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Non-commutative Geometry, Index Theory and Mathematical Physics

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Abstract. Non-commutative geometry today is a new but mature branch of mathematics shedding light on many other areas from number theory to operator algebras. In the 2018 meeting two of these connections were highlighted. For once, the applications to mathematical physics, in particular quantum field theory. Indeed, it was quantum theory which told us first that the world on small scales inherently is non-commutative. The second connection was to index theory with its applications in differential geometry. Here, non-commutative geometry provides the fine tools to obtain higher information.

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Introduction by the Organisers

The 2018 meeting on non-commutative geometry (NCG) was organized by R. Nest, T. Schick, G. Yu and A. Connes. We had chosen to focus the conference on the link of NCG with index theory and its link with physics. There were an average of five regular talks per day. The variety of topics is demonstrated for instance by the program of the Tuesday whose morning was concentrated on index theory and KK theory while the afternoon dealt with physics and in particular on the recent advances in QFT on non-commutative spaces. There was a very strong demand from the participants to give talks and the way we handled it is exemplified by the program of the Thursday which besides three regular talks contained a one hour talk shared between two speakers (due to the common ground they treated) and also a one hour session for a "gong show" which featured 6 talks of 8 minutes each where young researchers described their work. This happened to be very successful and the interaction with the audience worked very well. The speakers at the "gong show" were Mehran Seyedhosseini, Alexander Engel, Eske Ewert, Simone Cecchini, Mayuko Yamashita, and Valerio Proietti. The meeting was attended by 51 participants from all over the world and the emphasis on young researchers was quite efficient.

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Workshop: Non-commutative Geometry, Index Theory and Mathematical Physics

Table of Contents

Abstracts

Pairing cyclic theory with Higson-Roe's surgery exact sequence

Magnus Goffeng
(joint work with Robin Deeley)

This talk is based on the work [3, 4, 5] joint with Robin Deeley. We describe a geometric (in the sense of Baum-Douglas) model for Higson and Roe's analytic structure group [6]. We pair delocalized cyclic cocycles with the analytic structure group via a Chern character defined on the geometric model giving a coherent framework for several known secondary invariants such as relative η-invariants, L^2-ρ invariants and Lott's delocalized η-invariants.

There are several analytic approaches to Higson and Roe's analytic structure group: Higson-Roe's original definition [6] using coarse geometry, Xie-Yu's definition [9] using Yu's localization algebras and Zenobi's definition [10] using adiabatic deformations. The two main virtues of the analytically defined models are that they are "easy to map into" (geometric data gives rise to canonical cycles) and that they are structurally more stable coming with various K-theoretical tools. The geometric model is easier to "map out of" in the sense that there are fewer and better behaved cycles on which a Chern character is defined. This fact makes the geometric model highly suitable for geometric invariants.

1. Geometric models

In the 80's, Baum and Douglas [1] gave a geometric model $K_*^{\text{geo}}(X)$ for the K-homology of a space X using geometric cycles (M, E, φ); here M is a closed spin^c-manifold, $E \to M$ is a complex vector bundle and $\varphi : M \to X$ is a continuous mapping. The relation imposed on the geometric cycles is generated by direct sum/disjoint union, vector bundle modification and bordism. We shall work in a slightly more general setting by picking a dense $*$-subalgebra $\mathcal{A} \subseteq C_r^*(\Gamma)$ containing the group algebra $\mathbb{C}[\Gamma]$. We can define a geometric group $K_*^{\text{geo}}(X; \mathcal{A})$ as above by using geometric cycles $(M, \mathcal{E}_\mathcal{A}, \varphi)$ where $\mathcal{E}_\mathcal{A} \to M$ is an $\mathcal{A}$-bundle (i.e. a locally trivial bundle of finitely generated projective $\mathcal{A}$-modules). If X is a finite CW-complex and $\mathcal{A}$ is closed under holomorphic functional calculus, there is an explicit isomorphism $K_*^{\text{geo}}(X; \mathcal{A}) \cong KK_*(C(X), C_r^*(\Gamma))$.

In analogy with K-homology we ask for a Baum-Douglas model of the analytic structure group $\mathcal{S}_*(X; \Gamma)$. For simplicity, we assume that Γ acts freely on $\tilde{X}$ and set $X := \tilde{X}/\Gamma$. The geometric model for the analytic surgery group (given in [3]), relies on an explicit description of the free assembly mapping. The free assembly mapping uses the Mishchenko bundle $\mathcal{L}_{X,\mathcal{A}} := \tilde{X} \times_\Gamma \mathcal{A} \to X$ and is given by

$$\mu : K_*^{\text{geo}}(X) \to K_*^{\text{geo}}(pt; \mathcal{A}), \quad \mu(M, E, \varphi) := (M, E \otimes \varphi^* \mathcal{L}_{X,\mathcal{A}}). \tag{1}$$

If $\mathcal{A}$ is holomorphically closed, μ can be identified with the assembly mapping. Starting from this geometric description of assembly, we can define a geometric model. A cycle for $\mathcal{S}^{\text{geo}}_*(X;\mathcal{A})$ as a triple (W,ξ,f) where W is a compact spin^c-manifold with boundary, $f:\partial W\to B\Gamma$ is a continuous mapping and ξ is a relative K-theory cocycle for $(W,\partial W,f^*\mathcal{L}_{X,\mathcal{A}})$. The cocycle ξ contains three pieces of data: $\mathcal{A}$-bundle data $\xi_{\mathcal{A}}$ over W, vector bundle data $\xi_{\mathbb{C}}$ over ∂W and an isomorphism α identifying $\xi_{\mathcal{A}}|_{\partial W}$ with $\xi_{\mathbb{C}}\otimes f^*\mathcal{L}_{X,\mathcal{A}}$ on ∂W. Heuristically, a cycle for $\mathcal{S}^{\text{geo}}_*(X;\mathcal{A})$ is a cycle for $K^{\text{geo}}_{*-1}(B\Gamma)$ *with a prescribed reason for its vanishing under assembly.*

Theorem 1 ([3, 5])**.** *The abelian group $\mathcal{S}^{\text{geo}}_*(X,\mathcal{A})$ fits into an exact sequence*

$$\cdots\to\mathcal{S}^{\text{geo}}_{*+1}(X,\mathcal{A})\xrightarrow{\delta}K^{\text{geo}}_*(X)\xrightarrow{\mu}K^{\text{geo}}_*(pt;\mathcal{A})\xrightarrow{r}\mathcal{S}^{\text{geo}}_*(X,\mathcal{A})\xrightarrow{\delta}K^{\text{geo}}_{*-1}(X)\to\cdots,$$

with μ, δ and r defined at the level of cycles. This short exact sequence maps explicitly into Higson-Roe's analytic surgery exact sequence via higher index theory. If X is a (locally) finite CW-complex and $\mathcal{A}$ is holomorphically closed, these mappings are isomorphisms.

2. Chern characters

The geometric model comes with a Chern character. This construction requires a Frechet algebra structure on $\mathcal{A}$. Following Wahl's and Lott's work on higher index theory, we use topological noncommutative de Rham homology of $\mathcal{A}$ (see more in [5, 8]). The abelianized topological de Rham complex of $\mathcal{A}$ is denoted by $\hat{\Omega}^{\text{ab}}_*(\mathcal{A})$. The closed subcomplex of forms localized at the identity $e\in\Gamma$ is denoted by $\hat{\Omega}^{\langle e\rangle}_*(\mathcal{A})$. The associated topological homology groups are denoted by $\hat{H}^{\text{dR}}_*(\mathcal{A})$ and $\hat{H}^{\langle e\rangle}_*(\mathcal{A})$, respectively. The mapping cone complex of the inclusion $\hat{\Omega}^{\langle e\rangle}_*(\mathcal{A})\to\hat{\Omega}^{\text{dR}}_*(\mathcal{A})$ defines the *delocalized homology group* $\hat{H}^{\text{del}}_*(\mathcal{A})$.

Theorem 2 ([5])**.** *Chern-Weil theory for total connections allow for definitions of Chern characters making the following diagram commutative with exact rows:*

(2)
$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & K^{\text{geo}}_*(X) & \xrightarrow{\mu} & K^{\text{geo}}_*(pt;\mathcal{A}) & \xrightarrow{r} & \mathcal{S}^{\text{geo}}_*(X;\mathcal{A}) & \xrightarrow{\delta} & K^{\text{geo}}_{*-1}(X) & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \text{ch}_X} & & \downarrow{\scriptstyle \text{ch}_{\mathcal{A}}} & & \downarrow{\scriptstyle \text{ch}^{\text{del}}} & & \downarrow{\scriptstyle \text{ch}_X} & & \\
\cdots & \longrightarrow & \hat{H}^{\langle e\rangle}_*(\mathcal{A}) & \xrightarrow{\mu_{\text{dR}}} & \hat{H}^{\text{dR}}_*(\mathcal{A}) & \xrightarrow{r_{\text{dR}}} & \hat{H}^{\text{del}}_*(\mathcal{A}) & \xrightarrow{\delta_{\text{dR}}} & \hat{H}^{\langle e\rangle}_{*-1}(\mathcal{A}) & \longrightarrow & \cdots
\end{array}.$$

The full details of the construction of Chern characters can be found in [5]. An important feature of the delocalized Chern character is its connection with Wahl's higher η-invariant $\hat{\eta}(W,\xi,f,A_{\xi_{\mathbb{C}}})\in\hat{\Omega}^{\text{dR}}_*(\mathcal{A})$. The following result follows from the construction and Wahl's higher Atiyah-Patodi-Singer index theorem [8].

Theorem 3 ([5])**.** *Assume that $\mathcal{A}=\varprojlim\mathcal{A}_k$ for a projective system $(\mathcal{A}_k)_{k\in\mathbb{N}}$ where $\mathcal{A}_k\subseteq C^*_r(\Gamma)$ is a Banach *-algebra closed under holomorphic functional calculus. We let $p:\hat{H}^{\text{del}}_*(\mathcal{A})\to\hat{H}_*(\hat{\Omega}^{\text{dR}}_*(\mathcal{A})/\hat{\Omega}^{\langle e\rangle}_*(\mathcal{A}))$ denote the canonical projection. The image of the delocalized Chern character under p can be written as:*

$$p\left(\text{ch}^{\text{del}}(W,\xi,f)\right)=\left[\text{ch}_{\mathcal{A}}\left(\text{ind}^{APS}_{C^*_r(\Gamma)}(D^W_{\xi_{C^*_r(\Gamma)}},A_{\xi_{C^*_r(\Gamma)}})+\text{cs}(\xi,A_{\xi_{C^*_r(\Gamma)}},A_{\xi_{\mathbb{C}}})\right)\right]+[\hat{\eta}(W,\xi,f,A_{\xi_{\mathbb{C}}})].$$

3. Delocalized invariants and relations to higher index theory

The delocalized Chern character on the geometric model for the analytic surgery group allow us to pair with delocalized cyclic cocycles. If $c \in C^k_\lambda(\mathcal{A})$ is a continuous cyclic cocycle, we say that c is *delocalized* if

$$c(\gamma_0, \gamma_1, \dots, \gamma_k) = 0, \quad \text{whenever } \gamma_0\gamma_1 \cdots \gamma_k = e.$$

Theorem 4. *Assume that $c \in C^k_\lambda(\mathcal{A})$ is a continuous delocalized cyclic cocycle. Then there is an associated delocalized index character $\tau_c : \mathcal{S}^{\mathrm{geo}}_*(X, \mathcal{A}) \to \mathbb{C}$ such that $\tau_c \circ r(x) = \langle [c], \mathrm{ch}_\mathcal{A}(x)\rangle$ for $x \in K^{\mathrm{geo}}_*(pt; \mathcal{A})$. Moreover, τ_c factors over p and if $\mathcal{A}$ satisfies the assumption of Theorem 3 we have that*

$$\tau_c[(W,\xi,f)] = \left\langle [c], \mathrm{ch}_\mathcal{A}\left(\mathrm{ind}^{APS}_{C^*_r(\Gamma)}(D^W_{\xi_{C^*_r(\Gamma)}}, A_{\xi_{C^*_r(\Gamma)}}) + \mathrm{cs}(\xi, A_{\xi_{C^*_r(\Gamma)}}, A_{\xi_\mathbb{C}})\right)\right\rangle + \langle c, \hat{\eta}(W,\xi,f,A_{\xi_\mathbb{C}})\rangle.$$

A closed manifold (M,g) with a positive scalar curvature metric (PSC) has an associated ρ-class $\rho(M,g)$ in the analytic structure group. Its pairing with a delocalized continuous cyclic cocycle c can be computed when $\mathcal{A}$ satisfies the assumption of Theorem 3. Indeed, $\tau_c \circ \rho(M,g) = \langle c, \hat{\eta}(M,g)\rangle$, where $\hat{\eta}(M,g)$ denotes the higher eta invariant of the spin Dirac operator on M defined from g. Theorem 3 allows us to describe the special case when $c \in C^0_\lambda(\mathcal{A})$ satisfies $c(1) = 0$:

$$\tau_c \circ \rho(M,g) = \frac{2}{\sqrt{\pi}} \int_0^\infty (\mathrm{Tr} \otimes c)(\tilde{D}_g \mathrm{e}^{-s^2 \tilde{D}_g^2}) \mathrm{d}s,$$

where $\tilde{D}_g$ is the spin Dirac operator on the universal cover of M. This gives rise to invariants $\tau_c \circ \rho$ on Stolz' PSC-group. *It is an open problem to compute the pairing of delocalized continuous cyclic cocycles with the topological structure set.*

Let us consider some examples of continuous delocalized cocycles. All our examples are traces. *It would be interesting to exhibit some examples of higher delocalized cocycles. However, there are substantial technical difficulties with continuity of higher cocycles.* If $c : \mathcal{A} \to \mathbb{C}$ is a continuous trace, c is delocalized if $c(1) = 0$.

- For two representations $\sigma_1, \sigma_2 : \Gamma \to U(N)$, we can define a delocalized conintuous 0-cocycle $c := \mathrm{Tr}_{\sigma_1} - \mathrm{Tr}_{\sigma_2}$ on $C^*(\Gamma)$. The associated delocalized index character is the relative η-invariant. See [4, 5].
- For the trivial representation $\epsilon : \Gamma \to U(1)$ and the L^2-trace Tr_{L^2}, we can define the delocalized continuous 0-cocycle $c := \mathrm{Tr}_{L^2} - \mathrm{Tr}_\epsilon$ on $C^*(\Gamma)$ whose associated delocalized index character is the L^2-ρ-invariant. See [2, 4].
- If $g \in \Gamma$ is a non-trivial element with associated conjugacy class $\langle g\rangle$, we define the delocalized trace $\mathrm{Tr}_{\langle g\rangle}(\sum_{\gamma\in\Gamma} a_\gamma \gamma) := \sum_{\gamma\in\langle g\rangle} a_\gamma\gamma$. If $\langle g\rangle$ has polynomial growth, Xie-Yu showed that $\mathrm{Tr}_{\langle g\rangle}$ is continuous on the Connes-Moscovici algebra. It follows from Theorem 3 that the associated delocalized index character extends Lott's delocalized η-invariant to the analytic structure group.

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Singular structures, groupoids and K-theory invariants of Dirac operators

Paolo Piazza
(joint work with Vito Felice Zenobi)

Let ${}^{\mathrm{S}}X$ be a Thom-Mather stratified pseudomanifold and let X^{reg} be its regular part. The goal of this talk was to report on recent results, in collaboration with Vito Felice Zenobi, concerning analytic, geometric and topological invariants of Dirac operators on X^{reg} [4]. In fact, in addition to stratified spaces we considered others, more singular structures, such as manifolds X with a foliated boundary $(\partial X, \mathcal{F}_\partial)$, or foliated manifolds with boundary $(X, \mathcal{H})$ with a foliation $\mathcal{H}$ which is transverse to the boundary and degenerates on the boundary to a foliation $\mathcal{F}_\partial$ of higher codimension.

In the first part of the talk we considered a smooth compact manifold M with fundamental group Γ and with universal cover M_Γ. We illustrated two approaches to K-theory invariants for Dirac operators, the classic one due to Higson and Roe and the more recent one, due to Zenobi [5], based on the adiabatic deformation of the groupoid $M_\Gamma \times_\Gamma M_\Gamma$ over M, with source map $s[x,y] = y$ and range map $r[x,y] = x$. We explained in particular how given a Dirac operator $D : C^\infty(M,E) \to C^\infty(M,E)$ and its Γ-equivariant lift $D_\Gamma : C^\infty(M_\Gamma, E_\Gamma) \to C^\infty(M_\Gamma, E_\Gamma)$, it is possible to define the fundamental class $[D] \in K_*(M)$ and the index class $\mathrm{Ind}(D_\Gamma) \in K_*(C^*_r\Gamma)$ and how these fit into the analytic surgery sequence of Higson and Roe. We then introduced the rho class $\rho(D_\Gamma)$ of an invertible operator, an element in the K-theory of the C^*-algebra $D^*(M_\Gamma)^\Gamma$ obtained as the closure of the Γ-equivariant, finite propagation, pseudolocal operators on $L^2(M_\Gamma, E_\Gamma)$. We also stated the delocalized Atiyah-Patodi-Singer index theorem [3], a fundamental result relating the index class of a Dirac operator on a manifold with boundary with invertible boundary operator and the rho class of the boundary operator. We explained how the fundamental class, the index class and the rho class can be defined in the adiabatic deformation approach, and stated compatibility results relating the two sets of invariants. We also reported on the possibility of developing the theory for a general Lie groupoid and how in this generality Zenobi has proved a delocalized Atiyah-Patodi-Singer index theorem. We ended this introductory part by explaining some important geometric applications of these invariants, for example the mapping of the surgery sequence of Browder, Novikov, Sullivan and Wall to the analytic surgery sequence, and the mapping of the Stolz' surgery sequence for positive scalar curvature metrics to the analytic surgery sequence. In particular, we explained how the rho class gives well-defined maps

(1)

$$\rho : \pi_0(\mathcal{R}^+(M)) \to K_{\dim M+1}(D^*(M_\Gamma)^\Gamma), \;\; \rho : \widetilde{\pi}_0(\mathcal{R}^+(M)) \to K_{\dim M+1}(D^*(M_\Gamma)^\Gamma)$$

with $\mathcal{R}^+(M)$ denoting the space of positive scalar curvature metrics on X and $\widetilde{\pi}_0(\mathcal{R}^+(X))$ the associated set of concordance classes.

In the second part of the talk we tackled the following problem: how can one define these three K-theory classes (i.e. the fundamental class, the index class and the rho class) for the singular structures introduced above ? We started with a Thom-Mather pseudomanifolds of depth 1 with singular stratum Y and link L_Y. To any such space one can associated its resolution X, a smooth manifold with boundary $H := \partial X$ and with H the total space of a fibration $L_Y \to H \xrightarrow{\phi} Y$ with base Y and with typical fiber the link L_Y. There is a natural identification between the interior of X, $\mathring{X}$, and ${}^{\mathrm{S}}X^{\mathrm{reg}}$. We introduced a metric structure on our singular space by endowing ${}^{\mathrm{S}}X^{\mathrm{reg}}$, or, equivalently $\mathring{X}$, with a riemannian metric g. There are many different types of metrics that can be considered; the ones that were singled out in this talk were the fibered boundary metrics of Mazzeo and Melrose [2]. These are metrics that in a tubular neighbourhood of the singularity Y, or equivalently, in a collar neighborhood of the boundary of X, can be written in the following special form

$$\frac{dx^2}{x^4} + \frac{\phi^* g_Y}{x^2} + h_{H/Y}$$

with x a boundary defining function for ∂X and $h_{H/Y}$ a fiber metric on the boundary fibration $L_Y \to H \equiv \partial X \xrightarrow{\phi} Y$. The vector fields dual to the above metric are given by

$$\mathcal{V}_\Phi(X) = \{\xi \in \mathcal{V}_b(X) \;|\; \xi|_{\partial X} \text{ is tangent to the fibers of } \phi \text{ and } \xi x \in x^2 C^\infty(X)\}$$

where $\mathcal{V}_b(X)$ is the Lie algebra of vector fields that are tangent to the boundary. Notice that $\mathcal{V}_\Phi(X)$ is a Lie algebra and is finitely generated and projective as a $C^\infty(X)$-module. According to Serre-Swan, there exists a smooth vector bundle ${}^\Phi TX \to X$, the Φ-tangent bundle, and a natural map $\iota_\Phi \colon {}^\Phi TX \to TX$ such that $\mathcal{V}_\Phi(X) = \iota_\Phi C^\infty(X, {}^\Phi TX)$. These data define an algebroid $({}^\Phi TX, \iota_\Phi)$. Thanks to a theorem of Debord this algebroid can be integrated; the corresponding groupoid G_Φ, considered first in the work of Debord-Lescure-Rochon [1], is explicitly given by $X^{\mathrm{reg}} \times X^{\mathrm{reg}}$ over X^{reg} and $H \times_Y TY \times_Y H \times \mathbb{R}$ over H. There is also a Γ-equivariant version of it, G^Γ_Φ, again a groupoid over $X \times [0,1]$. Let us denote by $(G^\Gamma_\Phi)^0_{ad}$ the restriction of the adiabatic deformation of G^Γ_Φ to $X \times [0,1)$. The C*-algebra of this groupoid fits into the following exact sequence

$$0 \to C^*_r(X^{reg}_\Gamma \underset{\Gamma}{\times} X^{reg}_\Gamma \times (0,1)) \to C^*_r((G^\Gamma_\Phi)^0_{ad}) \to C^*_r(T^{\mathrm{NC}}X) \to 0. \tag{2}$$

with $T^{\mathrm{NC}}X$ a groupoid over $X^{reg} \times \{0\} \cup H \times [0,1)$ given explicitly by the disjoint union ${}^\Phi TX \cup (H \times_Y TY \times_Y H \times \mathbb{R}) \times (0,1)$. We denote by δ the connecting homomorphism in the K-theory long exact sequence associated to (2).

Let us now assume that the Φ-tangent bundle endowed with the given Φ-metric g admits a spin structure and let us choose one. Let us concentrate on the associated spin-Dirac operator $\not{D}_g$. Using the groupoid pseudodifferential calculus for G_Φ one can show that under the additional hypothesis that the singular stratum is spin and that the links have positive scalar curvature, the Dirac operator $\not{D}_g$ defines a class $\sigma_{\mathrm{nc}}(\not{D}_g) \in K_*(C^*_r(T^{\mathrm{NC}}X))$ and thus a class $\delta(\sigma_{\mathrm{nc}}(\not{D}_g))$ in $K_*(C^*_r(X^{reg}_\Gamma \times_\Gamma X^{reg}_\Gamma)) = K_*(C^*_r\Gamma)$. A fundamental result, ultimately due to Debord, Lescure and Rochon, states that $C({}^{\mathrm{S}}X)$, the algebra of continuous functions on the stratified pseudomanifold, is K-dual to the C*-algebra $C^*_r(T^{\mathrm{NC}}X)$: thus $K_*({}^{\mathrm{S}}X) \cong K_*(C^*_r(T^{\mathrm{NC}}X))$. Moreover, under this isomorphism the class $\sigma_{\mathrm{nc}}(\not{D}_g)$ correspond to $[\not{D}_g] \in K_*({}^{\mathrm{S}}X)$, the latter class defined through a parametrix construction in the Mazzeo-Melrose pseudodifferential calculus. In this talk we presented this as a compatibility result, relating an adiabatic invariant with a classic invariant. We then stated a second compatibility result, due to Piazza and Zenobi, asserting that the class $\delta(\sigma_{\mathrm{nc}}(\not{D}_g))$ is equal to the index class defined by the Γ-equivariant operator associated to $\not{D}_g$ via a Γ-equivariant parametrix construction.

Summarizing: at this point of the talk we have defined the fundamental class and the index class in the adiabatic context and we have related them to more classic invariants. Assume now that g has positive scalar curvature everywhere on X; then the class $\delta(\sigma_{\mathrm{nc}}(\not{D}_g))$ vanishes and we can define a class $\rho(g) \in K_*(C^*_r((G^\Gamma_\Phi)^0_{ad}))$ as a natural specific lift of the class $[\sigma_{nc}(\not{D}_g)]$ in the K-theory long exact sequence associated to (2). This is our rho-class. Thanks to the delocalized APS index theorem for groupoids, due to Zenobi, this rho class gives a well-defined map from $\widetilde{\pi}^{fb}_0(X)$, the set of concordance classes of fibered boundary metrics on X, to $K_*(C^*_r((G^\Gamma_\pi)^0_{ad}))$. We ended this talk by explaining how these groupoid techniques can be applied with little further work to the more singular structures mentioned at the beginning of this extended abstract; this is indeed one of the motivations for giving a groupoid treatment of the three K-theoretic invariants.

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Higher rho invariants, delocalized eta invariants, and the Baum-Connes conjecture

Zhizhang Xie

(joint work with Guoliang Yu)

Let X be a complete manifold of dimension n with a discrete group Γ acting on it properly and cocompactly through isometries. Each Γ-equivariant elliptic differential operator D on X gives rise to a higher index class $\mathrm{Ind}_\Gamma(D) \in K_n(C^*_r(\Gamma))$. Here $C^*_r(\Gamma)$ is the reduced group C^*-algebra of Γ. This higher index class is an obstruction to the invertibility of D. It is a far-reaching generalization of the classical Fredholm index and plays a fundamental role in the studies of many problems in geometry and topology such as the Novikov conjecture, the Baum-Connes conjecture and the Gromov-Lawson-Rosenberg conjecture. Higher index classes are often referred to as primary invariants. When the higher index class of an operator vanishes, a secondary index theoretic invariant naturally arises. One such example is the associated Dirac operator $\widetilde{D}$ on the universal covering $\widetilde{M}$ of a closed spin manifold M, which is equipped with a positive scalar curvature metric g. In this case, it follows from the Lichnerowicz formula that the higher index of the Dirac operator vanishes. And there is a natural secondary higher invariant – introduced by Higson and Roe [8, 9, 10, 18] – called the higher rho invariant of $\widetilde{D}$ (with respect to the metric g). This higher rho invariant is an obstruction to the inverse of the Dirac operator being local, and has important applications to geometry and topology.

On the other hand, for the same Dirac operator $\widetilde{D}$ above, Lott introduced the following delocalized eta invariant $\eta_{\langle h\rangle}(\widetilde{D})$ [16]:

$$\eta_{\langle h\rangle}(\widetilde{D}) := \frac{2}{\sqrt{\pi}}\int_0^\infty \mathrm{tr}_h(\widetilde{D}e^{-t^2\widetilde{D}^2})dt, \tag{1}$$

under the condition that the conjugacy class $\langle h\rangle$ of $h \in \pi_1 M$ has polynomial growth. Here $\pi_1 M$ is the fundamental group of M, and tr_h is the following trace map:

$$\mathrm{tr}_h(A) = \sum_{g \in \langle h \rangle} \int_{\mathcal{F}} A(x, gx) dx$$

on Γ-equivariant Schwartz kernels $A \in C^\infty(\widetilde{M} \times \widetilde{M})$, where $\mathcal{F}$ is a fundamental domain of $\widetilde{M}$ under the action of Γ.

This talk is based on joint work with Guoliang Yu on our conceptual K-theoretic approach to precise connections between Higson-Roe's K-theoretic higher rho invariants and Lott's delocalized eta invariants. More precisely, we have the following theorem.

Theorem 1. *Let M be a closed odd-dimensional spin manifold equipped with a positive scalar curvature metric g. Suppose $\widetilde{M}$ is the universal cover of M, $\tilde{g}$ is the Riemannnian metric on $\widetilde{M}$ lifted from g, and $\widetilde{D}$ is the associated Dirac operator on $\widetilde{M}$. Suppose the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \pi_1 M$ has polynomial growth, then we have*

$$\tau_h(\rho(\widetilde{D}, \widetilde{g})) = \frac{1}{2}\eta_{\langle h \rangle}(\widetilde{D}),$$

where $\rho(\widetilde{D}, \widetilde{g})$ is the K-theoretic higher rho invariant of $\widetilde{D}$ with respect to the metric $\tilde{g}$, and τ_h is a canonical determinant map associated to $\langle h \rangle$.

While the definition of Lott's delocalized eta invariant requires certain growth conditions on $\pi_1 M$ (e.g. polynomial growth on a conjugacy class), the K-theoretic higher rho invariant can be defined in complete generality, without any growth conditions on $\pi_1 M$. We give a generalization of Lott's delocalized eta invariant without imposing any growth conditions on $\pi_1 M$, provided that the strong Novikov conjecture holds for $\pi_1 M$. This is achieved by using the Novikov rho invariant introduced in [22, Section 7].

As an application of Theorem 1 above, we have the following algebraicity result concerning the values of delocalized eta invariants.

Theorem 2. *With the same notation as above, if the (rational) Baum-Connes conjecture holds for Γ, and the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, then the delocalized eta invariant $\eta_{\langle h \rangle}(\widetilde{D})$ is an algebraic number. Moreover, if in addition h has infinite order, then $\eta_{\langle h \rangle}(\widetilde{D})$ vanishes.*

This theorem follows from the construction of the determinant map τ_h and a L^2-Lefschetz fixed point theorem of B.-L. Wang and H. Wang [20, Theorem 5.10]. When Γ is torsion-free and satisfies the Baum-Connes conjecture, and the conjugacy class $\langle h \rangle$ of a non-identity element $h \in \Gamma$ has polynomial growth, Piazza and Schick have proved the vanishing of $\eta_{\langle h \rangle}(\widetilde{D})$ by a different method [17, Theorem 13.7].

In light of this algebraicity result, we propose the following question.

Question. *What values can delocalized eta invariants take in general? Are they always algebraic numbers?*

In particular, if a delocalized eta invariant is transcendental, then it will lead to a counterexample to the Baum-Connes conjecture [3, 4, 6]. Note that the above question is a reminiscent of Atiyah's question concerning rationality of ℓ^2-Betti numbers [1]. Atiyah's question was answered in negative by Austin, who showed that ℓ^2-Betti numbers can be transcendental [2].

Our work is inspired by previous work of Lott, Leichtnam, Piazza and Schick [15, 16][17][14]. A key new ingredient of our approach is the construction of an explicit determinant map τ_h on $K_1(C^*_{L,0}(\widetilde{M})^{\pi_1 M})$ for each non-identity conjugacy class $\langle h \rangle$ with polynomial growth. Here $C^*_{L,0}(\widetilde{M})^{\pi_1 M}$ is a certain geometric C^*-algebra and its K-theory $K_1(C^*_{L,0}(\widetilde{M})^{\pi_1 M})$ is the receptacle of secondary higher index theoretic invariants. Each such determinant map is induced by the corresponding trace map tr_h on $K_0(C^*_r(\pi_1 M))$, and our construction is inspired by the work of de la Harpe and Skandalis [7] and Keswani [12]. In fact, combined with finite propagation speeds of wave operators, our K-theoretic approach above can also be used to give a uniform treatment of various vanishing results and homotopy invariance results for delocalized eta variants in [21, 12, 13, 17, 11, 5]. These details will appear in a separate paper [19].

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Traces on Pseudodifferential Operators on the Noncommutative Torus

Carolina Neira Jimenez
(joint work with Cyril Levy, Sylvie Paycha)

Pseudodifferential calculus is a very useful tool in analysis and geometry. On smooth manifolds, this calculus is performed via symbols which are locally defined concepts. On manifolds equipped with certain symmetry (through the action of a Lie group) it is possible to develop a notion of the global symbol of a pseudodifferential operator following [3]. In this talk we consider such a notion and use it to define a global pseudodifferential calculus on the noncommutative torus. Analogously to the closed manifold case, we present a characterization of traces on this calculus in terms of the leading symbol trace, the noncommutative residue and the canonical trace. At the end, we show two instances where these traces appear, namely, to define zeta regularized traces and to give an interpretation of the scalar curvature of the noncommutative torus.

If θ is an $n \times n$ antisymmetric real matrix, we consider the 2–cocycle

$$c(k,l) = e^{-\pi i \langle k, \theta l \rangle}, \qquad \text{for all } k, l \in \mathbb{Z}^n.$$

For all $k \in \mathbb{Z}^n$ let U_k be the *Weyl elements* which satisfy $U_k U_l = e^{-2\pi i \langle k, \theta l\rangle} U_l U_k$. The set of Schwartz functions on the noncommutative torus is the set

$$\mathcal{A}_\theta := \left\{ \sum_{k \in \mathbb{Z}^n} a_k U_k \in L^1(\mathbb{Z}^n, c) : \ (a_k)_k \in \mathcal{S}(\mathbb{Z}^n) \right\}.$$

Considering the infinitesimal generators of an action of the commutative torus on $C^*(\mathbb{Z}^n, c)$ it is possible to equip $\mathcal{A}_\theta$ with a structure of Frechet $*$–algebra, and with forward difference operators the notion of $\mathcal{A}_\theta$–valued toroidal symbols on $\mathbb{Z}^n$ is introduced [2, Section 3].

A toroidal pseudodifferential operator on $\mathcal{A}_\theta$ is defined via the quantization map

$$\mathrm{Op}_\theta(\sigma)(a) := \sum_{k\in\mathbb{Z}^n} a_k\,\sigma(k)\,U_k,$$

where σ is an $\mathcal{A}_\theta$–valued toroidal symbol on $\mathbb{Z}^n$, and $a = \sum_{k\in\mathbb{Z}^n} a_k U_k \in \mathcal{A}_\theta$. The map Op_θ is a topological and algebraic isomorphism from the space of $\mathcal{A}_\theta$–valued toroidal symbols to the space of toroidal pseudodifferential operators on $\mathcal{A}_\theta$, compatible with the filtration given by the order of the symbols and the order of the operators.

An extension of a symbol σ on $\mathbb{Z}^n$ is a symbol on $\mathbb{R}^n$ whose restriction to $\mathbb{Z}^n$ is equal to σ. Extension maps can be used to introduce the notion of noncommutative classical toroidal symbols on $\mathbb{Z}^n$ and classical toroidal pseudodifferential operators on $\mathcal{A}_\theta$ [2, Section 4].

A linear form on a subset of $\mathcal{A}_\theta$–valued classical toroidal symbols is said to be

(1) exotic if it vanishes on symbols of order less than $-n$,
(2) singular if it vanishes on smoothing symbols,
(3) ℓ^1–continuous if it is continuous for the $\ell^1(\mathbb{Z}^n, \mathcal{A}_\theta)$–topology,
(4) trace if it vanishes on commutators.

By using the inverse of the map Op_θ, we also have such linear forms on classical pseudodifferential operators on $\mathcal{A}_\theta$. The noncommutative residue, the leading symbol trace and the canonical trace are examples of those linear forms. If the order of the symbols (operators) is an integer greater than or equal to $-n$, exotic traces are linear combinations of leading symbol traces and the noncommutative residue. If the order of the symbols (operators) is not an integer or if it is less than $-n$, ℓ^1–continuous traces are proportional to the canonical trace [2, Section 6].

With complex powers of an appropriate operator, zeta regularized traces on pseudodifferential operators on $\mathcal{A}_\theta$ can be defined by means of the canonical trace [2, Section 7]. The inverse Mellin transform of the zeta regularized trace of a pseudodifferential operator on $\mathcal{A}_\theta$ produces a heat kernel expansion, whose coefficients can be written in terms of the (extended) noncommutative residue. In particular, one recovers an analogue of the scalar curvature on the noncommutative torus [1, Section 4].

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An algebraic approach to locality for geometric and renormalisation purposes

Sylvie Paycha
(joint work with Pierre Clavier, Li Guo, Bin Zhang and Sara Azzali)

Disclaimer

These are informal notes based on a talk delivered at the meeting *Non-commutative Geometry, Index Theory and Mathematical Physics*, July 8-14th 2018. This short report on various papers in collaboration does not claim to have the level of rigour of the original articles listed in the bibliography.

Abstract

We propose a mathematical framework underlying the concept of locality in classical and quantum field theory as well as in geometry. We develop a machinery taylored to preserve locality during the renormalisation procedure. This provides an algebraic formulation of the conservation of locality while renormalising and applies to renormalise higher zeta functions at poles, such as conical (modelled on cones), branched (modelled on trees) zeta functions which generalise multizeta functions.

1. From locality sets to locality algebras

Our starting point is to view locality as a symmetric binary relation comprising all pairs of independent events. A **locality set** is a couple $(X,\top)$ where X is a set and $\top \subseteq X\times X$ is a symmetric relation on X, referred to as the **locality relation** (or **independence relation**) of the locality set. So for $x_1,x_2\in X$, denote $x_1\top x_2$ if $(x_1,x_2)\in\top$. When the underlying set X needs to be emphasised, we use the notation $X\times_\top X$ or $\top_X$ for $\top$. Here is a basic yet fundamental example.

Example 1.1. *The orthogonality relation* $\perp^Q$ *between vectors or subsets in an Euclidean vector space* (E,Q) *equips* E *or the power set* $\mathcal{P}(E)$ *with the structure of a locality set.*

Multivariate meromorphic germs at zero with linear poles are very useful for renormalisation purposes. Let $\mathbb{R}^\infty=\cup_{k=1}^\infty\mathbb{R}^k$ be equipped with an inner product Q compatible with the filtration.

For $k\in\mathbb{N}$, let $\mathcal{M}(\mathbb{C}^k)=\left\{\frac{h(z_1,\cdots,z_k)}{\prod_{i=1}^m L_i^{s_i}(z_1,\cdots,z_k)},\quad L_i\in\left(\mathbb{R}^k\right)^*,\quad s_i\in\mathbb{C}\right\}$, be the space of meromorphic germs at 0 with linear poles and rational coefficients, so h is a holomorphic germ at zero and for $i\in\{1,\cdots,m\}$, $L_i\in\left(\mathbb{R}^k\right)^*$ are linear forms with rational coefficients, s_i are positive integers. The set $\mathcal{M} := \bigcup_{k=1}^{\infty} \mathcal{M}(\mathbb{C}^k)$ is filtered by $\mathbb{N}$.

We now equip $\mathcal{M}$ with a locality structure.

Definition 1.2. *Two meromorphic germs with rational coefficients f and f' are* Q**- orthogonal (or independent)** *which we denote by $f \perp^Q f'$ if the sets of variables on which they depend span Q-orthogonal spaces. Let $(\mathcal{M}, \perp^Q)$ denote the resulting locality set.*

One easily checks that the locality set $\left(\mathcal{M}, \perp^Q, \cdot\right)$ is a (filtered) locality algebra as defined in [CGPZ1].

2. Locality linear forms on pseudodifferential operators

M stands for a closed smooth manifold of dimension n. Let $\Psi_{\rm phg}(M)$ denote the algebra of classical (polyhomogeneous) pseudodifferential operators acting on $C^\infty(M)$; it contains $\Psi_{\rm phg}^{-\infty}(M)$ as a subalgebra. The action of $C^\infty(M, \mathbb{C})$ via composition by multiplication operators extends to $\Psi_{\rm phg}(M)$ which is a $C^\infty(M)$-module. We want to study "linear"[1] forms on $\Psi_{\rm phg}^{\Gamma}(M) := \{A \in \Psi_{\rm phg}(M), {\rm ord}(A) \in \Gamma\}$, for a given subset $\Gamma \subset \mathbb{C}$, where ${\rm ord}(A)$ denotes the order of the operator A. Since multiplication operators have order zero, the set $\Psi_{\rm phg}^{\Gamma}(M)$ is a $C^\infty(M)$- operated subset of $\Psi_{\rm phg}(M)$.

Definition 2.1. *A linear form $\Lambda : \Psi_{\rm phg}^{\Gamma}(M) \longrightarrow \mathbb{C}, \quad A \longmapsto \Lambda(A)$, is $\top^0$-local if and only if for any $A \in \Psi_{\rm phg}^{\Gamma}(M)$ and for any $(\phi_1, \phi_2) \in C^\infty(M) \times C^\infty(M)$*

$$\phi_1 \top^0 \phi_2 \Longrightarrow \Lambda(\phi_1\, A\, \phi_2) = 0.$$

Here are two well-known examples of $\top^0$-local linear forms:

Example 2.2. *For $A \in \Psi_{\rm phg}(M)$ of order a, let*

$$\sigma(A) \sim \sum_{j=0}^{\infty} \sigma_{a-j} \tag{1}$$

denote the asymptotic expansion of the polyhomogneous symbol $\sigma(A)$ of A. Both the

(1) **canonical trace**

$${\rm TR} : \Psi_{\rm phg}^{\mathbb{C}\setminus\mathbb{Z}}(M) \longrightarrow \mathbb{C}; \quad A \longmapsto \frac{1}{(2\pi)^n} \int_M \left(\;\;\; -\!\!\!\!\!\!\int_{T_x^*M} \sigma(A)(x,\xi)\, d\xi \right) dx,$$

where the cut-off integral $-\!\!\!\!\!\!\int_{T_x^*M}$ on the cotangent space T_x^*M stands for the Hadamard finite part as $R \to \infty$ of the integral $\int_{\|\xi\| \le R}$,

[1]By linear form on a set which might not be a vector space, we mean a map that sends any linear combination in the set to a linear combination of their images.

(2) *and the* **Wodzicki residue**

$$\mathrm{Res}: \Psi^{\mathbb{Z}}_{\mathrm{phg}}(M) \longrightarrow \mathbb{C}; \quad A \longmapsto \frac{1}{(2\pi)^n} \int_M \left(\int_{S^*_x M} \sigma_{-n}(A)(x,\xi)\, d_S\xi \right) dx,$$

where $\sigma_{-n}(A)$ is the component of degree minus the dimension in the polyhomogeneous expansion (1) of the symbol $\sigma(A)$,

are cyclic and hence $\top^0$-local linear forms.

We now specialise to $\Gamma = \mathbb{Z}$ and $\Gamma = \mathbb{R} \setminus \mathbb{Z}$ and characterise local linear forms on $\Psi_{\mathrm{phg}}(M)$. Here is a result adapted from [AP, Theorem 2.10].

Theorem 2.3. *Any $\top^0$-local linear form on $\mathcal{A} := \Psi^{\mathbb{Z}}_{\mathrm{cl}}(M,E)$ (resp. $\mathcal{A} := \Psi^{\notin\mathbb{Z}}_{\mathrm{cl}}(M,E)$) which for any $A \in \mathcal{A}$, induces continuous linear forms $\Lambda_A : \phi \longmapsto \Lambda(\phi A)$ w.r. to the supremum norm topology, is proportional to the Wodzicki residue* Res *(resp. the canonical trace* TR*).*

3. Multivariate minimal subtraction as a locality projection

In order to extend the set (resp. semigroup, resp. monoid, resp. algebra) category to the locality set (resp. semigroup, resp. monoid, resp. algebra) category we need to require that the maps (resp. morphisms) preserve locality.

Definition 3.1. *Two maps $\phi, \psi : (X, \top_X) \to (Y, \top_Y)$ from a locality set $(X, \top_X)$ to a locality set $(Y, \top_Y)$ are called* **independent** *if $(\phi \times \psi)(\top_X) \subseteq \top_Y$. We write $\phi \top \psi$. A* **locality map** *from a locality set $(X, \top_X)$ to a locality set $(Y, \top_Y)$ is a map $\phi : X \to Y$ such that $\phi \top \phi$.*

The morphisms should also preserve the locality structures.

Definition 3.2. *Locality morphisms of locality semigroups (resp. locality monoids, resp. locality algebras) are locality maps which preserve the partial product (resp. as well as the unit, resp. as well as linear combinations).*

An element of $\mathcal{M}$ is called a **polar germ** if it can be written in the form

$$\frac{h(\ell_1, \cdots, \ell_m)}{L_1^{s_1} \cdots L_n^{s_n}}, \tag{2}$$

where h is a holomorphic germ with rational coefficients in linear forms $\ell_1, \cdots, \ell_m \in \mathbb{Q}^k$, $L_1, \cdots, L_n$ are linearly independent linear forms in $\mathbb{Q}^k$ and $s_1, \cdots, s_n$ are in $\mathbb{Z}_{>0}$, such that $\ell_i \perp^Q L_j$ for all $i \in \{1, \cdots, m\}, j \in \{1, \cdots, n\}$.

There is a direct sum decomposition $\mathcal{M} = \mathcal{M}_+ \oplus \mathcal{M}_-^Q$, where $\mathcal{M}_+$ is the subspace of holomorphic germs and $\mathcal{M}_-^Q$ is the subspace spanned by polar germs defined by Eq. (2).

Proposition 3.3. *The projection $\pi_+^Q : \mathcal{M} \to \mathcal{M}_+$ is a locality algebra homomorphism.*

Let $\mathrm{ev}_0 : \mathcal{M}_+ \to \mathbb{C}$ stand for the evaluation at zero. The composition $\mathrm{ev}_0\, \pi_+^Q$ defines what we call a **generalised evaluator** on $\left(\mathcal{M}, \perp^Q\right)$, namely a locality character on the locality algebra $\left(\mathcal{M}, \perp^Q\right)$ which coincides with the evaluation at zero ev_0 on $\mathcal{M}_+$.
The following straightforward corollary is the backbone of a multivariate minimal subtraction scheme which respects locality .

Theorem 3.4. *A locality morphism $\Phi : (\mathcal{A}, \top) \longrightarrow \left(\mathcal{M}, \perp^Q\right)$, on a locality algebra $(\mathcal{A}, \top)$ gives rise to a locality morphism $\Phi_+^Q := \pi_+^Q\, \Phi : (\mathcal{A}, \top) \longrightarrow \left(\mathcal{M}_+, \perp^Q\right)$, and hence to a renormalised map Φ_{ren}^Q:*

$$\Phi_{\mathrm{ren}}^Q := \mathrm{ev}_0 \Phi_+^Q : (\mathcal{A}, \top) \longrightarrow \mathbb{C},$$

which defines a locality character.

In particular, with the above notations, the renormalised map respects locality :

$$a_1 \top a_2 \Longrightarrow \Phi_{\mathrm{ren}}^Q(a\, b) = \Phi_{\mathrm{ren}}^Q(a)\, \Phi_{\mathrm{ren}}^Q(b).$$

This renormalisation procedure by means of locality morphisms serves various renormalisation purposes, namely to renormalise

- discrete Laplace transforms of characterstic functions of convex cones, leading to **conical zeta functions** [GPZ2];
- higher zeta functions associated with trees, which we call **branched zeta functions** [CGPZ2];
- nested integrals associated with trees, such as in Kreimer's toy model for Feynman graphs (work in progress).

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Hecke modules and spectral triples for arithmetic groups

Bram Mesland
(joint work with M.H. Şengün)

The cohomology of arithmetic groups as a Hecke module is a central object of study in the theory of automorphic forms. In recent joint work with M.H. Şengün (Sheffield) [3, 4] we have shown that the double coset Hecke ring of Shimura homomorphically maps into any KK-ring associated to a crossed product involving an arithmetic group. In the case of a boundary action associated to real hyperbolic n-space, integral operators built from harmonic measures give K-homologically non trivial spectral triples on the boundary crossed product algebra. During the workshop I discussed our results and presented an outlook for future work.

The setting is as follows: Let G be a semisimple Lie group, $\Gamma \subset G$ a lattice and write $\Gamma_g := \Gamma \cap g\Gamma g^{-1}$. The lattice Γ is said to be *arithmetic* in G, if the *commensurator subgroup*

$$C_G(\Gamma) := \{g \in G : \Gamma_g \text{ and } \Gamma_{g^{-1}} \text{ have finite index in } \Gamma\} \subset G,$$

is dense in G. For a subsemigroup $S \subset C_G(\Gamma)$ the free abelian group $\mathbb{Z}[\Gamma, S]$ on the double cosets $\Gamma g\Gamma$ becomes a ring under the Shimura product [5]. It is well known that elements $g \in C_G(\Gamma)$ act on the cohomology of Γ via *Hecke operators* T_g:

$$\begin{array}{ccc} H^*(\Gamma,\mathbb{Z}) & \xrightarrow{T_g} & H^*(\Gamma,\mathbb{Z}) \\ {\scriptstyle \text{res}}\downarrow & & \uparrow{\scriptstyle \text{cores}} \\ H^*(\Gamma_g,\mathbb{Z}) & \xrightarrow{\text{Ad}g} & H^*(\Gamma_{g^{-1}},\mathbb{Z}) \end{array} \tag{1}$$

The corestriction map is well defined because $[\Gamma : \Gamma_{g^{-1}},]$ is finite. In this way $H^*(\Gamma, \mathbb{Z})$ becomes a module over $\mathbb{Z}[\Gamma, S]$.

Now let B be a $C_G(\Gamma)$-C^*-algebra. In [3], we associated to each $g \in C_G(\Gamma)$ a bimodule defining a class in the ring $KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma)$. In our second paper [4] we proved the following structural result:

Theorem 1.1. *For any subsemigroup $S \subset C_G(\Gamma)$ and S-C^*-algebra B the map*

$$\mathbb{Z}[\Gamma, S] \to KK_0(B \rtimes_r \Gamma, B \rtimes_r \Gamma), \quad \Gamma g^{-1}\Gamma \mapsto T_g,$$

is a ring-homomorphism

Now consider the case $B = C_0(X)$ and Γ acts on X freely and properly with $M := X/\Gamma$. Any $g \in C_G(\Gamma)$ induces a *Hecke correspondence* $M \xleftarrow{\tau_g} M_g \xrightarrow{\pi_g} M$. The Hecke correspondence defines a bimodule T_g^M and a class in $KK_0(C_0(M), C_0(M))$ as well. In [3] we proved that in this case the Hecke bimodules are closely related.

Proposition 1.2. *Let X be a free and proper Γ-space, $M := X/\Gamma$ and E the associated $(C_0(X) \rtimes \Gamma, C_0(M))$ Morita equivalence bimodule. There are unitary $(C_0(X) \rtimes \Gamma, C_0(M))$-bimodule isomorphisms*

$$E \otimes_{C_0(M)} T_g^M \xrightarrow{\sim} T_g \otimes_{C_0(X)\rtimes\Gamma} E.$$

In particular $[E] \otimes [T_g^M] = [T_g] \otimes [E] \in KK_0(C_0(X) \rtimes \Gamma, C_0(M))$.

The Hecke correspondences act on manifold cohomology in a natural way and the action is compatible with the isomorphism $H^*(B\Gamma, \mathbb{Z}) \simeq H^*(\Gamma, \mathbb{Z})$. Moreover, the Hecke module structure on topological K-theory recovers the classical Hecke modules, as was shown in [4].

Theorem 1.3. *Let X be a free and proper Γ-space and $M := X/\Gamma$. The Chern character homomorphism*

$$K^i(M) \to \bigoplus_{n\geq 0} H^{2n+i}(M, \mathbb{Q}),$$

is Hecke equivariant.

Let $\mathbf{G}$ be a semisimple algebraic group over $\mathbb{Q}$ and $G = \mathbf{G}(\mathbb{R})$ and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Let K denote a maximal compact subgroup of G and let $X = K\backslash G$ denote the associated symmetric space. Assume that X is of non-compact type with geodesic boundary ∂X. Then the C^*-algebras $C_0(M)$, with $M := X/\Gamma$, the reduced group C^*-algebra $C_r^*\Gamma$ and the boundary crossed product $C(\partial X) \rtimes \Gamma$ are of particular interest.

In the case of real hyperbolic n-space $X = \mathbb{H}_n$, we have shown in [3] that the exact sequences in KK-theory induced from the G-equivariant extension

$$0 \to C_0(\mathbb{H}_n) \to C(\mathbb{H}_n \cup \partial\mathbb{H}_n) \to C(\partial\mathbb{H}_n) \to 0. \tag{2}$$

relating these C^*-algebras (see [2]) are Hecke equivariant. The connection map

$$\partial : K^*(C_0(M)) \to K^{*+1}(C(\partial\mathbb{H}_n) \rtimes \Gamma)$$

can be computed at the level of unbounded cycles, as we now discuss.

Consider the harmonic measures ν_x and associated metrics d_x on $\partial\mathbb{H}_n$ based at $x \in \mathbb{H}$. Let $T_1\mathbb{H}_n = \mathbb{H}_n \times \partial\mathbb{H}_n$ be the unit tangent bundle of $\mathbb{H}_n$ and $L^2(T_1\mathbb{H}_n, \nu_x)$ the associated C^*-module completion. The integral operators

$$\Delta\Psi(x,\xi) = \int_{\partial\mathbb{H}_n} \frac{\Psi(x,\xi) - \Psi(x,\eta)}{d_x(\xi,\eta)^{n-1}} d\nu_x\eta, \quad p\Psi(x,\xi) = \int_{\partial\mathbb{H}_n} \Psi(x,\eta) d\nu_x\eta,$$

are G-invariant and the multiplication operator $\rho\Psi(x,\xi) = d_{\mathbb{H}_n}(0,x)\Psi(x,\xi)$ commutes with G boundedly in $L^2(T_1\mathbb{H}, \nu_x)$. In [3] we prove:

Theorem 1.4. *The operators* Δ, ρ *and* p *assemble into a* G*-equivariant unbounded Kasparov module*

$$(C(\partial\mathbb{H}_n), L^2(T_1\mathbb{H}_n, \nu_x), -\Delta + (2p-1)\rho),$$

representing the class of the G*-equivariant extension* (2).

An embedded hypersurface $(N, \partial N) \subset (\overline{M}, \partial\overline{M})$ in the Borel-Serre compactification [1] defines a geometric cycle for M and an element $K^*(C_0(M))$. Their Dirac operators interact surprisingly well with the integral operator Δ, which gives rise to the following construction of spectral triples on the purely infinite simple C^*-algebra $C(\partial\mathbb{H}_n) \rtimes \Gamma$.

Theorem 1.5. *A self-adjoint Dirac operator* $\not{D}_{\mathring{N}}$ *associated to an embedded hypersurface* $(N, \partial N) \subset (\overline{M}, \partial\overline{M})$ *and the integral operator* $S := -\Delta + (2p-1)\rho$ *assemble into an unbounded Kasparov product*

$$(C(\partial\mathbb{H}_n) \rtimes \Gamma, L^2(T_1\mathbb{H}_n) \otimes_{C_0(M)} L^2(\mathring{N}, \not{S}_{\mathring{N}}), S \otimes \sigma + 1 \otimes_\nabla \not{D}_{\mathring{N}}).$$

If $\dim M = 3$*, such spectral triples exhaust the group* $K^1(C(\partial\mathbb{H}_3) \rtimes \Gamma)$*.*

It is worth noting that the same construction works with the Dirac operator $\not{D}_M$ on M.

The above results lay the groundwork for the study of automorphic forms in the context of KK-theory. In future work we intend to investigate the following questions:

- Summability properties of the (un)bounded Kasparov modules constructed in [3] and computation of their spectral zeta functions. Investigation of any relationship between the latter and the arithmetic of Bianchi modular forms.
- Investigation of whether the K-homology of the arithmetic C^*-algebras as Hecke modules can be accounted for by automorphic forms as is the case for cohomology of arithmetic groups. Since K-homology is directly defined in terms analytic data, we wish to associate a K-homology class to an automorphic form.
- The relation of the Hecke module structure of K-theory to the Baum-Connes assembly map. One of the simplest cases in which the Baum-Connes conjecture is open is the arithmetic group $SL(3, \mathbb{Z})$. We have recently shown that for such groups the Baum-Connes assembly map is Hecke equivariant and we investigate the implications of this fact.

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Another look at discrete series and the Dirac operator

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(joint work with Tsuyoshi Kato)

This is a report on a project still in progress with Tsuyoshi Kato that concerns the discrete series of real reductive groups. It is part of a broader effort to re-examine aspects of Harish-Chandra's theory from the perspective of noncommutative geometry.

The Dirac operator is involved in our approach to the discrete series, as it is in prior works of Parthasarathy, Atiyah-Schmid and Lafforgue, but it arises here in a somewhat different way. The starting point is an elementary result based on the following definition:

Definition. Let G be a real reductive group with a given Haar measure and with a given maximal compact subgroup K. The *compact ideal* in the reduced C^*-algebra $C^*_r(G)$ is

$$C^*_r(G)_{\mathrm{cpt}} = \left\{ f \in C^*_r(G) \,:\, \lambda(f) \text{ acts as a compact operator on every } L^2(G)^\sigma \right\}.$$

Here λ is the left-regular representation of G, σ is an irreducible representation of K and $L^2(G)^\sigma \subseteq L^2(G)$ is the σ-isotypical subspace for the right regular representation of K on $L^2(G)$.

Lemma. *The discrete series representations of G are precisely those irreducible representations of $C^*_r(G)$ that restrict to nonzero (irreducible) representations of the compact ideal.*

The rough idea is to use the operator traces on the compact ideal coming from the actions of $C^*_r(G)_{\mathrm{cpt}}$ on the Hilbert spaces $L^2(G)^\sigma$ to access information about the discrete series. More precisely we want to consider the supertraces

$$\mathrm{Trace}_\lambda(f) = \mathrm{SuperTrace}\Big(\lambda(f)\otimes 1\colon \big[L^2(G)\otimes S\big]^K \longrightarrow \big[L^2(G)\otimes S\big]^K\Big)$$

on the compact ideal, as S ranges over the indecomposable K-equivariant representations of the Clifford algebra $\mathrm{Cliff}(\mathfrak{p})$, where $\mathfrak{g} = \mathfrak{k}\oplus\mathfrak{p}$ is the Cartan decomposition associated to $K \subseteq G$. The spaces S carry $\mathbb{Z}/2$ gradings, but we should caution that it is not always possible to pick a *canonical* $\mathbb{Z}/2$-grading; see below.

Assume for simplicity that G is linear and connected, and that each maximal torus $T \subseteq K$ is a Cartan subgroup of G (the latter is Harish-Chandra's condition for the existence of discrete series). Then the spinor spaces S above are parametrized by infinitesimal weights $\lambda \in i\mathfrak{t}^*$ that are

- Shifted-integral for G (that is, analytically integral after shifting by the half-sum of the positive roots for G);
- Dominant for K (that is, lying in a chosen Weyl chamber for K); and
- Nonsingular for K (that is, fixed by no element of the Weyl group for K).

Let us call these *Dirac parameters.*

Harish-Chandra showed that the discrete series are parametrized by the smaller subset of *Harish-Chandra parameters*, which are infinitesimal weights $\lambda \in i\mathfrak{t}^*$ that are

- Shifted-integral for G;
- Dominant for K; and
- Nonsingular for G (not merely nonsingular for K).

Assume for a moment that G is compact. Denote by $d(\lambda)$ the polynomial function on $i\mathfrak{t}^*$ given by Weyl's dimension formula for the irreducible representations of G, whose value on a G-shifted integral, G-dominant and G-nonsingular weight is the dimension of the associated irreducible representation of G. In addition, denote by Trace_G the canonical (unbounded) trace on $C_r^*(G)$ given by evaluation of a function on G at the identity. The following formula is essentially due to Bott; it follows rather easily from Weyl's character formula.

Theorem. *In the compact case, if $f \in C_r^*(G)$, then*

$$|W_G|/|W_K| \cdot \mathrm{Vol}(G) \cdot \mathrm{Trace_G}(f) = (-1)^{\dim(G/K)/2} \sum_{\lambda} d(\lambda) \cdot \mathrm{Trace}_\lambda(f)$$

where the sum is over the Dirac parameters.

(Strictly speaking we should consider only f belonging to the minimal dense ideal of $C_r^*(U)$, to ensure all the traces are finite.)

Let us comment further on the issue of $\mathbb{Z}/2$ gradings on the spinor space S_λ associated to the Dirac parameter λ. If λ is G-nonsingular then there is a canonical $\mathbb{Z}/2$-grading associated to λ. On the other hand, if λ is G-singular, then $d(\lambda) = 0$. So overall the right-hand side of the formula is well-defined.

Our main observation is that the above trace formula corresponds very closely to a formula of Harish-Chandra in orbital integrals, using the "dictionary"

$$\mathrm{Trace}_\lambda(f) = \widehat{F}_f(\lambda),$$

when G is noncompact. Here F_f is Harish-Chandra's orbital integral for the Cartan subgroup T, and $\widehat{F}_f(\lambda)$ is the Fourier coefficient associated to λ. The proof of this identity when f lies in the (minimal dense ideal of) the compact ideal $C_r^*(G)_{\mathrm{cpt}}$ (so that $\mathrm{Trace}_\lambda(f)$ is well-defined) is a simple computation.

Harish-Chandra's orbital integral formula translates into the following trace formula, which is very close to Bott's formula:

Theorem. *In the noncompact case, if $f \in C_r^*(G)_{\mathrm{cpt}}$, then*

$$\mathrm{Vol}(U) \cdot \mathrm{Trace_G}(f) = (-1)^{\dim(G/K)/2} \sum_{\lambda} d(\lambda) \cdot \mathrm{Trace}_\lambda(f)$$

where the sum is over the Dirac parameters, where U is the compact form of G, and where the Haar measure on U is the one associated to the given Haar measure on G.

One way to think about the disappearance of the term $|W_G|/|W_K|$ is to write

$$|W_G|/|W_K| = \chi(G/K)$$

in Bott's formula for the compact case (χ is the Euler characteristic) and recall that in the noncompact case G/K is contractible, so that $\chi(G/K) = 1$.

Harish-Chandra's formula played a crucial role in his classification of the discrete series. Note, for instance, that if π is a discrete series representation, and if p is a diagonal matrix coefficient function for π, then formula immediately implies that $\mathrm{Trace}_\lambda(p)$ is nonzero for some λ. This means that π occurs in the index of a Dirac operator associated to a Harish-Chandra parameter. Moreover if p is normalized so as to be a projection in $C^*_r(G)$, then the normalizing factor, which is the formal dimension of π, is necessarily an integer, assuming we normalize Haar measure so that $\mathrm{vol}(U) = 1$. This is because each $\mathrm{Trace}_\lambda(p)$ is an integer, as is each $d(\lambda)$.

To complete the classification of the discrete series, Harish-Chandra used his theorem on the local integrability of distribution characters. As an alternative, it is possible to use instead some rather more elementary facts about tempered representation theory, plus the *injectivity* of the Connes-Kasparov index map (which is the easy part of the Connes-Kasparov isomorphism) to complete the classification. So it becomes interesting to ask for an geometric approach to the trace formula for noncompact groups. This is work in progress, but let us sketch our line of attack.

In the compact case there is a natural inclusion of G-Hilbert spaces

$$\bigoplus_\lambda d(\lambda) \cdot [L^2(G) \otimes S_\lambda]^K \longrightarrow L^2(G) \otimes \Omega_{L^2}(G/K)$$

whose image is the kernel of the identity tensored with the de Rham operator. The inclusion immediately proves Bott's formula.

The inclusion involves computations of Kostant on the kernel of the cubic Dirac operator, together with the simple identification

$$(G \times G)/K \cong G \times G/K.$$

In the noncompact case there is a more subtle identification

$$(U \times G)/K \cong G/K \times U$$

that comes from the dressing action of G on U. We aim to show that this leads to an inclusion

$$\bigoplus_\lambda d(\lambda) \cdot [L^2(G) \otimes S_\lambda]^K \longrightarrow L^2(U) \otimes \Omega_{L^2}(G/K)$$

whose image is the kernel of the identity tensored with Witten's (index one) perturbation of the de Rham operator. This would suffice.

Representations of *-algebras by unbounded operators

Ralf Meyer

This lecture is an invitation to my article [2].

Let $X \subseteq \mathbb{R}^n$ be an affine variety. We may describe it through the *-algebra $\mathcal{P}(X)$ of polynomial functions on X, which is a certain quotient of the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, or through the C*-algebra $\mathrm{C}_0(X)$. If X is not compact, say, $X = \mathbb{R}^n$, then $\mathcal{P}(X)$ is not contained in $\mathrm{C}_0(X)$. Nevertheless, we would like to say that $\mathrm{C}_0(X)$ is "the" C*-completion or better C*-hull of $\mathcal{P}(X)$. Woronowicz [7, 6] has found a way to make this precise, speaking of unbounded affiliated multipliers of a C*-algebra and what it means for a C*-algebra to be generated by a set of such unbounded multipliers. Similar situations for non-commutative algebras are very common in the theory of quantum groups, where it is usually quite hard to find a suitable C*-algebra.

A prototypical case for the theory is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$. It generates $\mathrm{C}^*(G)$, where G is the simply connected Lie group that integrates $\mathfrak{g}$. This does not say, however, that $\mathfrak{g}$ and $\mathrm{C}^*(G)$ have the same representation theory. In fact, Nelson's Theorem describes completely which representations of $\mathfrak{g}$ come from a representation of $\mathrm{C}^*(G)$ by differentiation: this happens if and only if Nelson's Laplacian acts by an essentially self-adjoint operator (see [5]). Here Nelson's Laplacian is $\sum_{j=1}^n x_j^2$ for a basis $x_1, \dots, x_n$ of $\mathfrak{g}$. This is a general feature of the representation theory of *-algebras: we must restrict attention to a subclass of representations, called "integrable" representations, to get a well behaved theory. Already the representations of the polynomial algebra $\mathbb{C}[x_1, x_2]$ are quite wild (see [5]).

The starting point of my work on representations by unbounded operators is an article by Savchuk and Schmüdgen [4]. They study a unital *-algebra A that is graded by a discrete group G, say, $G = \mathbb{Z}$. That is, $A = \bigoplus_{g \in G} A_g$ with vector subspaces $A_g \subseteq A$ that satisfy $A_g \cdot A_h \subseteq A_{gh}$, $A_g^* = A_{g^{-1}}$, and $1 \in A_e$ for all $g, h \in G$, where $e \in G$ is the unit element. For instance, let A be generated by an element a of degree 1 subject to the relation $aa^* = a^*a + 1$. This is naturally $\mathbb{Z}$-graded, and the unit fibre is the polynomial algebra generated by $N = a^*a$.

If A is graded by G, one would expect a C*-hull B of A to be graded as well. Such a grading on a C*-algebra is equivalent to an action of G on the unit fibre B_e by partial Morita–Rieffel equivalences (see [1]). Savchuk and Schmüdgen assume A_e to be commutative and certain line bundles to be trivial, so that the grading becomes a partial group action on a locally compact Hausdorff space. Then the crossed product for this partial action is a C*-hull for the graded algebra.

A generalisation of this result to graded algebras with non-commutative unit fibre is quite interesting because it forces us to make the notion of C*-hull much more precise. We want a C*-hull B_e for A_e to "induce" a C*-hull B for A. If our notion of C*-hull is too weak, then the information about B_e does not suffice to get B; if it is too strong, we may ask too much of B.

A C*-algebra generated by unbounded affiliated multipliers in Woronowicz' sense is not unique. The defining condition for the C*-hull in [2] is that its representations are "equivalent" to the "integrable" representations of the *-algebra A. Representations on Hilbert space do not suffice to determine a C*-algebra uniquely up to isomorphism. Therefore, I consider representations on Hilbert modules. In the context of commutative algebras or Lie algebra representations, it is known which representations integrate to the appropriate C*-algebra. Roughly speaking, one must ask operators to be *regular* and self-adjoint instead of just self-adjoint.

The "equivalence" of representation theories also requires that isometric intertwiners are the same in both worlds. If an isometry intertwines two representations of a C*-algebra, then its adjoint does so, too. For representations by unbounded operators, this may fail: restrict a representation to a subset of its domain. The identical inclusion of the representation with smaller domain is an isometric intertwiner, but the adjoint is not an intertwiner. In fact, the condition about having the same isometric intertwiners in both cases says that such extension phenomena do not happen among "integrable" representations. This very subtle condition is needed for the Induction Theorem for C*-hulls. A couterexample shows that the theorem becomes false if this condition is dropped.

The main theorem in [2] is the Induction Theorem. In the situation of a graded *-algebra, it describes a C*-hull for the "integrable" representations of A, given a C*-hull B_e for the "integrable" representations of the unit fibre A_e. Here a representation of A is called "integrable" if its restriction to A_e is integrable. More precisely, the C*-hull of A is the section C*-algebra of a Fell bundle over the group G, whose unit fibre is a certain quotient of B_e.

For instance, let $A = \mathbb{C}\langle a \mid aa^* = a^*a + 1\rangle$ be graded as above. We call a representation of A on a Hilbert module integrable if the closure of $N = a^*a$ is regular and self-adjoint. The C*-hull for these representations of the polynomial algebra $\mathbb{C}[N]$ is $B_e = \mathrm{C}_0(\mathbb{R})$. Most representations of B_e are not contained in any representation of A. The ones that appear are the ones where induction to A gives a representation on a vector space with positive-definite inner product. In the case at hand, this replaces B_e by the quotient $B_e^+ = \mathrm{C}_0(\mathbb{N})$. That is, only those characters that map N to an integer may be induced to a representation of A. The unit fibre of the relevant Fell bundle is B_e^+. In this case, the Fell bundle comes from a partial action of $\mathbb{Z}$ on $\mathbb{N}$. This is simply the restriction of the translation action on $\mathbb{Z}$ to the non-invariant subset $\mathbb{N}$. The section C*-algebra of the Fell bundle is the algebra of compact operators on the Hilbert space $\ell^2(\mathbb{N})$. This is the expected result.

The machinery above does not work so well for commutative algebras with infinitely many generators. Here the most natural class of "integrable" representations has no C*-hull. So the approach in [2] does not help much, say, to get a C*-algebra that is related to the canonical commutation relations. To build such a C*-algebra, we must impose some technical extra conditions for representations to make the space of integrable characters on the unit fibre locally compact.

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Lambert-W solves the noncommutative Φ^4-model

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(joint work with Erik Panzer)

This abstract is based on [1] where we give strong evidence for

Conjecture 1. *The non-linear integral equation for a function $G_\lambda : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$,*

$$
\begin{aligned}
(1+a+b)G_\lambda(a,b) &= 1 + \lambda \int_0^\infty dp \left(\frac{G_\lambda(p,b) - G_\lambda(a,b)}{p-a} + \frac{G_\lambda(a,b)}{1+p} \right) \\
&\quad + \lambda \int_0^\infty dq \left(\frac{G_\lambda(a,q) - G_\lambda(a,b)}{q-b} + \frac{G_\lambda(a,b)}{1+q} \right) \\
&\quad - \lambda^2 \int_0^\infty dp \int_0^\infty dq \; \frac{G_\lambda(a,b)G_\lambda(p,q) - G_\lambda(a,q)G_\lambda(p,b)}{(p-a)(q-b)} ,
\end{aligned}
\tag{1}
$$

is for any real coupling constant $\lambda > -1/(2\log 2) \approx -0.721348$ solved by

$$
G_\lambda(a,b) = G_\lambda(b,a) = \frac{(1+a+b)\exp(N_\lambda(a,b))}{\big(b + \lambda W\big(\tfrac{1}{\lambda} e^{(1+a)/\lambda}\big)\big)\big(a + \lambda W\big(\tfrac{1}{\lambda} e^{(1+b)/\lambda}\big)\big)} , \quad \textit{where} \tag{2}
$$

$$
N_\lambda(a,b) := \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} dt \; \log\left(1 - \frac{\lambda \log(\frac{1}{2} - \mathrm{i}t)}{b + \frac{1}{2} + \mathrm{i}t}\right) \frac{d}{dt} \log\left(1 - \frac{\lambda \log(\frac{1}{2} + \mathrm{i}t)}{a + \frac{1}{2} - \mathrm{i}t}\right) . \tag{3}
$$

Here, W denotes the Lambert function, more precisely its principal branch W_0 for $\lambda > 0$ and the other real branch W_{-1} for $-1 < \lambda < 0$ of the solution of $W(z)e^{W(z)} = z$. The function $N_\lambda(a,b)$ defined for $\lambda > -1/(2\log 2)$ has a perturbative expansion into Nielsen polylogarithms.

Equation (1) arises from the Dyson-Schwinger equation for the 2-point function of the $\lambda\phi^{\star 4}$-model with harmonic propagation on 2-dimensional noncommutative Moyal space in a special limit where the matrix size and the deformation parameter are simultaneously sent to infinity. We refer to [1, 2] for details and treat here only the solution of (1).

Starting point is the observation that (1) is equivalent to a boundary value problem. Define by

$$\Psi_\lambda(z,w) := 1+z+w-\lambda\log(-z)-\lambda\log(-w)+\lambda^2\int_0^\infty dp\int_0^\infty dq\,\frac{G_\lambda(p,q)}{(p-z)(q-w)}$$

a function holomorphic on $(\mathbb{C}\setminus[0,\Lambda^2])^2$. Then (1) is equivalent to

$$\Psi_\lambda(a+\mathrm{i}\epsilon,b+\mathrm{i}\epsilon)\Psi_\lambda(a-\mathrm{i}\epsilon,b-\mathrm{i}\epsilon)=\Psi_\lambda(a+\mathrm{i}\epsilon,b-\mathrm{i}\epsilon)\Psi_\lambda(a-\mathrm{i}\epsilon,b+\mathrm{i}\epsilon)\,. \tag{4}$$

Therefore, there is a real function $\tau_a(b)$ with

$$\Psi_\lambda(a+\mathrm{i}\epsilon,b+\mathrm{i}\epsilon)e^{-\mathrm{i}\tau_a(b)}=\Psi_\lambda(a+\mathrm{i}\epsilon,b-\mathrm{i}\epsilon)e^{\mathrm{i}\tau_a(b)}\,. \tag{5}$$

The Plemelj formulae give (after introducing a common cut-off Λ in the integrals (1)) two equations for the real and imaginary part of (5). Both are Carleman-type singular integral equations for $G_\lambda(a,b)$ and for $\mathcal{G}_\lambda(a,b):=\frac{1}{\lambda\pi}+\mathcal{H}_a[G_\lambda(\bullet,b)]$, where $\mathcal{H}_a[f(\bullet)]:=\frac{1}{\pi}⨍_0^{\Lambda^2}dp\,\frac{f(p)}{p-a}$ is the one-sided Hilbert transform (we denote by $⨍$ the Cauchy principal value). The equation for $G_\lambda(a,b)$ is easily solved by

$$G_\lambda(a,b)=\frac{\sin\tau_a(b)}{\lambda\pi}e^{\mathcal{H}_b[\tau_a(\bullet)]}\,. \tag{6}$$

The solution for $\mathcal{G}_\lambda(a,b)$ is the symmetric partner $a\leftrightarrow b$ of the $G_\lambda(a,b)$-equation provided that

$$\begin{aligned}&\lambda\pi\cot\tau_b(a)=1+a+b-\lambda\log a+I_\lambda(a)\,,\quad\text{where}\\&I_\lambda(a):=\frac{1}{\pi}\int_0^\infty dp\,\Big(e^{-\mathcal{H}_p[\tau_a(\bullet)]}\sin\tau_a(p)-\frac{\lambda\pi}{1+p}\Big)\,.\end{aligned} \tag{7}$$

A solution of (7) as formal power series in λ leads surprisingly far. Using the HyperInt package [3] we convinced ourselves that whereas $\mathcal{H}_p[\tau_a(\bullet)]$ recursively evaluates to polylogarithms and more complicated hyperlogarithms, $I_\lambda(a)$ itself remains extremely simple and only contains powers of $\log(1+a)$. The results for $I_\lambda(a)$ are of such striking simplicity and structure that we could obtain an explicit formula. Concretely,

$$\begin{aligned}I_\lambda(a)&=-\lambda\log(1+a)+\sum_{n=1}^\infty\lambda^{n+1}\Big(\frac{(\log(1+a))^n}{na^n}+\frac{(\log(1+a))^n}{n(1+a)^n}\Big)\\&+\sum_{n=1}^\infty\frac{(n-1)!\lambda^{n+1}}{(1+a)^n}\sum_{j=1}^{n-1}\sum_{k=0}^{n}(-1)^j\frac{s_{j,n-k}}{k!j!}\Big(\Big(\frac{1+a}{a}\Big)^{n-j}+1\Big)\big(\log(1+a)\big)^k\end{aligned} \tag{8}$$

correctly reproduces the first 10 terms of the expansion in λ. We conjecture that it holds true to all orders. By $s_{n,k}$ we denote the Stirling numbers of the first kind, with sign $(-1)^{n-k}$. Using generating functions of Stirling numbers, (8) is simplified to

$$I_\lambda(a)=\sum_{n=1}^\infty\frac{\lambda^n}{n!}\frac{d^{n-1}}{da^{n-1}}(-\log(1+a))^n-\lambda\sum_{n=1}^\infty\frac{\lambda^n}{n!}\frac{d^{n-1}}{da^{n-1}}\frac{(-\log(1+a))^n}{a}\,. \tag{9}$$

This structure is covered by the *Lagrange inversion theorem* which shows that the first sum in (9) is the inverse $w(\lambda) = \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{dw^{n-1}}(\phi(w))^n\big|_{w=0}$ of the function $\lambda(w) = \frac{w}{\phi(w)}$ if we set $\phi(w) = -\log(1+a+w)$. On the other hand, $\lambda(w) = -\frac{w}{\log(1+a+w)}$ is easily inverted to the Lambert-W function which solves $W(z)e^{W(z)} = z$. The second sum in (9) (without the preceeding $-\lambda$), written as $\sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \frac{d^{n-1}}{dw^{n-1}}\big(H'(w)\phi(w)^n\big)\big|_{w=0}$ for $H(w) = \log(1+w/a)$, is by the *Lagrange-Bürmann formula* equal to $H(w(\lambda))$ for the same $w(\lambda)$ as above. Putting everything together, we we have resummed (9) to

$$I_\lambda(a) = \lambda W\Big(\frac{1}{\lambda}e^{\frac{1+a}{\lambda}}\Big) - \lambda \log\Big(\lambda W\Big(\frac{1}{\lambda}e^{\frac{1+a}{\lambda}}\Big) - 1\Big) - 1 - a + \lambda \log a\,. \tag{10}$$

A closer discussion shows that for $\lambda \geq 0$ the principal branch W_0 of the Lambert function is selected and for $-1 < \lambda < 0$ the other real branch W_{-1}. It can be shown that (10) is analytic at any $\lambda > -1$.

This result gives $\tau_b(a)$ via (7). For $G_\lambda(a,b)$ we need according to (6) the Hilbert transform of that function. A lengthy calculation leads to

$$\begin{aligned}\mathcal{H}_a[\tau_b(\bullet)] &= \log\sqrt{(b + \lambda W(\tfrac{1}{\lambda}e^{(1+a)/\lambda}) - \lambda\log(\lambda W(\tfrac{1}{\lambda}e^{(1+a)/\lambda}) - 1))^2 + (\lambda\pi)^2} \\ &\quad + \log\Big(\frac{(1+a+b)\exp(N_\lambda(a,b))}{(b + \lambda W(\frac{1}{\lambda}e^{(1+a)/\lambda}))(a + \lambda W(\frac{1}{\lambda}e^{(1+b)/\lambda}))}\Big)\,, \\ N_\lambda(a,b) &:= \frac{1}{2\pi\mathrm{i}}\int_{\gamma_\epsilon} dz \log\Big(1 - \frac{\lambda\log(-z)}{1+b+z}\Big)\frac{d}{dw}\log\Big(1 - \frac{\lambda\log(1+z+w)}{1+a-(1+z+w)}\Big)\Big|_{w=0}\,,\end{aligned} \tag{11}$$

where γ_ϵ is the curve in the complex plane which encircles the positive real axis clockwise at distance ϵ. Equation (11) holds for $\lambda > -1$ and can be rearranged for $\lambda > -\frac{1}{2\log 2}$ into (3). In particular, formula (2) follows.

Further information is obtained from the generating function $R_{\alpha,\beta}(a,b;w)$ defined by $N_\lambda(a,b) = \sum_{m,n=1}^{\infty} \frac{(-\lambda)^{m+n}}{m!n!} \partial_a^{m-1}\partial_b^{n-1}\partial_\alpha^m\partial_\beta^n\partial_w R_{\alpha,\beta}(a,b;w)\big|_{\alpha=\beta=w=0}$,

$$\begin{aligned}R_{\alpha,\beta}(a,b;w) &= \frac{(1+w)^{\alpha+\beta}}{(1+a+b-w)}\frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)}\Big\{-1 \\ &\quad + \frac{(1+b)^\beta}{(1+w)^\beta}\,{}_2F_1\Big(\begin{matrix}-\alpha,\beta\\1-\alpha\end{matrix}\Big|\frac{w-b}{1+w}\Big) + \frac{(1+a)^\alpha}{(1+w)^\alpha}\,{}_2F_1\Big(\begin{matrix}-\beta,\alpha\\1-\beta\end{matrix}\Big|\frac{w-a}{1+w}\Big)\Big\}\,.\end{aligned} \tag{12}$$

The hypergeometric function generates precisely the Nielsen polylogarithms

$$\begin{aligned}{}_2F_1\Big(\begin{matrix}-x,y\\1-x\end{matrix}\Big|z\Big) &= 1 - \sum_{n,p\geq 1} S_{n,p}(z)x^n y^p\,, \\ S_{n,p}(z) &:= \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 dt\,\frac{(\log(t))^{n-1}(\log(1-zt))^p}{t}\,,\end{aligned} \tag{13}$$

and $\frac{\Gamma(1-\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} = \exp\big(\sum_{k=2}^{\infty}((\alpha+\beta)^k - \alpha^k - \beta^k)\frac{\zeta(k)}{k}\big)$ gives rise to Riemann zeta values.

Our result now permits to complete the exact solution of the whole $\lambda\phi^{\star 4}$-model on Moyal space [2]. Moreover, all experience shows that solving a non-linear problem such as (1) by generalised radicals (here $W(z), N_\lambda(a,b)$) can only be expected in presence of a hidden symmetry. We consider it worthwhile to explore the corresponding integrable structure.

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Random Tensors

Vincent Rivasseau

There are three main approaches to quantum gravity in two dimensions. The first one is the discretized random geometric approach provided by matrix models. Their Feynman graphs are *combinatorial maps* dual to surfaces. The corresponding perturbative expansion is structured into non-trivial layers by 'tHooft topological $1/N$ expansion [1]. Lower genera dominate when the size N of the matrix becomes big. By the Gauss-Bonnet theorem, the genus of the surface is related to the integral of the scalar curvature, hence matrix models perform a sum over discrete geometries *pondered by the Einstein-Hilbert action* [2].

The two other approaches to quantum gravity in two dimensions are through Liouville continuous field theory and through integrals over moduli spaces. The last approach, when suitably decorated through additional fields, is also the entry door for string and superstring theory. It turns out that these three different approaches to quantum gravity in two dimensions are essentially equivalent [3, 4].

In more than two dimensions the situation is much more complicated. One can wonder what are the analogs of these three successful approaches. Perelman's proof of the Poincaré conjecture and Thurston uniformization program, now completed, provide some understanding of the geometry of three dimensional manifolds roughly similar to the level of understanding reached by Poincaré and Koebe for two-dimensions more than a century ago. However theoretical physicists have certainly not fully incorporated yet the physical consequences of these mathematical breakthroughs. In four dimensions geometry is even richer, as the category of smooth/piecewise-linear manifolds becomes distinct from that of topological manifolds. The *smooth Poincaré conjecture* and the classification of smooth structures on four dimensional manifolds remain today major open geometric problems.

In view of these difficulties, for the time being the simplest and most straightforward path towards higher dimensional quantum gravity seems to be the generalization to higher dimensions of the discretized random geometric approach provided by matrix models. Matrix models are tensor models of rank 2. At higher rank $r \geq 3$ the Feynman expansion of tensor models is made of *stranded graphs* dual to random piecewise-linear quasi-manifolds of dimension r [5]. This Feynman expansion is again naturally pondered by a discretized analog of the dimension r Einstein-Hilbert action [6]. However tensor models for about twenty years lacked the additional structure provided by a $1/N$ expansion generalizing 'tHooft expansion. This expansion was found in 2010-11 [7]. It is led by a very simple class of series-parallel graphs, the so called melonic graphs [8]. The corresponding revived approach to quantum gravity in more than two dimensions has been nicknamed the *tensor track* to quantum gravity [9]. It led to a flurry of publications (see [10] for reviews), and in particular to the definition and renormalization of tensor analogs of non commutative field theories [11, 12].

The melonic series, i.e. the tensor model expansion restricted to the leading melonic graphs, displays a critical point and an associated *single scaling limit* leading to a continuous random geometric phase. When equipped with the graph distance this continuous limit identifies with the so-called branched polymer phase [14] corresponding to Aldous continuous random tree [15]. The latter certainly does not look like our smooth 4-dimensional universe. This fact has been taken by some theoretical physicists as an argument to dismiss the whole tensor track approach. However we think that more physical phases of random geometry probably hide in the structure of subdominant $1/N$ tensor contributions. This structure is much more complex in the tensor than in the matrix case, as it mixes in a non-trivial way topology and triangulation complexity. In matrix models the sub-leading $1/N$ contributions are incorporated through a single step, called the double scaling limit. In the tensor case the double scaling limit also exists, but it incorporates still a very small fraction of the tensor perturbative expansion [13]. It will certainly require a much finer mathematical analysis than in the matrix case to discover all what the $1/N$ tensor expansion has to tell us about quantum gravity.

In 2015 a bridge was found between tensor models and holography through the quantum mechanical Sachdev-Ye-Kitaev model [16]. This condensed matter model has a melonic large N limit which exhibits features of the much sought after near-AdS_2/CFT_1 correspondence. In 2016 the ordinary SYK model, which contains a quenched tensor field, has been generalized into *field theoretic models* [17] which share the same melonic leading term, but do contain the rich geometric structure of true tensor models in their subdominant $1/N$ expansion.

Other current developments tend to show that tensor models, just like their matrix model parents, have many unexpected mathematical as well as physical applications outside the initial quantum gravity motivation, for instance in enumerative combinatorics [18], in statistical mechanics [19, 20] and for understanding energy cascade in randomized non-linear PDE's [21]. Our Oberwolfach talk will partly review this rapidly expanding subject.

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The term a_4 in the heat kernel expansion of noncommutative tori

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(joint work with Alain Connes)

For a general metric in the canonical conformal structure on the noncommutative two torus $\mathbb{T}^2_\theta$, we calculate and study properties of the *local* geometric term a_4 that has explicit information about the noncommutative analog of the Riemann curvature tensor. The term a_4 appears in the small time heat kernel expansion of the Laplacian of the curved metric. We derive an explicit formula for this term by making use of heat kernel methods [8], the pseudodifferential calculus developed in [1], and suitable techniques to handle challenges posed by the noncommutativity of the algebra $C^\infty(\mathbb{T}^2_\theta)$ of the noncommutative torus. The presence of noncommutativity has in fact the significant effect of having complicated dependence in the final formulas for the geometric invariants, such as a_4, on the modular automorphism of the state representing the volume form of the curved metric.

Stimulated by the analog of the Gauss-Bonnet theorem and the scalar curvature for $\mathbb{T}^2_\theta$ [5, 6, 4, 7], this line of research has attained remarkable attention in recent years and its origins can be traced back to classical facts in spectral geometry. That is, let us consider the Laplacian Δ_g of a Riemannian metric g on a closed manifold M, which acts on $C^\infty(M)$. There are densities $a_{2n}(x, \Delta_g)\, dx$ on M that can be derived locally from the Reimann curvature tensor and its contractions and covariant derivatives such that for any $f \in C^\infty(M)$, as $t \to 0^+$, there is an asymptotic expansion of the following form:

$$\text{(1)} \qquad \mathrm{Trace}\,(f \exp(-t\Delta_g)) \sim t^{-\dim(M)/2} \sum_{n=0}^{\infty} t^n \int_M f(x)\, a_{2n}(x, \Delta_g)\, dx.$$

Due to the rapid growth in the complexity of the expressions for the $a_{2n}(x, \Delta_g)\, dx$, in the literature one can find formulas only up to a_{10}. For example, one has [8]: $a_0(x, \Delta_g) = (4\pi)^{-1}, a_2(x, \Delta_g) = (4\pi)^{-1}(-R(x)/6)$, and

$$a_4(x, \Delta_g) = (4\pi)^{-1}(1/360)(-12\Delta_g R(x) + 5R(x)^2 - 2|Ric(x)|^2 + 2|Riem(x)|^2),$$

where $Riem, Ric, R$ denote the Riemann curvature tensor, Ricci tensor and scalar curvature of the metric, respectively. This fact shows that fundamental geometric information is encoded in geometric operators such as Laplacians. Hence, in noncommutative geometry, one allows a *noncommutative geometric manifold* to have a noncommutative algebra of functions while the metric information is encoded in a geometric operator. The depth of this idea is fully illustrated in [3] by showing that the Dirac operator of any Riemannian spin^c manifold (which squares to a Laplace-type operator) contains the full metric information.

We now turn our focus back to the noncommutative torus whose algebra $C(\mathbb{T}^2_\theta)$ is the C^*-algebra generated by two unitaries U and V satisfying the commutation relation $VU = e^{2\pi i\theta}UV$ for an arbitrary $\theta \in \mathbb{R}$. The canonical trace φ_0 on $C(\mathbb{T}^2_\theta)$ is defined by $\varphi_0(1) = 1$ and $\varphi_0(U^m V^n) = 0$ if $(m, n) \neq (0, 0)$, and the canonical derivations (analogs of partial differentiation) δ_1 and δ_2 on $C^\infty(\mathbb{T}^2_\theta)$ are defined by $\delta_1(U) = U$, $\delta_1(V) = 0$, $\delta_2(U) = 0$, $\delta_2(V) = V$. The trace φ_0 is viewed as integration against the volume form of the flat metric. Therefore, by using a fixed self-adjoint element $\ell \in C^\infty(\mathbb{T}^2_\theta)$, the functional $\varphi(a) = \varphi_0\left(ae^{-2\ell}\right)$ for $a \in C(\mathbb{T}^2_\theta)$ plays the role of the volume form of the conformal perturbation of the flat metric by $e^{-2\ell}$ [5]. This functional is a state and the logarithm of its modular automorphism is given by $\nabla(a) = 2(-\ell a + a\ell)$. By performing a heavy computer aided calculation we find that the analog of the term a_4 in (1) for the Laplacian Δ_ℓ of the perturbed metric on $\mathbb{T}^2_\theta$ is of the following form (which belongs to $C^\infty(\mathbb{T}^2_\theta)$):

$$
\begin{gathered}
a_4(\ell) = \\
-e^{2\ell}\Big(K_1(\nabla)\left(\delta_1^2\delta_2^2(\ell)\right) + K_2(\nabla)\left(\delta_1^4(\ell) + \delta_2^4(\ell)\right) + K_3(\nabla,\nabla)\left((\delta_1\delta_2(\ell))\cdot(\delta_1\delta_2(\ell))\right) + \\
K_4(\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_2^2(\ell) + \delta_2^2(\ell)\cdot\delta_1^2(\ell)\right) + K_5(\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_1^2(\ell) + \delta_2^2(\ell)\cdot\delta_2^2(\ell)\right) + \\
K_6(\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1^3(\ell) + \delta_1(\ell)\cdot\left(\delta_1\delta_2^2(\ell)\right) + \delta_2(\ell)\cdot\delta_2^3(\ell) + \delta_2(\ell)\cdot\left(\delta_1^2\delta_2(\ell)\right)\right) + \\
K_7(\nabla,\nabla)\left(\delta_1^3(\ell)\cdot\delta_1(\ell) + \left(\delta_1\delta_2^2(\ell)\right)\cdot\delta_1(\ell) + \delta_2^3(\ell)\cdot\delta_2(\ell) + \left(\delta_1^2\delta_2(\ell)\right)\cdot\delta_2(\ell)\right) + \\
K_8(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2^2(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1^2(\ell)\right) + \\
K_9(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_2(\ell)\cdot(\delta_1\delta_2(\ell)) + \delta_2(\ell)\cdot\delta_1(\ell)\cdot(\delta_1\delta_2(\ell))\right) + \\
K_{10}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot(\delta_1\delta_2(\ell))\cdot\delta_2(\ell) + \delta_2(\ell)\cdot(\delta_1\delta_2(\ell))\cdot\delta_1(\ell)\right) + \\
K_{11}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_2^2(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_1^2(\ell)\cdot\delta_2(\ell)\right) + \\
K_{12}(\nabla,\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell) + \delta_2^2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\right) + \\
K_{13}(\nabla,\nabla,\nabla)\left((\delta_1\delta_2(\ell))\cdot\delta_1(\ell)\cdot\delta_2(\ell) + (\delta_1\delta_2(\ell))\cdot\delta_2(\ell)\cdot\delta_1(\ell)\right) + \\
K_{14}(\nabla,\nabla,\nabla)\left(\delta_1^2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell) + \delta_2^2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\right) + \\
K_{15}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1^2(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2^2(\ell)\right) + \\
K_{16}(\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1^2(\ell)\cdot\delta_1(\ell) + \delta_2(\ell)\cdot\delta_2^2(\ell)\cdot\delta_2(\ell)\right) + \\
K_{17}(\nabla,\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell) + \delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\right) + \\
K_{18}(\nabla,\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell) + \delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)\right) +
\end{gathered}
$$

$$K_{19}(\nabla,\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_1(\ell)+\delta_2(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_2(\ell)\right)+$$
$$K_{20}(\nabla,\nabla,\nabla,\nabla)\left(\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)\cdot\delta_1(\ell)+\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\cdot\delta_2(\ell)\right)\Big).$$

We find that all of the functions $K_1, \ldots, K_{20}$ are smooth quotients of exponential polynomials, which have lengthy expressions, in particular the 3 and 4 variable functions $K_8, \ldots, K_{20}$. This invites one to use methods from the algebraic geometry of exponential polynomials familiar in transcendence theory [9] to determine the general structure of the noncommutative geometric invariants. In order to confirm the accuracy of our calculations we check that $K_1, \ldots, K_{20}$ satisfy a highly non-trivial family of functional relations that we derive theoretically. This is done by comparing the outcomes of two different calculations of the gradient of the functional that sends any self-adjoint element $\ell \in C^\infty(\mathbb{T}^2_\theta)$ to $\varphi_0\left(a_4(\ell)\right)$: first, by using a fundamental identity proved in [4] and, second, by direct calculations in terms of finite differences. By restricting the functional relations to certain hyperplanes we find a partial differential system that admits the action of cyclic groups of order 2, 3 and 4. This helps us to find expressions in terms of our calculated functions that possess certain symmetries with respect to the action of these groups. Moreover, the partial differential system admits a very simple and natural flow that allows us to express the system in a simplified manner. Finally, as a corollary of our main calculation, we conveniently obtain an expression for the term a_4 of the 4-dimensional product geometry of the form $\mathbb{T}^2_{\theta_1} \times \mathbb{T}^2_{\theta_2}$ whose metric is not conformally flat. In this case, two modular automorphisms are involved and this motivates further systematic research on *twistings* that involve two dimensional modular structures, cf. [2].

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Harmonic functions on quantum trees

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(joint work with Sara Malacarne)

The study of harmonic functions on trees has a long history. In the early 1960s Dynkin and Malyutov [4] considered nearest neighbor random walks on free groups and obtained an analogue of the Poisson formula for them by identifying the Martin boundary of such a group with the space of ends of its Cayley graph. This result was then generalized by a number of authors, among others by Cartier [2], who considered nearest neighbor, but not necessarily homogeneous, random walks on trees. Finite range random walks were subsequently studied by Derriennic [3] in the homogeneous case and by Picardello and Woess [8] in general. In both cases the result was the same as before: the Martin boundary of a tree coincides with its space of ends. This was later generalized to hyperbolic graphs by Ancona [1] who considered finite range random walks on such graphs and showed that the corresponding Martin boundaries coincide with the Gromov boundaries. A related result was also obtained by Kaimanovich [5].

The natural quantum analogues of free groups are the duals of free quantum groups of Van Daele and Wang [11]. They are defined as follows.

Fix a natural number $n \geq 2$ and a matrix $F \in \mathrm{GL}_n(\mathbb{C})$ such that $\mathrm{Tr}(F^*F) = \mathrm{Tr}((F^*F)^{-1})$. The compact free unitary quantum group $A_u(F)$ is defined as the universal unital C*-algebra with generators u_{ij}, $1 \leq i,j \leq n$, such that the matrices $U = (u_{ij})_{i,j}$ and FU^cF^{-1} are unitary, where $U^c = (u_{ij}^*)_{i,j}$, equipped with the comultiplication

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

From now on we assume that F is not a unitary 2-by-2 matrix. An interpretation of the dual Γ of $A_u(F)$ as vertices of a quantum tree was proposed by Vergnioux [12], and an analogue of the end compactification was defined by Vaes and Vergnioux [10] in the free orthogonal case and by Vaes and Vander Vennet in the free unitary case [9]. It is defined as follows.

Let I be the set of isomorphism classes of irreducible representations of $A_u(F)$. Then

$$\ell^\infty(\Gamma) \cong \ell^\infty\text{-}\bigoplus_{x \in I} B(H_x).$$

The set I is a free monoid on letters α and β. Consider the tree with vertex set I such that different elements x and y of I are connected by an edge if and only if one of them is obtained from the other by adding (or removing) one letter on the left. Denote by $\bar{I}$ the corresponding end compactification of I. The elements of $\bar{I}$ are words in α and β that are either finite or infinite on the left, and the boundary $\partial I = \bar{I} \setminus I$ is the set of infinite words. The algebra $C(\bar{I})$ of continuous functions on $\bar{I}$ can be identified with the algebra of functions $f \in \ell^\infty(I)$ such that

$$|f(yx) - f(x)| \to 0 \text{ as } x \to \infty, \text{ uniformly in } y \in I.$$

Now, for all $x, y \in I$, fix an isometry $V(xy, x \otimes y) \in \mathrm{Mor}(xy, x \otimes y)$. Define ucp maps

$$\psi_{yx,x} \colon B(H_x) \to B(H_{yx}), \quad T \mapsto V(yx, y \otimes x)^*(1 \otimes T)V(yx, y \otimes x).$$

They do not depend on any choices. Define

$$\mathcal{B} = \{a \in \ell^\infty(\Gamma) : \|a_{yx} - \psi_{yx,x}(a_x)\| \to 0 \text{ as } x \to \infty, \text{ uniformly in } y \in I\}.$$

It can be shown that this is a unital C*-subalgebra of $\ell^\infty(\Gamma)$ containing $c_0(\Gamma)$. It can therefore be considered as the algebra of continuous functions on a compactification of Γ, which should be thought of as a quantum end compactification of Γ. The (noncommutative) algebra of continuous functions on the corresponding quantum space of ends is defined by $\mathcal{B}_\infty = \mathcal{B}/c_0(\Gamma)$.

The following is our main result.

Theorem 1 [6] *Consider a free unitary quantum group $G = A_u(F)$, with F not a unitary 2-by-2 matrix, and a generating finitely supported probability measure μ on I. Then the Martin compactification $C(\bar{\Gamma}_{M,\mu})$ of the discrete quantum group $\Gamma = \hat{G}$ with respect to μ, as defined in* [7]*, coincides with the compactification $\mathcal{B}$. Therefore the Martin boundary $C(\partial\Gamma_{M,\mu})$ coincides with $\mathcal{B}_\infty$.*

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Sums of linear operators in Hilbert C*-modules

Matthias Lesch
(joint work with Bram Mesland)

This is a report on the paper [LeMe18].

A well-known problem in functional analysis is to describe the domain and the spectral properties of the sum of two densely defined closed operators. In general nothing can be said as the intersection of the domains can be just $\{0\}$. The problem has a rich history which we are going to review briefly before summarizing our main results.

1. Banach space history of the problem

Given two densely defined unbounded operators A, B in some Banach space X with a joint ray, e.g. $i(0,\infty)$ or $(-\infty, 0)$, in the resolvent set. A basic problem is to give criteria which ensure the following to hold:

(1) $\overline{A+B}+\lambda$ is invertible for $-\lambda$ in the said ray and large enough.
(2) $A+B$ is a closed operator with domain $\mathscr{D}(A)\cap\mathscr{D}(B)$.

One of the first comprehensive papers on the problem [DPGr75] was motivated by evolution equations

$$\underbrace{-\partial_t^2}_{A} u + \underbrace{\Lambda(t)}_{B} u + \lambda u = f,$$

with $\Lambda(t)$ being a family of partial differential operators parametrized by t.

The validity of (1) means that the equation $Ax + Bx + \lambda x = y$ is *weakly* solvable for λ large, that is given y there is a sequence $x_n \in \mathscr{D}(A)\cap\mathscr{D}(B)$ such that $x_x \to x$ and $(A+B+\lambda)x_n \to y$. (1) and (2) together mean that the equation $Ax + Bx + \lambda x = y$ is *strongly* solvable for λ large, that is given y there exists a solution $x \in \mathscr{D}(A)\cap\mathscr{D}(B)$.

One, and essentially the only approach to the problem in the Banach space context rests on the idea of viewing $A+B+\lambda$ as a (operator valued) function of B and writing the resolvent $(A+B+\lambda)^{-1}$ as the Dunford integral

$$P_\lambda := \frac{1}{2\pi i}\int_\Gamma (z+\lambda+A)^{-1}\cdot(z-B)^{-1}dz, \tag{1}$$

where Γ is a suitable contour encircling the spectrum of B. This approach works well only for *sectorial* operators with spectral angle $< \pi/2$. Eq. (1) equals the resolvent only if A and B are resolvent commuting and so it is not surprising that in the literature certain commutator conditions are formulated to ensure that Eq. (1) gives an appropriate approximation to the resolvent [DPGr75, DoVe87, LaTe87, Fuh93, MoPr97, KaWe01, PrSi07, Roi16].

2. KK-theory history of the problem

In the completely different context of KK-theory [Kas80] one encounters the problem of regular sums of operators when one tries to construct the notoriously complicated Kasparov product at the level of unbounded cycles [Mes14, BMvS16, MeRe16, KaLe12, KaLe13].

Here, the operators in question act on a Hilbert-(A, B)-bimodule E, which is a complete inner product module over the C^*-algebra B. For an unbounded B-linear operator T in E it makes sense to talk about self-adjointness and hence one might be tempted to believe that everything is as nice as in a Hilbert space. This, unfortunately (or fortunately), is not the case as the axiom of *regularity* does not come for free: analogously as in the Banach space context above an unbounded self-adjoint B-linear operator S in E is called regular if $S \pm \lambda$ has dense range for one and hence for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. If $B = \mathbb{C}$ then regularity is equivalent to self-adjointness. In general, it is an additional feature, *cf.* [BaJu83, Wor91, Pie06, KaLe12].

An *unbounded Kasparov module* is a triple $(\mathscr{A}, E, D)$ consisting of a Hilbert (A, B)- bimodule E and a self-adjoint regular operator D, with compact resolvent, that commutes with the dense subalgebra $\mathscr{A} \subset A$ *up to bounded operators.* In the construction of the tensor product of two such modules $(\mathscr{A}, X, S_X)$ and $(\mathscr{B}, Y, T_Y)$ one encounters two problems.

The first one is the definition of the operator $T = 1 \otimes_\nabla T_Y$ on the module $E := X \otimes_B Y$. Since T does not commute with B, one needs to incorporate extra data in the form of a *connection* ∇. This is discussed in great generality in [MeRe16] and in this paper we will not be concerned with this construction.

Once a well-defined self-adjoint and regular connection operator T on E has been constructed fromT_Y, the second problem that needs to be addressed is self-adjointness and regularity of the sum $D = S + T$, where $S = S_X \otimes 1$. The goal is then to formulate an appropriate smallness condition on the graded commutator $ST + TS$ such that $S + T$ is self-adjoint and regular on $\mathscr{D}(S) \cap \mathscr{D}(T)$.

The Banach space results mentioned in the previous paragraph do not (at least not a priori) apply to this situation as in general self-adjoint operators are sectorial with spectral angle $\pi/2$. Hence the sum of the spectral angles of S and T is π which is exactly the threshold for the validity of the above mentioned regularity results for sectorial operators. The methods in the Hilbert module case therefore resemble much more the methods known from Hilbert space theory.

3. The main results

Here we offer the following result which contains all previously known results in this context as special cases [Mes14, KaLe12, MeRe16].

Theorem 1. *Let S, T be self-adjoint and regular operators in the Hilbert-B-module E. Assume that*

(1) *there are constants $C_0, C_1, C_2 > 0$ such that the form estimate*

$$\langle [S, T]x, [S, T]x\rangle \leq C_0 \cdot \langle x, x\rangle + C_1 \cdot \langle Sx, Sx\rangle + C_2 \cdot \langle Tx, Tx\rangle \tag{2}$$

holds for all $x \in \mathscr{F} := \mathscr{F}(S,T) = \{x \in \mathscr{D}(S) \cap \mathscr{D}(T) \mid Sx \in \mathscr{D}(T), Tx \in \mathscr{D}(S)\}$*. This is an inequality in the* C^**-algebra* B*.*

(2) *There is a core* $\mathscr{E} \subset \mathscr{D}(T)$ *such that* $(S+\lambda)^{-1}(\mathscr{E}) \subset \mathscr{F}(S,T)$ *for* $\lambda \in i\mathbb{R}$*,* $|\lambda| \geq \lambda_0$*.*

Then $S+T$ *is self-adjoint and regular on* $\mathscr{D}(S) \cap \mathscr{D}(T)$*. That is for* $z \in \mathbb{C} \setminus \mathbb{R}$ *and* $y \in E$ *the equation*

$$Sx + Tx + z \cdot x = y$$

has a unique (strong) solution $x \in \mathscr{D}(S) \cap \mathscr{D}(T)$*.*

Our main application of the Theorem is to the calculation of the Kasparov product of unbounded cycles in KK-theory.

Historically, the main tool for handling the Kasparov product has consisted of a guess-and-check procedure pioneered by Connes-Skandalis [CoSk84], and later refined by Kucerovsky [Kuc97]. This entails checking a set of three sufficient conditions to determine whether a cycle $(\mathscr{A}, E, D)$ is the product of the cycles $(\mathscr{A}, X, S_X)$ and $(\mathscr{B}, Y, T_Y)$. Although this avoids the aforementioned hard problems, it leaves one with the burden of coming up with a good guess for D in every particular instance, as well as proving that $(\mathscr{A}, E, D)$ is a cycle.

In recent years, significant progress has been made on the constructive approach to finding D. In this setting, the first sufficient condition of Kucerovsky is satisfied whenever $D = S + T$ and T is a connection operator relative to T_Y. The second condition will be satisfied whenever $\mathscr{D}(S+T) \subset \mathscr{D}(S)$. In previous work the condition

$$\langle [S,T]x, [S,T]x\rangle \leq C(\langle x,x\rangle + \langle Sx, Sx\rangle),$$

was imposed to ensure self-adjointness of the sum $S+T$. This condition implies that

$$\langle (S+T)x, Sx\rangle + \langle Sx, (S+T)x\rangle \geq -\kappa\langle x,x\rangle,$$

for some $\kappa > 0$, which is the third sufficient condition appearing in [Kuc97, Theorem 13]. The form estimate (2) is in general not compatible with Kucerovsky's estimate. In [LeMe18] we prove that it is nonetheless sufficient to construct the Kasparov product.

Theorem 2. *Let* $(\mathscr{A}, X, S_X)$ *and* $(\mathscr{B}, Y, T_Y)$ *be unbounded Kasparov modules for* (A,B) *and* (B,C) *respectively and let* $E := X \otimes_B Y$ *and* $S := S_X \otimes 1$*. Suppose that* $T : \mathscr{D}(T) \to E$ *is an odd self-adjoint regular connection operator for* T_Y *such that*

(1) *for all* $a \in \mathscr{A}$ *we have* $a : \mathscr{D}(T) \to \mathscr{D}(T)$ *and* $[T,a] \in \mathscr{L}(E)$*;*

(2) (S,T) *is a weakly anticommuting pair.*

Then $(\mathscr{A}, E, S+T)$ *is an unbounded Kasparov module that represents the Kasparov product of* (X, S_X) *and* (Y, T_Y)*.*

We note that the statement that the sum operator $D = S + T$ is a KK-cycle is part of this result. The proof consists of showing that weak anticommutation implies a weakened version of the sufficient conditions of Connes-Skandalis. In the constructive setting, this supersedes the result of Kucerovsky and covers a wider range of examples, provided that we construct our operator as a sum.

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Non-commutative spectral triples for space-time

John W. Barrett
(joint work with James Gaunt)

Following the success of spectral triples in describing the internal space of particle physics [1], it is natural to suggest that space-time itself may also be non-commutative using the same mathematical framework. Mathematically, one can think of this as the approximation of (commutative) manifolds by non-commutative spaces, in such a way that suitable limits of non-commutative spaces become commutative. Since the key physical requirement is a cut-off in energy (at the Planck scale), compact spaces will be approximated by finite spectral triples (i.e., a finite-dimensional Hilbert space), and non-compact ones should have a finite number of fermion states 'per unit volume'.

More than just approximation, however, one wants to learn what aspects of commutative geometry survive as analogues in the relevant non-commutative case (and converge to the commutative notions). After introducing the general framework, the example of the fuzzy torus as a finite spectral triple is explained (in joint work with James Gaunt). Finite spectral triples are very restrictive and it is well known that the standard construction of the Dirac operator on the non-commutative torus does not work with matrices. However, there is an alternative starting point that is very fruitful and yields a Dirac operator on the fuzzy torus that does indeed square to the Laplace operator.

The framework of a finite spectral triple [2] is a finite-dimensional Hilbert space $\mathcal{H}$ with a faithful representation λ of a $*$-algebra $\mathcal{A}$ in $\mathcal{H}$ and a Dirac operator D. The real structure is an anti-unitary operator J so that $a \mapsto J\lambda(a^*)J^{-1}$ is a right action of $\mathcal{A}$ that commutes with the left action λ.

Two key definitions are the operators $v(a) = \lambda(a) - J\lambda(a^*)J^{-1}$ and $x(a) = \lambda(a) + J\lambda(a^*)J^{-1}$ in $\mathcal{H}$. It is helpful to write the left and right actions in abbreviated notation so

$$v(a)\psi = [a, \psi] = a\psi - \psi a, \qquad x(a) = \{a, \psi\} = a\psi + \psi a,$$

for $\psi \in \mathcal{H}$. The operator $v(a)$ is the non-commutative version of a Hamiltonian vector field. I propose the interpretation of the anti-commutator $x(a)$ is that it is the non-commutative version of the operator of multiplication by a coordinate function determined by a. Indeed in a suitable limiting process as $\dim \mathcal{H} \to \infty$, the $x(a)$, after a rescaling, become commutative and converge (pointwise in $\mathcal{H}$) to the coordinate ring of a limiting manifold. The $v(a)$ are not scaled and converge to vector fields on this manifold. This is analogous to the formalism of the contraction of representations of a Lie algebra [3]. It is worth noting that in the only examples I know how to construct, the spinor bundle of the limiting manifold is enhanced to a trivial bundle (e.g., via an embedding into $\mathbb{R}^n$). One suspects this is a generic feature.

This limiting construction does throw some light onto the differences between the commutative and non-commutative definitions of a real spectral triple. The first-order condition for a non-commutative triple reads

$$\big[[D, x(a) + v(a)], x(b) - v(b)\big] = 0.$$

In the commutative limit, the v operators become vanishingly small compared to the x operators and so one is left with the correct first-order condition in the commutative case,

$$\big[[D, x(a)], x(b)\big] = 0.$$

In addition, it explains the additional axiom one has in the commutative case

$$Jx(a)^* J^{-1} = x(a).$$

This follows automatically in the limit because it is true for the $x(a)$ in the non-commutative case; but it is of course *not* true that $Ja^*J^{-1} = a$.

The fuzzy torus [4] illustrates the geometric interpretation of the anticommutators. Picking $U, V \in \mathcal{A}$ such that

$$UV = qVU,$$

with $q \in \mathbb{C}$ a root of unity, one can define the Laplacian [5]

$$\Delta\psi = \frac{-1}{q^{1/2} - q^{-1/2}}\Big([U, [U^*, \psi]] + [V, [V^*, \psi]]\Big).$$

This has eigenvalues given by the q-number analogues of eigenvalues of commutative flat tori, with a variety of possible shapes for the tori, depending on the choices of U and V.

The Dirac operator is defined using four gamma matrices that commute with the algebra. The formula is

$$D\psi = \frac{1}{q^{1/4} - q^{-1/4}} \sum_i \gamma^i \otimes [X_i, \psi] + \frac{1}{q^{1/4} + q^{-1/4}} \sum_{i<j<k} \gamma^i\gamma^j\gamma^k \otimes \{X_{ijk}, \psi\}$$

with

$$X_1 = X_{234} = -\frac{1}{4}(U + U^*), \qquad X_2 = -X_{134} = -\frac{i}{4}(U^* - U),$$
$$X_3 = -X_{124} = \frac{1}{4}(V + V^*), \qquad X_4 = X_{123} = \frac{i}{4}(V^* - V).$$

The eigenvalues are again q-number analogues of the commutative ones, so that D converges pointwise to the commutative operator in the large matrix limit, in which $q \to 1$.

The interpretation of this formula is that the commutator terms are analogues of partial derivatives and the anti-commutator terms are the analogues of spin connection terms. Indeed, the corresponding commutative formula is exactly the usual Dirac operator on a flat torus but expressed using a rotating frame, giving non-zero connection coefficients but vanishing curvature. The rotation is a map $T^2 \to \mathrm{SO}(4)$ and encodes a spin structure according to $\pi_1(T^2) \to \pi_1(SO(4))$. In cases where this is trivial (which is the non-bounding 'Lie' spin structure), one can calculate that the non-commutative D^2 is unitarily equivalent to Δ.

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Time to think about time?

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(joint work with F. D'Andrea, A. Devastato, S. Farnsworth, M.A. Kurkov and P. Martinetti in various combinations)

Most of noncommutative geometry [1], being spectral, is based on elliptic operators, the archetype of them being the Dirac operator, or the Laplacian defined on a Riemannian compact manifold. Physical spacetime, however, is noncompact and with a Lorentzian signature. The compactness issue is usually considered a minor issue, since usually the problems for quantum field theory come form short distance (ultraviolet) divergences[1]. The implementation of a "Lorentzian" noncommutative geometry seems to be a rather difficult problem. One direction has been to use Krein Spaces, which generalise the Hilbert space structure [3, 4, 5, 6], another is a covariant approaches [7], Wick rotations on pseudo-Riemanninan structures [8], or algebraic characterizations of causal structures [9, 10, 11], and others. Although the mathematics coming out of these approach is very interesting, we are still far away from a full understanding of the theory from a physical point of view. It is fair to say that noncommutative geometry has a *time* problem. It is probably time to confront this, both from a mathematical and a physical point of view.

Physicists themselves are no less culpable than mathematicians. To regularize quantum field theory one of the most useful methods is to consider the theory in Euclidean space, performing what is called a *Wick rotation*, whereby the time coordinate is rotated, in a complex plane, to its imaginary counterpart, therefore adjusting the signature of spacetime to an Euclidean one. After the rotation, usually divergent (path) integrals become convergent and it is possible perform calculations, for example scattering amplitudes. At the end another Wick (anti)rotation is performed to obtain the answer in the space with Minkowskian signature. It must be said that this procedure is often ill defined, especially in curved spacetime [12]. Nevertheless, the answers are in agreement with experiments.

We have analyzed in [13] the issue of Wick rotation in the Chamseddine-Connes noncommutative approach to the standard model [14, 15] (for a recent review

[1]This too may be optimistic, the large distance (infrared) divergences are becoming relevant, (for a review see [2]) and they may require a mathematical rethinking as well.

see [16]). The issue is intimately related with the extra degrees of freedom of the model known as *fermion doubling* feature of the model. We have given a precise prescription for the Wick rotation from the Euclidean theory to the Lorentzian one, eliminating the extra degrees of freedom. This requires not only projecting out mirror fermions, which leads to the correct Pfaffian, but also the elimination of the remaining extra degrees of freedom. The remaining doubling has to be removed in order to recover the correct Fock space of the physical (Lorentzian) theory. In order to get a Spin(1,3) invariant Lorentzian theory from a Spin(4) invariant Euclidean theory such an elimination must be performed *after* the Wick rotation. These considerations point to a deep connection between the spectral action and the signature of spacetime

There is another cunning connection between Lorentzian signature and the standard model. In [17] we have show how twisting the spectral triple [18] of the Standard Model of elementary particles naturally yields the Krein space associated with the Lorentzian signature of spacetime. The twist is necessary to introduce a"Grand Symmetry"[19], which is one of the solutions of the Higgs mass problem in the model, but is otherwise interesting [20, 21]. We established the fact that *the required twist corresponds to a Wick rotation.* More precisely, we show that the twist turns the inner product of the Hilbert space of (Euclidean) spinors into a Krein product. The latter is precisely the inner product associated with spinors on a Lorentzian manifold. In a sense the twist is actually the square of the Wick rotation. The picture that emerges is that twisted geometries may provide an appropriate framework from which to facilitate the description of non-Euclidean signatures in NCG.

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Higher rho-invariants for the signature operator – A survey and perspectives

Charlotte Wahl

Analytical rho-invariants associated to Dirac operators have been introduced by Atiyah–Patodi–Singer. Their definition is based on the eta-invariant, however they have better invariance properties, at least in geometrically interesting cases: For the spin Dirac operator for manifolds with positive scalar curvature and for the signature operator they do not depend on the metric. More generally, they are invariant under suitable bordism relations, which makes them an important tool for the study of bordism classes of psc-metrics and of the surgery structure set, respectively.

Along with the generalization of index theory first to an L^2-setting and later to a higher setting, allowing the pairing of the index with cocycles on the fundamental group, appropriate generalizations of the APS rho-invariant where found and studied in work of Cheeger–Gromov, Lott, Leichtnam–Piazza and others. APS and Cheeger–Gromov rho-invariants have been exploited for classification results of psc metrics and of differential structures in dimensions $4k-1$. Higher rho-invariants yield nontrivial results also in other, in particular even dimensions. A crucial property for applications are product formulas of the type $\rho(M \times N) = \rho(M)\,\mathrm{sign}(N)$.

While here we are focussing on numerical invariants, other generalizations based on and inspired by work of Higson–Roe lead to K-theoretic rho-invariants. A precise relation between the numerical and K-theoretic higher invariants has not yet been proven.

The basic ingredient for the definition of higher rho-invariants for a closed oriented Riemannian manifold M is the Mishchenko $C^*\Gamma$-vector bundle $\mathcal{F} = \tilde{M} \times_\Gamma C^*\Gamma$. Here $\tilde{M} \to M$ is a Galois covering with deck transformation group Γ. The invariants are derived from the twisted signature operator $D_{\mathcal{F}}$. Since the higher eta-invariant is only defined for invertible Dirac operators, one has to find a "canonical" invertible perturbation of $D_{\mathcal{F}}$ in order to construct higher rho-invariants. In the case of psc manifolds invertibility is automatically guaranteed. In the case of the signature operator an appropriate perturbation has been found in two situations (for simplicity we restrict to odd-dimensional M):

(1) If $\dim M = 2m-1$ and the m-th Novikov–Shubin invariant of M is ∞^+, let I be the involution which is 1 on forms of degree $< m$ and -1 on the complement. Then for $t > 0$ small $D_{\mathcal{F}} + tI$ is invertible. By a standard spectral flow argument, the higher rho-invariant does not depend on t.
(2) Instead of a single manifold one considers a smooth orientation preserving homotopy equivalence $f\colon M \to N$ between odd-dimensional oriented closed Riemannian manifolds. Let $\mathcal{F}_N = \tilde{N} \times_\Gamma C^*\Gamma$ be the Mishchenko bundle on N and $\mathcal{F}_M = f^*\mathcal{F}_N$ the induced Mishchenko bundle on M. The signature operator $N \cup M^{op}$ twisted by $\mathcal{F} := \mathcal{F}_N \cup \mathcal{F}_M$ has a special class of invertible perturbations which was defined by Hilsum–Skandalis and further studied in the context of rho-invariants by Piazza–Schick. Again, the induced higher rho-invariant does not depend on the choice of the perturbation.

In both cases the rho-invariants have the expected properties: invariance under changes of the metrics, product formulas, well-definedness on the (smooth) surgery structure set, compability with the action of the L-groups on the surgery structure set ...

In joint work (in progress) with Sara Azzali we are considering the generalization of rho-invariants to an almost flat setting. Here several frameworks are relevant:

(i) the case of twisted group C^*-algebras associated to a group 2-cocycle,
(ii) almost flat vector bundles,
(iii) quasirepresentations of the fundamental group.

In [1] we established the necessary tools for the definition of (higher) rho-invariants in the case of (i) for spin Dirac operators on psc manifolds. In this case the deformed C^*-algebras assemble to an upper semi-continuous field. The invertibility of the perturbed signature operator from (1) is then guaranteed for small parameters.

In cases (ii) and (iii), which include (i), this argument does not work. Additional conditions such as the restriction to completely positive asymptotic representations of $C^*\Gamma$ might be necessary. Note that these two frameworks are closely related by results of Dardalat. However, Dardalat focussed on the induced K-theory classes whereas for the study of secondary invariants the connections also have to be taken into account. Framework (iii) seems more suited than (ii) for the establishment of geometric properties like bordism invariance which are formulated in terms of Galois coverings and their classifying maps.

In the situation of (2) all three frameworks should lead to a reasonable definition of rho-invariants since the Hilsum–Skandalis perturbation was also constructed in an almost flat setting.

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Discrete group actions and a weak form of the Baum–Connes conjecture

Sara Azzali

(joint work with Paolo Antonini, Georges Skandalis)

Let Γ be a discrete group. In this talk, we use KK-theory with $\mathbb{R}$-coefficients to construct a Baum–Connes map localised at the unit element of Γ. The localisation is defined by a distinguished idempotent $[\tau]$ of the commutative ring $KK^{\Gamma}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$ which is canonically associated to Γ via its standard trace. We construct a natural Baum–Connes type morphism μ_τ between the τ*-parts* of the usual left and right hand side. We show that the τ-form of the Baum–Connes conjecture is weaker than the classical one, but still implies the strong Novikov conjecture.

1. KK-theory with coefficients in $\mathbb{R}$ and the τ-part

The groups $KK_{\mathbb{R}}$ are defined as inductive limits over II_1-factors. Let A and B be Γ-C^*-algebras, let N be a II_1-factor with trivial Γ-action. The functor $N \to KK^{\Gamma}_{*}(A, B\otimes N)$ from the category of II_1-factors N acting on a separable Hilbert space H, with morphism given by unital embeddings, to the category of groups has a limit [2] and we set

$$KK_{\mathbb{R}}(A,B) := \lim_{\mathcal{F}_{\mathrm{II}_1}(H)} (N \to KK(A, B\otimes N)) \ .$$

The tensor product $\otimes$ is the minimal one. Hence an element x of $KK^{\Gamma}_{\mathbb{R}}(A,B)$ is represented by a class $x_0 \in KK^{\Gamma}(A, B\otimes N)$, where N is a II_1-factor with trivial Γ-action. Note that if A and B are in the bootstrap class then $KK^{\Gamma}(A, B\otimes N)$ is independent of the II_1-factor N.

1.1. **$KK_{\mathbb{R}}$-classes coming from traces.** Let $\mathrm{tr} : D \to \mathbb{C}$ be a tracial state. There exists a II_1-factor N and a unital trace preserving morphism $\phi : D \to N$. The class $[\phi] \in KK_{\mathbb{R}}(D,\mathbb{C})$ only depends on the trace tr and will be denoted by [tr].

Let $C^*\Gamma$ denote the maximal group C^*-algebra of the discrete group Γ. Then the standard group trace $\tau\colon C^*\Gamma \to \mathbb{C}$ defined by $\tau(\sum a_\gamma u_\gamma) = a_e$ gives a class in $KK_{\mathbb{R}}(C^*\Gamma,\mathbb{C})$. As the latter group is isomorphic to $KK^{\Gamma}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$, we obtain a class

$$[\tau] \in KK^{\Gamma}_{\mathbb{R}}(\mathbb{C},\mathbb{C})$$

which is canonically associated to Γ.

1.2. **The element** $[\tau]$**.** In the ring $KK_{\mathbb{R}}^{\Gamma}(\mathbb{C},\mathbb{C})$, $[\tau]$ is a projection. Moreover it is central when acting on $KK_{\mathbb{R}}^{\Gamma}(A,B)$ by exterior product:

Lemma 1.1. (a) $[\tau]\otimes[\tau]=[\tau] \in KK_{\mathbb{R}}^{\Gamma}(\mathbb{C},\mathbb{C})$;
(b) $[\tau]\otimes x = x\otimes[\tau]$, $\forall x\in KK_{\mathbb{R}}^{\Gamma}(A,B)$, $\forall A,B$ Γ*-algebras.*

Proof. (a): one shows more generally that if t,s are tracial states on $C^*\Gamma$, then the product $[t]\otimes[s]$ is the class of the trace $t.s:=(t\otimes s)\circ\delta$. Note that $t.s(g)=t(g)s(g)$ for any group element $g\in\Gamma$, so that in particular $\tau.t=t$. (see [2, Remark 2.4]); (b) follows from the commutativity of the exterior product in $KK_{\mathbb{R}}$ ([3]). □

Definition 1.2. The image of the projector $[\tau]$ acting $KK_{\mathbb{R}}^{\Gamma}(A,B)$ is called *the* τ*-part of* $KK_{\mathbb{R}}^{\Gamma}(A,B)$ and denoted

$$KK_{\mathbb{R}}^{\Gamma}(A,B)_\tau := \{x\otimes[\tau];\ x\in KK_{\mathbb{R}}^{\Gamma}(A,B)\}\ .$$

To clarify what the τ-part contains, let us compute for example $1_{A,\mathbb{R}}^{\Gamma}\otimes[\tau]$, where $1_{A,\mathbb{R}}^{\Gamma}$ is the unit of $KK_{\mathbb{R}}^{\Gamma}(A,A)$, for a given Γ-algebra A. Say the Γ-action is denoted by β. Fix a II_1-factor N with a trace preserving map $\lambda: C_r^*\Gamma\to N$. Then the class $1_{A,\mathbb{R}}^{\Gamma}\otimes[\tau]$ is represented by the $(A,A\otimes N)$-bimodule $A\otimes N$, where Γ acts by $\beta\otimes\lambda$. This implies the following crucial fact.

Remark 1.3. Let $\mathcal{A}=C_0(Y)$, for a free and proper Γ-space Y. Then $KK_{\mathbb{R}}^{\Gamma}(\mathcal{A},\mathcal{A})_\tau = KK_{\mathbb{R}}^{\Gamma}(\mathcal{A},\mathcal{A})$. In fact, let β be the action of Γ on $\mathcal{A}$ by translations. For any II_1-factor N with trace preserving $\lambda: C_r^*\Gamma\to N$, the $(A,A\otimes N)$-bimodule $A\otimes N$ with Γ-action $\beta\otimes\lambda$ represents the module of sections of a flat bundle over Y/Γ with fibre N. This can be always trivialised [1, Prop. 5.1] so that it is equivalent in KK^{Γ} to the $(A,A\otimes N)$-bimodule $A\otimes N$ with Γ-action $\beta\otimes 1$. As a consequence, τ acts as the identity on the K-homology with Γ-compact support $K_{*,\mathbb{R}}^{\Gamma}(E\Gamma)$ – where $E\Gamma$ is the classifying space for free and proper actions.

What we have seen so far will be used to define the left hand side of a τ-Baum–Connes map.

Let A be a Γ-algebra. For the right hand side, it is natural to define the τ-part of $K_{\mathbb{R}}(A\rtimes_r\Gamma)$ by letting $[\tau]$ act via descent *i.e.* by right multiplication with the idempotent element $J_r^{\Gamma}(1_A\otimes[\tau])$ of the ring $KK_{\mathbb{R}}(A\rtimes_r\Gamma, A\rtimes_r\Gamma)$. Here J_r^{Γ} is the descent morphisms $KK_{\mathbb{R}}^{\Gamma}(A,A)\to KK_{\mathbb{R}}(A\rtimes_r\Gamma,A\rtimes_r\Gamma)$. The analogous for the maximal crossed product will be denoted J^{Γ}. We set

$$[\tau]_r := J_r^{\Gamma}([\tau])\in KK_{\mathbb{R}}^{\Gamma}(A\rtimes_r\Gamma, A\rtimes_r\Gamma)\ ,\ \ [\tau]_m := J_r^{\Gamma}([\tau])\in KK_{\mathbb{R}}^{\Gamma}(A\rtimes\Gamma, A\rtimes\Gamma)\ .$$

Definition 1.4. The τ-parts of the crossed product are defined as:

$$K_{*,\mathbb{R}}(A\rtimes_r\Gamma)_\tau := \{x\otimes[\tau]_r;\ x\in K_{*,\mathbb{R}}(A\rtimes_r\Gamma)\},$$
$$K_{*,\mathbb{R}}(A\rtimes\Gamma)_\tau := \{x\otimes[\tau]_m;\ x\in K_{*,\mathbb{R}}(A\rtimes\Gamma)\}\ .$$

The τ-part does not distinguish between reduced and maximal crossed product:

Theorem 1.5. *There is an isomorphism*

$$K_{*,\mathbb{R}}(A\rtimes\Gamma)_\tau \xrightarrow{\simeq} K_{*,\mathbb{R}}(A\rtimes_r\Gamma)_\tau$$

Proof. Let $\lambda^A : A \rtimes_r \Gamma \to A \rtimes \Gamma$ be the natural map. Via coproduct we have a map $A \rtimes_r \Gamma \to A \rtimes \Gamma \otimes C^*_r \Gamma \to A \rtimes \Gamma \otimes N$ which gives a class $[\Delta^A] \in KK_{\mathbb{R}}(A \rtimes_r \Gamma, A \rtimes \Gamma)$. It holds $[\Delta^A] \otimes [\lambda^A] = [\tau]_r$ and $[\lambda^A] \otimes [\Delta^A] = [\tau]_m$. □

2. The localised Baum–Connes maps in $KK_{\mathbb{R}}$

For a Γ-algebra A, let $\mu^A : K^{\mathrm{top}}_*(\Gamma; A) \longrightarrow K_*(A \rtimes_r \Gamma)$ be the classical Baum–Connes morphism, where $K^{\mathrm{top}}_*(\Gamma; A)$ is the K-homology with compact support of the classifying space for proper actions [5]. The collection of maps $\mu^{A \otimes N}$, where N ranges over II_1-factors with trivial Γ-action, defines a map $\mu_{\mathbb{R}} : K^{\mathrm{top}}_{*,\mathbb{R}}(\Gamma; A) \to K_{*,\mathbb{R}}(A \rtimes_r \Gamma)$ which in turn descends to a map

$$\mu_\tau : K^{\mathrm{top}}_{*,\mathbb{R}}(\Gamma; A)_\tau \longrightarrow K_{\mathbb{R}}(A \rtimes_r \Gamma)_\tau$$

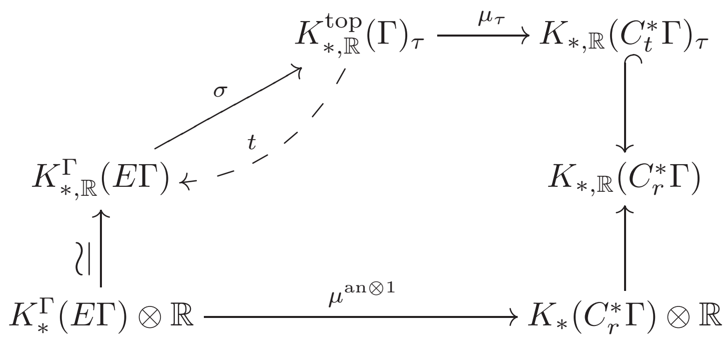
between the τ-parts given by $\mu_\tau(x \otimes [\tau]) := \mu_{\mathbb{R}}(x) \otimes [\tau]_r$ for all $x \in K^{\mathrm{top}}_{*,\mathbb{R}}(\Gamma; A)$. The τ-form of the Baum–Connes conjecture with coefficients in a Γ-algebra A is then the statement that μ_τ is an isomorphism.

Theorem 2.1. *If the Baum–Connes assembly map is injective (*resp. *surjective) for $A \otimes N$ for every II_1-factor N then μ_τ is injective (*resp. *surjective) for A.*

Let $\sigma : K^\Gamma_*(E\Gamma) \to K^{\mathrm{top}}_*(\Gamma)$ be the natural map.

Theorem 2.2. *If μ_τ is injective, then the analytic assembly map $\mu^{an} := \mu \circ \sigma \colon K_*(B\Gamma) \to K_*(C^*_r\Gamma)$ is rationally injective.*

For the complete proofs of the above, we refer to [3]. In particular, Theorem 2.2 is a consequence of the following **Claim**: *there is a map $t\colon K^{top}_{\mathbb{R},*}(\Gamma; A) \longrightarrow K^\Gamma_{*,\mathbb{R}}(E\Gamma)$ which is inverse to σ on the τ-part.* Whence the commutative diagram:



To construct the map t, we show that we can find a Γ-space X such that:

(I) X is a compact, and every torsion element of Γ acts freely. (Such a space will be called (TAF) (*torsion acts freely*). It satisfies: if Y is proper and X is (TAF), then $X \times Y$ is free and proper w.r.t. the diagonal Γ-action).

(II) X has a Γ-invariant probability measure.

From (II) we get trace $t_X \colon C(X) \rtimes \Gamma \to \mathbb{C}$, and its class $[t_X] \in KK^\Gamma_{\mathbb{R}}(C(X), \mathbb{C}))$.

For every pair $[\![Y, y]\!]$, where $Y \in \underline{E}\Gamma$, $y \in K^\Gamma(C_0(Y), \mathbb{C})$, the map defined by $t([\![Y, y]\!]) := [\![Y \times X, y \otimes [t_X]]\!]$ provides an inverse of σ on the τ-parts, as $\sigma \circ t = [\tau]$,

$t \circ \sigma = [\tau]$. This shows the claim as $[\tau] = 1$ on $K^{\Gamma}_{*,\mathbb{R}}(E\Gamma)$. More generally, the claim identifies the τ-part of $K^{\mathrm{top}}_{*,\mathbb{R}}(\Gamma; A)$ with $KK^{\Gamma}_{\mathbb{R}}(E\Gamma, A)$.

In [3, Section 7] we show that the construction of [9] for group actions using the "Gromov Monster" still provides counterexamples to the bijectivity of the τ-Baum–Connes map μ_τ. Again the failure of exactness is the source of counterexamples, so that μ_τ cannot be an isomorphism for every Γ-algebra.

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Group cocycles on loop groups

Jens Kaad

(joint work with Ryszard Nest, Jesse Wolfson)

The Connes-Karoubi multiplicative character is an invariant of higher algebraic K-theory associated to the geometric data contained in any finitely summable Fredholm module, [5, 9]. The construction of the multiplicative character relies on a long exact sequence of abelian groups relating the algebraic K-theory and the topological K-theory of any Banach algebra (in fact any Fréchet algebra) by means of a relative K-group. When the underlying geometric object is of spectral dimension one, this secondary invariant can be computed using central extensions and group 2-cocycles, [5]. Due to the work of Carey-Pincus this links the multiplicative character to the tame symbol of a pair of meromorphic functions on a Riemann surface, [3, 4]. As an example of the computation carried out by Carey and Pincus one may consider the 2-summable Fredholm module

$$\mathcal{F} = \big(C^\infty(\mathbb{T}), L^2(\mathbb{T}), 2P - 1\big),$$

where $P : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ denotes the orthogonal projection onto Hardy-space. In this case, the multiplicative character provides a group homomorphism

$$M_{\mathcal{F}} : K_2^{\mathrm{alg}}(C^\infty(\mathbb{T})) \to \mathbb{C}^*$$

and one may evaluate this group homomorphism on the Steinberg symbol $\{f, g\} \in K_2^{\mathrm{alg}}(C^\infty(\mathbb{T}))$ arising from a pair of smooth invertible functions on the circle $f, g : \mathbb{T} \to \mathbb{C}^*$. In this setting we have the local formula

$$M_{\mathcal{F}}\big(\{f, g\}\big) = \exp\Big(\frac{1}{2\pi i}\int_{\mathbb{T}} \log(f) d\log(g)\Big) \cdot g(1)^{w(f)},$$

where $w(f) \in \mathbb{Z}$ denotes the winding number and where the logarithms are taken with respect to the choice of basepoint $1 \in \mathbb{T}$.

The first main result of this talk is a category theoretic description of the Connes-Karoubi multiplicative character on the second algebraic K-group. Starting from a 2-summable Fredholm module $\mathcal{F} = (\mathcal{A}, H, F)$, we show how to construct a category $\mathfrak{C}_{\mathcal{F}}$ with an action of the general linear group $GL(\mathcal{A})$. Using results of Brylinski this $GL(\mathcal{A})$-category provides us with a group 2-cohomology class, which computes the Connes-Karoubi multiplicative character in this one-dimensional setting. Fundamental ingredients in our construction are graded determinant lines for Fredholm operators and canonical isomorphisms of these graded determinant lines associated with compositions and trace class perturbations of Fredholm operators. The graded determinant line of a Fredholm operator $T : H \to G$ is given by the pair

$$\mathrm{Det}(T) = \big(\Lambda^{\mathrm{top}}(\mathrm{Ker}(T)) \otimes \Lambda^{\mathrm{top}}(\mathrm{Coker}(T))^*, \mathrm{Index}(T)\big),$$

which should be considered as an object in the Picard category of graded complex lines. Our constructions are related to results of Arbarello, de Concini, and Kac, [1], but, contrary to these authors, we are working in the analytic context of Hilbert spaces and trace class perturbations.

The second (and much more substantial) main result of this talk concerns the two-dimensional version of the Connes-Karoubi multiplicative character (coming from a 3-summable Fredholm module), which is a secondary invariant of the third algebraic K-theory relating to group 3-cocycles on the general linear group over an algebra. Using ideas going back to Street, [10], we expose a higher category theoretic framework for constructing group 3-cohomology classes out of group actions. The core part of our work is however to explicitly construct examples of the relevant higher category theoretical structures from "realistic" geometric data. In order to achieve this goal, we assume that the geometry factorizes in a vertical and a horizontal component and this gives rise to the following notion of a bipolarized group representation:

Suppose that we have a discrete (but typically uncountable) group G that acts on a separable Hilbert space H (by invertible operators). We say that two bounded idempotents P and $Q : H \to H$ provide a bipolarization of this group representation when the following holds for all $g, h \in G$:

(1) The commutator $[gPg^{-1}, hQh^{-1}]$ is of trace class;

(2) The product of commutators $[g, P] \cdot [h, Q]$ is of trace class.

Interesting examples of bipolarized group representations come from the geometry of the noncommutative 2-torus. In this case, the group G in question is the general linear groups over the smooth functions on the noncommutative 2-torus whereas the two idempotents P and Q are given by spectral projections coming from the two first order differential operators $\frac{\partial}{\partial\theta_1}$ and $\frac{\partial}{\partial\theta_2}$.

Starting from a bipolarized group representation we are able to explicitly build the relevant higher category theoretic framework so that any bipolarized group representation gives rise to a group 3-cohomology class on the group in question and with values in the complex multiplicative group. In particular, we have a group 3-cohomology class

$$[c_{P,Q}] \in H^3(\mathrm{GL}(C^\infty(\mathbb{T}^2_\theta)), \mathbb{C}^*)$$

on the noncommutative 2-torus coming from the above mentioned bipolarization. In the case of the commutative 2-torus, we are also able to verify that this group 3-cohomology class is non-trivial.

It is conjectured that our group 3-cocycles compute the Connes-Karoubi multiplicative character and that they are linked to the tame symbol of triples of meromorphic functions on a 2-dimensional complex manifold.

Our constructions are related to the work of several authors: Frenkel and Zhu; Gorchinskiy and Osipov; Braunling, Groechenigg and Wolfson. However, contrary to these authors, we are working in the analytic setting of Hilbert spaces and Schatten ideals, and this allows us to treat examples such as the actual commutative 2-torus, $C^\infty(\mathbb{T}^2)$, instead of the formal 2-torus, $\mathbb{C}((s))((t))$, given by formal Laurent series in two variables, [6, 7, 2].

The results of this talk are available as a preprint, which will hopefully soon appear on the arXiv, [8].

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Solutions to the Yang–Baxter equation and related deformations

Giovanni Landi

The duality between spaces and algebras of functions on the spaces is at the basis of noncommutative geometry. One gives up the commutativity of the algebras of functions while replacing them by appropriate classes of noncommutative associative algebras which are considered as 'algebras of functions' on (virtual) 'noncommutative spaces'. A possibility is to consider noncommutative associative regular algebras generated by coordinates functions that satisfy relations other than the commutation between them, thus generalizing the polynomial algebras.

In this framework in the papers [4, 5] there were defined noncommutative (products of) finite-dimensional Euclidean spaces. One starts with an algebra $\mathcal{A}_R$ generated by two sets of hermitian elements $x = (x_1, x_2) = (x_1^\lambda, x_2^\alpha)$, with $\lambda \in \{1, \ldots, N_1\}$ and $\alpha \in \{1, \ldots, N_2\}$, subject to relations

$$(1) \quad x_1^\lambda x_1^\mu = x_1^\mu x_1^\lambda \ , \quad x_2^\alpha x_2^\beta = x_2^\beta x_2^\alpha, \quad x_1^\lambda x_2^\alpha = R^{\lambda\alpha}_{\beta\mu}\, x_2^\beta x_1^\mu \ , \quad x_2^\alpha x_1^\lambda = \overline{R}^{\lambda\alpha}_{\beta\mu}\, x_1^\mu x_2^\beta$$

for a 'matrix' $(R^{\lambda\alpha}_{\beta\mu})$. Thus $\mathcal{A}_R = \oplus_{n\in\mathbb{N}}(\mathcal{A}_R)_n$ is the quadratic $*$-algebra generated by the hermitian elements x_1^λ and x_2^α with the relations (1) and by duality is thought to be the algebra of coordinate functions on the noncommutative vector space $\mathbb{R}^{N_1} \times_R \mathbb{R}^{N_2}$ (the commutative solution is $(R_0)^{\lambda\alpha}_{\beta\mu} = \delta^\lambda_\mu \delta^\alpha_\beta$ and $\mathcal{A}_{R_0}$ is the coordinate algebra over the product $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \simeq \mathbb{R}^{N_1+N_2}$).

If we collect together the coordinates, defining the x^a for $a \in \{1, 2, \ldots, N_1+N_2\}$ by $x^\lambda = x_1^\lambda$ and $x^{\alpha+N_1} = x_2^\alpha$, the relations (1) can be written in the form

$$x^a x^b = \mathcal{R}^{a\,b}_{c\,d}\, x^c x^d \ .$$

The endomorphism $\mathcal{R} = (\mathcal{R}^{ab}_{cd})$ of $(\mathcal{A}_R)_1 \otimes (\mathcal{A}_R)_1$ turns out to be involutive: $\mathcal{R}^2 = I \otimes I$ and one next imposes that $\mathcal{R}$ satisfies the Yang-Baxter equation

$$(\mathcal{R} \otimes I)(I \otimes \mathcal{R})(\mathcal{R} \otimes I) = (I \otimes \mathcal{R})(\mathcal{R} \otimes I)(I \otimes \mathcal{R}).$$

Finally, additional conditions comes by requiring that both quadratic elements $(x_1)^2 = \sum_{\lambda=0}^{N_1} (x_1^\lambda)^2$ and $(x_2)^2 = \sum_{\alpha=0}^{N_2} (x_2^\alpha)^2$ of $\mathcal{A}_R$ be central. The general solution of these conditions was given in [5] as follows. By setting $\widehat{R}^{\lambda\alpha}_{\mu\beta} = R^{\lambda\alpha}_{\beta\mu}$ for the endomorphism $\widehat{R} = (\widehat{R}^{\lambda\alpha}_{\mu\beta})$ of $\mathbb{R}^{N_1} \otimes \mathbb{R}^{N_2}$ one has the representation

$$(2) \qquad \widehat{R} = \sum\nolimits_r A_r \otimes B_r + \mathrm{i} \sum\nolimits_a C_a \otimes D_a$$

with A_r real symmetric and C_a real anti-symmetric $N_1 \times N_1$ matrices, and B_r real symmetric and D_a real anti-symmetric $N_2 \times N_2$ matrices. The matrices A's commutes among themselves and with the C's and these also commute among themselves; similarly for the matrices B's and D's, with a normalization condition

$$\sum\nolimits_{r,s} A_r A_s \otimes B_r B_s + \sum\nolimits_{a,b} C_a C_b \otimes D_a D_b = 1_{N_1} \otimes 1_{N_2}.$$

Being the quadratic elements $(x_1)^2$ and $(x_2)^2$ central, the quotient algebra $\mathcal{A}_R / \big((x_1)^2 - 1, (x_2)^2 - 1\big)$ defines the noncommutative product $\mathbb{S}^{N_1-1} \times_R \mathbb{S}^{N_2-1}$ of the classical (commutative) spheres $\mathbb{S}^{N_1-1}$ and $\mathbb{S}^{N_2-1}$. Furthermore, the quotient algebra $\mathcal{A}_R / \big((x_1)^2 + (x_2)^2 - 1\big)$ defines the noncommutative $(N_1 + N_2 - 1)$-sphere $\mathbb{S}_R^{N_1+N_2-1}$, a noncommutative spherical manifold in the sense of [2] and [1].

When $N_1 = N_2 = 4$, explicit solutions for the matrix $R^{\lambda\alpha}_{\beta\mu}$ were given in [4] and [5] using quaternions. The space of quaternions $\mathbb{H}$ is identified with $\mathbb{R}^4$:

$$\text{(3)} \quad \mathbb{H} \ni q = x^0 1 + x^1 e_1 + x^2 e_2 + x^3 e_3 \quad \longmapsto \quad x = (x^\mu) = (x^0, x^1, x^2, x^3) \in \mathbb{R}^4,$$

with imaginary units e_a obeying the multiplication rule of $\mathbb{H}$. With this, left and right multiplication of quaternions are represented by matrices acting on $\mathbb{R}^4$: $L_{q'} q := q'q \to E^+_{q'}(x)$ and $R_{q'} q := qq' \to E^-_{q'}(x)$. In particular, the basis matrices $J^+_a := E^+_a$ and $J^-_a := -E^-_a$ obey the algebra $J^\pm_a J^\pm_b = -\delta_{ab} 1 + \sum_{c=1}^3 \varepsilon_{abc} J^\pm_c$ and $J^+_a J^-_b = J^-_b J^+_a$. With the identification $U_1(\mathbb{H}) \simeq SU(2)$, they are representations of the Lie algebra $su(2)$ corresponding to commuting $SU(2)$ actions on $\mathbb{R}^4$.

We get antisymmetric matrices $J^+_{\mathbf{u}} := u^1 J^+_1 + u^2 J^+_2 + u^3 J^+_3$ with any vector $\mathbf{u} = (u^1, u^2, u^3) \in \mathbb{R}^3$. Then the matrix $R^{\lambda\alpha}_{\beta\mu} = u^0\, \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}\, (J^+_{\mathbf{v}})^\lambda_\mu\, (J^+_{\mathbf{u}})^\alpha_\beta$, satisfy all required commutation relations. The action of $SO(3)$ rotate $\mathbf{v}$ to a fixed direction $\widehat{\mathbf{u}}$ and a residual gauge freedom, a rotation around $\widehat{\mathbf{u}}$, removes one component of $\mathbf{u}$. We get families of noncommutative spaces governed by the deformation matrix

$$\text{(4)} \qquad R^{\lambda\alpha}_{\beta\mu} = u^0 \delta^\lambda_\mu \delta^\alpha_\beta + \mathrm{i}\, (J^+_1)^\lambda_\mu\, (u^1 J^+_1 + u^2 J^+_2)^\alpha_\beta,$$

and parameters constrained by $(u^0)^2 + \mathbf{u}^2 = 1$, a two-sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{S}^3/\mathbb{S}^1 = \mathbb{S}^2$.

These are quaternionic generalisations of the toric four-dimensional noncommutative spaces described in [2] for which the space of deformation parameter is $\mathbb{P}^1(\mathbb{R}) = \mathbb{S}^1/\mathbb{Z}_2 = \mathbb{S}^1$. In parallel to the complex case where there is an action of the torus $\mathbb{T}^2$, there is now an action of the torus $T^2_{\mathbb{H}} = U_1(\mathbb{H}) \times U_1(\mathbb{H}) = SU(2) \times SU(2)$ by $*$-automorphisms of the algebra $\mathcal{A}_R$. The mappings $x_1 \mapsto J^-_a x_1$, $x_2 \mapsto J^-_b x_2$ for $a, b \in \{1, 2, 3\}$ leave the algebra relations alone and thus define $*$-automorphisms of the $*$-algebra $\mathcal{A}_R$. This action of $U_1(\mathbb{H}) \times U_1(\mathbb{H})$ on $\mathcal{A}_R$ passes to the quotient by the ideal generated by the central elements $(x_1)^2$, $(x_2)^2$ giving an action of the quaternionic torus $U_1(\mathbb{H}) \times U_1(\mathbb{H})$ by $*$-automorphisms of the coordinate algebra

$$\mathcal{A}((T^2_{\mathbb{H}})_R) = \mathcal{A}_R / \big((x_1)^2 - 1, (x_2)^2 - 1\big)$$

of a noncommutative 'quaternionic torus' $(T^2_{\mathbb{H}})_R$. The action also passes to the quotient by the ideal generated by the central element $(x_1)^2 + (x_2)^2$ and thus defines an action of $U_1(\mathbb{H}) \times U_1(\mathbb{H})$ by $*$-automorphisms of the coordinate algebra $\mathcal{A}(\mathbb{S}^7_R) = \mathcal{A}_R / \big((x_1)^2 + (x_2)^2 - 1\big)$ of a noncommutative seven-sphere $\mathbb{S}^7_R$.

As shown in [3], with the diagonal action of $U_1(\mathbb{H}) \subset U_1(\mathbb{H}) \times U_1(\mathbb{H})$ on $\mathcal{A}(\mathbb{S}^7_R)$ one gets a $SU(2)$-principal bundles $\mathbb{S}^7_R \to \mathbb{S}^4_R$ on a noncommutative four-sphere.

Indeed, in parallel with (3) consider the two quaternions $x_1 = x_1^\mu e_\mu, x_2 = x_2^\alpha e_\alpha$, with commutation relations among their components governed by a matrix $R^{\lambda\alpha}_{\beta\mu}$ as in (1). Then, on the sphere $\mathbb{S}^7_R$ the vector-valued function

$$|\psi\rangle = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \tag{5}$$

has norm $\langle\psi,\psi\rangle = ||x_1||^2 + ||x_2||^2 = 1$ and we get a projection $p = |\psi\rangle\langle\psi|$, that is $p = p^* = p^2$. Define coordinate functions $Y = Y^0 e_0 + Y^k e_k$ and Y^4 by

$$Y^4 = ||x_2||^2 - ||x_1||^2 \qquad \text{and} \qquad \tfrac{1}{2}Y = x_2 x_1^*. \tag{6}$$

The condition $p^2 = p$ gives that Y^4 is central and leads to the conditions

$$-(Y^{0*}Y^k - Y^{k*}Y^0) + \varepsilon_{kmn}Y^{m*}Y^n = 0, \quad Y^0Y^{k*} - Y^kY^{0*} + \varepsilon_{kmn}Y^mY^{n*} = 0$$

for $k, r, m = 1, 2, 3$ and totally antisymmetric tensor ε_{krm}, and sphere relations

$$\sum\nolimits_{\mu=0}^{3} Y^{\mu*}Y^\mu + (Y^4)^2 = 1 = \sum\nolimits_{\mu=0}^{3} Y^\mu Y^{\mu*} + (Y^4)^2.$$

These also give that $\sum_{\mu=0}^3 Y^{\mu*}Y^\mu$ and $\sum_{\mu=0}^3 Y^\mu Y^{\mu*}$ are central. The elements Y^μ generate the $*$-algebra $\mathcal{A}(\mathbb{S}^4_R)$ of a four-sphere $\mathbb{S}^4_R$. This four-sphere $\mathbb{S}^4_R$ is the suspension (by the central element Y^4) of a three-sphere $\mathbb{S}^3_R$.

With $|\psi\rangle$ the vector-valued function in (5), let the action of a unit quaternion $w \in U_1(\mathbb{H}) \simeq SU(2)$ on $\mathbb{S}^7_R$ be obtained from the following action on the generators:

$$\alpha_w(|\psi\rangle) = |\psi\rangle w = \begin{pmatrix} x_2 w \\ x_1 w \end{pmatrix}. \tag{7}$$

Clearly, the projection p and then the algebra $\mathcal{A}(\mathbb{S}^4_R)$ are invariant for this action.

In general the action (7) is not by $*$-automorphisms of the coordinate algebra $\mathcal{A}(\mathbb{S}^7_R)$ since it does not preserve the commutation relations of $\mathbb{S}^7_R$. On the other hand, for the quaternionic deformations governed by the matrix in (4) and in particular for the corresponding noncommutative seven-sphere, as we have seen, there is a compatible action of $U_1(\mathbb{H}) \times U_1(\mathbb{H})$ by $*$-automorphisms of the corresponding coordinate algebra. It is shown in [3] that the corresponding algebra inclusion $\mathcal{A}(\mathbb{S}^4_R) \subset \mathcal{A}(\mathbb{S}^4_R)$ is a noncommutative $SU(2)$-principal bundle.

With commutation relations for the x's by the matrix (4), one has $Y^{\mu*} = \Lambda^\mu{}_\nu Y^\nu$ for $\Lambda \in \mathbb{M}_4(\mathbb{C})$ a symmetric unitary matrix. This matrix can be diagonalized by a real rotation S and further normalized by a factor of modulus one. The corresponding rodefinition of the generators gives that the sphere $\mathbb{S}^4_R \simeq \mathbb{S}^4_\theta$ that is it is isomorphic to a θ-deformation sphere, as the one in [2]

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Factorizations of spectral triples in unbounded KK-theory

Walter D. van Suijlekom
(joint work with Jens Kaad)

We start by explaining the natural link between noncommutative geometry and non-abelian gauge theories. This is mainly due to the fact that any noncommutative (involutive) algebra $\mathcal{A}$ gives rise to a non-abelian group of invertible (unitary) elements in $\mathcal{A}$: the gauge group. This has given rise to many applications in physics, such as to Yang–Mills theories [4, 5] and to the Standard Model of elementary particles [6].

Even though these examples deal with gauge theories on *commutative* background spaces, the gauge group and the gauge fields are defined along the general lines of [9] (*cf.* the more recent [8], valid for any real spectral triple for a C^*-algebra A. For instance, the gauge group is given in terms of the unitary elements $\mathcal{U}(A)$ in A, independent of a classical background space. In the physical applications of [4, 5, 6, 12, 7] —including extensions of them to the topologically non-trivial case [2, 3, 1]— the elements in $\mathcal{U}(A)$ are realized as automorphisms of a principal bundle, in perfect agreement with the usual description of gauge theories. However, in the general case the geometric picture appears to be less clear.

Basing ourselves on [18] we will explain that the gauge theory derived from any real spectral triple for the C^*-algebra A can always be described by means of bundles on a commutative background space. We will identify a subalgebra A_J in the center of A which by Gelfand duality is isomorphic to $C(X)$ for some compact Hausdorff topological space X. This turns A into a so-called $C(X)$-algebra [15] for which it is well-known that it can be identified with the C^*-algebra of continuous sections of a bundle $\mathfrak{B}$ of C^*-algebras on X (in general, this is an upper semi-continuous C^*-bundle, see [16, Appendix C] and references therein). This bundle $\mathfrak{B}$ will set the stage for the generalized gauge theory. We will show that the gauge group $\mathcal{U}(A)/\mathcal{U}(A_J)$ derived from the real spectral triple acts by vertical bundle automorphisms on this bundle, which agrees with the action of it on A by inner automorphisms. Moreover, under some additional conditions, we identify a group bundle whose space of continuous sections coincides with the gauge group. The gauge fields can be considered as sections of a bundle $\mathfrak{B}_\Omega$ constructed in much the same way as $\mathfrak{B}$, also carrying an action of the gauge group which agrees with the usual gauge transformation for gauge fields.

Besides the applications to Yang–Mills theory we consider the interesting class of toric noncommutative manifolds. They were obtained in [11] (see also [10]) by deformation quantization of a Riemannian spin manifold M along a torus action and are real spectral triples for the deformed C^*-algebras $C(M_\theta)$ derived by Rieffel in [17]. We identify the base space of our C^*-bundle with the orbit space for the torus action on M, and characterize the fiber C^*-algebra as noncommutative tori or subalgebras thereof). We show that the C^*-bundle is always continuous, as opposed to merely upper semi-continuous. Moreover, if the orbit space is simply connected, then the gauge group is isomorphic to the space of continuous sections of a group bundle on that orbit space that we explicitly determine, which in turn is isomorphic to the group of inner automorphisms. We end by a concrete study of two examples: the toric noncommutative spheres $\mathbb{S}^3_\theta$ and $\mathbb{S}^4_\theta$.

In the second part of this talk we will lift the above *topological* bundle picture to the *geometric* level, working in the setting of unbounded KK-theory. We do this by showing that also the Dirac operator D_{M_θ} on M_θ can be decomposed into a vertical operator D_V acting on the Hilbert module $E_{C_0(X_0)}$ of continuous sections of a Hilbert bundle, and a horizontal Dirac operator on the (principal) orbit space $X_0 := M_0/\mathbb{T}^n$ where $M_0 \subseteq M$ is the principal stratum for the torus action. This is based on [14] and is in line with the recent paper [13] in which we deal with factorizations of Dirac operators on almost-regular fibrations.

The final result can then be summarized as follows:

Up to unitary equivalence of C^*-correspondences (from $C_0((M_0)_\theta)$ to $\mathbb{C}$), we have the equality of selfadjoint operators

$$D_V \otimes 1 + \gamma \otimes_\nabla D_{X_0} = \overline{D_{M_0} - \frac{i}{8} c_{M_0}(\Omega)},$$

where $D_{M_0} : \Gamma^\infty_c(M_0, \mathcal{S}_{M_0}) \to L^2(M_0, \mathcal{S}_{M_0})$ is the Dirac operator and $c_{M_0}(\Omega) : \Gamma^\infty_c(M_0, \mathcal{S}_{M_0}) \to L^2(M_0, \mathcal{S}_{M_0})$ denotes Clifford multiplication (*cf.* [13, Section 3.3] for the precise formula) by the curvature 2-form $\Omega : \Lambda^2(T_H M_0) \otimes T_V M_0 \to M_0 \times \mathbb{C}$ defined by

$$\Omega(X, Y, Z) = \langle [X, Y], Z \rangle_{M_0},$$

for all real horizontal vector fields X, Y and every real vertical vector field Z.

Moreover, if $\imath : M_0 \to M$ denotes the (torus-equivariant) inclusion of the principal stratum we have the identity

$$\imath^*[D_{M_\theta}] = [(D_V)_\theta] \widehat{\otimes}_{C_0(X_0)} [D_{X_0}]$$

in the KK-group $KK(C_0((M_0)_\theta), \mathbb{C})$. The curvature 2-form thus arises as an obstruction for having a tensor sum decomposition in unbounded KK-theory which can not be detected at the level of bounded KK-theory.

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The minimal exact crossed product

Rufus Willett
(joint work with Alcides Buss, Siegfried Echterhoff)

The Baum-Connes conjecture (for a group G, with arbitrary coefficients) posits that for every G-C^*-algebra A, a certain *assembly map*

$$\mu : K_*^{\mathrm{top}}(G; A) \to K_*(A \rtimes_r G)$$

is an isomorphism. The conjecture is intrinsically interesting in its own right, and has many applications in algebra, geometry, topology, and representation theory.

For fixed G, the left and right hand sides can both be considered as functors from the category of G-C^*-algebras to the category of (graded) abelian groups. Now, the left hand side of the Baum-Connes conjecture is always an exact functor, meaning that it takes short exact sequences of G-C^*-algebras to long exact sequences of abelian groups. Higson, Lafforgue, and Skandalis [4] were able to show, however, that for certain highly pathological groups constructed by Gromov (see Osajda's work [6] for a more refined result, and proof), the left hand side of the Baum-Connes conjecture is not an exact functor. This implies in particular that the Baum-Connes conjecture with coefficients fails for such groups.

Now, Paul Baum suggested that the it might be possible to 'fix' the Baum-Connes conjecture by replacing the reduced crossed product $\rtimes_r$ with some other crossed product functor with better exactness properties. This program was carried out by Baum, Guentner, and myself [2] where we showed that if one takes the minimal crossed product functor $\rtimes_{\mathcal{E}}$ amongst all of those that are exact and are compatible with Morita equivalences in a suitable sense, then it can indeed be used to 'fix' the Baum-Connes conjecture: precisely there are no counterexamples to the reformulated conjecture, and some previous counterexamples become confirming examples.

However, the reformulated Baum-Connes conjecture with associated assembly map

$$K_*^{\mathrm{top}}(G;A) \to K_*(A \rtimes_{\mathcal{E}} G)$$

remained mysterious. In particular, we did not know what the exotic group algebra $C^*_{\mathcal{E}}(G) := \mathbb{C} \rtimes_{\mathcal{E}} G$ was: one would like it to be the reduced group C^*-algebra (as one would like not to change the conjecture more than necessary, and there are no known counterexamples in this case); however, even this was not clear. Moreover, although the crossed product $A \rtimes_{\mathcal{E}} G$ is well-defined, it was not at all clear how it might be concretely realised, or computed, in particular cases.

The point of my talk was to exposit recent work with Buss and Echterhoff where we remedy some of these deficiencies. The key idea (developed from a suggestion of Ozawa) is to show that $A \rtimes_{\mathcal{E}} G$ can be defined as the completion of the algebraic crossed product $C_c(G,A)$ for the largest possible norm coming from the canonical embedding

$$C_c(G,A) \hookrightarrow \frac{B \rtimes_r G}{I \rtimes_r G},$$

where B and I range over all possible $G-C^*$-algebras appearing in a short exact sequence $0 \to I \to B \to A \to 0$ with A appearing on the right.

The key step in proving that this works is a proof that, for a crossed product functor $\rtimes_\mu$, the following are equivalent:

(1) $\rtimes_\mu$ is half exact;
(2) for any G-C^*-algebra, the natural map $C_c(G,A^{**}) \to (A \rtimes_\mu G)^{**}$ extends to $A^{**} \rtimes_\mu G$;
(3) $\rtimes_\mu G$ is exact.

This result is new even for $\rtimes_\mu = \rtimes_r$ (although partly inspired by work of Matsumura [5] in that case), and seems interesting in its own right; it is a close analogue of work of Archbold and Batty on tensor products [1].

Once we know the equivalences above, it is not too difficult to show that the new description gives $\rtimes_{\mathcal{E}}$. Several interesting consequences follow fairly directly: in particular that $C^*_{\mathcal{E}}(G)$ is indeed equal to the reduced group C^*-algebra $C^*_r(G)$.

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Orbit Integrals and the Connes-Kasparov Conjecture

Xiang Tang

(joint work with Nigel Higson and Yanli Song)

Let G be a real reductive group and K be its maximal compact subgroup. Let $X = G/K$ be the associated homogeneous space. We assume that X has even dimension, and is equipped with a spin structure with the associated spinor bundle $\mathcal{S}^{\pm}$. Let V_μ be an irreducible K-representation, and $\mathcal{V}_\mu := (G \times V_\mu)/K$ be the associated G-equivariant vector bundle over X. In this talk, we study the index theory associated the twisted Dirac operator D_μ on X,

$$D_\mu : \mathcal{S}^+ \otimes \mathcal{V}^\mu \to \mathcal{S}^- \otimes \mathcal{V}_\mu.$$

The kernel and cokernel of D_μ are naturally unitary G-representations. The index of D_μ is defined to be the element

$$\mathrm{Ind}(D_\mu) \in K_0(C^*_r(G)),$$

where $C^*_r(G)$ is the reduced group C^*-algebra of G.

Let $\mathrm{Rep}(K)$ be the representation ring of the compact group K. The Connes-Kasparov isomorphism theorem states that the index map

$$\mathrm{Ind} : \mathrm{Rep}(K) \to K_0(C^*_r(G)),\ V_\mu \mapsto \mathrm{Ind}(D_\mu),$$

is an isomorphism of abelian groups. With the contribution of many authors, we know that this isomorphism holds true for all almost connected Lie groups, [4, 6, 7] . In this talk, we present some results using cyclic theory to study the above isomorphism.

Let tr be the standard trace on $C^*_r(G)$ defined by $\mathrm{tr}(f) = f(e)$, where e is the identity of G. tr defines a linear functional on $K_0(C^*_r(G))$. In [5], Connes and Moscovici obtained a topological formula for $\mathrm{tr}(\mathrm{Ind}(D_\mu))$, and furthermore they proved that $\mathrm{tr}(\mathrm{Ind}(D_\mu))$ is equal to the formal degree of the irreducible G-representation on the kernel $\mathrm{Ker}(D_\mu)$ of the operator D_μ. When μ is regular, the kernel $\mathrm{Ker}(D_\mu)$ is not trivial, and $\mathrm{tr}(\mathrm{Ind}(D_\mu))$ is not zero. However, when μ is singular, the kernel $\mathrm{Ker}(D_\mu)$ is trivial, and $\mathrm{tr}(\mathrm{Ind}(D_\mu))$ vanishes. Our main idea is to use orbit integrals from representation theory, [1]-[3], to detect such an element $\mathrm{Ind}(D_\mu)$ in $K_0(C_r^*(G))$.

Assume that H is a compact maximal cartan subgroup of G. Fix a Haar measure on G. For $f \in C_c(G)$, consider the following orbit integrals

$$F_f^H(h) := \Delta_H(h) \int_{G/H} f(ghg^{-1})dh,$$

where h is a regular element in H, and $\Delta(h)$ is the Weyl denominator defined by positive roots, i.e.

$$\Delta_H(h) = \prod_{\alpha \text{ positive root}} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}).$$

When h is regular, the orbit integral $f \mapsto F_f^H(h)$ defines a trace on the Harish-Chandra Schwartz algebra $\mathcal{S}(G)$, and therefore induces a linear functional on $K_0(C_r^*(G))$, as $\mathcal{S}(G)$ is a dense subalgebra of $C_r^*(G)$ closed under holomorphic functional calculus. We prove the following theorem about orbit integrals.

Theorem. (Higson-Song-Tang) *When G is equal rank, and G/K is even dimensional and spin, the orbit integrals F^H defines an isomorphism*

$$F^H : K_0(C_r^*(G)) \to \mathrm{Rep}(K).$$

As a corollary, we obtain a new proof of the following isomorphism theorem.

Corollary. *Under the same assumptions as the above Theorem, the composition of* Ind *and F^H is the identity modulo a sign factor, i.e.*

$$F^H \circ \mathrm{Ind} = \pm I.$$

Therefore, the index map $\mathrm{Ind} : \mathrm{Rep}(K) \to K_0(C_r^*(G))$ *is an isomorphism with the inverse defined by the orbit integral F^H.*

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Inertia Groupoids, their singularity structure and Hochschild homology

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(joint work with C. Farsi, Ch. Seaton, H.B. Posthuma, X. Tang)

The loop space $\Lambda_0 G$ of a Lie groupoid $G \rightrightarrows M$ is defined as the space of loops $\{g \in G \mid s(g) = t(g)\}$, where $s, t : G \to M$ denote the source and target map, respectively. The groupoid acts on this space by conjugation. The corresponding action groupoid ΛG is called the *inertia groupoid* of G and is the object of our study. As one checks immediately, its space of arrows is given by $\Lambda_1 G = \{(h, g) \in G \times \Lambda_0 G \mid s(h) = t(g)\}$, the source map $s^{\Lambda G}$ is projection onto the second coordinate and the target map $t^{\Lambda G}$ is given by $(h, g) \mapsto hgh^{-1}$. In the particular case where the underlying groupoid G is the action groupoid of a compact Lie group H on a manifold M, the loop space is given by $\Lambda_0(H \ltimes M) = \{(h, p) \in H \times M \mid h \cdot p = p\}$, i.e. as the union of all $\{h\} \times M^h$, where M^h denotes the fixed point manifold of the group element h. In the case where H is a finite group, $\Lambda_0(H \ltimes M)$ thus is the disjoint finite union of manifolds with possibly different dimensions. The inertia groupoid $\Lambda(H \ltimes M)$ then is again, like $H \ltimes M$, a proper étale Lie groupoid. In the non-finite case, though, $\Lambda_0(H \ltimes M)$ is in general a singular space. By application of the slice theorem for a compact Lie group action it has been shown in [5] that locally $\Lambda_0(H \ltimes M)$ has the structure of a semialgebraic set. This then entails that $\Lambda_0(H \ltimes M)$ possesses a minimal Whitney stratification. More generally, it has been shown in [6] that for every proper Lie groupoid the associated inertia groupoid has the structure of a so-called differentiable stratified groupoid. Each stratum of such a groupoid is a Lie groupoid itself. One particular difficulty in regard to the singularity structure of inertia groupoids is that the well-known stratification of a manifold with a compact Lie group action by orbit types does not work for inertia groupoids. Using the concept of Cartan subgroups an explicit Whitney stratification of the loop space and the inertia groupoid of a proper Lie groupoid was given in [6]. In several examples this stratification coincides with the minimal Whitney stratification. We call it the Cartan orbit type stratification of the loop space. In addition, a de Rham theorem for loop spaces of proper Lie groupoids has been derived in [6]. Note that in general a de Rham theorem does not hold for singular (differentiable) stratified spaces.

The orbit space of the inertia groupoid of a Lie groupoid $G \rightrightarrows M$, i.e. the quotient $\Lambda_0 G/G$, is called the inertia space of G and denoted $\widetilde{M/G}$. This space inherits from the loop space the Cartan orbit type stratification, see [6]. In the case where the underlying groupoid is proper and étale, the inertia space $\widetilde{M/G}$ is, like the orbit space M/G, itself an orbifold and is called the inertia orbifold associated to M/G. Such inertia orbifolds appear at various places in the literature, even though sometimes under a different or not a particular name. For example, the inertia orbifold of an orbifold serves as a bookkeeping device for the contribution of singularities to the analytic or algebraic index over the orbifold. In both cases the analytic respectively algebraic index can be identified with a topological orbifold index that is the integral of a topologically or geometrically defined form over the (cotangent bundle) of the inertia orbifold. See [7] for the original work on the index theorem for orbifolds, and [8] for details on the algebraic orbifold index theorem. The inertia orbifold also appears naturally in the computation of the Hochschild and cyclic homology of the convolution algebra of a proper étale Lie groupoid, see [3, 4]. The question now arises in how far these results can be generalized to the proper Lie groupoid case.

In his paper [2] and unpublished preprint [1], Brylinski outlined a path to compute the Hochschild and cyclic homology of the (smooth) convolution algebra $\mathcal{A}$ of a smooth H-action on the manifold M. Brylinski claimed in [1] that the Hochschild homology group $HH_k(\mathcal{A})$ coincides with the space of basic relative forms. By definition, this space consists of smooth and H-invariant families $\omega = (\omega_h)_{h\in H}$ of forms $\omega_h \in \Omega^k(M^h)$ on the fixed point manifolds M^h such that each ω^h is horizontal. The latter hereby means that contractions with fundamental vector fields of elements of the Lie algebra $\mathrm{Lie}(H^h)$ vanish.

In view of a possible generalization to Lie groupoids, Brylinskis's basic relative forms have been interpreted in [9] as particular Grauert–Grothendieck forms on the loop space. Let us explain this in some more detail. Denote by $\mathcal{J} \subset \mathcal{C}^\infty(H \times M)$ the ideal of smooth functions vanishing on the loop space $\Lambda_0(H \ltimes M)$. The complex of Grauert–Grothendieck forms on the loop space is now defined as the differential graded algebra

$$\Omega^\bullet_{\mathrm{GG}}(\Lambda_0(H \ltimes M)) = \Omega^\bullet(H \times M)/\mathcal{J}\Omega^\bullet(H \times M) + d\mathcal{J} \wedge \Omega^{\bullet-1}(H \times M)\ .$$

Next one needs a relative version of that complex. It is defined as

$$\Omega^\bullet_{\mathrm{rel}}(\Lambda_0(H \ltimes M)) = \Gamma^\infty(\wedge^\bullet s^*T^*M)/\mathcal{J}\Gamma^\infty(\wedge^\bullet s^*T^*M) + d_{\mathrm{rel}}\mathcal{J} \wedge \Gamma^\infty(\wedge^{\bullet-1} s^*T^*M),$$

where $s : H \times M \to M$ is projection onto the second coordinate and d_{rel} is the relative exterior derivative, see [9]. Horizontal relative forms are those elements of $\Omega^\bullet_{\mathrm{rel}}(\Lambda_0(H \ltimes M))$ which are images of sections of alternating powers of the pull-back of the co-normal bundle on T^*M via the source map s. The subspace of H-invariant horizontal relative Grauert–Grothendieck forms finally is called the space of basic relative Grauert–Grothendieck forms. Brylinski's conjecture can be reformulated in these terms as follows:

Conjecture. *The Hochschild homology $HH_\bullet(\mathcal{A})$ of the (smooth) convolution algebra $\mathcal{A}$ of the transformation groupoid $H \ltimes M$ is given by $\Omega^\bullet_{\text{bas-rel}}(\Lambda_0(H \ltimes M))$, the space of basic relative Grauert–Grothendieck forms on the loop space.*

Using methods of real algebraic geometry we succeeded in [10] to prove the conjecture for the case of the standard circle action on the plane or in other words for the action groupoid $S^1 \ltimes \mathbb{R}^2$. Together with the orbifold case, where the claim is verified due to the papers [3, 4], it thus follows that the conjecture holds true for all S^1-actions. Using the Cartan orbit type stratification of loop spaces from [6] it appears feasible to verify the conjecture for arbitrary compact Lie group actions.

Finally, a generalization of the conjecture to proper Lie groupoids comes within reach by the concept of relative Grauert–Grothendieck forms.

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