Mathematisches Forschungsinstitut Oberwolfach

Enumerative Combinatorics

Organised by
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13 May – 19 May 2018

ABSTRACT. Enumerative Combinatorics focusses on the exact and asymptotic counting of combinatorial objects. It is strongly connected to the probabilistic analysis of large combinatorial structures and has fruitful connections to several disciplines, including statistical physics, algebraic combinatorics, graph theory and computer science. This workshop brought together experts from all these various fields, including also computer algebra, with the goal of promoting cooperation and interaction among researchers with largely varying backgrounds.

Introduction by the Organisers

Four years ago, in March 2014, we (MIREILLE BOUSQUET-MÉLOU (Bordeaux), MICHAEL DRMOTA (Vienna), CHRISTIAN KRATTENTHALER (Vienna), and MARC NOY (Barcelona)) had organised a *Workshop on "Enumerative Combinatorics"* here at the Mathematische Forschungsinstitut at Oberwolfach. It was apparently the first of its kind. Now, four years later, it was about time to meet again and assess the developments which have taken place since then, in particular, to examine the impact of the previous workshop, and to witness and discuss the recent trends and most exciting developments in Enumerative Combinatorics. Among the participants of the two workshops there was of course a non-trivial intersection. However, many "new" and younger researchers were among us this time, as particularly "observed" by some of the organisers close to or beyond age 60 . . .

Indeed, the impact of the last workshop could be felt in several ways. As expected, the presentations and discussions from 2014 resulted in several new collaborations, and in several papers, as could be witnessed on the $\mathrm{ar}\chi\mathrm{iv}$. Moreover, as part of the long-term impact so-to-speak, it was not only once that a speaker of the present workshop opened her/his talk by saying "Four years ago I learnt about the problem that I am talking about here; I will now present you the progress that I made since then."

This workshop took place May 13–19, 2018. There were over 50 participants from the US, Canada, Australia, New Zealand, Japan, Korea, India and various European countries. The program consisted of 11 one hour lectures, accompanied by 18 shorter contributions and the special session of presentations by three Oberwolfach Leibniz graduate fellows. (One more gave a contributed talk.) There was also an extensive and inspiring problem session extremely efficiently organised and moderated by BRENDAN MCKAY. Three of the one-hour lectures were designated "keynote lectures" — given by GUILLAUME CHAPUY, GRÉGORY MIERMONT, and IGOR PAK — which provided overviews of recent exciting developments in probabilistic graph theory, in the theory of random maps, and on linear extensions of posets, respectively.

In general, the lecturers in this workshop presented the state of the art in various areas in and/or related to Enumerative Combinatorics, together with relevant new results. The lectures and short talks ranged over a wide variety of topics including classical enumerative problems, algebraic combinatorics, asymptotic and probabilistic methods, statistical physics, methods from computer algebra, among others. Special attention was paid throughout to providing a platform for younger researchers to present themselves and their results. This report contains extended abstracts of the talks and the statements of the problems that were posed during the problem session.

This was the second workshop held on Enumerative Combinatorics. The goal of the workshop was to bring together researchers from different fields with a common interest in enumeration, whether from an algebraic, analytic, probabilistic, geometric or computational angle, in order to enhance collaboration and new research projects. The organizers believe this goal was amply achieved, as demonstrated by the strong interaction among the participants and the lively discussions in and outside the lecture room during the whole week.

On behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere.

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Workshop: Enumerative Combinatorics

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Abstracts

From mafia expansion to analytic functions in percolation theory
Agelos Georgakopoulos

(joint work with John Haslegrave and Christoforos Panagiotis)

In [2] we introduced a random graph that admits diverse equivalent definitions. One of those definitions is as a long range Bernoulli percolation process (see [4] for definitions) on a group, namely the direct sum $\bigoplus_{i \in \mathbb{N}} \mathbb{Z}_2$ of infinitely many copies of the group of two elements. This random graph has only finite clusters almost surely, thus providing the first example of a long range Bernoulli percolation process on a group non-quasi-isometric with $\mathbb{Z}$ that is subcritical for all values of the parameter.

The second equivalent definition is given by the following

Proposition 1. *For every $\lambda \in \mathbb{R}_+$, there is a unique rooted connected random multi-graph $(G(\lambda), o)$ with finite average degree which is invariant under the following operation.*

(1)

Replace each vertex v of $G(\lambda)$ (including the root o) by two vertices v_1, v_2, and join v_1 to v_2 with a random number of edges with distribution $Po(\lambda)$. Moreover, replace each edge uv of $G(\lambda)$, with one of the four edges $v_i u_j, i, j \in \{1, 2\}$, chosen uniformly at random. All these random experiments are made independently from each other.
Choose the root of the resulting graph to be each of o_1, o_2 with probability $\frac{1}{2}$.

A third equivalent definition follows the lines of a more general construction of [1].

In [2] it is proved that the expected size $\chi(\lambda)$ of the component of the root admits a lower bound exponential in λ and an upper bound doubly exponential in λ, while simulations suggest that the right order is $\lambda^{c\lambda}$.

Trying to understand $\chi(\lambda)$ in this particular random graph lead us to consider analyticity properties in more general percolation processes. Generalising a result (and the technique) of Kesten [5], we prove in [3] that the expected size of the cluster in the subcritical regime of any long- or short-range invariant Bernoulli percolation model on a countable group is an analytic function of the parameter.

The technique is applicable to other functions than $\chi(\lambda)$, although usually additional arguments are needed. The main result of [3] is

Theorem 1. *For Bernoulli bond percolation with parameter p on any quasi-transitive lattice, the percolation density $\theta(p)$ is an analytic function in the interval $(p_c, 1]$.*

Here, $\theta(p)$ is the probability that the component of a fixed vertex is infinite. It is a well-known question whether $\theta(p)$ is analytic above p_c for 'nearest neighbour' percolation on $\mathbb{Z}^d$, see e.g. [4]. Theorem 1 answers this for $d = 2$.

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Asymptotic number and properties of graphs on surfaces

MIHYUN KANG

(joint work with Michael Moßhammer and Philipp Sprüssel)

For $g \in \mathbb{N} \cup \{0\}$ we let $\mathbb{S}_g$ be the orientable surface of genus g and $\mathcal{S}_g(n, m)$ be the class of all graphs with vertex set $[n] = \{1, \ldots, n\}$ and m edges that are embeddable on $\mathbb{S}_g$ without crossing edges. Let $S_g(n, m)$ be a graph chosen uniformly at random from $\mathcal{S}_g(n, m)$. For each $i \in \mathbb{N}$ let $L_i = L_i(G)$ denote the i-th largest component of $G = S_g(n, m)$.

We show that $S_g(n, m)$ undergoes two phase transitions. The first phase transition mirrors the classical phase transition in the Erdős–Rényi random graph and it takes place when the giant component emerges.

Theorem 1. *Let $m = \left(1 + \lambda n^{-1/3}\right) \frac{n}{2}$, where $\lambda = \lambda(n) = o(n^{1/3})$.*

(1) If $\lambda \to -\infty$, then for every $i \geq 1$ whp L_i is a tree of order

$$(2 + o(1)) \frac{n^{2/3}}{\lambda^2} \log(-\lambda^3).$$

(2) If $\lambda \to c$ for a constant $c \in \mathbb{R}$, then the probability that G has complex components is bounded away both from 0 and 1. For $i \geq 1$, whp the i-th largest component has order

$$\Theta_p(n^{2/3}).$$

(3) If $\lambda \to \infty$, then whp L_1 has genus g, is complex, and has order

$$\lambda n^{2/3} + O_p(n^{2/3}).$$

The rest $G \setminus L_1$ of the graph is planar whp and has $O_p(1)$ complex components, each of which has order $O_p(n^{2/3})$.

For $i \geq 2$, we have $|L_i| = \Theta_p(n^{2/3})$. The probability that G has at least i complex components is bounded away both from 0 and 1.

The second phase transition occurs when the giant component covers almost all vertices of the graph. This kind of phenomenon is strikingly different from the Erdős–Rényi random graph and has only been observed for graphs on surfaces.

Theorem 2. *Let $m = \left(2 + \zeta n^{-2/5}\right) \frac{n}{2}$, where $\zeta = \zeta(n) = o(n^{2/5})$. Then whp the largest component L_1 has genus g, is complex, and*

$$n - |L_1| = \begin{cases} (1 + o(1)) \, |\zeta| \, n^{3/5} & \text{if } \zeta \to -\infty, \\ \Theta\left(n^{3/5}\right) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta\left(\zeta^{-3/2} n^{3/5}\right) & \text{if } \zeta \to \infty, \text{ but } \zeta = o((\log n)^{-2/3} n^{2/5}). \end{cases}$$

Whp all other components of G are planar and for $i \geq 2$, we have

$$|L_i| = \begin{cases} \Theta_p\left(|\zeta|^{2/3} \, n^{2/5}\right) & \text{if } \zeta \to -\infty, \\ \Theta_p\left(n^{2/5}\right) & \text{if } \zeta \to c \in \mathbb{R}, \\ \Theta_p\left(\zeta^{-1} n^{2/5}\right) & \text{if } \zeta \to \infty, \text{ but } \zeta = o((\log n)^{-2/3} n^{2/5}). \end{cases}$$

When the number of edges is between the regimes of the two phase transitions, that is, the average degree of the graph is between one and two, the largest component is complex, has genus g, and its order is linear both in n and in the average degree of the graph.

Theorem 3. *Let $m = d\frac{n}{2}$, where $d = d(n)$ converges to a constant in $(1, 2)$. Then whp the largest component L_1 has genus g, is complex, and has order*

$$|L_1| = (d - 1)\, n + O_p\left(n^{2/3}\right).$$

Whp all other components of G are planar and for $i \geq 2$, we have $|L_i| = \Theta_p(n^{2/3})$.

For $m = d\frac{n}{2}$ with $d > 1$, the Erdős–Rényi random graph $G(n, m)$ whp has a largest component of order $(1 + o(1))\beta n$, where β is the unique positive solution of the equation

$$1 - \beta = e^{-\alpha \beta}.$$

The components of $G(n, m)$ can be explored via a Galton-Watson branching process with offspring distribution $\text{Po}(d)$; the survival property of such a process is given by β above, yielding order $(1 + o(1))\beta n$ of the largest component. For graphs on surfaces, however, there is no such simple approach to explore components.

More precisely, the local structure of $G(n, d\frac{n}{2})$ converges to that of a Galton-Watson tree with offspring distribution $\text{Po}(d)$ in the sense of Benjamini-Schramm local weak convergence. For $S_g(n, m)$, the additional constraint of the graph being embeddable on $\mathbb{S}_g$, exploration via a simple Galton-Watson type process is not possible. This naturally raises the question if the local structure of $S_g(n, m)$ can be described in terms of the Benjamini-Schramm local weak convergence.

Problem 1. *What is the limit of the local structure of $S_g(n, m)$ in the sense of the Benjamini-Schramm local weak convergence?*
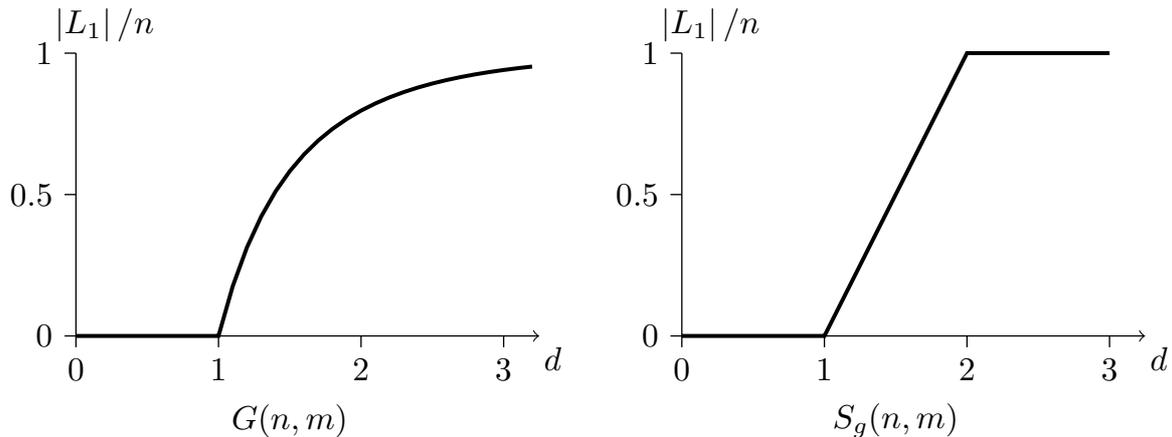
FIGURE 1. Rescaled order of the largest component of $G(n,m)$ and of $S_g(n,m)$.

From exact enumeration to asymptotics in Algebraic Combinatorics

GRETA PANOVA

(joint work with Alejandro H. Morales, Igor Pak and Damir Yelliussizov)

The dimension of the irreducible representations of the Symmetric Group (similarly GLn) are given by the hook-length formulas (hook-content formulas), which as compact explicit product formulas have allowed for extensive both from enumerative/bijective Combinatorics but also asymptotic/probabilistic characterizations for the "typical" representations and limit behavior. However, no such explicit product formulas are known for any other quantities in the world of Algebraic Combinatorics (skew shape SYTs and SSYTs, Kostka, Littlewood-Richardson and Kronecker coefficients, Schubert polynomial evaluations etc).

Recently, Ikeda-Naruse discovered a generalization of the hook-length formula for the number of skew SYTs:

$$(1) \qquad f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ is the set of *excited diagrams* of λ/μ, obtained by pushing the boxes of the Young diagram of μ along diagonals within the Young diagram of λ.

In a series of papers [1, 2, 3] we developed and generalized this formula further, giving different proofs (from a purely algebraic, through direct bijection, to lattice-path LGV formula interpretations), and further used it to observe many interesting phenomena like explicit product formulas for certain skew SYTs and Schubert polynomials, lozenge tilings under multivariate weights, etc.

We also used the compact formulation of this formula to derive asymptotics of the number of skew SYTs in [4]. In particular using the immediate upper and

lower bounds from equation (1) for the largest term given by

$$F(\lambda/\mu) := n! \prod_{u \in \lambda/\mu} \frac{1}{h_u},$$

we obtain that $F(\lambda/\mu) \leq f^{\lambda/\mu} \leq |\mathcal{E}(\lambda/\mu)|F(\lambda/\mu)$. In various regimes the higher order term is $F(\lambda/\mu)$ which gives the first order asymptotics of $f^{\lambda/\mu}$:

Theorem 1. *Let $\omega, \pi : [0, a] \to [0, b]$ be continuous non-increasing functions, and suppose that $area(\omega/\pi) = 1$. Let $\{\lambda^{(n)}/\mu^{(n)}\}$ be a sequence of skew shapes with the stable shape ω/π, i.e. $[\lambda^{(n)}]/\sqrt{n} \to \omega$, $[\mu^{(n)}]/\sqrt{n} \to \pi$. Then*

$$\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \frac{1}{2}n\log n \quad as \quad n \to \infty.$$

Suppose $(\sqrt{N} - L)\omega \subset [\lambda^{(n)}](\sqrt{N} + L)\omega$ for some $L > 0$, similarly $\mu^{(n)}$ wrt π, then

$$-\big(1 + c(\omega/\pi)\big)n + o(n) \leq \log f^{\lambda^{(n)}/\mu^{(n)}} - \frac{1}{2}n\log n$$
$$\leq -\big(1 + c(\omega/\pi)\big)n + \log \mathcal{E}(\lambda^{(n)}/\mu^{(n)}) + o(n),$$

as $n \to \infty$, where

$$c(\omega/\pi) = \iint_{\omega/\pi} \log h(x, y)dxdy,$$

where $h(x, y)$ is the hook length from (x, y) to ω.

Similar bounds can be obtained for other regimes: in the Thoma-Vershik-Kerov limit with linearly growing Frobenius coordinates, then $\log f^{\lambda/\mu} \sim Cn$, where C can be computed explicitly from the limit shapes. When the limit shape is "thin" then $\log f^{\lambda/\mu} \sim n\log(n)$.

Skew SYTs are closely related to straight-shape SYTs via the Littlewood-Richardson rule. The connections prompt an exploration of the next step – the asymptotics of Littlewood-Richardson and Kronecker coefficients. Stanley used certain summation formulas to deduce that the maximal Kronecker coefficient and the maximal Littlewood-Richardson coefficient are asymptotically

$$\max_{\lambda \vdash n} \max_{\mu \vdash n} \max_{\nu \vdash n} g(\lambda, \mu, \nu) = \sqrt{n!}\, e^{-O(\sqrt{n})},$$

$$\max_{0 \leq k \leq n} \max_{\lambda \vdash n} \max_{\mu \vdash k} \max_{\nu \vdash n-k} c_{\mu,\nu}^{\lambda} = 2^{n/2 - O(\sqrt{n})}.$$

In [6] we gave answers to when (for which triples of partitions) these coefficients are asymptotically maximal, answering Stanley's conjecture. In particular, using various different summation formulas involving combinations of dimensions (i.e. the numbers f^{λ} etc) and these coefficients we conclude:

Theorem 2. *Let $\{\lambda^{(n)} \vdash n\}$, $\{\mu^{(n)} \vdash n\}$, $\{\nu^{(n)} \vdash n\}$ be three partitions sequences, such that*

(2) $$g\big(\lambda^{(n)}, \mu^{(n)}, \nu^{(n)}\big) = \sqrt{n!}e^{-O(\sqrt{n})}.$$

Then all three partition sequences are Plancherel (i.e. VKLS shape, equivalently asymptotically maximal dimensions $\sim \sqrt{n!}$). Conversely, for every two Plancherel partition sequences $\{\lambda^{(n)} \vdash n\}$ and $\{\mu^{(n)} \vdash n\}$, there exists a Plancherel partition sequence $\{\nu^{(n)} \vdash n\}$, s.t. (2) holds.

We further determine, in a similar spirit, the families of partitions for which the Littlewood-Richardson coefficients are asymptotically maximal.

Finally, inspired by Stanley's "Schubert Shenanigans", in [5] also study the asymptotics of the evaluation at 1s of the Schubert polynomials, in particular looking for their maximal values and obtain a new lower bound, which is a conjectured maximum. Let $\mathfrak{S}_w(x_1, \ldots)$ be the Schubert polynomial for a permutation $w \in S_n$. Let $u(n) := \max_{w \in S_n} \mathfrak{S}_w(1^n)$. Stanley conjectured that $\lim_{n \to \infty} \frac{\log_2 u(n)}{n^2}$ should exist and showed that if it does it is in $[1/4, 1/2]$. Motivated by a conjecture of Merzon-Smirnov that the asymptotic maximum is achieved at a "layered" (Richardson) permutation

$$w(b_k, b_{k-1}, \ldots, b_1) := \big(w_0(b_k), b_k + w_0(b_{k-1}), \ldots, n - b_1 + w_0(b_1)\big),$$

in [5] we found the sequences $(b_1, \ldots)$ giving the asymptotic maxima among those permutations, showing that

$$\lim_{n \to \infty} \max_{b_1 + \cdots = n} \frac{\log_2 \mathfrak{S}_w(b_1, \ldots)(1^n)}{n^2} = \frac{\gamma}{\log 2} \approx 0.2932362762,$$

where γ is the root of a given explicit equation. This also improves Stanley's lower bound.

This area, of asymptotic study of quantities from Algebraic Combinatorics like degrees and structure constants, is barely explored and we have just scratched its mysterious surface. The current findings pose numerous further question and show a need for new methods allowing us to extract asymptotics without having explicit formulas.

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Asymptotics of skew standard Young tableaux

Jehanne Dousse

(joint work with Valentin Féray)

The Young diagram of a partition $n = \lambda_1 + \cdots + \lambda_s$ is a diagram made of s rows of left-justified boxes, the i-th row having λ_i boxes. A standard Young tableau (SYT) is a filling of the boxes of a Young diagram of size n with the numbers 1 to n, such that the rows and columns are increasing. The hook-length formula of Frame, Robinson and Thrall [1] allows one to compute the number of SYTs of a certain shape. This has many applications in algebraic combinatorics, discrete probability and representation theory.

SYTs of skew shapes (a diagram obtained by removing a Young diagram of size k from the top left corner of a larger Young diagram of size n) have also been a subject of interest, but there is no efficient exact formula for their number. But it is also interesting to study them from an asymptotic point of view. Recently, Morales, Pak and Panova [2] have showed that in certain particular cases (with k of the order of n), the hook-length formula gives a good approximation for the number of skew-SYTs.

In joint work with Valentin Fray, we used a different approach based on bounds for characters of the symmetric group to study the asymptotics in other cases. So far we have been focusing mainly on the case where the Young diagrams are balanced, i.e. the largest part and number of parts of λ (resp. μ) are at most $L\sqrt{n}$ (resp. $L\sqrt{k}$). Our main result is an asymptotic expansion at any order when $k = o(n^{\frac{1}{3}})$:

Theorem 1. *Let $\lambda \vdash n$ and $\mu \vdash k$ be balanced, with $k = o\left(n^{1/3}\right)$. Then for any natural integer r (not depending on k and n), we have as n tends to infinity,*

$$|\mu|! \frac{f^{\lambda/\mu}}{f^\lambda f^\mu} = \sum_{\substack{\sigma \in S_k, \\ |\sigma| \le r}} \frac{\chi^\lambda(\sigma)}{f^\lambda} \frac{\chi^\mu(\sigma)}{f^\mu} + O\left(\left(k^{\frac{3}{2}} n^{-\frac{1}{2}}\right)^{r+1}\right).$$

We also get a sharp asymptotic bound in the case $k \le C\sqrt{n}$, where C is some constant. For $k > C\sqrt{n}$, we obtain an asymptotic bound, but we do not think it is sharp. Very likely several other techniques will be needed to treat other cases.

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A bijective proof of the hook-length formula for skew shifted shapes
Matjaž Konvalinka

The celebrated *hook-length formula* gives an elegant product expression for the number of standard Young tableaux of fixed shape λ:

$$f^\lambda = \frac{|\lambda|!}{\prod_{u \in [\lambda]} h(u)}$$

Here $h(u)$ is the hook length of the cell u.

The formula gives dimensions of irreducible representations of the symmetric group. The formula was discovered by Frame, Robinson and Thrall in [4] based on earlier results of Young [23], Frobenius [5] and Thrall [22]. Since then, it has been reproved, generalized and extended in several different ways, and applied in a number of fields ranging from algebraic geometry to probability, and from group theory to the analysis of algorithms.

Interestingly, an identical formula (when hook lengths are defined appropriately) also holds for strict partitions and standard Young tableaux of shifted shape. Even though it was discovered before the formula for straight shapes [22], it is typically considered the "lesser" of the two: less interesting, and with more complicated proofs. Shortly after the famous Greene-Nijenhuis-Wilf probabilistic proof of the ordinary hook length formula [7], Sagan [20] extended the argument to the shifted case. The proof needs a careful analysis of special cases and a delicate double induction, and therefore lacks the intuitiveness of the original hook-walk proof. In 1995, Krattenthaler [13] provided a bijective proof. While short, it is very involved, as it needs a variant of Hillman-Grassl algorithm, a bijection that comes from Stanley's (P, ω)-partition theory, and the involution principle of Garsia and Milne. A few years later, Fischer [3] gave the most direct proof of the formula, in the spirit of Novelli-Pak-Stoyanovskii's [18] bijective proof of the ordinary hook-length formula. At almost 50 pages in length, the proof is very involved. Bandlow [1] gave a short proof via interpolation, and there is a variant of the hook-walk proof in [12]; again, special cases need to be considered, and the bijection is hard to describe succinctly. There are also many generalizations of both formulas, such as the q-version of Kerov [9], and its further generalizations and variations (see [6, 10] and also [2]). There are also a great number of proofs of the more general Stanley's hook-content formula (see e.g. [21, Corollary 7.21.4]), see for example [19, 14, 15]. Morales, Pak and Panova wrote a series of the papers on the topic of Naruse's formulas, the most relevant to this paper being [16].

There is no (known) product formula for the number of standard Young tableaux of skew shape (straight or shifted), even though some formulas have been known for a long time. In 2014, Hiroshi Naruse [17] presented and outlined a proof of a remarkable cancellation-free generalization for skew shapes, both straight and shifted. Here we present one of the (two) formulas for shifted shapes.

An *excited move (of type B)* means that we move a cell (i, j) of a shifted diagram to position $(i+1, j+1)$, provided that the cells $(i+1, j)$, $(i, j+1)$ and $(i+1, j+1)$ are not in the diagram.

Let $\mathcal{E}(\lambda/\mu)$ denote the set of all *excited diagrams* of shifted shape λ/μ, diagrams in $[\lambda]$ obtained by taking the diagram of μ and performing series of excited moves in all possible ways. They were introduced by Ikeda and Naruse [8].

Naruse's formula says that

$$(1) \qquad f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)} = \frac{|\lambda/\mu|!}{\prod_{u \in [\lambda]} h(u)} \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in D} h(u)$$

where all the hook lengths are evaluated in the diagram of λ.

Our main result is the following formula, valid for strict partitions λ, μ and for commutative variables x_i:

$$\left(\sum_{k \in \mathcal{W}(\mu, \lambda)} x_k \right) \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,i) \in D} x_i \prod_{\substack{(i,j) \in D \\ i < j}} (x_i + x_j)$$

$$= \sum_{\mu \lessdot \nu \subseteq \lambda} \sum_{D \in \mathcal{E}(\lambda/\nu)} \prod_{(i,i) \in D} x_i \prod_{\substack{(i,j) \in D \\ i < j}} (x_i + x_j),$$

where $\mathcal{W}(\mu, \lambda)$ is a certain finite subset of positive integers. The formula specializes to a recursive versions of equation (1).

Like in [11], the proof of the formulas is bijective and uses a simple bumping algorithm.

The fact that the algorithm is so easy to describe (it takes less than a page!) is quite remarkable. One would hope that this fact will help the shifted hook-length formula overcome its "lesser" status; indeed, how can it be lesser if it is a straightforward generalization with a bijective proof that is not more complicated in any way?

Of course, simplicity of definition does not imply that an algorithm is computationally efficient. It can be proved that the number of steps needed can be $2^{\Omega(\sqrt{n})}$ for partitions λ, μ of size at most n.

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Monotone Triangles and Operator Formulae – A Combinatorial Approach

Hans Höngesberg

A *Gelfand–Tsetlin pattern* of order n is a triangular array of integers of the following form

$$
\begin{array}{ccccccccc}
& & & & a_{1,1} & & & & \\
& & & a_{2,1} & & a_{2,2} & & & \\
& & \cdot\cdot & & \cdots & & \ddots & & \\
& a_{n-1,1} & & \cdots & & \cdots & & a_{n-1,n-1} & \\
a_{n,1} & & a_{n,2} & & \cdots & & a_{n,n-1} & & a_{n,n}
\end{array}
$$

which is weakly increasing along northeast and southeast diagonals, i.e. $a_{i,j} \leq a_{i-1,j} \leq a_{i,j+1}$ for $1 \leq j < i \leq n$. Gelfand–Tsetlin patterns were originally introduced in [5, p. 655] to enumerate the irreducible representations of the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$.

If we fix a sequence $(k_1, \ldots, k_n)$ of weakly increasing integers $k_1 \leq \cdots \leq k_n$, we denote the number of Gelfand–Tsetlin patterns of order n with bottom row $(k_1, \ldots, k_n)$ by $\mathrm{GT}(n; k_1, \ldots, k_n)$. It is known that these patterns are enumerated by the following formula:

$$
\mathrm{GT}(n; k_1, \ldots, k_n) = \prod_{1 \leq i < j \leq n} \frac{k_j - k_i + j - i}{j - i}.
$$

This can be shown via a bijection between Gelfand–Tsetlin patterns and certain semistandard Young tableaux (see [7, pp. 313–314]) or via a translation into a enumeration problem of non-intersecting lattice paths (see [3, Appendix A]).

We are interested in a special type of Gelfand–Tsetlin patterns, namely *monotone triangles*. These are Gelfand–Tsetlin patterns with strictly increasing rows in contrast to weakly increasing rows in ordinary Gelfand–Tsetlin patterns. The number of monotone triangles of order n with a prescribed bottom row $(k_1, \ldots, k_n)$ with $k_1 < \cdots < k_n$ is denoted by $\mathrm{MT}(n; k_1, \ldots, k_n)$.

Monotone triangles are of special interest since they easily generalise *alternating sign matrices*. An alternating sign matrix of order n is a square matrix of order n with entries -1,0 and +1 such that in each row and each column the entries sum up to 1 and the non-zero entries alternate in sign. As already shown in [6, pp. 354–355], alternating sign matrices of order n are in one-to-one correspondence with monotone triangles of order n and bottom row $(1, 2, \ldots, n)$.

Monotone triangles with prescribed bottom row $(k_1, \ldots, k_n)$ can be enumerated by Fischer's *operator formula* (see [1, Theorem 1]):

$$
\left(\prod_{1 \leq s < t \leq n} \left(\mathrm{E}_{x_s} + \mathrm{E}_{x_t}^{-1} - \mathrm{E}_{x_s} \mathrm{E}_{x_t}^{-1} \right) \prod_{1 \leq i < j \leq n} \frac{x_j - x_i + j - i}{j - i} \right) \Bigg|_{(x_1, \ldots, x_n) = (k_1, \ldots, k_n)}
$$

where E_x denotes the *shift operator*: $\mathrm{E}_x f(x) := f(x+1)$.

The proof of the operator formula (as in [1, 2]) is so far by no means combinatorial. It is rather computational based on induction. However, the shorthand notation

$$\mathrm{MT}(n; k_1, \ldots, k_n) = \prod_{1 \le s < t \le n} \left(\mathrm{E}_{k_s} + \mathrm{E}_{k_t}^{-1} - \mathrm{E}_{k_s} \mathrm{E}_{k_t}^{-1} \right) \mathrm{GT}(n; k_1, \ldots, k_n)$$

suggests that there is a combinatorial argument connecting Gelfand–Tsetlin patterns and monotone triangles via shift operators.

Indeed, using the following local rule (due to the inclusion-exclusion principle)

we decompose the monotone triangles of order n into a signed sum of $3^{\binom{n}{2}}$ Gelfand–Tsetlin patterns with additional inequality conditions between their entries. In most cases, we are able to associate a sum of shift operators that enumerate these arrays if applied to $\mathrm{GT}(n; k_1, \ldots, k_n)$. Let us give an example:

All the arrays of the following form with bottom row $(k_1, \ldots, k_n)$ and $k_1 < \cdots < k_n$ such that
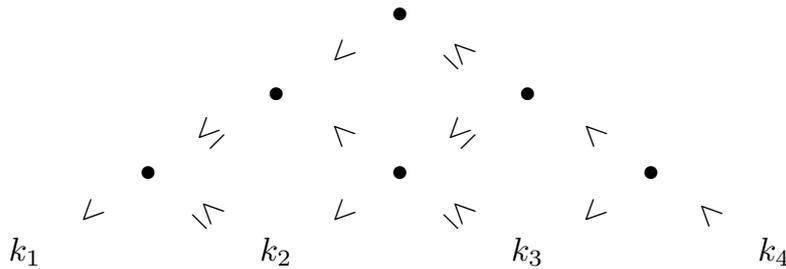
are enumerated by $\mathrm{E}_{k_1}^3 \mathrm{E}_{k_2}^2 \mathrm{E}_{k_3} \mathrm{E}_{k_4}^{-1} \mathrm{GT}(n; k_1, \ldots, k_n)$.

Although we are not able to recover the sought-after operator

$$\prod_{1 \le s < t \le n} \left(\mathrm{E}_{k_s} + \mathrm{E}_{k_t}^{-1} - \mathrm{E}_{k_s} \mathrm{E}_{k_t}^{-1} \right)$$

in this manner, we can, for instance, combinatorially prove operator formulae like

$$\mathrm{MT}(n; k_1, k_2, k_3) = \left(\mathrm{E}_{k_1} - \mathrm{E}_{k_1}^2 + \mathrm{E}_{k_3}^{-1} - \mathrm{E}_{k_3}^{-2} + \mathrm{E}_{k_1} \mathrm{E}_{k_3}^{-1} \right) \mathrm{GT}(n; k_1, k_2, k_3).$$

In fact, we know that there is not only one specific operator but rather an entire ideal of operators that solve the enumeration problem. This is a consequence of the following theorem due to Fischer (see [4, Lemma 2.5]):

$$e_r(\mathrm{E}_{k_1} - \mathrm{Id}, \ldots, \mathrm{E}_{k_n} - \mathrm{Id}) \mathrm{GT}(n; k_1, \ldots, k_n) = 0$$

for $1 \le r \le n$ where e_r denotes the rth *elementary symmetric polynomial*.

Hence, in order to obtain a combinatorial proof of the operator formula, it will be sufficient to find a Laurent polynomial P by combinatorial means such that $\mathrm{MT}(n; k_1, \ldots, k_n) = P(\mathrm{E}_{k_1}, \ldots, \mathrm{E}_{k_n}) \mathrm{GT}(n; k_1, \ldots, k_n)$. Then it must be shown that $P(\mathrm{E}_{k_1}, \ldots, \mathrm{E}_{k_n}) - \prod_{1 \le s < t \le n} \left(\mathrm{E}_{k_s} + \mathrm{E}_{k_t}^{-1} - \mathrm{E}_{k_s} \mathrm{E}_{k_t}^{-1} \right)$ lies in the ideal generated by $e_r(\mathrm{E}_{k_1} - \mathrm{Id}, \ldots, \mathrm{E}_{k_n} - \mathrm{Id})$. This project is still work in progress.

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Asymptotic enumeration of 4-regular planar graphs

Clément Requilé

(joint work with Marc Noy and Juanjo Rué)

A graph on n vertices is said to be *labelled* when its vertex set is $\{1, \ldots, n\}$, *simple* when it has no loop nor multiple edge and *planar* when it is embeddable on the plane without any edges crossing. The enumeration of labelled simple planar graphs has been recently the subject of much research; see [10] for a survey on the area. The problem of counting planar graphs was first solved by Giménez and Noy [7], while cubic planar graphs where enumerated by Bodirsky, Kang, Löffler and McDiarmid in [1] (see also [11] for an update). On the other hand, the enumeration of simpler classes of planar graphs, such as series-parallel graphs and, more generally, subcritical classes of graphs is easier and well understood [5].

One of the open problems in this area is the enumeration of labelled *4-regular* simple planar graphs. There are several references on the exhaustive generation of 4-regular planar graphs. Starting with a collection of basic graphs one shows how to generate all graphs in a certain class starting from the basic pieces and applying a sequence of local modifications. This was first done for the class of 4-regular planar graphs by Lehel [8], using as basis the graph of the octahedron. For 3-connected 4-regular planar graphs a similar generation scheme was shown by Boersma, Duijvestijn and Göbel [3]; by removing isomorphic duplicates they were able to compute the numbers of 3-connected 4-regular planar graphs up to 15 vertices. It is also the approach of the more recent work by Brinkmann, Greenberg, Greenhill, McKay, Thomas and Wollan [2] for generating planar quadrangulations of several types. The authors of [2] use several enumerative formulas to check the correctness of their generation procedure. However this does not include the class of 3-connected quadrangulations, which by duality correspond to 3-connected 4-regular planar graphs, a class for which no enumeration scheme was known until now.

In this talk, we present the first asymptotic estimate of the number of labelled 4-regular planar graphs with n vertices, as $n \to \infty$. As a biproduct, we can also enumerate 3-connected and 2-connected 4-regular planar graphs, as well as simple 4-regular planar maps. The proof is based on an estimation of the growth of the coefficients of the associated generating function $G(x) = \sum_n g_n \frac{x^n}{n!}$, where g_n is the number of 4-regular planar graphs with n vertices. To that end, we first derive a polynomial functionnal equation satisfied by the generating function $C^\bullet(x)$, counting connected 4-regular planar graphs rooted at a vertex, via a decompostion of connected graphs following their 3-connected components. From there and using classical methods from analytic combinatorics [6], we can deduce the asymptotic estimate for g_n.

The main difficulty resides in that we need to access the generating function of 3-connected 4-regular planar multigraphs with two variables: one marking simple edges and another double edges. This extra variable marking double edges is necessary to encode the symmetries inherent to the decomposition. We were able to obtain it by tracking the second variable all the way from quadrangulations, thus adapting a scheme of Mullin and Schellenberg in [9], to 4-regular planar maps. By also decomposing them following their 3-connected components, then using Whitney's unique embedding theorem [12], we could finally obtain the bivariate generating function counting 3-connected 4-regular planar multigraphs rooted at an edge.

The estimate we obtain is of the form $g_n \sim g \cdot n^{-7/2} \cdot \rho^{-n} \cdot n!$, where the exponential growth ρ is computable explicitly as the root of a univariate polynomial of degree 14. Our scheme however, does not allow us yet to compute the multiplicative constant g. For that, one would need to access the bivariate generating function counting 3-connected 4-regular planar multigraphs, either unrooted or rooted at a vertex, to then apply the dissymmetry theorem for tree-decomposable classes (see [4]).

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Random tableaux and sorting networks

Robin Sulzgruber

(joint work with Svante Linusson and Samu Potka)

Figure 1. A shifted SYT of staircase shape.

Shifted diagrams and shifted standard Young tableaux (SYT) appear frequently in combinatorics and related fields. For example, shifted diagrams correspond to strict partitions and index irreducible projective representations of the symmetric group [9]. In the theory of partially ordered sets shifted diagrams appear as an infinite family of d-complete posets [8]. Moreover, shifted diagrams are order filters in the root poset of type B_n.

Shifted SYT can be seen as linear extensions of shifted diagrams. They are enumerated by elegant product formulas, called shifted hook-length formulas, and can be sampled efficiently using hook-walks or jeu de taquin [4]. Moreover, shifted SYT are in bijection with chains of maximal length in the Tamari lattice, and play an important role in the analysis of reduced words in the Coxeter group of type B_n [5].

In joint work with S. Linusson and S. Potka [6] we prove a limit shape theorem for shifted SYT of staircase shape chosen uniformly at random. For $n \in \mathbb{N}$ let $N = \binom{n}{2}$, let Δ_n denote the shifted staircase with N cells, let X_n denote the set of shifted SYT of shape Δ_n, and let $\mathbb{P}_n$ denote the uniform probability measure on X_n.

Theorem 1. *There exists a surface* $L : [0,1] \times [0,1] \to \mathbb{R}$ *given by explicit level curves such that for all* $\varepsilon > 0$ *we have*

$$\lim_{n \to \infty} \mathbb{P}_n \left(T \in X_n : \max_{(i,j) \in \Delta_n} \left| \frac{T(i,j)}{N} - L\left(\frac{i}{n}, \frac{j}{n} \right) \right| > \varepsilon \right) = 0 \,.$$

The surface L was found by B. Pittel and D. Romik as the limit shape of SYT of square shape [7]. Our proof uses the ideas and results of [7].

(Non-shifted) SYT of staircase shape are in bijection with sorting networks (reduced words of the reverse permutation) via the Edelman–Greene correspondence [3]. The set X_n can be viewed as a subset of the set of SYT of staircase shape. The restriction of the Edelman–Greene correspondence yields a bijection between X_n and 132-avoiding sorting networks, that is, sorting networks in which all intermediate permutations are 132-avoiding.

The study of random sorting networks was initiated by O. Angel, A. Holroyd, D. Romik and B. Virág [1]. Their paper contains several intriguing conjectures on the limit shapes for intermediate permutations and trajectories. Proofs for these (and other) conjectures were recently announced by D. Dauvergne [2].

We use the Edelman–Greene bijection and the limit shape for shifted SYT of staircase shape to derive several results on random 132-avoiding sorting networks, including limit shapes for intermediate permutations and trajectories. Moreover we show that on average each row and each column of a shifted SYT of staircase shape contains precisely one adjacency (that is, entries i and $i+1$ in neighbouring cells of the tableau), and we state conjectures on the distribution of adjacencies in random sorting networks.

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Symbolic evaluation of determinants and rhombus tilings of holey hexagons

Christoph Koutschan

(joint work with Thotsaporn Thanatipanonda)

The following determinant, that counts descending plane partitions, was famously evaluated by George Andrews [1]:

$$\det_{1 \leqslant i,j \leqslant n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j - 1} \right),$$

where $\delta_{i,j}$ denotes the Kronecker delta, i.e., $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. The same determinant is also mentioned in [7, Thm. 32] (where μ was replaced by 2μ). One year later, Andrews [2, page 105] came up with a curious determinant which is a variation of the above:

$$D(n) := \det_{1 \leqslant i,j \leqslant n} \left(\delta_{i,j} + \binom{\mu + i + j - 2}{j} \right).$$

There a closed-form formula for the quotient $D(2n)/D(2n - 1)$ was conjectured. It was mentioned (and popularized) again as Problem 34 in [8, page 47], and it was proven, for the first time, in [5]. However this shows only "half" of the formula for $D(n)$. The quotient $D(2n + 1)/D(2n)$ remained mysterious, due to an increasingly large "ugly" (i.e., irreducible) polynomial factor that is always shared between two consecutive determinants. Thus the determinant $D(n)$ does not completely factor into linear polynomials, while many similar determinants do. Not fully satisfied with this situation, the first-named author made a *monstrous* conjecture [5, Conj. 6] of the full formula of $D(n)$. As our main theorem, we present a nicer formula for $D(n)$.

Theorem 1. *Let μ be an indeterminate and let ρ_k be defined as $\rho_0(a,b) = a$ and $\rho_k(a,b) = b$ for $k > 0$. If n is an odd positive integer then $D(n)$ equals*

$$\sum_{k=0}^{(n+1)/2} \rho_k\big(4(\mu - 2), \tfrac{1}{(2k-1)!}\big) \frac{(\mu - 1)_{3k-2}}{2\big(\tfrac{\mu}{2} + k - \tfrac{1}{2}\big)_{k-1}}$$

$$\times \left(\prod_{j=1}^{k-1} \frac{(\mu + 2j + 1)_{j-1}\big(\tfrac{\mu}{2} + 2j + \tfrac{1}{2}\big)_{j-1}}{(j)_{j-1}\big(\tfrac{\mu}{2} + j + \tfrac{1}{2}\big)_{j-1}} \right)^2$$

$$\times \left(\prod_{j=k}^{(n-1)/2} \frac{(\mu + 2j)_j^2\big(\tfrac{\mu}{2} + 2j - \tfrac{1}{2}\big)_j\big(\tfrac{\mu}{2} + 2j + \tfrac{3}{2}\big)_{j+1}}{(j)_j\,(j+1)_{j+1}\big(\tfrac{\mu}{2} + j + \tfrac{1}{2}\big)_j^2} \right).$$

If n is an even positive integer then $D(n)$ equals

$$\sum_{k=0}^{n/2} \rho_k\left(4(\mu-2), \tfrac{1}{(2k-1)!}\right) \frac{(\mu-1)_{3k-2}}{2\left(\frac{\mu}{2}+k-\frac{1}{2}\right)_{k-1}} \left(\prod_{j=1}^{k-1} \frac{(\mu+2j+1)_{j-1}\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_{j-1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)^2$$

$$\times \left(\prod_{j=k}^{n/2} \frac{(\mu+2j)_j\left(\frac{\mu}{2}+2j+\frac{1}{2}\right)_{j-1}}{(j)_j\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j-1}}\right)\left(\prod_{j=k}^{n/2-1} \frac{(\mu+2j)_j\left(\frac{\mu}{2}+2j+\frac{3}{2}\right)_{j+1}}{(j+1)_{j+1}\left(\frac{\mu}{2}+j+\frac{1}{2}\right)_{j}}\right).$$

The proof of Theorem 1 is based on the Desnanot-Jacobi identity, and of computer proofs of some related determinants (see Definition 2). These computer proofs follow a strategy proposed by Zeilberger [11], that he called the "holonomic ansatz". Under certain conditions, the proof of a conjectured determinant evaluation can be reduced to the proof of several holonomic function identities. Such identities can be proven automatically by employing specialized computer programs, such as the author's HolonomicFunctions package [4]. The detailed computations can be found in the electronic material [6]. We also showed that Theorem 1 it is equivalent to our monstrous conjecture.

Definition 2. *For $n, s, t \in \mathbb{Z}$, $n \geqslant 1$, and μ an indeterminate, we define $D_{s,t}(n)$ to be the following $(n \times n)$-determinant:*

$$D_{s,t}(n) := \det_{\substack{s \leqslant i < s+n \\ t \leqslant j < t+n}} \left(\delta_{ij} + \binom{\mu+i+j-2}{j}\right), \qquad n \geqslant 1.$$

Note that Andrews's determinant is a special case of $D_{s,t}(n)$, namely $D(n) = D_{1,1}(n)$. The first observation is that $D_{s,t}(n)$ can be written as a sum of minors. For example, by iterated Laplace expansion, on gets that

$$D_{s,t}(n) = \sum_{I \subseteq \{1,\ldots,n-s+t\}} (-1)^{(s-t)\cdot|I|} \det\left(M_{I+s-t}^{I}\right) \qquad (s \geqslant t),$$

where $I + x = \{i + x \mid i \in I\}$ and where M_J^I denotes the matrix that is obtained by deleting all rows with indices in I and all columns with indices in J from

$$B_{s,t}(n) := \left(\binom{\mu+i+j+s+t-4}{j+t-1}\right)_{1 \leqslant i,j \leqslant n}$$

i.e., the matrix $D_{s,t}(n)$ without the Kronecker deltas. For $t > s$ there is an analogous formula.

The second observation is that, by the Lindström–Gessel–Viennot lemma [10, 3], $\det\left(B_{s,t}(n)\right)$ counts n-tuples of non-intersecting paths in the integer lattice $\mathbb{N}^2$: the starting points are $(0,t), (0,t+1), \ldots, (0,t+n-1)$, the end points are $(\mu + s - 2, 0), \ldots, (\mu + s + n - 3, 0)$, and the allowed steps are $(1,0)$ and $(0,-1)$. Note that the number of such paths from $(0, t+j-1)$ to $(\mu + s + i - 3, 0)$ is given by $\binom{\mu+i+j+s+t-4}{j+t-1}$, which is precisely the (i,j)-entry of $B_{s,t}(n)$ (see Figure 1). Then, $\det\left(M_J^I\right)$ with $|I| = |J|$ counts the $(n - |I|)$-tuples of non-intersecting paths when the starting points with indices I and the end points with indices J are omitted.
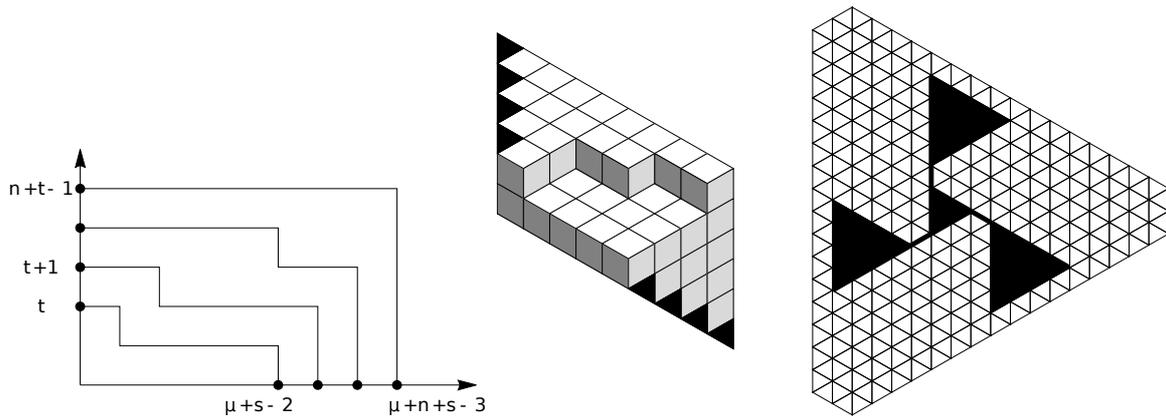
FIGURE 1. Lattice paths, rhombus tilings, and holey hexagon

In the case $s = t$, the expression $\sum_{I \subseteq \{1,\dots,n\}} \det\left(M_I^I\right)$ counts all tuples of non-intersecting paths for all subsets of starting points (and the same subset of end points). If $s > t$ then we count only tuples of non-intersecting paths which include the first $s - t$ starting points and the last $s - t$ end points (and vice versa for $t > s$).

Third, one observes that the previously described non-intersecting lattice paths are in bijection with rhombus tilings of a lozenge-shaped region, where some triangles on the border are cut out. These triangles correspond to the starting and end points. The two types of steps (right and down) correspond to two orientations of the rhombi, while rhombi of the third possible orientation fill the areas which are not covered by paths; see the middle part of Figure 1. Note that a row of small black triangles induces a larger triangular hole, because the tiling on that large triangle is uniquely determined and hence may be omitted.

By rotating this lozenge by 120° and by 240°, and by putting the three copies together in a suitable way, one obtains a region (Figure 1 right) whose cyclically symmetric rhombus tilings are counted by the determinant $D_{s,t}(n)$, provided that $s - t$ is even. In the other case the count is weighted by $+1$ and -1, according to the length of the tuples of paths. This combinatorial interpretation is due to [9].

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The Z-Dirac and massive Laplacian operators in the Z-invariant Ising model.

BÉATRICE DE TILIÈRE

We consider Baxter's Z-invariant Ising model. We prove that certain key quantities of the Ising model, *i.e.*, the partition function and probabilities of occurrence of edges in contour configurations, are explicitly expressed as a function of the Z-invariant massive Laplacian and its inverse, the massive Green function, introduced in [1]. This establishes a deep relation between classical models of statistical mechanics: the Ising model, rooted spanning forests, random walks. In proving these results, we introduce the Z-Dirac operator and relate it to the Z-massive Green function, extending to the full Z-invariant case results proved by Kenyon at criticality [2]. Proofs consist in establishing matrix relations allowing to compare matrix inverses and also, after extra combinatorial work, determinants.

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Enumeration of cyclic orders and consecutive coordinates polytopes

MATTHIEU JOSUAT-VERGÈS

(joint work with Arvind Ayyer and Sanjay Ramassamy)

Computing the volume of a lattice polytope $\mathcal{P} \subset \mathbb{R}^n$ can be seen as an enumerative problem. Here *lattice polytope* means that all vertices of $\mathcal{P}$ have integer coordinates. Indeed, we can consider a triangulation of the polytope, if possible with unit simplices all having volume $\frac{1}{n!}$, then computing the volume is done by enumerating simplices. Numerous examples of polytopes leads to such enumeration problem. A famous example is the Chan-Robbins-Yuen polytope [2].

In this work, we are interested in polytopes introduced by Stanley in an exercise of [3]. They are defined by inequalities: $B_{k,n}$ is the set of vectors $(x_i)_{1 \le i \le n} \in \mathbb{R}^n$ such that:

- $x_i \ge 0$ for $1 \le i \le n$ (coordinates are nonnegative),
- $x_i + x_{i+1} + \cdots + x_{i+k-1} \le 1$ for $1 \le i \le n+1-k$ (the sum of any k consecutive coordinates is less than 1).

It is elementary to check that the vertices of the polytopes are vectors with entries in $\{0, 1\}$, so that this is a lattice polytope.

To describe the combinatorics associated with these polytopes, we need to consider total cyclic orders on the set $\{0, 1, \ldots, n\}$. These can be seen as permutations of $\{0, 1, \ldots, n\}$ with only one cycle, or configurations of $n+1$ points labelled $\{0, 1, \ldots, n\}$ on a circle (one can find the permutation with one cycle by reading the labels clockwise). Then, we define a set $A_{k,n}$ of total cyclic orders on $\{0, 1, \ldots, n\}$ defined by the condition that for all $0 \le i \le n - k$, the tuple $(i, i+1, \ldots, i+k)$ is an oriented cycle, which means starting from i the other elements appear in this order when reading labels on the circle in a clockwise manner.

Then our fist result is that the volume of $B_{k,n}$ is $\frac{1}{n!} \# A_{k,n}$.

More generally, we are interested in refinements coming from Ehrhart theory of counting integer points in lattice polytopes [1, 3]. The Ehrhart polynomial of $B_{k,n}$ is defined, for any nonnegative integer t, by:

$$E(B_{k,n}, t) = \#\big((t \cdot B_{k,n}) \cap \mathbb{Z}^n\big)$$

where $t \cdot B_{k,n}$ is the t-times dilated polytope. It is not completely obvious from the definition that it is polynomial in t, but it is rather straight forward to see that its leading coefficient is the volume of the polytope. For example for the hypercube $[0, 1]^n$ we get $(t + 1)^n$.

The Ehrhart polynomial might have negative coefficients, and in general they are rational numbers. From the combinatorial point of view, the interesting quantity is the Ehrhart h^*-polynomial, defined as follows:

$$E^*(B_{k,n}, z) = (1 - z)^{n+1} \sum_{t \ge 0} E(B_{k,n}, t) z^t.$$

Indeed, knowing that $E(B_{k,n}, t)$ is a polynomial of degree n, the sum gives a rational function, and $(1 - z)^{n+1}$ clears the denominator so that the result is again a polynomial. It is known that the coefficients of $E^*(B_{k,n}, z)$ are nonnegative integers. So, from the combinatorial point of view it is natural to look for a refined enumeration of $A_{k,n}$ giving this polynomial. It turns out that the appropriate statistic is the number of descents. Reading clockwise the numbers in a total cyclic order, a descent is a number i followed by j with $j < i$, then we have

$$E^*(B_{k,n}, z) = \sum_{\sigma \in A_{k,n}} z^{\text{des}(\sigma)}.$$

There are a number of interesting particular cases of these polynomials:

- For $k = 1$, $B_{1,n}$ is the n-dimensional hypercube and its h^*-polynomial is the nth Eulerian polynomial.

- For $k = 2$, $B_{2,n}$ can be seen as the chain polytope (see [4]) of the so-called zigzag poset, so that its h^*-polynomial is a refinement of Euler numbers.
- For $i \geq 0$, the polytopes $B_{n+i,2n+i}$ can be seen again as chain polytopes, so that their h^*-polynomial are identified as the n-th Narayana polynomial (in particular it does not depend on i).

Just as the Eulerian and Narayana polynomials, $E^*(B_{k,n}, z)$ is a palindromic polynomial for any value of k and n, which can be proved by checking that $B_{k,n}$ is a Gorenstein polytope (the dilated and translated polytope $(k+1) \cdot B_{k,n} - (1, \ldots, 1)$ is reflexive).

In general, proving the combinatorial formula for $E^*(B_{k,n}, z)$ can be done via a triangulation of the polytopes: we decompose it as a union of unit simplices, with possible intersection only at their boundaries. Doing that in an appropriate manner permits to see that each simplex contribute to some power of z to the h^*-polynomial.

Let us finish this abstract with an open problem. As a variant of the polytope $B_{2,n}$, let us consider the polytope defined by the inequalities $0 \leq x_i \leq 1$, and either $x_i + x_{i+1} \leq 1$ or $x_i + x_{i+1} \geq 1$ according to some choice of a map $\{1, \ldots, n-1\} \to \{\pm\}$ (a sign sequence). The problem is to understand what is the combinatorics of the h^*-polynomial of these polytopes.

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Eumeration of partitions with prescribed successive rank parity blocks

AE JA YEE

(joint work with Seunghyun Seo)

A partition of a positive integer n is a weakly decreasing sequence of positive integers whose sum equals n. Let $p(n)$ be the number of partitions of n for $n \geq 1$ with $p(0) = 1$.

In 1944, F. Dyson defined the rank of a partition as the largest part minus the number of parts and then conjectured that the rank statistic would account for the following Ramanujan partition congruences combinatorially [8]:

$$p(5n + 4) \equiv 0 \pmod{5},$$
$$p(7n + 5) \equiv 0 \pmod{7}.$$

Dyson's conjecture was proved by A. O. L. Atkin and H. P. F. Swinnerton-Dyer [6]. In search of the crank statistic for

$$p(11n + 6) \equiv 0 \pmod{11},$$

whose existence was conjectured by Dyson, Atkin introduced successive ranks and showed how they would replace the rank statistic in various algebraic expressions [5].

Successive ranks of a partition are the difference of the i-th row and the i-th column in the Ferrers graph [2]. In 70's, G. E. Andrews introduced the concept of oscillations of successive ranks to obtain various interesting Rogers-Ramanujan type partition identities using partition sieves [1]. Andrews' sieve methods and results were generalized by many mathematicians including Andrews-Bressoud-Baxter-Burge-Forrester-Viennot [4], Bressoud [7], and Gessel-Krattenthaler [9].

Recently, in the study of singular overpartitions [3], Andrews revisited successive ranks and parity blocks, which are equivalent to the concept of oscillations. Motivated by the work of Andrews, we investigate partitions with prescribed successive rank parity blocks. Surprisingly, it turns out that the enumeration of such partitions involves the enumeration of a certain type of tableaux and lattice paths.

In this talk, I will discuss the following results. This is joint work with S. Seo from Kangwon National University [10].

For a partition λ, if the i-th rank is positive or nonnegative, we say it has positive or negative parity, respectively. We now divide the successive ranks of λ into parity blocks. These are sets of contiguous ranks maximally extended along the main diagonal, where all the ranks have the same parity. We shall say that a block is positive or negative if it contains positive or negative ranks, respectively.

For positive integers d and m with $d \geq m$, let $a_m^+(n; d)$ and $a_m^-(n; d)$ be the numbers of partitions of n into exactly d successive ranks and m parity blocks, where the last block is positive and negative, respectively. The our main result is as follows: For $d \geq m \geq 1$,

$$\sum_{n=1}^{\infty} a_m^+(n; d) q^n = \frac{q^{d^2 + d + \binom{m}{2}}}{(q; q)_{2d}} \frac{1 - q^m}{1 - q^d} \begin{bmatrix} 2d \\ d + m \end{bmatrix}$$

and

$$\sum_{n=1}^{\infty} a_m^-(n; d) q^n = \frac{q^{d^2 + \binom{m}{2}}}{(q; q)_{2d}} \frac{1 - q^m}{1 - q^d} \begin{bmatrix} 2d \\ d + m \end{bmatrix}$$

Here, we employ the customary q-series notation:

$$(a; q)_n := (1 - a)(1 - aq) \cdots (1 - aq^{n-1}),$$

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{cases} \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, & \text{if } n \geq k \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

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Random planar maps: peeling, slicing and layering
Grégory Miermont

In the past decade, a lot of effort has been done in trying to understand the properties of distances in various models of large random planar maps. In this talk, we survey three different approaches that are relevant to the study of geodesic distances in random planar maps with controlled degrees (in most of the talk those are assumed to be even, which simplifies many discussions). Fix a sequence of non-negative weights $\mathbf{q} = (q_1, q_2, \ldots)$ and consider the "Boltzmann measure" [23] on the set of rooted bipartite maps given by

$$w_\mathbf{q}(\mathbf{m}) = \prod_{f \in \mathrm{Faces}(\mathbf{m})} q_{\deg(f)/2} \cdot$$

We say that $\mathbf{q}$ is *admissible* if the total mass $Z(\mathbf{q})$ of $w_\mathbf{q}$ is finite, in which case a map with distribution $P_\mathbf{q} = w_\mathbf{q}/Z(\mathbf{q})$ is called a $\mathbf{q}$-Boltzmann map. We note that $Z(\mathbf{q})$, which is the generating series of bipartite maps where each face of degree $2k$ is counted by q_k, is also the partition function of the so-called 1-matrix model in random matrix theory [20, Chapter 3]. For instance, when $q_i = a\delta_{ip}$ for some $p \geq 2$ and $a > 0$, $w_\mathbf{q}$ is the measure putting mass a^n on every $2p$-angulation of the sphere with n faces.

The $\mathbf{q}$-Boltzmann maps form a rich model that is also relevant in the study of more complicated models of decorated random maps, including $O(n)$ loop models or FK percolation on random planar maps, [22, 5, 6, 7], for which the determination of the order of distances is a well-known open problem that takes its origin in work of Watabiki [28], with interesting recent progress by Gwynne, Holden and Sun [16, 17]. We note that all three approaches described below play important roles, in different guises, in the program of Miller-Sheffield to construct the Brownian map metric from Liouville quantum gravity [25, 26, 27].

The first approach ("slicing") allows one to control the usual graph distances in the map. Using a celebrated generalization by Bouttier-Di Francesco-Guitter [8] of the bijection of Cori-Vauquelin-Schaeffer [13] between maps and labeled trees, it is possible to relate geodesic distances (for the usual graph distance) in a $\mathbf{q}$-Boltzmann map M to spatial branching processes [12, 23, 18]. Depending on whether the offspring distribution of this branching process has finite variance (coined the *generic case* in [5]) or is in the domain of attraction of a stable law of parameter $\alpha \in (1, 2)$ (the *non-generic case*, with the generic case corresponding to $\alpha = 2$), the geometric properties of the map M conditioned on having n vertices are very different as $n \to \infty$. In the generic case, such a large map with graph distances renormalized by $n^{1/4}$ converges in distribution in the Gromov-Hausdorff topology to the so-called Brownian map [24, 21], while in the non-generic case, the typical distances are of order $n^{1/2\alpha}$ and one-parameter family of limiting "stable maps" arises [22]. The slicing approach takes its name from the fact the Bouttier-Di Francesco-Guitter bijection naturally enumerates *slices*, that are maps with a boundary made of two geodesic paths of equal length. It was noted by Bouttier-Guitter [9] that slices play an important role in the enumeration of planar maps with a boundary, an idea that was used in [4] to show scaling limit results for generic $\mathbf{q}$-Boltzmann maps with a boundary of size 2ℓ, with distances renormalized by $\sqrt{\ell}$, with a limiting metric space called the Brownian disk.

The second approach ("peeling") is more directly inspired from the Tutte's decomposition (a.k.a. Schwinger-Dyson equation in random matrix theory). It was introduced by [29, 2], but one had to wait for the more recent work [10, 1, 11, 15] to realize that peeling process can be mathematically used to get a very precise estimation of the distances, but in the *dual* $\mathbf{q}$-Boltzmann maps (equivalently, of the usual graph distance in a model of random maps with controlled vertex degrees). This led in particular Bertoin, Budd, Kortchemski and Curien [11, 3] to show that dual graph distances in a $\mathbf{q}$-Boltzmann map with an order of n vertices converging to a stable map with exponent $\alpha \in (3/2, 2]$ are of order $n^{1-3/2\alpha}$, while distances shrink to subpolynomial orders for $\alpha \in (1, 3/2]$. In fact, in this situation, it is more natural to study maps with a boundary, and one should choose the perimeter of order $n^{1/\alpha}$ to get a map with an order of n vertices.

In the generic case $\alpha = 2$, one sees that distances are of similar order both for usual and dual graph distances, and it is natural to expect that the rescaled distanes converge in both cases to the Brownian map. This was indeed proved by Curien and Le Gall [14] (along with other types of local modifications of distances, including first-passage percolation) in the case of uniform triangulations of the sphere. The proof builds on the third approach ("layering"), which was initially due to Krikun [19], and decomposes bi-pointed map into a sequence of independent layers that can be encoded with a branching process whose offspring distribution is in the domain of attraction of a stable random variable with exponent $3/2$. A drawback of this approach is that it seems to apply well only for specific models where the face degrees are small.

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Geometry of large Boltzmann outerplanar maps

BENEDIKT STUFLER

(joint work with Sigurður Örn Stefánsson)
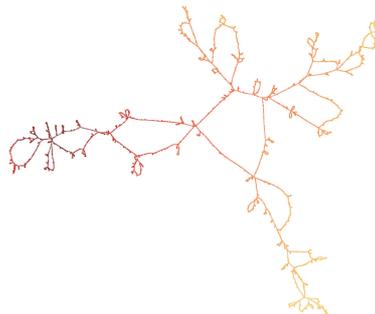
FIGURE 1. A simulation of the $\alpha = 1.25$ stable looptree.

We study the phase diagram of random outerplanar maps sampled according to non-negative Boltzmann weights that are assigned to each face of a map. We prove that for certain choices of weights the map looks like a rescaled version of its boundary when its number of vertices tends to infinity. The Boltzmann outerplanar maps are then shown to converge in the Gromov-Hausdorff sense towards the α-stable looptree introduced by Curien and Kortchemski [1], with the parameter α depending on the specific weight-sequence. See Figure 1 for a simulation of this random object. This allows us to describe the transition of the asymptotic geometric shape from a deterministic circle to the Brownian tree [2].

In ongoing joint work with Delphin Sénizergues and Sigurður Örn Stefánsson we construct novel limit objects that are created by blowing up the branch points of the stable trees into arbitrary spaces. The universality class of these objects include Boltzmann outerplanar maps that are equipped with certain block weights.

Boltzmann outerplanar maps may also be studied from a local point of view and a complete classification of Benjamini–Schramm limits was given in [3].

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An update on matters discussed at Oberwolfach by Ian Macdonald in May 1977 and David Robbins in May 1982

ROGER BEHREND

Oberwolfach Workshop 1820 on *Enumerative Combinatorics*, at which this talk was given, ran from 13 to 19 May 2018, almost exactly 41 years after Oberwolfach Workshop 7719 on *Kombinatorik*, which ran from 8 to 14 May 1977, and almost exactly 36 years after Oberwolfach Workshop 8219 on *Kombinatorik*, which ran from 9 to 15 May 1982. At the May 1977 workshop, a talk was given by Ian Macdonald, with the title *Plane partitions*, and at the May 1982 workshop, two talks were given by David Robbins, with titles *Proof of the Macdonald conjecture* and *Alternating sign matrices and descending plane partitions*. The essential contents of these talks can inferred from several sources, including the workshop reports and handwritten abstracts in the Oberwolfach Digital Archive [11, 12], citations to the talks (in particular, those by Andrews [1, 2] for the talk by Macdonald), papers on which the talks were based (in particular, those by Mills, Robbins and Rumsey [8, 9] for the talks by Robbins), and first-hand written accounts of the talks (such as those by Bressoud [4] and Zeilberger [14] for the talks by Robbins).

It is clear that the talks included detailed discussions of *cyclically symmetric plane partitions* (CSPPs), *descending plane partitions* (DPPs) and *alternating sign matrices* (ASMs). An n-CSPP is a plane partition whose 3-dimensional Ferrers diagram is contained in an $n \times n \times n$ box and is invariant under cyclic rotations of coordinates. For example, there are five 2-CSPPs: $\emptyset$, (1), $\left(\begin{smallmatrix} 2 & 1 \\ 1 & \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & 2 \\ 2 & 2 \end{smallmatrix}\right)$. An n-DPP is a column strict shifted plane partition in which the largest part is at most n, and the first part of any row is larger than the number of parts in that row but (except for the first row) no larger than the number of parts in the row above. For example, there are seven 3-DPPs: $\emptyset$, (2), (3), $(3\ 1)$, $(3\ 2)$, $(3\ 3)$, and $\left(\begin{smallmatrix} 3 & 3 \\ & 2 \end{smallmatrix}\right)$. An n-ASM is an $n \times n$ matrix for which each entry is 0, 1 or -1, the sum of entries in each row and column is 1, and the nonzero entries alternate in sign along each row and column. For example, there are seven

3-ASMs: $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ and

$\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Note that every $n \times n$ permutation matrix is an n-ASM.

In Macdonald's May 1977 talk, he conjectured that the generating function for CSPPs is given by a simple product formula involving q-numbers and q-factorials:

$$
(1) \qquad \sum_{n\text{-CSPPs } C} q^{\sum_{ij} C_{ij}} \;=\; \prod_{i=0}^{n-1} \frac{[3i+2]_q \, [3i+1]_{q^3}!}{[3i+1]_q \, [n+i]_{q^3}!}.
$$

The $q = 1$ case of Macdonald's conjecture was proved in [1, 2] by Andrews. In [1, 2], Andrews also introduced DPPs, conjectured that their generating function is given by a product formula similar to that for CSPPs, i.e.,

$$
(2) \qquad \sum_{n\text{-DPPs } D} q^{\sum_{ij} D_{ij}} \;=\; \prod_{i=0}^{n-1} \frac{[3i+1]_q!}{[n+i]_q!},
$$

and proved the $q = 1$ case of this conjecture. Furthermore, in [1, 2], Andrews defined (in a slightly different form) an (n, k)-*column strict shifted plane partition* (CSSPP) to be a column strict shifted plane partition in which the largest part is at most $n + k$, and the first part of any row exceeds the number of parts in that row by exactly k, and showed that

$$
\begin{aligned}
(3) \quad (\# \text{ of } (n, k)\text{-CSSPPs}) \;=\; 2 \prod_{i=1}^{\lfloor (n-1)/2 \rfloor} & \frac{(2i + k + 2)_i \, (2i + (k+3)/2)_{i-1}}{(i)_i \, (i + (k+3)/2)_{i-1}} \\
& \times \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{(2i + k)_{i-1} \, (2i + (k+1)/2)_i}{(i)_i \, (i + (k+1)/2)_{i-1}},
\end{aligned}
$$

where the Pochhammer symbol has been used. The definition of CSSPPs was motivated by the facts that $(n, 0)$-CSSPPs are in simple bijection with n-CSPPs, and that $(n-1, 2)$-CSSPPs are in simple bijection with n-DPPs. Accordingly, the $q = 1$ case of (1) is the $k = 0$ case of (3), and the $q = 1$ case of (2) is the $k = 2$ case, with n replaced by $n - 1$, of (3). It was subsequently shown by Ciucu and Krattenthaler [5] that (n, k)-CSSPPs are in simple bijection with cyclically symmetric rhombus tilings of a hexagon with alternating sides of lengths n and $n + k$, and a central equilateral triangular hole of side length k.

The general cases (i.e., with q arbitrary) of (1) and (2) were proved in [8] by Mills, Robbins and Rumsey, with the proofs being discussed by Robbins in his first May 1982 talk. ASMs were introduced, and various conjectures for their enumeration were made, by Mills, Robbins and Rumsey in [8, 9], with these conjectures being discussed by Robbins in his second May 1982 talk. In particular, it was conjectured that

$$
(4) \qquad\qquad (\# \text{ of } n\text{-ASMs}) \;=\; (\# \text{ of } n\text{-DPPs}),
$$

and that a refinement of this equality, involving three ASM statistics and three DPP statistics, also holds.

In my talk, I outlined the matters above, and provided some updates (including a description of current work in progress), as follows. The conjecture (4) was first proved in 1996 by Zeilberger [13]. Shortly thereafter, a different proof was obtained by Kuperberg [7]. The three-statistic refinement of (4) was proved in 2012 by myself, Di Francesco and Zinn-Justin [3]. All of these proofs are nonbijective.

Despite the confirmation of the validity of (4), several difficult problems remain unresolved. The most important of these is probably that of finding a bijective proof of (4). Stanley [10, Prob. 226] described this (together with associated problems) as *"one of the most intriguing open problems in the area of bijective proofs"*, while Krattenthaler [6] (in a Festschrift for Stanley) wrote that *"the greatest, still unsolved, mystery concerns the question of what plane partitions have to do with ASMs"*.

Another open problem is that of finding a natural definition of (n,k)-ASMs, such that $(n-1,2)$-ASMs are the same as, or in simple bijection with, n-ASMs, and there is equality between the numbers of (n,k)-ASMs and (n,k)-CSSPPs. Hence, the case $k=2$ would be the equality (4). Some work related to this problem is currently being done by myself and Ilse Fischer. In particular, we have defined an (n,k)-*alternating sign trapezoid* (AST) to be an array of $n(n+k)$ entries arranged in n vertically-centred rows of lengths $2n+k-1$, $2n+k-3$, ..., $k+3$, $k+1$ (from top to bottom) such that, for $k \geq 1$: (i) Each entry is 0, 1 or -1; (ii) Along each row, the nonzero entries alternate in sign; (iii) The sum of entries in each row is 1; (iv) Moving down each column, the nonzero entries (if there are any) alternate in sign, starting with a 1; and (v) The sum of entries in columns $n+1$, ..., $n+k-1$ is 0. For $k=0$, (i), (ii) & (iv) should be satisfied, together with: (iii$_0$) The sum of entries in each row except row n is 1, while row n consists of 0 or 1. Our main results regarding ASTs, which are proved nonbijectively, are that

$$(5) \qquad (\# \text{ of } (n,k)\text{-ASTs}) \; = \; (\# \text{ of } (n,k)\text{-CSSPPs}),$$

and that a refinement of this equality, involving certain AST and CSSPP statistics, also holds. However, the problem stated above has not been completely solved, since we do not currently have a bijection between $(n-1,2)$-ASTs and n-ASMs.

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Enumeration of locally restricted compositions using de Bruijn graph and covering graph

Zhicheng Gao

(joint work with Andrew MacFie)

Let $(\Gamma, +)$ be a finite group. An m-composition over Γ is an m-tuple $(g_1, g_2, \ldots, g_m)$ over Γ. It is called an m-composition of g if $\sum_{j=1}^{m} g_j = g$. A family of compositions over Γ is called *locally restricted* if there is a positive integer σ such that any σ consecutive terms in a composition satisfy certain restrictions. Locally restricted compositions over Γ can be defined using walks in a de Bruijn graph. The de Bruijn graph over Γ with span σ, denoted by $B(\Gamma; \sigma)$, is a digraph whose vertices are σ-tuples such that there is an arc from $\mathbf{u} := (\mathbf{u}(1), \mathbf{u}(2), \ldots, \mathbf{u}(\sigma))$ to $\mathbf{v} := (\mathbf{v}(1), \mathbf{v}(2), \ldots, \mathbf{v}(\sigma))$ if $\mathbf{v}(j) = \mathbf{u}(j+1)$, $1 \leq j \leq \sigma - 1$. Let D be a subgraph of $B(\Gamma; \sigma)$. We associate with each directed walk $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $B(\Gamma; \sigma)$ a composition $\mathbf{c} = (\mathbf{v}_1(1), \ldots, \mathbf{v}_1(\sigma), \mathbf{v}_2(\sigma), \mathbf{v}_k(\sigma))$. That is, $\mathbf{c}$ is obtained from the walk by appending the last components of the subsequent vertices in the walk to the initial vertex of the walk. We denote this set of compositions by $\mathscr{C}(D)$. To keep track of the net sum of a composition in $\mathscr{C}(D)$, we make use of the derived graph of the voltage graph (D, α), where the voltage of the arc $(\mathbf{u}, \mathbf{v})$ is given by $\alpha(\mathbf{u}, \mathbf{v}) = \mathbf{v}(\sigma)$. Let D' denote the derived graph of (D, α). That is, the vertex set of D' is $V(D) \times \Gamma$, and there is an arc from $(\mathbf{u}, g)$ to $(\mathbf{v}, h)$ if and only if $(\mathbf{u}, \mathbf{v})$ is an arc in D and $h = g + \mathbf{v}(\sigma)$. Let $\mathscr{S}$ be the set of vertices in D' such that the second component is equal to the sum of the parts of the first component. It is easy to see that, for $m \geq \sigma$, an m-composition of g in $\mathscr{C}(D)$ corresponds to a walk in D' from $\mathscr{S}$ to a vertex whose second component is g. Fix an ordering of the vertices of D' and let T denote the corresponding transfer matrix of D'. That is, $T(i, j)$ is equal to 1 if there is an arc from $\mathbf{v}_i$ to $\mathbf{v}_j$, and zero otherwise.

Let $\vec{s}$ denote the $\{0, 1\}$ row vector such that its ith component is equal to 1 if and only if the corresponding vertex belongs to $\mathscr{S}$. Let $\vec{f_g}$ denote the $\{0, 1\}$ column vector such that its jth component is equal to 1 if and only if the corresponding vertex is of the form $(*, g)$. The following proposition is immediate.

Proposition 1. *For $m \geq \sigma$ and $g \in \Gamma$, the number of m-compositions of g in $\mathscr{C}(D)$ is equal to $\vec{s} M^{m-\sigma} \vec{f_g}$.*

Our main results are

Theorem 2. *Suppose D' is strongly connected and aperiodic. Then the number of m-compositions of g is equal to, as $m \to \infty$, $A \times B^m \left(1 + O(\exp(-\delta m))\right)$ for some positive constants A, B, and δ, which are independent of g.*

In this talk, we present some asymptotic results for the number of m-compositions, as $m \to \infty$, associated with some digraphs D and some finite group Γ. It will also be shown that the distribution of the number of occurrences of a given subword in a random locally restricted m-composition is asymptotically normal with mean and variance proportional to m.

The basic tools for deriving these results are covering graphs of de Bruijn graphs, Perron-Frobenius theorem and transfer matrix method. These results extend previous results on compositions over a finite abelian group which were obtained by counting compositions over integers and using the multisection formula.

Lecture hall tableaux
Sylvie Corteel, Jang Soo Kim

Lecture hall partitions are partitions satisfying certain conditions introduced by Bousquet-Mélou and Eriksson [1, 2]. Anti-lecture hall compositions are compositions satisfying similar conditions. Lecture hall partitions and anti-lecture hall compositions have been studied extensively in the last two decades. See the recent survey written by Savage [3]. In this talk we show that these objects are closely related to the little q-Jacobi polynomials $p_n^L(x; a, b; q)$.

For monic univariate orthogonal polynomials $p_n(x)$, the *mixed moment* $\mu_{n,k}$ and the *(normalized) moment* μ_n are defined by

$$x^n = \sum_{k=0}^{n} \mu_{n,k} p_k(x), \qquad p_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$$

In this talk we show that the mixed moments and the dual mixed moments of the little q-Jacobi polynomials are generating functions for anti-lecture hall compositions and lecture hall partitions respectively. We then extend this result to the multivariate little q-Jacobi polynomials.

Let $\mathcal{P}_n$ denote the set of partitions with at most n parts. In many cases, a family $\{p_n(x)\}_{n \geq 0}$ of univariate orthogonal polynomials generalizes naturally to a family $\{p_\lambda(x_1, \ldots, x_n)\}_{\lambda \in \mathcal{P}_n}$ of multivariate orthogonal polynomials via

$$(1) \qquad p_\lambda(x_1, \ldots, x_n) = \frac{\det\left(p_{\lambda_j + n - j}(x_i)\right)_{i,j=1}^{n}}{\Delta(x)},$$

where

$$\Delta(x) = \Delta(x_1, \ldots, x_n) = \prod_{1 \leq i < j \leq n} (x_i - x_j).$$

			9	4	3
	5	6	4	3	1
2	2	1	0		
1	0	0			

			$\frac{9}{8}$	$\frac{4}{9}$	$\frac{3}{10}$
	$\frac{5}{5}$	$\frac{6}{6}$	$\frac{4}{7}$	$\frac{3}{8}$	$\frac{1}{9}$
$\frac{2}{3}$	$\frac{2}{4}$	$\frac{1}{5}$	$\frac{0}{6}$		
$\frac{1}{2}$	$\frac{0}{3}$	$\frac{0}{4}$			
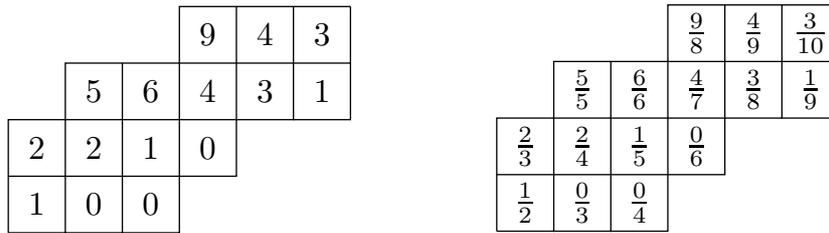
FIGURE 1. On the left is a lecture hall tableau $T \in \mathrm{LHT}_{(n,\geq,>)}(\lambda/\mu)$ for $n = 5$, $\lambda = (6,6,4,3)$ and $\mu = (3,1)$. The diagram on the right shows the number $T(i,j)/(n + c(i,j))$ for each entry $(i,j) \in \lambda/\mu$.

Considering $s_\lambda(x_1, \ldots, x_n)$ as a multivariate analog of x^i, we define the *mixed moment* $M_{\lambda,\mu}(n)$ and the *dual mixed moment* $N_{\lambda,\mu}(n)$ of $\{p_\lambda(x_1, \ldots, x_n)\}_{\lambda \in \mathcal{P}_n}$ by

$$s_\lambda(x_1, \ldots, x_n) = \sum_{\mu \in \mathcal{P}_n} M_{\lambda,\mu}(n) p_\mu(x_1, \ldots, x_n),$$

$$p_\lambda(x_1, \ldots, x_n) = \sum_{\mu \in \mathcal{P}_n} N_{\lambda,\mu}(n) s_\mu(x_1, \ldots, x_n).$$

The *multivariate little q-Jacobi polynomials* $p_\lambda^L(x_1, \ldots, x_n; a, b; q)$ are defined by the equation (1) using $p_n^L(x; a, b; q)$. It is known that $p_\lambda^L(x_1, \ldots, x_n; a, b; q)$ are multivariate orthogonal polynomials with explicit linear functional $\mathfrak{L}^L$ related to the q-Selberg integral. Therefore, we can consider their mixed moments $M_{\lambda,\mu}^L(n; a, b)$ and the dual mixed moments $N_{\lambda,\mu}^L(n; a, b)$. In this talk we give a combinatorial interpretation for these quantities using new combinatorial objects called lecture hall tableaux.

For an integer n and partitions λ and μ with $\mu \subseteq \lambda$ and $\ell(\lambda) \leq n$, a *lecture hall tableau* of shape λ/μ and of type $(n, \geq, >)$ is a filling T of the cells in the Young diagram λ/μ with nonnegative integers satisfying the following conditions:

$$\frac{T(i,j)}{n + c(i,j)} \geq \frac{T(i,j+1)}{n + c(i,j+1)}, \qquad \frac{T(i,j)}{n + c(i,j)} > \frac{T(i+1,j)}{n + c(i+1,j)},$$

where $c(i,j) = j - i$. We denote by $\mathrm{LHT}_{(n,\geq,>)}(\lambda/\mu)$ the set of such fillings and by $\mathrm{LHT}_{(n,<,\leq)}(\lambda/\mu)$ the set of fillings where the inequalities are changed to $<$ and $\leq$ respectively. See Figure 1 for an example of a lecture hall tableau.

Consider a sequence $\vec{x} = (x_0, x_1, \ldots)$ of variables. For T in $LS_{\lambda/\mu}^{(n,\geq,>)}$ or $LS_{\lambda/\mu}^{(n,<,\leq)}$, the *weight* $\mathrm{wt}(T)$ is defined by

$$\mathrm{wt}(T) = \prod_{s \in \lambda/\mu} x_{T(s)} u^{\lfloor T(s)/(n+c(s)) \rfloor} v^{o(\lfloor T(s)/(n+c(s)) \rfloor)},$$

where $o(m)$ is 1 if m is odd and 0 otherwise. For example, if T is the lecture hall tableau in Figure 1, its weight is

$$\mathrm{wt}(T) = x_0^3 x_1^3 x_2^2 x_3^2 x_4^2 x_5 x_6 x_9 u^3 v^3.$$

We define the *lecture hall Schur functions* of shape λ/μ and of types $(n, \geq, >)$ and $(n, <, \leq)$ by

$$LS^{(n,\geq,>)}_{\lambda/\mu}(\vec{x}; u, v) = \sum_{T \in \mathrm{LHT}_{(n,\geq,>)}(\lambda/\mu)} \mathrm{wt}(T),$$

$$LS^{(n,<,\leq)}_{\lambda/\mu}(\vec{x}; u, v) = \sum_{T \in \mathrm{LHT}_{(n,<,\leq)}(\lambda/\mu)} \mathrm{wt}(T).$$

These lecture hall Schur functions become the usual Schur functions when $n \to \infty$:

$$\lim_{n \to \infty} LS^{(n,\geq,>)}_{\lambda/\mu}(\vec{x}; u, v) = s_{\lambda/\mu}(\vec{x}), \qquad \lim_{n \to \infty} LS^{(n,<,\leq)}_{\lambda/\mu}(\vec{x}; u, v) = s_{\lambda'/\mu'}(\vec{x}),$$

where λ' is the conjugate of λ. We show that they also have Jacobi-Trudi type formulas.

Let $\vec{q} = (1, q, q^2, \dots)$ be the principal specialization of $\vec{x} = (x_0, x_1, x_2, \dots)$. In this talk we show that the mixed moments $M^L_{\lambda,\mu}(n; a, b; q)$ and the dual mixed moments $N^L_{\lambda,\mu}(n; a, b; q)$ for the multivariate little q-Jacobi polynomials $p^L_\lambda(x_1, \dots, x_n; a, b; q)$ are generating functions for lecture hall tableaux.

Theorem 1. *We have*

$$N^L_{\lambda,\mu}(n; -uv, -u/v; q) = (-1)^{|\lambda/\mu|} LS^{(n,<,\leq)}_{\lambda/\mu}(\vec{q}; u, v),$$

$$M^L_{\lambda,\mu}(n; -uv, -u/v; q) = LS^{(n,\geq,>)}_{\lambda/\mu}(\vec{q}; u, v).$$

Equivalently,

$$p^L_\lambda(x_1, \dots, x_n; -uv, -u/v; q) = \sum_{\mu \subseteq \lambda} (-1)^{|\lambda/\mu|} LS^{(n,<,\leq)}_{\lambda/\mu}(\vec{q}; u, v) s_\mu(x_1, \dots, x_n),$$

$$s_\lambda(x_1, \dots, x_n) = \sum_{\mu \subseteq \lambda} LS^{(n,\geq,>)}_{\lambda/\mu}(\vec{q}; u, v) p^L_\mu(x_1, \dots, x_n; -uv, -u/v; q).$$

Note that the *moments* $M^L_\lambda(n; a, b; q) := M^L_{\lambda,\emptyset}(n; a, b; q)$ and the *dual moments* $N^L_\lambda(n; a, b; q) := N^L_{\lambda,\emptyset}(n; a, b; q)$ are the generating functions for lecture hall tableaux of a normal shape $\lambda = \lambda/\emptyset$. We prove the following theorem, which shows that the moments and the dual moments have product formulas.

Theorem 2. *Given an integer n and a partition λ into at most n parts,*

$$LS^{(n,\geq,>)}_\lambda(\vec{q}; u, v) = \prod_{1 \leq i < j \leq n} \frac{q^{\lambda_j + n - j} - q^{\lambda_i + n - i}}{q^{i-1} - q^{j-1}} \prod_{i=1}^n \frac{(-uvq^{n-i+1}; q)_{\lambda_i}}{(u^2 q^{2n-i+1}; q)_{\lambda_i}},$$

$$LS^{(n,<,\leq)}_\lambda(\vec{q}; u, v)$$

$$= q^{n(\lambda) - n(\lambda')} \prod_{1 \leq i < j \leq n} \frac{q^{\lambda_j + n - j} - q^{\lambda_i + n - i}}{q^{i-1} - q^{j-1}} \prod_{i=1}^n \frac{(-uvq^{n-i+1}; q)_{\lambda_i}}{(u^2 q^{n-i+1+\lambda_i}; q)_{n-i+\lambda_i}}$$

$$\times \prod_{1 \leq i < j \leq n} (1 - u^2 q^{2n + \lambda_i + \lambda_j - i - j + 1}),$$

where $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ and $n(\lambda) = \sum_{i=1}^{\ell(\lambda)}(i-1)\lambda_i$.

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Slice decomposition of bicolored planar maps

JÉRÉMIE BOUTTIER

(joint work with Marie Albenque)
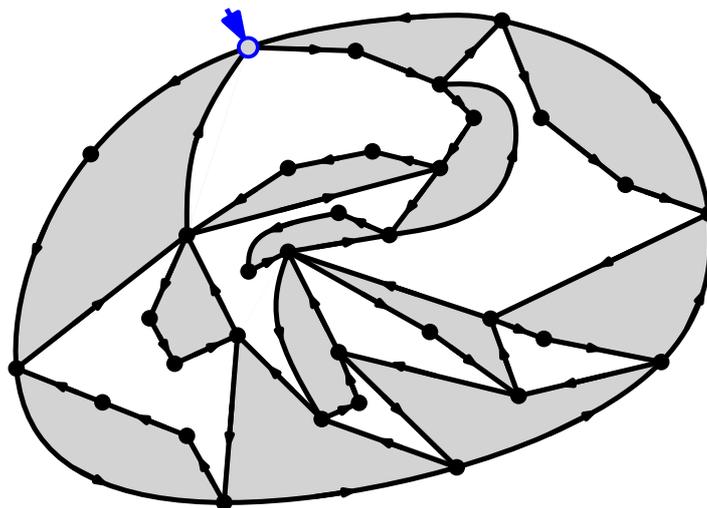
FIGURE 1. A rooted bicolored planar map with a white outer face.

A *rooted planar map* is a connected graph drawn in the plane without edge crossings, considered up to continuous deformation, and with a distinguished corner incident to the *outer face*. In this talk we consider Eulerian planar maps: all vertices have even degrees, or equivalently the faces may be colored in black and white in such a way that adjacent faces have opposite colors. If we fix the color of the outer face, then the colors of the all other faces are uniquely determined. We call *bicolored map* an Eulerian rooted planar map endowed with such a coloring.

To a bicolored map M, we assign a Boltzmann weight

$$(1) \qquad\qquad w(M) = t^{\#\text{vertices}} \prod_{f \text{ inner face}} t_{\text{degree}(f)}^{\text{color}(f)}$$

which depends on countably many parameters $t, t_1^\circ, t_1^\bullet, t_2^\circ, t_2^\bullet, \ldots$. For $p \geq 1$, we then define the generating function F_p° (resp. $F_p^\bullet$) as the sum of the weights of

all maps whose outer face is white (resp. black) and of degree p. By convention $F_0^\circ = F_0^\bullet = t$.

By specializing the variables we obtain several cases of interest, such as Eulerian triangulations (take $t_k^\bullet = t_k^\circ = 1$ for $k = 3$ and $= 0$ otherwise), the Ising model (as obtained by allowing bivalent faces and "collapsing" them), etc. The general case is known as the two-matrix model in theoretical physics, and was studied by several authors including Kazakov, Douglas, Eynard... In enumerative combinatorics, it was considered in 2002 by Bousquet-Mélou and Schaeffer (who treated the cases $p = 1, 2$ using a bijection with so-called blossom trees), then in 2004 by Di Francesco, Guitter and myself (we introduced another bijection with so-called mobiles). More recently, Bernardi and Fusy introduced in 2014 a generalized bijection that also handle girth constraints.

In the present work in progress with Marie Albenque [1], our purpose is to give a combinatorial (bijective) interpretation to some intriguing formulas for $F_p^{\circ/\bullet}$ which were obtained by Eynard and his collaborators. A good reference for these formulas is the last chapter of Eynard's book [2]. Let me state one of the fundamental results, which require to introduce some notations. We assume that the degrees are bounded (i.e. we let $t_k^\circ = 0$ for $k > d_\circ$ and $t_k^\bullet = 0$ for $k > d_\bullet$) and we introduce the quantities

$$(2) \qquad Y(x) = \sum_{k=1}^{d_\circ} t_k^\circ x^{k-1} + \sum_{p \geq 0} \frac{F_p^\circ}{x^{p+1}}, \qquad X(y) = \sum_{k=1}^{d_\bullet} t_k^\bullet y^{k-1} + \sum_{p \geq 0} \frac{F_p^\bullet}{y^{p+1}}$$

which are formal Laurent series in $1/x$ and $1/y$ respectively.

Theorem 1. *There exists a polynomial $E(x, y)$ such that*

$$(3) \qquad\qquad\qquad E(x, Y(x)) = E(X(y), y) = 0.$$

Furthermore the algebraic curve $\{E(x, y) = 0\}$ admits a rational parametrization of the form

$$(4) \qquad \begin{cases} x(z) = a_{-1}z + a_0 + a_1 z^{-1} + \cdots + a_m z^{-m}, & m := d_\bullet - 1, \\ y(z) = z^{-1} + b_0 + b_1 z + \cdots + b_n z^n, & n := d_\circ - 1. \end{cases}$$

where the coefficients a_k, b_k are series in $t, t_1^\bullet, t_1^\circ, t_2^\bullet, t_2^\circ, \ldots$ satisfying

$$(5) \qquad \begin{aligned} a_k &= t\delta_{k,-1} + \sum_{\ell=0}^{d_\bullet - 1} t_{\ell+1}^\bullet [z^{-k}] y(z)^\ell \qquad (k \geq -1), \\ b_k &= \sum_{\ell=0}^{d_\circ - 1} t_{\ell+1}^\circ [z^k] x(z)^\ell \qquad (k \geq 0). \end{aligned}$$

In other words we have $Y(x(z)) = y(z)$ and $X(y(z)) = x(z)$, viewing the left-hand sides as appropriate substitutions of Laurent series. These relations encode the solution to our counting problem in the following way: note that $x(z)$ admits a unique compositional inverse $z(x)$ (which is a Laurent series in $1/x$). Then, we have $Y(x) = y(z(x))$. The relations (5) simply tell that the leading terms of $Y(x)$

have their prescribed value as in (2). Then, by pushing the expansion further, the following terms yield the generating functions F_p° for $p \geq 1$. We obtain $F_p^\bullet$ by reversing the roles of x and y.

Let me point out that a combinatorial interpretation for the series a_k and b_k determined by (5) was already known: they match precisely the generating functions for blossom trees or mobiles mentioned above. What is missing is a combinatorial understanding of the substitution formula $Y(x) = y(z(x))$. For this we generalize to the bicolored setting the approach of *slice decomposition*, introduced by Guitter and myself in 2012 in the uncolored case.

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Exact Enumeration of Planar Eulerian Orientations

Andrew Elvey Price

(joint work with Mireille Bousquet-Mélou)

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In 2016, Bonichon, Bousquet-Mélou, Dorbec and Pennarun posed the problem of enumerating planar rooted Eulerian orientations with a given number of edges [2]. That is, rooted planar maps with directed edges such that each vertex has equal in and out degree. They also posed the problem in the quartic case, in which each vertex is restricted to having degree exactly 4. In physics, this problem is known as the ice model on a random lattice [5, 8]. In 2017, E. and Guttmann found a structural decomposition for each problem from which they obtained an elaborate system of functional equations to characterise the ordinary generating functions $\mathsf{G}(t)$, for Eulerian orientations, and $\mathsf{Q}(t)$, for quartic Eulerian orientations [4]. Subsequently, they computed around 100 coefficients of each series. Their analysis of these terms, using the method of differential approximants, led them to conjecture that the growth constants for the two problems are 4π and $4\pi\sqrt{3}$, respectively. More specifically, they predicted that the coefficients $g_n = [t^n]\mathsf{G}(t)$ behave like

$$g_n \sim \kappa_g \frac{(4\pi)^n}{n^2(\log n)^2},$$

while the coefficients $q_n = [t^n]\mathsf{Q}(t)$ behave like

$$q_n \sim \kappa_q \frac{(4\sqrt{3}\pi)^n}{n^2(\log n)^2},$$

where κ_g and κ_q are constants. For q_n, this asymptotic form was earlier predicted by Kostov and Zinn-Justin in the mathematical physics literature [5, 8].

In this work we prove the following two theorems.

Theorem 1. *Let* $\mathsf{R}(t) \equiv \mathsf{R}$ *be the unique formal power series with constant term* 0 *satisfying*

$$t = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} \binom{3n}{n} \mathsf{R}^{n+1}.$$

Then the generating function of quartic rooted planar Eulerian orientations, counted by vertices, is

$$\mathsf{Q}(t) = \frac{1}{3t^2} \left(t - 3t^2 - \mathsf{R}(t) \right).$$

Theorem 2. *Let* $\mathsf{S}(t) \equiv \mathsf{S}$ *be the unique formal power series with constant term* 0 *satisfying*

$$t = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n}^2 \mathsf{S}^{n+1}.$$

Then the generating function of rooted planar Eulerian orientations, counted by edges, is

$$\mathsf{G}(t) = \frac{1}{4t^2} \left(t - 2t^2 - \mathsf{S}(t) \right).$$

This work was initiated when we conjectured the above Theorems by searching for an exact form of each series which would match the asymptotic forms predicted by E. and Guttmann.

To prove the Theorems, we find simpler systems of functional equations to charaterise the series $\mathsf{G}(t)$ and $\mathsf{Q}(t)$. The system charaterising $\mathsf{Q}(t)$ is shown below

$$\mathsf{Q}(t) = [y]\mathsf{P}(t, y) - 1$$

$$\mathsf{P}(t, y) = \frac{1}{y}[x^1]\mathsf{C}(t, x, y)$$

$$\mathsf{D}(t, x, y) = \frac{1}{1 - \mathsf{C}\left(t, \frac{1}{1-x}, y\right)}$$

$$\mathsf{D}(t, x, y) = 1 + y\mathsf{D}(t, x, y)[y^1]\mathsf{D}(t, x, y) + y[x^{\geq 0}]\frac{1}{x}\mathsf{P}\left(t, \frac{1}{x}\right)\mathsf{D}(t, x, y)$$

$$[y^1]\mathsf{D}(t, x, y) = \frac{1}{1 - x}\left(1 + 2t[y^2]\mathsf{D}(t, x, y) - t([y^1]\mathsf{D}(t, x, y))^2\right).$$

We solve these equations by guessing the exact form of each series, then verifying that these guesses are consistent with the equations. In each case, the series involved are all D-algebraic.

For the problem of general planar rooted Eulerian orientations, our solution utilises a beautiful bijection of Ambjørn and Budd [1] and Miermont [6] between these orientations (counted by edges) and a subclass of quartic Eulerian orientations (counted by vertices). This bijection is a generalisation of a bijection of Schaeffer [7]. Finally, we observe that general Eulerian orientations are equinumerous with certain bicoloured trees, however, finding a bijective proof of this fact is an open problem.

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How surprising are Wilf-equivalences?

Michael Albert

(joint work with Mathilde Bouvel, Jinge Li, Vít Jelínek and Michal Opler)

Two collections of finite structures are said to be *Wilf-equivalent* if there is a size-preserving bijection between them. Put another way, they are Wilf-equivalent if they have the same generating function (counted according to the size of the structures). Particularly in the area of permutation classes there has been a widespread interest in Wilf-equivalent classes with an implicit implication that such equivalences are somehow "surprising" and require explanation (or the discovery of an appropriate bijection in the case where the Wilf-equivalence is observed based on the equality of two generating functions). But how surprising are these Wilf-equivalences really?

Let us put the problem into a general context. We work in some universe, $\mathcal{U}$ of finite structures, with a natural substructure relation $\preceq$. So we tend to think of $\mathcal{U}$ as a ranked poset, with the structures of size n in $\mathcal{U}$ denoted by $\mathcal{U}_n$ (this notation is also used more generally). For $\tau \in \mathcal{U}$ define the set of *τ-avoiding structures* to be:

$$\mathrm{Av}(\tau) = \{\theta \in \mathcal{U} \,:\, \tau \npreceq \theta\}$$

(the complement of this set in $\mathcal{U}$ is the set of *τ-containing structures*.)

Now given two structures $\pi, \sigma \in \mathcal{U}_k$ we say that π and σ are Wilf-equivalent and write $\pi \equiv_{\mathrm{WE}} \sigma$ if for all n, $|\mathrm{Av}(\pi)_n| = |\mathrm{Av}(\sigma)_n|$.

To prove that two structures π and σ are Wilf-equivalent it suffices to:

- compute the generating functions of $\mathrm{Av}(\pi)$ and $\mathrm{Av}(\sigma)$ and show they are the same, or
- exhibit a size-preserving bijection between the sets $\mathrm{Av}(\pi)$ and $\mathrm{Av}(\sigma)$, or
- exhibit a size-preserving bijection between the sets $\mathcal{U}\backslash\mathrm{Av}(\pi)$ and $\mathcal{U}\backslash\mathrm{Av}(\sigma)$, or

- exhibit a size-preserving bijection from $\mathcal{U}$ to $\mathcal{U}$ which restricts to bijections of the previous two types.

In practice, any one of these four possibilities may be the most natural way to pursue such a proof.

We are interested in the behaviour of the sequence $w_n = |\mathcal{U}_n/ \equiv_{\mathrm{WE}}|$ in relation to the sequence $u_n = |\mathcal{U}_n|$. If $w_n = o(u_n)$ we say that $\mathcal{U}$ exhibits a *Wilf-collapse*, and if $w_n = o(c^n u_n)$ for some $c < 1$ we say that $\mathcal{U}$ exhibits an *exponential Wilf-collapse*. If it turns out to be the case that (exponential) Wilf-collapse is common behaviour, then we should not be surprised at the discovery of Wilf-equivalences.

In [1] we considered a particular class $\mathcal{U}$ enumerated by the Catalan numbers. This can be thought of as the class of 231-avoiding permutations, of non-crossing arch systems with n arches, of plane forests with n nodes (where, when a node is deleted its ordered list of children replace it in the ordered list of children of its parent), or many other interpretations.

In this case we were able to show by means of bijections that the number of Wilf-equivalence classes for structures of size n is not greater than the number of ordinary (i.e., non-plane) forests with n nodes. In particular this implies an exponential Wilf-collapse since the exponential growth rate for plane forests is 4, while for ordinary forests it is less than 3. In fact there's an additional special rule that allows the replacement of any binary tree (within another tree) by any other binary tree with the same number of leaves (and the same subtrees attached to those leaves). This reduces the exponential growth rate of the number of Wilf-equivalence classes still further to something less than 2.5.

In [3] we considered even smaller universes $\mathcal{U}$ – specifically taking the sets of permutations avoiding two patterns of length three. Ignoring the trivial case of $\mathrm{Av}(123, 321)$ (which is finite) and symmetry, the results for these classes are as follows:

$\mathcal{U}$	u_n	w_n
$\mathrm{Av}(123, 312)$	$\binom{n}{2}$	2
$\mathrm{Av}(312, 132)$	2^{n-1}	1
$\mathrm{Av}(231, 312)$	2^{n-1}	$p_1(n)$ (see below)
$\mathrm{Av}(312, 321)$	2^{n-1}	$p_2(n)$ (see below)

where $p_1(n)$ is the number of partitions of n having at most one part of size 1, and $p_2(n)$ is the number of two-coloured (say red and blue) partitions of n such that the number of red parts is not greater than one plus the number of blue parts, and all the blue parts are at least two. Clearly we have a Wilf-collapse in all four cases, and exponential Wilf-collapse in the latter three (it being impossible in the first one of course!)

In forthcoming work ([2]) we will consider a generalisation of the results of [1] and the third and fourth cases above. In all of these cases the universe $\mathcal{U}$ can be identified with $\mathcal{A}^*$ the set of words over an alphabet $\mathcal{A}$ of "indecomposable structures" and the relation $\preceq$ on $\mathcal{U}$ is the transitive closure of the subword order on $\mathcal{A}^*$ along with some specific relations $w \preceq a$ for certain $w \in \mathcal{A}^*$ and $a \in \mathcal{A}$. In this context we are able to demonstrate some quite general criteria sufficient

for exponential Wilf-collapse. Say that $a, b \in \mathcal{A}$ are *incompatible* if $ab \npreceq c$ for any $c \in \mathcal{A}$. We are in the *supercritical* domain, if the equation $A(t) = 1$ (where A is the ordinary generating function of $\mathcal{A}$) has a solution inside the radius of convergence of A. If there exists an incompatible pair of letters and we are in the supercritical domain, then there is an exponential Wilf-collapse in $\mathcal{U}$.

Notably though, every example we have considered of this type (whether the conditions above are met or not) has exhibited an exponential Wilf-collapse (if possible). We are led to believe that this really is the normal state of affairs, and that we should not be surprised at discovering Wilf-equivalences.

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Enumerative parameters of trees

STEPHAN WAGNER

(joint work with Dimbinaina Ralaivaosaona and Matas Šileikis)

We are interested in parameters of trees that have an enumerative flavour. Some examples of this type are the number of leaves, the number of vertices of a given degree, the number of fringe subtrees of a given shape (a fringe subtree of a rooted tree consists of a vertex and all its descendants), the total number of subtrees (all induced subgraphs that are again trees, not necessarily only fringe subtrees), as well as the number of independent sets, matchings, linear extensions or automorphisms, and there are many more.

Our main interest is in the *distribution* of such parameters in random trees: specifically, we would like to obtain (exact or asymptotic) formulas for mean, variance and higher moments, and also determine—if possible—the limiting distribution. There are several natural models of randomness in the context of trees, and there are various results on the aforementioned parameters for different models. Examples include labelled and unlabelled trees, plane trees, binary trees, and recursive trees.

For the purpose of this extended abstract, let us specifically mention two general models of randomness:

Simply generated trees/Galton-Watson trees. *Simply generated trees* can be seen as weighted plane trees: on the set of all rooted ordered (plane) trees, we impose a weight function by first specifying a sequence $1 = w_0, w_1, w_2, \ldots$ and then setting
$$w(T) = \prod_{i \geq 0} w_i^{N_i(T)},$$

where $N_i(T)$ is the number of vertices of outdegree i in T. Then we pick a tree of given order n at random, with probabilities proportional to the weights.

A different, more probabilistic point of view is that of branching processes. A classical model to generate random trees is the *Galton-Watson tree model*: fix a probability distribution on the set $\{0, 1, 2, \ldots\}$.

- Start with a single vertex, the root.
- At time t, all vertices at distance t from the root produce a number of children, independently at random according to the fixed distribution.

A random Galton-Watson tree of order n is obtained by conditioning the process. It is well known that simply generated trees and Galton-Watson trees are essentially equivalent.

Increasing trees. Another random model that produces very different shapes uses the following simple process, which generates *random recursive trees*:

- Start with the root, which is labelled 1.
- The n-th vertex is attached to one of the previous vertices, uniformly at random.

In this way, the labels along any path that starts at the root are increasing. Clearly, there are $(n-1)!$ possible recursive trees of order n, and there are indeed interesting connections to permutations. The model can be modified by not choosing a parent uniformly at random, but depending on the current outdegrees (to generate, for example, binary increasing trees or plane-oriented recursive trees).

Additive invariants. A very useful concept that covers many different parameters of trees is that of an *additive invariant*. An invariant $F(T)$ defined for rooted trees T is called *additive* if it satisfies a recursion of the form

$$F(T) = \sum_{i=1}^{k} F(T_i) + f(T),$$

where $T_1, \ldots, T_k$ are the branches of the tree and $f(T)$ is a so-called toll function, which often only depends on specific aspects of T, such as its size.

Examples that are covered include the number of leaves, more generally the number of occurrences of a fixed rooted tree as a fringe subtree, and the number of vertices whose outdegree is a fixed number k. But also more complicated invariants are covered. In the context of enumeration, one often has to take the logarithm first. For example, if $\mathrm{le}(T)$ denotes the number of linear extensions of a tree T, then $\log(|T|!/\mathrm{le}(T))$ becomes an additive invariant with toll function $f(T) = \log(|T|)$. Similar examples include the number of subtrees that contain the root and the number of automorphisms.

For additive invariants, rather general results on the distribution can be obtained under different technical assumptions on the toll function $f(T)$. Typically, one obtains a statement of the following type:

- There exist constants μ and σ^2 such that mean and variance of $F(T_n)$ for a random tree T_n are $\mu_n \sim \mu n$ and $\sigma_n^2 \sim \sigma^2 n$.

- Moreover, the renormalised random variable

$$X_n = \frac{F(T_n) - \mu n}{\sqrt{\sigma^2 n}}$$

converges weakly to a standard normal distribution (provided $\sigma \neq 0$).

Examples of "suitable technical conditions" include the following for simply generated trees/Galton-Watson trees (assuming that the offspring distribution has finite variance), where T_n always denotes a random tree with n vertices:

- f is bounded and $\mathbb{E}|f(T_n)| = O(c^n)$ for some $c < 1$ (see [6]).
- $\mathbb{E}|f(T_n)| = O(1)$, $\mathbb{E}(f(T_n)^2) = o(1)$, and

$$\sum_{n \geq 1} \frac{\sqrt{\mathbb{E}\, f(T_n)^2}}{n} < \infty,$$

  see [3].
- f is bounded and "local" (determined by a fixed neighbourhood of the root, see [3]).
- $f(T) \leq C \deg(T)^\alpha$ (where deg denotes the root degree and α is a positive real number for which the $(2\alpha+1)$-th moment of the offspring distribution exists) and f is "almost local", which intuitively means that the values of the toll function can be approximated well (at least with high probability) if a fixed neighbourhood of the root is known (see [4] for details).

These schemes cover, among other things, the number of fringe subtrees of a given type, the number of vertices whose outdegree is a prescribed number, but also more complicated examples such as the number of subtrees or the number of matchings.

For increasing families of trees, analogous results under similar conditions are known as well:

- f is bounded and $\mathbb{E}|f(T_n)| = O(c^n)$ for some $c < 1$ (for recursive trees and binary increasing trees, see [6]).
- $\mathbb{V}\, f(T_n) = o(n)$,

$$\sum_{n \geq 1} \frac{\sqrt{\mathbb{V}\, f(T_n)}}{n^{3/2}} < \infty, \quad \sum_{n \geq 1} \frac{(\mathbb{E}\, f(T_n))^2}{n^2} < \infty,$$

  again only for recursive and binary increasing trees (see [1]).
- f is finitely supported, for d-ary increasing trees, recursive trees and generalised plane-oriented recursive trees (see [2]).
- f is bounded, $\mathbb{E}|f(T_n)| = o(1)$ and

$$\sum_{n \geq 1} \frac{\sqrt{\mathbb{E}|f(T_n)|}}{n} < \infty,$$

  for d-ary increasing trees and generalised plane-oriented recursive trees (see [5]).

Again, these conditions cover many natural examples, such as again the number of fringe subtrees of a given type, but also for instance the number of subtrees or the number of automorphisms of d-ary increasing trees.

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3D positive lattice walks and spherical triangles

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(joint work with Beniamin Bogosel, Vincent Perrollaz and Amélie Trotignon)

The enumeration of lattice walks is an important topic in combinatorics. In addition to having various applications, it is connected to other mathematical fields such as probability theory. Recently, lots of consideration has been given to the enumeration of walks confined to cones, in particular to $\mathbb{N} = \{0, 1, 2, \ldots\}$, $\mathbb{N}^2$ and $\mathbb{N}^3$. Positive walks in 1D and 2D are now well understood, see [2, 6]. On the other hand, much less is known on 3D lattice walks confined to the non-negative octant $\mathbb{N}^3$. An intrinsic difficulty lies in the number of models to handle: more than 11 millions of small step models (see [4])! (Compare with 79 quadrant models [6].)

We assume that the step set $\mathcal{S} \subset \{-1, 0, 1\}^3$, which describes the set of possible jumps of the walk, is not included in any half-space $\{y \in \mathbb{R}^3 : \langle x, y \rangle \geq 0\}$, with $x \in \mathbb{R}^3 \setminus \{0\}$. Introduce $o_{A \to B}(n)$, the number of n-step walks in the octant starting (resp. ending) at $A \in \mathbb{N}^3$ (resp. $B \in \mathbb{N}^3$). It is proved in [7] that if A and B are far enough from the boundary, as $n \to \infty$,

$$(1) \qquad o_{A \to B}(pn) = \kappa(A, B) \cdot \rho^{pn} \cdot n^{-\lambda} \cdot (1 + o(1)),$$

where $\kappa(A, B) > 0$ is some constant, $\rho \in (0, |\mathcal{S}|]$ is the exponential growth, $\lambda > 0$ is the critical exponent and $p = \gcd\{n \in \mathbb{N} : o_{A \to B}(n) > 0\}$ is the period of the model.

Applying the results of [7] (see in particular Equation (12) there) readily shows the following expression for the critical exponent:

$$(2) \qquad \qquad \lambda = \sqrt{\lambda_1 + \frac{1}{4}} + 1,$$

where λ_1 is the smallest (or principal) eigenvalue of the Dirichlet problem

$$(3) \qquad \qquad \begin{cases} \Delta_{\mathbb{S}^2} m & = -\Lambda m & \text{in } T, \\ m & = 0 & \text{in } \partial T, \end{cases}$$

$\Delta_{\mathbb{S}^2}$ being the Laplace-Beltrami operator on the sphere and $T = T(\alpha, \beta, \gamma)$ being a spherical triangle, which can be computed algorithmically (and easily) in terms of the model $\mathcal{S}$.

Our main objective is to relate combinatorial properties of the step set (structure of the so-called group of the walk, existence of a Hadamard factorization, existence of differential equations satisfied by the generating functions) **to geometric or analytic properties of the associated spherical triangle** (remarkable angles, tiling properties, existence of an exceptional closed-form formula for the principal eigenvalue). Let us now detail our main contributions.

- We give the exact value of the angles α, β, γ, which are arccosines of algebraic numbers. We prove that the cosine matrix of the angles is strongly related to the Coxeter matrix of the group, and can also be interpreted as a Gram matrix.
- We then show that the spherical triangle captures a lot of combinatorial information about the model from which it is constructed, in the following sense:
  - Finite group models correspond to triangular tilings of the sphere $\mathbb{S}^2$. The simplest example is the so-called simple walk

    $$\mathcal{S} = \{(\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1)\}.$$

    Its triangle has three right angles, namely $\alpha = \beta = \gamma = \frac{\pi}{2}$. A second example is 3D Kreweras model, with step set

    $$\mathcal{S} = \{(-1, 0, 0), (0, -1, 0), (0, 0, -1), (1, 1, 1)\}.$$

    The associated triangle is also equilateral, with angles $\frac{2\pi}{3}$; this corresponds to the tetrahedral tiling of the sphere.
  - By definition, Hadamard models are those whose jump polynomial $\sum_{(i,j,k) \in \mathcal{S}} x^i y^j z^k$ may be composed as

    $$U(x) + V(x)T(y, z) \quad \text{or} \quad U(x, y) + V(x, y)T(z).$$

    These models are quite special for combinatorial reasons, as explained in [4]. We prove that Hadamard models have birectangular triangles (i.e., with two right angles). Finite group (resp. infinite) Hadamard models correspond to angles β such that $\frac{\pi}{\beta} \in \mathbb{Q}$ (resp. $\frac{\pi}{\beta} \notin \mathbb{Q}$).
  - We can also see the dimensionality of the model on the triangle. In the case of 2D models, the triangles degenerate into a spherical digon.

- Our next result is the study of Hadamard models (mostly with infinite group, as finite group Hadamard walks are solved in [4]). They are special not only for combinatorial reasons, but also for the Laplacian: to the best of our knowledge, their birectangular triangles are the only ones (with the exception of a few tiling triangles) for which one can compute in closed-form the first eigenvalue!

  We deduce the critical exponent λ and show that (most of the) infinite group Hadamard models are non-D-finite. This is the first result on the non-D-finiteness of truly 3D models.

- We classify the models with respect to their triangle and the associated principal eigenvalue, and compare our results with the classification in terms of the group and the Hadamard property obtained in [4, 8]. We exhibit some exceptional models, which do not have the Hadamard property but for which, remarkably, one can compute an explicit form for the eigenvalue; this typically leads to non-D-finiteness results.

- Our last result is about generic infinite group models. Even if no closed-form formula exists for λ_1, we may consider λ_1 as a special function of the triangle T (or equivalently of its angles α, β, γ), and with numerical analysis methods (finite element method), obtain approximations of this function when evaluated at particular values. The techniques developed here are completely different from the rest of the paper.

  Notice that for some cases, approximate values of the critical exponents have been found by Bostan and Kauers [5], Bacher, Kauers and Yatchak [1]. In these articles the method is to compute a certain amount of terms of the generating function and then to estimate the exponents via different ideas. Our technique has the advantage of being applicable to any spherical triangle, not necessarily related to a 3D model.

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First-order logic for permutations

Mathilde Bouvel

(joint work with Michael Albert and Valentin Féray)

In the combinatorics literature, there are two different ways of defining a permutation (say, of size n). One is as a bijection from $\{1, 2, \ldots, n\}$ to itself, or as an element of the permutation group S_n. The other is as a word containing exactly once each letter in $\{1, 2, \ldots, n\}$, or as a diagram (*i.e.*, an $n \times n$ grid containing one element per row and per column). Depending on the definition that is chosen, the results that are proved are usually of different nature, and these two points of view are believed to be rather opposite. The purpose of this talk is to provide mathematical evidence of this "belief".

For each of these points of view, we define a first-order logical theory, whose models are the permutations seen as bijections in one case, as words in the other case. We then investigate which properties of permutations are expressible in each of these theories. It is no surprise that the theory associated with permutations seen as bijections (called TOOB – the Theory Of One Bijection) can express statements about the cycle decomposition of permutations, while the theory associated with permutations seen as words (called TOTO – the Theory Of Two Orders) is designed to express pattern-related concepts. This can be illustrated with many examples.

But we are also interested in describing which properties are *not* expressible in each of these theories. For instance, we prove that TOTO cannot express that an element of a permutation is a fixed point, while TOOB cannot express that a permutation contains the pattern 231.

The result that we put forward in the talk is that we completely characterize the properties that are expressible in both theories. As "expected" based on the "belief" explained earlier, these properties are trivial in some sense. More precisely, such a property is either identically true on all permutations of sufficiently large support, or identically false on all permutations of sufficiently large support. In the proofs, a key tool that we use is the theory of Ehrenfeucht-Fraïssé games.

Additional results that we provide are a characterization of permutation classes where TOTO can express (unlike in the general case) some information on the cycle-types, and a proof that TOTO is more suited than excluded patterns to express sortability by stacks in the sense of West.

Combinatorics of the asymmetric exclusion process on a ring, and Macdonald polynomials

Lauren Williams

(joint work with Sylvie Corteel and Olya Mandelshtam)

The asymmetric simple exclusion exclusion process (ASEP) is a model of particles hopping on a one-dimensional lattice of n sites. It was introduced around 1970 [1, 2], and since then has been extensively studied by researchers in statistical

mechanics, probability, and combinatorics. Recently the ASEP on a lattice with open boundaries has been linked to Koornwinder polynomials [3, 4], and the ASEP on a ring has been linked to Macdonald polynomials [5].
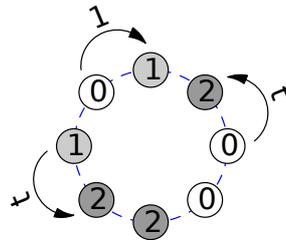
FIGURE 1. The two-species ASEP on a lattice with 8 sites. There are three holes (0's), two light particles (1's), and three heavy particles (2's).

In joint work with Sylvie Corteel and Olya Mandelshtam, we study the two-species asymmetric simple exclusion process (ASEP) on a ring, in which two kinds of particles ("heavy" and "light"), as well as "holes," can hop both clockwise and counterclockwise (at rates 1 or t depending on the particle types) on a ring of n sites. We introduce some new tableaux on a cylinder called *cylindric rhombic tableaux* (CRT), and use them to give a formula for the stationary distribution of the two-species ASEP – each probability is expressed as a sum over all CRT of a fixed type. When λ is a partition in $\{0, 1, 2\}^n$, we then give a formula for the non-symmetric Macdonald polynomial E_λ and the symmetric Macdonald polynomial P_λ by refining our tableaux formulas for the stationary distribution.
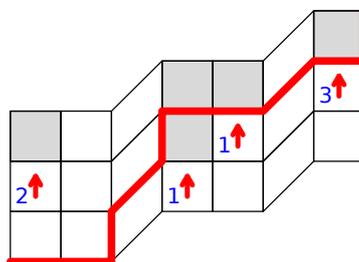
FIGURE 2. A cylindric rhombic tableau T of type 0212201022 with a chosen arrow ordering σ.

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Tableaux formulae for probabilities of the 2-ASEP on a ring and Macdonald polynomials

Olya Mandelshtam

(joint work with Sylvie Corteel and Lauren Williams)

The multispecies ASEP on a ring is a model describing particles of different species hopping right and left on a 1D lattice with n sites with periodic boundary conditions, at rates 1 and t respectively. In the 2-ASEP, particles can be of types 1 or 2 (with 2 having priority over 1).
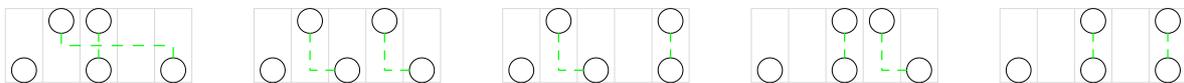
FIGURE 1. All five multiline queues of type 12020. By the result of [1], the stationary probability of the state 12020 is proportional to the number of corresponding multiline queues.

The well-known *multiline queues* of Ferrari and Martin give formulae for the steady state probabilities of the multispecies ASEP on a ring at $t = 0$: there is a Markov chain on the multiline queues which projects to the ASEP [1] (such a formula exists for any number of species of particles). In the 2-species case, each multiline queue is some placement of "balls" on a 2 by n lattice, and there is a unique way of associating every such placement to a particular state of the 2-ASEP by forcing each ball on the top row to pair with the first unpaired ball below and weakly to its right (with wrapping allowed if necessary), such as in Figure 1. The probability of each state is proportional to the number of multiline queues associated to it.

In joint work with Sylvie Corteel and Lauren Williams, we introduce new tableaux on a cylinder which we call *cylindric rhombic tableaux*. By enumerating all tableaux corresponding to a given state, we obtain a formula for steady state probabilities for the 2-ASEP on a ring for general t. Each tableau is some placement of arrows and some ordering σ chosen for those arrows. The weight of a tableau is a generating function in t that depends on σ, as in Figure 2.
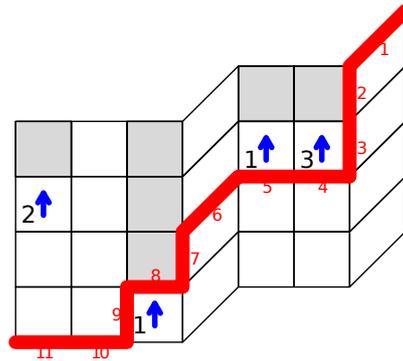
FIGURE 2. A cylindric rhombic tableau of type 2010021200 (read from the thickened path) and enhanced weight $\frac{t^{12}q^3}{[7]_{qt}[6]_{qt}[5]_{qt}[4]_{qt}}x_2 x_3 x_4^2 x_5 x_6 x_7 x_8 x_9^2 x_{10}^2$ where $[j]_{qt} = (1-qt^j)/(1-t)$.

Using recent results of [2], we further enhance these tableaux by associating to each one a weight in q and $x_1, \ldots, x_n$, from which we obtain formulae for nonsymmetric Macdonald polynomials E_λ (where λ is a partition) and for symmetric Macdonald polynomials P_λ (from summing over all cylindric rhombic tableaux corresponding to some permutation of λ), when λ is a partition in $\{0, 1, 2\}^n$.

At $t = 0$, we show a bijection from the multiline queues to cylindric rhombic tableaux (with a trivial arrow ordering). Through a natural extension, we obtain a bijection from cylindric rhombic tableaux with an arrow ordering σ to multiline queues in which there is now some choice σ for the pairing of balls between rows. From this bijection, we obtain multiline queues whose weights are functions of t, q, $x_1, \ldots, x_n$.

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Weak order on monotone triangles and Terwilliger's poset

VICTOR REINER

(joint work with Zachary Hamaker)

Let ASM_n denote the set of $n \times n$ *alternating sign matrices*, that is, matrices whose rows are *alternating* $\{0, \pm 1\}^n$-vectors, in the sense that their non-zero values alternate in sign, beginning and ending with 1. These matrices have been an object of intense study since their introduction by Mills, Robbins, and Rumsey [2]; see also Bressoud [3] for details and history.

There is an easy bijection between ASM_n and the set MT_n of *monotone triangles of size n*, which are the triangular arrays whose rows are subsets of $\{1, 2, \ldots, n\}$

of each cardinality $1, 2, \ldots, n$ and which weakly *interlace*– the bijection sends the alternating sign matrix to the triangle whose m^{th} row from the top is the subset whose $\{0,1\}^n$-characteristic vector is the sum of the first m rows of the matrix.

For example, the matrix $\begin{bmatrix} 0 & +1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ +1 & -1 & +1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & +1 & 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 & +1 & 0 \end{bmatrix}$ in ASM_6 corresponds to the triangle $\begin{smallmatrix} & & & 2 & & & \\ & & 2 & & 4 & & \\ & 1 & & 3 & & 6 & \\ & 1 & & 3 & 4 & & 6 \\ 1 & & 3 & & & 5 & 6 \\ 1 & & 2 & 3 & & 5 & 6 \end{smallmatrix}$ in MT_6.

We use this identification with MT_n to introduce a *weak order* on the set ASM_n, extending the weak order on permutations $\mathfrak{S}_n$, and having several properties:

- The weak order on $\mathfrak{S}_n$ is a principal order ideal inside the weak order on ASM_n.
- This weak order on ASM_n is weaker than the componentwise order on ASM_n arising naturally as the Dedekind-MacNeille completion of the strong Bruhat order on $\mathfrak{S}_n$.
- Linear extensions of weak order on ASM_n all give *shelling orders* on the maximal chains in Terwilliger's recently introduced [1] poset Φ_n extending the Boolean algebra of subsets of $\{1, 2, \ldots, n\}$. Among these linear extensions are a family of shellings arising from an EL-labeling of Φ_n.
- These shellings all give rise to the same notion of *descent set* for elements of ASM_n.
- The weak order on ASM_n encodes an action of the 0-*Hecke monoid* of type A.

To illustrate some of this for $n = 3$, we depict here the Boolean algebra of subsets of $\{1, 2, 3\}$, together with the one extra order relation in Φ_3 shown dotted, then the order complex for the proper part of Φ_3 with the one extra facet shown dotted, and finally the weak order on MT_3, with the one extra monotone triangle outside $\mathfrak{S}_3$ related to others by dotted edges:

Terwilliger's Φ_3 Order complex of $\Phi_3 \setminus \{\hat{0}, \hat{1}\}$

$$
\begin{array}{c}
3 \\
2\ 3
\end{array}
$$

π_2 π_1

$$
\begin{array}{ccc}
3 & 2 & 2 \\
1\ 3 & 1\ 3 & 2\ 3
\end{array}
$$

π_1 π_1 π_2 π_2

$$
\begin{array}{cc}
1 & 2
\end{array}
$$

$$
\begin{array}{cc}
1\ 3 & 1\ 2
\end{array}
$$

π_2 π_1

$$
\begin{array}{c}
1 \\
1\ 2
\end{array}
$$
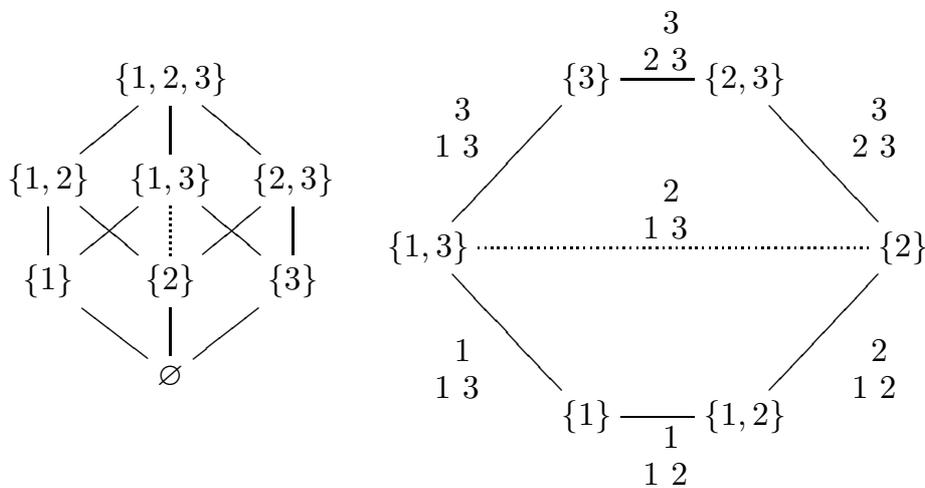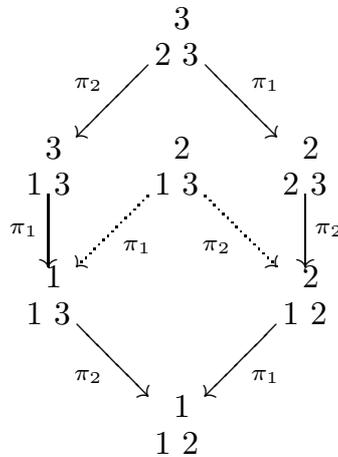
Weak order on MT_3

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Schur-positive grid classes and cyclic descents of SYT

SERGI ELIZALDE

(joint work with Ron Adin and Yuval Roichman)

To each set of permutations one can associate a quasisymmetric function by adding the fundamental quasisymmetric functions indexed by the descent sets of the permutations in the set. A long-standing problem in algebraic combinatorics is to characterize sets of permutations whose associated quasisymmetric function is symmetric and Schur-positive [7].

Classical Schur-positive sets include Knuth classes, conjugacy classes, and inverse descent classes. We give a general method to construct new Schur-positive (multi)sets of permutations [5]. The method relies on pattern-avoiding sets called geometric grid classes [3]. We also show that horizontal rotations of Schur-positive sets of permutations are always Schur-positive, by applying a cyclic action on standard Young tableaux (SYT) of certain skew shapes and a *jeu-de-taquin* type straightening algorithm [6]. As a byproduct, we obtain a notion of cyclic descents on these tableaux, in analogy to Cellini's definition of cyclic descents of permutations [4].

$$\phi$$

| 6 | | 1 | | 2 | | 3 | | 4 | | 5 |

| 1 | 3 | 5 | $\overset{\phi}{\mapsto}$ | 2 | 4 | 6 | $\overset{\phi}{\mapsto}$ | 1 | 3 | 5 | $\overset{\phi}{\mapsto}$ | 1 | 2 | 4 | $\overset{\phi}{\mapsto}$ | 1 | 3 | 5 | $\overset{\phi}{\mapsto}$ | 1 | 2 | 4 |
| 2 | 4 | | | 3 | 5 | | | 4 | 6 | | | 5 | 6 | | | 2 | 6 | | | 3 | 6 |

$\{1,3,6\}$ $\{1,2,4\}$ $\{2,3,5\}$ $\{3,4,6\}$ $\{1,4,5\}$ $\{2,5,6\}$
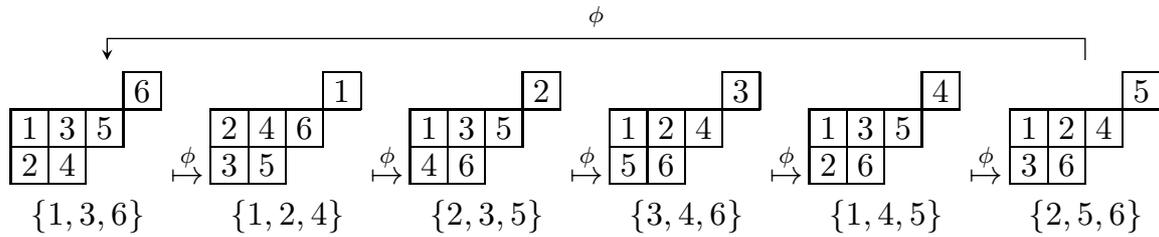
This motivates the question of whether a well-behaved notion of cyclic descents exists for SYT of arbitrary shape. Adin, Reiner and Roichman have recently proved that such a notion exists for SYT of any skew shape that is not a connected ribbon, using nonnegativity properties of Postnikov's toric Schur polynomials [2]. Unfortunately, the proof does not provide an explicit definition of the cyclic descent set for a specific tableau.

We present explicit combinatorial descriptions of cyclic descent sets of SYT of rectangular shape (due to Rhoades [8]), two-row SYT (both straight and skew), and SYT consisting of a hook plus an additional cell. In some cases, we also describe an action on SYT that shifts the cyclic descent set. Detailed descriptions of these constructions are given in [1].

It remains an open problem to find an explicit combinatorial description of the cyclic descent set of a SYT of arbitrary non-ribbon shape, as well as a explicit cyclic action on SYT of given shape that shifts the cyclic descent set.
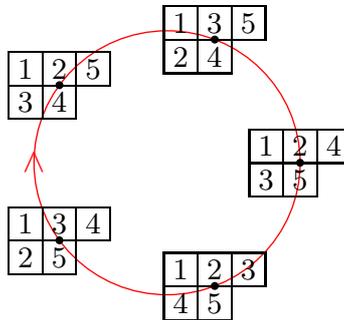
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1324 pattern-avoiding permutations

ANTHONY J. GUTTMANN

(joint work with Andrew R. Conway and Paul Zinn-Justin)

In an earlier paper [4], ARC and AJG gave five new coefficients and a detailed analysis of the generating function for 1324 pattern-avoiding permutations (PAPs). That analysis led them to conjecture that, unlike the known length-4 PAPs, notably the classes $Av(1234)$ [8] and $Av(1342)$ [2], the OGF for $Av(1324)$ included a stretched exponential term. That is to say, if p_n denotes the number of n-step $Av(1324)$ permutations, then $p_n \sim B \cdot \mu^n \cdot \mu_1^{\sqrt{n}} \cdot n^g$, where estimates of the parameters were given.

In this talk, we give a new, substantially improved algorithm that allows us to give 14 further terms[1].

This stretched exponential behaviour is not without precedent. There are a number of models in mathematical physics whose coefficients possess this more complex asymptotic structure. In particular, Duplantier and Saleur [6] and Duplantier and David [5] studied the case of *dense* polymers in two dimensions, and found the partition functions had the asymptotic form $const \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g$. In [9], Owczarek, Prellberg and Brak investigated an exactly solvable model of interacting partially-directed self-avoiding walks (IPDSAW), and found the coefficients behaved with this asymptotic form, and estimated $\sigma = 1/2$, $g = -3/4$, while the sub-exponential growth constant μ_1 was found to more than 5 digit accuracy. From [3] the value of μ is exactly known. For self-avoiding walks and polygons attached to a surface and pushed toward the surface by a force applied at their top vertex, Beaton *et al* [1] gave probabilistic arguments for stretched exponential behaviour, but with growth $\mu_1^{n^{3/7}}$.

1. THE ALGORITHM

The algorithm like many is based upon recursive solution of a set of equations

$$f(S) = \sum_{s \in n(S)} f(s)$$

where $n(S)$ is a set (or possibly multiset if the same s appears with multiplicity) of possible substates of S, culminating in some final states for which $f(S) = 1$. These states correspond to the build up of permutations one entry at a time, with each pass through the equation corresponding to adding one extra entry.

As an example, one could use this formalism to enumerate all n length permutations, saying the state is the number n of as yet unchosen elements. Then state n would have n substates, each $n-1$. This reduces to the normal factorial recurrence $f(n) = nf(n-1)$. To enumerate PAPs, a more complex state is needed.

In the prior paper [4], the state consisted of a series of numbers being the length of contiguous series of unchosen elements of the permutation, together with

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[1]The principal limit to obtaining additional terms is computer memory. The present calculation required about 4.2TB of (distributed) memory.

brackets to store sufficient information to prevent a 1324 pattern. For instance, the state 4(2)1 means that there are four contiguous numbers left to be chosen, then two contiguous numbers that may not be chosen until all numbers after them have been chosen, then 1 number. Each pass through the equation reduced the sum of the available numbers by one.

The big insight is that this is unnecessarily fine grained. If one adds together all states with a given pattern of brackets and numbers, ignoring what the numbers actually are other than their total sum, the equations still work. The start state and end state only contain a single slot containing a number, so summing the contents of that one slot has no effect, so the equations involving them can safely be summed over. This significantly reduces the total number of states, and thus the running time and memory use of the enumeration.

A second insight is that by tracking states by what is taken rather than by what is available, the states can be represented by Dyck paths, or link patterns, or by any set in bijection with these, and enumerated by the Catalan numbers [11]; here we choose to use link patterns, as they provide a convenient graphical description of the algorithm. Since explicit bijections of these objects in length say $2k$ to $\{1, \ldots, \mathrm{Catalan}_k = \frac{(2k)!}{k!(k+1)!}\}$ are known, we can encode them as integers. The reader should be warned that k here is *not* the length of the permutation; it can vary from state to state, with the upper bound $2k \leq n$.

The intuitive motivation of this way of tracking states comes from considering a prefix P of a 1324 avoiding permutation of $1 \ldots n$, and considering what constraints it puts on subsequent elements of the permutation. If P contains n, then this cannot be part of a 1324 pattern where the 4 is in the suffix, so the n is irrelevant as a constraint on the future, and can be safely ignored, turning the problem into a permutation of $1 \ldots n - 1$. This process is repeated until all numbers in P are lower than the largest number remaining in the suffix. The remaining numbers in P must be 132 avoiding, as otherwise the largest number, now in the suffix, would cause a 1324 pattern. 132 PAPs of a given length are enumerated by the Catalan numbers, and are readily bijectable to link patterns.

1.1. **Running.** We wrote a C program using message passing to run on a distributed system, and ran it in the Spartan [10] cluster at the High Performance Computing Centre at the University of Melbourne on 168 cores with 20GB per core. The computation was performed nine times, each with computations performed modulo a number close to 2^{16}, so that only 16 bits of storage were needed for each state. Each run took several hours. The actual answers were then reconstituted using the Chinese remainder theorem. This produced the series up to length 50 permutations.

2. Ratio analysis

Our primary tool is based on the behaviour of the ratio of successive coefficients. We also make use of the approximate ratios and coefficients, as calculated by the method of series extension.

In the case of a simple power-law singularity with the asymptotic form of the coefficients given by $a_n \sim const. \cdot \mu^n \cdot n^g$, the ratio of the coefficients is

$$(1) \qquad r_n = \frac{a_n}{a_{n-1}} = \mu\left(1 + \frac{g}{n} + O(\frac{1}{n^2})\right).$$

If on the other hand the coefficients behave as

$$b_n \sim B \cdot \mu^n \cdot \mu_1^{n^\sigma} \cdot n^g,$$

then the ratio of successive coefficients $r_n = b_n/b_{n-1}$, is

$$(2) \quad r_n = \mu\left(1 + \frac{\sigma \log \mu_1}{n^{1-\sigma}} + \frac{g}{n} + \frac{\sigma^2 \log^2 \mu_1}{2n^{2-2\sigma}} + \frac{(\sigma - \sigma^2)\log \mu_1 + 2g\sigma \log \mu_1}{2n^{2-\sigma}}\right.$$
$$\left. + \frac{\sigma^3 \log^3 \mu_1}{6n^{3-3\sigma}} + O(n^{2\sigma-3}) + O(n^{-2})\right).$$

In order to determine the nature of the asymptotic form of the coefficients of the $Av(1324)$ OGF, we first plot the ratios of successive coefficients $r_n = p_n/p_{n-1}$ against $1/n$. In this and subsequent plots we have used the first 50 exact ratios and the subsequent 200 predicted ratios. Significant curvature is observed. This is inconsistent with an algebraic singularity, as can be seen from eqn. (1). We next plot the same ratios against $1/\sqrt{n}$, and the plot is then visually linear, implying, from eqn. (2) that $\sigma \approx 1/2$. Linear extrapolation implies a limiting value as $n \to \infty$ around 11.60.

We can significantly improve on this estimate by considering the sequence of extrapolants defined by successive pairs of points. That is to say, one can simply linearly extrapolate successive pairs of ratios (r_k, r_{k+1}) with k increasing up to 240. A plot of successive extrapolants against $1/n$ appears to be approaching a limit of around 11.60, or slightly below. We also take $\sigma = 1/2$ as our (initial) conjectured value.

In order to more accurately estimate the value of the exponent σ, we note from (2) that

$$(r_n/\mu - 1) \sim const.n^{\sigma-1}.$$

A log-log plot of $(1 - r_n/\mu)$ against $\log n$, where we have taken 11.60 as the value of μ, is an uninteresting linear plot. However if we calculate the gradient from successive pairs of points, then the negative of this gradient is an estimator of the exponent $1 - \sigma$. We plotted these estimators against $1/n$, which gave compelling evidence that $\sigma = 1/2$. In subsequent analysis, we assumed this value. Repeating this analysis with various values of μ around 11.60, we find that a value slightly below this, around $\mu \approx 11.598$ is most consistent with $\sigma = 1/2$.

Assuming then that $\sigma = 1/2$, from (2) it follows that

$$r_n/\mu = 1 + \frac{\log \mu_1}{2\sqrt{n}} + \frac{g + \frac{1}{8}\log^2 \mu_1}{n} + O(n^{-3/2}).$$

In order to estimate μ_1 and g, we solve, sequentially, the equations

$$(3) \qquad r_j/\mu = 1 + \frac{c_1}{\sqrt{j}} + \frac{c_2}{j} + \frac{c_3}{j^{3/2}},$$

for $j = k - 1$, $j = k$ and $j = k + 1$, with k ranging from 3 up to 49.

These, and more elaborate numerical methods, allowed us to conjecture that

$$s_n(1324) \sim B \cdot \kappa^n \cdot \mu_1^{\sqrt{n}} \cdot n^g,$$

where $\kappa = 11.600 \pm 0.003$, $\mu_1 = 0.0400 \pm 0.0005$, $g = -1.1 \pm 0.1$.

3. Conclusion

Our strongest conclusion is the conjecture that the $Av(1324)$ generating function has a stretched exponential term in the asymptotics. As the coefficients are integers, it follows [7] that the generating function cannot be D-finite.

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Universal limits of substitution-closed classes

Valentin Féray

(joint work with F. Bassino, M. Bouvel, L. Gerin, M. Maazoun and A. Pierrot)

Introduction: The general problem we are interested in is the following: consider a permutation class $\mathcal{C}$, and take, for each $n \geq 1$, a random permutation $\boldsymbol{\sigma_n}$, uniformly among permutations of $\mathcal{C}$ of size n, i.e. $\boldsymbol{\sigma_n} \in_u (\mathcal{C} \cap S_n)$. We aim at describing the asymptotic properties of $\boldsymbol{\sigma_n}$ when n tends to infinity.

This is a very vast question, and the kind of answer we might expect depends very much on the class and on the asymptotic properties we are considering. In this work:

(1) we do not focus on a single permutation class (as in many previous works), but consider a family of classes defined by a closure property, namely classes closed by *substitution*. These classes (there are uncountably many of them) are solvable models, giving a starting point for the asymptotic analysis of random elements.

(2) rather than considering a specific parameter (fixed points, probability that $\boldsymbol{\sigma_n}(i_n) = j_n$ for given i_n and j_n, number of occurrences of patterns) as often done in the literature, we study the scaling limit of $\boldsymbol{\sigma_n}$, in the sense of *permutons*.

Informally, our main result is that, under a quite general analytic assumption, a uniform random permutation $\boldsymbol{\sigma_n}$ in a substitution-closed class converges towards a universal limit, which is a random permuton constructed from the Brownian excursion.

Substitution-closed classes: In the permutation literature, a permutation class is a set of permutations defined by the avoidance of a given set B of patterns (possibly infinite), or equivalently a set stable by taking patterns.

The substitution operation consists in replacing each dot of the diagram of a permutation by the diagram of another permutation. This is better understood on a picture, rather than with a formal definition.
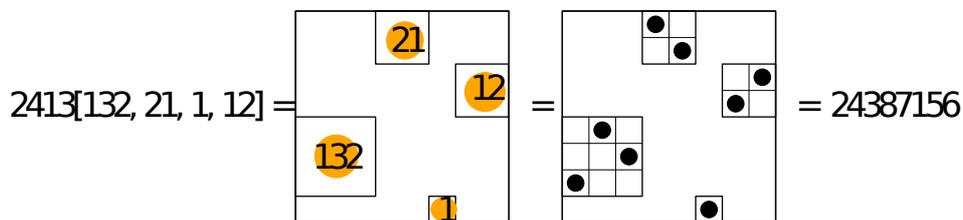

All permutations can be obtained by iterating substitutions operations, starting from "indecomposable elements", which are called simple permutations in this context [1]. This allows to represent permutations as *substitution trees*, whose internal nodes are labelled by simple permutations. Again, we prefer to show an example rather than to give a formal statement.
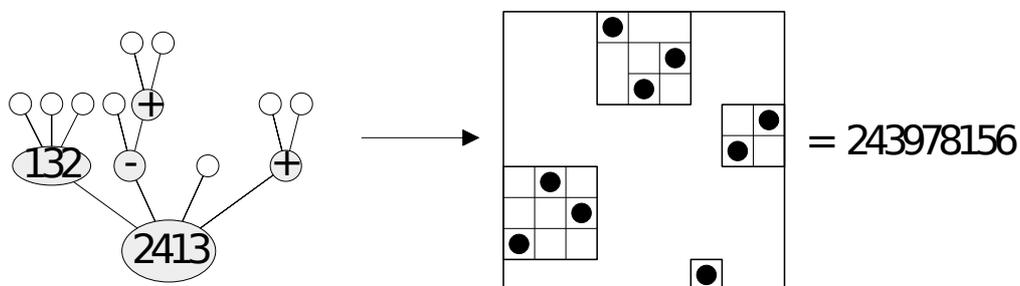
For a substitution-closed class $\mathcal{C}$, a permutation is in $\mathcal{C}$ if and only if all simple permutations in its substitution tree are in $\mathcal{C}$. This allows to relate in a simple way the generating series $C(z)$ of the class to that $S(z)$ of simple permutation in the class.

Permutons: To study limits of combinatorial objects, we need to embed them in some natural way in a (Polish) metric space. For permutations (which we want to identify with their diagrams), it is natural to associate a measure on the unit square $[0,1]^2$. On an example of size $n = 5$:

$$\pi = 5\,2\,4\,1\,3 = \quad\boxed{\qquad\qquad} \quad \mapsto \quad \mu_\pi = \boxed{\qquad\qquad}$$
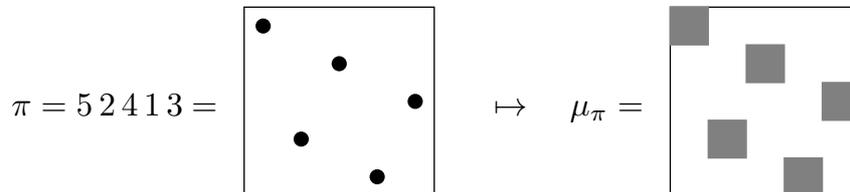
(Each gray square on the right has a total weight $1/n$, i.e. density n.) The limiting objects are then some probability measures on $[0,1]^2$ (with an additional property, namely having uniform marginals), called *permutons*. This notion was first considered by Hoppen and coauthors, in [3], in analogy with graphons, aka dense graph limits considered by Borgs and co-authors..

We then say that a sequence of (random) permutations converge if the associated (random) measures converge (in distribution) for the weak topology. The following criterion makes this convergence more concrete.

Proposition 1 (BBFGMP, building on a result of Hoppen et al. in the deterministic case). *Let $\boldsymbol{\sigma_n}$ be a sequence of random permutations. Then the following assertions are equivalent:*

(1) *There exists a random permuton $\boldsymbol{\mu}$ such that $\boldsymbol{\sigma_n}$ converges to $\boldsymbol{\mu}$;*

(2) *For each $k \geq 2$, and each pattern τ of size k, the following sequence has a limit:*

$$\mathbb{E}\left(\frac{\#\text{occurrences}(\tau, \boldsymbol{\sigma_n})}{\binom{n}{k}}\right).$$

This turns a statement of convergence of random measures into convergence of numbers! Moreover, in the case we are interested in, these expectations can be studied by combinatorial means (counting permutations in the class with a marked occurrence of a given pattern).

Our main result:

Theorem 1. *Let $\mathcal{C}$ be a substitution-closed class and take $\boldsymbol{\sigma_n} \in_u (\mathcal{C} \cap S_n)$ (for each $n \geq 1$). Assume $S'(R_S) > \frac{2}{(1+R_S)^2} - 1$, where S is the OGF of simple permutations in $\mathcal{C}$ and R_S its radius of convergence. Then $\boldsymbol{\sigma_n}$ tends to the biased Brownian separable permuton $\boldsymbol{\mu}^{(p)}$ for some p in $(0,1)$.*

The limit only depends on the class $\mathcal{C}$ through a parameter p that can be computed from some associated generating series. We do not present the construction of the limit object here, but it can be obtained from a Brownian excursion or from the Brownian continuum random tree [4]. (which is the limit of the substitution tree of $\boldsymbol{\sigma_n}$). We also have some results in the so-called stable and condensation regimes.

Final comments: Most substitution-closed classes we have considered fulfill our analytic assumption, with the notable exception of Av(2413). The proof of Theorem 1 uses the above convergence criterion and analytic combinatorics. Here are some pictures of large permutations in substitution-closed classes, given by their set $\mathcal{S}$ of simple permutations.
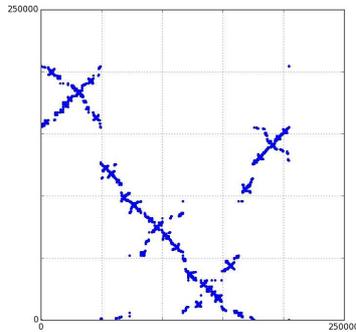
Simulation of $\boldsymbol{\sigma}_n$ with
$\mathcal{S} = \emptyset$
(separable permutations)
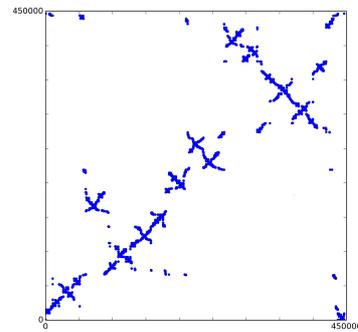
Simulation of $\boldsymbol{\sigma}_n$ with
$\mathcal{S} = \{2413, 3142, 24153\}$

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Complex Martingales and Asymptotic Enumeration
Brendan D. McKay
(joint work with Mikhail Isaev)

Consider a martingale $Z_0, Z_1, \ldots, Z_n$ of complex-valued random variables. We give several explicit bounds on $\mathbb{E}\, e^{Z_n}$.

An important special case is given by the Doob martingale of a complex function $f(X_1, \ldots, X_n)$ of independent random variables $X_1, \ldots, X_n$. For $1 \le k \le n$, define

$$\alpha_k(f) = \sup |f(\boldsymbol{x}^k) - f(\boldsymbol{x})|,$$

where the supremum is over $\boldsymbol{x}, \boldsymbol{x}^k$ that differ only in the k-th coordinate. Similarly, for $j \ne k$, define

$$\Delta_{jk}(f) = \sup |f(\boldsymbol{x}) - f(\boldsymbol{x}^j) - f(\boldsymbol{x}^k) + f(\boldsymbol{x}^{jk})|,$$

where the supremum is over $\boldsymbol{x}, \boldsymbol{x}^k, \boldsymbol{x}^j, \boldsymbol{x}^{jk}$ such that $\boldsymbol{x}, \boldsymbol{x}^k$ differ only in the k-th component, $\boldsymbol{x}, \boldsymbol{x}^j$ differ only in the j-th component, $\boldsymbol{x}^j, \boldsymbol{x}^{jk}$ only in the k-th component, and $\boldsymbol{x}^k, \boldsymbol{x}^{jk}$ only in the j-th component. Define vectors $\boldsymbol{X} = (X_j)$, $\boldsymbol{\alpha} = (\alpha_j)$ and the matrix $\boldsymbol{\Delta} = (\Delta_{jk})$.

For any complex random variable U, define $\mathbb{V}\, U = \mathbb{E}\,(U - \mathbb{E}\, U)^2$ (which is usually called the pseudovariance in distinction to the variance $\mathbb{E}\,|U - \mathbb{E}\, U|^2$). Then we have

$$\mathbb{E}\, e^{f(\boldsymbol{X})} = e^{\mathbb{E}\, f(\boldsymbol{X}) + \frac{1}{2}\mathbb{V} f(\boldsymbol{X})}\bigl(1 + L e^{\frac{1}{2}\mathbb{V}\,\Im f(\boldsymbol{X})}\bigr)$$

where

$$|L| \le \exp\biggl(\frac{1}{6}\sum_{k=1}^n \alpha_k^3 + \frac{1}{6}\boldsymbol{\alpha}^T\boldsymbol{\Delta}\boldsymbol{\alpha} + \frac{5}{8}\sum_{k=1}^n \alpha_k^4 + \frac{5}{16}\boldsymbol{\alpha}^T\boldsymbol{\Delta}^2\boldsymbol{\alpha}\biggr).$$

An example of a complex Doob martingale is provided by the variable-by-variable integration of a multivariable complex function. In many examples that occur in the asymptotic enumeration of combinatorial objects, the dominant part of such integrals can be written as the integral over an axis-aligned cuboid of the exponential of a polynomial in independent truncated normal random variables. See [1] for a long list of published examples.

As first applications, we considerably strengthen the results of Barvinok and Hartigan [2] regarding the distribution of edges within a graph of given degrees and count subgraphs isomorphic to a given graph in a graph with given degrees [3].

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Open Problem Session
BRENDAN MCKAY
(joint work with the many authors who presented open problems)

A curious determinantal identity

Authors: Jérémie Bouttier, Mark Bowick, Emmanuel Guitter and Monwhea Jeng

For n a positive integer, let us consider a square grid of size $(n+1) \times n$ to which we add n *winding* oriented edges connecting, for each $k = 1, \ldots, n$, the $(k+1)$-th vertex on the bottom row (numbered starting from left) to the k-th vertex on the left row (numbered starting from bottom). We also add an extra *phantom* vertex i_0 connected to the bottom-left vertex. Let us denote by G_n the resulting (multi)graph, G_4 being displayed on Figure 1(b).

|	
(a)	(b)
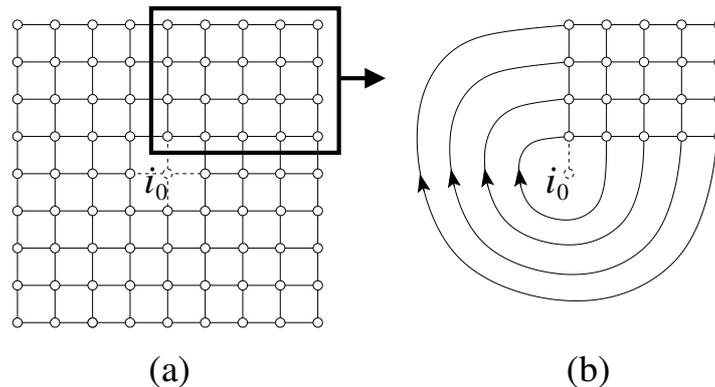
FIGURE 1. The graph (b) is obtained from the square grid of odd size (a) by identifying the vertices that differ by $\pi/2$ rotations.

For two vertices i, j of G_n distinct from i_0, we set

$$n_{ij} := \begin{cases} 1 & \text{if there is a normal (non-winding) edge between } i \text{ and } j, \\ 0 & \text{otherwise,} \end{cases}$$

$$w_{ij} := \begin{cases} \alpha & \text{if there is a winding edge oriented from } i \text{ to } j, \\ \alpha^{-1} & \text{if there is a winding edge oriented from } j \text{ to } i, \\ 0 & \text{otherwise.} \end{cases}$$

We then define the square matrix $\bar{\Delta}^{(n)}$ of size $n(n+1)$ by

$$\bar{\Delta}^{(n)}_{i,j} := \begin{cases} \deg(i) & \text{for } i = j, \\ -n_{ij} - w_{ij} & \text{for } i \neq j, \end{cases}$$

where $\deg(i)$ denotes the degree of i in G_n (note that the two leftmost vertices on the bottom row are connected by two edges, one normal and one winding, and it is important that we sum their both contributions in $\bar{\Delta}$). Finally, we set

$$P_n(\alpha) := \det \bar{\Delta}^{(n)}.$$

For $\alpha = 1$, $\bar{\Delta}^{(n)}$ is the minor of the Laplacian matrix of G_n obtained by removing the row and column of index i_0. By the matrix-tree theorem, $P_n(1)$ counts the number of spanning trees of G_n. The first values of $P_n(1)$ read

$$2, 60, 21112, 81608976, 3376585316896, 1476304297181272000, \ldots$$

This sequence does not appear in the Online Encyclopedia of Integer Sequences (OEIS) and does not seem to admit a nice product formula due to large prime factors.

For general α, $P_n(\alpha)$ is a polynomial in $\alpha + \alpha^{-1}$, and is known to count so-called *spanning webs* [1]. We computed it up to $n = 60$ and the first values read

$$P_1(\alpha) = 4 - \left(\alpha + \frac{1}{\alpha}\right), \qquad P_2(\alpha) = 178 - 60\left(\alpha + \frac{1}{\alpha}\right) + \left(\alpha^2 + \frac{1}{\alpha^2}\right),$$

$$P_3(\alpha) = 82128 - 31667\left(\alpha + \frac{1}{\alpha}\right) + 1160\left(\alpha^2 + \frac{1}{\alpha^2}\right) - \left(\alpha^3 + \frac{1}{\alpha^3}\right).$$

Our conjecture concerns the special value $\alpha = \mathrm{i}$ (imaginary unit). The first values of $P_n(\mathrm{i})$ read

(1) $4, 176, 79808, 372713728, 17931360207872, 8887976555024756736, \ldots$

In [1], we remarked that these numbers seem to count the number of perfect matchings of the graph $C_{2n+1} \times P_{2n}$ (i.e. a cylindrical square grid with circumference $2n + 1$ and height $2n$). This is the conjecture that I presented during the problem session.

After the problem session I was led to check again my numbers. Then, I realized that the sequence (1) actually appears in the OEIS under the reference A127606, which we did not notice when writing our article. It was added in April 2007, probably after we made our conjecture[1]. Sequence A127606 is defined through the formula

$$a_n = 2^{2n^2} \prod_{k=1}^{n} \prod_{\ell=1}^{n} \left(\cos\left(\frac{k\pi}{2n+1}\right)^2 + \sin\left(\frac{\ell\pi}{2n+1}\right)^2 \right)$$

and the entry does not mention a combinatorial interpretation (yet), but it may be checked using the Kasteleyn Pfaffian method (which I did not know that well in 2007) that a_n indeed counts perfect matchings of $C_{2n+1} \times P_{2n}$. The conjecture can be then reformulated as the fact that $P_n(\mathrm{i}) = a_n$ for all $n \geq 1$. After I brought this fact to Emmanuel Guitter's attention, he made the observation that the individual eigenvalues of $\bar{\Delta}^{(n)}$ at $\alpha = \mathrm{i}$ seem to be $4\cos\left(\frac{k\pi}{2n+1}\right)^2 + 4\sin\left(\frac{\ell\pi}{2n+1}\right)^2$ for $k = 1, \ldots, n$ and $\ell = 0, \ldots, n$. This may be checked by explicitly constructing the eigenvectors, a moderately difficult exercise (as a bonus we obtain similar product formulas for $P_n(1)$ and $P_n(-1)$). In this sense my conjecture can be regarded as solved. It would be however interesting to have a more combinatorial proof of it. Guillaume Chapuy and Matjaž Konvalinka informed me that they have made some progress in this direction.

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[1] We made our conjecture because the numbers in (1) appear individually in other OEIS sequences – namely A0284(78,80,...,86) – that count perfect matchings of $C_{2m+1} \times P_{2n}$ with fixed m and varying n.

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Voronoi-like decomposition of $\mathbb{S}^1$ with randomness

Author: Agelos Georgakopoulos

Pick k points $p_1, \ldots, p_k$ on the unit circle $\mathbb{S}^1$ uniformly at random (or equidistantly), and start an independent Brownian motion at each p_i. Let $B_i \subset \mathbb{S}^1$ be the set of points $x \in \mathbb{S}^1$ such that the first particle that visited x was the one started at p_i, and let $|B_i|$ denote its Lebesgue measure.

Problem 1. *Determine the probability distribution of the vector* $(|B_1|, \ldots, |B_k|)$.

This problem is unpublished, and nothing is known.

Rank sizes of Differential Posets

Author: Pritam Majumder

A poset P is said to be r-**differential** if it satisfies the following conditions
- P is graded, locally finite and has a unique minimal element
- $|C_+(x)| = |C_-(x)| + r$ for all $x \in P$
- $|C_+(x) \cap C_+(y)| = |C_-(x) \cap C_-(y)| = 1$ for all $x, y \in P$ and $x \neq y$,

where $C_+(t)$ is the set of elements in P covering t and $C_-(t)$ is the set of elements in P covered by t.

Now let p_n be the size of the nth rank of a r-differential poset P. Then we have the following conjecture due to Stanley: For every $n \geq 2$,

$$p_n \leq r p_{n-1} + p_{n-2}.$$

Note that the above conjecture implies that $p_n \leq F_r(n)$, where $F_r(n)$ is the r-Fibonacci number, which are recursively defined by $F_r(0) = 1, F_r(1) = r$, and $F_r(n) = r F_r(n-1) + F_r(n-2)$. This was proved by P. Byrnes in [1] (see Theorem 1.2 and its proof in Chapter 5). For proving this, he also proved the following inequality

$$p_n \leq 1 + r \sum_{k=0}^{n-2} p_k + (r-1) p_{n-1}.$$

We ask whether the above inequality can be improved to get us closer towards the inequality of Stanley's conjecture, which is still open.

Our next problem is about the strict rank growth of differential poset. In [2], A. Miller proved that the rank sizes of differential posets are strictly increasing, i.e. $p_n > p_{n-1}$ for every n. This was proved by computing the last smith entry of the Smith Normal Form of $DU + kI$ (where D and U are the usual up and down operators). But a combinatorial proof of this fact is open.

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Matrices over finite fields and Higman's conjecture

Author: Alejandro H. Morales

A problem relating matrices over finite fields and polynomiality is the celebrated Higman's conjecture from 1960 [2]:

Conjecture 1 (Higman [2]). *The number $k(U_n(\mathbb{F}_q))$ of conjugacy classes of the group $U_n(\mathbb{F}_q)$ of $n \times n$ upper triangular matrices over a finite field $\mathbb{F}_q$ is a polynomial in q.*

This conjecture came from Higman's study of p-groups and small calculations and was verified up to $n \leq 13$ by Vera-López and Arregi [7] and recently up to $n \leq 16$ by Pak and Soffer [4]. Also, Halasi and Pálfy [1] showed that Higman's conjecture fails for other classes of groups defined from partially ordered sets.

Kirilov [3] conjectured that for $q = 2, 3$ the sequence $\{k(U_n(\mathbb{F}_q))\}_{n \geq 1}$ is at least the *Euler numbers* E_n [5, A001111] and the *Springer numbers* S_n [5, A001586] respectively. Soffer [6] used the exact values of these sequences and improvements on a lower bound of $k(U_n(\mathbb{F}_q))$ by Higman to show these conjectures hold for $n \geq 43$ and $n \geq 30$, respectively. Computations with data in [4, Appendix A] suggest the following more general conjecture.

Conjecture 2. *Let $T = q - 1$ and $A_n(T)$ be defined as the coefficients of the generating function $\sum_{n=0}^{\infty} A_n(T)\frac{x^n}{n!} = (1 - \sin(Tx))^{-1/T}$, $(A_n(1) = E_n, A_n(2) = S_n)$ then for an infinite family of primes such that $k(U_n(\mathbb{F}_q))$ is a polynomial in q then $k(U_n(\mathbb{F}_{T+1})) - A_n(T) \in \mathbb{N}[T]$.*

This conjecture has been verified up to $n = 16$.

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3-cores of random planar graphs

Authors: Marc Noy and Lander Ramos

The k-core of a graph G is obtained by repeatedly removing vertices of degree less than k. It is well defined, as the k-core is the largest subgraph of G with minimum degree at least k.

It is shown in [2] that the 2-core of a random planar graph (or map) has linear expected size. It can be shown that the 3-core of a random planar graph has also linear expected size. The question is whether it has a connected component of linear size. We consider labelled planar graphs with the uniform distribution on graphs with n vertices.

Problem. Show that the 3-core of a random planar graph has a connected component of size cn, for some constant $c > 0$, with high probability (probability $\to 1$ as $n \to \infty$).

Comments. Simulations using the random generation algorithm of Fusy [1], strongly indicate that the 3-core of a random planar graph has a unique connected component of linear size. The same problem can be posed for planar maps. The main difficulty is to analyze the dynamic process of repeatedly deleting vertices of degree 2.

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Nick Early's conjecture for the h^*-vector of the hypersimplex

Author: Vic Reiner

Recall for a d-dimensional polytope P in $\mathbb{R}^n$ with vertices in some lattice $\Lambda \subset \mathbb{R}^n$, the Ehrhart function $\mathrm{Ehr}(P, m) := mP \cap \Lambda$ is a polynomial in m of degree d, and one can uniquely express its Ehrhart series

$$\sum_{m \geq 0} \mathrm{Ehr}(P, m) t^m = \frac{h^*(P, t)}{(1 - t)^{d+1}} \quad \text{for some polynomial} \quad h^*(P, t) = \sum_{i=0}^{d} h_i^*(P) t^i.$$

In `arXiv:1710.09507`, Nick Early conjectures the form for $h^*(P, t)$ for a **hypersimplex**

$$P = \left\{ x \in [0, 1]^n : \sum_i x_i = r \right\} = \mathrm{conv}\left(\mathbf{e}_S : S \in \binom{[n]}{r} \right),$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are standard basis vectors in $\mathbb{R}^n$, and $\mathbf{e}_S := \sum_{i \in S} \mathbf{e}_i$ for $S \subseteq [n] := \{1, 2, \ldots, n\}$. His conjecture is stated in terms of the following class of objects. A *decorated set partition of* $[n] = \{1, 2, \ldots, n\}$ is an ordered sequence

$$\pi = ((L_1)_{\ell_1}, (L_2)_{\ell_2}, \ldots, (L_N)_{\ell_N})$$

in which

- the $L_1, L_2, \cdots, L_N$ are subsets of $[n]$ that disjointly partition $[n] = \sqcup_{i=1}^N L_i$,
- the $\ell_1, \ell_2, \cdots, \ell_n$ are positive integers,
- the sequence π is considered only up to cyclic equivalence, that is,

$$\begin{aligned}
\pi &= ((L_1)_{\ell_1}, (L_2)_{\ell_2}, \ldots, (L_N)_{\ell_N}) \\
&= ((L_N)_{\ell_N}, (L_1)_{\ell_1}, (L_2)_{\ell_2}, \ldots, (L_{N-1})_{\ell_{N-1}}) \\
&= \cdots \\
&= ((L_2)_{\ell_2}, \ldots, (L_N)_{\ell_N}, (L_1)_{\ell_1})
\end{aligned}$$

If one draws the blocks $L_1, L_2, \ldots, L_N$ clockwise around a circle, then one defines the *winding number* $\mathrm{wind}(\pi)$ is the number of times in π that one winds around the circle when proceeding clockwise from 1 to 2, then 2 to 3, ..., then $n-1$ to n, and finally n to 1.

Conjecture 1. *The* **hypersimplex** $P = \Delta(n, r)$ *has*

$$(2) \qquad\qquad h^*(P, t) = \sum_\pi t^{\mathrm{wind}(\pi)}$$

where in the sum, π runs over the decorated set partitions of $[n]$ satisfying

- $\sum_{i=1}^N \ell_i = r$, *and*
- $1 \le \ell_i \le |L_i| - 1$ *for all i.*

A simple (?) problem in Permutation Patterns
Author: Andrea Sportiello

This problem is in the realm of "Permutation Patterns" (PP), that is, the counting of permutations $\sigma \in \mathfrak{S}_n$ which behave in some specific way w.r.t. the number of occurrences of certain patterns $\pi \in \mathfrak{S}_k$ (typically, they have no occurrences at all of the pattern). We say that π occurs in σ at positions $(i_1, \ldots, i_k)$ if $(\sigma_{i_1}, \ldots, \sigma_{i_k})$ is sorted by the permutation π^{-1}.

Normally, problems in PP are made "difficult" by the introduction of asymptotic notions: one wants to establish properties which hold in the limit $n \to \infty$, for example one wants to establish that, for a certain value $\alpha \in \mathbb{R}$, the number of permutations in $\mathfrak{S}_n$ avoiding a pattern π grows as $\exp(\alpha n + o(n))$, Here we present a conjecture which is conceptually simpler: we present two families of configurations, namely $\{\mathcal{A}_\lambda\}_\lambda$ and $\{\mathcal{B}_\lambda\}_\lambda$, and we conjecture that their cardinalities A_λ and B_λ satisfy $B_\lambda \ge A_\lambda$ for all λ. The problem is: prove this conjecture.

However, at difference with most (but not all!) of the works in PP, here we have two small extra ingredients:

(1) we count permutations contained in some (digitally-convex) shape λ of side n, instead of just in the $n \times n$ square;

(2) one of the two families is a family of 'coloured' permutations.

These notions are elementary but messy to define, so I will just define them in a sketchy way (and "by examples" in figure), and then pass to the definition of the two families in the conjecture.

Definitions. **q-coloured permutations.** $\mathfrak{S}_n^{(q)}$ is the set of size-n permutations in which each entry is coloured in one of q colours, thus $\mathfrak{S}_n^{(q)} \simeq \mathfrak{S}_n \times \{1, \ldots, q\}^n$. Here we only use ordinary (1-coloured) permutations, and 2-coloured ones, so that we just use "blue" and "red" instead of the set $\{1, 2\}$.

Permutations in a shape λ. A shape λ is a finite subset of $\mathbb{Z}^2$. It has side $n = n(\lambda)$ if it is boxed in a square of side n, but not in one of side $n - 1$. It is *digitally convex* if for all $x, y \in \lambda$ there exists a directed walk on the grid that connects x to y,[2] and is contained in λ. We use $\mathfrak{S}_\lambda$ for the set of permutations $\sigma \in \mathfrak{S}_{n(\lambda)}$ such that all points (i, σ_i) are in λ.

Patterns within a shape. Let π be a $k \times k$ matrix filled with k 'bullets', forming a permutation in $\mathfrak{S}_k$, and possibly some 'crosses'. For $\sigma \in \mathfrak{S}_\lambda$ we say that π occurs in σ at positions $(i_1, \ldots, i_k)$ if it occurs as a permutation, and, for all cross positions $(i, j) \in \{1, \ldots, k\}^2$, the position (i, σ_j) is in λ. We call $\mathscr{A}\nu_\lambda^{(q)}(\pi)$ the set of $\sigma \in \mathfrak{S}_\lambda^{(q)}$ avoiding π, and $\mathrm{Av}_\lambda^{(q)}(\pi)$ its cardinality (superscript $\cdot^{(q)}$ is omitted in the uncoloured case, $q = 1$). We call $\mathscr{A}\nu_\lambda^{(q)}(\pi_1, \ldots, \pi_s)$ the analogous notion in which none of the patterns π_a occurs in σ.

The conjecture. Call $\mathcal{A}_\lambda = \mathscr{A}\nu_\lambda \left(\begin{smallmatrix} \bullet & & \times \\ & \bullet & \\ \times & & \bullet \end{smallmatrix} \right)$ and $\mathcal{B}_\lambda = \mathscr{A}\nu_\lambda^{(2)} \left(\begin{smallmatrix} \bullet & \\ \times & \bullet \end{smallmatrix}, \begin{smallmatrix} \bullet & \times \\ & \bullet \end{smallmatrix} \right)$.


Conjecture: For all digitally-convex shapes λ, $B_\lambda \geq A_\lambda$.

Remark: If true, this is not completely trivial. Indeed it is not true that for every $\sigma \in \mathcal{A}_\lambda$ there exists at least one 2-colouring such that the coloured permutation σ' is in $\mathcal{B}_\lambda$. The following configuration is a counterexample: . Indeed, it is easily seen that a good colouring must alternate red and blue, however none of the two possibilities is valid.
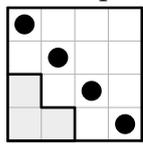

[2]I.e. a walk that uses at most two out of the four "north", "south", "east" and "west" types of steps.
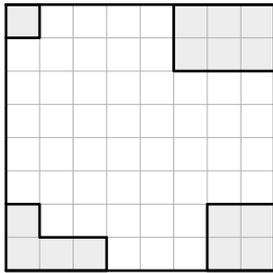
A digitally-convex shape λ
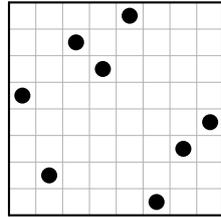
A permutation $\sigma \in \mathfrak{S}_n$, which is also in $\mathfrak{S}_\lambda$, and in $\mathcal{A}_\lambda$.

$\pi:$
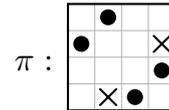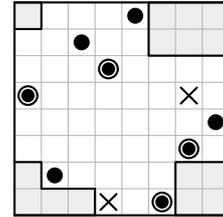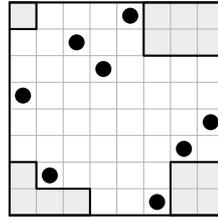
Occurrence of a pattern π in σ.
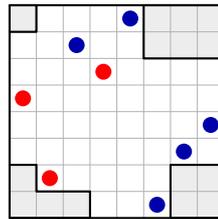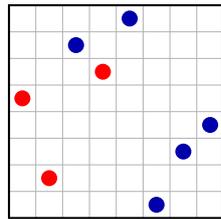
A 2-coloured permutation $\sigma' \in \mathfrak{S}_n^{(2)}$, which is also in $\mathfrak{S}_\lambda^{(2)}$, and in $\mathcal{B}_\lambda$.

A determinant related to set partitions

Authors: Elmar Teufl and Stephan Wagner

In the enumeration of spanning trees of ladder-like graphs, one naturally encounters certain transfer matrices whose determinants exhibit a strange pattern. They can be described in terms of set partitions.

Let $\mathcal{P}$ be a set partition of the set of vertices $X = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{Q}$ a set partition of the set of vertices $Y = \{y_1, y_2, \ldots, y_n\}$. A subset S of the set $\{x_i y_j : 1 \le i, j \le n\}$ of edges between X and Y is a *transition* from $\mathcal{P}$ to $\mathcal{Q}$ if the following holds:

If we take (arbitrary) spanning trees on all blocks $P_1, P_2, \ldots, P_k$ of $\mathcal{P}$ and add the edges in S to their union, we obtain a spanning forest of $X \cup Y$ such that each component of this spanning forest contains at least one vertex of Y and the components of the spanning forest induce the partition $\mathcal{Q}$ on Y.

The weight of a transition S is $w(S) = \prod_{x_i y_j \in S} a_{ij}$. We define a square matrix T whose dimension is the n-th Bell number B_n by its entries $t_{\mathcal{P},\mathcal{Q}}$ associated to pairs of set partitions (rows and columns are ordered in the same way):

$$t_{\mathcal{P},\mathcal{Q}} = \sum_{S:\, \text{transition from } \mathcal{P} \text{ to } \mathcal{Q}} w(S).$$

For example, when $n = 2$, we have

$$T = \begin{bmatrix} a_{11}a_{22} + a_{11}a_{21} + a_{12}a_{22} + a_{12}a_{21} & a_{11}a_{12}a_{21} + a_{11}a_{12}a_{22} + a_{11}a_{21}a_{22} + a_{12}a_{21}a_{22} \\ a_{11} + a_{12} + a_{21} + a_{22} & a_{11}a_{22} + a_{11}a_{12} + a_{12}a_{21} + a_{21}a_{22} \end{bmatrix}$$

The first row corresponds to the partition $\{x_1\}, \{x_2\}$, the first column to the partition $\{y_1\}, \{y_2\}$.

If A denotes the matrix whose entries are a_{ij}, then we notice that $\det T = (\det A)^{B_n}$ for $n = 1, 2, 3$; for example,

$$\det T = (a_{11}a_{22} - a_{12}a_{21})^2 = (\det A)^2$$

in the case $n = 2$ shown above. The pattern does not continue, however: for $n = 4$, we have

$$\det T = (\det A)^{14} \cdot \operatorname{per} A.$$

For $n = 5$, it seems (experimentally) that

$$\det T = (\det A)^{42} \cdot P_1 \cdot P_2,$$

where P_1, P_2 are homogeneous polynomials whose total degrees are 20 and 30 respectively.

Problem 1. *Is it true that* $(\det A)^{C_n}$ *is always a factor of* $\det T$*, where* C_n *is the Catalan number* $\frac{1}{n+1}\binom{2n}{n}$*?*

We have experimental evidence that this is true for $n = 6$, but nothing beyond this point (since the matrices get too big).

Conjectured Uniform Presentation for Pure Braid Groups

Authors: Jon McCammond and Nathan Williams

Notation: given a set X, let $\overline{X}$ be the relation expressing the equality of all elements of X.

Braid Groups: Fix a finite Coxeter group W with simple reflection S and longest element $w_\circ$. Write $\operatorname{Red}_S(w_\circ)$ for the set of reduced words in simple reflections for $w_\circ$. Then results of Brieskorn-Saito and Deligne (independently) show that the braid group for W has presentation

$$B(W) = \langle S : \overline{\operatorname{Red}_S(w_\circ)} \rangle.$$

Dual Braid Groups: On the other hand, building on work of Birman-Ko-Lee, Bessis and Brady-Watt (also independently) both gave a different presentation of the braid group. Let T be the set of reflections of W, and fix a Coxeter element c (a product of the simple reflections, in some order). Write $\operatorname{Red}_T(c)$ for the set of reduced words in reflections for c. Then associated to W and c is the dual braid group, with presentation

$$B_c(W) = \langle T : \overline{\operatorname{Red}_T(c)} \rangle.$$

Pure Braid Groups: The pure braid group $P_c(W)$ is generated by $\mathbb{T}$—the squares of the generators of $B_c(W)$. To my knowledge, no uniform presentation has been written down. Write $\mathbb{c}$ for the generator of the center of $P_c(W)$, which is known as the full twist. Let $\operatorname{Red}_\mathbb{T}(\mathbb{c})$ be the set of reduced words in the squares of the reflections for $\mathbb{c}$.

Conjecture 1.

$$P_c(W) = \langle \mathbb{T} : \overline{\operatorname{Red}_\mathbb{T}(\mathbb{c})} \rangle.$$

The conjecture is easily generalized to well-generated finite complex reflection groups.

We also have an explicit conjectural description of $\text{Red}_{\mathbb{T}}(\mathbb{c})$. The choice of c cyclically orients the reflections of the rank 2 parabolic subgroups of W—we believe that the reduced words in $\mathbb{T}$ for $\mathbb{c}$ are exactly those permutations of $\mathbb{T}$ that respect this orientation for each rank 2 parabolic subgroup.

Reporter: Robin Sulzgruber

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