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Subgroups of Cremona Groups

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Abstract. There have been recent breakthroughs in two related questions: the study of Cremona groups, and the problem of rationality of algebraic varieties. Our workshop brought together algebraic geometers, who discussed and tried to solve problems that are relevant to the classification of finite subgroups of Cremona groups. Priority was given to the following four related areas: automorphisms of algebraic varieties, birational geometry of Mori fibre spaces, and rationality problems.

Mathematics Subject Classification (2010): Primary: 14E07, Secondary: 14E05, 14E08, 14J45.

Introduction by the Organisers

The workshop *Subgroups of Cremona Groups* has been organized by Fabrizio Catanese, Ivan Cheltsov, Julie Deserti and Yuri Prokhorov. Unfortunately Julie Deserti was unable to participate.

The workshop was well attended with 26 participants with broad geographic representation from France, Germany, Great Britain, Japan, Korea, Switzerland, Russia and United States: Hamid Ahmadinezhad, Artem Avilov, Ingrid Bauer, Jeremy Blanc, Christian Bohning, Serge Cantat, Fabrizio Catanese, Ivan Cheltsov, Adrien Dubouloz, Alexander Duncan, Anne-Sophie Kaloghiros, Igor Krylov, Anne Lonjou, Frederic Mangolte, Mirko Mauri, Lucy Moser-Jauslin, Keiji Oguiso, Jihun Park, Yuri Prokhorov, Victor Przyjalkowski, Julia Schneider, Costya Shramov, Christian Urech, Egor Yasinsky and Susanna Zimmermann.

Among the participants there were 6 women (Bauer, Kaloghiros, Lonjou, Moser-Jauslin, Schneider, Zimmermann), 5 postdocs (Avilov, Lonjou, Krylov, Urech, Yasinsky), and 2 PhD students (Mauri, Schneider). All of them (with the exception of Yasinsky) gave talks, PhD students gave short talks, and all talks were excellent.

The complex projective plane $\mathbb{P}^2$ and the projective space $\mathbb{P}^3$, while being the most basic objects of geometry, especially concerning their linear geometry, still provide source for intriguing questions.

One such question is the algebraic structure of the group of their birational transformations, called the Cremona groups, and denoted respectively $\mathrm{Cr}_2(\mathbb{C})$ and $\mathrm{Cr}_3(\mathbb{C})$.

The study of the plane Cremona group, respectively of the space Cremona group,shows that these are two exceptionally complicated objects. The group $\mathrm{Cr}_2(\mathbb{C})$ has been studied intensively over the last two centuries, after the pioneering work of Noether and Castelnuovo in the nineteenth century, and many facts about it were established until now. For example, Serge Cantat and Stephane Lamy proved in 2013 that the group $\mathrm{Cr}_2(\mathbb{C})$ is not simple. The structure of the group $\mathrm{Cr}_3(\mathbb{C})$ is much more complicated and mysterious. Until know it resisted all the attempts to study its global structure. Nevertheless, during our workshop, Jeremy Blanc and Susanna Zimmermann announced

Theorem (Blanc, Lamy, Zimmermann)**.** The group $\mathrm{Cr}_n(\mathbb{C})$ is not simple for $n \geqslant 3$.

This came as a big surprise. Because of this, we asked them to give two talks about the proof of this beautiful result.

One approach to study Cremona groups is by means of their finite subgroups. The complete classification of finite subgroups in the plane Cremona group $\mathrm{Cr}_2(\mathbb{C})$ was obtained by Blanc, Dolgachev and Iskovskikh. Recent achievements in three-dimensional birational geometry allowed Prokhorov to classify finite simple subgroups in $\mathrm{Cr}_3(\mathbb{C})$. His classification became possible thanks to a general observation that a birational action of a finite group G on the projective space can be regularized, that is, replaced by a regular action of this group on some more complicated rational threefold. This transfers the problem into the rich world of rational threefolds with prescribed symmetry groups, which is an inseparable part of the much more natural world of rationally connected threefolds.

Our workshop carried together 26 mathematicians actively working on automorphisms of algebraic varieties, classification of Fano varieties, birational geometry of Mori fibre spaces, and rationality problems. Its main goal was to understand how (finite) groups can act on rational three-dimensional algebraic varieties. In this respect, the workshop has been very successful. The atmosphere has been lively and very collaborative. During every talk, many questions have been posed and interesting problems pointed out. The active presence of young participants has been especially remarkable. All of the talks presented top-level results, we do not have time to comment on all of them, but we would like to mention one important breakthrough by a younger participant: Igor Krylov announced the proof of a twenty years old conjecture posed by Corti in a famous paper published in 1996 on Annals of Mathematics, about the existence of good birational models of fibrations in del Pezzo surfaces of degree 1.

During the week, 18 one-hour long lectures have been given by the participants, and PhD students, Mirko Mauri and Julia Schneider, gave half-hour talks. This report contains extended abstracts of all the talks (Jeremy Blanc and Susanna Zimmermann prepared one extra long joint abstract for their talks).

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Subgroups of Cremona Groups

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Abstracts

Conjugacy classes of the Klein simple group in rank 3 Cremona group and geometry of del Pezzo fibrations

Hamid Ahmadinezhad
(joint work with Igor Krylov)

We word over the field of complex numbers.
The Klein simple group $\mathrm{PSL}_2(\mathbb{F}_7)$ is the automorphism group of the Klein quartic curve C defined as the vanishing of

$$Q = x^3y + y^3z + z^3x$$

in $\mathbb{P}^3$. The action of this group on the projective plane that leaves C invarient defines an embedding of $\mathrm{PSL}_2(\mathbb{F}_7)$ into $\mathrm{Cr}(2)$, the group of birational automorphisms of $\mathbb{P}^2$. Another embedding can be obtained by the action of $\mathrm{PSL}_2(\mathbb{F}_7)$ on the surface$S$, where S is the double cover of $\mathbb{P}^2$ branched along C. The surface S is a smooth del Pezzo surface of degree 2, hence it is rational resulting in an embedding of the Klein group in $\mathrm{Cr}(2)$. These two embeddings are however not conjugate in $\mathrm{Cr}(2)$ as both $\mathbb{P}^2$ and S are $\mathrm{PSL}_2(\mathbb{F}_7)$-birationally rigid [4, Theorem B.8]. The same actions can be lifted to $\mathbb{P}^2 \times \mathbb{P}^m$ and $S \times \mathbb{P}^m$, for any $m \geq 1$, by acting trivially on the second component. It is natural to ask whether the two induced embeddings of $\mathrm{PSL}_2(\mathbb{F}_7)$ in $\mathrm{Cr}(m+2)$ are conjugate, that is to ask whether the two embeddings of the Klein group in the plane Cremona group are stably conjugate. Following [3], it can be shown that stable conjugacy of the two embeddings of $\mathrm{PSL}_2(\mathbb{F}_7)$ as above would imply

$$\mathrm{H}^1(G, \mathrm{Pic}(\mathbb{P}^2)) = \mathrm{H}^1(G, \mathrm{Pic}(S))$$

for any subgroup G of $\mathrm{PSL}_2(\mathbb{F}_7)$. But this fails if $G = \mathbb{Z}_2$ that fixes a line in $\mathbb{P}^2$ and an elliptic curve in S [1, §2], hence these two embeddings of the Klein simple group in $\mathrm{Cr}(2)$ are stably non-conjugate. A particularly interesting case is when $m = 1$, that is to study conjugacy classes of $\mathrm{PSL}_2(\mathbb{F}_7)$ in $\mathrm{Cr}(3)$. By running equivariant resolution of singularities followed by equivariant minimal model programme, an action of a finite group on $\mathbb{P}^3$ can be lifted to a faithful action on a rational Mori fibre space. Hence, the study of finite subgroups of $\mathrm{Cr}(3)$ is replaced by the study of finite subgroups of the automorphism groups of Mori fibre spaces of dimension 3. There are three possibilities for rational Mori fibre spaces in dimension 3: Fano varieties, del Pezzo fibrations over $\mathbb{P}^1$, and conic bundles over rational surfaces. The study of conic bundles that admit an action of $\mathrm{PSL}_2(\mathbb{F}_7)$ is rather complicated. On the other hand, embeddings of $\mathrm{PSL}_2(\mathbb{F}_7)$ into $\mathrm{Cr}(3)$ coming from a faithful action on Fano 3-folds has been studied in [5]. There remains the study of del Pezzo fibrations that admit a $\mathrm{PSL}_2(\mathbb{F}_7)$ action and whether they are rational.
For each $n \in \mathbb{N}$, let $\mathcal{X}_n$ be a hypersurface in a toric variety T of Picard number two, constructed as follows. Let the coordinate ring of T be a $\mathbb{Z}^2$-graded ring with variables u, v, x, y, z, t, and grading $(1,0), (1,0)$ for u and v, and $(0,1), (0,1), (0,1)$,

and $(-n, 2)$ for x, y, z, t, and the irrelevant ideal $(u, v) \cap (x, y, z, t)$. Suppose $\mathcal{X}_n$ is a degree $(0, 4)$ hypersurface defined by

$$\alpha(u, v)t^2 + Q(x, y, z) = 0,$$

where α is a general homogeneous polynomial of degree $2n$. The singular locus of T is $\mathbb{P}^1_{u:v} \times \frac{1}{2}(1, 1, 1)$ quotient singularity. This locus is cut out by $\mathcal{X}_n$ in $2n$ points, the solutions of $\alpha = 0$ in $\mathbb{P}^1$, so that $\mathcal{X}_n$ has $2n$ singular points of type $\frac{1}{2}(1, 1, 1)$. For each n, the 3-fold $\mathcal{X}_n$ is a $\mathrm{PSL}_2(\mathbb{F}_7)$-del Pezzo fibrations of degree 2. See [1, §3] for details of the construction. It was shown in [6] that these are the only del Pezzo fibrations admitting an action of $\mathrm{PSL}_2(\mathbb{F}_7)$. Clearly, $\mathcal{X}_0$ is the same variety as $\mathbb{P}^2 \times \mathbb{P}^1$. In [1] I showed that $\mathcal{X}_1$ is $\mathrm{PSL}_2(\mathbb{F}_7)$-equivariantly birational to $S \times \mathbb{P}^1$ and conjectured that $\mathcal{X}_n$ is birationally rigid, in particular irrational, for $n \geq 2$, which implies that there are only two (stably non-conjugate) embeddings of $\mathrm{PSL}_2(\mathbb{F}_7)$ in $\mathrm{Cr}(3)$ coming from an action on a del Pezzo fibration. This conjecture, verified for $n \geq 3$ in [6], comes from a general expectation in the subject:

Conjecture. Let X be a semi-stable del Pezzo fibration of degree $1, 2,$ or 3 that is a Mori fibre space. Then X is birationally rigid if and only if $-K_X \notin \overline{\mathrm{Mob}}(X)^o$.

Unfortunately, this conjecture is out of reach at the moment. However, several theorems have been made that get us closer to the statement above. The "right" notion of semi-stability is the subject of an upcoming joint work of the author with Maksym Fedorchuk and Igor Krylov. In a recent joint work with Igor Krylov we proved the following theorem [2], which stands as the closest result to the conjecture above in degree 2.

Theorem. Let $\pi \colon X \to \mathbb{P}^1$ be a del Pezzo fibration of degree 2. Suppose X is a general quasismooth hypersurface of bi-degree $(4, \ell)$ in a $\mathbb{P}(1, 1, 1, 2)$-bundle over $\mathbb{P}^1$ satisfying the K^2-total condition. Then X is birationally rigid.

In the theorem above, quasi-smooth means that no singularities come from the defining equation of X, hence all singularities are indeed of type $\frac{1}{2}(1, 1, 1)$ and inherited from the ambient toric variety. The K^2-condition requires that $K_X^2 \notin \overline{\mathrm{NE}}(X)^o$, which is slightly weaker than the K-condition in the conjecture above. For every singular point Q of X there is a Sarkisov link starting by blowing up X at Q and results in a new quasi-smooth model of X, that is square birational. Let N be the number of singularities of X and denote the singular points by Q_i for $I \subset \{1, \ldots, N\}$. Denoting by X_I the model acquired by combining the elementary links corresponding to Q_i, $i \in I$, we say X satisfies the K^2-total condition if for every $I \subset \{1, \ldots, N\}$ the model X_I satisfies the K^2-condition. Let F be the quartic surface in $\mathbb{P}(1_x, 1_y, 1_z, 2_w)$ given by

$$wq_2(x, y, z) + q_4(x, y, z) = 0,$$

where q_2 and q_4 are homogeneous polynomials of degrees 2 and 4 respectively. The generality condition in the theorem asks that the intersection

$$q_2(x, y, z) = q_4(x, y, z) = 0$$

on $\mathbb{P}^2$ is 8 distinct points.

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G-birational rigidity of *G*-del Pezzo threefolds

Artem Avilov

In this talk we work over an angebraically closed field k of characteristic 0. Recall that a *G-variety* is a pair (X, ρ), where X is an algebraic variety and $\rho : G \to \mathrm{Aut}(X)$ is an injective homomorphism of groups. We say that G-variety X has *$G\mathbb{Q}$-factorial singularities* if every G-invariant Weil divisor of X is $\mathbb{Q}$-Cartier.

Let X be a G-variety with at most $G\mathbb{Q}$-factorial terminal singularities and $\pi : X \to Y$ be a G-equivariant morphism. We call π a *G-Mori fibration* if $\pi_*\mathcal{O}_X = \mathcal{O}_Y$, $\dim X > \dim Y$, the relative invariant Picard number $\rho^G(X/Y)$ is equal to 1 (in this case we say that G is *minimal*) and the anticanonical class $-K_X$ is π-ample. If Y is a point then X is a *$G\mathbb{Q}$-Fano variety.* If in addition the anticanonical class is a Cartier divisor then X is a *G-Fano variety.*

Let X be arbitrary normal projective G-variety of dimension 3. Resolving the singularities of X and applying the G-equivariant minimal model program we reduce X either to a G-variety with nef anticanonical class, or to a G-Mori fibration (see e.g. [10, §3]). So such fibrations (and $G\mathbb{Q}$-Fano varieties in particular) form a very important class in the birational classification. A projective n-dimensional variety X is a *del Pezzo variety* if it has at most terminal Gorenstein singularities and the anticanonical class $-K_X$ is ample and divisible by $n-1$ in the Picard group $\mathrm{Pic}(X)$. If a G-Fano variety X is a del Pezzo variety, then we say that X is a *G-del Pezzo* variety.

$G\mathbb{Q}$-factorial G-minimal three-dimensional G-del Pezzo varieties were partially classified by Yu. Prokhorov in [11]. The main invariant of a del Pezzo threefold X is the *degree* $d = (-\frac{1}{2}K_X)^3$, it is an integer in the interval from 1 to 8. In this talk we consider the case $d = 4$, 3 and 2. If $d = 8$ then X is a projective space. In this case equivariant birational geometry were studied by I. Cheltsov and C. Shramov in the paper [3]. If $d > 4$ then X is smooth (cf. [11]) while smooth del Pezzo threefolds and their automorphism groups are known well. For other types of G-Fano threefolds there are only some partial results.

Classification of finite subgroups of the Cremona group $\mathrm{Cr}_3(\mathrm{k})$ is one of the motivations of this research. The Cremona group $\mathrm{Cr}_n(\mathrm{k})$ is the group of birational automorphisms of the projective space $\mathbb{P}^n_{\mathrm{k}}$. Finite subgroups of $\mathrm{Cr}_2(\mathrm{k})$ were completely classified by I. Dolgachev and V. Iskovskikh in [9]. The core of their method is the following. Let G be a finite subgroup of $\mathrm{Cr}_2(\mathrm{k})$. The action of G can be regularized in the following sence: there exists a smooth projective G-variety Z and an equivariant birational morphism $Z \to \mathbb{P}^2$. Then we apply the equivariant minimal model program to Z and obtain a G-Mori fibration which is either a G-conic bundle over $\mathbb{P}^1$ (which is a blowing up of a Hirzebruch surface at some points), or a G-minimal del Pezzo surface. Dolgachev and Iskovskikh classified all minimal subgroups in automorphism groups of del Pezzo surfaces and conic bundles and so they obtained the full list of finite subgroups of $\mathrm{Cr}_2(\mathrm{k})$. But quite often two subgroups from such list are conjugate in $\mathrm{Cr}_2(\mathrm{k})$, so it is natural to identify them. One can see that G-varieties Z_1 and Z_2 give us conjugate subgroups if and only if there exists a G-equivariant birational map $Z_1 \dashrightarrow Z_2$. So we need to classify all rational G-Mori fibrations and birational maps between them as well.

Following this program in the three-dimemsional case one can reduce the question of classification of all finite subgroups in $\mathrm{Cr}_3(\mathrm{k})$ to the question of classification of all rational $G\mathbb{Q}$-Mori fibrations and birational equivariant maps between them. Such program was realized in some particular cases: simple non-abelian groups which can be embedded into $\mathrm{Cr}_3(\mathbb{C})$ (see [13], see also [4], [5], [6], [7]) and p-elementary subgroups of $\mathrm{Cr}_3(\mathbb{C})$ (see [12], [14]).

For applications to Cremona groups we are mostly interested in classification of *rational* del Pezzo varieties. Thus if degree of X is equal to 3 then we assume that x is singular (every smooth cubic threefold is not rational due to the classical result of Clemens and Griffiths [8]). The rationality of del Pezzo threefolds of degree 2 were stydied by I. Cheltsov, V. Przyjalkowski and C. Shramov in [2].

In this talk we are interested in the following problem: classify rational G-birationally rigid G-del Pezzo threefolds of degree less than 5. We give a partial answer for this question.

In this talk we use the following notation:

- $\mathfrak{C}_n$ is a cyclic group of order n;
- $\mathfrak{D}_{2n}$ is a dihedral group of order $2n$;
- $\mathfrak{S}_n$ is a symmetric group of degree n;
- $\mathfrak{A}_n$ is an alternating group of degree n.

The main our results is the following theorems:

Theorem 1 ([1])**.** Let X be a G-del Pezzo threefold of degree 4. Assume that X is G-birationally rigid. Then X is one of the following varieties:

(1) intersection of two quadrics in $\mathbb{P}^5$ with $\mathrm{rk}\,\mathrm{Cl}(X) = 5$. Such a variety is unique and his automorphism group is isomorphic to $(\mathbb{C}^* \rtimes \mathfrak{C}_2)^3 \rtimes \mathfrak{S}_3$;
(2) smooth intersection of two quadrics. In this case we have the following possibilities:
   (i) $\mathrm{Aut}(X) \simeq \mathfrak{C}_2^5 \rtimes \mathfrak{C}_5$;

(ii) $\mathrm{Aut}(X) \simeq \mathfrak{C}_2^5 \rtimes \mathfrak{D}_{12}$;
(iii) $\mathrm{Aut}(X) \simeq \mathfrak{C}_2^5 \rtimes \mathfrak{D}_6$;
(iv) the group $\mathrm{Aut}(X)$ fits in an exact sequence

$$0 \to \mathfrak{C}_2^5 \to \mathrm{Aut}(X) \to \mathfrak{S}_4 \to 0.$$

In cases (2, i), (2, ii) and (2, iv) such a variety X is unique up to isomorphism. In the case (2, iii) such varieties X form a one-parametric family.

In the case (2, i) the variety X is G-birationally rigid if and only if $G = \mathrm{Aut}(X)$ or $\mathfrak{C}2^4 \rtimes \mathfrak{C}_5$. In cases (1), (2, ii) and (2, iv) variety X is G-birationally superrigid with respect to $\mathrm{Aut}(X)$.

Theorem 2 ([2]). Let $X = X_3 \subset \mathbb{P}^4$ be a singular cubic hypersurface and G be a finite subgroup of $\mathrm{Aut}(X)$. Suppose that X is G-birationally rigid. Then there is only the following possibilities for X and G:

1. $X = \left\{ \sum_{i=0}^{5} x_i = \sum_{i=0}^{5} x_i^3 = 0 \right\} \subset \mathbb{P}^5$, i.e. X is the Segre cubic, and G is $\mathrm{Aut}(X) = \mathfrak{S}_6$, $\mathfrak{A}_6$, $\mathfrak{S}_5$ or $\mathfrak{A}_5$, two last subgroups are standard;
2. $X = \{x_0x_1x_2 - x_3x_4x_5 = \sum_{i=0}^{5} x_i = 0\} \subset \mathbb{P}^5$ and G is $\mathrm{Aut}(X) = \mathfrak{S}_3^2 \rtimes \mathfrak{C}_2$, $\mathfrak{S}_3^2$ (which acts transitively on the set of singularities) or $\mathfrak{C}_3^2 \rtimes \mathfrak{C}_4$.

All G-varieties of the first type are G-birationally superrigid and the variety of the second type is birationally superrigid with respect to the whole automorphism group.

Proposition 3. Let X be a G-del Pezzo threefold of degree 2. Assume that X has 13, 14 or 15 nodes and no other singularities. Then X is G-birationally rigid if and only if X has the following equations in $\mathbb{P}(2, 1, 1, 1, 1, 1)$:

$$y^2 = 4\sum_{i=1}^{5} x_i^4 - (\sum_{i=1}^{5} x_i^2)^2, \ \sum_{i=1}^{5} x_i = 0$$

and G is isomorphic to $\mathrm{Aut}(X) = \mathfrak{S}_5 \times \mathfrak{C}_2$ or $\mathfrak{S}_5$ (non-standard subgroup of $\mathrm{Aut}(X)$).

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Symmetries and equations of Del Pezzo surfaces and applications

Ingrid Bauer
(joint work with Fabrizio Catanese)

We consider an arrangement of lines $\mathcal{L} := \{L_0, L_1, \dots, L_r\}$ of $(r+1)$ lines in $\mathbb{P}^2$, not passing through one point. For any $n \geq 2$ we take the Galois covering $X(n, \mathcal{L})$ of $\mathbb{P}^2$ branched over the union of these $(r+1)$ lines with Galois group $(\mathbb{Z}/n\mathbb{Z})^r$. The *Hirzebruch-Kummer covering of exponent n associated to $\mathcal{L}$* is the minimal resolution $S(n, \mathcal{L})$ of $X(n, \mathcal{L})$. The simplest interesting example occurs when $r = 5$ and $\mathcal{L}$ is the complete quadrangle $\mathcal{CQ}$, the union of the sides and of the medians of a triangle (in other words, the six lines joining pairs of points of a projective basis P_1, P_2, P_3, P_4).
In this case $S(n) := S(n, \mathcal{CQ})$ is a smooth ramified Galois covering of the Del Pezzo surface Y_5 of degree 5, the blow-up of the plane in the points P_1, P_2, P_3, P_4. In (cf. [1]) it is shown that $S(5)$ is a ball quotient.
In particular $S(5)$ enjoys the following properties:

(1) $S(5)$ is rigid;
(2) $S(5)$ admits a Hermitian metric of strongly negative curvature;
(3) $S(5)$ is a projective classifying space (indeed $S(5)$ has a contractible universal cover $\tilde{S}(5) \cong \mathbb{B}_2 := \{z \in \mathbb{C}^2 || z| < 1\}$);
(4) the universal cover $\tilde{S}(5)$ of $S(5)$ is Stein.

A natural question ([2]) is whether these properties extend, for exponent n sufficiently large, to Hirzebruch-Kummer coverings $S(n, \mathcal{L})$ associated to rigid line configurations $\mathcal{L}$.
Motivated by these and other considerations, we analysed in [2] the first property in the particular case of $\mathcal{CQ}$, establishing the following result:

Theorem 1. The surface $S(n, \mathcal{CQ})$ is rigid (indeed, infinitesimally rigid) if and only if $n \geq 4$.

We raised therefore the following conjecture:

Conjecture 2. Given a rigid line configuration $\mathcal{L}$, then the surface $S(n, \mathcal{L})$ is rigid for n sufficiently large.

The proof of theorem 1 is quite long and technically involved, and makes use of the $\mathfrak{S}_5$-symmetries of the Del Pezzo surface Y_5 and vanishing theorems for twisted sheaves of logarithmic forms. The proof does not use the deformation invariance of the fibrations onto generalized Fermat curves induced by the projection of the plane with centre one of the singular points of the configuration.

In order to investigate the properties (2) and (3) for $S(n)$ it should be useful to have "good" equations for $S(n)$. Therefore we consider the Del Pezzo surface of degree 5: $Y = Y_5 := \hat{\mathbb{P}}^2(P_1, P_2, P_3, P_4)$, where $P_1, \dots, P_4$ is a projective basis of $\mathbb{P}^2$. Then there are 5 fibrations $\varphi_i : Y \to \mathbb{P}^1$, induced, for $1 \leq i \leq 4$, by the projection with centre P_i, and, for $i = 5$, by the pencil of conics through the 4 points. We have the following result:

Theorem 3.
1) Let $\Sigma \subset (\mathbb{P}^1)^4 =: Q$, with coordinates

$$(v_1 : v_2), (w_1 : w_2), (z_1 : z_2), (t_1 : t_2),$$

be the image of the Del Pezzo surface Y via $\varphi_1 \times \cdots \times \varphi_4$. Then the equations of Σ are given by the four 3×3-minors of the following Hilbert-Burch matrix:

$$A := \begin{pmatrix} t_2 & -t_1 & t_1 + t_2 \\ v_1 & v_2 & 0 \\ w_2 & 0 & w_1 \\ 0 & -z_1 & z_2 \end{pmatrix}. \tag{4}$$

In particular, we have a Hilbert-Burch resolution:

$$0 \to (\mathcal{O}_Q(-\sum_{i=1}^{4} H_i))^{\oplus 3} \to \bigoplus_{j=1}^{4}(\mathcal{O}_Q(-\sum_{i=1}^{4} H_i + H_j)) \to \mathcal{O}_Q \to \mathcal{O}_\Sigma \to 0, \tag{5}$$

where H_i is the pullback to Q of a point in $\mathbb{P}^1$ under the i-th projection.
2) The equations of $S(n) \subset C(n)^4$ (where $C(n)$ is the Fermat curve of degree n, $C(n) = \{Y_1^n + Y_2^n + Y_3^n = 0\} \subset \mathbb{P}^2$) are given by the four 3×3-minors of the following matrix:

$$A' := \begin{pmatrix} T_2 & -T_1 & -T_3 \\ V_1 & V_2 & 0 \\ W_2 & 0 & W_1 \\ 0 & -Z_1 & Z_2 \end{pmatrix}, \tag{6}$$

and the linear syzygies among the four equations are given by the columns of the matrix A'.

If one instead considers the anticanonical embedding $\varphi_{|-K_Y|} : Y \to \mathbb{P}^5$ of the del Pezzo surface of degree 5, then it is wellknown that the equations of Y are given by the (4×4) - Pfaffians of an anti-symmetric (5×5) - matrix of linear forms. In order to find a matrix which is equivariant under the $\mathfrak{S}_5$-action on Y, we choose as basis for $H^0(Y, \mathcal{O}_Y(-K_Y)) \cong \{F \in \mathbb{C}[x_1,x_2,x_3]_3 : F(P_1) = \ldots = F(P_4) = 0\}$:

$$s_{ij} := x_i x_j (x_j - x_k),\ 1 \leq i \neq j \leq 3.$$

Then we can prove the following result:

Theorem 7. Let Y be the del Pezzo surface of degree 5, embedded anticanonically in $\mathbb{P}^5$. Then the ideal of Y is generated by the 4×4 - Pfaffians of the $\mathfrak{S}_5$ - invariant anti-symmetric 6×6-matrix

$$\begin{pmatrix}
0 & \begin{array}{c} s_{21}+s_{23}- \\ -s_{31}-s_{32} \end{array} & \begin{array}{c} s_{12}+s_{31}- \\ -s_{32}-s_{21} \end{array} & -s_{13}-s_{21} & s_{31}+s_{32} & \begin{array}{c} s_{21}+s_{32}- \\ -s_{12}-s_{23} \end{array} \\
\hline
\begin{array}{c} -s_{21}-s_{23}+ \\ +s_{31}+s_{32} \end{array} & 0 & s_{12}+s_{23} & \begin{array}{c} s_{31}+s_{23}- \\ -s_{13}-s_{32} \end{array} & \begin{array}{c} s_{13}+s_{21}- \\ -s_{23}-s_{31} \end{array} & -s_{12}-s_{31} \\
\hline
\begin{array}{c} -s_{12}-s_{31}+ \\ s_{32}+s_{21} \end{array} & -s_{12}-s_{23} & 0 & \begin{array}{c} s_{12}+s_{13}- \\ -s_{32}-s_{31} \end{array} & \begin{array}{c} s_{12}-s_{21}+ \\ s_{31}-s_{13} \end{array} & s_{23}+s_{31} \\
\hline
s_{13}+s_{21} & \begin{array}{c} -s_{31}-s_{23}+ \\ s_{13}+s_{32} \end{array} & \begin{array}{c} -s_{12}-s_{13}+ \\ s_{32}+s_{31} \end{array} & 0 & -s_{21}-s_{32} & \begin{array}{c} s_{23}+s_{12}- \\ -s_{13}-s_{32} \end{array} \\
\hline
-s_{31}-s_{32} & \begin{array}{c} -s_{13}-s_{21}+ \\ s_{23}+s_{31} \end{array} & \begin{array}{c} -s_{12}+s_{21}- \\ -s_{31}+s_{13} \end{array} & s_{21}+s_{32} & 0 & \begin{array}{c} s_{12}+s_{13}- \\ -s_{12}-s_{23} \end{array} \\
\hline
\begin{array}{c} -s_{21}-s_{32}+ \\ s_{12}+s_{23} \end{array} & s_{12}+s_{31} & -s_{23}-s_{31} & \begin{array}{c} -s_{23}-s_{12}+ \\ s_{13}+s_{23} \end{array} & \begin{array}{c} -s_{12}-s_{13}+ \\ s_{12}+s_{23} \end{array} & 0
\end{pmatrix}$$

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Non-simplicity of the Cremona groups in dimension ≥ 3

Jérémy Blanc and Susanna Zimmermann

(joint work with Stéphane Lamy)

This is an abstract for two talks on a common work given by Susanna Zimmermann and then Jérémy Blanc in the workshop "subgroups of Cremona groups" held in Oberwolfach in June 2018.

The *Cremona group of rank n*, denoted by $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$, or simply $\mathrm{Bir}(\mathbb{P}^n)$ when the ground field $\mathbf{k}$ is implicit, is the group of birational transformations of the projective space.

The classical case is $n = 2$, where the group is already quite complicated but well described when $\mathbf{k}$ is algebraically closed. In this case the Noether-Castelnuovo Theorem [Cas01, Alb02] asserts that $\mathrm{Bir}(\mathbb{P}^2)$ is generated by $\mathrm{Aut}(\mathbb{P}^2)$ and a single quadratic transformation. This fundamental result together with the strong factorisation of birational maps between surfaces helps to have a good understanding of the group.

The dimension $n \geq 3$ is more difficult, as we do not have any analogue of the Noether-Castelnuovo Theorem and also no strong factorisation. Here is an extract from the article "Cremona group" in the Encyclopedia of Mathematics, written by V. Iskovskikh in 1987 (who uses the notation $\mathrm{Cr}(\mathbb{P}^n_{\mathbf{k}})$ for the Cremona group):

> *One of the most difficult problems in birational geometry is that of describing the structure of the group* $\mathrm{Cr}(\mathbb{P}^3_{\mathbf{k}})$*, which is no longer generated by the quadratic transformations. Almost all literature on Cremona transformations of three-dimensional space is devoted to concrete examples of such transformations. Finally, practically nothing is known about the structure of the Cremona group for spaces of dimension higher than* 3. [Isk87]

In fact, 30 years later there is still almost no result on the group structure of $\mathrm{Bir}(\mathbb{P}^n)$, but only about some of its subgroups. In arbitrary dimension there are descriptions of the algebraic subgroups of rank n [Dem70], and much more recently of their lattices [CX18]. For $n = 3$, there is also a classification of the connected algebraic subgroups [Ume85], and partial classification of finite subgroups (e.g. [Pro11, Pro12, PS16, BZ17]). There are also numerous articles devoted to the study of particular classes of examples of elements in $\mathrm{Bir}(\mathbb{P}^n)$, especially for n small (we do not attempt to start a list here, as it would always be very far from exhaustive).

Nevertheless, using modern tools such as Sarkisov links and Minimal model program, we are able, in this work, to give insight on the structure of the Cremona groups $\mathrm{Bir}(\mathbb{P}^n)$ and of its quotients.

0.1. **Normal subgroups.** The question of the non-simplicity of $\mathrm{Bir}(\mathbb{P}^n)$ for each $n \geq 2$ was also mentioned in the article of V. Iskovskikh in the Encyclopedia

> *It is not known to date* (1987) *whether the Cremona group is simple.* [Isk87]

but was in fact asked much earlier. It is explicitly mentioned in a book by F. Enriques in 1895:

> *Tuttavia altre questioni d'indole gruppale relative al gruppo Cremona nel piano* (*ed a più forte ragione in* S_n $n > 2$) *rimangono ancora insolute; ad esempio l'importante questione se il gruppo Cremona contenga alcun sottogruppo invariante* (*questione alla quale sembra probabile si debba rispondere negativamente*). [Enr95, p. 116][1]

[1] "However, other group-theoretic questions related to the Cremona group of the plane (and, even more so, of $\mathbb{P}^n$, $n > 2$) remain unsolved; for example, the important question of whether the Cremona group contains a non-trivial normal subgroup (a question which seems likely to be answered negatively)."

The feeling expressed by Enriques that the Cremona group should be simple was maybe supported by the analogy with biregular automorphism groups of projective varieties, such as $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{k})$. In fact in the trivial case of dimension $n = 1$, we have $\mathrm{Bir}(\mathbb{P}^1) = \mathrm{Aut}(\mathbb{P}^1) = \mathrm{PGL}_2(\mathbf{k})$, which is indeed a simple group when the ground field $\mathbf{k}$ is algebraically closed. Pushing further the analogy with algebraic groups, it was proved by J. Blanc that when considered as a topological group, the Cremona group $\mathrm{Bir}(\mathbb{P}^2)$ is indeed simple, in the sense that any proper Zariski closed normal subgroup must be trivial [Bla10]. This result was recently extended to arbitrary dimension by J. Blanc and S. Zimmermann [BZ18].

The non-simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ as an abstract group was proven, over an algebraically closed field, by S. Cantat and S. Lamy [CL13]. The idea of proof was to apply small cancellation theory to an action of $\mathrm{Bir}(\mathbb{P}^2)$ on a hyperbolic space. A first instance of roughly the same idea was [Dan74], in the context of plane polynomial automorphisms. The modern small cancellation machinery as developed in [DGO17] allowed A. Lonjou to prove the non simplicity of $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary field, and the fact that every countable group is a subgroup of a quotient of $\mathrm{Bir}(\mathbb{P}^2)$ [Lon16]. It turns out that all the normal subgroups produced by this technique have infinite index, and for instance the group $\mathrm{PGL}_2(\mathbf{k}(T))$ of Jonquières maps embeds in all associated quotients, which are in particular infinite non-abelian.

Another source of normal subgroups for $\mathrm{Bir}(\mathbb{P}^2)$, of a very different nature, was discovered by S. Zimmermann, when the ground field is $\mathbb{R}$ [Zim18]. In contrast with the case of an algebraically closed field where the Cremona group of rank 2 is a perfect group, she proved that the abelianisation of $\mathrm{Bir}_{\mathbb{R}}(\mathbb{P}^2)$ is an uncountable direct sum of $\mathbb{Z}/2$. Here the main idea is to use an explicit presentation by generators and relations. In fact a presentation of $\mathrm{Bir}(\mathbb{P}^2)$ over an arbitrary perfect field is available since [IKT93], but because they insist in staying inside the group $\mathrm{Bir}(\mathbb{P}^2)$, they obtain a very long list. However, if one accepts to consider birational maps between non-isomorphic varieties, the Sarkisov program provides more tractable lists of generators. Using this idea together with results of Kaloghiros [Kal13], the existence of abelian quotients for $\mathrm{Bir}(\mathbb{P}^2)$ was extended to the case of many non-closed perfect fields by S. Lamy and S. Zimmermann [LZ17].

The work presented in this abstract is a further extension in this direction, this time in arbitrary dimension, and over any ground field $\mathbf{k}$ which is abstractly isomorphic to a subfield of $\mathbb{C}$ (this includes any field of rational functions of any algebraic variety defined over a subfield of $\mathbb{C}$). Our main result is the following:

Theorem 1. For each subfield $\mathbf{k} \subseteq \mathbb{C}$ and $n \geq 3$, there is a group homomorphism

$$\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n) \twoheadrightarrow \bigoplus_I \mathbb{Z}/2$$

where the indexing set I has the same cardinality as $\mathbf{k}$, and such that the restriction to the subgroup given locally by

$$\{(x_1, \dots, x_n) \mapsto (x_1\alpha(x_2, \cdots, x_n), x_2, \dots, x_n) \mid \alpha \in \mathbf{k}(x_2, \dots, x_n)^*\}$$

is surjective. Moreover, $\mathrm{Aut}(\mathbb{P}^n) = \mathrm{PGL}_{n+1}(\mathbf{k})$ is contained in the kernel of the homomorphism. In particular, $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^n)$ is not perfect and thus not simple.

0.2. **Generators.** As already mentioned, the Noether-Castelnuovo theorem provides simple generators of $\mathrm{Bir}(\mathbb{P}^2)$ when $\mathbf{k}$ is algebraically closed. In dimension $n \geq 3$, we do not have a complete list of all Sarkisov links and thus are far from having an explicit list of generators for $\mathrm{Bir}(\mathbb{P}^n)$. The lack of an analogue to the Noether-Castelnuovo Theorem for $\mathrm{Bir}(\mathbb{P}^n)$ and the question of finding good generators was already cited in the article of the Encyclopedia above, in [HM13, Question 1.6], and also in the book of Enriques:

> *Questo teorema non è estendibile senz'altro allo S_n dove $n > 2$; resta quindi insoluta la questione capitale di assegnare le più semplici trasformazioni generatrici dell'intiero gruppo Cremona in S_n per $n > 2$.* [Enr95, p. 115][2]

A classical result, due to H. Hudson and I. Pan [Pan99], shows that $\mathrm{Bir}(\mathbb{P}^n)$, for $n \geq 3$, is not generated by $\mathrm{Aut}(\mathbb{P}^n)$ and finitely many elements. The reason is that one needs at least, for each irreducible variety Γ of dimension $n-2$, one birational map that contracts a hypersurface birational to $\Gamma \times \mathbb{P}^1$. These contractions can be realised in $\mathrm{Bir}(\mathbb{P}^n)$ by *Jonquières elements*, i.e. elements that preserve a family of lines through a given point, which form a subgroup

$$\mathrm{PGL}_2(\mathbf{k}(x_1, \ldots, x_n)) \rtimes \mathrm{Bir}(\mathbb{P}^{n-1}) \subseteq \mathrm{Bir}(\mathbb{P}^n).$$

Hence, it is natural to ask whether the group $\mathrm{Bir}(\mathbb{P}^n)$ is generated by $\mathrm{Aut}(\mathbb{P}^n)$ and by Jonquières elements (a question for instance asked in [PS15]).

We answer the question in dimension 3 by the negative, in the following stronger form:

Theorem 2. Let $\mathbf{k}$ be a subfield of $\mathbb{C}$. Then the normal subgroup of the Cremona group $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$ generated by $\mathrm{Aut}_{\mathbf{k}}(\mathbb{P}^3)$, by all Jonquières elements and by any finite set of elements is a strict subgroup of $\mathrm{Bir}_{\mathbf{k}}(\mathbb{P}^3)$.

It is interesting to make a parallel between this statement and the classical Tame Problem in the context of the affine Cremona group $\mathrm{Aut}(\mathbb{A}^n)$, or group of polynomial automorphisms (which is one of the "challenging problems" on the affine spaces, described in the Bourbaki seminar [Kra96]). The Tame Problem asks whether $\mathrm{Aut}(\mathbb{A}^n)$ is generated by the group $\mathrm{PGL}_{n+1}(\mathbf{k}) \cap \mathrm{Aut}(\mathbb{A}^n)$ of affine automorphisms and by the group of elementary automorphisms of the form $(x_1, \ldots, x_n) \mapsto (x_1 + P(x_2, \ldots, x_n), x_2, \ldots, x_n)$, analogue of the $\mathrm{PGL}_2(\mathbf{k}(x_1, \ldots, x_n))$ factor in the Jonquières group. It was solved in dimension 3 over a field of characteristic zero in [SU04], and is an open problem for $n \geq 4$.

[2] "This theorem can not be easily extended to $\mathbb{P}^n$ where $n > 2$; therefore, the main question of finding the most simple generating transformations of the entire Cremona group of $\mathbb{P}^n$ for $n > 2$ remains open."

On the one hand, one could say that our Theorem 2 is much stronger, since we consider the *normal* subgroup generated by these elements, and we allow some extra generators of bounded degree. It is not known (even if not very likely) whether one can generate $\mathrm{Aut}(\mathbb{A}^3)$ with linear automorphisms, elementary automorphisms and one single automorphism, and *a fortiori* neither whether the normal subgroup generated by these is the whole group $\mathrm{Aut}(\mathbb{A}^3)$.

On the other hand, even in dimension 3 we should stress that Theorem 2 does not recover a solution to the Tame Problem. The reason is that the group $\mathrm{Aut}(\mathbb{A}^3)$, or more generally the group $\mathrm{Bir}_0(\mathbb{P}^3)$ of birational maps that contract only rational hypersurfaces (birational maps of genus 0 in the sense of Frumkin [Fru73, Lam14]), lies in the kernel of all group homomorphisms produced by Theorem 1.

0.3. **Overwiew of the strategy.** By the Minimal model program, every rationally connected variety Z is birational to a Mori fibre space, and every birational map between two Mori fibre spaces is a composition of simple birational maps, called *Sarkisov links*. We associate to such a variety Z the groupoid $\mathrm{BirMori}(Z)$ of all birational maps between Mori fibre spaces birational to Z. We then concentrate on some special Sarkisov links, called *Sarkisov links of conic bundles of type* II.

To each Sarkisov link of conic bundles of type II, we associate a marked conic, which is a pair $(X/B, \Gamma)$, where X/B is a conic bundle (a terminal Mori fibre space with $\dim B = \dim X - 1$) and $\Gamma \subseteq B$ is an irreducible hypersurface, and measure the base-locus and the conic bundle associated to the link. We then define a natural equivalence relation between marked conics, and associate each of these Sarkisov links χ an integer $\mathrm{covgon}(\chi) = \mathrm{covgon}(\Gamma)$ that measures the degree of irrationality of Γ. For each variety Z, we denote by $\mathcal{C}_Z$ the set of equivalence classes of conic bundles X/B with X birational to Z, and for each class of conic bundles $C \in \mathcal{C}_Z$ we denote by $\mathcal{M}_C$ the set of equivalence classes of marked conics $(X/B, \Gamma)$, where C is the class of X/B.

The Sarkisov program is established in every dimension [HM13] and relations among them are described in [Kal13]. Inspired by the latter, we define *rank* r *fibrations* X/B ; rank 1 fibrations are Mori fibres spaces and rank 2 fibrations dominate Sarkisov links. We prove that the relations among Sarkisov links are generated by *elementary relations*, which are relations dominated by rank 3 fibrations.

The BAB conjecture, proven in [Bir16a] and [Bir16b], tells us that the set of weak Fano terminal varieties of dimension n form a bounded family and the degree of their images by a (universal) multiple of the anticanonical system is bounded by a (universal) integer d. As a consequence, we show that any Sarkisov link χ of conic bundles of type II appearing in an elementary relation over a base of small dimension has $\mathrm{covgon}(\chi) \leq d$. This and the description of the elementary relations over a base of maximal dimension and including a Sarkisov link of conic bundles of type II allow us to prove the following statement

Theorem 3. Let $n \geq 3$. There is an integer $d \geq 2$ such that for every conic bundle X/B, where X is a rationally connected variety of dimension n, we have a groupoid homomorphism

$$\mathrm{BirMor}i(X) \to \mathop{\ast}_{C\in\mathcal{C}_X} \left(\bigoplus_{\mathcal{M}_C} \mathbb{Z}/2 \right)$$

that sends each Sarkisov link of conic bundles χ of type II with $\mathrm{covgon}(\chi) \geq d$ onto the generator indexed by its marked conic, and all other Sarkisov links and all automorphisms of Mori fibre spaces onto zero. Moreover it restricts to group homomorphisms

$$\mathrm{Bir}(X) \to \mathop{\ast}_{C\in\mathcal{C}_X} \left(\bigoplus_{\mathcal{M}_C} \mathbb{Z}/2 \right), \qquad \mathrm{Bir}(X/B) \to \bigoplus_{\mathcal{M}_{X/B}} \mathbb{Z}/2.$$

In order to deduce Theorem 1 and Theorem 2, we study the image of the group homomorphism from $\mathrm{Bir}(X)$ and $\mathrm{Bir}(X/B)$ provided by Theorem 3, for some conic bundle X/B. We give a criterion to compute the image and then apply this criterion to show that the image is very large if the generic fibre of X/B is $\mathbb{P}^1$ (or equivalently if X/B has a rational section, or is equivalent to $(\mathbb{P}^1 \times B)/B$). This allows us to prove Theorem 1. We finish by studying the more delicate case where the generic fibre X/B is not $\mathbb{P}^1$ (or equivalently if X/B has no rational section). We manage to give such examples where X is a rational threefold and where the image of the group homomorphism is big enough to deduce Theorem 2.

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Universal CH_0, the Brauer group and conic bundle threefolds in characteristic 2

Christian Böhning
(joint work with Asher Auel, Alessandro Bigazzi and Hans-Christian Graf von Bothmer)

The talk was an overview of the main results of the two recent preprints [ABBB18-1], [ABBB18-2].

The degeneration method due to Voisin [Voi15], Colliot-Thélène, Pirutka [CT-P16] et al. can be summarised as follows. Let R be a discrete valuation ring with algebraically closed residue field k and field of fractions K. Let $\pi\colon \mathcal{X} \to \operatorname{Spec} R$ be a flat surjective projective morphism with integral geometric fibres, smooth geometric generic fibre $X_{\bar{K}}$, and mildly singular special fibre X_k: the latter means that there should exist a CH_0-universally trivial resolution of singularities $\tilde{X}_k \to X_k$. Then it holds that if $X_{\bar{K}}$ has universally trivial Chow group of zero cycles, so does $\tilde{X}_k$. In particular, since smooth stably rational varieties have universally trivial CH_0, it follows that if one can produce an obstruction to $X := \tilde{X}_k$ having universally trivial CH_0, then one can conclude that $X_{\bar{K}}$ is not stably rational.

Such obstructions have traditionally been of two types: (1) nonzero sections in some $H^0(X, \Omega^i_X)$, $i > 0$, for R of unequal characteristic; (2) unramified elements in $M_*(k(X))$, where M_* is some Rost cycle module.

We gave an indication of the proof of the following result.

Theorem 1. Suppose X is a smooth projective variety over any (not necessarily algebraically closed) field k. If X has universally trivial CH_0, then the Brauer group $\mathrm{Br}(X)$ is reduced to $\mathrm{Br}(k)$.

This is new only if the torsion order of the Brauer classes equals the characteristic of k, otherwise it had previously been proven by Merkurjev.

We are particularly interested to study the case when X is birationally a conic bundle over a smooth projective surface S and the ground field k is algebraically closed of characteristic 2. The reason for this is that in tame cases the Brauer group is still governed by the arrangement and double covers of discriminant components, but those are now subject to Artin-Schreier theory (not Kummer theory any more), and the usual formalism of Gersten complexes to ascertain which discriminant profiles are realised by actual conic bundles breaks down in characteristic 2: hence we expect that there will be new and interesting ways to degenerate.

More precisely, for k of char. 2, we call a flat surjective projective morphism $f\colon Y \to S$ a conic bundle if each geometric fibre is isomorphic to a plane conic, and if the geometric generic fibre is a smooth conic: hence wild conic bundles in the sense of Mori are excluded for the time being. In addition, to be able to formulate our result concisely, let us call a conic bundle *good* if (a) all the irreducible components of its discriminant are reduced, (b) the geometric generic fibre of the conic bundle restricted to each component of the discriminant is a cross of lines, (c) the conic bundle induces a nontrivial $\mathbb{Z}/2$ Galois extension of the function field of each discriminant component, i.e., the double cover over it is non-split.

Theorem 2. Let $Y \to S$ and $Y' \to S$ be good conic bundles over a smooth projective surface S and $k = \bar{k}$ of char. 2. Suppose that

(1) the discriminant Δ_Y of Y can be written as

$$\Delta_Y = \Delta_{Y'} \cup \Delta''$$

where $\Delta_{Y'}$ is the discriminant of $Y' \to S$, and Δ'' is the union of the discriminant components of $Y \to S$ that are not discriminant components of $Y' \to S$. Also assume that Δ'' is not empty;

(2) the $\mathbb{Z}/2$ Galois extensions induced by $Y \to S$ and $Y' \to S$ are isomorphic for every irreducible component of $\Delta_{Y'}$;

(3) whenever p is a point of $\Delta_{Y'} \cap \Delta''$, the fibre of $Y' \to S$ over p is a cross of lines.

Then for every resolution of singularities $\tilde{Y} \to Y$, the Brauer group $\mathrm{Br}(\tilde{Y})[2]$ is not trivial.

In [ABBB18-2] this is applied to show that a certain concretely given conic bundle, which is a divisor of bidegree $(2, 2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ and defined over $\mathbb{Z}$, is not stably rational over $\mathbb{C}$. If one reduces mod. 2, the preceding theorem applies in this case. However, if one reduces modulo any other prime, the Brauer group is trivial, and we know of no other method that yields this result other than reducing modulo 2 and applying the theorem.

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Parabolic elements of the Cremona group

Serge Cantat
(joint work with Yves de Cornulier, Julie Déserti and Junyi Xie)

The Cremona group in two variables, over the field of complex numbers, is the group $Cr_2(\mathbf{C})$ of all birational transformations of the projective plane $\mathbb{P}^2_{\mathbf{C}}$. This group acts faithfully by isometries on an infinite dimensional hyperbolic space $\mathbb{H}^\infty$, which is obtained by taking the limit of all Néron-Severi groups $NS(X;\mathbf{R})$ of all rational surfaces

$$\pi\colon X \to \mathbb{P}^2_{\mathbf{C}},$$

π being a birational morphism (see the book "Cubic Forms" of Yuri Manin). The limit of these Néron-Severi groups comes with an intersection form of signature $(1,\infty)$, and $\mathbb{H}^\infty$ is the connected component of the set $\{u \mid u\cdot u = 1\}$ that contains the class of a line $\ell \subset \mathbb{P}^2_{\mathbf{C}}$.

This representation $Cr_2(\mathbf{C}) \to Isom(\mathbb{H}^\infty)$ is useful to describe algebraic properties of $Cr_2(\mathbf{C})$. For instance, a birational transformation f of $\mathbb{P}^2_{\mathbf{C}}$ acts as a parabolic isometry on $\mathbb{H}^\infty$ if and only if f satisfies one of the following two properties:

(1) $\deg(f^n) \simeq n$, and then f preserves a unique pencil of rational curves;
(2) $\deg(f^n) \simeq n^2$ and then f preserves a unique pencil of curves of genus 1.

Here $\deg(f)$ is the degree of the total transform of a line by f, f^n is the n-th iterate of f, and $u_n \simeq v_n$ if there is a positive constant α such that $|u_n - \alpha v_n|/|u_n|$ converges towards 0 as n goes to $+\infty$. This result, due to Gizatullin and to Diller and Favre, provides a path from hyperbolic geometry to algebraic geometry, and vice versa.

In collaboration with Yves de Cornulier, and with Julie Déserti and Junyi Xie, we push the interplay between hyperbolic and algebraic geometry farther, and show the following results. Let f be a parabolic element of $Cr_2(\mathbf{C})$, with quadratic degree growth. Firstly, f is not distorted in $Cr_2(\mathbf{C})$. This means that, given any finite subset S of $Cr_2(\mathbf{C})$ which generates a subgroup $\langle S\rangle$ containing f, the word-length of f^n with respect to S grows linearly with n. Secondly, one can bound the degree of the invariant pencil (of curves of genus 1 in $\mathbb{P}^2_{\mathbf{C}}$) by a function of $\deg(f)$. Thirdly, the conjugacy class of f in $Cr_2(\mathbf{C})$ is constructible: for every degree k, the intersection of the conjugacy class in the space of birational transformations of degree k is constructible (in the sense of Chevalley, i.e. in the Zariski topology). The first and second properties extend to all parabolic elements of $Cr_2(\mathbf{C})$, but not the third.

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Algebraic vector bundles on the 2-sphere and smooth rational varieties with infinitely many real forms

Adrien Dubouloz
(joint work with Gene Freudenburg and Lucy Moser-Jauslin)

A classical problem in real algebraic geometry is the classification of real forms of a given real algebraic variety X, that is, real algebraic varieties Y non isomorphic to X but whose complexifications $Y_{\mathbb{C}}$ are isomorphic to $X_{\mathbb{C}}$ as complex algebraic varieties. For example, the smooth real affine algebraic surfaces $\mathbb{S}^2 = \{x^2+y^2+z^2=1\}$ and $X = \{uv+z^2=1\}$ in $\mathbb{A}^3_{\mathbb{R}}$ have isomorphic complexifications, an explicit isomorphism being simply given by the linear change of complex coordinates $u = x+iy$ and $v = x-iy$, but are non isomorphic. This follows for instance from the fact that the set of real points of $\mathbb{S}^2$ is the usual euclidean 2-sphere $S^2 \subset \mathbb{R}^3$ whereas the set of real points of X is not compact for the euclidean topology.

Examples of smooth real projective varieties admitting infinitely many pairwise non-isomorphic real forms were only found very recently successively by Lesieutre [7] in dimension ≥ 6 and by Dinh-Oguiso [3] in every dimension ≥ 2. These are obtained as a by-product of clever constructions of smooth complex projective algebraic varieties defined over $\mathbb{R}$ with discrete but non finitely generated automorphism groups containing infinitely many conjugacy classes algebraic involutions. All their examples are non geometrically rational and the question of existence of rational real algebraic varieties, projective or not, with infinitely many real forms was left open. Our first main result explicitly fills this gap for smooth real affine fourfolds:

Theorem 1. The smooth rational real affine fourfold $\mathbb{S}^2 \times \mathbb{A}^2_{\mathbb{R}}$ has at least countably infinitely many pairwise non-isomorphic real forms.

In contrast with the examples found by Lesieutre and Dinh-Oguiso, which rely on constructions of special classes of complex projective varieties by techniques of birational geometry, ours are inspired by basic results on the classification of topological vector bundles on the real sphere $S^2 \subset \mathbb{R}^3$. Our construction can indeed be interpreted as a sort of "algebraization" of the property that the complexification $E \otimes_{\mathbb{R}} \mathbb{C}$ of any topological real vector bundle $\pi : E \to S^2$ of rank 2 on S^2 is isomorphic, as a topological real vector bundle of rank 4, to the trivial bundle $S^2 \times \mathbb{R}^4$. More precisely, we show that the topological real vector bundles of rank 2 on S^2, which are nothing but the underlying real vector bundles of the complex line bundles $\mathcal{O}_{\mathbb{CP}^1}(n)$, $n \geq 0$, over $\mathbb{CP}^1 \simeq S^2$, admit algebraic models in the form of algebraic vector bundles $p_n : V_n \to \mathbb{S}^2$ of rank 2 on $\mathbb{S}^2$ with pairwise non-isomorphic total spaces, but whose complexifications $p_{\mathbb{C}} : V_{n,\mathbb{C}} \to \mathbb{S}^2_{\mathbb{C}}$ are all isomorphic to the trivial bundle $\mathbb{S}^2_{\mathbb{C}} \times \mathbb{A}^2_{\mathbb{C}}$.

The existence of such algebraic models was known in the form of certain projective modules on the coordinate ring of the affine surface $\mathbb{S}^2$ after successive works of Fossum [4] and Moore [9] and, later on, of Swan [10]. Nevertheless, we give a new alternative geometric construction which provides models of these bundles in the form of restrictions to $\mathbb{S}^2$ of natural algebraic vector bundles on the real projective quadric $\mathbb{Q}^2 = \left\{X^2 + Y^2 + Z^2 - T^2 = 0\right\}$ in $\mathbb{P}^3_{\mathbb{R}}$.

It is worth noticing that by a result of Kambayashi [6], $\mathbb{A}^2_{\mathbb{R}}$ has no nontrivial real form. One can check along the same lines using the fact that similarly as to $\mathrm{Aut}(\mathbb{A}^2_{\mathbb{C}})$, the automorphism group $\mathrm{Aut}(\mathbb{S}^2_{\mathbb{C}})$ of $\mathbb{S}^2_{\mathbb{C}} \simeq X_{\mathbb{C}}$ has a structure of a free product of two subgroups amalgamated along their intersection [2, 8], that X is the unique nontrivial real form of $\mathbb{S}^2$. So while $\mathbb{A}^2_{\mathbb{R}}$ and $\mathbb{S}^2$ both have finitely many real forms, the total spaces of the algebraic vector bundles $p_n : V_n \to \mathbb{S}^2$ provide an infinite countable family of real forms of $\mathbb{S}^2 \times \mathbb{A}^2_{\mathbb{R}}$ which are by construction pairwise locally isomorphic over $\mathbb{S}^2$, but globally pairwise non-isomorphic as real algebraic varieties. In contrast, reminiscent of the fact that for every $r \geq 3$ there exists a unique nontrivial topological real vector bundle of rank r on S^2, it follows in particular from a surprising result of Barge and Ojanguren [1] that the varieties $V_n \times \mathbb{A}^{r-2}_{\mathbb{R}}$, $n \geq 0$, give rise to a unique class of nontrivial real form of $\mathbb{S}^2 \times \mathbb{A}^r_{\mathbb{R}}$. This observation generalizes in passing a famous counter-example to the Zariski Cancellation Problem constructed by Hochster [5].

Our construction thus does not directly yield higher dimensional families of examples by simply taking product with affine spaces. Nevertheless, a suitable adaptation of the technique used by Dinh-Oguiso [3], consisting in our situation of taking products of the V_n with well-chosen real rational affine varieties of log-general type, allows us to derive the following general existence result:

Theorem 2. For every $d \geq 4$, there exist smooth rational real affine varieties of dimension d which have at least countably infinitely many pairwise non-isomorphic real forms.

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Automorphisms of del Pezzo surfaces in positive characteristic

Alexander Duncan
(joint work with Igor Dolgachev)

Let k be an algebraically closed field of arbitrary characteristic p. I reported on progress towards a full classification of the automorphism groups of del Pezzo surfaces over k. Our work is a first step towards understanding all finite subgroups of the plane Cremona group in positive characteristic.

In [3], Dolgachev and Iskovskikh (almost) classified all finite subgroups of the plane Cremona group over the complex numbers $\mathbb{C}$. More specifically, finite subgroups of the plane Cremona group are precisely those with faithful actions on minimal rational G-surfaces. By the classification of Manin [6] and Iskovkikh [5], we know that these are either del Pezzo surfaces or conic bundle surfaces.

Recall that a del Pezzo surface is a smooth projective surface X with an ample anticanonical bundle $-K_X$. The degree of a del Pezzo surface $d = K_X^2$ is an integer with values $1 \leq d \leq 9$. Except for $\mathbb{P}^1 \times \mathbb{P}^1$, they are isomorphic to blow ups of $9-d$ points in general position on the plane $\mathbb{P}^2$.

There is unique del Pezzo surface X of degree $d \geq 5$, except for $d = 8$ where there are exactly two. Their automorphism groups $\mathrm{Aut}(X)$ are listed in Table 1, where $B_n(\mathbb{P}^2)$ denotes $\mathbb{P}^2$ blown up at n general points and $\overline{\mathcal{M}_{0,5}}$ is the moduli space of stable curves of genus 0 with 5 marked points.

Degree	X	$\mathrm{Aut}(X)$
9	$\mathbb{P}^2$	$\mathrm{PGL}_3(k)$
8	$\mathbb{P}^1 \times \mathbb{P}^1$	$\mathrm{PGL}_2(k)^2 \rtimes 2$
	$B_1(\mathbb{P}^2)$	subgroup of $\mathrm{PGL}_3(k)$
7	$B_2(\mathbb{P}^2)$	subgroup of $\mathrm{PGL}_3(k)$
6	$B_3(\mathbb{P}^2)$	$(k^\times)^2 \rtimes (\mathfrak{S}_3 \times \mathfrak{S}_2)$
5	$\overline{\mathcal{M}_{0,5}}$	$\mathfrak{S}_5$

Table 1. Automorphism groups of del Pezzo surfaces of degree ≥ 5.

Despite appearances, the automorphism groups can vary dramatically as abstract groups as the field varies. For example, in characteristic 0, the only non-abelian simple subgroups that appear are the alternating groups $\mathfrak{A}_5$ and $\mathfrak{A}_6$, and the group $\mathrm{PSL}_2(7)$ of order 168. In contrast, there are infinitely many non-abelian simple subgroups of the form $\mathrm{PSL}_2(q)$ and $\mathrm{PSL}_3(q)$ for $q = p^n$ as $n \to \infty$.

A del Pezzo surface of degree 4 is a smooth complete intersection of two quadrics in $\mathbb{P}^5$. Since 5 general points sit in a unique conic, del Pezzo surfaces of degree 4 are in bijection with 5-tuples of distinct points on $\mathbb{P}^1$. A classical result of Cauchy and Jacobi tells us that when $p \neq 2$, the quadrics can be simultaneously diagonalized. This is impossible when $p = 2$, so we determined a new normal form that works in this case [1]. Using the normal forms, we prove the following, which is new in characteristic 2:

Theorem 1. If X is a del Pezzo surface of degree 4, then:

$$\mathrm{Aut}(X) = R \rtimes G$$

where $R \cong 2^4$ is an elementary abelian 2-group generated by 5 reflections in canonical bijection with the 5 points on $\mathbb{P}^1$ and G is isomorphic to the subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ which leaves invariant the set of 5 points.

The specific automorphism groups $\mathrm{Aut}(X)$ that occur can now be obtained by enumerating the embedded automorphism groups of 5 points in $\mathbb{P}^1$. They are listed in Table 2.

char(k)	Aut(X)	Order
any	2^4	16
$\neq 2$	$2^4 \rtimes 2$	32
2	$2^4 \rtimes 2^2$	64
$\neq 2, 5$	$2^4 \rtimes 4$	64
$\neq 2, 3$	$2^4 \rtimes \mathfrak{S}_3$	96
$\neq 2, 5$	$2^4 \rtimes D_{10}$	160
5	$2^4 \rtimes (5 \rtimes 4)$	320
2	$2^4 \rtimes \mathfrak{A}_5$	960

Table 2. Automorphism groups of del Pezzo surfaces of degree 4.

A del Pezzo surface of degree 3 is a smooth cubic surface in $\mathbb{P}^3$. Over $\mathbb{C}$, automorphism groups were classified by Kantor, Wiman and Segre but it seems the first error-free clasification is due to Hosoh [4] in 1997! We determine a full list of possible automorphism groups for positive characteristics in [2], which I list in Table 3. We also find normal forms for cubic surfaces realizing each abstract automorphism group as well as determine the dimensions of their corresponding strata in the moduli space.

We do not yet have a full classification for del Pezzo surfaces of degree 2 and 1 in positive characteristic.

char(k)	Aut(X)	Order
any	1	1
any	2	2
$\neq 2$	2^2	4
$\neq 2$	4	4
any	$\mathfrak{S}_3$	6
$\neq 2, 3$	8	8
$\neq 2$	$\mathfrak{S}_3 \times \mathfrak{S}_2$	12
2	2^4	16
$\neq 2$	$\mathfrak{S}_4$	24
any	$\mathcal{H}_3(3) \rtimes 2$	54
$\neq 2, 3$	$\mathcal{H}_3(3) \rtimes 4$	108
$\neq 2, 5$	$\mathfrak{S}_5$	120
2	$2^3 \rtimes \mathfrak{S}_4$	192
3	$\mathcal{H}_3(3) \rtimes 8$	216
$\neq 2, 3$	$3^3 \rtimes \mathfrak{S}_4$	648
2	$\mathrm{PSU}_4(2)$	25920

Table 3. Automorphism groups of cubic surfaces.

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Birational geometry and mirror symmetry of Calabi–Yau pairs

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(joint work with Alessio Corti)

A Calabi–Yau (CY) pair (X, D_X) consists of a normal projective variety X and a reduced sum of integral Weil divisors D_X such that $K_X + D_X \sim_{\mathbb{Z}} 0$.

The pair (X, D_X) has (t, dlt) (resp. (t, lc)) singularities if X is terminal and (X, D_X) divisorially log terminal (resp. log canonical).

A birational map $(X, D_X) \overset{\varphi}{\dashrightarrow} (Y, D_Y)$ is called volume preserving if for any geometric valuation E with centre on X and on Y, the equality $a_E(K_X + D_X) = a_E(K_Y + D_Y)$ holds.

One can define an invariant of the volume preserving class of a (t, lc) CY pair as follows. Recall that the dual complex of a dlt pair $(Z, D_Z = \sum D_i)$ is a regular cell complex constructed by attaching an $(|I|-1)$-dimensional cell for every irreducible component of $\bigcap_{i \in I} D_i$ a non-empty intersection of components of D_Z. De Fernex, Kollár and Xu show that the PL homeomorphism class of the dual complex is a volume preserving birational invariant of a dlt pair [1]. By [2], a (t, lc) CY pair (X, D_X) has a volume preserving (t, dlt) modification $(\widetilde{X}, D_{\widetilde{X}}) \to (X, D_X)$. Abusing notation, I call dual complex of (X, D_X) the PL homeomorphism class of the dual complex of a volume preserving (t, dlt) modification. The dual complex is denoted $\mathcal{D}(X, D_X)$; it is an invariant of the volume preserving birational equivalence class of (X, D_X).

The underlying varieties of CY pairs range from Calabi–Yau to Fano varieties, but X being Fano is not a volume preserving birational invariant of (X, D_X). A CY pair (X, D_X) has *maximal intersection* if $\dim \mathcal{D}(X, D_X) = \dim X - 1$. In other words, (X, D_X) has maximal intersection if a volume preserving (t, dlt) modification has a 0-dimensional log canonical centre. Since the dual complex is a volume preserving birational invariant, so is the property of having maximal intersection. Pairs with maximal intersection have some *Fano-type* properties, in a sense made precise by the following result.

Theorem 1 ([3]). Let (X, D_X) be a CY pair with maximal intersection. Then, there is a volume preserving birational map $(X, D_X) \dashrightarrow (Z, D_Z)$ to a CY pair whose boundary fully supports a big and semiample divisor.

Note that having maximal intersection is a "degenerate" condition; a CY pair (X, D_X) whose underlying variety is Fano does not have maximal intersection in general.

Example 2. A toric pair (X_Σ, D_Σ) is a CY pair formed by a toric variety and the reduced sum of its toric invariant divisors. A volume preserving birational map to a toric pair is called a toric model. Any (t, lc) CY pair with a toric model has maximal intersection.

Example 3. In dimension 2, this is an equivalence: CY pairs with maximal intersection are precisely those with a toric model.

The existence of a toric model for a pair is difficult to determine. The results of [5] state criteria that characterise toric pairs, but it is not clear whether or how such criteria could be extended to characterise CY pairs with a toric model.

A motivation to understand better the birational geometry of CY pairs and their relation to toric pairs comes from mirror symmetry. Most known constructions of mirror partners make use of toric features of the varieties or pairs considered, such as the existence of a toric model. In an exciting development, Gross, Hacking and Keel propose a construction of the mirror partner of CY pairs with maximal intersection; they conjecture:

Conjecture 4 ([4]). Let (Y, D_Y) be a simple normal crossings CY pair with maximal boundary such that D_Y supports an ample divisor (in particular $U =$

$Y \setminus D_Y$ is affine). Let $R = k[\mathrm{Pic}(Y)^\times]$, Ω the canonical volume form on U and

$$U^{\mathrm{trop}}(\mathbb{Z}) = \{ \text{ divisorial valuations: } k(U) \setminus \{0\} \to \mathbb{Z} | v(\Omega) < 0\} \cup \{0\}.$$

Then, denoting by V the free R-module with basis $U^{\mathrm{trop}}(\mathbb{Z})$, V has a natural finitely generated R-algebra structure whose structure constants are non-negative integers determined by counts of rational curves on U. The associated fibration $p\colon \mathrm{Spec(V)} \to \mathrm{Spec(R)} = \mathrm{T_{Pic(Y)}}$ is a flat family of affine log CY varieties with maximal boundary. Letting $K = \mathrm{Ker}\{\mathrm{PicY} \to \mathrm{Pic(U)}\}$, the map p is T_K-equivariant. The quotient family $\mathrm{Spec(V)/T_K} \to \mathrm{T_{Pic(U)}}$ depends only on U and is the mirror family to U.

Versions of this conjecture are proved for cluster varieties in [6], but relatively few examples are known.

In this talk, I present some examples of (t, lc) CY pairs with maximal intersection which do not have a toric model because their underlying varieties are birationally rigid.

One can construct volume preserving (t, dlt) modifications of these pairs that have relatively mild singularities and for which one expects to be able to compute the punctured Gromov-Witten invariants appearing in the conjecture. There is no known construction of mirror partners for these pairs, and they would be natural examples on which to test and study the conjecture.

Example 5. Consider the pair (X, D_X) where:

$$X = \{x_0^4 + x_1^4 + x_2^4 + x_0x_1x_2x_3 + x_4(x_3^3 + x_4^3) = 0\} \text{ and } D_X = X \cap \{x_4 = 0\}.$$

The quartic X is rigid because it is nonsingular. The unique singular point $p = (0{:}0{:}0{:}1{:}0)\}$ of D_X is locally analytically equivalent to a $T_{4,4,4}$ singular point $0 \in \{x^4 + y^4 + z^4 + xyz = 0\}$. A volume preserving (t, dlt) modification of (X, D_X) is obtained by taking a resolution of the cusp singularity of D_X.

Example 6. (Example due to R. Svaldi) Consider the smooth cubic 3-fold

$$X = \{x_0x_1x_2 + x_1^3 + x_2^3 + x_3q + x_4q' = 0\},$$

where q, q' are general conics in $x_0, \cdots, x_4$ with $(q(1,0,0,0,0), q'(1,0,0,0,0)) \neq (0,0)$. Let $\Pi = \{x_3 = x_4 = 0\}$ and $D = \{x_3 = 0\} + \{x_4 = 0\}$. The section $\Pi \cap X$ is a cubic with a node at $p = (1{:}0{:}0{:}0{:}0)$; $p \in D_X$ is locally analytically equivalent to $0 \in \{x^2y^2 - z^2 = 0\}$. A volume preserving (t, dlt) modification of (X, D_X) can be constructed, showing that (X, D_X) has maximal intersection.

Example 7. Consider the pair (X, D_X) where:

$$X = \{x_1^2x_2^2 + x_1x_2x_3l + x_3^2q + x_4f_3 = 0\}, D_X = X \cap \{x_4 = 0\},$$

where l is a general linear form and q a general conic in $x_0, \cdots, x_3$ and f_3 is a general cubic in $x_0, \cdots, x_4$. The surface D_X is non normal as it has multiplicity 2 along $L_1 = \{x_1 = x_3 = x = 4 = 0\}$ and $L_2 = \{x_2 = x_3 = x = 4 = 0\}$; the point $p = L_1 \cap L_2$ is locally analytically equivalent to $0 \in \{x^2y^2 - z^2 = 0\}$. The quartic X has three ordinary double points lying on $L_1 \cap \{f_3 = 0\}$ and three ordinary double points lying on $L_2 \cap \{f_3 = 0\}$. As X has less than 9 ordinary double points, X

is birationally rigid. A volume preserving (t, dlt) modification of (X, D_X) can be constructed, showing that (X, D_X) has maximal intersection.

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Stability over rings and good models

Igor Krylov

(joint work with Hamid Ahmadinezhad and Maksym Fedorchuk)

The birational classification of algebraic varieties is one of the key problems which fueled the development of algebraic geometry since the very beginning. The idea is to find and classify the good representatives in the birational class and then to describe how varieties relate to the good representatives. The breakthrough was achieved by Mori who has developed the *Minimal Model Program*. It describes how to get a good representative from any variety. This good representative may not be unique and in many situations one can find "best" representatives.

For now assume that the base field is $\mathbb{C}$. Let $X \to S$ be a conic bundle over a surface. It is well known that X is birational to a smooth variety X' such that $X' \to S'$ is a conic bundle for some surface S' [2]. Thus it is enough to study only smooth conic bundles. On the other hand there are singular Fano varieties which are unique Mori fiber spaces in their birational classes [1]. Thus we have to study singular Fano varieties.

There is one more type of three-dimensional Mori fiber spaces: del Pezzo fibrations. With regards to existence of good models the del Pezzo fibrations are between Fano varieties and conic bundles. If the degree of the fibration over $\mathbb{P}^1$ is $\geqslant 4$, then there is a Mori fiber space with a smooth total space in the birational class. If the degree is $\leqslant 3$, then we can improve singularities, but often we cannot get rid of them altogether.

Theorem 1 ([C96]). Let $X \to \mathbb{P}^1$ be a flat morphism and suppose its general fiber X_η is a del Pezzo surface.

(1) If the degree of X_η is 3, then X is birational to a Gorenstein variety X' such that there is a del Pezzo fibration $X' \to \mathbb{P}^1$ of degree 3.

(2) If the degree of of X_η is 2, then X is birational to a 2-Gorenstein variety X' such that there is a del Pezzo fibration $X' \to \mathbb{P}^1$ of degree 2.

The *Gorenstein* singularities are the hypersurface singularities, they are smoothable. The 2-Gorenstein singularity is either Gorenstein or is a quotient of hypersurface singularity by a group of order 2. In the latter case the 2-Gorenstein singularity is a degeneration of the *half-point*, that is a singularity analytically isomorphic to $\mathbb{C}^3/\langle \mathrm{diag}(-1,-1,-1)\rangle$. Thus we expect that a general del Pezzo fibration of degree 3 has smooth total space and singularities of the total space space of a general del Pezzo fibration of degree 2 are only half-points.

It is expected that a similar result to hold for del Pezzo fibrations of degree 1.

Conjecture 2 ([C96])**.** Let $X \to \mathbb{P}^1$ be a flat morphism and suppose its general fiber X_η is a del Pezzo surface of degree 1. Then X is birational to a 6-Gorenstein variety X' such that there is a del Pezzo fibration $X' \to \mathbb{P}^1$ of degree 2.

To prove the Theorem 1 Corti used the equations of the fibrations and constructed all the birational maps explicitly. The result on degree 3 may also be proven using stability conditions [K97]. Kollar introduced a notion of stability for of a hypersurface over a ring. This notion interpolates the notion of GIT-stability of the general and the central fibers of a hypersurface fibration. Given a fibration over an affine curve with a GIT-stable general fiber Kollar gives an algorithm to construct a model stable over the coordinate ring of the base curve thus improving the singularities of the total space. Moreover this techinque works in a much broader setting. Together with Ahmadinezhad and Fedorchuk we have extended this to apply it to fibrations of degree 1 and 2.

Fibrations of degree 3 can be embedded into $\mathbb{P}^3$-bundle and $\mathrm{Aut}\,\mathbb{P}^3$ is reductive, thus the GIT techniques work very well. The del Pezzo surfaces of degree 1 and 2 are the hypersurfaces in weighted projective spaces $\mathbb{P}(1,1,2,3)$ and $\mathbb{P}(1,1,1,2)$, respectively. The automorphism groups of these weighted projective spaces are non-reductive which makes it difficult to use stability conditions. On the other hand they have natural formal stability conditions which tell as a lot about singularities. From now on assume that A is a PID, typically $A = \mathbb{Z}$ or $A = \Bbbk[Z]$, where Z is a curve. Let K be the fraction field of A.

Definition 3. We say (x,w) is a *weight system* on $\mathbb{P}(a_0,\dots,a_n)$ if $x = (x_0,\dots,x_n)$ is a choice of coordinates on the weighted projective space and $w = (w_0,\dots,w_n)$ is an n-tuple of integers.

Let F be a polinomial of degree d on $\mathbb{P}(a_0,\dots,a_n)$. Let $p \in A$ be prime. Then we say that (x,w) is *stable* (resp. *semistable*, *unstable*) on F at p over A if

$$\mathrm{mult}_p\, F(p^w x) = \mathrm{mult}_p\, F(p^{w_1}x_0,\dots,p^{w_n}x_n) < (\leqslant, >) \frac{d}{\sum a_i}\cdot \sum w_i.$$

We say that F is *stable* (resp. *semistable*) over A if every weight system is stable (resp. semistable) on F at every prime $p \in A$. We say that F is *unstable* over A if there is a weight system which is unstable on F at some p.

Note that if A is a field and $a_i = 1$ for all i, this definition becomes Hilbert-Mumford criterion of stability of hypersurfaces. Del Pezzo surfaces of degree 2 are quartics in $\mathbb{P}(1,1,1,2)$ and del Pezzo surface of degree 1 are sextics in $\mathbb{P}(1,1,2,3)$. This notion of stability is quite useful because of the following theorem.

Theorem 4. Let Z be and affine curve and let $\pi : X \to Z$ be a flat morphism such that X is a semistable del Pezzo surface of degree 1 (resp. degree 2) over $\mathbb{C}[Z]$. Then the total space of π is a terminal 6-Gorenstein (resp. 2-Gorenstein) variety. Also if a fiber over $o \in Z$ is reducible, then the local equation of X in $\mathbb{P}(1,1,2,3)$-bundle over Z can be written as

$$G_1(x_1, x_2, x_3)G_2(x_1, x_2, x_3) + t(\dots) = 0,$$

where t is a parameter on the base and $G_i = 0$ defines a smooth curve of degree 3 on $\mathbb{P}(1,1,2)$ (resp. a smooth conic on $\mathbb{P}^2$).

The degree 2 case follows from the computations of Corti and the degree 1 case is our computation.

Remark 5. If the semistable model has reducible fibers it can further be improved by unprojection in $\mathbb{P}^6$ for degree 2 and in $\mathbb{P}^{22}$ for degree 1.

Theorem 6. Let X_η be a smooth del Pezzo surface of degree 1 or 2 over K. Then there exists semistable del Pezzo surface $\pi : X \to \operatorname{Spec} A$ over A such that its extension to K is X_η.

This theorem together with the remark prove Corti's conjecture and allow us to find good models for del Pezzo surfaces of degree 1 and 2 in any characteristic. Also setting $A = \mathbb{Z}$ we find good reductions to finite characteristic.

The idea of the proof is to transfer the problem to the appropriate projective space. In case of degree 2 we do it by embedding $\mathbb{P}(1,1,1,2)$ and therefore X into $\mathbb{P}^6$ over A. Then we construct an space W of pairs (V, Q) such that $V \cap Q$ is a del Pezzo surface of degree 2 and find a good notion of stability on W.

Lemma 7. There exists a line bundle L on W such that X is stable over A if and only if the corresponding pair (V, Q) is stable over A with respect to L on W.

This key lemma together with usual GIT tricks proves Theorem 6.

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Cremona group and Voronoï tesselation

Anne Lonjou

A key tool for studying the Cremona group of rank 2 is the hyperboloid H in the Picard-Manin space. The Picard-Manin space associated to $\mathbb{P}^2$ is the direct limit of the Picard groups of surfaces obtained by blowing up any finite sequences of points in $\mathbb{P}^2$ or infinitely near. The intersection form is of signature $(1, \infty)$. Taking a hyperboloid sheet, gives us an infinite dimensional hyperbolic space, denoted by H (see for instance [3] and [4]). Guided by the analogy of the modular group acting on the hyperbolic plane, we exhibit a fundamental domain of the action of the Cremona group over an algebraically closed field $\mathbf{k}$ on H using Voronoï tessellation (which is a way of discretizing a metric space). We consider the Voronoï cells associated to the orbit of the class of the line $\ell \in$ H by the action of $\mathrm{Bir}(\mathbb{P}^2)$. The cell associated to ℓ and denoted by $\mathcal{V}(\mathrm{id})$ corresponds to classes of H which are closer to ℓ than to any other points of the orbit. We characterize the classes in $\mathcal{V}(\mathrm{id})$. The following statement says that it is enough to check that classes are closer to ℓ than to any other element of the orbit of ℓ under the action of de Jonquières element. We do not need to test all the orbits of ℓ.

Theorem 1. A class c belongs to $\mathcal{V}(\mathrm{id})$ if and only if for every Jonquières transformation $\mathfrak{j}$,

$$\mathrm{dist}(\mathfrak{j}_\#(c), \ell) \geq \mathrm{dist}(c, \ell).$$

In fact, the theorem is more precise. It is enough to check the inequality for a finite number of Jonquières transformations.

We also study the geometry of this tessellation by determining the cells which share a class with the cell $\mathcal{V}(\mathrm{id})$ called "adjacent cells" ([5]). Sites of such cells can be de Jonquières transformations or of another form. The other form is Cremona transformations that have at most 8 base-points in almost general position. A set of points $\{p_0, p_1, \ldots, p_r\}$ is said in almost general position if two conditions are satisfied. The first one is that for every $0 \leq i \leq r$ the point p_i lies either in $\mathbb{P}^2$ or in a surface obtained by blowing up $\mathbb{P}^2$ in a subset of points of $\{p_0, \ldots, p_r\}$. The second one is that none of the three following conditions is satisfied: four points of this set are aligned, seven points of this set lie on a conic, two points of this set belong to the exceptional divisor obtained by blowing up another point of this set.

Theorem 2. The following Cremona transformations:

- Jonquières transformations,
- applications having at most 8 base-points in almost general position

are the sites of cells adjacent to $\mathcal{V}(\mathrm{id})$.

We study also the cells sharing a point at infinity with $\mathcal{V}(\mathrm{id})$ called "quasi-adjacent cells". The applications which are sites of adjacent cells are also the sites of quasi-adjacent cells. Moreover applications having at most 9 base-points in almost general position are the only transformations which are sites of quasi-adjacent cells but not sites of adjacent cells.

This gives us a natural way of building two metric graphs: the adjacency graph and the quasi-adjacency graph. This leads us to two natural questions. Do they correspond to graphs already built? Are they Gromov hyperbolic graphs? (A metric space is Gromov hyperbolic if all its triangles are uniformly thin). The second question comes from the analogy with the action of $\mathrm{PSL}(2,\mathbb{Z})$ on the Poincaré's half plane because in the case of a positive answer it would be an analogous to the Bass-Serre tree.

We prove in [6] that the adjacent graph is quasi-isometric to a graph already introduced by Wright in [7]. It is also quasi-isometric to a Cayley graph of the Cremona group using the family of Jonquières and $\mathrm{PGL}(3,\mathbf{k})$ as generating set. Using a result of J. Blanc and J-P Furter about the minimal decomposition in Jonquières maps of a given Cremona transformation ([1]), we prove that this graph is not Gromov hyperbolic. However, we prove that the quasi-adjacent graph has this property.

Theorem 3. The quasi-adjacent graph is Gromov hyperbolic.

For this we use the Bowditch's criterion [2] and the Gromov hyperbolicity of H even if this two metric spaces are not quasi-isometric.

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Algebraic models of the line in the real affine plane

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(joint work with Adrien Dubouloz)

It is a standard consequence of the Jordan–Schoenflies Theorem that every two smooth closed embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ are ambient diffeotopic. Every algebraic closed embedding of the real affine line $\mathbb{A}^1_{\mathbb{R}}$ into the real affine plane $\mathbb{A}^2_{\mathbb{R}}$ induces a smooth embedding of the real locus $\mathbb{R}$ of $\mathbb{A}^1_{\mathbb{R}}$ into the real locus $\mathbb{R}^2$ of $\mathbb{A}^2_{\mathbb{R}}$. Given two such algebraic embeddings $f, g : \mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$, the famous Abhyankar-Moh Theorem [1], which is valid over any field of characteristic zero [23, § 5.4], asserts the existence of a polynomial automorphism ϕ of $\mathbb{A}^2_{\mathbb{R}}$ such that $f = \phi \circ g$. This implies in particular that the smooth closed embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ induced by f and g are equivalent under composition by a polynomial diffeomorphism of $\mathbb{R}^2$.

In this talk, we consider a similar problem in a natural category intermediate between the real algebraic and the smooth ones. Our main object of study consists of smooth embeddings of $\mathbb{R}$ into $\mathbb{R}^2$ induced by rational algebraic maps $\mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ defined on the real locus of $\mathbb{A}^1_{\mathbb{R}}$ and whose restrictions to this locus induce smooth closed embeddings of $\mathbb{R}$ into $\mathbb{R}^2$. We call these maps *rational smooth embeddings*, and the question is the classification of these embeddings up to *birational diffeomorphisms* of $\mathbb{A}^2_{\mathbb{R}}$, that is, diffeomorphisms of $\mathbb{R}^2$ which are induced by birational algebraic endomorphisms of $\mathbb{A}^2_{\mathbb{R}}$ containing $\mathbb{R}^2$ in their domains of definition and admitting rational inverses of the same type.

A first natural working question in this context is to decide whether any rational smooth embedding is equivalent up to birational diffeomorphism to the standard regular closed embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$ as a linear subspace. Since every rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ uniquely extends to a morphism $\mathbb{P}^1_{\mathbb{R}} \to \mathbb{P}^2_{\mathbb{R}}$ birational onto its image, a rational smooth embedding which can be rectified to a linear embedding by a birational diffeomorphism defines in particular a rational plane curve C that can be mapped onto a line by a birational automorphism of $\mathbb{P}^2_{\mathbb{R}}$. By classical results of Coolidge [6], Iitaka [14] and Kumar-Murthy [17], complex curves with this property are characterized by the negativity of the logarithmic Kodaira dimension of the complement of their proper transform in a minimal resolution of their singularities. Building on these ideas and techniques, we show the existence of *non-rectifiable* rational smooth embedding. In particular, we obtain the following result:

Theorem. For every integer $d \geq 5$ there exists a non-rectifiable rational smooth embedding of $\mathbb{A}^1_{\mathbb{R}}$ into $\mathbb{A}^2_{\mathbb{R}}$ whose associated projective curve $C \subset \mathbb{P}^2_{\mathbb{R}}$ is a rational nodal curve of degree d.

The existence of non-rectifiable rational smooth embeddings motivates the search for weaker properties which can be satisfied by rational smooth embeddings. To this end we observe that the Abhyankar-Moh Theorem implies that the image of a regular closed embedding $\mathbb{A}^1_{\mathbb{R}} \hookrightarrow \mathbb{A}^2_{\mathbb{R}}$ is a real fiber of a structure of trivial $\mathbb{A}^1$-bundle $\rho : \mathbb{A}^2_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}}$ on $\mathbb{A}^2_{\mathbb{R}}$. In the complex case, this naturally leads to a "generalized Abhyankar-Moh property" for closed embeddings of the affine line in affine surfaces S equipped with $\mathbb{A}^1$-fibrations over $\mathbb{A}^1_{\mathbb{C}}$, i.e. morphisms $\pi : S \to \mathbb{A}^1_{\mathbb{C}}$ whose general fibers are affine lines, which was studied for certain classes of surfaces in [11]: the question there is whether the image of every regular closed embedding of $\mathbb{A}^1_{\mathbb{C}}$ in such a surface is an irreducible component of a fiber of an $\mathbb{A}^1$-fibration. The natural counterpart in our real birational setting consists in shifting the focus to the question whether the image of a rational smooth embedding is actually a fiber of an $\mathbb{A}^1$-fibration $\pi : S \to \mathbb{A}^1_{\mathbb{R}}$ on a suitable real affine surface S birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$, but possibly non biregularly isomorphic to it. A rational smooth embedding with this property is said to be *biddable*.

Being a fiber of an $\mathbb{A}^1$-fibration on a surface birationally diffeomorphic to $\mathbb{A}^2_{\mathbb{R}}$ imposes strong restrictions on the scheme-theoretic image $f_*(\mathbb{A}^1_{\mathbb{R}})$ of a rational smooth embedding $f : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$. We show in particular that the *real Kodaira dimension* [4] $\kappa_{\mathbb{R}}(\mathbb{A}^2_{\mathbb{R}} \setminus f_*(\mathbb{A}^1_{\mathbb{R}}))$ of the complement of the image has to be negative, with the consequence for instance that none of the rational smooth embedding mentioned in the theorem above is actually biddable. In contrast, a systematic study of small degree embeddings reveals the existence of non-rectifiable biddable rational smooth embeddings whose images are in a natural way smooth fibers of $\mathbb{A}^1$-fibrations on some *fake real planes*, a class of real birational models of $\mathbb{A}^2_{\mathbb{R}}$ recently introduced and studied in the series of papers [7, 8]. These are smooth real surfaces S non isomorphic to $\mathbb{A}^2_{\mathbb{R}}$ whose real loci are diffeomorphic to $\mathbb{R}^2$ and whose complexifications have trivial reduced rational singular homology groups.

We therefore develop a collection of geometric techniques to tackle the classification of equivalence classes of biddable rational smooth embeddings up to birational diffeomorphisms. As a result, we obtain in particular the following synthetic criterion:

Theorem. For $i = 1, 2$, let $f_i : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$, be a biddable rational smooth embedding and let $\alpha_i : \mathbb{A}^2_{\mathbb{R}} \dashrightarrow S_i$ be a birational diffeomorphism onto an $\mathbb{A}^1$-fibered fake real plane $\pi_i : S_i \to \mathbb{A}^1_{\mathbb{R}}$ such that $\alpha_i \circ f_i : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow S_i$ is a closed immersion as the support of a *smooth* fiber of π_i.

Then f_1 and f_2 are not rectifiable. Furthermore, the following conditions are equivalent:

a) $f_1 : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ and $f_2 : \mathbb{A}^1_{\mathbb{R}} \dashrightarrow \mathbb{A}^2_{\mathbb{R}}$ are equivalent rational smooth embeddings

b) There exists a birational diffeomorphism $\beta : S_1 \dashrightarrow S_2$ and an automorphism γ of $\mathbb{A}^1_{\mathbb{R}}$ such that $\gamma \circ \pi_1 = \pi_2 \circ \beta$.

As an application of this characterization, we derive in particular the existence of infinitely many equivalence classes of biddable rational smooth embeddings.

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The dual complex of log Calabi-Yau pairs on Mori fibre spaces

Mirko Mauri

A dual complex is a cell complex, encoding the combinatorial data of how the irreducible components of a simple normal crossing or a dlt boundary intersect. These objects have raised the interest of many scholars in different fields. For instance, the homeomorphism type of the dual complex of a minimal dlt modification is an interesting invariant of a singularity, see [2]. In mirror simmetry, the dual complex of the special fibre of a good minimal dlt degeneration of Calabi-Yau varieties has recently been proved to be the basis of a non-archimedean SYZ fibration, see [6] and [7].

In both these examples, a neighbourhood of any cell of the dual complex is a cone over the dual complex of a new dlt pair (X, Δ), which satisfies the additional property that $K_X + \Delta \sim_{\mathbb{Q}} 0$, provided that the singularity is log canonical and the degeneration semistable. These pairs are called log Calabi-Yau, in brief logCY. Their dual complexes have been deeply studied in [5]. In that paper, the authors have posed the question whether the dual complex of a logCY pair of dimension n is the quotient of a sphere $\mathbb{S}^k$ of dimension $k \leq n$ for some finite group $G \subset \mathrm{O}_{k+1}(\mathbb{R})$. With the techniques developed, they were able to provide a positive answer in dimension ≤ 4 and in dimension $= 5$, under the additional hypothesis that (X, Δ) has simple normal crossings. It is worthy to remark that an affirmative answer to this question would imply, for instance, that the basis of a SYZ fibration has the structure of a topological orbifold.

In this talk, we answer positively the question for a special class of dlt logCY pairs (Y, Δ), endowed with a morphism $\pi\colon Y \to Z$ of relative Picard number one. This hypothesis is inspired by the following observation. If (X, Δ_X) is a logCY pair with maximal intersection, *i.e.* the pair admits a 0-dimensional lc centre, then X is rationally connected, see [5, §18]. By [1], a K_X-MMP with scaling $f\colon X \to Y$ terminates with a Mori fibre space $\pi\colon Y \to Z$ and the pair $(Y, \Delta := f_*\Delta_X)$ is still logCY. It sounds sensible to us to check first whether the dual complexes of these special pairs are finite quotients of spheres, under the dlt assumption.

The main results of our work are collected in the following statement.

Main Theorem. Let (Y, Δ) be a dlt pair such that:

(1) Y is a $\mathbb{Q}$-factorial projective variety of dimension $n + 1$;
(2) (Mori fibre space) $\pi\colon Y \to Z$ is a Mori fibre space of relative dimension r;
(3) (logCY) $K_Y + \Delta \sim_{\mathbb{Q}} 0$.

If $\rho(Y) = 1$, then the dual complex $\mathcal{D}(Y, \Delta)$ is PL-homeomorphic either to a ball $\mathbb{B}^m$ of dimension $m \leq n$ or to the sphere $\mathbb{S}^n$.

If $\rho(Y) = 2$, then $\mathcal{D}(Y, \Delta)$ is PL-homeomorphic to a ball $\mathbb{B}^m$ of dimension $m \leq n$, a sphere $\mathbb{S}^m$ of dimension $m = r - 1, n - r$ or n.

If $\dim Z = 2$, then $\mathcal{D}(Y, \Delta)$ is PL-homeomorphic to a ball $\mathbb{B}^m$ of dimension $m \leq n$, a sphere $\mathbb{S}^m$ of dimension $m = 1, n - 2, n - 1$ or n, or the quotient $\mathbb{P}^2(\mathbb{R}) * \mathbb{S}^{n-3}$.

All these cases occur.

The Main Theorem can be summarised in Table 4. In particular observe that

$\rho(Y)$	$\dim(Z)$	PL-homeomorphism type of $\mathcal{D}(Y, \Delta)$
1		$\mathbb{B}^m, \mathbb{S}^n$
2		$\mathbb{B}^m, \mathbb{S}^{r-1}, \mathbb{S}^{n-r}, \mathbb{S}^n$
	1	$\mathbb{B}^m, \mathbb{S}^0, \mathbb{S}^{n-1}, \mathbb{S}^n$
	2	$\mathbb{B}^m, \mathbb{S}^1, \mathbb{S}^{n-2}, \mathbb{S}^{n-1}, \mathbb{S}^n, \mathbb{P}^2(\mathbb{R}) * \mathbb{S}^{n-3}$

Table 4. Dual complex of logCY pairs on Mori fibre spaces.

all these dual complexes are quotients of spheres, compatibly with the prediction [5, Question 4].

The key ingredients of the proof of the Main Theorem are various connectivity theorems. The first of them is the Hodge index theorem: ample divisors always intersect, provided that they have dimension at least one. This fact allows to list all the triangulations of $\mathcal{D}(Y, \Delta)$ under the assumption $\rho(Y) = 1$. The naive idea for the next step, namely the case of $\rho(Y) = 2$, would be to build $\mathcal{D}(Y, \Delta)$ out of the contribution of vertical divisors together with the information provided by horizontal divisors. Indeed, the pushforward of the former determines a logCY pair (Z, B) of Picard number one, while the latter cut out a logCY pair $(F_{\text{gen}}, \Delta_{\text{gen}})$ on the general fibre F_{gen}, which in turns behaves like a logCY pair of Picard number one. The special pairs for which this program works are here called *of combinatorial product type*. The proof of the Main Theorem consists precisely in understanding how far the general pair (Y, Δ) is from this ideal arrangement.

As a measure of what can go wrong, observe that there could be strata of (Y, Δ) which do not dominate Z, but which are not contained in any vertical divisor of Δ. The existence of such a stratum rigidifies the configuration of horizontal divisors and it allows to describe explicitly the triangulation of $\mathcal{D}(Y, \Delta)$. Another issue is represented by horizontal strata which map two-to-one to Z. These strata are responsible for instance for the occurrence of the homeomorphism type $\mathbb{P}^2(\mathbb{R}) * \mathbb{S}^{n-3}$.

We point out that the proof of the Main Theorem highly relies on the connectivity theorems [4, Proposition 4.37] and [4, Theorem 4.40] and the canonical bundle formula [3, Theorem 8.5.1]. Observe finally that for a statement which does not involve non-trivial quotients of spheres, our hypothesis on the Picard number is the sharpest possible. Indeed, we construct logCY pairs on Mori fibre spaces of Picard rank three such that $\mathcal{D}(Y, \Delta) \simeq \mathbb{P}^2(\mathbb{R}) * \mathbb{S}^{n-3}$.

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Families of inequivalent real circle actions on affine four-space

Lucy Moser-Jauslin

In this talk, we will show that there exist infinite families of inequivalent equivariant real forms of linear $\mathbb{C}^*$-actions on affine four-space. We consider the real form of $\mathbb{C}^*$ whose fixed point is a circle. In 2004, in collaboration with G. Freudenburg [1], we constructed one example of a non-linearizable circle action. This was done using an example of a non-trivial $O_2(\mathbb{C})$-bundle constructed by G. Schwarz in 1989 [4]. Here, we generalize this result by using classifications of different equivariant

$O_2(\mathbb{C})$-bundles from [4], [3] and [5], and developing a new approach which allows us to adapt methods to compare different real forms.

Note that in 2013, M. Koras and P. Russell [2] proved that over any field of characteristic zero, all actions of any forms of a $\mathbb{G}_m$-action on affine three-space are linearizable. In particular, dimension four is minimal for finding non-linearizable circle actions of affine space.

We start by defining real circle actions. A real form of an algebraic linear complex group is defined by an antiholomorphic involution σ on G which is a group homomorphism. A (G, σ)-real form of Y, or a real form of Y which is compatible with σ, is given by an antiholomorphic involution μ on Y such that $\mu(gy) = \sigma(g)\mu(y)$ for all $g \in G$ and $y \in Y$. Two such real forms are equivalent if they are conjugate by a G-automorphism of Y. If one real equivariant form μ_0 exists, then for any other real equivariant form μ, there exists $\varphi \in Aut_G(Y)$ such that $\mu = \varphi \circ \mu$. Moreover, the condition that μ is an involution means that $\mu_0\varphi\mu_0 = \varphi^{-1}$. Also $\varphi_1\mu_0$ is equivalent to $\varphi_2\mu_0$ if and only if there exists a G-equivariant automorphism ψ of Y such that $\psi \circ \varphi_1 \circ (\mu_0\psi^{-1}\mu_0) = \varphi_2$. In other worlds, he set of all real equivariant forms can be described by a cohomology set. More precisely, let $\Gamma = Gal(\mathbb{C}/\mathbb{R}) = \{1, \gamma\}$, and consider the action of Γ on $Aut_G(Y)$ by ${}^{\gamma}\varphi = \mu_0\varphi\mu_0$. Then the set of real equivariant (G, σ)-forms of Y are described by $H^1(\Gamma, Aut_G(Y))$.

For the cases studied here we fix $G = \mathbb{C}^*$ and $\sigma(t) = \overline{t}^{-1}$, and set Y to be complex affine four space, endowed with a linear action of weights $(2, -2, 2m + 1, -(2m + 1))$. The linear $(\mathbb{C}^*, \sigma)$-real form is given by μ_0: If a, b, x, y are the coordinates of Y with respective weights $2, -2, 2m + 1, -(2m + 1)$, then

$$\mu_0(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) = (\begin{pmatrix} \overline{b} \\ \overline{a} \end{pmatrix}, \begin{pmatrix} \overline{y} \\ \overline{x} \end{pmatrix}).$$

In order to construct families of inequivalent real circle forms on Y, we use some of the classifications of $O_2(\mathbb{C})$-vector bundles of G. Schwarz. Remember that $O_2(\mathbb{C})$ is a semi-direct product of $\mathbb{C}^*$ with $\{1, \tau\}$, where τ is an involution. Consider the irreducible two-dimensional representations W_k, for $k \in \mathbb{N}$, where $\mathbb{C}^*$ acts linearly with weights k and $-k$ and the involution τ exchanges the two $\mathbb{C}^*$-eigenspaces. Schwarz showed that the set of isomorphism classes of $O_2(\mathbb{C})$-vector bundles with base W_2 and zero fiber W_{2m+1} is a moduli space isomorphic to $\mathbb{C}^m$ for $m \geq 1$. Moreover all of these equivariant vector bundles are constructed as trivial $\mathbb{C}^*$-vector bundles on which one acts by an involution. By construction, if this involution commutes with complex conjugation, it defines a real circle form on the total space.

This allows us to construct large families of real circle forms. In order to distinguish different circle forms, new methods have to be developed. Indeed, two involutions giving non-isomorphic vector bundles do not necessarily give non-equivalent real circle actions. For details of how to do this, see [6].

The main result is as follows:

Let $T = ab$ and let $n = 2m+1$. Given $h \in \mathbb{R}[T]$, let φ_h be the automorphism of Y:

$$\varphi_h(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}) = (\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} 1 - Th^2 & a^n h^n \\ -b^n h^n & \sum_{j=0}^{n-1} (Th^2)^j \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}).$$

Note that φ_h is $\mathbb{C}^*$ equivariant: $\varphi_h \in Aut_{\mathbb{C}^*}(Y)$.

Theorem 1. Let $h \in \mathbb{R}[T]$ be a real polynomial, and let $\mu_h = \varphi_h \mu_0$.

(i) μ_h defines a real circle form on $Y = W_2 \times W_{2m+1}$;

(ii) μ_h is equivalent to $\mu_{h'}$ if and only if there exists $r \in \mathbb{R}^*$ such that

$$h(T) \equiv r h'(r^2 T) \mod (T^m).$$

In particular, if $m \geq 2$, one finds infinitely many distinct real circle forms of the linear $\mathbb{C}^*$-action on $\mathbb{C}^4$.

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A surface with discrete and non-finitely generated automorphism group

Keiji Oguiso

(joint work with Tien-Cuong Dinh)

We work in the category of projective varieties defined over the complex number field $\mathbf{C}$. In my talk, I presented the following theorem with an idea of proof, which is the main result of my joint work with Professor Tien-Cuong Dinh [DO17]:

Theorem 1. For each integer $d \geq 2$, there is a smooth projective variety V of dimension d and of Kodaira dimension $d-2$ such that

(1) the automorphism group $\mathrm{Aut}\,(V)$ is discrete, i.e., its identity component $\mathrm{Aut}^0\,(V)$ is $\{id_V\}$;

(2) $\mathrm{Aut}\,(V)$ is not finitely generated; and

(3) V admits infinitely many real forms which are mutually non-isomorphic over $\mathbf{R}$.

Theorem 1 is inspired by a remarkable construction, due to Lesieutre, of such a variety in dimension 6 over any field of characteristic zero [Le17]. We refer to this reference for history on the finite generation problem of automorphism groups of smooth projective varieties as well as its relation with their real forms.

For dimension $d = 1$, it is well-known that no smooth projective curve satisfies the above properties. The existence of surfaces of Kodaira dimension 0 with infinitely many non-isomorphic real forms is unexpected and gives the final answer to a longstanding open problem. In fact, the same property fails for abelian varieties by Borel-Serre [BS64, Cor.6.3] and all surfaces of Kodaira dimension ≥ 1 and all *minimal* surfaces of Kodaira dimension 0, see Degtyarev-Itenberg-Kharlamov [DIK00, Appendix D]. For the reader's convenience, we recall now the notion of real form.

Let V be a variety defined over $\mathbf{C}$. Let $V'_{\mathbf{R}}$ be a variety defined over $\mathbf{R}$ and V' the complex variety associated to it, i.e.,

$$V' := V'_{\mathbf{R}} \times_{\operatorname{Spec}\mathbf{R}} \operatorname{Spec}\mathbf{C}\ .$$

We say that $V'_{\mathbf{R}}$ is *a real form* of V if V' is isomorphic to V over $\mathbf{C}$. Two real forms $V'_{\mathbf{R}}$ and $V''_{\mathbf{R}}$ of V are said to be *isomorphic* if they are isomorphic over $\mathbf{R}$.

The main new idea to construct varieties of low dimension satisfying Theorem 1 is an effective use of a projective K3 surface together with suitable blow-ups. As our projective K3 surface is minimal and defined over $\mathbf{C}$, the canonical representation of its automorphisms group is finite, see [Ue75, Th.14.10], and blow-ups do not produce new automorphisms. This is an advantage of using a K3 surface, which is not available for the rational surface used in Lesieutre's construction.

In my talk, I first presented a general strategy to construct a variety V from a K3 surface S and a smooth rational curve $C \subset S$ and a point $P \in C$ with a special property, called *very special triple* [DO17] and a smooth projective variety M of dimension $d-2$ with special properties. Next, I explained an explicit example of (S, C, P) with properties required, i.e., of a very special triple, and get a variety satisfying the properties (1) and (2) in Theorem 1. We will use the theory of elliptic surfaces due to Kodaira [Ko63]. Finally, by specifying some parameters in our construction, we get the property (3) of Theorem 1. A criterium for a variety to have infinitely many non-isomorphic real forms, due to Serre and Lesieutre, is crucial in our approach, see [Se02, Chapter 5] and [Le17, Lemma 13].

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Automorphism groups of the complements of hypersurfaces

Jihun Park

(joint work with Ivan Cheltsov and Adrien Dubouloz)

Let S_d be a smooth hypersurface of degree d in $\mathbb{P}^n$, where $n \geq 2$. It is easy to see that the automorphism group of the hypersurface S_d coincides with the automorphism group of $\mathbb{P}^n$ that keeps S_d fixed, i.e.,

$$\mathrm{Aut}\,(S_d) = \mathrm{Aut}\left(\mathbb{P}^3, S_d\right).$$

We consider this phenomenon inside out. To be precise, we consider the complement of the hypersurface S_d in $\mathbb{P}^n$ and ask a question,

$$\text{When } \mathrm{Aut}\,(\mathbb{P}^n \setminus S_d) = \mathrm{Aut}\left(\mathbb{P}^3, S_d\right)?$$

The results on non-ruledness of the hypersurfaces by Clemens-Griffith, Iskovskikh-Manin, Kollár, Pukhlikov, and de Fernex ([4, 5, 8, 10, 12, 13]) yield partial answers to this question. Using techniques based on cylinder, additive action and locally nilpotent derivation ([6]), one can also show that the equality does not hold for quadric hypersurfaces.

In fact, the question above has been motivated by the following long-standing conjecture by Gizatullin ([7]):

Conjecture 1. For a smooth cubic surface S in $\mathbb{P}^3$,

$$\mathrm{Aut}\left(\mathbb{P}^3 \setminus S\right) = \mathrm{Aut}\left(\mathbb{P}^3, S\right).$$

At the current stage, Gizatullin's conjecture is far away from answer. However, based on the results in [1], [2] and[3], we may extend the conjecture to del Pezzo surfaces that are hypersurfaces in weighted projective spaces.

Let S be a del Pezzo surface with at worst Du Val singularities. We suppose that the surface S is a hypersurface in a weighted projective space $\mathbb{P}$. This means that the surface S is one of the following:

(1) a hypersurface of degree $d \leq 3$ in $\mathbb{P}^3$;
(2) a hypersurface of degree 4 in $\mathbb{P}(1,1,1,2)$;
(3) a hypersurface of degree 6 in $\mathbb{P}(1,1,2,3)$.

Denote by $\mathbb{P}$ the ambient weighted projective space of the hypersurface S. It can be $\mathbb{P}^3$, $\mathbb{P}(1,1,1,2)$ or $\mathbb{P}(1,1,2,3)$ depending on the del Pezzo surface S.

An $\mathbb{A}^1$-cylinder is a variety isomorphic to $Z \times \mathbb{A}^1$ for some affine variety Z. A cylindrical affine variety W is an affine variety that contains a principal Zariski open subset isomorphic to an $\mathbb{A}^1$-cylinder. For a $\mathbb{Q}$-divisor M on a normal projective variety X, an M-polar cylinder in X is an open subset

$$U = X \setminus \mathrm{Supp}(D)$$

defined by an effective $\mathbb{Q}$-divisor D on X with $D \sim_{\mathbb{Q}} M$ and isomorphic to an $\mathbb{A}^1$-cylinder (see [9]).

It is surprising that anticanonically polarised cylinders on Fano hypersurfaces in weighted projective spaces turn out to have a strong connection to unipotent group actions on their complements in the weighted projective spaces.

Theorem 2 ([3])**.** If S contains a $(-K_S)$-polar cylinder, then $\mathrm{Aut}(\mathbb{P} \setminus S)$ contains a unipotent subgroup. In particular, $\mathrm{Aut}(\mathbb{P} \setminus S) \neq \mathrm{Aut}(\mathbb{P}, S)$.

It is known that a smooth del Pezzo surface of anticanonical degree at most 3 never contains any $(-K_S)$-polar cylinder. For the singular case, we obtain a complete description for $(-K_S)$-polar cylinders from [2, Theorem 1.5].

Theorem 3. The surface S does not contain any $(-K_S)$-polar cylinder if and only if one of the following conditions is satisfied:

(1) Its anticanonical degree is 1 and it has only singular points of types A_1, A_2, A_3, D_4 if any;
(2) Its anticanonical degree is 2 and it allows only singular points of type A_1 if any;
(3) Its anticanonical degree is 3 and it allows no singular point.

When Cheltsov, Dubouloz and the author proved Theorem 2 in [3], they also proposed a conjecture ([3, Conjecture 4.12]) as follows:

> *The surface S does not contain any $(-K_S)$-polar cylinder if and only if the affine variety $\mathbb{P} \setminus S$ is not cylindrical.*

The conjecture has been simply verified by the author,

Theorem 4 ([11])**.** If S contains no $(-K_S)$-polar cylinder, then $\mathbb{P} \setminus S$ is not cylindrical. In particular, $\mathrm{Aut}(\mathbb{P} \setminus S)$ contains no unipotent subgroup.

It seems reasonable to propose the following conjecture based on various interesting phenomena and evidences:

Conjecture 5. The surface S contains no $(-K_S)$-polar cylinder if and only if

$$\mathrm{Aut}(\mathbb{P}\setminus S)=\mathrm{Aut}(\mathbb{P},S).$$

In the case when S is a smooth cubic surface, it does not contain any $(-K_S)$-polar cylinder ([1]). Therefore, Conjecture 5 claims that $\mathrm{Aut}\,(\mathbb{P}\setminus S)=\mathrm{Aut}\,(\mathbb{P},S)$ for a smooth cubic surface S, which is Gizatullin's conjecture.

Together with Theorems 2 and 3, the following theorem summarizes the current knowledge towards the structure of the automorphism groups of the complements of smooth del Pezzo hypersurfaces.

Theorem 6 ([3])**.** Suppose that S is smooth. If its anticanonical degree is 1, then

$$\mathrm{Aut}\,(\mathbb{P}\setminus S)=\mathrm{Aut}\,(\mathbb{P},S)\,.$$

In particular, $\mathrm{Aut}\,(\mathbb{P}\setminus S)$ is a finite group. If its anticanonical degree is either 2 or 3, then $\mathrm{Aut}(\mathbb{P}\setminus S)$ does not contain nontrivial connected algebraic groups.

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Katzarkov–Kontsevich–Pantev conjectures in lower dimensions

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(joint work with Ivan Cheltsov)

Mirror Symmetry came from physics, where physicists, describing elementary particles as Calabi–Yau threefolds, observed that for any such threefold X there exists another Calabi–Yau threefold Y such that $h^{p,q}(X) = h^{3-p,q}(X)$. In mathematics mirror symmetry conjecture got several formulations. In particular, Mirror Symmetry conjecture of variations of Hodge structures describes the duality as a relation between genus 0 Gromov–Witten invariants (expected numbers of rational curves lying on the variety) with periods of the dual family. The most well-known conjecture, Homological Mirror Symmetry conjecture, formulated by Kontsevich in 1994, states the duality in terms of derived categories. Unfortunately, it is hard to prove this conjecture at the moment, it is known only several rare examples.

The dual object to a Fano variety is not a variety again but a Landau–Ginzburg model — a certain open variety with a complex-valued function on it called superpotential (often Landau–Ginzburg models are considered as families of fibers of the superpotential). The mirror conjecture for Hodge numbers can not be generalized straightforward for Fano varieties since dual objects for them are not compact varieties. However Katzarkov, Kontsevich, and Pantev, based on Homological mirror symmetry intuition, in [7] gave generalizations of Hodge numbers for this case. More precise, for a Landau–Ginzburg model $w\colon Y \to \mathbb{C}$ in loc. cit. three sets of numbers were defined: the numbers $f^{p,q}(Y,w)$ via w-adopted logarithmic forms, the numbers $h^{p,q}(Y,w)$ via weight filtration of a monodromy at infinity, and the numbers $i^{p,q}(Y,w)$ via sheaves of vanishing cycles at singular fibers. The two natural conjectures are formulated.

Conjecture 1 ([7])**.** $f^{p,q}(Y,w) = h^{p,q}(Y,w) = i^{p,q}(Y,w)$.

Conjecture 2 ([7])**.** If (Y,w) is a Landau–Ginzburg model for a Fano variety X of dimension n, then $h^{p,q}(X) = f^{n-p,q}(Y,w)$.

To test these conjectures one needs to construct Landau–Ginzburg models for Fano varieties. For two-dimensional case this is done in [1].

Theorem 3 ([9])**.** (Slightly corrected) Conjectures 1 and 2 hold for del Pezzo surfaces (and their Landau–Ginzburg models constructed in [1]) and the numbers $f^{p,q}(Y,w)$, $h^{p,q}(Y,w)$; they do not hold for numbers $i^{p,q}(Y,w)$.

Hodge numbers of del Pezzo surfaces are given just by their degrees. However the situation in the threefold case is more complicated, because the dimension of the third cohomology can not be computed straightforward. Moreover, Homological Mirror symmetry is not proven in this case. So to test the conjectures in dimension three one needs to construct Landau–Ginzburg models for smooth Fano threefolds. This can be done via the toric Landau–Ginzburg models theory.

Definition 4 (see [11, §6])**.** *A toric Landau–Ginzburg model* for a smooth Fano variety X of dimension n is a Laurent polynomial $f \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ which satisfies the following.

- **Period condition:** One has $I_f = \widetilde{I}_0^X$, where I_f is the *main period* for f and $\widetilde{I}_0^X$ is *the constant term of regularized I-series* for X.
- **Calabi–Yau condition:** There exists a relative compactification of the family $f\colon (\mathbb{C}^*)^n \to \mathbb{C}$ whose total space is a (non-compact) smooth Calabi–Yau variety Y. Such compactification is called *a Calabi–Yau compactification.*
- **Toric condition:** There exists a degeneration $X \rightsquigarrow T_X$ to a toric variety T_X whose fan polytope coincides with the Newton polytope for f.

Definition 5. A compactification of the family $f\colon (\mathbb{C}^*)^n \to \mathbb{C}$ to a family $f\colon Z \to \mathbb{P}^1$, where Z is smooth and $-K_Z = f^{-1}(\infty)$, is called a *log Calabi–Yau compactification* (cf. a notion of tame compactified Landau–Ginzburg model in [7]).

We consider Mirror Symmetry as a correspondence between Fano varieties and Laurent polynomials. (In other words, we look for Landau–Ginzburg models which are algebraic tori $(\mathbb{C}^*)^n$, so superpotentials are represented by the Laurent polynomials.) That is, a strong version of Mirror Symmetry of variations of Hodge structures conjecture states the following.

Conjecture 6 (see [11, Conjecture 38])**.** Any pair of a smooth Fano variety and a divisor on it has a toric Landau–Ginzburg model.

The existence of toric Landau–Ginzburg models has been shown for Fano threefolds (see [11] and [6] for Picard rank one case and [2], [5],and [12] for the general case). For complete intersections in projective spaces see [11] and [6]. Aside from that only partial results are known.

The natural wish is to compute invariants of Fano varieties in terms of their toric Landau–Ginzburg models. Say, by [8] one can see rationality of Picard rank one Fano threefold via monodromy of the cental fiber of its toric Landau–Ginzburg model. Hodge numbers of Fano threefolds are given by their Picard numbers and the dimensions of the intermediate Jacobians. Thus the Hodge numbers mirror symmetry conjecture is almost given by the following.

Conjecture 7 (see [13, Conjecture 1.1])**.** Let X be a smooth Fano variety of dimension n and let f_X be its toric Landau–Ginzburg model. Let k_{f_X} be a number of all components of all reducible fibers of a Calabi–Yau compactification for f_X minus the number of reducible fibers. One has $h^{1,n-1}(X) = k_{f_X}$.

This conjecture is proven for Fano threefolds of Picard rank one (see [11]) and for complete intersections (see [13]).

The case of ant Picard rank is more complicated. Let X be a smooth Fano threefold and let f be its toric Landau–Ginzburg model *of Minkowski type.* Then, using the toric variety dual to one associated with f, in [12] a log Calabi–Yau compactification Z of f was constructed. In particular, this compactification gives a *tame compactified Landau–Ginzburg model* $w\colon Y \to \mathbb{C}$, so Landau–Ginzburg Hodge numbers defined for them.

In the paper [4], Harder, under some assumptions, proved Conjecture 1 for $f-$ and $h-$numbers. By [12], these conditions hold for Z, so Conjecture 1 holds for X. Moreover, Harder showed how to compute $f^{p,q}(Y,w)$ using the global geometry of Z. He applied this in the case when X is a smooth Fano threefold. In this case $w\colon Y \to \mathbb{C}$ is a fibration into $K3$ surfaces by [12]. Harder proved that

$$f^{1,1}(Y,w) = f^{2,2}(Y,w) = \sum_{P\in\mathbb{C}^1} (\rho_P - 1),$$

where ρ_P is the number of irreducible components of the fiber $w^{-1}(P)$. Similarly, he proved that

$$f^{1,2}(Y,w) = f^{2,1}(Y,w) = \dim\left(\operatorname{coker}\Big(H^2(Z,\mathbb{R}) \to H^2(F,\mathbb{R})\Big)\right) - 2 + h^{1,2}(Z),$$

where F is a general fiber of w.

In most of cases one can construct the Calabi–Yau compactification in another, more straightforward way. Using the natural embedding $\mathbb{C}[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}] \hookrightarrow \mathbb{P}[x,y,z,t]$, one can compactify the pencil of fibers for f to a pencil of quadrics in $\mathbb{P}^3$. Then one needs to resolve the base locus of the pencil. Doing this by blowing up points and smooth curves, one can keep track the required invariants.

Theorem 8 (Cheltsov, Przyjalkowski)**.** Conjecture 2 hold smooth Fano threefolds.

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Boundedness properties for finite groups of birational selfmaps

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(joint work with Yuri Prokhorov)

1. Known results

Although groups of birational selfmaps of algebraic varieties can be very far from linear algebraic groups, they sometimes exhibit similar features on the level of finite subgroups. The group-theoretic property we will be mostly interested in is described as follows.

Definition 1 (see [4, Definition 1])**.** A group Γ is called *Jordan* if there is a constant J such that for any finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J.

The first result concerning Jordan property (and motivating the modern terminology) is an old theorem by C. Jordan that asserts this property for the group $\mathrm{GL}_n(\Bbbk)$ over a field $\Bbbk$ of characteristic zero. J.-P. Serre noticed that Jordan property sometimes holds for groups of birational automorphisms.

Theorem 2 ([9, Theorem 5.3])**.** The group of birational automorphisms of $\mathbb{P}^2$ over a field of characteristic zero is Jordan.

In [6, Theorem 1.8], Yu. Prokhorov and C. Shramov generalized Theorem 2 to the case of rationally connected varieties of arbitrary dimension (actually, their results were initially obtained modulo boundedness of terminal Fano varieties, which was recently proved by C. Birkar in [3, Theorem 1.1]). Jordan property was also proved for groups of birational automorphisms (and similar types of automorphism groups) in many other cases, see e.g. [8] and references therein. However, there are varieties whose groups of birational automorphisms are not Jordan. Yu. Zarhin showed in [10] that a product of a positive-dimensional abelian variety and a projective line over an algebraically closed field of characteristic zero has a non-Jordan group of birational automorphisms.

Keeping in mind the example of Zarhin, it seems natural to try to understand the groups of birational automorphisms for varieties with maximal rationally connected fibration of small relative dimension. note that the case when the relative dimension is 0 (that is, the MRC fibration is birational or, equivalently, the variety is not uniruled) is settled by [5, Theorem 1.8(ii)]. To study fiberwise birational automorphism in the case of MRC fibration of relative dimension 1, T. Bandman and Yu.Zarhin proved the following.

Theorem 3 ([2, Corollary 4.11])**.** Let $\mathbb{K}$ be a field of characteristic zero that contains all roots of 1. Let C be a conic over $\mathbb{K}$. Assume that C is not $\mathbb{K}$-rational, i.e., that $C(\mathbb{K}) = \varnothing$. Then every non-trivial element of finite order in $\mathrm{Aut}(C)$ has order 2, and every finite subgroup of $\mathrm{Aut}(C)$ has order at most 4.

Theorem 4 ([2, Theorem 1.6])**.** Let X be a variety over a field of characteristic zero, and $\phi\colon X \dashrightarrow Y$ be its maximal rationally connected fibration. Suppose that the relative dimension of ϕ equals 1, and that ϕ has no rational sections. Then the group of birational automorphisms of X is Jordan.

A partial two-dimensional generalization of Theorem 3 is as follows.

Theorem 5 ([7, Theorem 1.6])**.** Let $\mathbb{K}$ be a field of characteristic zero that contains all roots of 1, and S be a geometrically rational surface over $\mathbb{K}$. Assume that S is not $\mathbb{K}$-rational but has a $\mathbb{K}$-point. Then the group of birational automorphisms of S has bounded finite subgroups.

Theorem 6 ([7, Lemma 4.6])**.** Let X be a variety over a field of characteristic zero, and $\phi\colon X \dashrightarrow Y$ be its maximal rationally connected fibration. Suppose that the relative dimension of ϕ equals 2, and that ϕ *has* a rational section. Suppose also that X is not birational to $Y \times \mathbb{P}^2$. Then the group of birational automorphisms of X is Jordan.

2. Work in progress

The following assertion that can be interpreted as a two-dimensional generalization of Theorem 3 and that applies both to surfaces with $\mathbb{K}$-points and without them is a work in progress with V. Vologodsky.

Theorem 7. Let $\mathbb{K}$ be a field of characteristic zero that contains all roots of 1. Let X be a geometrically rational surface over $\mathbb{K}$. Then the group of birational automorphisms of S has unbounded finite subgroups if and only if X is $\mathbb{K}$-birational to $\mathbb{P}^1 \times C$, where C is a conic.

Using Theorem 7, one can obtain the following result that is somewhat similar to Theorems 4 and 6.

Proposition 8. Let X be a variety over a field $\Bbbk$ of characteristic zero, and $\phi\colon X \dashrightarrow Y$ be its maximal rationally connected fibration. Suppose that the relative dimension of ϕ equals 2, and that ϕ has no rational sections. Suppose also that X is not birational to $Z \times \mathbb{P}^1$, where Z is a conic bundle over Y. Then the group of birational automorphisms of X is Jordan.

3. Further directions

The proof of Theorem 7 is based on studying boundedness properties for linear algebraic groups that do not have subgroups isomorphic to $\mathbb{G}_{\mathrm{m}}$. In general, it seems that many properties of finite groups of birational selfmaps of rationally connected varieties can be deduced from the properties of linear algebraic groups acting on (other) rationally connected varieties. In particular, I suspect that the answer to the following question is positive.

Question 9. Can one give an alternative proof of the Jordan property for rationally connected varieties over a field of characteristic zero, following the approach of [9] instead of [6]?

Another question that may become accessible after properly studying certain linear algebraic groups of automorphisms is as follows. We know from the example of Zarhin that Jordan property does not always hold for groups of birational selfmaps. Making obvious changes in Definition 1, one can introduce the *nilpotent Jordan property* of groups.

Question 10. Does nilpotent Jordan property always hold for groups of birational automorphisms of varieties over a field of characteristic zero?

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Maximal A_k-singularities of plane curves of fixed bidegree

Julia Schneider

We study algebraic curves (not necessarily reduced) on the affine plane $\mathbb{A}^2(\mathbb{C})$ that have a singularity of type A_k, which means that there is an analytical local isomorphism such that the curve is given by

$$y^2 - x^{k+1} = 0$$

in a neighbourhood of the singular point. We ask:

Question 1. For $d \geq 1$, what is the maximal k such that there exists a curve of degree d that has an A_k-singularity?

We denote this by $N(d)$ and can give answers for small d:

d	1	2	3	4	5	6	7...
$N(d)$	0	1	3	7	12	19	?

,

where the result for $d = 6$ is by Yang, who gave a classification of all simple singularities of sextic curves in [5]. (Note that the answers of $N(2)$, $N(3)$ and $N(4)$ differs if we consider only irreducible curves.) The difficulty of the question increases rapidly for larger values of d, so the asymptotic behaviour is studied and bounds for

$$\alpha = \limsup \frac{2\,N(d)}{d^2}$$

are wanted, where we multiplied by 2 to obtain nicer numbers, as it is often done in the literature. Gusein-Zade and Nekhoroshev [3] found in 2000 that $1.5 \geq \alpha \geq \frac{15}{14} \simeq 1.07142$ and in the same year, Cassou-Nogues and Luengo [1] refined the lower bound to $8 - 4\sqrt{3} \simeq 1.07179$. A decade passed until Orevkov [4] improved it even further to $\frac{7}{6} = 1.1\overline{6}$ in 2012.
Question 1 can also be approached through fixing a bidegree instead of the degree. We say that a polynomial F (or equivalently, the curve in $\mathbb{A}^2(\mathbb{C})$ defined by its zero set) has *bidegree* (a, b) if its Newton polygon lies in the triangle spanned by $(a, 0)$, $(0, 0)$ and $(0, b)$. For example, all polynomials of degree at most d are of bidegree (d, d). So we generalize Question 1:

Question 2. For $(a, b) \in \mathbb{N}^2$, what is the maximal k such that there is a curve in $\mathbb{A}^2(\mathbb{C})$ of bidegree (a, b) with an A_k-singularity?

Similar to above, we denote this by $N(a, b)$. For instance, one finds $N(1, b) = 0$ for all b, and fixing $a = 2$ yields $N(2, 2n - 1) = 2n - 2$ for odd b, and for even b we get $N(2, 2n) = 2n - 1$. We have studied the case where $b = 3$ and found the following values of $N(3, b)$:

Theorem 3. For small b, $N(3, b)$ is given by the following table:

b	3	4	5	6	7	8	9	10	11	12
$N(3, b)$	3	5	7	8	10	12	13	15	17	18

Moreover, for $b \geq 4$ there are irreducible polynomials that achieve the maximal singularities.

Studying polynomials of bidegree (a, b) is interesting on its own, however it could also help to determine the asymptotical behaviour of $N(d)$, thanks to the following result.

Proposition 4 (Orevkov [4])**.** If $N(a, b) \geq k - 1$, then $\alpha \geq \frac{2k}{ab}$.

And in fact, it *does* help: Luengo found $N(4, 6) \geq 13$, Orevkov applied this proposition and got $\alpha \geq \frac{7}{6}$. Initially, we hoped to improve this bound, but the best we get with our results is $N(3, 11) = 17$ yielding $\alpha \geq \frac{12}{11} \simeq 1.09$. Anyway, a better bound than Orevkov's is not expected using $N(3, b)$: A result in knot theory by Feller [2] about the existence of algebraic cobordisms between the torus knots $T_{2,k+1}$ and $T_{3,b}$ gives $\frac{5b-4}{3}$ as an upper bound for $N(3, b)$, at least for b coprime to 3. Hence, the best that might be found with $N(3, b)$ is $\frac{2N(3,b)}{3b} \simeq \frac{10}{9} < \frac{7}{6}$, asymptotically.

Idea of the proof of Theorem 3. Let m be the integer such that $b = 3m - r$ for some $r \in 0, 1, 2$. By embedding $\mathbb{A}^2(\mathbb{C})$ into the Hirzebruch surface $\mathbb{F}_m$ in a certain way, a curve as in Question 2 is mapped onto a curve $C \subset \mathbb{F}_m$ that has an A_k-singularity and that satisfies $C \sim 3S_+$ and does not intersect S_-, where S_+ is a curve with self-intersection m and S_- is the curve with self-intersection $-m$.
By blowing up the A_k-singularity and contracting the strict transform of the fiber passing through the A_k-singularity, we obtain a curve $C_1 \subset \mathbb{F}_{m_1}$ that has an A_{k-2}-singularity and that intersects the push-forward S_1 of S_- with intersection multiplicity 1, where $m_1 \in \{m+1, m-1\}$. We can repeat this $n = \lceil \frac{k}{2} \rceil$ times and we get a smooth curve $C_n \subset \mathbb{F}_{m_n}$ and a rational curve $S_n \subset \mathbb{F}_{m_n}$ that contain a point with intersection multiplicity n.
In fact, we can prove that this configuration lies in $\mathbb{F}_1$ (or $\mathbb{F}_0$) and by contracting it onto $\mathbb{P}^2$ we can determine whether the configuration exists or not. If it exists, we find $N(3, b) \geq k$, and otherwise $N(3, b) \leq k$. In case of existence we provide explicit polynomials for the configuration in $\mathbb{P}^2$. □

We stop at $b = 12$ because the computations get more and more tedious. It would be interesting to have a family of curves of bidegree $(3, b)$ with increasing b that have maximal A_k-singularity.

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Subgroups of elliptic elements of the plane Cremona group

Christian Urech

The main source for the content of this abstract are the two papers [7] and [8]. The *Cremona group* $\mathrm{Cr}_2(\mathbb{C})$ is the group of birational transformation of the complex projective plane. One of the key techniques for studying the group theoretical properties of infinite subgroups of the complex plane Cremona group $\mathrm{Cr}_2(\mathbb{C})$ has been an action by isometries on an infinite dimensional hyperboloid $\mathbb{H}^\infty(\mathbb{P}^2)$ (see [3] for an overview and references). Recall that there are three types of isometries of hyperbolic spaces:

- *elliptic isometries*, which are the isometries that fix a point in $\mathbb{H}^\infty(\mathbb{P}^2)$,
- *parabolic isometries*, which are the isometries that do not fix any point in $\mathbb{H}^\infty(\mathbb{P}^2)$, but fix exactly one point in the boundary $\partial\,\mathbb{H}^\infty(\mathbb{P}^2)$,

- *loxodromic isometries*, which are the isometries that do not fix any point in $\mathbb{H}^\infty(\mathbb{P}^2)$, but fix exactly two points in $\partial\,\mathbb{H}^\infty(\mathbb{P}^2)$.

We call an element $f \in \mathrm{Cr}_2(\mathbb{C})$ *elliptic, parabolic* or *loxodromic*, if the isometry of $\mathbb{H}^\infty(\mathbb{P}^2)$ induced by f is elliptic, parabolic or loxodromic respectively. This notion is linked to the dynamical behavior of f.

We consider subgroups of $\mathrm{Cr}_2(\mathbb{C})$ consisting only of elliptic elements. The main result is that the group theoretical structure of these subgroups is not more complicated than the structure of algebraic subgroups of $\mathrm{Cr}_2(\mathbb{C})$:

Theorem 1 ([8])**.** Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a subgroup of elliptic elements. Then one of the following is true:

(1) G is contained in an algebraic subgroup;
(2) G preserves a rational fibration;
(3) G is a torsion subgroup.

Theorem 2 ([8])**.** Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a torsion subgroup. Then G is isomorphic to a bounded subgroup of $\mathrm{Cr}_2(\mathbb{C})$.

In combination with the classification of maximal algebraic subgroups (see [1]), Theorem 1 and Theorem 2 give an explicit description of groups of elliptic elements. This allows to give new descriptions of arbitrary subgroups of $\mathrm{Cr}_2(\mathbb{C})$.

Theorem 1 and Theorem 2 can now be used to prove structure theorems on general subgroups of $\mathrm{Cr}_2(\mathbb{C})$. Given a subgroup G of $\mathrm{Cr}_2(\mathbb{C})$ one can consider the following three cases:

(1) G contains a loxodromic element;
(2) G contains no loxodromic element but a parabolic element;
(3) G is a subgroup of elliptic elements.

In case (1), the group G can be understood by using tools from hyperbolic geometry and geometric group theory, in case (2) it is known that G preserves a rational or elliptic fibration and case (3) can be treated with the help of Theorem 1 and Theorem 2. Let us explain two results that can be proved with this strategy.

1. The Tits alternative

Recall the following definition:

Definition 3.

(1) A group G satisfies the *Tits alternative* if every subgroup of G is either virtually solvable or contains a non-abelian free subgroup.
(2) A group G satisfies the *Tits alternative for finitely generated subgroups* if every finitely generated subgroup of G either is virtually solvable or contains a non-abelian free subgroup.

Cantat established the Tits alternative for finitely generated subgroups of $\mathrm{Cr}_2(\mathbb{C})$ ([2]). Theorem 1 and Theorem 2 yield the results needed to generalize this result:

Theorem 4 ([8])**.** The plane Cremona group $\mathrm{Cr}_2(\mathbb{C})$ satisfies the Tits alternative.

2. Simple subgroups of the plane Cremona group

It had been a long-standing open question, whether the plane Cremona group is simple as a group until Cantat and Lamy showed in 2012 that it is not ([4]). The main idea to prove this result was to use techniques from small cancellation theory, an approach that has been refined by Shepherd-Barron and Lonjou (see [6], [5]). These results are a starting point for the following classification of all simple subgroups of the plane Cremona group:

Theorem 5 ([7]). Let $G \subset \mathrm{Cr}_2(\mathbb{C})$ be a simple group. Then:

(1) G does not contain loxodromic elements.
(2) If G contains a parabolic element, then G is conjugate to a subgroup of $\mathcal{J}$.
(3) If all elements in G are elliptic, then either G is a simple subgroup of an algebraic subgroup of $\mathrm{Cr}_2(\mathbb{C})$, or G is conjugate to a subgroup of $\mathcal{J}$.

With the help of Theorem 5 all simple groups that act non-trivially by birational transformations on compact complex Kähler surfaces can be described:

Theorem 6 ([7]). Let G be a simple group. Then

(1) G acts non-trivially by birational transformations on a rational complex projective surface if and only if G is isomorphic to a subgroup of $\mathrm{PGL}_3(\mathbb{C})$.
(2) G acts non-trivially by birational transformations on a non-rational compact complex Kähler surface of negative Kodaira dimension if and only if G is finite or isomorphic to a subgroup of $\mathrm{PGL}_2(\mathbb{C})$.
(3) G acts non-trivially by birational transformations on a compact complex Kähler surface S of non-negative Kodaira dimension if and only if G is finite.

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