp\varphi.\nabla)L\varphi \qquad (E)$$

where $L\varphi = \Delta\varphi - \varepsilon x.\nabla\varphi$ is the Ornstein-Ulhenbeck operator in $L^2_{\sigma dx}$.

Concerning classical solutions we have:

Theorem. Given ω_0 with $\rho\omega_0 \in L^1 \cap L^\infty$, there exists $T > 0$ such that equation (E) has a weak solution $\omega(t)$ defined for $t < T$, with $\rho\omega = L\varphi \in L^\infty([0,T]; L^1 \cap L^\infty)$.

Now consider the probability measure formally given by

$$d\mu_\sigma(\varphi) = \frac{1}{Z} e^{-\frac{1}{2}(\|L\varphi\|^2_{L^2_\sigma} + \|\varphi\|^2_{L^2_\sigma})} \mathcal{D}\varphi$$

More rigorously, it is the law of the random variable

$$\Gamma(\omega, x) := \sum_k \frac{W_k(\omega)}{1 + |k|} H_k(x),$$

where H_k are Hermite polynomials on $\mathbb{R}^2$, forming an o.n. basis of $L^2_{\sigma dx}$.

$$d\mu_\sigma(\varphi) = \prod_k \frac{(1 + |k|)^2}{2\pi} e^{-\frac{1}{2}(1+|k|)^2 |\varphi_k|^2} d\varphi_k$$

The vorticity equation for the modified Euler equation, with $\varphi(t, x) = \sum_{k \geq 0} \varphi_k(t) H_k(x)$, reads

$$\frac{d}{dt}\varphi(t) = \widetilde{B}(\varphi(t))$$

where

$$\widetilde{B}(\varphi) = -\sum_{|p|\geq 0} \sum_{0<|k|<2|p|} \sum_{0<|q|<|p|} \frac{1}{|k|}(|p| - |q|)A(p, q, k)\varphi_p\varphi_q H_k,$$

$A(p, q, k) = [-\sqrt{p_2 q_1}\Theta(p_1, q_1 - 1, (p_1 + q_1 - 1 - k_1)/2)\Theta(p_2 - 1, q_2, (p_2 + q_2 - 1 - k_2)/2) + \sqrt{p_1 q_2}\Theta(p_1 - 1, q_1, (p_1 + q_1 - 1 - k_1)/2)\Theta(p_2, q_2 - 1, (p_2 + q_2 - 1 - k_2)/2)],$

$$\Theta(n, m, r) = \left[\binom{n}{r}\binom{m}{r}\binom{n + m - 2r}{n - r}\right]^{1/2}.$$

We have the following regularity results:

For all $r \geq 1$, $\beta > 0$, $\gamma > 0$,

$$\widetilde{B} \in L^r_{\mu_\sigma}(H^{-\gamma}_\sigma(\mathbb{R}^2); H^\beta_\sigma(\mathbb{R}^2))$$

$$\nabla\widetilde{B} \in L^r_{\mu_\sigma}(H^{-\gamma}_\sigma(\mathbb{R}^2); HS(H^2(\mathbb{R}^2); H^\beta_\sigma(\mathbb{R}^2)))$$

$$\nabla^2\widetilde{B} \in L^r_{\mu_\sigma}(H^{-\gamma}_\sigma(\mathbb{R}^2); HS(H^2(\mathbb{R}^2) \otimes H^2(\mathbb{R}^2; H^\beta_\sigma(\mathbb{R}^2)))$$

$$\exists \lambda > 0 : \int \left(\exp\lambda \|\nabla\widetilde{B}\|_{HS(H^2_\sigma; H^\beta_\sigma)} + \exp\lambda |\mathrm{div}_{\mu_\sigma}\widetilde{B}| d\mu_\sigma\right) d\mu_\sigma < +\infty$$

Here ∇ and ∇^2 are taken in the Malliavin calculus sense.

Using Malliavin Calculus on abstract Wiener spaces these assumptions are sufficient to prove the existence of a flow for the modified vorticity equation starting a.e. on the support of μ_σ, which is $H_{\sigma dx}^{-\gamma}(\mathbb{R}^2) \cap L_{loc}^p(\mathbb{R}^2)$.

We see that one can define solutions starting from functions and not only distributions as in the standard Euler case. In this modified case the Gaussian measure is not invariant for the flow but only quasi-invariant (the divergence is not equal to zero). Denoting the density of the law of the flow $\varphi(t)$ with respect to μ_σ by k_t, we have $k_t \in L_{\mu_\sigma}^r \ \forall r \geq 1$.

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Invariant measures for the periodic derivative nonlinear Schrödinger equation

Renato Lucà

(joint work with G. Genovese, D. Valeri)

The periodic DNLS equation

$$
(1) \qquad \begin{cases} i\partial_t \psi + \psi'' & = & i\beta \left(\psi|\psi|^2\right)' \\ \psi(x,0) & = & \psi_0(x), \quad x \in \mathbb{T}, \end{cases}
$$

where $\psi(x,t) : \mathbb{T} \times \mathbb{R} \to \mathbb{C}$, $\psi_0(x) : \mathbb{T} \to \mathbb{C}$, $\psi'(x,t)$ denotes the derivative of ψ with respect to x, and $\beta \in \mathbb{R}$ is a real parameter is a dispersive nonlinear model describing the motion along the longitudinal direction of a circularly polarized wave, generated in a low density plasma by an external magnetic field. This is an integrable system, in the sense that there is an infinite sequence of linearly independent quantities (integrals of motion) which are conserved by the flow.

In our previous work [GLV16] we constructed a family of weighted Gaussian measures, supported on Sobolev spaces of increasing regularity, associated to the integrals of motion of the DNLS equation. The measure associated to the energy (namely at H^1 level) had already been constructed in [TT10] . Here we construct a sequence of weighted Gaussian measures invariant along the flow of a gauged version of the equation (GDNLS). The pull-back of these measures are invariant under the flow of DNLS and are presumably the measures constructed in [GLV16].

The DNLS equation has interesting transformation properties with respect to a group of gauge maps which will be now discussed. For $\alpha \in \mathbb{R}$ let $\mathscr{G}_\alpha : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ be defined by

$$(2) \qquad \mathscr{G}_\alpha(f)(x) := e^{i\alpha\mathcal{I}(f(x))} f(x) \,.$$

where

$$(3) \qquad \mathcal{I}(f(x)) := \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_\theta^x \left(|f(y)|^2 - \frac{\|f\|^2_{L^2(\mathbb{T})}}{2\pi} \right) dy \,.$$

One can easily check that the (real) function $\mathcal{I}(f(x))$ is the unique zero average $(2\pi$-periodic) primitive of $|f(x)|^2 - (2\pi)^{-1}\|f\|^2_{L^2}$.

Let ψ be a solution of the DNLS equation (1). For any $\alpha \in \mathbb{R}$, we set for brevity

$$(4) \qquad \varphi := \mathscr{G}_\alpha(\psi) \,.$$

Clearly we have $\psi = \mathscr{G}_{-\alpha}(\varphi)$. Even if φ depends on the choice of the parameter α, see (4), we omit these dependence to simplify the notations. If ψ is a solution of the DNLS equation then, for any $\alpha \in \mathbb{R}$, the function φ solves the GDNLS

$$(5) \quad i\partial_t\varphi + \varphi'' + 2i\alpha\mu\varphi' = ic_1|\varphi|^2\varphi' + ic_2\varphi^2\bar{\varphi}' + c_3|\varphi|^4\varphi + c_4\mu|\varphi|^2\varphi + \Gamma[\varphi]\varphi \,,$$

where

$$(6) \quad c_1 = 2(\alpha + \beta)\,, \qquad c_2 = 2\alpha + \beta\,, \qquad c_3 = -\alpha^2 - \frac{\alpha\beta}{2}\,, \qquad c_4 = -\alpha\beta$$

and

$$(7) \qquad \Gamma[f] = \left(\frac{3\alpha\beta}{4\pi} + \frac{\alpha^2}{\pi} \right) \|f\|^4_{L^4} - \alpha^2\mu[f]^2 + \frac{i\alpha}{\pi} \int_\mathbb{T} f'\bar{f} \,.$$

The GDNLS equation admits a countable family of integral of motions. The simplest are:

$$\mathcal{E}_0[\varphi] = \frac{1}{2}\|\varphi\|^2_{L^2}\,,$$

$$(8) \qquad \mathcal{E}_{\frac{1}{2}}[\varphi] = \frac{i}{2} \int \varphi'\bar{\varphi} + \frac{1}{4}(2\alpha + \beta)\|\varphi\|^4_{L^4} - \pi\alpha\mu^2\,,$$

$$\mathcal{E}_1[\varphi] = \frac{1}{2}\|\varphi\|^2_{\dot{H}^1} + i\alpha\mu \int \varphi\bar{\varphi}' + \frac{i}{4}(4\alpha + 3\beta) \int |\varphi|^2\varphi'\bar{\varphi}$$

$$+ \pi\alpha^2\mu^3 - \frac{\alpha}{4}(4\alpha + 3\beta)\mu\|\varphi\|^4_{L^4} + \frac{1}{4}(\alpha + \beta)(2\alpha + \beta)\|\varphi\|^6_{L^6}\,,$$

where $\mu = \mu[\varphi] := \frac{1}{2\pi}\|f\|^2_{L^2}$. When $k \geq 2$ the general form of the integrals of motion is

$$\mathcal{E}_k[\varphi]$$

$$= \frac{1}{2}\|\varphi\|^2_{\dot{H}^k} + ik\alpha\mu \int \bar{\varphi}^{(k)}\varphi^{(k-1)} - \frac{1}{2}\left((2k+2)\alpha + (2k+1)\beta\right) \Im \int \varphi^{(k)}\bar{\varphi}^{(k-1)}|\varphi|^2,$$

plus remainders, which are lower differential degree polynomials in φ, $\bar{\varphi}$ and their derivatives. Notice that, setting

$$\alpha = -\frac{2k+1}{2k+2}\beta \,,$$

we reduce to

$$\mathcal{E}_k[\varphi] = \frac{1}{2}\|\varphi\|_{\dot{H}^k}^2 - ik\frac{2k+1}{2k+2}\beta\mu\int \bar{\varphi}^{(k)}\varphi^{(k-1)} + \text{remainders} \,.$$

We ca define a measure on $L^2(\mathbb{T})$ as follows

$$(9)\quad \widetilde{Q}_k[f] := \mathcal{E}_k[f] - \frac{1}{2}\|f\|_{\dot{H}^k}^2, \quad \widetilde{\rho}_k(A) = \int_A \prod_{m=0}^{k-1} \chi_{R_m}\left(\mathcal{E}_m(f)\right)\exp(-\widetilde{Q}_k[f])\gamma_k(df),$$

for any $A \in \mathscr{B}(L^2(\mathbb{T}))$, where $R_0, \ldots, R_{k-1}$ are positive real parameter with $R_0 \ll 1$ and χ_{R_m} are positive cut off functions of the intervals $[-R_m, R_m]$. Here γ_k denotes the standard (infinite-dimensional) Gaussian measure with covariance associated to the H^k scalar product on $\mathbb{T}$. The condition $R_0 \ll 1$ ensures the integrability of the density in (9) w.r.t. γ_k. $\widetilde{Q}_k$ and so $\widetilde{\rho}_k$ depends on α.

The goal is to prove that $\widetilde{\rho}_k$ is invariant under Φ_{t,α_k}, that is the flow associated to the GDNLS equation (5) for our choice of α. Then we obtain an invariant measure for DNLS letting $\widehat{\rho}_k(A) := (\widetilde{\rho}_k \circ \mathscr{G}_{\alpha_k})(A)$. Namely

Theorem 1 ([GLV18])**.** *Let $k \geq 2$ and let R_0 be sufficiently small. Then there exists a probability measure $\widehat{\rho}_k$ on $(L^2(\mathbb{T}), \mathscr{B}(L^2(\mathbb{T})))$ such that the flow-map Φ_t associated to DNLS is measure preserving in $(L^2(\mathbb{T}), \mathscr{B}(L^2(\mathbb{T})), \widehat{\rho}_k)$.*

The low regularity case $k = 1$ had been already treated in [NOR-BS12] and [NR-BSS11]. In the first paper the authors show the invariance of the measures under the flow for the choice $\alpha = -\beta$. In the second paper they prove that the pull-back measure is indeed the measure constructed in [TT10] for the DNLS equation.

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Gibbs measures and parabolic SPDEs

HENDRIK WEBER

Stochastic quantisation, first proposed by Parisi and Wu in the early 80s [1] is a method by which complex measures arising in field theories are sampled as the invariant measures of suitable distribution valued Markov processes. The Euclidean Φ^4 theories constitute an interesting test case for the mathematically rigorous implementation of this idea. They are given by measures on the space of distributions, and are formally described by the expression

$$(1) \qquad \mu(dX) \propto \exp\left(-2\int\left[\frac{1}{2}|\nabla X(x)|^2 - \frac{1}{4}X(x)^4 + \frac{1}{2}\infty X(x)^2\right]dx\right)dX,$$

where the term $\frac{1}{2}\infty X(x)^2$ indicates that a renormalisation procedure has to be performed in the construction of the measure. In this case the associated distribution valued Markov process is (at least formally) given by the stochastic PDE

$$(2) \qquad \partial_t X = \Delta X - (X^3 - \infty X) + \xi,$$

where ξ is space-time white noise.

A rigorous interpretation for (2) was not known until Hairer's ground-breaking work on Regularity Structures [2] where this interpretation was given for the first time and short time well posedness on a finite domain for (2) was shown. Hairer's work triggered a lot of activity, including [3], [4] where alternative arguments for this short time well posedness were given.

In this talk recent work on the large scale behaviour of solutions was reviewed. The main result presented was the a priori bound

$$(3) \qquad \mathbb{E}\left[\sup_{0<t\leq 1}\sup_{X_0\in\mathcal{B}_{\infty,\infty}^{-\frac{3}{5}}}\left(\sqrt{t}\,\|X(t)\|_{\mathcal{B}_{\infty,\infty}^{-\frac{1}{2}-\varepsilon}}\right)^p\right] < \infty.$$

for solutions of (2) over the three dimensional torus obtained in [5]. This bound holds for all choices of $p < \infty$ and $\varepsilon > 0$, and $\|\cdot\|_{\mathcal{B}_{\infty,\infty}^\alpha}$ stands for the norm of the Hölder-Besov space of regularity $\alpha \in \mathbb{R}$. The bound expresses the strong non-linear damping coming from the cubic term X^3 in the sense that solutions at any positive time t can be controlled uniformly over all possible choices of initial datum X_0. This fact can be used, for example, to give a dynamic construction of the invariant measure (1).

In the (simpler) two-dimensional case a similar bound holds. This was one of the key ingredients in [6] to prove a strong equilibration result for solutions of (2). There it was shown that there exists a $\lambda > 0$ such that for all $t > 0$ the Markov transition semigroup P_t for (2) satisfies

$$\sup_x \|P_t(x) - \mu\|_{\mathrm{TV}} \leq (1 - \lambda)^t,$$

where $\|\cdot\|_{\mathrm{TV}}$ denotes the total variation metric.

In the end of the talk, ongoing work with A. Moinat [7] was discussed. As a toy problem we investigate the one-dimensional stochastic reaction diffusion equation

$$(4) \qquad (\partial_t - \partial_x^2)u = -|u|^{m-1}u + \xi, \qquad m > 1$$

and show a space-time version of the "coming down from infinity" property

$$\|u\|_{P_R} \leq C(m.\alpha, \varepsilon) \max\left\{ R^{-\frac{2}{m-1}-\varepsilon}, [\xi]_{\alpha-2,P_0}^{\frac{2}{2+(m-1)\alpha}} \right\}$$

for $\varepsilon > 0$. Here $\|u\|_{P_R}$ denotes the supremum norm of u restricted to the cylinder

$$P_R := \{(t,x) \colon t \in [R^2, 1], \ |x| \leq 1 - R\},$$

and $[\xi]$ denotes a local Besov-Hölder norm of ξ restricted to P_0.

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Almost sure scattering for the one dimensional nonlinear Schrödinger equation

Laurent Thomann

(joint work with N. Burq)

1. Introduction

We present the main results of our article [2]. We study long time dynamics for the one-dimensional nonlinear Schrödinger equation

$$(1) \qquad \begin{cases} i\partial_s U + \partial_y^2 U = |U|^{p-1}U, & (s,y) \in \mathbb{R} \times \mathbb{R}, \quad 1 < p < 5, \\ U(0, \cdot) = U_0, \end{cases}$$

where U_0 is a random initial condition, with low Sobolev regularity. The distribution of U_0 will be given by a Gaussian measure and we will study its evolution under the nonlinear flow of (1), denoted by $\Sigma(s)$ and compare it with the evolution under the linear flow $\Sigma_{lin}(s) = e^{is\partial_y^2}$.

When working on compact manifolds M instead of $\mathbb{R}_x$, there exists natural Gaussian measures μ supported in some Sobolev spaces $H^\sigma(M)$ which are invariant

by the flow $\Sigma_{lin}(s)$ of the linear equation. At some particular scales of regularity these measures can be suitably modified to ensure that they are invariant by the non linear flow.

In our context the situation is different, since dispersion prohibits the existence of measures invariant by the linear flow. First we define mesures on the space of initial data for which we can describe precisely the non trivial evolution by the linear flow. Second, we prove that the non linear evolution of these measures is absolutely continuous with respect to these linear evolutions, and finally we get benefit from this precise description to prove almost sure scattering of our solutions of (1) for $p > 3$.

This work extends previous results from [3], were the case $p \geq 5$ is considered. When $p \geq 5$, monotonicity properties allow to simplify the proof and a complete description of the non linear evolution of the measures is unnecessary.

2. Statement of the results

We denote by $H = -\partial_x^2 + x^2$, the harmonic oscillator in one space dimension, and by $\{e_n, \ n \geq 0\}$ the Hermite functions its L^2-normalised eigenfunctions, $He_n = \lambda_n^2 e_n = (2n+1)e_n$. Recall that the family $\{e_n, \ n \geq 0\}$ forms a Hilbert basis of $L^2(\mathbb{R})$.

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{p})$ and let $\{g_n, \ n \geq 0\}$ be a sequence of independent complex standard Gaussian variables. Let $\sigma > 0$, we define the probability measure μ on $\mathcal{H}^{-\sigma}(\mathbb{R})$ as the image of the probability measure $\mathbf{p}$ on Ω by the map

$$\omega \longmapsto \gamma^\omega = \sum_{n=0}^{+\infty} \frac{1}{\lambda_n} g_n(\omega) e_n, \qquad \mu = \mathbf{p} \circ \gamma^{-1}.$$

For $\sigma > 0$, denote by

$$\mathcal{H}^\sigma(\mathbb{R}) = \big\{ u \in L^2(\mathbb{R}) : |D_x|^\sigma u \in L^2(\mathbb{R}), |x|^\sigma u \in L^2(\mathbb{R}) \big\}.$$

and define $\mathcal{H}^{-\sigma}(\mathbb{R})$ as its dual space.

The measure μ satisfies $\mu\big(\mathcal{H}^\sigma(\mathbb{R})\big) = 0$ and $\mu\big(\mathcal{H}^{-\sigma}(\mathbb{R})\big) = 1$, for all $\sigma > 0$.

To begin with, we have the following global well-posedness result.

Theorem 1. *Let $1 < p < 5$. Then Equation (1) has for μ-almost every initial data a unique global solution satisfying for some $\sigma > 0$,*

$$U(s,\cdot) - e^{is\partial_y^2} U_0 \in C\big(\mathbb{R}; \mathcal{H}^\sigma(\mathbb{R})\big),$$

with uniqueness in a space continuously embedded in $C\big(\mathbb{R}; \mathcal{H}^\sigma(\mathbb{R})\big)$. This defines a flowmap $\Sigma(s)$, and for all $s \in \mathbb{R}$ we have $U(s) = \Sigma(s)U_0$.

We are now able to study the evolution of μ under the nonlinear flow of (1) defined in Theorem 1.

Theorem 2. *Let $1 < p < 5$.*

(1) *For all $s \in \mathbb{R}$, the measures $\Sigma(s)\#\mu$ and $(e^{is\partial_y^2})\#\mu$ are equivalent (they have the same zero measure sets).*
(2) *For all $t \neq s$, the measures $\Sigma(t)\#\mu$ and $\Sigma(s)\#\mu$ are mutually singular.*

A natural question is the long time behaviour of the solution to (1). Actually, using a more quantitative version of Theorem 2, we prove

Theorem 3. *Let $3 < p < 5$. The solution to (1) constructed in Theorem 1 scatters in the following sense: there exists $\sigma > 0$ and there exists μ a.s. states $G_\pm \in \mathcal{H}^\sigma(\mathbb{R})$ so that*

$$\|U(s, \cdot) - e^{is\partial_y^2}(U_0 + G_\pm)\|_{\mathcal{H}^\sigma(\mathbb{R})} \longrightarrow 0, \quad when \quad s \longrightarrow \pm\infty.$$

In the case $p \leq 3$, Barab [1] showed that a non trivial solution to (1) never scatters, thus even with a stochastic approach one can not have scattering in this case. Therefore the condition $p > 3$ in Theorem 3 is optimal. In [4], Tsutsumi and Yajima proved a scattering result in $L^2(\mathbb{R})$, but assuming additional regularity on the initial conditions.

In the argument, we use the lens transform which allows to work with the Schrödinger equation with harmonic potential for finite times, and it is enough to prove finite time results for this latter equation to infer scattering for (1).

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Ergodicity for Nonlinear Stochastic Hyperbolic Equations with damping

LEONARDO TOLOMEO

Consider the equation

(SDNLW) $$u_{tt} + u_t + u + (-\Delta)^{\frac{s}{2}}u + u^3 = \xi,$$

posed on the d-dimensional torus $\mathbb{T}^d$, where ξ is the space-time white noise, and $s > d$. Our goal is to study the long-time behaviour of the flow of this equation.

First of all, we notice the existence of an invariant measure, given formally by

$$\text{(G)} \qquad \exp\Big(-\frac{1}{2}\int(u+\Delta^{\frac{s}{4}}u)^2 - \frac{1}{4}\int u^3\Big)du \, \exp\Big(-\frac{1}{2}\int u_t^2\Big)du_t.$$

This can be seen formally by dividing the infinitesimal flow of the equation into the flow of the *Hamiltonian* equation

$$u_{tt} + u + (-\Delta)^{\frac{s}{2}}u + u^3 = 0,$$

which is expected to preserve the *Gibbs* measure (G), and the Ornstein-Uhlenbeck process

$$u_{tt} + u_t = \xi,$$

which is known to preserve the Gaussian measure

$$\exp\Big(-\frac{1}{2}\int u_t^2\Big)du_t,$$

and so, since the flow of this process does not depend on u, preserves the measure (G) as well.

After proving invariance of the measure with respect to the flow of (SDNLW), we want to study convergence to equilibrium starting from an arbitrary initial data. The standard recipe for showing this convergence relies on three ingredients: Strong Feller property of the flow, good a priori estimates, and irreducibility of the flow. If all of these ingredients are available, one would hopefully be able to show a *spectral gap* property for the flow and exponential convergence to equilibrium.

In the case of (SDNLW), while we do have good a priori estimates and irreducibility, we can actually show that the flow is *not* Strong Feller.

Therefore, we proceed in a different way. We show that the flow of (SDNLW) is Strong Feller in a much stronger topology, which is not good enough to conclude ergodicity of (G). Hence, we mix this property with a completely algebraic argument, which allows us to show uniqueness of the invariant measure (G). Unfortunately, this does not provide convergence to equilibrium of the flow, nor any quantitative estimate for the convergence of the time averages.

However, this argument has the benefit to be completely local (in time) with regards to showing ergodicity (in the dynamical system sense) of the measure (G), so it may prove useful in situations for which good long-time estimates are unavailable and/or finite time blowup is possible.

Non-equilibrium invariant measures for the resonant NLS
ANDREA R. NAHMOD
(joint work with Z. Hani, J. Mattingly, L. Rey-Bellet, G. Staffilani)

In this talk we describe joint work with Z. Hani, J. Mattingly, L. Rey-Bellet and G. Staffilani on the construction of non-equilibrium invariant measures associated to the resonant cubic nonlinear Schrödinger equation (NLS) on $\mathbb{T}^2$.

There are by now several examples of semilinear Schrödinger and wave equations defined $\mathbb{R}^d$ for which it is mathematically proven that dispersion sets in and, after a time long enough, solutions settle into a purely linear behavior. This phenomena is often referred to as *scattering* (asymptotic stability). For linear solutions, energy at any given frequency does not migrate to higher or lower frequencies, that is there is no *forward* or *backward cascade*. As a consequence of scattering then, certain nonlinear solutions in $\mathbb{R}^d$ also will avoid these cascades. We also say that linear solutions are in equilibrium and scattering solutions are asymptotically in equilibrium. The situation is believed to be different on compact domains, where dispersion is weak and does not translate into time decay of the solutions, so much less is known in this regard. For example, on periodic domains $\mathbb{T}^d$, $d \geq 2$, many different long-time behaviors can occur; in particular, equilibrium solutions are not expected to be stable, giving rise to *out of equilibrium* dynamics. How to analytically describe this expected out-of-equilibrium behavior is one of the most intriguing questions in the study of the long time dynamics of dispersive equations in this case, with only few partial answers given so far, as will be mentioned below.

Wave/weak turbulence theory seeks to obtain a statistical description of the out-of-equilibrium dynamics for Hamiltonian nonlinear dispersive equations by for example deriving effective equations that track the long time evolution for the *energy distribution* of the system at hand. Putting this theory on a rigorous mathematical foundation is a challenging and difficult proposition. There are two main aspects pertaining a rigorous mathematical study of wave turbulence. One entails deriving the fundamental equations (wave kinetic equations) governing the dynamics and energy distribution of the dispersive equation under consideration. Some work in this direction has been done in recent years, see for example works by Faou-Germain-Hani, Germain-Hani-Thomann, and Buckmaster-Germain-Hani-Shatah. The other is that of proving some of its dynamical conclusions[1] such as the energy cascade phenomenon[2]. One approach to the latter is by exhibiting dynamics that reflect the conclusions of the theory, and by constructing -and proving convergence of the system to- a non-equilibrium invariant measure with positive *entropy production* (energy transfer), that the theory predicts. Our work pertains this approach.

We focus on the 2D periodic cubic NLS equation. The stationary non-equilibrium states (SNS)- also known as the Kolmogorov-Zakharov spectra of the wave kinetic equation (c.f. Nazarenko's book and reference therein) in the theory of wave turbulence- correspond to non-equilibrium steady states[3] or invariant measures for NLS in which energy (or mass) cascades through frequency scales at a constant flux. Such flux of energy through scales cannot be a final steady state for a Hamiltonian system –without violating the conservation laws– unless there is a

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[1]Independently of a complete justification of the formalism.

[2]That is, the transfer of energy from low to high modes, or vice-versa as the initial data evolves.

[3]The well-known white noise and Gibbs measures are equilibrium steady states (equilibrium spectra).

source pumping energy into the system at low frequencies and a sink dissipating energy from it at high frequencies (for forward cascade).

Even for stochastically forced systems, proving the existence and uniqueness of non-equilibrium invariant measures is very hard in the context of PDEs. However, we have recent developments on non-equilibrium statistical mechanics of open classical systems aimed at understanding analogous questions for some ODE systems modeling *heat transfer* in a finite chain of anharmonic oscillators *with nearest neighbor couplings*. This line of research started with the works of Eckmann, Pillet and Rey-Bellet, Rey-Bellet and Thomas up to more recent progress by Hairer and Mattingly. In these works, energy is fed into and dissipated from the system using so-called *heat baths*, one at a temperature T_1 and another at temperature $T_2 > T_1$. The interaction with heat baths is modeled by standard Langevin dynamics. Naturally, one expects the system to converge to a non-equilibrium invariant measure in which energy moves at a constant flux from the low-temperature heat bath towards the high temperature one. This is the content of the works mentioned above, along with rates on the convergence to those steady states.

We follow those developments in order to shed some light on the non-equilibrium dynamics for resonant NLS. As a first attempt at this, we wish to construct invariant measures that correspond to positive and constant energy fluxes for finite sub-systems of the resonant NLS..

We consider the reduced *Toy Model* first derived by Colliander-Keel-Staffilani-Takaoka-Tao whose Hamiltonian H has interactions of N particles depending not just on their relative distance but also on the momenta of each particle and that of its neighbors. Colliander-Keel-Staffilani-Takaoka-Tao showed that this toy model approximates the cubic NLS on the 2D torus over certain timescales (in the weak nonlinearity regime) and used it to exhibit smooth solutions (to the defocusing cubic NLS on $\mathbb{T}^2$) for which the support of the conserved energy moves to higher Fourier modes. This behavior is quantified by the growth of higher Sobolev norms (as suggested by Bourgain). This implies there exist arbitrarily large, but finite, energy cascades, thus showing 'weakly turbulent dynamics'.

Together with Z. Hani, J. Mattingly, L. Rey-Bellet and G. Staffilani, we construct a suitable finite dimensional Hamiltonian stochastic ODE model - out of the toy model- where two heat baths, respectively at temperatures T_1 and T_N, are attached at the first and last mode[4]. That is, we inject energy into the lowest mode in the form of *dissipation* plus *fluctuation* (white noise) and suitably absorb it from the last mode. We call this model the stochastic resonant system (SRS) and its Hamiltonian H. At equilibrium, that is, when the baths temperatures are equal ($T_1 = T_N = T$), the system has a unique Gibbs (equilibrium) invariant measure of the form $e^{-\frac{1}{T}H}dx$.

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[4]This is a mechanism to stochastically add and dissipate energy from the system in a controlled way.

In this talk we describe how to then construct a unique ergodic invariant non-equilibrium measure associated to the system SRS with positive entropy production when $N = 3$. Our aim also entails obtaining estimates on the rate of convergence of the system to such stationary non-equilibrium state (relaxation rate). This measure correspond to invariant measures for (a finite subsystem of) the resonant NLS with positive energy-fluxes. We expect them to give insight into the non-equilibrium statistics of NLS and be a strong manifestation of the conjectured cascade phenomena for NLS.

The resolution of our problem entails proving 1) the existence of an invariant measure, 2) its uniqueness, 3) its ergodicity and finally 4) the positive entropy production. Our setting however, is considerably more complex than the anharmonic chain considered in previous works by J.P. Eckman, M. Hairer, J. Mattingly, C.A. Pillet, L. Rey-Bellet, L. Thomas, as already seen in step 1), when a suitable continuous and piecewise C^2 Lyapunov function V must be constructed, which penalizes the region where the second mode has small and high energy. Such construction gives an upper bound on the hitting time of the good region (compact set) where the dynamics spends most of time. A natural candidate for V is to use a coercive conserved quantity of the starting Hamiltonian system. But this does not work in the whole space. We need to chop our phase space in several regions and a deep/detailed understanding of the dynamics is needed. In fact we need to solve suitable Poisson equations for V with appropriate boundary conditions and check a 'convexity' condition as in Herzog-Mattingly'a work. We study the behavior of the phases and in some regions prove that asymptotically they get locked.

As far as we know, these measures are new even from the physics point of view.

Multi-state condensation in Berlin–Kac spherical models

Jani Lukkarinen

In 1952, Berlin and Kac proposed [1] a *spherical model* as a modification of the Ising model of a ferromagnet, replacing discrete spin variables by continuum variables, i.e., by real numbers. Their goal was to find simple models were phase transitions could be studied fairly explicitly, including also the physically relevant case of three dimensions. By explicit computation of the large volume limit of the partition function of their model, they found a phase transition related to the constraint of fixed total "spin" density. The mechanism behind the phase transition is similar to Bose–Einstein condensation in quantum statistical mechanics.

In a personal communication, Herbert Spohn pointed out that the Berlin-Kac spherical model is in fact closely related to a model studied in our earlier joint work, namely, to the discrete nonlinear Schrödinger equation (DNLS). In [2], we proved that standard kinetic theory is valid, at least for short kinetic times, for certain time-correlations of fields following the DNLS evolution assuming that the initial field values are distributed according to a grand canonical ensemble. The grand canonical ensembles yield a two-parameter family of probability measures invariant under the DNLS evolution, and in the weak coupling limit they roughly

correspond to the "Gaussian models" studied in [1]. The spherical model was introduced as a generalization of the Gaussian models which allows for arbitrary initial values of energy and spin density. Analogously, it turns out that the version of the spherical model involving complex-valued fields allows for more general DNLS initial data with well-defined weak coupling limits.

We begin by considering a finite d-dimensional periodic lattice Λ_L of side length L, $\Lambda_L = \{-L/2+1, \ldots, L/2\}^d$ for even L. The dynamical variables are then given by the field values $\psi_x \in \mathbb{C}$, $x \in \Lambda_L$, and we determine the initial values of the fields via randomly distributed Fourier components Φ_k, $k \in \Lambda_L^* = \Lambda_L/L$: we set $\psi_x = \int_{\Lambda_L^*} \mathrm{d}k\, \mathrm{e}^{\mathrm{i}2\pi x \cdot k} \Phi_k$, where "$\int_{\Lambda_L^*} \mathrm{d}k \cdots$" $:= \frac{1}{V} \sum_{k \in \Lambda^*} \cdots$ with $V = |\Lambda_L| = L^d$.

The distribution of the initial data is assumed to be given by the spherical model. The probability measure for the Fourier modes then reads

$$\mu_0[\mathrm{d}\Phi] := \frac{1}{Z_\rho} \mathrm{e}^{-H[\Phi]} \delta(N[\Phi] - \rho V) \prod_{k \in \Lambda^*} [\mathrm{d}\Phi_k^* \mathrm{d}\Phi_k]$$

where $\mathrm{d}\Phi_k^* \mathrm{d}\Phi_k := \mathrm{d}(\mathrm{Re}\,\Phi_k)\mathrm{d}(\mathrm{Im}\,\Phi_k)$, Z_ρ normalizes the integral to one, and

$$H[\Phi] := \int_{k \in \Lambda^*} \mathrm{d}k\, \omega(k)|\Phi_k|^2\,, \qquad N[\Phi] := \int_{k \in \Lambda^*} \mathrm{d}k\, |\Phi_k|^2 = \sum_{x \in \Lambda} |\psi_x|^2\,.$$

Here $\rho > 0$ is some given density parameter and, for notational simplicity, we do not introduce any explicit temperature factors β but rather assume that this has already been included in the definition of $\omega(k)$. Various choices of the "energies" $\omega(k)$ are possible here, but for the measure to be invariant under the DNLS evolution, they need to match with the dispersion relation of its free, wave evolution part. In practise, one is most often interested in a case where a fixed smooth dispersion relation $\omega : \mathbb{T}^d \to \mathbb{R}$ is given and the values of $\omega(k)$ are defined by restriction to $k \in \Lambda_L^*$.

The original Berlin–Kac paper considers nearest neighbour interactions with $\omega(k) = -\beta \sum_{i=1}^d \cos(2\pi k_i)$, $\beta > 0$, and their results are consistent with the scenario that for supercritical initial densities the extra mass condenses into the lowest energy mode with $k = 0$ as $L \to \infty$. In a forthcoming work [3], more general dispersion relations are considered, in particular, allowing for several minima. It is shown that there is a way of splitting the Fourier modes into "condensate modes", $k \in \Lambda_0^*$, and "normal fluid modes", $k \in \Lambda_+^* = \Lambda^* \setminus \Lambda_0^*$, so that these two collections become approximately independent random variables. The normal fluid modes can also be approximated by mean zero, mutually independent, Gaussian random variables. However, the structure of the fluctuations of the "condensate" can be fairly complicated and could depend in a nontrivial way on the lattice size L.

Explicitly, define $\omega_0 := \min_{k \in \Lambda^*} \omega(k)$ and $e_k := \omega(k) - \omega_0 \geq 0$, $k \in \Lambda^*$. In [3], the aim is to prove that if the measure μ_0 is supercritical, $\rho > \sup_L \rho_\mathrm{c}(L)$, where $\rho_\mathrm{c}(L) := \int_{k \in \Lambda_+^*} \mathrm{d}k\, e_k^{-1}$, and the dispersion relation $\omega : \mathbb{T}^d \to \mathbb{R}$ has only finitely many non-degenerate minima (i.e., with an invertible Hessian), then the Wasserstein distance between μ_0 and a measure μ_1 with the above stated fluctuation

properties satisfies

$$W_2(\mu_0, \mu_1) = O(L^{\frac{d}{2} - \kappa}), \qquad L \to \infty,$$

for some $\kappa > 0$. The condensate wave number set Λ_0^* will consist of a certain number of modes which have the lowest excess energies e_k, and their number $|\Lambda_0^*|$ remains bounded as $L \to \infty$. In the special case where all $k \in \Lambda_0^*$ have the minimal energy, i.e., if $e_k = 0$ for all $k \in \Lambda_0^*$, the measure μ_1 has a simplified form given by

$$\mu_1[\mathrm{d}\Phi] := \frac{1}{Z_1} \prod_{k \in \Lambda_+^*} [\mathrm{d}\Phi_k^* \mathrm{d}\Phi_k] \, \mathrm{e}^{-E_+[\Phi]} \prod_{k \in \Lambda_0^*} [\mathrm{d}\Phi_k^* \mathrm{d}\Phi_k] \, \delta(\rho_0[\Phi] - \Delta),$$

where $E_+[\Phi] := \int_{k \in \Lambda_+^*} \mathrm{d}k \, e_k |\Phi_k|^2$ and $\rho_0[\Phi] := V^{-1} \int_{k \in \Lambda_0^*} \mathrm{d}k \, |\Phi_k|^2$.

The proof is based on a construction of a suitable coupling between the two measures, proving the stated Wasserstein distance bound. This will turn out to be sufficient for the error between expectations of any finite moments of the field ψ_x to vanish as $L \to \infty$. The construction of the coupling is a generalization of the one used by Saksman and Webb in Appendix B of [4].

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Gibbs measures of nonlinear Schrödinger equations as limits of many-body quantum states in dimensions $d \leq 3$

ANTTI KNOWLES

(joint work with Jürg Fröhlich, Benjamin Schlein, Vedran Sohinger)

An invariant Gibbs measure $\mathbb{P}$ of a nonlinear Schrödinger equation (NLS) is, at least formally, defined as a probability measure on the space of fields φ that takes the form

$$(1) \qquad \mathbb{P}(\mathrm{d}\varphi) = \frac{1}{Z} \mathrm{e}^{-H(\varphi)} \, \mathrm{d}\varphi,$$

where Z is a normalization constant, H is the Hamilton function, and $\mathrm{d}\varphi$ is the (nonexistent) Lebesgue measure on the space of fields. Here the field φ is a function on $\mathbb{T}^d$ or $\mathbb{R}^d$, for $d = 1, 2, 3$. We are interested in Hamilton functions of the form

$$(2) \quad H(\varphi) := \int \mathrm{d}x \left(|\nabla \varphi(x)|^2 + V(x)|\varphi(x)|^2 \right) + \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, |\varphi(x)|^2 \, w(x - y) \, |\varphi(y)|^2,$$

where $V \geq 0$ is a one-body potential and w is a repulsive interaction potential. The Hamiltonian dynamics associated with (2) is the time-dependent nonlinear

Schrödinger equation (NLS). At least formally, the Gibbs measure (1) is invariant under the flow generated by the NLS.

Our goal here is to obtain Gibbs measures of the form (1), (2) as limits of quantum many-body thermal states. In order to construct the classical field φ with law given by (1), we use the spectral decomposition of $h := -\Delta + V = \sum_{k \in \mathbb{N}} \lambda_k u_k u_k^*$, and set $\varphi = \sum_{k \in \mathbb{N}} \frac{\omega_k}{\sqrt{\lambda_k}} u_k$, where (ω_k) is a family of i.i.d. standard complex normal random variables. It is easy to see that, in an appropriate Sobolev space, the series converges almost surely. Defining the classical interaction

$$(3) \qquad W := \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, |\varphi(x)|^2 \, w(x - y) \, |\varphi(y)|^2 \,,$$

we introduce the classical state (or expectation)

$$(4) \qquad \rho(X) := \frac{\int X \, \mathrm{e}^{-W} \, \mathrm{d}\mu}{\int \mathrm{e}^{-W} \, \mathrm{d}\mu} \,,$$

where X is a random variable.

The quantum many-body problem is formulated on Bosonic Fock space $\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathfrak{H}^{(n)}$, where $\mathcal{F}^{(n)}$ is the Hilbert space of wave functions of n variables that are symmetric under permutation of the variables. On the sector $\mathfrak{H}^{(n)}$, the Hamiltonian reads

$$(5) \qquad H^{(n)} := \sum_{i=1}^n h_i + \lambda \sum_{1 \leq i < j \leq n} w(x_i - x_j) \,.$$

We introduce the *temperature* τ, and define the Hamiltonian divided by the temperature as

$$(6) \qquad \begin{aligned} H_\tau &:= \frac{1}{\tau} \bigoplus_{n \in \mathbb{N}} H^{(n)} = \int \mathrm{d}x \, \mathrm{d}y \, \varphi_\tau^*(x) \, h(x; y) \, \varphi_\tau(y) \\ &\quad + \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, \varphi_\tau^*(x) \varphi_\tau^*(y) \, w(x - y) \, \varphi_\tau(x) \varphi_\tau(y) \,, \end{aligned}$$

where we introduced the bosonic canonical annihilation and creation operators φ_τ and φ_τ, which have been rescaled by $\tau^{-1/2}$ so that $[\varphi_\tau(x), \varphi_\tau^*(y)] = \frac{1}{\tau} \delta(x - y)$. Then the quantum state is given by

$$(7) \qquad \rho_\tau(\mathcal{A}) := \frac{\mathrm{Tr}(\mathcal{A} P_\tau)}{\mathrm{Tr}(P_\tau)} \,, \qquad P_\tau := \mathrm{e}^{-H_\tau} \,.$$

It was proved by Lewin, Nam, and Rougerie [2] that for $d = 1$ the correlation functions of the state (7) converge (in trace norm) to those of (4) as $\tau \to \infty$. Our first result [1] is another proof of this theorem using a completely different method.

For $d > 1$, the situation becomes more complicated. This is manifested by the fact that the classical field φ is almost surely not a function but a distribution of negative regularity. Hence, already the definition (3) does not make sense. The remedy is well known: one has to perform a renormalization by Wick ordering,

formally replacing (3) with

$$W = \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, (|\varphi(x)|^2 - \infty) \, w(x - y) \, (|\varphi(y)|^2 - \infty) \,,$$

where the infinities are carefully chosen using a truncation procedure of the field φ.

On the quantum many-body side, the singularity for $d \geq 2$ is manifested by the fact that the particle number grows much faster than the temperature τ. Again, one has to renormalize by replacing the interaction term in (6) with

$$(8) \qquad \frac{1}{2} \int \mathrm{d}x \, \mathrm{d}y \, (\varphi_\tau^*(x)\varphi_\tau(x) - \varrho_\tau(x)) \, w(x - y) \, (\varphi_\tau^*(y)\varphi_\tau(y) - \varrho_\tau(y)) \,,$$

where $\varrho_\tau(x)$ is the expected quantum density, defined as the expectation of the density operator $\varphi_\tau^*(x)\varphi_\tau(x)$ in the free (i.e. $w = 0$) quantum state.

Our second, and main, result is that, for $d = 2, 3$, the quantum correlation function converge again (in Hilbert-Schmidt norm) to the classical ones, provided all interactions are appropriately renormalized. For technical reasons, instead of using the thermal state $P_\tau = \mathrm{e}^{-H_\tau} = \mathrm{e}^{-H_{\tau,0} - W_\tau}$, we use a modified thermal state

$$P_\tau^\eta := \mathrm{e}^{-\eta H_{\tau,0}} \mathrm{e}^{-(1-2\eta)H_{\tau,0} - W_\tau} \mathrm{e}^{-\eta H_{\tau,0}}$$

for some fixed $\eta > 0$.

For $d = 2, 3$, the relationship between the physical many-body Hamiltonian (6) and its renormalized version with interaction (8) is nontrivial; it is governed by the so-called *counterterm problem*, which can be formulated as a nonlinear integral equation

$$(9) \qquad\qquad u \; = \; V + w * (\varrho_\tau^{\kappa+u} - \bar\varrho_\tau^\kappa) \,,$$

for the *dressed one-body potential u*. Here $\varrho_\tau^{\kappa+u}$ is the free quantum density associated with the one-body Hamiltonian $h = -\Delta + \kappa + u$. In [1], we formulate this problem precisely and solve the resulting equation using a fixed point argument.

The proof of our main theorem, the convergence of correlation function for $d = 2, 3$, is based on ideas from field theory, using a perturbative expansion in the interaction, organized by using a diagrammatic representation, and on Borel resummation of the resulting series.

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Quasi-invariance of Gaussian measures under the ∂NLS gauge

GIUSEPPE GENOVESE

(joint work with R. Lucà, D. Valeri)

The derivative nonlinear Schrödinger equation (∂NLS, see R. Lucà's talk) admits a one-parameter group of gauge transformation $\mathscr{G}_\alpha : L^2(\mathbb{T}) \to L^2(\mathbb{T})$, $\alpha \in \mathbb{R}$, which in the periodic setting (i.e. $t \in \mathbb{T}$) reads as [1]

(1)
$$(\mathscr{G}_\alpha y)(t) := e^{i\alpha \mathcal{I}(y(t))} y(t), \quad \mathcal{I}(y(t)) := \frac{1}{2\pi} \int_0^{2\pi} d\theta \int_\theta^t \left(|y(s)|^2 - \frac{\|y\|_{L^2(\mathbb{T})}^2}{2\pi} \right) ds.$$

The gauge can be equivalently defined in terms of the Cauchy problem

(2)
$$\frac{d}{d\alpha}(\mathscr{G}_\alpha y)(t) = i\mathcal{I}[(\mathscr{G}_\alpha y)(t)](\mathscr{G}_\alpha y)(t), \quad (\mathscr{G}_0 y)(t) = y(t).$$

Here γ_k denotes the Gaussian measure on $L^2(\mathbb{T})$ with covariance $(1+(-\Delta)^k)^{-1}$ and $\gamma_{k,\alpha} := \gamma_k \circ \mathscr{G}_\alpha$ and it is interesting to study the absolute continuity of these two measures. The main difficulty is that the gauge is *anticipative*, i.e. $(\mathscr{G}_\alpha y)(t)$ is computed using the values $y(s)$ also for $s > t$ (see (1)). However in [2] the exact change of variable formula under $\mathscr{G}_\alpha$ for γ_1 was established. The typical process under γ_1 is the complex Brownian bridge and the proof of for $k = 1$ relies crucially on the conditional independence of modulus and phase of the complex Brownian bridge after a time-change. Since the gauge is a *good* phase-shift (in the sense of [2]) depending only on the modulus, Girsanov theorem and the change of time formula yield the correct Jacobi formula. This nice trick does not generalise for $k \geq 2$ as the typical processes for γ_k get more correlated and a functional analytic proof seems to be more viable. Anticipative transformations with Hilbert-Schmidt differentials have been widely studied [3]. However a direct computation shows the gauge not to belong to this class, therefore the classical results do not apply. Even though the determination of the correct density in the Jacobi formula is still an open problem, one can use a soft argument introduced by Tzvetkov [4] to prove the quasi-invariance of γ_k under the action of $\{\mathscr{G}_\alpha\}_{\alpha \in \mathbb{R}}$, i.e. $\gamma_k \ll \gamma_{k,\alpha}$ for any $\alpha \in \mathbb{R}$. The main interesting consequence of this result is the absolute continuity of the invariant measure associated to the k-th integral of motion of ∂NSL w.r.t. γ_k [5]. Of course one would expect the invariant measure to be the Gibbs measure, but to prove it one really needs the precise form of the density w.r.t. γ_k of $\gamma_{k,\alpha}$.

The form (2) is particularly suitable to define for $N \in \mathbb{N}$ a truncated gauge-group $\mathscr{G}_\alpha^N : E_N \to E_N$ (here $E_N := \mathrm{span}_\mathbb{C}\{e^{int} : |n| \leq N\} \subset L^2(\mathbb{T})$) by

(3)
$$\frac{d}{d\alpha}(\mathscr{G}_\alpha^N y)(t) = iP_N \mathcal{I}[\mathscr{G}_\alpha^N y](t)(\mathscr{G}_\alpha^N y)(t), \quad (\mathscr{G}_0^N y)(t) = y(t),$$

where P_N is the projector on E_N. $\{\mathscr{G}_\alpha^N\}_{\alpha \in \mathbb{R}}$ is still a one-parameter group of transformations and it approximates $\{\mathscr{G}_\alpha\}_{\alpha \in \mathbb{R}}$ as $N \to \infty$ in the topology of H^s,

$s \geq 0$. The finite-dimensional change of variables hence reads as

$$\int_A \gamma_k(d\mathscr{G}_\alpha^N x) = \int_A |\det D\mathscr{G}_\alpha^N(y)| \exp\left(\frac{1}{2}\|y\|_{H^k}^2 - \frac{1}{2}\|\mathscr{G}_\alpha^N y\|_{H^k}^2\right) \gamma_k(dy).$$

Since one can show $\mathscr{G}_\alpha^N$ to be asymptotically divergence-free, from now on the determinant will be neglected. The group property gives for $\bar{\alpha} \in \mathbb{R}$

$$\frac{d}{d\alpha}\gamma_{\alpha,k}(A)\Big|_{\alpha=\bar{\alpha}} = \frac{d}{d\alpha}\int_{\mathscr{G}_\alpha^N(A)} \gamma_k(dy)\Big|_{\alpha=\bar{\alpha}} = \frac{d}{d\alpha}\gamma_{\alpha,k}(\mathscr{G}_{\bar{\alpha}}^N(A))\Big|_{\alpha=0}.$$

for any measurable $A \subset E_N$. Differentiating in α the term to estimate is thus

$$\frac{d}{d\alpha}\frac{1}{2}\|\mathscr{G}_\alpha^N y\|_{H^k}^2\Big|_{\alpha=0}$$

and a fundamental observation is that

$$\frac{d}{d\alpha}\|\mathscr{G}_\alpha^N y\|_{H^k}\Big|_{\alpha=0} = \frac{d}{d\alpha}\|\mathscr{G}_\alpha y\|_{H^k}\Big|_{\alpha=0}.$$

This identity is important in view of the following formula (see [5, Lemma 2.9])

$$(4)$$
$$\|\mathscr{G}_\alpha y\|_{\dot{H}^k}^2$$
$$= \|y\|_{\dot{H}^k}^2 + 2ik\alpha\mu[y]\int y^{(k-1)}\bar{y}^{(k)} + i(k+1)\alpha\int |y^2|\left(y^{(k)}\bar{y}^{(k-1)} - y^{(k-1)}\bar{y}^{(k)}\right),$$

plus reminders $\int r_k$. Here $r_k \in \mathcal{V}_{k-1}$ and $\mu[y] := \|y\|_{L^2(\mathbb{T})}/(2\pi)$, where $\mathcal{V} = [\psi^{(n)}, \bar{\psi}^{(n)} \mid n \in \mathbb{N}_0]$ denotes the algebra of differential polynomials in the variables ψ and $\bar{\psi}$ and $\mathcal{V}_n = \{f \in \mathcal{V} \mid \frac{\partial f}{\partial u^{(m)}} = 0, \text{ for every } m > n, u = \psi \text{ or } \bar{\psi}\}$. Formula (4) provides a tight control on the growth of the Sobolev norms under the gauge evolution, allowing one to compute explicitly the derivative in $\alpha = 0$ and to prove the necessary estimates in order to apply the argument of [4].

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A microscopic derivation of time-dependent correlation functions of the $1D$ cubic nonlinear Schrödinger equation

Vedran Sohinger

(joint work with J. Fröhlich, A. Knowles, B. Schlein)

1. Setup of the problem

In [2] we give a microscopic derivation of time-dependent correlation functions of the $1D$ cubic nonlinear Schrödinger equation (NLS) from many-body quantum theory. This is a time-dependent extension of $1D$ results in [1, 4]. On $\Lambda = \mathbb{T}$ or $\Lambda = \mathbb{R}$, we study the NLS

$$(1) \quad \begin{cases} \mathrm{i}\partial_t \varphi_t(x) + \big(\Delta - \kappa\big)\varphi_t(x) \;=\; v(x)\,\varphi_t(x) + \int \mathrm{d}y\, w(x-y)\,|\varphi_t(y)|^2\,\varphi_t(x) \\ \varphi_0(x) \;=\; \Phi(x) \in H^s(\Lambda)\,. \end{cases}$$

Here $\kappa > 0, v : \Lambda \to [0, +\infty)$, w is either in $L^\infty(\Lambda)$ and *pointwise nonnegative* or $w = \delta$. Assume that $h := -\Delta + \kappa + v$, has a compact resolvent and satisfies $\operatorname{Tr} h^{-1} < \infty$. On $\mathfrak{H}^{(n)} := L^2_{\mathrm{sym}}(\Lambda^n)$ we consider the *n-body Hamiltonian*

$$H^{(n)} \;:=\; \sum_{i=1}^n \big(-\Delta_{x_i} + \kappa + v(x_i)\big) + \frac{1}{n}\sum_{1 \le i < j \le n} w(x_i - x_j)\,.$$

For fixed $\tau > 0$ (temperature), consider

$$H_\tau \;:=\; \frac{1}{\tau}\bigoplus_{n \in \mathbb{N}} H^{(n)}\,, \quad P_\tau \;:=\; \mathrm{e}^{-H_\tau}\,,$$

acting on the bosonic Fock space $\mathcal{F} := \bigoplus_{n \in \mathbb{N}} \mathfrak{H}^{(n)}$. Given $\mathcal{A}$ a closed operator on $\mathcal{F}$, we define its expectation in the quantum state by $\rho_\tau(\mathcal{A}) := \operatorname{Tr}_{\mathcal{F}}(\mathcal{A}P_\tau)/\operatorname{Tr}_{\mathcal{F}}(P_\tau)$. Given ξ a closed linear operator on $\mathfrak{H}^{(p)}$, we define the *lift* of ξ to $\mathcal{F}$ by

$$(2) \quad \Theta_\tau(\xi) \;:=\; \int \mathrm{d}x_1 \cdots \mathrm{d}x_p\, \mathrm{d}y_1 \cdots \mathrm{d}y_p\, \xi(x_1, \ldots, x_p; y_1, \ldots, y_p)$$
$$\times\, \varphi_\tau^*(x_1)\cdots\varphi_\tau^*(x_p)\varphi_\tau(y_1)\cdots\varphi_\tau(y_p)\,,$$

where $\varphi_\tau := \tau^{-1/2}b, \varphi_\tau^* := \tau^{-1/2}b^*$, for the standard annihilation and creation operators b, b^* on $\mathcal{F}$. Furthermore, $\Psi_\tau^t\Theta_\tau(\xi) := \mathrm{e}^{\mathrm{i}t\tau H_\tau}\Theta_\tau(\xi)\mathrm{e}^{-\mathrm{i}t\tau H_\tau}$. In the classical problem, $\varphi \equiv \varphi(\omega)$ denotes the *free classical field*

$$\varphi \;:=\; \sum_{k \in \mathbb{N}} \frac{\omega_k}{\sqrt{\lambda_k}}\, u_k\,, \quad \omega_k \text{ are i.i.d. complex Gaussians}\,,$$

where $h = \sum_{k \in N} \lambda_k u_k u_k^*$ for $u_k \in L^2(\Lambda)$ normalized. We define $\Theta(\xi)$ similarly as in (2) by replacing occurrences of $\varphi_\tau, \varphi_\tau^*$ with $\varphi, \bar{\varphi}$ respectively. We also define $\Psi^t\Theta(\xi)$ by evolving $\varphi, \bar{\varphi}$ up to time t according to (1). Finally, let $\rho(\cdot) := \mathbb{E}_\mu(\cdot)$, for $\mathrm{d}\mu$ the Gibbs measure corresponding to (1). We can now state our main result.

Theorem 1. *Let $w \in L^\infty(\Lambda)$ and $w \geq 0$ pointwise. For $m \in \mathbb{N}$, $p_1, \ldots, p_m \in \mathbb{N}$, $\xi^1 \in \mathcal{L}(\mathfrak{H}^{(p_1)}), \ldots, \xi^m \in \mathcal{L}(\mathfrak{H}^{(p_m)})$ and $t_1, \ldots, t_m \in \mathbb{R}$, we have*

$$\lim_{\tau \to \infty} \rho_\tau \Big(\Psi_\tau^{t_1} \Theta_\tau(\xi^1) \cdots \Psi_\tau^{t_m} \Theta_\tau(\xi^m) \Big) = \rho \Big(\Psi^{t_1} \Theta(\xi^1) \cdots \Psi^{t_m} \Theta(\xi^m) \Big).$$

2. Ideas of the proof

We apply a series expansion similar to that used in the work [3, Section 3.4] on the lattice. Let $\mathcal{N} := \|\varphi\|_{L^2}^2, \mathcal{N}_\tau := \int \mathrm{d}x\, \varphi_\tau(x) \varphi_\tau^*(x)$. We reduce to proving that for suitable $F \in C_c^\infty(\mathbb{R})$ and $\xi \in \mathcal{L}(\mathfrak{H}^{(p)})$ we have

$$(3) \qquad \lim_{\tau \to \infty} \rho_\tau \big(\Theta_\tau(\xi) F(\mathcal{N}_\tau) \big) = \rho \big(\Theta(\xi) F(\mathcal{N}) \big).$$

The presence of F in (3) does not allow us to directly apply the methods from [1]. We overcome this difficulty by complex analytic methods. When $w = \delta$, we obtain a partial analogue of Theorem 1 by using $X^{s,b}$ spaces.

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