Mathematisches Forschungsinstitut Oberwolfach

Report No. 36/2018

DOI: 10.4171/OWR/2018/36

Mathematical General Relativity

Organised by
Carla Cederbaum, Tübingen
Mihalis Dafermos, Princeton
Jim Isenberg, Eugene
Hans Ringström, Stockholm

5 August – 11 August 2018

Abstract. General Relativity is one of the triumphs of twentieth century physics. Its spectacular predictions include gravitational waves, black holes, and spacetime singularities. The mathematical study of this theory leads to deep problems connecting the areas of partial differential equations, geometry and topology with physics. The talks of the workshop illustrated the rapid progress in this subject over the last few years.

Mathematics Subject Classification (2010): 83-XX, 35-XX, 53-XX.

Introduction by the Organisers

The workshop Mathematical Aspects of General Relativity was organised by Carla Cederbaum (Tübingen), Mihalis Dafermos (Cambridge/Princeton), Jim Isenberg (Eugene) and Hans Ringström (Stockholm). The over 50 participants represented a wide selection of different research areas connected to the general theory of relativity, and roughly half of them gave talks at the workshop.

Proving nonlinear stability for solutions of the vacuum Einstein equations (without symmetry assumptions) has been a long-term goal of the field, and the progress reported in previous workshops is now finally converging to a complete solution. In this direction, Martin Taylor discussed a proof (in progress, joint with Holzegel, Rodnianski and Dafermos) of the full nonlinear stability of the Schwarzschild solution, the first non-trivial asymptotically flat black hole solution to be discovered. Andras Vasy described his proof (joint with Hintz) of stability of very slowly rotating Kerr-de Sitter spacetimes (for the case of positive cosmological constant). Peter Hintz discussed a new proof of the non-linear stability of Minkowski space.

Finally, on this theme, Pau Figueras presented beautiful numerical work exhibiting various novel *instabilities* of higher dimensional black holes. These numerics may serve as inspiration for mathematical work to be presented at future workshops!

The Einstein–Vlasov and Vlasov–Poisson systems are of central interest in physics, since they can be used to model, e.g., galaxies. One central question concerning solutions to these systems is whether they are stable or not. This question was addressed in a talk by Mahir Hadzic. In particular, he described an instability theory for self gravitating relativistic matter systems.

In addition to stability problems, important progress has been made in understanding solutions of the vacuum Einstein equations with no (or limited) symmetry in various singular regimes. Jonathan Luk presented joint work with Van de Moortel concerning the interaction of impulsive gravitational waves with *three* singular fronts, while Y. Shlapentokh-Rothman presented joint work with Rodnianski concerning asymptotically self-similar solutions.

As discussed in several previous workshop of this series, Anti-de Sitter space has been conjectured to be *unstable*, in stark contrast to Minkowski and de Sitter space. Though numerical evidence has more recently been given for the validity of this conjecture, it has remained elusive to prove because the instability mechanism is a purely non-linear effect. Georgios Moschidis discussed the first rigorous proof of this instability, accomplished in the context of the Einstein–Vlasov system under spherical symmetry. On the other hand, Stephen Green discussed numerical work connected to so called "islands of stability".

Black hole interiors remain one of the most intriguing domains for general relativity. Sung-Jin Oh discussed his joint work with Luk concerning *weak null singularities* in black hole interiors, giving finally a complete proof of a suitable formulation of Strong Cosmic Censorship for the Einstein–Maxwell-scalar field eqations under spherical symmetry. Christoph Kehle presented joint work with Shlapentokh-Rothman concerning a scattering theory for the wave equation on Reissner–Nordström.

The formulation of strong cosmic censorship requires the notion of "maximal Cauchy development". In his talk, Jan Sbierski explained the subtleties in defining this object for some of the most classical equations of mathematical physics.

The causal structure plays a central role in the understanding of the asymptotic behaviour of solutions to Einstein's equations. In the standard models of the universe, particle horizons form in the direction of the big bang. However, already in 1969, Misner suggested that in, e.g., Bianchi type IX solutions, particle horizons would not form. In recent work, presented during the workshop, Bernhard Brehm demonstrated that for Lebesgue almost all Bianchi type IX initial data, particle horizons form. However, he also presented heuristics indicating that for Baire generic initial data, particle horizons do not form.

Back reaction is a topic of current interest in physics. During the workshop it was discussed in the presentation by Cécile Huneau. In joint work with Jonathan Luk, she has explored the relation between weak limits of solutions to Einstein's vacuum equations and solutions to the Einstein null dust system. In particular, Huneau and Luk have demonstrated that (under suitable assumptions) solutions to the Einstein null dust system can be obtained as weak limits of families of solutions to Einstein's vacuum equations. One of the basic examples of this phenomenon is obtained in the class of polarised Gowdy solutions. A generalisation of this class is the T^2-symmetric solutions. The future asymptotics of solutions to this family was discussed in the presentation of Adam Layne. In joint work with Beverly Berger and James Isenberg, he has identified an attractor for the future asymptotics. However, the attractor does not represent a known explicit solution.

Gregory Galloway presented a result joint with Eric Ling giving an interesting connection between the topology of cosmological spacetimes and the occurrence of singularities. Roughly speaking, the result states that in a globally hyperbolic spacetime satisfying the null energy condition and having a smooth compact Cauchy hypersurface with a positive definite second fundamental form, the following holds. Either the Cauchy hypersurfaces are spherical spaces or the spacetime is past causally geodesically incomplete. In a related talk, Melanie Graf presented singularity theorems for $C^{1,1}$-metrics. A topological understanding of photon spheres in Kerr spacetimes was presented by Sophia Jahns, relying on a lift to the phase space and using methods related to some of those of central importance in the result presented by Galloway (joint with Carla Cederbaum).

Henri Roesch discussed his result on the Penrose inequality for (rather) general null hypersurfaces in 4-dimensional spacetimes. His method of proof heavily relies on a new quasi-local notion of mass that is tailored to converge to Bondi mass and has very useful monotonicity properties.

A number of talks focused on initial data for Einstein's equations with prescribed asymptotic behavior. Many talks focused on the special case of Riemannian manifolds, typically assuming that those have non-negative scalar curvature, corresponding to the dominant energy condition in the spacetime generated from the Riemannian manifold, assuming time symmetry.

In this context, Richard Schoen presented his groundbreaking result with Shing-Tung Yau settling the positive mass theorem for asymptotically flat Riemannian manifolds obeying the dominant energy condition (non-negative scalar curvature, in this setup) in arbitrary dimension without any topological conditions. Working instead in the asymptotically hyperbolic context, but still in arbitrary dimension, Romain Gicquaud showed that certain algebraic ideas can be used to define novel, mass-like covariants to be determined from the asymptotics of the metric (joint with Julien Cortier and Mattias Dahl).

Christina Sormani (joint work with a number of co-authors) and Jeffrey Jauregui (joint work with Dan Lee) each presented subtle results on stability / almost rigidity of general relativistic inequalities for Riemannian manifolds such as the positive mass theorem, using the relatively new, tailor-made notion of intrinsic flat convergence suggested by Sormani and Wenger.

Working with methods from Riemannian geometry and also with those developed in the context of curvature flows, Martín Reiris presented an extensive classification result for static spacetimes with an axisymmetry condition but without any asymptotic assumptions.

Furthermore, Armando Cabrera Pacheco presented a gluing construction for Riemannian extensions allowing to give an upper estimate on the quasi-local Bartnik mass of a 2-dimensional surface with prescribed metric and mean curvature, defined as the infimum of the ADM-masses of all "admissible" Riemannian manifolds realizing the given geometric data on the 2-surface. This construction extends and refines a construction suggested by Christos Mantoulidis and Richard Schoen for minimal surfaces in the asymptotically flat case both to the context of constant mean curvature surfaces and to the asymptotically hyperbolic case (joint with Carla Cederbaum, Stephen McCormick, Pengzi Miao).

More generally, for arbitrary asymptotically flat initial data sets satisfying the dominant energy condition, several results were presented that demonstrate the progress the field has made in the last years: Both Ye Sle Cha (partially joint work with Marcus Khuri) and Eugenia Gabach Clement (partially joint work with Sergio Dain and also with others) discussed recent progress made towards proving geometro-physical inequalities such as the quasi-local area-angular momentum and global angular momentum-mass inequalities for black holes.

Anna Sakovich presented a novel foliation near infinity for asymptotically flat initial data sets in three dimensions that generalizes the canonical foliation by constant mean curvature surfaces initially studied by Gerhard Huisken and Shing-Tung Yau and that is intimately linked with the mathematically consistent definition of the center of mass of an isolated system (joint with Carla Cederbaum).

The participants of the workshop benefitted a lot from an open problem session led by Robert Wald, where he discussed the subtle intricacies of well-posing perturbative corrections to surprisingly simple equations.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Gregory Galloway in the "Simons Visiting Professors" program at the MFO.

Workshop: Mathematical General Relativity

Table of Contents

Abstracts

The positive mass theorem again

Richard Schoen
(joint work with Shing-Tung Yau)

The purpose of this talk is to describe and present some features of the paper [5] which extends the minimal hypersurface proof of the positive mass theorem to all dimensions. We refer the reader to [5] for a more complete history and background on the problem.

We consider an asymptotically flat Riemannian manifold (M^n, g) $(n \geq 3)$. By this we mean that there is a compact subset K of M so that the complement $M \setminus K$ is diffeomorphic to the exterior of a ball in $\mathbb{R}^n$ in the sense that there are coordinates $x^1, \ldots, x^n$ on $M \setminus K$ so that the metric coefficients g_{ij} in these coordinates satisfy the falloff

$$g_{ij} = \delta_{ij} + O_{2,\alpha}(|x|^{-p}), \quad R_g = O_{0,\alpha}(|x|^{-q})$$

where $p > (n-2)/2$, $q > n$, $\alpha \in (0,1)$ and the notation $O_{k,\alpha}(|x|^{-s})$ means that the indicated function is bounded by a constant times $|x|^{-s}$, its first k partial derivatives and the α-Hölder coefficient of the kth order derivatives decay at the corresponding rates.

For such a manifold it is possible to define the ADM mass by the expression

$$m = \lim_{\sigma \to \infty} c_n \int_{S_\sigma} (\partial_j g_{ij} - \partial_i g_{jj}) \nu^i \, da$$

where $S_\sigma = \{x : \ |x| = \sigma\}$ and ν is the outward unit normal to S_σ with respect to the euclidean metric. The positive constant c_n is chosen so that the n-dimensional Schwarzschild metric $g_m = (1 + \frac{m}{2|x|^{n-2}})^{4/(n-2)}\delta_{ij}$ has ADM mass m. For example $c_3 = 1/(16\pi)$.

The simplest form of the positive mass theorem is the following.

Theorem 1. *Let (M^n, g) be an asymptotically flat Riemannian manifold with $R_g \geq 0$. We then have $m \geq 0$ and $m = 0$ if and only if (M, g) is isometric to $(\mathbb{R}^n, \delta)$, the euclidean space of dimension n.*

The original conjecture was the case $n = 3$, but the statement makes sense for any $n \geq 3$ and the higher dimensional case is of both mathematical and physical importance. There are three methods which have been successful for proving the result. The first was the minimal hypersurface approach developed by the speaker and S-T. Yau [3]. The second method uses the Dirac operator and was introduced by E. Witten [6]. There is also a third method, the inverse mean curvature flow proposed by R. Geroch and rigorously developed by G. Huisken and T. Ilmanen [1], which gives a stronger result, but only seems to work for $n = 3$.

In general dimensions both the minimal hypersurface and Dirac operator arguments have limitations. The Dirac operator argument requires the condition that M be a spin manifold while the minimal hypersurface argument only works directly for $n \leq 7$ because of the possibility of singularities in the minimal hypersurfaces in higher dimensions. There is a separate argument which extends the method to the $n = 8$ case (see the discussion in [5]).

There is a technical simplification based on the early work [4], which simplifies the asymptotics, and the work of J. Lohkamp [2]. The result is a compactification argument.

Theorem 2. *Let M_0^n be any closed n-manifold and let $M = M_0 \# T^n$. Then M cannot carry a metric with positive scalar curvature.*

The conclusion of [4] and [2] is that Theorem 2 implies Theorem 1. Technically Theorem 2 only implies the weak form of Theorem 1 which says that $m \geq 0$. The strong form can then be derived by well known arguments.

We outline the argument for the proof of Theorem 2 by introducing the notion of minimal k-slicings. We may assume that M is orientable and we observe that there are closed 1-forms $\omega^1, \ldots, \omega^n$ such that $\int_M \omega^1 \wedge \ldots \wedge \omega^n = 1$. These are gotten by pulling back the basic 1-forms on T^n by a degree one map from M to T^n which collapses M_0 to a point. We may then construct a nested family of oriented hypersurfaces

$$\Sigma_k \subset \Sigma_{k+1} \subset \ldots \subset \Sigma_{n-1} \subset M$$

for $1 \leq k \leq n-1$. We construct these hypersurfaces by minimization of a weighted volume functional. First construct Σ_{n-1} by minimizing volume among oriented hypersurfaces Σ of M satisfying $\int_\Sigma \omega^1 \wedge \ldots \wedge \omega^{n-1} = 1$. We then choose a positive lowest eigenfunction u_{n-1} of the second variation form $S_{n-1}(\varphi, \varphi)$ and let $\rho_{n-1} = u_{n-1}$ be a weight function. Construct Σ_{n-2} by minimizing the weighted volume $V_{\rho_{n-1}}$ over oriented hypersurfaces Σ in Σ_{n-1} subject to the condition that

$$\int_\Sigma \omega^1 \wedge \ldots \wedge \omega^{n-2} = 1.$$

Let u_{n-2} be a positive first eigenfunction of the second variation form S_{n-2} and set $\rho_{n-2} = u_{n-1}\rho_{n-1}$. We then continue this argument down to dimension k. To summarize, Σ_j is a minimizer of $V_{\rho_{j+1}}$ subject to the homological constraint, and $\rho_j = u_j \rho_{j+1}$. Such a family of hypersurfaces is called a *minimal k-slicing.*

The basic geometric theorem about such objects is the following.

Theorem 3. *If M has positive scalar curvature and*

$$\Sigma_k \subset \ldots \subset \Sigma_{n-1} \subset M$$

is a minimal k-slicing, then Σ_k with its induced metric as a submanifold of M is Yamabe positive. In particular if $k = 2$, then Σ_2 is homeomorphic to a union of 2-spheres.

A basic example of a minimal 2-slicing in the manifold $M = S^2 \times T^{n-2}$ is given by

$$S^2 \subset S^2 \times S^1 \subset S^2 \times T^2 \subset \ldots \subset S^2 \times T^{n-3} \subset M.$$

We see that M has positive scalar curvature if we take the product metric.

The proof of Theorem 2 follows from Theorem 3 because the condition $\int_{\Sigma_2} \omega^1 \wedge \omega^2 = 1$ is not possible if Σ_2 is simply connected (each ω^i would be exact on Σ_2).

For $n \geq 8$ there is the possibility that Σ_{n-1} could have a closed singular set $\mathcal{S}_{n-1}$ of Hausdorff dimension at most $n-8$. The strategy for extending the proof to all dimensions is to construct the minimal slicings in the presence of singularities and to show that the two dimensional slice is completely regular. The proof involves an existence and a regularity theorem. If we denote by $\mathcal{S}_k$ the singular set of Σ_k, the regularity theorem says that $dim(\mathcal{S}_k) \leq k-3$. The proof involves construction of 'tangent' slicings at singular points and the regularity of homogeneous minimal 2-slicings.

As to the existence part, we must construct the eigenfunctions which comprise the weights, and the slices which minimize allowing the possibility of small singular sets. To construct the eigenfunctions it seems necessary to modify the second variation forms to make them more coercive. For example if we take a volume minimizing hypersurface Σ in a manifold M, its second variation form is

$$S(\varphi, \varphi) = \int_\Sigma [\|\nabla\varphi\|^2 - (Ric_M(\nu,\nu) + \|A\|^2)\varphi^2]\ d\mu$$

where A is the second fundamental form of Σ in M. In order to diagonalize this form on the unit sphere of L^2 we must show that a sequence φ_i with $S(\varphi_i, \varphi_i)$ and $\|\varphi_i\|_{L^2}$ bounded has a subsequence $\varphi_{i'}$ which converges in L^2 norm. What must be shown is that the φ_i cannot concentrate on the singular set $\mathcal{S}$. This seems difficult to do since the two terms $\|\nabla\varphi\|^2$ and $\|A\|^2\varphi^2$ can both become large and cancel keeping S bounded. The way this is handled is to observe that only half of the term $\|A\|^2\varphi^2$ term is needed for the geometric conclusions, so we can replace S by a form $Q = S + \alpha\int_\Sigma \|A\|^2\varphi^2\ d\mu$ with $\alpha \in (0, 1/2)$. Having a bound on $Q(\varphi, \varphi)$ then gives a bound on $\int_\Sigma \|A\|^2\varphi^2\ d\mu$ and hence a bound on $\int_\Sigma \|\nabla\varphi\|^2\ d\mu$. Since minimal hypersurfaces have a uniform Sobolev constant it would then follow that there is a bound on $\int_\Sigma \varphi^{2n/(n-2)}\ d\mu$. Such a bound prevents the L^2 norm from concentrating on sets of small volume. There is an analogous way of modifying the forms S_j for $j < n-1$ so that they can be diagonalized and so that the geometric conclusions remain correct.

References

[1] G. Huisken, T. Ilmanen, *The Inverse Mean Curvature Flow and the Riemannian Penrose Inequality*, J. Diff. Geom **59** (2001), 353–437.

[2] J. Lohkamp, *Scalar curvature and hammocks*, Math. Ann. **313** (1999), 385–407.

[3] R. Schoen, S-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), 45–76.

[4] R. Schoen, S-T. Yau, *Positivity of the total mass of a general space-time*, Phys. Rev. Lett. **43** (1979), 1457–1459.

[5] R. Schoen, S-T. Yau, *Scalar curvature and minimal hypersurface singularities*, arXiv:1704.05490.

[6] E. Witten, *A New Proof of the Positive Energy Theorem*. Comm. Math. Phys. **80** (1981), 381–402.

The stability of Kerr–de Sitter black holes

Andras Vasy
(joint work with Peter Hintz)

According to Einstein's theory of General Relativity, a vacuum spacetime with cosmological constant $\Lambda \in \mathbb{R}$ is a $(3+1)$-dimensional manifold M equipped with a Lorentzian metric g satisfying the Einstein vacuum equation

$$\mathrm{Ric}(g) + \Lambda g = 0. \tag{1}$$

A Kerr–de Sitter (KdS) spacetime, discovered by Kerr [Ker63] and Carter [Car68], models a stationary, rotating black hole within a universe with $\Lambda > 0$: far from the black hole, the spacetime behaves like de Sitter space with cosmological constant Λ, and close to the event horizon of the black hole like a Kerr black hole. Fixing $\Lambda > 0$, a $(3+1)$-dimensional KdS spacetime (M°, g_b) depends, up to diffeomorphism equivalence, on two real parameters, namely the mass $M_\bullet > 0$ of the black hole and its angular momentum a, which we combine into a single parameter $b = (M_\bullet, a)$. The Kerr–de Sitter family of black holes is then a smooth family g_b of stationary Lorentzian metrics, parameterized by $b = (M_\bullet, a)$, on a fixed 4-dimensional manifold $M^\circ \cong \mathbb{R}_{t_*} \times (0,\infty)_r \times \mathbb{S}^2$ solving the equation (1). The Schwarzschild–de Sitter (SdS) family is the subfamily of the KdS family with $a = 0$; a SdS black hole describes a static, non-rotating black hole. We point out that according to the currently physically accepted ΛCDM model, the cosmological constant is indeed positive in our universe.

The equation (1) is a non-linear second order partial differential equation for the metric tensor g. Due to the diffeomorphism invariance of this equation, the formulation of a well-posed initial value problem is more subtle than for (non-linear) wave equations. This was first accomplished by Choquet-Bruhat [CB52], who with Geroch [CBG69] proved the existence of maximal globally hyperbolic developments for sufficiently smooth initial data. The initial data are a triple (Σ_0, h, k), consisting of i) a 3-manifold Σ_0, ii) a Riemannian metric h on Σ_0, iii) a symmetric 2-tensor k on Σ_0, subject to the *constraint equations*, which are the Gauss–Codazzi equations on Σ_0 implied by (1). Fixing Σ_0 as a submanifold of M°, a metric g satisfying (1) is then said to solve the initial value problem with data (Σ_0, h, k) if i) Σ_0 is spacelike with respect to g; ii) h is the Riemannian metric on Σ_0 induced by g; iii) k is the second fundamental form of Σ_0 within M°. Our main result concerns the *global non-linear asymptotic stability of the KdS family as solutions of the initial value problem for* (1); we prove this for slowly rotating black holes, i.e. near $a = 0$. To state the result in the simplest form, let us fix a SdS spacetime (M°, g_{b_0}), and within it a compact spacelike hypersurface $\Sigma_0 \subset \{t_* = 0\} \subset M^\circ$ extending slightly beyond the event horizon $r = r_-$ and the cosmological horizon $r = r_+$; let (h_{b_0}, k_{b_0}) be the initial data on Σ_0 induced by g_{b_0}. Denote by Σ_{t_*} the translates of Σ_0 along the flow of ∂_{t_*}, and let $\Omega^\circ = \bigcup_{t_* \geq 0} \Sigma_{t_*} \subset M^\circ$ be the spacetime region swept out by these. Since we only consider slow rotation speeds, it suffices to consider perturbations of SdS initial data, as this includes (perturbations of) slowly rotating KdS initial data.

Theorem 1 (Stability of the Kerr–de Sitter family for small a; informal version, [HV18]). *Suppose (h, k) are smooth initial data on Σ_0, satisfying the constraint equations, which are close to the data (h_{b_0}, k_{b_0}) of a Schwarzschild–de Sitter spacetime in a high regularity norm. Then there exist a solution g of (1) attaining these initial data at Σ_0, and black hole parameters b which are close to b_0, so that*

$$g - g_b = \mathcal{O}(e^{-\alpha t_*})$$

for a constant $\alpha > 0$ independent of the initial data; that is, g decays exponentially fast to the KdS metric g_b. Moreover, g and b are quantitatively controlled by (h, k).

In particular, *we do not require any symmetry assumptions on the initial data.* Above, we measure the pointwise size of tensors on the spacetime M° by means of a fixed smooth stationary Riemannian metric g_R on M°. The norms we use for $(h - h_{b_0}, k - k_{b_0})$ on Σ_0 and of $g - g_b$ on Σ_{t_*} are then high regularity Sobolev norms; any two choices of g_R yield equivalent norms. If (h, k) are smooth and sufficiently close to (h_{b_0}, k_{b_0}) in a fixed high regularity norm, the solution g we obtain is smooth as well, and in a suitable Fréchet space of smooth symmetric 2-tensors on M° depends smoothly on (h, k), as does b. In terms of the maximal globally hyperbolic development (MGHD) of the initial data (h, k), the Theorem states that the MGHD contains a subset isometric to Ω° on which the metric decays at an exponential rate to g_b. We stress that a *single* member of the KdS family is *not* stable, rather we have *orbital stability*: small perturbations of the initial data of, say, a SdS black hole, will in general result in a solution which decays to a KdS metric with slightly different mass and non-zero angular momentum.

Earlier global non-linear stability results for the Einstein equation include Friedrich's work [Fri86] on the stability of $(3+1)$-dimensional de Sitter space, the monumental proof by Christodoulou–Klainerman [CK93] of the stability of $(3+1)$-dimensional Minkowski space. Partial simplifications and extensions of these results include those of Lindblad–Rodnianski [LR10] and Bieri–Zipser [BZ09] on Minkowski space, as well as many others. For $\Lambda = 0$, linear stability of the Schwarzschild spacetime (i.e. with $a = 0$) was proved recently by Dafermos, Holzegel and Rodnianski [DHR16] and they also obtained results for the Teukolsky equation [DHR17]. Our Theorem is the first result for the Einstein equation proving an orbital stability statement, and our flexible techniques allow for investigations of many further orbital stability questions, such as for the Kerr–Newman–de Sitter family of rotating and charged black holes accomplished by Hintz [Hin16b] after the work being reported on. More recently [HV17] we showed that our methods yield yet another proof of the stability of Minkowski space, together with a rather precise analysis of the asymptotic behavior of the solutions.

The proof of the Theorem uses a generalized wave coordinate gauge adjusted 'dynamically' (from infinity) by finite-dimensional gauge modifications. The key tool is the precise analysis of the linearized problem around a SdS metric; we develop a robust framework that has powerful stability properties with respect to perturbations. This completes in a sense a series of works starting with [Vas13], including [HV15, Hin16a, HV16], with the works of Wunsch–Zworski [WZ11] and

Dyatlov [Dya16] as an external input to handle the trapped set. The restriction to small angular momenta in the Theorem is then due to the fact that the required algebra, mostly concerning resonances, is straightforward for linear equations on a SdS background, but gets rather complicated for non-zero angular momenta. Our framework builds on a number of recent advances in the global geometric microlocal analysis of black hole spacetimes. For solving the non-linear problem, we use a Nash–Moser iteration scheme, which proceeds by solving a *linear* equation *globally* at each step and is thus rather different in character from bootstrap arguments.

References

[BZ09] L. Bieri and N. Zipser. *Extensions of the stability theorem of the Minkowski space in general relativity*, volume 45. American Mathematical Society, 2009.

[Car68] B. Carter. Hamilton–Jacobi and Schrödinger separable solutions of Einstein's equations. *Communications in Mathematical Physics*, 10(4):280–310, 1968.

[CB52] Y. Choquet-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta mathematica*, 88(1):141–225, 1952.

[CBG69] Y. Choquet-Bruhat and R. Geroch. Global aspects of the Cauchy problem in general relativity. *Communications in Mathematical Physics*, 14(4):329–335, 1969.

[CK93] D. Christodoulou and S. Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[DHR16] M. Dafermos, G. Holzegel, and I. Rodnianski. The linear stability of the Schwarzschild solution to gravitational perturbations. *Preprint, arXiv:1601.06467*, 2016.

[DHR17] M. Dafermos, G. Holzegel and I. Rodnianski. Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: the case $|a| \ll M$. *Preprint, arXiv:1711.07944*, 2017.

[Dya16] S. Dyatlov. Spectral gaps for normally hyperbolic trapping. *Ann. Inst. Fourier (Grenoble)*, 66(1):55–82, 2016.

[Fri86] H. Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. *Comm. Math. Phys.*, 107(4):587–609, 1986.

[Hin16a] P. Hintz. Global well-posedness of quasilinear wave equations on asymptotically de Sitter spaces. *Annales de l'Institut Fourier*, 66(4):1285–1408, 2016.

[Hin16b] P. Hintz. Non-linear stability of the Kerr–Newman–de Sitter family of charged black holes. *Annals of PDE, to appear; arXiv:1612.04489.*

[HV15] P. Hintz and A. Vasy. Semilinear wave equations on asymptotically de Sitter, Kerr–de Sitter and Minkowski spacetimes. *Anal. PDE*, 8(8):1807–1890, 2015.

[HV16] P. Hintz and A. Vasy. Global Analysis of Quasilinear Wave Equations on Asymptotically Kerr–de Sitter Spaces. *Int. Math. Res. Notices*, 2016(17):5355–5426, 2016.

[HV17] P. Hintz and A. Vasy. A global analysis proof of the stability of Minkowski space and the polyhomogeneity of the metric. *Preprint, arXiv:1711.0019*, 2017.

[HV18] P. Hintz and A. Vasy. The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta Math.*, 220: 1–206, 2018.

[Ker63] R. P. Kerr. Gravitational field of a spinning mass as an example of algebraically special metrics. *Physical Review Letters*, 11(5):237, 1963.

[LR10] H. Lindblad and I. Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.

[Vas13] A. Vasy. Microlocal analysis of asymptotically hyperbolic and Kerr–de Sitter spaces (with an appendix by Semyon Dyatlov). *Invent. Math.*, 194(2):381–513, 2013.

[WZ11] J. Wunsch and M. Zworski. Resolvent estimates for normally hyperbolic trapped sets. In *Annales Henri Poincaré*, volume 12, pages 1349–1385. Springer, 2011.

Interaction of three impulsive gravitational waves under polarized $\mathbb{U}(1)$ symmetry

Jonathan Luk
(joint work with Maxime van de Moortel)

An *impulsive gravitational wave* is a (weak) solution $(\mathcal{M}, g)$ to the Einstein vacuum equations

$$\text{(EinVac)} \qquad Ric(g) = 0$$

in which the Riemann curvature tensor of g admits a delta singularity supported on a null hypersurface. An explicit solution featuring such an impulsive gravitational wave was first considered in [6]. Impulsive gravitational waves can be thought of as an idealized representation of gravitational waves emitting from strongly gravitating objects and as such it is interesting to understand their interaction (which is geometrically represented by the intersection of the corresponding singular hypersurfaces).

Previously, there have been two important works regarding the interaction of two impulsive gravitational waves:

(1) An *explicit* example in plane symmetry was discovered by Khan–Penrose [3] and Szekeres [7] featuring the interaction of two plane symmetric impulsive gravitational waves (see also [2] for a survey on the vast literature on related explicit examples). In the example of Khan–Penrose and Szekeres, locally immediately after the interaction, the spacetime metric remains smooth; nevertheless, the interaction creates a focusing effect so that the spacetime eventually terminates with a spacelike singularity.

(2) A *local* theory for the interaction of two impulsive gravitational waves for general wavefronts without any symmetry assumptions has been developed by Luk–Rodnianski [4, 5]. They showed in particular that in general, locally after the interaction of two impulsive gravitational waves, the spacetime metric remains smooth away from the impulsive wavefronts.

Despite the above developments, there has been no known example for a spacetime solution to (EinVac) featuring the transversal interaction of three (or more) impulsive gravitational waves. In particular, in [4, 5], the interaction of two impulsive gravitational waves was treated by a geometric construction (based on the double null foliation gauge) which exploits the fact that "the spacetime is more regular (in fact smooth) in two directions". The methods in [4, 5] break down once there is a third transversally interacting impulsive gravitational wave; indeed one may even expect that the structure of the singularity is qualitatively different with three impulsive waves (cf. Remark 1 below). Understanding the transversal interaction of three impulsive gravitational waves will therefore necessarily introduce new ideas to study the dynamics of (EinVac).

Our main result is a construction of a large class of solutions to (EinVac) featuring the interaction of three impulsive gravitational waves. For this we do not work in full generality, but instead impose a polarized $\mathbb{U}(1)$-symmetry, i.e. we consider a spacetime $(\mathcal{M}, {}^{(4)}g)$ which $\mathcal{M} = [0,1] \times \mathbb{R}^2 \times \mathbb{S}^1$, where ${}^{(4)}g$ takes the form (for some $(2+1)$-dimensional Lorentzian metric g and some real-valued function ϕ)

$$^{(4)}g = e^{-2\phi} g + e^{2\phi}(dx^3)^2.$$

Under this ansatz, (EinVac) reduces to the following $(2+1)$-dimensional problem

$$\text{(EinU(1))} \qquad \Box_g \phi = 0, \quad Ric(g) = 2d\phi \otimes d\phi.$$

We set up our problem by considering the initial value problem for (EinU(1)) such that ϕ and its derivatives are compactly supported and small in L^∞. The derivatives of ϕ are imposed to have a small jump discontinuity along three curves. Moreover, the singularities are arranged to that they propagate towards each other.

Theorem 1 (Luk–Van de Moortel, forthcoming)**.** *Given a polarized $\mathbb{U}(1)$ symmetric initial data set corresponding to three appropriately small impulsive gravitational waves propagating towards each other, there exists a solution to the Einstein vacuum equations corresponding to the given data up to and beyond the interaction of these waves. Moreover, in the solution, the metric is everywhere Lipschitz and is $C^{1,\alpha}$ for some $\alpha > 0$ away from the three null hypersurfaces corresponding to impulsive gravitational waves.*

Remark 1. *Our theorem does not give uniqueness, nor does it give higher regularity beyond $C^{1,\alpha}$ away from the impulsive gravitational waves. Both of these aspects are different from the known result for the interaction of two impulsive gravitational waves established in* [5]*. Nevertheless, in view of the picture established even in some much simpler semilinear model problems* [1]*, it seems reasonable to conjecture that the spacetime metric is in general not smooth away from the impulsive gravitational waves, in the sense that some weaker singularities are generated by the interaction of the impulsive gravitational waves.*

We now discuss some key ideas of the proof.

(1) (A δ-approximate problem) The general strategy is to smooth out the singularity at scale δ and to prove uniform bounds as $\delta \to 0$. In the δ-approximate problem, the initial singularities are localized to a length scale $\sim \delta$ and obey the estimates $|\phi| \sim 1$, $|\partial\phi| \sim 1$ and $|\partial^2\phi| \sim \delta^{-1}$ (in L^∞).

(2) (Geometric coordinate systems) To close the estimates we introduce multiple systems of coordinates: one global elliptic coordinate system together with three null coordinate systems, where each null coordinate system is adapted to one of singularities. The elliptic coordinate system is used so that the metric components satisfy semilinear elliptic PDEs and we can obtain maximal spatial regularity for them. On the other hand, we use the null coordinate systems to capture exactly the directions that the singularities propagate. This allows us to show that derivatives in some directions remain more regular. However, while the use of multiple coordinate systems is advantageous for proving regularity, it also necessarily creates the challenge of controlling the transformation between different coordinate

systems. In the process, we show that all the transformations give only "lower order" contributions in an appropriate sense.

(3) (Wave estimates) When controlling the wave part ϕ, we cannot hope for an estimate in the L^2-based Sobolev spaces better than $\|\phi\|_{H^1} \sim 1$ and $\|\phi\|_{H^2} \sim \delta^{-\frac{1}{2}}$ — such an estimate is way too weak! Instead, we need to show that in fact (1) the H^2 norm of ϕ is only large in a "small region" of length scale $\sim \delta$ and (2) that it is only derivatives of ϕ in some directions that are large, and that the derivatives in some other directions (defined with respect to the null coordinates) remain better controlled.

In order to achieve points (1) and (2) above, we decompose ϕ into a "regular" and a "singular" part (constructed via solving an auxiliary characteristic-Cauchy problem). The regular part of ϕ has bounded H^2 norm. The singular part ϕ_{sing} of ϕ, while necessarily has $\|\phi_{\text{sing}}\|_{H^2} \sim \delta^{-\frac{1}{2}}$, is localized to scale δ, and has the additional property that the L^2 norms of its lower order derivatives, as well as derivatives in the "good" directions along the singularities, are in fact *small* in terms of δ.

We can then propagate a *hierarchy* of δ-dependent estimates for ϕ_{sing}. This is reminiscent of Christodoulou's short pulse method for proving the formation of trapped surfaces. In particular, such estimates can be achieved with only relatively weak control of the geometry.

(4) (Higher order regularity and harmonic analytic estimates) In order to show that the solution is in fact Lipschitz everywhere and $C^{1,\alpha}$ away from the impulsive gravitational waves, we introduce an additional commutation of a fractional derivative with respect to the elliptic coordinate systems. Constructing these commuting fractional derivatives in the elliptic coordinate system in particular allows us to exploit the spatial regularity of the metric components to control the commutator terms. In order to implement and utilize this, we need to additionally bound the transformations of fractional derivatives between different coordinate systems, and prove certain anisotropic Sobolev embedding and Sobolev extension results. All of these are achieved using tools from harmonic analysis.

References

[1] M. Beals, *Propagation and interaction of singularities in nonlinear hyperbolic problems*, Progress in Nonlinear Differential Equations and their Applications, 3., 1989.

[2] J. B. Griffiths, *Colliding plane waves in general relativity*, Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1991.

[3] K. A. Khan and R. Penrose, *Scattering of two impulsive gravitational plane waves*, Nature 229: 185-186, 1971.

[4] J. Luk and I. Rodnianski, *Local propagation of impulsive gravitational waves*, Comm. Pure and Appl. Math., 68(4):511-624, 2015.

[5] J. Luk and I. Rodnianski, *Nonlinear interaction of impulsive gravitational waves for the vacuum Einstein equations*, Cambridge Journal of Math. 5(4): 435-570, 2017.

[6] R. Penrose. The geometry of impulsive gravitational waves. In General relativity (papers in honour of J. L. Synge), 101–115. Clarendon Press, Oxford, 1972.

[7] P. Szekeres, *Colliding plane gravitational waves*, J. Math. Phys. 13: 286-294, 1972.

Towards uniqueness of near-horizon geometries

Piotr T. Chruściel
(joint work with Sebastian Szybka and Paul Tod)

Degenerate Killing horizons are, by definition, those Killing horizons on which the surface gravity vanishes. The metric induced on a section of a degenerate Killing horizon $\mathscr{H}$ by a vacuum space-time metric satisfies the set of equations

$$R_{AB} - (\mathscr{D}_A\omega_B + \mathscr{D}_B\omega_A + 2\omega_A\omega_B) = 0 \ , \tag{1}$$

known as the *near-horizon-geometry equations*, where ω_A is a field of one-forms on the section. Equation (1) implies a nonlinear elliptic system of equations for both ω_A and the metric.

The fact that there are no solutions of these equation on S^2 on horizons arising from static space-times [2] implies that *there are no static vacuum black holes with degenerate horizons.*

The fact that [5, 8] (compare [7, 10])

(2) all axisymmetric solutions (g_{AB}, ω_A) of (1) on a two-dimensional sphere arise from degenerate Kerr metrics

plays a key role in the proof that *all connected degenerate stationary axisymmetric vacuum black holes are Kerr* [1] (see also [4, 9]).

It is expected that

Conjecture 1. (2) *holds without the axisymmetry assumption.*

In my talk I described the computer-assisted proof, presented in [3], that the conjecture holds in a neighborhood of the Kerr near-horizon geometries:

Theorem 1. *Let $\mathcal{S}$ denote the set of pairs (g_{AB}, ω_A) on S^2 satisfying* (1)*, let $\mathcal{S}_{\mathrm{Kerr}} \subset \mathcal{S}$ denote the set of such pairs arising from some Kerr solution. There exists a neighborhood $\mathscr{U}$ of $\mathcal{S}_{\mathrm{Kerr}}$ in the set of all pairs (g_{AB}, ω_A) such that*

$$\mathcal{S} \cap \mathscr{U} = \mathcal{S}_{\mathrm{Kerr}} \, .$$

A useful fact in this context, also proved in [3], and likely to be useful in further analyses of the problem, is

Theorem 2. *Consider a smooth solution of* (1) *on a two-dimensional sphere S^2. Then $\omega = \omega_A dx^A$ has exactly two zeros, each of index one.*

The proof of Theorem 1 proceeds by showing that the operator obtained by linearising (1) has no kernel other than the fields obtained by varying the mass and the direction of the angular momentum of the extreme Kerr solutions. The analysis uses a parametrisation of solutions of (1) introduced by Jezierski and Kamiński in [6]. There the linearised problem is reduced to a coupled system of second order elliptic equations for a $\mathbb{R}^2$-valued function, say ϕ. The system can be handled by separately analysing each of the equations satisfied by the coefficients ϕ_k of the Fourier series of ϕ with respect to the longitudinal angle on S^2. It is elementary to show that all modes except those with $|k| \in \{1, \dots, 7\}$ vanish. Further

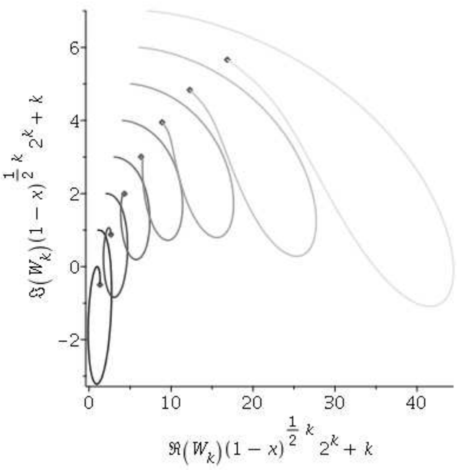


FIGURE 1. Solution curves $W_k(x)$, $1 < |k| < 8$, $x = \cos\theta$, in the complex plane, regular at the north pole.

manipulations of the equations reduce the problem to showing that a family of second-order Fuchsian ordinary differential equations, degenerating at the poles, for complex-valued functions $W_k(\cos\theta)$, $k \in \{1, \ldots, 7\}$, has no solutions which are regular both at the north and south pole. This last property is established by numerically solving the equations. The solution curves W_k are shown in Figure 1: non-trivial smooth solutions would lead to closed curves in the complex plane, which is clearly not the case.

REFERENCES

[1] P.T. Chruściel and L. Nguyen, *A uniqueness theorem for degenerate Kerr-Newman black holes*, Ann. H. Poincaré **11** (2010), 585–609, arXiv:1002.1737 [gr-qc]. MR 2677740 (2011i:53123)

[2] P.T. Chruściel, H.S. Reall, and K.P. Tod, *On non-existence of static vacuum black holes with degenerate components of the event horizon*, Class. Quantum Grav. **23** (2006), 549–554, arXiv:gr-qc/0512041. MR MR2196372 (2007b:83090)

[3] P.T. Chruściel, S.J. Szybka, and P. Tod, *Towards a classification of near-horizons geometries*, Class. Quantum Grav. **35** (2017), 015002, 14 pp., arXiv:1707.01118 [gr-qc]. MR 3747353

[4] P. Figueras and J. Lucietti, *On the uniqueness of extremal vacuum black holes*, Class. Quantum Grav. **27** (2010), 095001, arXiv:0906.5565 [hep-th].

[5] P. Hájíček, *Three remarks on axisymmetric stationary horizons*, Commun. Math. Phys. **36** (1974), 305–320. MR MR0418816 (54 #6852)

[6] J. Jezierski and B. Kamiński, *Towards uniqueness of degenerate axially symmetric Killing horizon*, Gen. Rel. Grav. **45** (2013), 987–1004, arXiv:1206.5136 [gr-qc]. MR 3046208

[7] H.K. Kunduri and J. Lucietti, *Classification of near-horizon geometries of extremal black holes*, Living Rev. Rel. **16** (2013), 8, arXiv:1306.2517 [hep-th].

[8] J. Lewandowski and T. Pawłowski, *Extremal isolated horizons: A local uniqueness theorem*, Class. Quantum Grav. **20** (2003), 587–606, arXiv:gr-qc/0208032.

[9] R. Meinel, M. Ansorg, A. Kleinwächter, G. Neugebauer, and D Petroff, *The Kerr metric as the solution to a boundary value problem*, Relativistic Figures of Equilibrium, Cambridge University Press, 2008, pp. 108–113.

[10] P. Nurowski and M. Randall, *Generalized Ricci solitons*, Jour. Geom. Anal. **26** (2016), 1280–1345, arXiv:1409.4179 [math.DG]. MR 3472837

On geometric foliations and the center of mass of isolated systems in general relativity

Anna Sakovich

(joint work with Carla Cederbaum)

The notion of the center of mass of a physical system is one of the most fundamental concepts in mathematical physics and geometry, as understanding its position and motion is often the first step towards understanding of the overall dynamics of the system. In this talk we present a novel definition of the center of mass of initial data for general relativistic isolated systems.

Let $\varepsilon \in (0, \frac{1}{2}]$ and let $\mathcal{I} = (M^3, g, K)$ be a smooth initial data set, consisting of a smooth Riemannian 3-manifold (M^3, g) equipped with a smooth symmetric 2-tensor K to be thought of as the second fundamental form of the initial data set inside some ambient Lorentzian spacetime. We say that $\mathcal{I}$ is an *asymptotically Euclidean initial data set* if there is a smooth diffeomorphism $\Phi\colon M^3 \setminus \mathcal{B} \to \mathbb{R}^3 \setminus \overline{B_R(0)}$ defined in the region exterior to a compact set $\mathcal{B} \subset M^3$ such that in the coordinates $\vec{x} = (x^1, x^2, x^3)$ induced by this diffeomorphism on $\mathbb{R}^3 \setminus \overline{B_R(0)}$, we have the following pointwise estimates

$$\begin{aligned} |g_{ij} - \delta_{ij}| + |\vec{x}||\partial_k g_{ij}| + |\vec{x}|^2|\partial_k\partial_l g_{ij}| &\leq C|\vec{x}|^{-\frac{1}{2}-\varepsilon} \\ |K_{ij}| + |\vec{x}||\partial_k K_{ij}| &\leq C|\vec{x}|^{-\frac{3}{2}-\varepsilon} \\ |\mu| + |J_i| &\leq C|\vec{x}|^{-3-\varepsilon} \end{aligned}$$

where the Kronecker delta δ_{ij} denotes the components of the Euclidean metric with respect to the coordinates $\vec{x}$. The local energy density μ and the local momentum density J of the initial data can be computed using the standard constraint equations. The case when $K \equiv 0$ is called the *Riemannian case*; in this case μ is the scalar curvature of (M, g) and $J \equiv 0$.

Asymptotically Euclidean initial data sets as described above are well-known to have well-defined total energy E and linear momentum $\vec{P}$ in the sense of Arnowitt, Deser, and Misner [1]. However, the so-called *Beig-Ó Murchadha center of mass*, a notion developed in [2] using the Hamiltonian formalism, will not be well-defined unless certain asymptotic symmetry conditions (such as e.g. the so-called Regge-Teitelboim condition) are additionally imposed. This being a rather restrictive additional assumption, an alternative strategy (first suggested by Christodoulou and Yau in [4]) is to define the center of mass of an asymptotically Euclidean initial data set in terms of geometric foliations. A very satisfying definition of this kind was given in the Riemannian case by Huisken and Yau in [5], where the center of mass was related to the unique *CMC-foliation*, i.e. foliation by surfaces of constant mean curvature. The main result of this work proving the existence of such foliation near infinity of a Riemannian asymptotically Schwarzschildean 3-manifold with positive energy $E > 0$ has subsequently been generalized to the case of asymptotically Euclidean initial data sets as defined above, with $K \equiv 0$ and $E \neq 0$. Moreover, by now it is known that the coordinate center of the CMC-foliation coincides with the Beig-Ó Murchadha center of mass, provided that the later is well-defined. For further details, see [6] and references therein.

We note yet another generalization of Huisken and Yau's result. In [9], Metzger introduced foliations of asymptotically Euclidean initial data sets $\mathcal{I} = (M, g, K)$ by surfaces of constant expansion. A surface $\Sigma \hookrightarrow \mathcal{I}$ is said to have *constant expansion* if either $H_\Sigma + \mathrm{tr}_\Sigma K = \mathrm{const}$ or $H_\Sigma - \mathrm{tr}_\Sigma K = \mathrm{const}$ holds along Σ, where H_Σ is the mean curvature of $\Sigma \hookrightarrow (M, g)$ with respect to the outward pointing unit normal and $\mathrm{tr}_\Sigma K$ denotes the partial trace of K with respect to the induced metric on $\Sigma \hookrightarrow (M, g)$. As observed in [9] and [7], this foliation will only exist under certain smallness assumptions on K. Furthermore, in general, the coordinate centers of its leaves will drift away towards infinity ($|\vec{x}| \to \infty$) in the direction of the linear momentum $\vec{P}$. Consequently, this foliation is not related to the center of mass of the initial data set $\mathcal{I}$, unless K falls of very fast.

As the second fundamental form K does not play a role in defining the Huisken-Yau foliation by surfaces of constant mean curvature, the associated notion of center of mass will not necessarily behave in accordance with a point particle in Special Relativity. In particular, explicit examples of initial data sets for Schwarzschild spacetime may be constructed showing that this notion of center of mass does not transform equivariantly under the asymptotic Poincaré group; in particular, it is not boost covariant. In fact, as pointed out in [8], there is a similar issue related to the Beig-Ó Murchadha center of mass.

In [3], we propose a foliation which may be viewed as a covariant relativistic generalization of the Huisken-Yau CMC-foliation. Given an asymptotically Euclidean initial data set $\mathcal{I} = (M, g, K)$ with non-vanishing energy $E \neq 0$, we prove that outside a compact set M is foliated by 2-surfaces of constant spacetime mean curvature (STCMC), and that this STCMC-foliation is unique. Recall that if Σ^2 is a 2-surface in a 4-dimensional spacetime $(\mathcal{L}^4, \gamma)$ then its *spacetime mean curvature* is defined as the length with respect to γ of the mean curvature vector $\vec{\mathcal{H}}$. If, in addition, it is known that Σ lies in an initial data set $\mathcal{I} = (M^3, g, K)$ for the spacetime $(\mathcal{L}^4, \gamma)$, and that $\vec{\mathcal{H}}$ is spacelike, then the spacetime mean curvature can be computed as

$$|\vec{\mathcal{H}}|_\gamma = \sqrt{(H_\Sigma)^2 - (\mathrm{tr}_\Sigma K)^2}.$$

Clearly, in the Riemannian case $K \equiv 0$ the notion of STCMC-surface coincides with the notion of CMC-surface. Note also that the STCMC-condition $|\vec{\mathcal{H}}|_\gamma \equiv \mathrm{const}$ is boost covariant. Moreover, this condition is "independent" of the initial data set in the following sense: if $\Sigma \hookrightarrow \mathcal{I}_i = (M_i, g_i, K_i)$, $i = 1, 2$, then Σ is STCMC with respect to $\mathcal{I}_1$ if and only if it is STCMC with respect to $\mathcal{I}_2$. This indicates that there is a plethora of STCMC 2-spheres in a neighborhood of spatial infinity of any asymptotically flat spacetime.

Our proof of existence and uniqueness of the STCMC-foliation builds upon ideas of Metzger [9] and Nerz [6, 7]. It is based on a method of continuity argument, using the established existence and uniqueness of CMC-foliations in the Riemannian setting. In particular, we introduce a non-self-adjoint STCMC-stability operator and analyze the asymptotic behavior of its lowest (a priori real) eigenvalues and the respective eigenfunctions. This operator resembles the CMC-stability operator except for an additional non-self-adjoint term which turns out to have sufficiently good fall-off properties.

The leaves of the foliation are proven to be asymptotic to large coordinate spheres. As for their coordinate centers, assuming a slightly faster fall-off $K = O(|x|^{-2})$ we obtain the formula

$$\left| \vec{z}_1^{\,\sigma} - \vec{z}_0^{\,\sigma} - \frac{1}{32\pi E} \int_{|\vec{x}|=\sigma} \frac{\left(\sum_{k,l} \pi_{kl} x^k x^l\right)^2 \vec{x}}{\sigma^3} \, d\mu^\delta \right| \leq \frac{C}{\sigma^\varepsilon},$$

where $\pi = K - (\mathrm{tr}_g K)g$. Here $\vec{z}_1^{\,\sigma}$ denotes the coordinate center of the STCMC-surface Σ_1^σ with spacetime mean curvature $\frac{2}{\sigma}$ in the initial data set $\mathcal{I}_1 = (M, g, K)$, $\vec{z}_0^{\,\sigma}$ denotes the coordinate center of the (ST)CMC-surface Σ_0^σ with (spacetime) mean curvature $\frac{2}{\sigma}$ in the initial data set $\mathcal{I}_0 = (M, g, 0)$, and $\sigma > 0$ is assumed to be sufficiently large. This formula indicates that the CMC-foliation and the STCMC-foliation of the same initial data set (M, g, K) will in general not have the same coordinate center and that the second fundamental form of the initial data set can in a sense "compensate" for the diverging coordinate center of the CMC-foliation.

Finally, we prove that the coordinate center of the STCMC foliation evolves in time under the Einstein evolution equations like a point particle in Special Relativity. In fact, we obtain estimates which show that the individual leaves of the foliation evolve in a way more and more close to a translation with velocity $\frac{\vec{P}}{E}$.

References

[1] R. Arnowitt, S. Deser, and C.W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. (2) **122** (1961), 997–1006.

[2] R. Beig and N. Ó Murchadha, *The Poincaré group as the symmetry group of canonical general relativity*, Ann. Physics **174** (1987), no. 2, 463–498.

[3] C. Cederbaum and A. Sakovich, *On center of mass and foliations by constant spacetime mean curvature surfaces for isolated systems in General Relativity*, in preparation.

[4] D. Christodoulou and S.-T. Yau, *Some remarks on the quasi-local mass*, Mathematics and general relativity, Contemp. Math., 71, Amer. Math. Soc., Providence, RI, 1988.

[5] G. Huisken, S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliation by stable spheres with constant mean curvature*, Invent. Math. **124** (1996), no. 1-3, 281–311.

[6] C. Nerz, *Foliations by stable spheres with constant mean curvature for isolated systems without asymptotic symmetry*, Calc. Var. Partial Differential Equations **54** (2015), no. 2, 1911–1946.

[7] C. Nerz, *Foliations by stable spheres with constant expansion for isolated systems without asymptotic symmetry*, J. Differential Geom. **109** (2018), no. 2, 257–289.
[8] L. Szabados, *The Poincaré structure and the centre-of-mass of asymptotically flat spacetimes*, Analytical and numerical approaches to mathematical relativity, 157–184, Lecture Notes in Phys., 692, Springer, Berlin, 2006.
[9] J. Metzger, *Foliations of asymptotically flat 3-manifolds by 2-surfaces of prescribed mean curvature*, J. Differential Geom. **77** (2007), no. 2, 201–236.

Critical phenomena in the general spherically symmetric Einstein-Yang-Mills system

Oliver Rinne
(joint work with Maciej Maliborski)

The Einstein–Yang–Mills (EYM) equations form a rich dynamical system already in spherical symmetry due to the existence of nontrivial static solutions (Bartnik–McKinnon solitons and coloured black holes). After exploiting the residual gauge freedom, the most general spherically symmetric YM connection with gauge group SU(2) can be written as

$$\mathcal{A} = u\tau_3 \mathrm{d}t + (w\tau_1 + \omega\tau_2)\,\mathrm{d}\theta + (\cot\theta\tau_3 + w\tau_2 - \omega\tau_1)\sin\theta \mathrm{d}\phi, \tag{1}$$

where u, w and ω are functions of t and r only and τ_i form a standard basis of SU(2). There are two degrees of freedom, w and ω (the third field u obeys a constraint equation analogous to the Gauss constraint in electromagnetism). Most numerical work so far has assumed the *magnetic ansatz* $\omega = 0$, which is self-consistent in that if the initial data satisfy $\omega = 0$ then this will hold at all times. The aim of this work is to investigate critical phenomena in gravitational collapse in the general system, including nonzero ω. We refer the reader to our paper [1] for a more detailed presentation and list of references.

In the magnetic sector three different types of critical collapse have been observed. We begin by discussing the first two. In Type I, black hole formation turns on at a finite mass, and the critical solution at the threshold between dispersal and black hole formation is the lowest member X_1 of the family of Bartnik–McKinnon solitons. In Type II, the black hole mass vanishes at the threshold and scales as $M \sim (p - p_*)^\gamma$ with a universal (independent of the family of initial data) exponent γ. The critical solution is universal and discretely self-similar, i.e. in logarithmic coordinates

$$\rho = \ln r, \quad \tau = \ln(T_0^* - T_0), \tag{2}$$

where T_0 is proper time at the origin and T_0* the so-called accumulation time, any scale-free variable Z of the critical solution obeys

$$Z(\tau - \Delta, \rho - \Delta) = Z(\tau, \rho) \tag{3}$$

with a universal echoing exponent Δ. We illustrate this in the left panel of Fig. 1, where we plot five echoes of the scale-free variable $w' := \mathrm{d}w/\mathrm{d}r$, which nicely

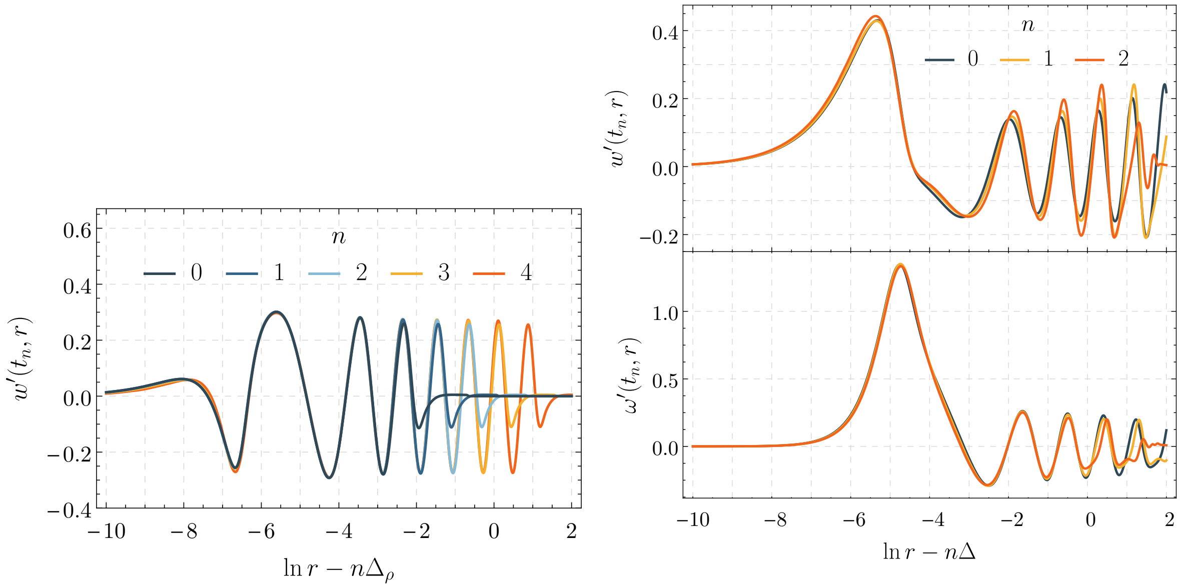


FIGURE 1. Discrete self-similarity of the critical solution in the magnetic ansatz (left) and (lack thereof) in the general system (right).

overlap. Our numerical results are in good agreement with previous numerical studies [2].

When the *sphaleronic sector* (i.e. the second potential ω) is turned on, X_1 can no longer be a critical solution because it has a second unstable mode in the general EYM system. Hence Type I critical collapse disappears and we generically observe Type II. However we have some indications that neither w' nor ω' are exactly scale invariant in the extended system (Fig. 1). This is apparent even in manifestly gauge-invariant quantities [1]. Moreover, exact universality of the critical solution (i.e. independence of the family of initial data chosen) also appears to be lost. There are small but noticeable differences when we compare gauge-invariant quantities for the magnetic critical solution vs. the general one. We cannot rule out completely that there is a very slowly decaying mode about the Type II critical solution in the general system that would require us to tune much closer to the critical point than is currently feasible in order to see the true features of the critical solution.

When we add a small sphaleronic perturbation to initial data that would be Type I-critical in the magnetic sector, the Bartnik–McKinnon soliton X_1 appears as an intermediate attractor before the Type II critical solution is approached (left panel of Fig. 2). We observe quasi-normal mode decay to this unstable attractor and obtain good agreement with the quasi-normal mode frequency that we computed in linear perturbation theory.

Finally we briefly discuss the third type of critical collapse that occurs in the magnetic sector [3]. Here black holes form on both sides of the threshold but with different vacuum values of the YM potential, $w = \pm 1$. The dynamical evolutions

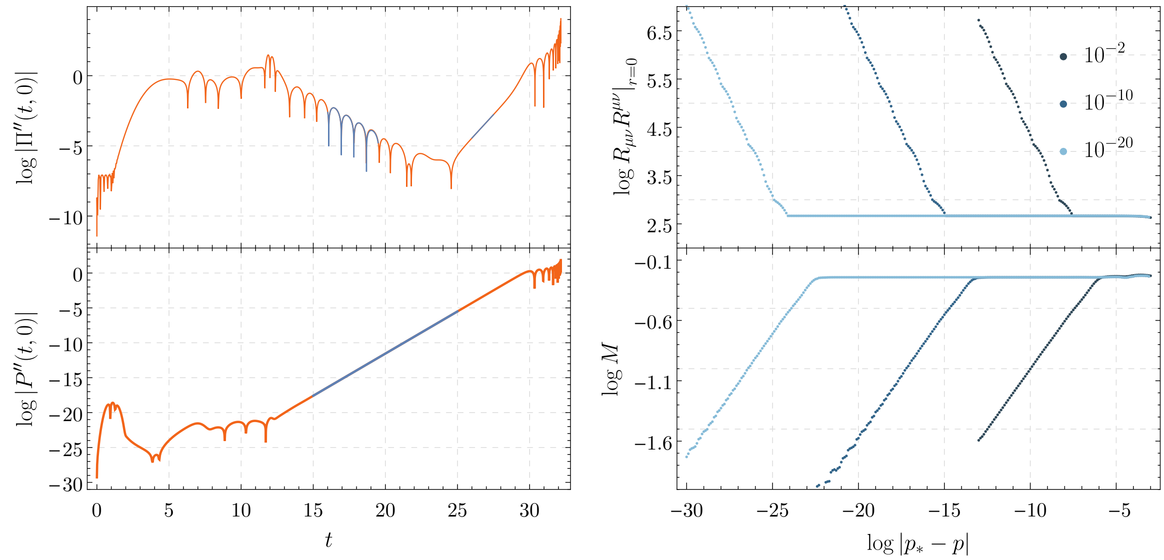


FIGURE 2. Adding a small sphaleronic perturbation to magnetic Type I-critical data: dynamical evolution (left) and onset of the polynomial mass/curvature scaling for different strengths of the perturbation as a function of critical parameter distance (right).

are markedly different and there is a gap in the final black hole mass as the threshold is crossed. In the general system, it is to be expected that this critical phenomenon disappears as well because the dichotomy of vacuum states $w^2 = 1$ is replaced with a continuum $w^2 + \omega^2 = 1$. Indeed we observe numerically that the discontinuous transition in the potential w across the threshold is replaced by a continuous one as soon as a small sphaleronic perturbation is added to the initial data (Fig. 3).

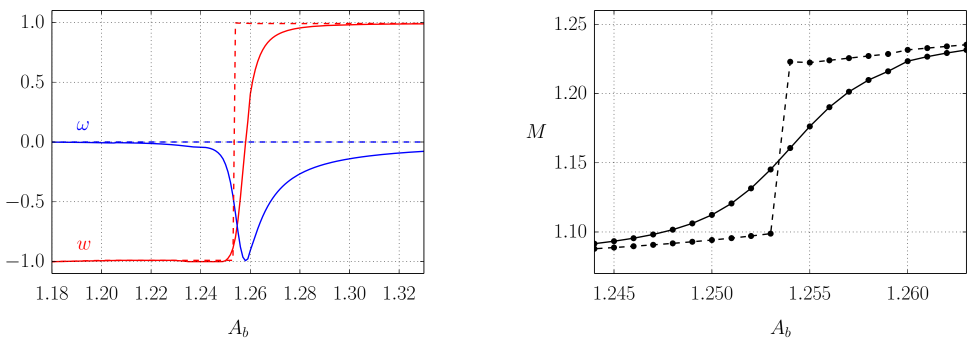


FIGURE 3. Final values of the YM potentials (left) and the black hole mass (right) along a family of magnetic initial data (dashed) and with a small sphaleronic perturbation added to the initial data (solid).

References

[1] M. Maliborski and O. Rinne, *Critical phenomena in the general spherically symmetric Einstein-Yang-Mills system*, Phys. Rev. D **97** (2018), 044053.
[2] M. W. Choptuik, T. Chmaj and P. Bizoń, *Critical Behavior in Gravitational Collapse of a Yang-Mills Field*, Phys. Rev. Lett. **77** (1996), 424.
[3] M. W. Choptuik, E. W. Hirschmann and R. Marsa, *New critical behavior in Einstein-Yang-Mills collapse*, Phys. Rev. D **60** (1999), 124011.

Stability of Minkowski space and polyhomogeneity of the metric

Peter Hintz
(joint work with András Vasy)

We prove the non-linear stability of (3 + 1)-dimensional Minkowski space as a solution (M°, g), g Lorentzian, of Einstein's vacuum equation

$$\mathrm{Ric}(g) = 0. \tag{1}$$

On a conceptual level, we show how some of the methods we developed for our proofs of black hole stability in cosmological spacetimes [12, 10] apply in this more familiar setting, studied by Friedrich [8], Christodoulou–Klainerman [5], Lindblad–Rodnianski [14, 15], Klainerman–Nicolò [13], Bieri [2], and many others. We also establish full *polyhomogeneous* expansions of g, in all asymptotic regions, i.e. near spacelike, null, and timelike infinity on a suitable compactification of $\mathbb{R}^4$. See [7, 6, 16, 1] for previous works in related contexts.

The simplest solution of (1) is the *Minkowski spacetime* $(M^\circ, g) = (\mathbb{R}^4, g_0) = (\mathbb{R}_t \times \mathbb{R}^3_x, dt^2 - dx^2)$. The far field of an isolated gravitational system (M°, g) with total (ADM) mass m is described by the *Schwarzschild metric*

$$g \approx g^S_m = \Big(1 - \frac{2m}{r}\Big)dt^2 - \Big(1 - \frac{2m}{r}\Big)^{-1} dr^2 - r^2 g_{\mathbb{S}^2} = g_0 + \mathcal{O}(r^{-1}), \quad r \gg 1.$$

Suitably interpreted, the field equation (1) has the character of a quasilinear wave equation, and we thus study its *initial value problem*: given a 3-manifold Σ° and symmetric 2-tensors $\gamma, k \in \mathcal{C}^\infty(\Sigma^\circ; S^2T^*\Sigma^\circ)$, with γ a Riemannian metric, one seeks a vacuum spacetime (M°, g) and an embedding $\Sigma^\circ \subset M^\circ$ such that

$$\mathrm{Ric}(g) = 0 \ \text{ on } M^\circ, \quad g|_{\Sigma^\circ} = -\gamma, \ \mathrm{II}_g = k \ \text{ on } \Sigma^\circ, \tag{2}$$

where II_g denotes the second fundamental form of Σ°. Choquet-Bruhat and Geroch [3, 4] proved that provided the *constraint equations* for γ and k hold,

$$R_\gamma + (\mathrm{tr}_\gamma k)^2 - |k|^2_\gamma = 0, \quad -\mathrm{div}_\gamma k + d\mathrm{tr}_\gamma k = 0, \tag{3}$$

there exists a unique (up to isometries) maximal globally hyperbolic development (M°, g) of (2). The *future development* is the causal future of Σ° within (M°, g). Our main theorem concerns the long time behavior of solutions of (2) with initial data close to those of Minkowski space:

Theorem 1 (Rough version [11])**.** *Let $b_0 > 0$. Suppose that (γ, k) are smooth initial data on $\mathbb{R}^3$ satisfying the constraint equations* (3) *which are small in the following sense: for some small $\delta > 0$, a cutoff $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^3)$ identically* 1 *near* 0*, and $\widetilde{\gamma} := \gamma - (1-\chi) g_m^S|_{\{t=0\}}$, where $|m| < \delta$, we have*

$$(4) \qquad \sum_{j \le N+1} \|\langle r\rangle^{-1/2+b_0} (\langle r\rangle \nabla)^j \widetilde{\gamma}\|_{L^2} + \sum_{j \le N} \|\langle r\rangle^{1/2+b_0} (\langle r\rangle \nabla)^j k\|_{L^2} < \delta_N,$$

for all $N \in \mathbb{N}$, and δ_{26} is small.

Then the future development of the data $(\mathbb{R}^3, \gamma, k)$ is future causally geodesically complete and decays to the Minkowski solution. More precisely, there exist a smooth manifold with corners M with boundary hypersurfaces Σ, i^0, $\mathscr{I}^+$, i^+, and a diffeomorphism of the interior M° with $\{t > 0\} \subset \mathbb{R}^4$, as well as an embedding $\mathbb{R}^3 \cong \Sigma^\circ$ of the Cauchy hypersurface, and a solution g of the initial value problem (2) *which is* conormal *on M and satisfies $|g - g_0| \lesssim (1 + t + |r|)^{-1+\epsilon}$ for all $\epsilon > 0$. See Figure 1. For fixed ADM mass m, the solution g depends continuously on $\widetilde{\gamma}$, k.*

If the initial data $(\widetilde{\gamma}, k)$ are in addition $\mathcal{E}$-smooth, i.e. polyhomogeneous *at infinity with index set $\mathcal{E}$ (see below), then the solution g is also polyhomogeneous on M, with index sets given explicitly in terms of $\mathcal{E}$.*

This allows for the initial data to be Schwarzschildean modulo $\mathcal{O}(r^{-1-\epsilon})$ for any $\epsilon > 0$. The assumption of $\mathcal{E}$-smoothness, i.e. polyhomogeneity with index set $\mathcal{E} \subset \mathbb{C} \times \mathbb{N}_0$, means, roughly speaking, that $\langle r\rangle \widetilde{\gamma}$ (similarly $\langle r\rangle^2 k$) has a full asymptotic expansion as $r \to \infty$ of the form

$$\langle r\rangle \widetilde{\gamma} \sim \sum_{(z,k) \in \mathcal{E}} r^{-iz} (\log r)^k \widetilde{\gamma}_{(z,k)}(\omega), \quad \omega = x/|x| \in \mathbb{S}^2, \ \widetilde{\gamma}_{(z,k)} \in \mathcal{C}^\infty(\mathbb{S}^2; S^2 T^* \mathbb{R}^3),$$

with $\operatorname{Im} z < -b_0$, where for any C, the number of $(z,k) \in \mathcal{E}$, $\operatorname{Im} z > -C$, is finite.

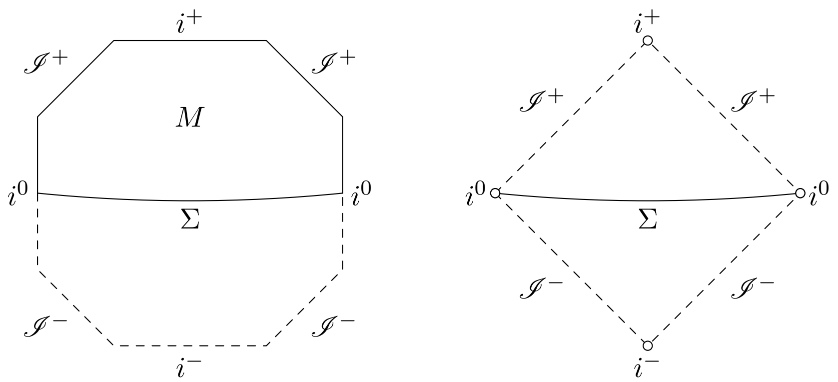


FIGURE 1. *Left:* the compact manifold M (solid boundary), pasted together with the past development (dashed boundary). M is closely related to the blow-up of a Penrose diagram (*right*) at timelike and spatial infinity.

The compactification M only depends on the ADM mass m of the initial data set; for the class of initial data considered here, the mass gives the only long range contribution to the metric that significantly (namely, logarithmically) affects the bending of light rays and thus the location of null infinity. We prove Theorem 1 using a Nash–Moser iteration scheme in which we a *linearization* of the gauge-fixed Einstein equation *globally* on M at each step; we use a wave map gauge which is a slight modification of the wave coordinate gauge. This gauge, which can be expressed as the vanishing of a certain 1-form $\Upsilon(g)$, fixes the long range part of g and hence the main part of the null geometry at $\mathscr{I}^+$. We implement *constraint damping* [9], already crucially used in [12]; here it ensures that throughout the iteration scheme, the 1-form Υ vanishes sufficiently fast so as to fix the long range part of g to be that of mass m Schwarzschild. In this gauge, we also identify the Bondi mass in terms of metric coefficients, and prove the Bondi mass loss law.

Our systematic approach is based on energy estimates for the linearized equations (using complete vector fields on M as multipliers)—including versions giving iterated regularity with respect to suitable vector fields—which are rather refined in terms of a splitting of the symmetric 2-tensor bundle (different components behave differently at null infinity). Both the relevant notion of regularity and the determination of the precise asymptotic behavior of the solution can be read off from the geometric and algebraic properties of the linearized equations; correspondingly, once M and the required function spaces are defined, the proof of stability itself is rather concise.

References

[1] Dean Baskin, András Vasy, and Jared Wunsch. Asymptotics of radiation fields in asymptotically Minkowski space. *Amer. J. Math.*, 137(5):1293–1364, 2015.

[2] Lydia Bieri and Nina Zipser. *Extensions of the stability theorem of the Minkowski space in general relativity*, volume 45. American Mathematical Society, 2009.

[3] Yvonne Choquet-Bruhat. Théorème d'existence pour certains systèmes d'équations aux dérivées partielles non linéaires. *Acta mathematica*, 88(1):141–225, 1952.

[4] Yvonne Choquet-Bruhat and Robert Geroch. Global aspects of the Cauchy problem in general relativity. *Communications in Mathematical Physics*, 14(4):329–335, 1969.

[5] Demetrios Christodoulou and Sergiu Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[6] Piotr T. Chruściel and Szymon Łęski. Polyhomogeneous solutions of nonlinear wave equations without corner conditions. *Journal of Hyperbolic Differential Equations*, 3(01):81–141, 2006.

[7] Piotr T. Chruściel and Roger T. Wafo. Solutions of quasi-linear wave equations polyhomogeneous at null infinity in high dimensions. *Journal of Hyperbolic Differential Equations*, 8(02):269–346, 2011.

[8] Helmut Friedrich. On the existence of n-geodesically complete or future complete solutions of Einstein's field equations with smooth asymptotic structure. *Comm. Math. Phys.*, 107(4):587–609, 1986.

[9] Carsten Gundlach, Gioel Calabrese, Ian Hinder, and José M. Martín-García. Constraint damping in the Z4 formulation and harmonic gauge. *Classical and Quantum Gravity*, 22(17):3767, 2005.

[10] Peter Hintz. Non-linear Stability of the Kerr–Newman–de Sitter Family of Charged Black Holes. *Annals of PDE*, 4(1):11, Apr 2018.

[11] Peter Hintz and András Vasy. A global analysis proof of the stability of Minkowski space and the polyhomogeneity of the metric. *Preprint, arXiv:1711.00195*, 2017.

[12] Peter Hintz and András Vasy. The global non-linear stability of the Kerr–de Sitter family of black holes. *Acta mathematica*, 220:1–206, 2018.
[13] Sergiu Klainerman and Francesco Nicolò. *The evolution problem in general relativity*, volume 25 of *Progress in Mathematical Physics*. Birkhäuser Boston, Inc., Boston, MA, 2003.
[14] Hans Lindblad and Igor Rodnianski. Global existence for the Einstein vacuum equations in wave coordinates. *Comm. Math. Phys.*, 256(1):43–110, 2005.
[15] Hans Lindblad and Igor Rodnianski. The global stability of Minkowski space-time in harmonic gauge. *Ann. of Math. (2)*, 171(3):1401–1477, 2010.
[16] Fang Wang. Radiation field for Einstein vacuum equations with spacial dimension $n \geq 4$. *Preprint, arXiv:1304.0407*, 2013.

Total mass and Sormani–Wenger intrinsic flat convergence

Jeffrey Jauregui
(joint work with Dan Lee)

Interest in the semicontinuity phenomenon of the total mass in general relativity (GR) arises from its connection with the positive mass theorem, as well as the study of Ricci flow of asymptotically flat metrics and of Bartnik's quasi-local mass. By total mass we mean the ADM mass of an asymptotically flat, totally geodesic spacelike slice of a spacetime; in the lower regularity setting, this is replaced with Huisken's isoperimetric mass concept. We present joint work in progress with Dan Lee on establishing the lower semicontinuity of total mass for pointed Sormani–Wenger intrinsic flat volume convergence.

The first result we recall is:

Theorem 1 ([8]). *If (M_j, g_j, p_j) is a sequence of pointed asymptotically flat 3-manifolds, each with nonnegative scalar curvature and with no horizons, converging in the pointed C^2 Cheeger–Gromov sense to a pointed asymptotically flat 3-manifold (N, h, q), then*

$$m_{ADM}(N, h) \leq \liminf_{j\to\infty} m_{ADM}(M_j, g_j). \tag{1}$$

Pointed C^k Cheeger–Gromov convergence essentially means C^k convergence of the metric tensors on compact sets, modulo diffeomorphisms.

Through a basic example of expanding spherical shells, one can see that equality in (1) can fail, and why the non-negativity of scalar curvature is necessary. Through a simple blow-up example, one can see that Theorem 1 actually recovers the positive mass theorem. The key ingredient in the proof of Theorem 1 is Huisken–Ilmanen's weak inverse mean curvature flow [7].

For applications in GR such as Bartnik's minimal mass extension conjecture [3], C^2 convergence is far too strong. We then consider C^0 convergence, with a C^0 limit space. For a smooth manifold M (with a possibly C^0) asymptotically flat Riemannian metric g, Huisken's isoperimetric mass [5], [6] is defined as:

$$m_{iso}(M, g) = \sup_{\{K_i\}} \limsup_{i\to\infty} \frac{2}{|\partial K_i|_g} \left(|K_i| - \frac{1}{6\sqrt{\pi}} |\partial K_i|^{3/2} \right),$$

where the supremum is taken over exhaustions $\{K_i\}$ of M by compact sets. (Here, "$|\cdot|$" is used to denote volumes and areas.) It is known that with nonnegative scalar curvature, if the metric is sufficiently regular, m_{ADM} and m_{iso} agree.

In joint work with Lee, we previously extended Theorem 1 to C^0 convergence:

Theorem 2 ([9]). *If (M_j, g_j, p_j) is a sequence of pointed asymptotically flat 3-manifolds, each with nonnegative scalar curvature and with no horizons converging in the pointed C^0 Cheeger–Gromov sense to a pointed C^0 asymptotically flat 3-manifold (N, h, q), then*

$$m_{iso}(N, h) \leq \liminf_{j\to\infty} m_{ADM}(M_j, g_j).$$

Significantly more technical work is involved in this proof. In addition to Huisken–Ilmmanen's work, we also require a modified weak mean curvature flow argument and Huisken's monotonicity of an isoperimetric defect quantity [6]. The modified flow freezes connected components once their area drops below a threshold.

Unfortunately, C^0 convergence is still too strong for GR applications; for example it is sensitive to "gravity wells" or "tentacles" of positive scalar curvature that can occur in asymptotically flat manifolds, even of arbitrarily small total mass. Thus, we turn our attention to the Sormani–Wenger intrinsic flat distance ($\mathcal{F}$) [13] that has shown promise to, for example, the "near rigidity" of the positive mass theorem; see [1], [4], [10], [11], [12], for example.

The $\mathcal{F}$-distance is defined using Ambrosio–Kirchheim's notion of currents in metric spaces [2]. Given two compact, oriented Riemannian n-manifolds M_1 and M_2, one considers all distance-preserving embeddings of M_1 and M_2 into a common space metric Z, and computes the flat distance between the corresponding pushed-forward integral currents. The $\mathcal{F}$-distance is defined by taking the infimum over all such metric spaces and embeddings. To handle spaces with infinite diameter, one may define pointed $\mathcal{F}$-convergence by considering $\mathcal{F}$-convergence of balls. Pointed $\mathcal{F}$-volume convergence means further that the volumes of the balls converge (as opposed to being merely lower semicontinuous).

The main result we present is:

Theorem 3 (J.–Lee, in progress). *Let (M_j, g_j, p_j) be a sequence of smooth, oriented asymptotically flat Riemannian 3-manifolds, with nonnegative scalar curvature and no horizons. Assume there exists a uniform positive lower bound for the isoperimetric constants of (M_j, g_j). If this sequence converges in the pointed intrinsic flat volume sense to a pointed asymptotically flat local integral current space $N = (X, d, g, T, q)$ of dimension 3, then*

$$\liminf_{j\to\infty} m_{ADM}(M_j, g_j) \geq m_{iso}(N).$$

Integral current spaces are the natural $\mathcal{F}$-limits of sequences of Riemannian manifolds, defined by Sormani–Wenger in [13]. The definition of asymptotically flat integral current space is new; in essence it requires a C^0 asymptotically flat end structure where the Riemannian metric is locally compatible with the underlying distance function. This implies the isoperimetric mass of the limit is well-defined.

We describe three of the main ingredients in the proof of Theorem 3. First, there is an estimate proved in [9] for the quasi-local isoperimetric mass that plays a critical role. If Ω is an outward-minimizing region in a (smooth) asymptotically flat 3-manifold M with nonnegative scalar curvature and no horizons, then

$$\frac{2}{|\partial\Omega|_g}\left(|\Omega| - \frac{1}{6\sqrt{\pi}}|\Omega|^{3/2}\right) \leq m_{iso}(M) + \frac{C}{\sqrt{|\partial\Omega|}} \tag{2}$$

for a well-controlled constant C.

Second, given a competitor $\Omega \subset X$ for the quasi-local isoperimetric mass in the limit space, one must construct "corresponding" regions Ω_j in M_j. (This was trivial for Cheeger–Gromov convergence, where embeddings of large regions in X into M_j is assured.) This is achieved, roughly, by considering embeddings of M_j and X into a common metric space Z (using a theorem of Sormani–Wenger [13]), viewing Ω as a sublevel set of a Lipschitz function, and eventually constructing, non-canonically, Lipschitz functions of M_j, which can be used to define Ω_j.

Third, and most difficult, is to control how the boundary masses (perimeters) of the Ω_j behave when passing to a limit. A priori they are lower semicontinuous, but unfortunately this is the wrong sign for lower semicontinuity of the quasi-local isoperimetric mass (i.e., the left-hand side of (2)). Roughly speaking, the strategy is to use the volume convergence and Ambrosio–Kirchheim's slicing theorem [2] to show that enough of the perimeters nearly converge.

Theorem 3 also provides further evidence that the $\mathcal{F}$-distance interacts well with general relativistic concepts such as scalar curvature and total mass.

References

[1] B. Allen, *Inverse Mean Curvature Flow and the Stability of the Positive Mass Theorem*, preprint `https://arxiv.org/abs/1807.08822`.

[2] L. Ambrosio and B. Kirchheim, *Currents in metric spaces*, Acta Math., **185** (2000), 1–80.

[3] R. Bartnik, *New definition of quasilocal mass*, Phys. Rev. Lett., **62** (1989), 2346-2348.

[4] L.-H. Huang, D. Lee and C. Sormani, *Intrinsic flat stability of the positive mass theorem for graphical hypersurfaces of Euclidean space*, J. Reine Angew. Math., to appear.

[5] G. Huisken, *An isoperimetric concept for mass and quasilocal mass*, Oberwolfach Reports **3** (2006), 87–88.

[6] G. Huisken, *An isoperimetric concept for the mass in general relativity*, `https://video.ias.edu/node/234`, (2009).

[7] G. Huisken and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom., **59** (2001), 353–437.

[8] J. Jauregui, *On the lower semicontinuity of the ADM mass*, Comm. Anal. Geom. **26** (2018), 85–111.

[9] J. Jauregui and D. Lee, *Lower semicontinuity of mass under C^0 convergence and Huisken's isoperimetric mass*, J. Reine Angew. Math., to appear.

[10] D. Lee and C. Sormani, *Stability of the Positive Mass Theorem for Rotationally Symmetric Riemannian Manifolds*, J. Reine Angew. Math., **686** (2014), 187–220.

[11] A. Sakovich and C. Sormani, *Almost Rigidity of the Positive Mass Theorem for Asymptotically Hyperbolic Manifolds with Spherical Symmetry*, General Relativity and Gravitation, to appear.
[12] C. Sormani and I. Stavrov, *Geometrostatic manifolds of small ADM Mass*, Comm. Pure Appl. Math., to appear.
[13] C. Sormani and S. Wenger, *The intrinsic flat distance between Riemannian manifolds and other integral current spaces*, J. Differential Geom., **87**, (2011), 117–199.

Topology and singularities in cosmological spacetimes

Gregory J. Galloway
(joint work with Eric Ling)

A theme of long standing interest in global General Relativity concerns the relationship between the topology of spacetime and the occurrence of singularities, as expressed in terms of causal geodesic incompleteness. Many such results have centered on the notion of *topological censorship*; see e.g. [2] (which includes a brief review of the topic) and references therein. The results on topological censorship apply to the asymptotically flat setting, or other such settings, which are not directly relevant to cosmology. The main aim of our talk is to present recent work with Eric Ling [3] concerning the relationship between topology and singularities in the cosmological setting. More specifically, we consider globally hyperbolic spacetimes with compact Cauchy surfaces in a setting compatible with the presence of a positive cosmological constant. (See e.g. [4, 6] for basic causal theoretic notions.)

The classical Hawking singularity theorem [4, p. 272] establishes past timelike geodesic incompleteness in spatially closed spacetimes that at some stage are future expanding. This singularity theorem requires the Ricci tensor of spacetime to satisfy the strong energy condition, $\mathrm{Ric}(X, X) \geq 0$ for all timelike vectors X. However, in spacetimes obeying the Einstein equations with positive cosmological constant, $\Lambda > 0$, this energy condition is not in general satisfied, and the conclusion then need not hold; de Sitter space, which is geodesically complete and (in $3+1$ dimensions) satisfies

$$\mathrm{Ric} = \Lambda g \qquad (\Lambda > 0) \tag{1}$$

is an immediate example. Although the strong energy condition fails, note that the null energy condition (NEC), $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X, holds. In fact the NEC is compatible with the presence of a cosmological constant, regardless of sign.

Dust filled FLRW spacetimes with positive cosmological constant provide further examples in which the conclusion to Hawking's theorem can fail, but now topology enters the picture. Consider FLRW models with compact Cauchy surfaces:

$$M = (0, \infty) \times V^3, \qquad ds^2 = -dt^2 + R^2(t) d\sigma_k^2$$

where $(V^3, d\sigma_k^2)$ is a compact Riemannian manifold of constant curvature $k = -1, 0, +1$. These three cases are topologically quite distinct. For instance, in the

$k = +1$ (spherical space) case, the Cauchy surfaces have finite fundamental group, while in the $k = 0, -1$ (toroidal and hyperbolic 3-manifold) cases, the fundamental group is infinite.

Assuming a collisionless perfect fluid (dust), one can solve the Einstein equations with $\Lambda > 0$, and analyze the behavior of the scale factor $R(t)$, which is done in many standard text books. One finds that in the case $k = -1$ or $k = 0$, there is aways a big bang singularity. Only in the $k = +1$ case, can the past big-bang singularity be avoided. As it turns out, this behavior holds in a much broader context.

In [1], this topology dependent behavior was studied (without symmetry assumptions) for spacetimes which are asymptotically de Sitter in the sense of admitting a regular spacelike conformal (Penrose) compactification. For example, in this more general setting, it was shown that past timelike geodesic incompleteness is related to the Yamabe type of future null infinity $\mathscr{I}^+$ (with negative or zero Yamabe type leading to incompleteness). Here we present a result in a similar vein (recently obtained in [3]) which strengthens various aspects of the results in [1]. In particular, the assumption that spacetime be future asymptotically de Sitter is not needed, and, apart from the NEC, no further curvature conditions are required.

Theorem 1. *Suppose V is a smooth compact spacelike Cauchy surface in a $3+1$ dimensional spacetime (M, g) that satisfies the null energy condition (NEC), $\mathrm{Ric}(X, X) \geq 0$ for all null vectors X. Suppose further that V is expanding in all directions (i.e. the second fundamental form of V is positive definite). Then either*

(i) V is a spherical space, or

(ii) M is past null geodesically incomplete.

In other words, if M is past null geodesically complete, the only possible Cauchy surface topology is that of a spherical space; any other topology leads to past null incompleteness. Here, by a spherical space, we mean that V is diffeomophic to a quotient of the 3-sphere S^3, $V = S^3/\Gamma$ (where Γ is isomorphic to a subgroup of $SO(4)$). By taking quotients of de Sitter space, we see that there are geodesically complete spacetimes satisfying the assumptions of the theorem, having Cauchy surface topology that of any spherical space.

We make some remarks about the proof. The proof makes use of fundamental existence results for minimal surfaces and takes advantage of our current understanding of the topology of 3-manifolds, specfically the positive resolution of Thurston's geometrization conjecture, and subsequent consequences of it. Ultimately, the proof relies on the Penrose singularity theorem. Underlying the proof is the following simple observation:

Lemma 1. *Let Σ be a compact minimal surface in a smooth spacelike Cauchy surface which is expanding in all directions, i.e. which has positive definite second fundamental form. Then Σ is a* past *trapped surface.*

Proof. One can express the null expansion scalars $\theta_\pm$ of Σ in terms of initial data on Σ; specifically one has

$$\theta_\pm = -\mathrm{tr}_\Sigma K \pm H \,, \tag{2}$$

where K is the second fundamental form Σ and H is the mean curvature of Σ in V. Since Σ is minimal ($H = 0$) and K is positive definite, one has $\theta_\pm < 0$. □

The strategy of the proof is then as follows. Assuming M is past null geodesically complete we show that V must be a spherical space. We can reduce to the case that V is orientable. Then by the prime decomposition theorem (and the positive resolution of the Poincaré conjecture!), V can be expressed as a connected sum,

$$V = V_1 \# V_2 \# \cdots \# V_k \,, \tag{3}$$

where for each $i = 1, ...k$,

(1) V_i is a spherical space, or
(2) V_i is diffeomorphic to $S^2 \times S^1$, or
(3) V_i is a $K(\pi, 1)$ manifold.

We can eliminate possibilities (2) and (3) briefly as follows. In these cases we can establish the existence of a compact minimal surface in a noncompact Cauchy surface $\tilde{V}$ for a covering spacetime $(\tilde{M}, \tilde{g})$ of (M, g), with $\tilde{V}$ expanding in all directions. (For case (2), we make use of an argument in [1, Theorem 4.3]. For case (3), we make use of the positive resolution of the *surface subgroup conjecture* [5] and a well-known existence result for minimal surfaces due to Schoen and Yau [7].) Lemma 1, together with an application of the Penrose singularity theorem, leads to the past null geodesic incompleteness of $(\tilde{M}, \tilde{g})$, and hence of (M, g), contrary to assumption. This shows that V is a connected sum of spherical spaces. Similar sorts of arguments can then be used to show that there can be only one spherical space.

Of course it is possible that, in Theorem 1, if V is other than a spherical space, the consequent past null completeness is due to the formation of a Cauchy horizon. As such, it would be desirable to weaken the assumption of global hyperbolicity.

References

[1] L. Andersson and G. J. Galloway, *dS/CFT and spacetime topology*, Adv. Theor. Math. Phys. **6** (2002), no. 2, 307–327.

[2] M. Eichmair, G. J. Galloway, and D. Pollack, *Topological censorship from the initial data point of view*, J. Differential Geom. **95** (2013), no. 3, 389–405.

[3] G. J. Galloway and E. Ling, *Topology and singularities in cosmological spacetimes obeying the null energy condition*, Commun. Math. Phys. **360** (2018), no. 2, 611–617.

[4] S. W. Hawking and G. F. R. Ellis, *The large scale structure of space-time*, Cambridge University Press, London, 1973, Cambridge Monographs on Mathematical Physics, No. 1.

[5] J. Kahn and V. Markovic, *Immersing almost geodesic surfaces in a closed hyperbolic three manifold*, Ann. of Math. (2) **175** (2012), no. 3, 1127–1190.

[6] B. O'Neill, *Semi-Riemannian geometry*, Pure and Applied Mathematics, vol. 103, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1983.

[7] R. Schoen and S.-T. Yau, *Existence of incompressible minimal surfaces and the topology of three-dimensional manifolds with nonnegative scalar curvature*, Ann. of Math. (2) **110** (1979), no. 1, 127–142.

Black hole instabilities and violation of the weak cosmic censorship conjecture in higher dimensional asymptotically flat spacetimes

Pau Figueras
(joint work with Markus Kunesch, Saran Tunyasuvuankool, Luis Lenher)

Proving or disproving the weak cosmic censorship conjecture (WCC) [1, 2] is perhaps one of the most important unresolved problems in classical general relativity. One can formulate this conjecture around any spacetime which is a solution of the Einstein equation, and in particular around black hole spacetimes. In the latter case, the conjecture relates to another open problem in general relativity, namely the stability problem for black hole spacetimes.

Even though proving the non-linear stability of the Kerr black hole is still an open problem, the recent linear stability results, (see e.g., [3, 4]), strongly suggest that such a proof is within reach. On the other hand, the situation is radically different in spacetime dimensions $D > 4$, where black holes in vacuum can be unstable to gravitational perturbations. This was first famously shown by Gregory and Laflamme (GL) [5] for the case of black strings and black p-branes in the asymptotically Kaluza–Klein spaces. This result has been recently extended to various types of asymptotically flat higher dimensional vacuum black holes [6, 7, 8, 9, 10].

Given that some black holes can be unstable, the natural question to ask is what is the endpoint of such instabilities. And, is the WCC around such spacetimes violated? Addressing this question involves evolving unstable black hole spacetimes into the fully non-linear regime of the Einstein equation, and the only tools available at the moment are numerical methods. Using numerical relaltivity, Lehner and Pretorious [11] showed that unstable black strings in $D = 5$ spacetime dimensions evolve, in a self-similar fashion, into dynamical black holes with fractal horizons consisting of spherical bulges connected by long and thin black strings on different scales. The black string segments are GL unstable, leading to a cascade of instabilities that results in the pinch off of the horizon, and hence a naked singularity, in finite asymptotic time. Since there is no fine tuning of the initial data, this result can constitute an example of a violation of the WCC conjecture in spacetimes with Kaluza–Klein asymptotics.

The result of [11] left some important open questions. Firstly, the dynamics of the GL instability is intimately related to the fact that the spacetime itself has a non-trivial topology. Indeed, black strings in $D = 5$ have a horizon topology of $S^1 \times S^2$, where the S^1 is the non-contractible Kaluza–Klein circle, and this non-contractible S^1 plays a crucial role in the GL dynamics. Second, the original WCC conjecture was formulated in asymptotically flat spaces, and therefore one would like to know whether the conjecture is true or not and under which circumstances in such spaces.

In our first paper [12] we considered the endpoint of the GL instability of asymptotically flat black rings in 5 spacetime dimensions.[1] Black rings are know explicitly in $D = 5$ [13] (see also [14] for a review), and the spatial sections of the horizons have topology $S^1 \times S^2$. The solution of [13] rotates along the S^1, which is a contractible circle since the topology of the spacetime is trivial. Black rings occur in two families, known as 'thin' or 'fat' according to their physical and geometrical properties. [10] showed that all thin rings are unstable under a GL type of instability, and hence it is natural to ask whether the WCC around such spacetimes can be violated, similarly to what happens with black strings. Note that in this case, the GL modes break the rotational symmetry along the S^1 and hence they will lead to gravitational wave emission in the non-linear regime. If the loss of angular momentum and mass through gravitational wave emission is efficient enough, unstable black rings can collapse into a spherical black holes (since the S^1 of the ring is contractible) and hence now violation of the WCC will occur.

In [12] we showed that thin but not so thin rings (in the language of [13]) also suffer from a new type of instability, which we called "elastic". This instability dominates for low enough angular momentum, and its endpoint is a rotating spherical black hole, with a lower angular momentum. Hence no violation of WCC occurs for this range of thin rings. On the other hand, for thin enough black rings, the GL instability dominates and its evolution is qualitatively similar to the black string case: the horizon develops a structure of bulges connected by long and thin strings. The latter are very unstable and their local dynamics should be the same as in [11]; in particular, they should pinch off on an exponentially fast time scale, as opposed to the collapse time, which is dominated by gravitational wave emission and hence polynomial in time. Therefore, the results presented in [12] provide strong evidence that sufficiently thin black rings will pinch off in finite time, thus resulting in a naked singularity and hence potentially violating the WCC in asymptotically flat spaces in $D = 5$ dimensions.

The results in [12] left a number of questions unanswered, most notably, 1) we could not estimate the timescale of the pinch off and, 2) we could not determine whether the process is self-similar. To address these two questions in the asymptotically flat context, in [15] we studied the endpoint of the ultraspinning instability of singly spinning Myers–Perry black holes in $D = 6$ spacetime dimensions. These instabilities preserve the two rotational symmetries of the background and hence the dynamics can be consistently truncated to a 2+1 system. The main results in [12] show that these black holes pinch off in finite time but the dynamics is no longer self-similar. The ultimate reason for this is conservation of angular momentum, which is redistributed along the horizon and hence different regions evolve on different time scales. It is precisely this local dynamics on the horizon that breaks self-similarity. Interestingly though, the minimum thickness of the horizon follows a scaling law.

[1] From a computational point of view, evolving black rings is effectively a $3+1$ problem, while the evolution of the GL instability of black string is a $2+1$ problem.

In our ongoing work [16] we study the evolution of non-axisymmetric instabilities of rotating black holes in $D = 6$, but for larger values of the angular momentum than [8]. Our results show that for sufficiently rapidly rotating black holes, even if they do not suffer from a GL type of instability, the non-linear dynamics leads to deformations of the horizon such that local GL instabilities should kick in. Therefore, for sufficiently large angular momentum, unstable spherical rotating black holes should also generically evolve into naked singularities in finite time.

Our results motivate us to make the following conjectures:

Conjecture 1: The Gregory–Laflamme instability is the only mechanism that GR has to change the horizon topology.

Conjecture 2: The only stable black hole in $D > 4$ spacetime dimensions is the Myers–Perry black hole with $J/M^{D-3} \leq O(1)$.

Resolving these conjectures should provide a general picture of the dynamics of black holes in $D > 4$ asymptotically flat spaces.

References

[1] R. Penrose, Riv. Nuovo Cim. **1** (1969) 252 [Gen. Rel. Grav. **34** (2002) 1141].

[2] D. Christodoulou, "On the global initial value problem and the issue of singularities" Class. Quant. Grav., **16** (1999), A23.

[3] M. Dafermos, I. Rodnianski and Y. Shlapentokh-Rothman, arXiv:1402.7034 [gr-qc].

[4] M. Dafermos, G. Holzegel and I. Rodnianski, "The linear stability of the Schwarzschild solution to gravitational perturbations," arXiv:1601.06467 [gr-qc].

[5] R. Gregory and R. Laflamme, Phys. Rev. Lett. **70** (1993) 2837 doi:10.1103/PhysRevLett.70.2837 [hep-th/9301052].

[6] O. J. C. Dias, P. Figueras, R. Monteiro, J. E. Santos and R. Emparan, Phys. Rev. D **80** (2009) 111701 doi:10.1103/PhysRevD.80.111701 [arXiv:0907.2248 [hep-th]].

[7] O. J. C. Dias, P. Figueras, R. Monteiro, H. S. Reall and J. E. Santos, JHEP **1005** (2010) 076 doi:10.1007/JHEP05(2010)076 [arXiv:1001.4527 [hep-th]].

[8] M. Shibata and H. Yoshino, Phys. Rev. D **81** (2010) 104035 doi:10.1103/PhysRevD.81.104035 [arXiv:1004.4970 [gr-qc]].

[9] . J. C. Dias, G. S. Hartnett and J. E. Santos, Class. Quant. Grav. **31** (2014) no.24, 245011 doi:10.1088/0264-9381/31/24/245011 [arXiv:1402.7047 [hep-th]].

[10] J. E. Santos and B. Way, Phys. Rev. Lett. **114** (2015) 221101 doi:10.1103/PhysRevLett.114.221101 [arXiv:1503.00721 [hep-th]].

[11] L. Lehner and F. Pretorius, Phys. Rev. Lett. **105** (2010) 101102 doi:10.1103/PhysRevLett.105.101102 [arXiv:1006.5960 [hep-th]].

[12] P. Figueras, M. Kunesch and S. Tunyasuvunakool, Phys. Rev. Lett. **116** (2016) no.7, 071102 doi:10.1103/PhysRevLett.116.071102 [arXiv:1512.04532 [hep-th]].

[13] R. Emparan and H. S. Reall, Phys. Rev. Lett. **88** (2002) 101101 doi:10.1103/PhysRevLett.88.101101 [hep-th/0110260].

[14] R. Emparan and H. S. Reall, Class. Quant. Grav. **23** (2006) R169 doi:10.1088/0264-9381/23/20/R01 [hep-th/0608012].

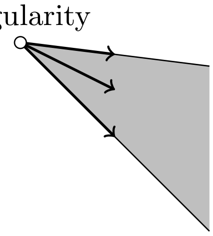
[15] P. Figueras, M. Kunesch, L. Lehner and S. Tunyasuvunakool, Phys. Rev. Lett. **118** (2017) no.15, 151103 doi:10.1103/PhysRevLett.118.151103 [arXiv:1702.01755 [hep-th]].

[16] H. Bantilan, P. Figueras, M. Kunesch, R. P. Macedo, *work in progress.*

The Asymptotically Self-Similar Regime for the Einstein Vacuum Equations

Yakov Shlapentokh-Rothman
(joint work with Igor Rodnianski)

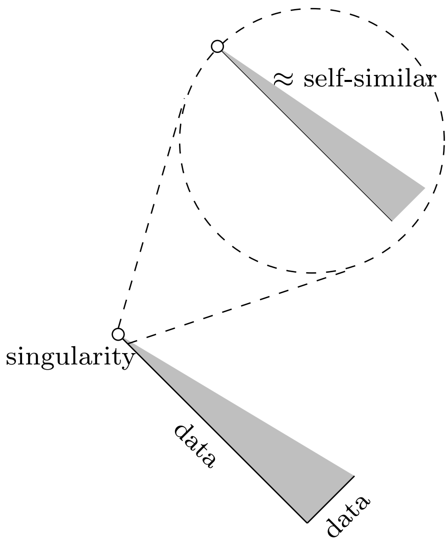
Loosely speaking, a singular solution to the Einstein vacuum equations $\mathrm{Ric}(g) = 0$ is called self-similar if the singularity is scale invariant in that there exists a vector field K which "vanishes" at the singularity and satisfies $\mathcal{L}_K g = 2g$:



Some motivation for studying self-similar solutions are as follows:

(1) There is numerical (and heuristic) evidence that (approximate and/or discrete) self-similar solutions play an important underlying role in many interesting dynamical situations. Two particularly relevant examples are "Type-II critical phenomena" in gravitational collapse [1, 6, 8] and the behavior near the endpoint of the Gregory–Laflamme instability [5, 7].
(2) In his work on the cosmic censorship conjecture for the spherically symmetric Einstein-scalar field system, Christodoulou established a well-posedness result for so-called "solutions of bounded variation" [2]. While these solutions may be quite singular on the axis of symmetry, the norms defining the space are scale invariant, and the individual solutions satisfy the property that rescalings around any point on the axis eventually converge to a self-similar solution.
(3) The ambient metrics of Fefferman–Graham [4, 3], which play such an important role in the classification of conformal invariants and the AdS-CFT correspondence, are formal expansions for a self-similar solution.

Our main result develops a local theory underlying the dynamical construction (in all spacetime dimensions) of "asymptotically self-similar solutions". By a "dynamical construction", we mean that our solutions arise from suitable characteristic Cauchy data, and by "asymptotically self-similar" we mean that successive rescalings around the singular point eventually converge to a self-similar solution.



A corollary of our main result is the construction of actual solutions corresponding to all of the formal power series expansions of Fefferman–Graham [3, 4].

References

[1] M. Choptuik *Universality and scaling in gravitational collapse of a massless scalar field*, Physical Review Letters **70** (1993), 9–12.
[2] D. Christodoulou *Bounded variation solutions of the spherically symmetric Einstein-scalar field equations*, Communications in Pure and Applied Mathematics **46** (1993), no. 8, 1131–1220.
[3] C. Fefferman and R. Graham *The Ambient Metric*, Annals of Mathematics Studies **178** (2012) Princeton University Press, Princeton, NJ, 113 pp.
[4] C. Fefferman and R. Graham *Conformal Invariants*, Astérisque (1984), 95–116.
[5] R. Gregory and R. Laflamme *Black strings and p-branes are unstable*, Physical Review Letters **70** (1993), no. 19, 2837–2840.
[6] C. Gundlach and J. Martín-García *Critical Phenomena in Gravitational Collapse*, Living Reviews in Relativity **10:5** (2007).
[7] L. Lehner and F. Pretorius *Black strings, low viscosity fluids, and violation of cosmic censorship*, Physical Review Letters **105** (2010), no. 10, 101102, 4 pp.
[8] A. Ori and T. Piran *Self-similar spherical gravitational collapse and the cosmic censorship hypothesis*, General Relativity and Gravitation **20** (1988), no. 1, 7–13.

On the smoothness of the critical sets of the cylinder at spatial infinity in vacuum spacetimes

Tim-Torben Paetz

Penrose [6] suggested an elegant geometric approach to define the notion of *asymptotic flatness* in general relativity. A spacetime is regarded as *asymptotically flat* supposing that it admits a *smooth conformal completion at infinity.* By this it is meant that, after a suitable conformal rescaling, one can attach a boundary $\mathcal{I}$ to the spacetime through which the rescaled metric admits a *smooth* extension.

We consider spacetimes which satisfy Einstein's vacuum field equations. Then $\mathcal{I}$ is a null hypersurface. We further assume the "natural" topology $\mathcal{I} \cong \mathbb{R} \times S^2$, which implies that the Weyl tensor vanishes on $\mathscr{I}$ (in fact, it shows a specific peeling behavior). An open issue is to what extent Penrose's notion of asymptotic flatness is compatible with the vacuum equations in the sense that a sufficiently large class of solutions to model the physical situations of interest is admitted.

The conformal field equations (CFE) [7, 9] provide a system of equations which is equivalent to Einstein's in regions where the conformal factor is positive, but remains regular where it vanishes. This permits the formulation of an *asymptotic Cauchy problem* where data are prescribed on either a portion of $\mathscr{I}^-$ and an incoming null hypersurface [12], or on $\mathscr{I}^-$ in a setting where it forms the future light-cone of a regular point i^- representing past timelike infinity [4]. While this shows that there is a large class of asymptotically flat vacuum spacetimes no insight is provided into how generic such spacetimes are.

For this, it is beneficial to study *hyperboloidal* and *characteristic Cauchy problems*. In both cases [8, 3] the emerging vacuum spacetime admits at least a piece of a smooth $\mathscr{I}$, supposing that the initial data admit a smooth extension through the 2-sphere S where the initial surface hits $\mathscr{I}$. The data need to fulfill constraint equations, solutions whereof can be constructed from suitable freely prescribable *seed data*. It turns out, though, that seed data which are smooth at S will in general produce solutions of the constraints which are only *polyhomogeneous* [1, 2, 5, 13]. However, one can get rid of all log-terms if the leading order expansion coefficients of the seed data are suitably adjusted. While, again, this shows that large classes of asymptotically flat vacuum spacetimes exist, it raises the question whether a polyhomogeneous $\mathscr{I}$ might be more natural than a smooth one.

To fully understand the restrictions which arise from a smooth $\mathscr{I}$, one needs to study a spacelike Cauchy problem with asymptotically Euclidean data which incorporates spatial infinity. Hintz & Vasy [11] proved that an appropriate class of data which are polyhomogeneous at spatial infinity produces a polyhomogeneous $\mathscr{I}$. As their gauge produces log-terms, too, it is not clear under which additional conditions on the data a gauge transformation yields a smooth $\mathscr{I}$.

To analyze the appearance of log-terms near spatial infinity it is convenient to use a blow-up to a cylinder I [10]. The CFE provide a symmetric hyperbolic system of evolution equations. However, hyperbolicity breaks down at the *critical sets* $I^\pm \cong S^2$ where the cylinder "touches" $\mathscr{I}^\pm$. One can construct data which extend smoothly through $I^0 \cong S^2$, where the Cauchy surface runs into I. The CFE become inner equations on I so that, in principle, on I, all fields, including derivatives of any order, can be computed from the data. Generically, though, the fields are merely polyhomogeneous at $I^\pm$ [10, 15], and one expects the log-terms to spread over to $\mathscr{I}$. A full characterization of Cauchy data which permit smooth extensions through $I^\pm$ turns out to be subtle as, in contrast to hyperboloidal and characteristic Cauchy problem, log-terms can arise at *any* order.

To gain a better understanding and establish geometric restriction on the data, it is beneficial to set up an asymptotic Cauchy problem with data on $\mathscr{I}^-$ (and some incoming null hypersurface) and analyze the appearance of log-terms at the critical set I^-. A main advantage of this approach is that I^- arises as the future boundary of the initial surface $\mathcal{I}^-$, whence it is easier to control the fields there.

For the analysis [14] we use a *conformal Gauss gauge* [9], adapted to a congruence of timelike conformal geodesics. We then show that if the radiation field, which represents the seed data on $\mathcal{I}^-$, vanishes at all orders at I^-, no log-terms are produced there. This requires an analysis of the constraints on $\mathcal{I}^-$ and their transverse derivatives of all orders. It turns out that only equations satisfied by certain components of the rescaled Weyl tensor $W_{ijkl} := \Theta^{-1}C_{ijkl}$ provide potential sources of log-terms. In our current setting they do not produce log-terms as the source has a specific structure,

$$r^{n+3}\partial_r(r^{-n-2}\partial_\tau^n W)|_{\mathcal{I}^-} = \mathcal{P}^{n+1} + O(r^\infty). \tag{1}$$

Here W is symbolic for certain components of the rescaled Weyl tensor. I^- is located at $r = 0$, ∂_τ is transverse to $\mathcal{I}^-$, and $\mathcal{P}^k$ is a polynomial in r of degree k.

The situation is very similar when approaching I^- from I, where an analysis of the transport equations (and their radial derivatives) shows that the source term has a somewhat complementary structure (I^- is located at $\tau = -1$),

$$(1+\tau)^{n-1}\partial_\tau[(1+\tau)^{2-n}\partial_r^n W)]|_I = O(1+\tau)^{n-1}. \tag{2}$$

One is led to the question whether the vanishing of the radiation field at all orders at I^- is also necessary for the non-appearance of log-terms there. To tackle this issue, we assume that the limit M of the Bondi mass aspect to I^- is constant (and non-zero), and that the same holds for the limit N of the Bondi dual mass aspect, which requires $N = 0$. In that case the vanishing of the radiation field at all orders at I^- is necessary. To prove this, one needs to compute the log-producing term of order $n + 2$ in (1) and require it to vanish. In fact, the corresponding *no-logs condition* can explicitly be expressed in terms of the asymptotic data. The proof works by induction. Assuming that the first m_0 expansion coefficients of the radiation field vanish at I^-, the first non-trivial no-logs condition adopts the form

$$\prod_{\ell=1}^{m_0}(\Delta_s + \ell(\ell+1))\partial_r^{m_0+1}W_{\mathrm{rad}}|_{I^-} = 0\,, \tag{3}$$

where Δ_s is the standard Laplacian on the round 2-sphere, and W_{rad} are the Hodge decomposition scalars of the radiation field. From this condition one cannot deduce the vanishing of W_{rad} which may contain spherical harmonics up to order m_0. When going to the next order, though, this Taylor coefficient appears in the source,

$$\prod_{\ell=1}^{m_0+1}(\Delta_s + \ell(\ell+1))\partial_r^{m_0+2}W_{\mathrm{rad}}|_{I^-} = M \cdot f(\partial_r^{m_0+1}W_{\mathrm{rad}}|_{I^-})\,, \tag{4}$$

and brings in precisely those spherical harmonics which destroy solvability of (4). A calculation shows that no cancellations occur and that $\partial_r^{m_0+1}W_{\mathrm{rad}}|_{I^-}$ needs to vanish. To conclude, while certain spherical harmonics don't produce log-terms immediately, they do produce them in the next order (this statement is wrong for the massless spin-2 equation as there is no coupling between the no-logs conditions).

Theorem 1 ([14]). *(i) Assume that a smooth vacuum spacetime with smooth $\mathcal{I}^-$, I and I^- has been given and assume further that $M = \text{const.} \neq 0$ and $N = 0$. Then the radiation field vanishes at I^- at any order.*

(ii) Conversely, the restriction of all the fields appearing in the CFE to both $\mathcal{I}^-$ and I, and all derivatives thereof, admit smooth extensions through I^- if the radiation field vanishes there at any order.

References

[1] L. Andersson, P.T. Chruściel: *On asympt. behavior of solutions of the constraint equations in general relativity with hyperboloidal boundary conditions*, Diss. Math. **355** (1996) 1–100.

[2] L. Andersson, P.T. Chruściel, H. Friedrich: *On the regularity of solutions to the Yamabe equation and the existence of smooth hyperboloidal initial data for Einstein's field equations*, Comm. Math. Phys. **149** (1992) 587–612.

[3] A. Cabet, P.T. Chruściel, R. Tagne Wafo: *On the characteristic initial value problem for nonlinear symmetric hyperbolic systems, including Einstein equations*, Diss. Math. **515** (2016) 1–72.

[4] P.T. Chruściel, T.-T. Paetz: *Solutions of the vacuum Einstein equations with initial data on past null infinity*, Class. Quantum Grav. **30** (2013) 235037.

[5] P.T. Chruściel, T.-T. Paetz: *Characteristic initial data and smoothness of Scri. I. Framework and results*, Ann. Henri Poincaré **16** (2015) 2131–2162.

[6] R. Penrose: *Zero rest-mass fields including gravitation: Asymptotic behavior*, Proc. R. Soc. Lond. A **284** (1965) 159–203.

[7] H. Friedrich: *On the regular and the asymptotic characteristic initial value problem for Einstein's vacuum field equations*, Proc. R. Soc. Lond. A **375** (1981) 169–184.

[8] H. Friedrich: *Cauchy problems for the conformal vacuum field equations in general relativity*, Comm. Math. Phys. **91** (1983) 445–472.

[9] H. Friedrich: *Einstein equations and conformal structure: Existence of Anti-de Sitter-type space-times*, J. Geom. Phys. **17** (1995) 125–184.

[10] H. Friedrich: *Gravitational fields near space-like and null infinity*, J. Geom. Phys. **24** (1998) 83–163.

[11] P. Hintz, A. Vasy: *A global analysis proof of the stability of Minkowski space and the polyhomogeneity of the metric*, (2017), arXiv:1711.00195 [math.AP].

[12] J. Kánnár: *On the existence of C^∞ solutions to the asymptotic characteristic initial value problem in general relativity*, Proc. R. Soc. Lond. A **452** (1996) 945–952.

[13] T.-T. Paetz: *Characteristic initial data and smoothness of Scri. II. Asymptotic expansions and construction of conformally smooth data sets*, J. Math. Phys. **55** (2014) 102503.

[14] T.-T. Paetz: *On the smoothness of the critical sets of the cylinder at spatial infinity in vacuum spacetimes*, 2018), arXiv:1804.05034 [gr-qc].

[15] J.A. Valiente Kroon: *Regularity Conditions for Einstein's Equations at Spatial Infinity*, Ann. Henri Poincaré **10** (2009) 623–671.

Instability theory for self-gravitating isolated relativistic galaxies and stars

Mahir Hadžić
(joint work with Zhiwu Lin, Gerhard Rein)

We consider compactly supported steady states of the spherically symmetric asymptotically flat Einstein–Vlasov (EV) system and Einstein–Euler (EE) system, describing steady galaxies and stars respectively. For the EV-system it is well-known that given a microscopic equation of state which prescribes the functional dependence of the phase spaces density on the local particle energy, one can find a 1-parameter family of solutions parametrised by the size of the central redshift of the steady galaxy. We prove that, for sufficiently high central redshifts such steady galaxies are necessarily unstable. Moreover, we show that the instability is driven by the existence of a growing mode and prove a linear exponential trichotomy result. For the former statement, we characterise the behaviour of steady galaxies in a suitably rescaled region around the centre of symmetry, by showing convergence to a singular limiting object known as the Bisnovatyi-Kogan–Zeldovich solution of the massless Einstein–Vlasov system. To prove the exponential trichotomy we capitalise on the manifestly Hamiltonian character of the linearised equation and use a recent theory developed by Lin and Zeng. An analogous statement applies to the spherically symmetric steady states of the EE-system.

Free S^1-symmetric static solutions

Martín Reiris

In 1917 H. Weyl published a seminal paper in Annalen der Physik, describing a simple method of solving the static axisymmetric vacuum Einstein equations. The virtue of the procedure rooted in casting the Einstein equations into a linear equation on a two dimensional domain plus another always solvable by quadratures. This influential 'generating technique' allowed Myers in 1987 (and later Korotkin-Nicolai in 1994) to 'superpose' infinitely many Schwarzschild holes along a common axis, where consecutive horizons are separated from each other by a characteristic length. Quotients of this 'periodic' solution by translations give asymptotically Kasner but metrically complete 'static' black hole solutions, with a finite number of compact horizons.

The Myers/Korotkin–Nicolai solutions play an important role in the understanding of metrically complete static solutions with compact horizons (shortly below, *static black holes*). In the papers arXiv:1806.00818, arXiv1806.00819 it was proved the following classification theorem of static vacuum black holes.

Theorem 1 (The classification Theorem)**.** *Any static black hole is either,*

(1) A Schwarzschild black hole, or,
(2) a Boost, or,
(3) is of Myers/Korotkin–Nicolai type.

As the theorem does not assume any a priori asymptotic, it is therefore a generalisation of the celebrated Schwarzschild's uniqueness theorem.

The Boosts are quotients of the Rindler wedge of the Minkowski space-time. Thus, they are flat and their topologies are $\mathrm{T}^2 \times [0,\infty)$ (see arXiv:1806.00819). On the other hand a solution is of Myers/Korotkin–Nicolai type if it has the same topology and Kasner asymptotic as the Myers/Korotkin–Nicolai black holes introduced above.

A large part of these two works was dedicated to show that metrically complete static black holes are either asymptotically flat or asymptotically Kasner. When the volume growth is subcubic, then it can be proved by scaling techniques that a S^1 or a T^2 symmetry forms asymptotically (one must use here the quadratic a priori curvature decay due to M. Anderson). It is not then a surprise that the study of the asymptotic of static ends required a previous understanding of spaces with a free S^1-symmetry.

This links to Weyl's work, or indeed to the larger field of static solutions with a free S^1 symmetry. Note that the perpendicular distribution of Killing fields generating free S^1-symmetries are not necessarily integrable, in contrast to the Killing fields of axisymmetry.

The reduced data $(S;q,U,V)$ of a free S^1-symmetric static data, consists of a two-manifold S, a two-metric q, the field $U = \ln N$ where N is the lapse, and the field V that is equal to the logarithm of the norm of the Killing generating the S^1-symmetry. We then prove the following theorem on the structure of static solutions with a free S^1-symmetry (see arXiv:1806.00818). This theorem is a crucial piece of information inside the proof of the classification Theorem 1.

Theorem 2. *Let $(S;q,U,V)$ be a metrically complete (reduced) static data set with S non-compact and ∂S compact. Then,*

(1) S has a finite number of ends each diffeomorphic to $\mathrm{S}^1 \times [0,\infty)$.
(2) The fields U and V, and the Gaussian curvature κ have the following decay,

$$|\nabla U|^2(p) \leq \frac{\eta}{d^2(p,\partial S)}, \qquad |\nabla V|^2(p) \leq \frac{\eta}{d^2(p,\partial S)}, \tag{1}$$

and,

$$\kappa(p) \leq \frac{\eta}{d^2(p,\partial S)}. \tag{2}$$

The theorem provides global information, on the topology and the fields, from which a large amount of information can be obtained. The proof involves using techniques á la Bakry-Émery, plus a careful understanding, in the case when the twist is non-zero, of the geometry of high-curvature regions, that turn of to be model by a special type of solutions called the cigars.

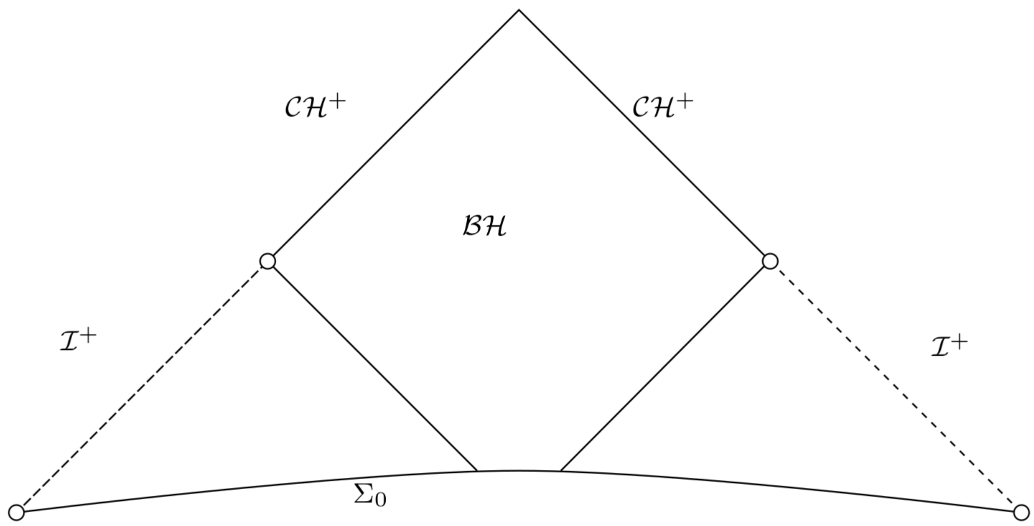
Strong cosmic censorship and generic mass inflation for charged black holes in spherical symmetry

Sung-Jin Oh
(joint work with Jonathan Luk)

The subject of this talk is the *strong cosmic censorship conjecture* and *mass inflation*, which concern the stability and instability properties of the Cauchy horizon in the interior of a rotating or charged black hole. Our main results are rigorous formulation and proof of these conjectures for a widely-studied model system, namely the *Einstein–Maxwell–Scalar Field system in spherical symmetry.*

1. Formulation of the problems

To introduce these problems, we focus on the simplest example, namely, the (subextremal) Reissner–Nordström spacetime; it is a spherical symmetric solution to the Einstein–Maxwell equation that describes an electrically charged, static black hole. The properties of our interest are most conveniently expressed in the Penrose diagram of the Reissner–Nordström spacetime[1]:



Here, Σ_0 is a complete spacelike Cauchy hypersurface, which has two asymptotically flat ends. Associated to each asymptotically flat end is (complete) future null infinity $\mathcal{I}^+$, a portion of the ideal future boundary of the Penrose diagram consisting of limits of radial future-directed null geodesics along which $r \to \infty$. The remainder of the ideal future boundary of the Penrose diagram is called the *Cauchy horizon* $\mathcal{CH}^+$, which is null and lies in the black hole interior $\mathcal{BH}$ (i.e., the complement of the causal past of $\mathcal{I}^+$).

[1]As we work with the initial value formulation, we only consider the forward maximal development of a complete Cauchy hypersurface Σ_0 inside the subextremal Reissner–Nordström spacetime.

The Cauchy horizon in a subextremal Reissner–Nordström spacetime turns out to be smooth (in fact, real-analytic)[2]. This feature leads to the troubling phenomenon of *breakdown of global uniqueness*: The spacetime is smoothly extendible past $\mathcal{CH}^+$, even as a solution to the Einstein–Maxwell equation, in many different ways that cannot be uniquely determined from the data.

An intriguing way out of this issue was suggested by Simpson–Penrose [9], who numerically observed linear instability of $\mathcal{CH}^+$ under the electromagnetic perturbations. Based on this instability, Penrose put forth the celebrated *strong cosmic censorship conjecture*, asserting that the failure of global uniqueness in the Reissner–Nordström spacetimes is a unstable, nongeneric phenomenon. In rough terms, the conjecture may be stated as follows:

Strong cosmic censorship. *The maximal forward development of "generic" asymptotically flat initial data under a "reasonable Einstein–matter model" leads to a spacetime that is inextendible as a "suitably regular" Lorentzian manifold.*

Moreover, through the study of a simple nonlinear model (Einstein–Maxwell–null dust system in spherical symmetry), Poisson–Israel [8] (see also Ori [6]) put forth a detailed nonlinear instability scenario dubbed *mass inflation*; for perturbations of the Reissner–Nordström spacetime, it may be formulated as follows:

Mass inflation scenario. *Consider the the maximal future development of a generic perturbation of the Reissner–Nordström spacetime.*

- *The null Cauchy horizon $\mathcal{CH}^+$ is nonempty, and the metric extends continuously past $\mathcal{CH}^+$.*
- *Nevertheless, the Hawking mass m, which is defined by the relation $1 - \frac{2m}{r} = g(\nabla r, \nabla r)$ in spherical symmetry blows up identically on $\mathcal{CH}^+$.*

2. Main results for the Einstein–Maxwell–Scalar Field system

To investigate the issues of strong cosmic censorship and mass inflation in an analytically simple yet interesting setting, we consider the *spherically symmetric Einstein–Maxwell–(real) Scalar Field* system:

$$\begin{cases} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 2(T^{M}_{\mu\nu} + T^{SF}_{\mu\nu}), \\ g^{\alpha\beta} \partial_\alpha F_{\nu\beta} = 0, \quad \mathrm{d}F = 0, \\ \Box_g \phi = 0, \end{cases}$$

where

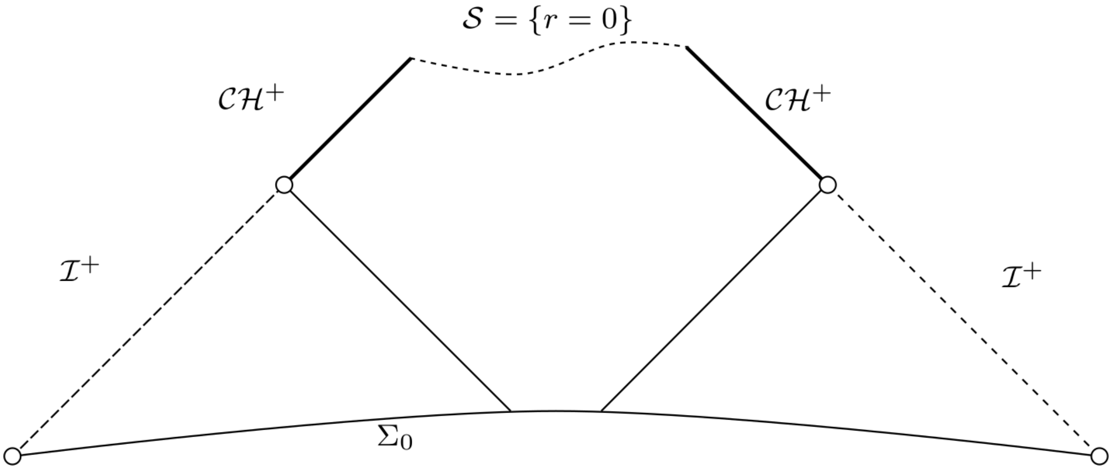
$$T^{M}_{\mu\nu} = F_{\mu}{}^{\alpha} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \quad T^{SF}_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi.$$

Importantly, this system admits the Reissner–Nordström spacetime as a solution ($\phi = 0$). However, as the Einstein–Maxwell in spherical symmetry is rigid,

[2] These notions of regularity must be interpreted in the sense of manifolds-with-boundaries, and take into account the nondegenerate bifurcation sphere, which is represented as a corner in the Penrose diagram.

the real scalar field ϕ is added to 'emulate' the dynamic degree of freedom in the full Einstein equation.

We start with a previous result:



Theorem 1 (Dafermos [1, 2], Dafermos–Rodnianski [3]). *The maximal forward development of an admissible initial data has the Penrose diagram above*[3]*. In particular, $\mathcal{CH}^+ \neq \emptyset$, and the solution (g, F, ϕ) is C^0 up to $\mathcal{CH}^+$.*

Therefore, the strong cosmic censorship conjecture is *false*(!), if 'suitably regular' means C^0 Lorentzian metric. Nevertheless, we show that the strong cosmic censorship conjecture holds in a different formulation:

Theorem 2 (Luk–O. [4, 5]). *There exists a generic*[4] *class of initial data sets, denoted by $\mathcal{G}$, such that the maximal forward development of a data set in $\mathcal{G}$ exhibits blow up near every point on $\mathcal{CH}^+$, in the sense that $\partial\phi \notin L^2(\mathcal{O})$ for any neighborhood $\mathcal{O}$ of $p \in \mathcal{CH}^+$. Moreover, the solution is inextendible as a spacetime with C^2 metric.*

Let us also announce a forthcoming work, in which we verify that the *mass inflation scenario* of Poisson–Israel and Ori is valid for the model at hand:

Theorem 3 (Luk–O., forthcoming). *There exists a Baire-generic*[5] *subclass $\widetilde{\mathcal{G}} \subset \mathcal{G}$ such that the maximal forward development of a data set in $\widetilde{\mathcal{G}}$ exhibits $m = \infty$ on each point of $\mathcal{CH}^+$.*

[3] We remark that for small perturbations of the Reissner–Nordström spacetime, $\mathcal{S} = \emptyset$.

[4] By generic, we mean open and dense in a natural topology for admissible data sets; for details, we refer to [4, 5].

[5] By Baire-generic, we mean the countable intersection of open and dense sets in a natural (complete) topology.

References

[1] M. Dafermos, *The interior of charged black holes and the problem of uniqueness in general relativity*, Comm. Pure Appl. Math., **58** (2005), 445–504.
[2] M. Dafermos, *Black holes without spacelike singularities*, Comm. Math. Phys., **332** (2014), 729–757.
[3] M. Dafermos and I. Rodnianski, *A proof of Price's law for the collapse of a self-gravitating scalar field*, Invent. Math., **162** (2005), 381–457.
[4] J. Luk and S.-J. Oh, *Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data I. the interior of the black hole region*, preprint (available at arXiv.org) (2017).
[5] J. Luk and S.-J. Oh, *Strong cosmic censorship in spherical symmetry for two-ended asymptotically flat initial data II. the exterior of the black hole region*, preprint (available at arXiv.org) (2017).
[6] A. Ori, *Inner Structure of a Charged Black Hole: An Exact Mass-Inflation Solution*, Phy. Rev. Lett., **67** (1991), 789–792.
[7] R. Penrose. *Gravitational collapse*, in C. Dewitt-Morette, editor, Gravitational Radiation and Gravitational Collapse, Vol. 64 of IAU Symposium, (1975), 82–91.
[8] E. Poisson and W. Israel, *Inner-horizon instability and mass inflation in black holes*, Phys. Rev. Lett., **63** (1989), 1663–1666.
[9] M. Simpson and R. Penrose, *Internal instability in a Reissner-Nordström black hole*, *Internat. J. Theoret. Phys.*, **7** (1973), 183–197.

A proof of the instability of AdS spacetime for the Einstein–massless Vlasov system

Georgios Moschidis

The simplest solution of the vacuum Einstein equations

$$Ric_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0 \tag{1}$$

with a *negative* cosmological constant Λ is Anti-de Sitter spacetime $(\mathcal{M}^{3+1}_{AdS}, g_{AdS})$. In the standard polar coordinate chart (t, r, θ, φ) on $\mathcal{M}_{AdS} \simeq \mathbb{R}^{3+1}$, the AdS metric g_{AdS} is expressed as

$$g_{AdS} = -\Big(1 - \frac{1}{3}\Lambda r^2\Big) dt^2 + \Big(1 - \frac{1}{3}\Lambda r^2\Big)^{-1} dr^2 + r^2\big(d\theta^2 + \sin^2\theta \cdot d\varphi^2\big).$$

The spacetime $(\mathcal{M}_{AdS}, g_{AdS})$ can be conformally identified with $(\mathbb{R} \times \mathbb{S}^3_+, -dt^2 + g_{\mathbb{S}^3})$, where $\mathbb{S}^3_+$ is the northern hemisphere of $\mathbb{S}^3$. Through this identification, the timelike boundary $\mathcal{I} = \mathbb{R} \times \partial\mathbb{S}^3_+$ of $\mathbb{R} \times \mathbb{S}^3_+$ can be naturally attached to $(\mathcal{M}_{AdS}, g_{AdS})$ as a conformal boundary at infinity. More generally, the conformal boundary $\mathcal{I}$ is a fundamental feature of the asymptotic geometry of any *asymptotically* AdS spacetime $(\mathcal{M}, g)$.

In view of the timelike character of $\mathcal{I}$, the right framework to study the dynamics of (1) in the asymptotically AdS setting is that of an *initial-boundary* value problem, with initial data (satisfying the vacuum constraint equations) prescribed on a complete spacelike hypersurface Σ terminating at $\mathcal{I}$, and boundary data imposed asymptotically on $\mathcal{I}$. Most physically relevant boundary conditions on $\mathcal{I}$ can be classified as either *reflecting* or *dissipative*.

The well-posedness of the initial-boundary value problem for (1) was first addressed by Friedrich [10]: Working in the conformally compactified picture and rewriting (1) as a hyperbolic system of renormalised variables which are regular up to $\mathcal{I}$, [10] established that the initial-boundary value problem for (1) is well-posed for a wide class of boundary conditions on $\mathcal{I}$ (including both reflecting and dissipative conditions).

Given the well-posedness of the initial-boundary value problem for (1), the question naturally arises whether the trivial solution $(\mathcal{M}_{AdS}, g_{AdS})$ is stable under perturbations of its initial data. In 2006, Dafermos and Holzegel [7, 6] suggested the following conjecture (see also [1]):

AdS instability conjecture. *There exist arbitrarily small perturbations to the initial data of $(\mathcal{M}_{AdS}, g_{AdS})$ which, under evolution by the vacuum Einstein equations (1) with a reflecting boundary condition on $\mathcal{I}$, lead to the development of black hole regions at late enough times. In particular, $(\mathcal{M}_{AdS}, g_{AdS})$ is non-linearly unstable.*

Remark 1. *The initial data norm $||\cdot||_{data}$, used to measure the size of the initial perturbations, is not specified by the conjecture; a natural minimal requirement for $||\cdot||_{data}$ is that the initial-boundary value problem for (1) for initial data with finite $||\cdot||_{data}$ norm is well-posed. Let us also remark that the choice of a reflecting boundary condition on $\mathcal{I}$ is important for the conjecture: In the case of a maximally dissipative boundary condition,* [12] *showed that, at the linear level, perturbations of $(\mathcal{M}_{AdS}, g_{AdS})$ decay in time at a superpolynomial rate.*

The numerical and heuristic study of the AdS instability conjecture has been mainly focused on the simpler setting of the spherically symmetric Einstein–scalar field system

$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\Big(\partial_\mu\varphi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}\partial^\alpha\varphi\partial_\alpha\varphi\Big), \\ \Box_g\varphi = 0. \end{cases} \tag{2}$$

In the pioneering work [3], Bizon–Rostworowski were the first to propose a mechanism driving the instability of AdS, suggesting, in particular, that the concentration of energy in small scales (which is necessary for a black hole to form) is triggered by a resonant mode mixing for the scalar field φ. Following [3], a vast number of numerical and heuristic works by several authors have been dedicated to the study of the AdS instability conjecture for the spherically symmetric Einstein–scalar field system (see, e. g. [8, 4, 14, 5, 2, 9, 11, 13]); however, no rigorous proof has been obtained so far in this setting.

An alternative Einstein–matter field system allowing for non-trivial spherically symmetric dynamics and retaining many of the qualititative properties of (1) is the Einstein–massless Vlasov system:

$$\begin{cases} Ric_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi \int_{T_x\mathcal{M}} p_\mu p_\nu f\,\sqrt{-g}dp, \\ p^\alpha\partial_{x^\alpha} f - \Gamma^\alpha_{\beta\gamma}p^\beta p^\gamma\partial_{p^\alpha} f = 0, \\ supp(f) \subset \big\{(x;p)\in T\mathcal{M}:\ g_{\alpha\beta}(x)p^\alpha p^\beta = 0\big\}. \end{cases} \tag{3}$$

In my talk, I presented the proof of the AdS instability conjecture in the spherically symmetric setting of the system (3) (which will appear in [15]):

Theorem 1. *There exists a one parameter family of spherically symmetric, asymptotically AdS initial data* $\mathcal{S}_\varepsilon = (\bar{g}_\varepsilon, k_\varepsilon; \bar{f}_\varepsilon)$, $\varepsilon \in (0,1]$, *for the Einstein–massless Vlasov system (3), such that the following hold:*

(1) As $\varepsilon \to 0$, *the initial data* $\mathcal{S}_\varepsilon$ *converge to the trivial data* $\mathcal{S}_0 = (\bar{g}_0, k_0; 0)$ *of pure AdS spacetime with respect to a rough initial data norm* $||\cdot||_{data}$, *for which the initial-boundary value problem for (3) with reflecting conditions on* $\mathcal{I}$ *is well-posed in spherical symmetry.*

(2) Let $(\mathcal{M}_\varepsilon, g_\varepsilon; f_\varepsilon)$ *be the maximal future development of* $\mathcal{S}_\varepsilon$ *for (3) with reflecting boundary conditions on* $\mathcal{I}$. *Then, for any* $\varepsilon \in (0,1]$, $(\mathcal{M}_\varepsilon, g_\varepsilon; f_\varepsilon)$ *contains a black hole region.*

Remark 2. *The initial data norm* $||\cdot||_{data}$ *measures the concentration of energy along the evolution of the Vlasov field* $f^{(\varepsilon)}_{AdS}$ *obtained by solving the uncoupled Vlasov equation on* $(\mathcal{M}_{AdS}, g_{AdS})$ *with* $f^{(\varepsilon)}_{AdS} = \bar{f}_\varepsilon$ *initially. In a certain sense,* $||\cdot||_{data}$ *is a minimal initial data norm for which (3) is well-posed in spherical symmetry. In the case* $\Lambda = 0$, *it can be shown that Minkowski spacetime is globally stable as a solution of (3) under spherically symmetric perturbations which are small with respect to (the analogue of)* $||\cdot||_{data}$.

In the proof of the theorem above, the initial data family $(\bar{g}_\varepsilon, k_\varepsilon; \bar{f}_\varepsilon)$ is constructed so that the Vlasov field f_ε is initially arranged into a large number of ingoing beams, with each successive Vlasov beam supported in an increasingly narrower region in phase space. The proof then proceeds by exploiting an asymmetry in the exchange of energy during the (non-linear) interaction of any pair of beams taking place close to the axis of symmetry and close to $\mathcal{I}$, respectively. In view of the specific configuration of the beams, this asymmetry leads, after many reflections off $\mathcal{I}$, to the gradual concentration of energy at the beams supported in the narrowest regions in phase space. Some of the technical challenges involved in designing this configuration and controlling the non-linear evolution up to the point of black hole formation were briefly highlighted in the talk.

While the technical details of the proof depend crucially on the structure of the system (3), the main instability mechanism is expected to be relevant for more general matter models admitting non-trivial spherically symmetric dynamics and allowing the arrangement of matter into nearly null beams, the Einstein–scalar field system being one particular example.

References

[1] M. Anderson, *On the uniqueness and global dynamics of AdS spacetimes*, Classical and Quantum Gravity **23** (2003), 6935–6954.

[2] P. Bizoń, M. Maliborski and A. Rostworowski *Resonant Dynamics and the Instability of Anti–de Sitter Spacetime*, Physical Review Letters **115** (2015), 081103.

[3] P. Bizoń and A. Rostworowski *Weakly turbulent tnstability of anti–de Sitter spacetime*, Physical Review Letters **107**(3) (2011), 031102.

[4] A. Buchel, L. Lehner and S. Liebling, *Scalar collapse in AdS spacetimes*, Physical Review D **86**(12) (2012), 123011.

[5] B. Craps, O. Evnin and J. Vanhoof, *Renormalization group, secular term resummation and AdS (in)stability*, Journal of High Energy Physics **48** (2014), doi:10.1007/JHEP10(2014)048.

[6] M. Dafermos, *The Black Hole Stability problem*, Talk at the Newton Institute, Cambridge, http://www-old.newton.ac.uk/webseminars/pg+ws/2006/gmx/1010/dafermos/ (2006).

[7] M. Dafermos and G. Holzegel, *Dynamic instability of solitons in 4+1 dimesnional gravity with negative cosmological constant*, https://www.dpmms.cam.ac.uk/.17ex∼md384/ADSinstability.pdf (2006).

[8] O. Dias, G. Horowitz and J. Santos, *Gravitational turbulent instability of anti-de Sitter space*, Classical and Quantum Gravity **29**(19) (2012), 194002.

[9] F. Dimitrakopoulos, B. Freivogel, M. Lippert and I-S. Yang, *Position space analysis of the AdS (in)stability problem*, Journal of High Energy Physics **77** (2015), doi:10.1007/JHEP08(2015)077.

[10] H. Friedrich, *Einstein equations and conformal structure: Existence of Anti-de Sitter-type space-times*, Journal of Geometry and Physics **17** (1995), 125–184.

[11] S. Green, A. Maillard, L. Lehner and S. Liebling, *Islands of stability and recurrence times in AdS*, Physical review D **92** (2015), 084001.

[12] G. Holzegel,J. Luk, J. Smulevici and C. Warnick, *Asymptotic properties of linear field equations in anti-de Sitter space*, arXiv preprint: arXiv:1502.04965 (2015).

[13] G. Horowitz and J. Santos, *Geons and the Instability of Anti-de Sitter Spacetime*, Surveys in Differential Geometry **20** (2015), 321–335.

[14] M. Maliborski and A. Rostworowski, *Time-periodic solutions in Einstein AdS - massless scalar field system*, Physical Review Letters **111** (2013), 051102.

[15] G. Moschidis, *A proof of the instability of AdS for the Einstein–massless Vlasov system*, in preparation.

Mass-like covariants for asymptotically hyperbolic manifolds

Romain Gicquaud

(joint work with Julien Cortier, Mattias Dahl)

The mass of an asymptotically hyperbolic manifold is an important covariant introduced by X. Wang in [7] and in a more general setting by P. Chruściel and M. Herzlich in [3].

As its Euclidean analog, it is defined in terms of the asymptotic geometry. Still, unlike the mass of an asymptotically Euclidean manifold which is a scalar invariant under the change of chart at infinity, this mass appears as a vector in Minkowski space that transforms covariantly under the change of chart at infinity. To be more precise, it is proven in [3, 4, 5] that the transition functions from one chart to another is at top order an isometry of the hyperbolic space (or equivalently a Lorentz transformation) that encodes the change of the mass vector.

Assume that the metric decays sufficiently fast towards the hyperbolic metric at infinity and that the scalar curvature is greater than or equal to that of the hyperbolic space. Then the positive mass theorem states that the mass vector is a timelike future pointing vector unless the manifold is isometric to the hyperbolic space. This theorem has now been proven under the assumption that the manifold is spin [3, 7] or under dimensional restrictions [1, 2].

The aim of the talk is to classify asymptotics objects that enjoy similar covariance properties. We show that these covariants belong to two different families each indexed by a non-positive integer.

To describe these covariants, we work in the conformal ball model of the hyperbolic space. Namely, it is described as the unit ball $B_1(0) \subset \mathbb{R}^n$ endowed with the metric

$$b = \rho^{-2}\delta$$

where δ denotes the Euclidean metric on $\mathbb{R}^n$ and

$$\rho = \frac{1 - |x|^2}{2}$$

is a defining function for the unit sphere $S_1(0)$ of $\mathbb{R}^n$. In a well chosen chart at infinity, a given asymptotically hyperbolic metric g can be written as

$$g = \rho^{-2}(\delta + \rho^k m + \text{h.o.t.}),$$

where m is a symmetric 2-tensor defined in a neighborhood of $S_1(0)$ that satisfies the so-called transversality condition $m_{ij}x^i = 0$. The integer $k > 0$ is the decay rate towards the hyperbolic metric.

The covariants are best described as invariant pairings between the space of mass-aspect tensors and certain representations of the Lorentz group $O(n, 1)$.

- Conformal covariants: Let $\mathcal{H}_p$ denote the set of wave harmonic homogeneous polynomials of degree p:

$$\mathcal{H}_p = \{P \in \mathbb{R}[X^0, X^1, \ldots, X^n],\ P \text{ homogeneous of degree } p,\ \Box P = 0\}.$$

  Then the p-th conformal mass is defined when $k = n - 1 + p$ and is given by

$$\mathcal{C}_p(m, P) = \int_{S_1(0)} P(1, x^1, \ldots, x^n) \operatorname{tr}^\sigma(m) d\mu^\sigma,$$

  where σ denotes the round metric on $S_1(0)$ and $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$ restruicted to $S_1(0)$.
- Weyl covariants: Let $\mathcal{W}_p$ denote the set of homogeneous Weyl tensors on $\mathbb{R}^{n,1}$, i.e. polynomial 4-tensors that satisfy the symmetry and trace free assumptions of the usual Weyl tensor together with the (linearized) second Bianchi identity. Then the p-th Weyl mass is defined for $k = n + 1 + p$ and is given as follows:

$$\mathcal{W}_p(m, P) = \int_{S_1(0)} \left\langle W(1, x^1, \ldots, x^n)(e_+, \cdots, e_+, \cdot), m \right\rangle_\sigma d\mu^\sigma,$$

where $e_+ = (1, x^1, \ldots, x^n)$ is the outgoing null vector to the sphere $S_1(0)$.

The usual mass is the conformal mass with $p = 1$. An extension of the definition of these covariants to a broader context (à la Chruściel–Herzlich) can be done by following a method described in [6] but finding a tight link between a given mass and a curvature operator, as the standard mass is bound to scalar curvature, is still an open question.

References

[1] L. Andersson, M. Cai, and G. J. Galloway, *Rigidity and positivity of mass for asymptotically hyperbolic manifolds*, Ann. Henri Poincaré **9** (2008), no. 1, 1–33.

[2] P. T. Chruściel and G. J. Galloway and L. Nguyen and Tim-Torben Paetz, *On the mass aspect function and positive energy theorems for asymptotically hyperbolic manifolds*, unpublished, arXiv:1801.03442.

[3] P. T. Chruściel and M. Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific J. Math. **212** (2003), no. 2, 231–264.

[4] P. T. Chruściel and G. Nagy, *The mass of spacelike hypersurfaces in asymptotically anti-de Sitter space-times*, Adv. Theor. Math. Phys. **5** (2001), no. 4, 697–754.

[5] J. Cortier and M. Dahl and R. Gicquaud, *Mass-like invariants for asymptotically hyperbolic metrics*, submitted, arXiv:1603.07952.

[6] B. Michel, *Geometric invariance of mass-like asymptotic invariants*, J. Math. Phys. **52** (2011), no. 5, 052504, 14.

[7] X. Wang, *The mass of asymptotically hyperbolic manifolds*, J. Differential Geom. **57** (2001), no. 2, 273–299.

Ill Posed Perturbative Corrections to Well Posed Equations

Robert M. Wald

It often happens in physical theories with well posed equations that the addition of a "perturbative correction" to the equations will drastically change the character of the equations—making them ill posed or introducing spurious additional degrees of freedom and/or instabilities. The perturbative corrections should, in principle, improve the accuracy of the description of the system, but it is not obvious how to make predictions with the perturbatively corrected equations.

A good example of this is the equation of motion for a particle in classical electrodynamics, when the effects of radiation reaction are taken into account. The equations of motion without radiation reaction are of the form of Newton's second law

$$\ddot{x} = F(x, \dot{x}). \tag{1}$$

Radiation reaction adds a perturbative correction of the form

$$\ddot{x} = F(x, \dot{x}) + \epsilon \dddot{x}\,. \tag{2}$$

Although this equation is still an ODE and is well posed, it requires the initial data $(x, \dot{x}, \ddot{x})$—rather than $(x, \dot{x})$—and it gives rise to unphysical "runaway solution" instabilities of form $x \sim \exp(t/\epsilon)$.

Another good example is the semiclassical Einstein equation

$$G_{ab} = 8\pi \langle T_{ab} \rangle \tag{3}$$

where the classical stress-energy tensor T_{ab} of the matter fields is replaced by the expectation value of the quantum stress tensor in some state. The difference between the classical stress energy and the quantum expectation value can be viewed as a perturbative correction to the ordinary Einstein equation. For a fixed initial quantum state, (3) becomes a nonlocal equation for the metric. The quantum field regularization procedure introduces dependence of $\langle T_{ab} \rangle$ on 4th derivatives of the metric, so this equation is effectively of higher differential order. It seems unlikely that (3) will have a well posed initial value formulation, and even if it did, the higher derivative dependence on the metric would undoubtedly introduce additional spurious degrees of freedom.

The issue I wish to address is how to treat perturbative corrections to the equations of motion such as the ones above. One certainly cannot treat the modified equations as literally correct, as they give rise to physically spurious solutions. One alternative would be to first solve the original unperturbed equation, substitute the solution in the perturbation term, and then treat this term as a "source term" to find the first order correction to the original solution. This will properly give a perturbative correction to the unperturbed solution. However, over long periods of time, a perturbative correction calculated in this way will typically become large, making it unreliable/inaccurate. Going to any finite higher order in perturbation theory will not help (or, at least, not help much). Is there a better way to take account perturbative corrections of the above sort?

In the case of ordinary differential equations such as (2), the answer is "yes." The *reduction of order* procedure provides a systematic way to modify an equation like (2) so that it remains as accurate to leading order in ϵ but does not introduce spurious degrees of freedom. To implement this procedure, we compute $\dddot{x}$ using the unperturbed equation (1)

$$\begin{aligned} \dddot{x} &= \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial \dot{x}}\ddot{x} \\ &= \frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial \dot{x}}F(x,\dot{x}) \end{aligned} \tag{4}$$

One then substitutes this in (2) to produce the modified equation

$$\ddot{x} = F(x,\dot{x}) + \epsilon \left[\frac{\partial F}{\partial x}\dot{x} + \frac{\partial F}{\partial \dot{x}}F(x,\dot{x}) \right] . \tag{5}$$

This equation has entirely satisfactory mathematical properties, and should accurately describe the leading order effects of radiation reaction on particle motion.

However, there is no known similar procedure for dealing with perturbative corrections to partial differential equations. As a much simpler example than (3) consider a classical scalar field ϕ in Minkowski spacetime that satisfies

$$\Box\phi - \phi^3 = \epsilon S^{abcd} \nabla_a \phi \nabla_b \phi \nabla_c \phi \nabla_d \phi \tag{6}$$

where the tensor field S^{abcd} appearing in the perturbative correction is smooth but does not have any other special properties. One could make a choice of coordinates and "reduce order" on the time derivatives as in (5). However, the reduced equations would then depend on the choice of coordinates. Worse yet, this procedure would introduce additional spatial derivatives, and the equations would still have an ill posed character.

Many proposals for modifying Einstein's equation yield equations that have a character like (6). It is far from clear whether these proposals make "predictions"—since this requires a well posed initial value formulation—and, if so, what these predictions are. This is an open problem that undoubtedly deserves more attention.

Singularity theorems for $\mathcal{C}^{1,1}$-metrics

Melanie Graf
(joint work with James D. E. Grant, Michael Kunzinger, Roland Steinbauer)

The classical singularity theorems of General Relativity show that a Lorentzian manifold with a C^2-metric that satisfies physically reasonable conditions cannot be geodesically complete. One of the questions left unanswered by the classical singularity theorems is whether one could extend such a spacetime with a lower regularity Lorentzian metric such that the extension still satisfies these physically reasonable conditions and does no longer contain any incomplete causal geodesics. In other words, the question is if a lower differentiability of the metric is still sufficient for the theorems to hold. A number of technical obstacles for lowering the regularity of the metric are discussed, e.g., in Sect. 6.1 of the review article [10].

A natural differentiability class to consider here is $C^{1,1}$. From the point of view of physics, the curvature of a metric being bounded but discontinuous, rather than blowing up, would, via the Einstein field equations, give rise to (or be generated by) a finite jump in the energy-momentum tensor, a scenario that arises in the classical example of the Oppenheimer–Snyder solution and the whole class of matched spacetimes. Mathematically, this regularity guarantees that the Levi-Civita connection is still locally Lipschitz, so the classical Picard–Lindelf theorem gives existence and uniqueness of solutions of the geodesic equations and these solutions depend continuously (in fact, Lipschitz continuously) on the initial data. However, while the curvature of the metric exists almost everywhere and is well-defined in L^∞_{loc}, it might be undefined along any given geodesic.

Recent progress in low-regularity Lorentzian geometry ([1], [8], [5] and others) has allowed one to tackle the many challenges in proving singularity theorems for $C^{1,1}$-metrics and show that, in fact, the classical singularity theorems of Penrose ([9]), Hawking ([3]), and even the more general theorem of Hawking–Penrose ([4]) remain valid in this regularity (cf. [7], [6], [2]).

The main difference between the Hawking–Penrose theorem and the earlier theorems lies in the introduction of the so-called genericity condition, which allows a significant weakening of some of the other assumptions in the Hawking and the Penrose theorem. One of the modern formulations of this condition is that for any inextendible causal geodesic $\gamma : I \to M$ there exists $t_0 \in I$ such that the operator

$$[R(t_0)] : [\dot\gamma(t_0)]^\perp \to [\dot\gamma(t_0)]^\perp, \;\; V \mapsto R(V, \dot\gamma(t_0))\dot\gamma(t_0),$$

where $[\dot\gamma]^\perp$ denotes $\dot\gamma^\perp \subseteq TM$ if γ is timelike and the quotient space $\dot\gamma^\perp/\mathbb{R}\dot\gamma$ if γ is null, is non-zero. This condition manifestly involves curvature along geodesics and as such any $C^{1,1}$-version of the Hawking–Penrose theorem needs to involve a suitable replacement. Further, while the classical proofs of both the Hawking and the Penrose theorem involve concrete and explicit estimates (in terms of the curvature and an initial condition) on the parameter time it takes geodesics to reach a focal point, no such estimates are found for conjugate points in the proofs of the Hawking–Penrose theorem.

As for previous results, the overarching strategy for the proof of the $C^{1,1}$-Hawking–Penrose theorem, [2, Thm. 2.5], is to follow the smooth proof but circumvent the need to use curvature estimates and Raychaudhuri arguments along g-geodesics by using approximating smooth metrics g_ε respecting the causal structure (cf. [1]) and leveraging estimates on when g_ε-geodesics stop being maximising to show that also g-geodesics must stop maximising.

To this end we derive the following for smooth metrics (cf. [2, Section 4]): Given a bound on the causal Ricci curvature, $\mathrm{Ric}(X, X) > -\delta$ for all causal vectors X in a compact subset $K \subseteq TM$, and any causal geodesic $\gamma : I \to M$ for which $\dot\gamma \in K$ and for which $(g(R(E_i, \dot\gamma)\dot\gamma, E_j))_{ij} > \mathrm{diag}(c, -C, \dots, -C)$ (where E_i denotes a parallel orthonormal frame of $[\dot\gamma]^\perp$ along γ) on $[-r, r]$, we show that for δ small enough (compared to c, r) there exists a bound on the maximal time T such that $\gamma : [-T, T] \to M$ does not contain conjugate points. This bound depends only on δ, c and r, not on γ or g, and, by the well-known relation between maximising properties and conjugate points, also bounds the interval on which γ maximises the distance between its endpoints.

We define the *genericity condition* for $C^{1,1}$-metrics by saying that the genericity condition holds along γ if there exists a neighbourhood U of $\gamma(t_0)$ and continuous vector fields X and V on U and a constant $c > 0$ such that $X(\gamma(t)) = \dot\gamma(t)$ and $V(\gamma(t)) \in \dot\gamma(t)^\perp$ for all $t \in I$ with $\gamma(t) \in U$ and $|g(R(V, X)X, V)| > c$ in $L^\infty(U)$ (cf. [2, Def. 2.2]) and the *strong energy condition* by $\mathrm{Ric}(X, X) \geq 0$ a.e. for all Lipschitz continuous causal local vector fields X. This guarantees that the approximating metrics satisfy the inequalities required in the previous paragraph (cf. [2, Section 3]) and the estimate on T for g_ε geodesics can be used to show that complete timelike g-geodesics contained in an open globally hyperbolic subset and complete null geodesics, that are not closed, cannot be maximising (cf. [2, Section 5]).

Dealing with the initial conditions (the existence of a compact achronal set without edge, a closed trapped surface or a trapped point (cf. [2, Def. 2.4]), similarly, one establishes the existence of an achronal set S such that $E^+(S)$ is compact (Cf. [2, Section 6]). This reduces the proof of the theorem to an argument involving only causality theory which remains unaffected by a drop of the regularity of the metric to $C^{1,1}$ (cf. [2, Section 7, Appendix A]).

This shows a $C^{1,1}$-version of the Hawking–Penrose singularity theorem. One question that is left unanswered by this approach is whether this Theorem continues to hold without excluding the existence of closed null curves additionally to excluding closed timelike curves: While excluding closed null curves is necessary for the given proof, the smooth version only excludes closed timelike curves and one might wonder if there exists a spacetime with a $C^{1,1}$-metric that contains closed null curves but satisfies all other conditions of the theorem *and* is causal geodesically complete. Another avenue for possible future exploration is whether it is possible to show some version of a singularity theorem for metrics of even lower regularity.

References

[1] P.T. Chruściel, J.D.E. Grant, *On Lorentzian causality with continuous metrics*, Classical Quantum Gravity **29** (2012), 145001.

[2] M. Graf, J.D.E. Grant, M. Kunzinger and R. Steinbauer, *The Hawking–Penrose singularity theorem for $C^{1,1}$-Lorentzian metrics*, Commun. Math. Phys. **360** (2018), 1009–1042.

[3] S.W. Hawking, *The occurrence of singularities in cosmology. III. Causality and singularities*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences **300** (1967), 187–201.

[4] S.W. Hawking, R. Penrose, *The singularities of gravitational collapse and cosmology*, Proc. Roy. Soc. Lond. A. **314** (1970), 529–548.

[5] M. Kunzinger, R. Steinbauer, M. Stojković, J.A. Vickers, *A regularisation approach to causality theory for $C^{1,1}$-Lorentzian metrics*, Gen. Relativ. Gravit. **46** (2014), 1738.

[6] M. Kunzinger, R. Steinbauer, M. Stojković, J.A. Vickers, *Hawking's singularity theorem for $C^{1,1}$-metrics*, Classical Quantum Gravity **32** (2015), 075012.

[7] M. Kunzinger, R. Steinbauer, J.A. Vickers, *The Penrose singularity theorem in regularity $C^{1,1}$*, Classical Quantum Gravity **32** (2015), 155010.

[8] E. Minguzzi, *Convex neighborhoods for Lipschitz connections and sprays*, Monatsh. Math. **177** (2015), 569–625.

[9] R. Penrose, *Gravitational collapse and space-time singularities*, Phys. Rev. Lett. **14** (1965), 57–59.

[10] J.M.M. Senovilla, *Singularity Theorems and Their Consequences*, Gen. Rel. Grav. **30** (1998), 701–848.

High-frequency backreaction for the Einstein equations under polarized $\mathbb{U}(1)$ symmetry

Cecile Huneau

(joint work with Jonathan Luk)

1. Introduction

In this report, I describe the article [5], which is a joint work with Jonathan Luk. Our article is concerned with backreation, which is the following non linear phenomenon. Let us consider a sequence of metrics g_λ, solutions of Einstein vacuum equations, that we recall here

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 0, \tag{1}$$

where $R_{\mu\nu}$ is the Ricci tensor and R the scalar curvature. Let us assume that

- g_λ converges strongly in L^2_{loc} when λ tends to zero toward a metric g_0 which is smooth,
- ∂g_λ is bounded in L^2_{loc}.

Then, since (1) contains nonlinear terms of the form $\partial g_\lambda \partial g_\lambda$, which may not converge, even weakly toward $\partial g_0 \partial g_0$, a priori we can only write that g_0 satisfies the equation

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = T_{\mu\nu}.$$

This phenomenon has been first studied by Isaacson in [6] and [7] in the context of high frequency gravitational waves. This study was continued by Burnett in the late eighties [1]. In this article, he conjectured that the only $T_{\mu\nu}$ which can be obtained with such a process correspond to massless Vlasov matter model. He also asked the reverse question which is the following : if we consider a solution g_0 to Einstein equations coupled to a massless Vlasov field, can we find a sequence of metric g_λ, satisfying Einstein vacuum equations, which converges toward g_0 in the sense described above? These two questions were then forgotten until Green and Wald replaced this issue in the context of small scale inhomogeneities in cosmology in [2] and [3]. Under additional assuptions, they prove that $T_{\mu\nu}$ should be traceless and satisfy the weak dominant energy condition. These conditions are compatible with Burnett's conjecture, but the road is still long toward the proof of it.

2. Main result

The aim of our article [5] is to address the reverse question of Burnett conjecture. Doing so in full generality requires to study solutions of Einstein equations which are only bounded in H^1, which is below the regularity known to ensure local wellposedness (see [8]). This makes the problem highly non trivial. Consequently, we reduced the difficulty in two ways

- by working in the context of polarized $U(1)$ symmetry,
- by approaching solutions of Einstein equations coupled to an arbitrary number of null dust, model which is a discrete analog of massless Vlasov.

2.1. **Polarized $U(1)$ symmetry.** We study solutions of the vacuum Einstein equations of the form $(I \times \mathbb{R}^3, {}^{(4)}g)$, where $I \subset \mathbb{R}$ is an interval, with

$${}^{(4)}g = e^{-2\phi} g + e^{2\phi}(dx^3)^2,$$

where $\phi : I \times \mathbb{R}^2 \to \mathbb{R}$ is a scalar function and g is a Lorentzian metric on $I \times \mathbb{R}^2$. The vector field ∂_{x_3} is Killing and hypersurface orthogonal. Then Einstein vacuum equations $R({}^{(4)}g)_{\mu\nu} = 0$ are equivalent to the following system for (g, ϕ):

$$\begin{cases} \Box_g \phi = 0, \\ R_{\mu\nu}(g) = 2\partial_\mu \phi \partial_\nu \phi. \end{cases} \tag{2}$$

2.2. **Null dusts.** We describe the system of Eistein coupled to N null dusts. We consider a quadruple $(g, \phi, F_{\mathbf{A}}, u_{\mathbf{A}})$, with $\mathbf{A} \in \mathcal{A}$ for some finite set $\mathcal{A}$ with $|\mathcal{A}| = N$, where g is a Lorentzian metric on $I \times \mathbb{R}^2$, $\phi : I \times \mathbb{R}^2 \to \mathbb{R}$ is a scalar function, $F_{\mathbf{A}} : I \times \mathbb{R}^2 \to \mathbb{R}_{\geq 0}$ is the *density* of the null dust for each $\mathbf{A}$ and $u_{\mathbf{A}} : I \times \mathbb{R}^2 \to \mathbb{R}$ is an eikonal function such that $(du_{\mathbf{A}})^\sharp$ is the *direction* of propagation of the null dust for each $\mathbf{A}$, which is a solution to the system

$$(3) \qquad \begin{cases} R_{\mu\nu}(g) = 2\partial_\mu\phi\partial_\nu\phi + \sum_{\mathbf{A}}(F_{\mathbf{A}})^2\partial_\mu u_{\mathbf{A}}\partial_\nu u_{\mathbf{A}}, \\ \Box_g\phi = 0, \\ 2(g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta F_{\mathbf{A}} + (\Box_g u_{\mathbf{A}})F_{\mathbf{A}} = 0, \\ (g^{-1})^{\alpha\beta}\partial_\alpha u_{\mathbf{A}}\partial_\beta u_{\mathbf{A}} = 0. \end{cases}$$

2.3. **Main theorem.** We state an unprecise version of our main theorem in [5]

Theorem 1. *Let* $(g_0, \phi_0, F_{\mathbf{A}}, u_{\mathbf{A}})$ *be a sufficiently small and sufficiently regular local-in-time asymptotically conic solution to* (3) *such that*

- *The initial hypersurface is maximal;*
- *The* $u_{\mathbf{A}}$*'s are angularly separated (meaning* $|\nabla u_{\mathbf{A}}.\nabla u_{\mathbf{B}}| \leq (1-\eta)|\nabla u_{\mathbf{A}}||\nabla u_{\mathbf{B}}|$ *for* $\mathbf{A} \neq \mathbf{B}$*)*
- *A genericity condition holds for the initial data.*

Then (g_0, ϕ_0) *can be weakly approximated by a* 1*-parameter family of solutions* $(g_\lambda, \phi_\lambda)$ *for* $\lambda \in (0, \lambda_0)$*,* $\lambda_0 \in \mathbb{R}$ *to* (2)*, i.e., in a suitable coordinate system, as* $\lambda \to 0$*,* $(g_\lambda, \phi_\lambda) \to (g_0, \phi_0)$ *uniformly on compact sets and the derivatives* $(\partial g_\lambda, \partial\phi_\lambda) \rightharpoonup (\partial g_0, \partial\phi_0)$ *weakly in* L^2 *(for each component).*

3. Ideas of proof

The proof is based on two main ingredients :

- In $2+1$ dimensions, we can write Einstein equations in a gauge where the principal symbol of the Ricci tensor is elliptic. Consequently, in (2), the $2+1$ metric g is more regular than the scalar field.
- We write ϕ as a sum of planes waves oscillating at frequency λ in the $u_{\mathbf{A}}$ direction and a remainder term, and we use a continuity argument for the remainder.

3.1. **Elliptic gauge.** We write the $(2+1)$-dimensional metric g on $\mathcal{M} := I \times \mathbb{R}^2$ in the form

$$g = -N^2dt^2 + \bar{g}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt).$$

Let $\Sigma_t := \{(s, x^1, x^2) : s = t\}$ and $e_0 = \partial_t - \beta^i\partial_i$, which is a future directed normal to Σ_t. We introduce the second fundamental form of the embedding $\Sigma_t \subset \mathcal{M}$

$$K_{ij} = -\frac{1}{2N}\mathcal{L}_{e_0}\bar{g}_{ij}.$$

We decompose K into its trace and traceless parts.

$$K_{ij} =: H_{ij} + \frac{1}{2}\bar{g}_{ij}\tau.$$

Here, $\tau := \mathrm{tr}_{\bar{g}}K$ and H_{ij} is therefore traceless with respect to $\bar{g}$.

We can introduce the following *gauge conditions*:

- $\bar{g}$ is conformally flat, i.e., for some function γ, $\bar{g}_{ij} = e^{2\gamma}\delta_{ij}$;
- The constant t-hypersurfaces Σ_t are maximal $\tau = 0$.

In this gauge, Einstein equations can be written as semi-linear elliptic equations for the metric coefficients N, β, γ. In [4] we proved the local wellposedness of Einstein and Einstein-null dust equations in elliptic gauge.

3.2. **Parametrix construction.** We want to introduce an ansatz for ϕ of the form

$$\phi_\lambda = \phi_0 + \sum_{\mathbf{A}} \lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \widetilde{\phi}_\lambda$$

where $\|\partial\widetilde{\phi}_\lambda\|_{L^2} \lesssim \lambda$. The fact that $u_{\mathbf{A}}$ satisfies the eikonal equation and $F_{\mathbf{A}}$ a transport equation yield a priori that $\sum_{\mathbf{A}} \lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right)$ is a good approximation of a solution of $\Box_{g_0}\phi = 0$. The metric perturbations induced by ϕ_λ have a better behaviour in λ than ϕ_λ itself but it is not sufficient to ensure that $\sum_{\mathbf{A}} \lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right)$ is a good approximation of a solution of $\Box_{g_\lambda}\phi = 0$. The two main ideas to overcome this difficulty are the following.

- We use a more precise parametrix for ϕ, which describe the terms of order λ^2. We give here the form of the parametrix we use
$$\begin{aligned}\phi_\lambda =& \phi_0 + \sum_{\mathbf{A}} \lambda F_{\mathbf{A}} \cos\left(\frac{u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}} \lambda^2 \widetilde{F}_{\mathbf{A}} \sin\left(\frac{u_{\mathbf{A}}}{\lambda}\right) \\ &+ \sum_{\mathbf{A}} \lambda^2 \widetilde{F}^{(2)}_{\mathbf{A}} \cos\left(\frac{2u_{\mathbf{A}}}{\lambda}\right) + \sum_{\mathbf{A}} \lambda^2 \widetilde{F}^{(3)}_{\mathbf{A}} \sin\left(\frac{3u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}_\lambda\end{aligned}$$
with
$$\|\partial\mathcal{E}_\lambda\|_{L^2} \lesssim \lambda^2.$$
The terms $F^{(2)}_{\mathbf{A}}$ and $F^{(3)}_{\mathbf{A}}$ can be computed directly from the solution of (3). The term $\widetilde{F}_{\mathbf{A}}$ however satisfies a transport equation with a source term given by g_λ.
- We use precisely Einstein equations to deal with the time derivative of the metric coefficients. Notice that for a model problem of the form $\Delta g = (\partial\phi)^2$, $\partial_t g$ is less regular than ∇g. However, for the actual system, we can use for instance the condition $\tau = 0$ which links $\partial_t\gamma$ to $div\beta$.

References

[1] Gregory A. Burnett, *The high-frequency limit in general relativity*, J. of Math. Phys. 30, 90–96 (1989).

[2] Stephen R. Green and Robert M. Wald, *New framework for analyzing the effects of small scale inhomogeneities in cosmology*, Phys. Rev. D 83, 084020, arXiv:1011.4920 (2011).

[3] Stephen R. Green and Robert M. Wald, *Examples of backreaction of small scale inhomogeneities in cosmology*, Phys. Rev. D 87, 124037, arXiv:1304.2318 (2013).

[4] Cécile Huneau and Jonathan Luk, *Einstein equations under polarized* $\mathbb{U}(1)$ *symmetry in an elliptic gauge*, arXiv preprint arXiv:1706.09499, to appear in Comm. Math. Phys. (2017).

[5] Cécile Huneau and Jonathan Luk, *High-frequency backreaction for the Einstein equations under polarized* $\mathbb{U}(1)$ *symmetry*, arXiv preprint arXiv:1706.09501. .

[6] Richard A. Isaacson, *Gravitational Radiation in the Limit of High Frequency. I. The Linear Approximation and Geometrical Optics*, Phys. Rev. 166, 1263–1271 (1968).
[7] Richard A. Isaacson, *Gravitational Radiation in the Limit of High Frequency. II. Nonlinear Terms and the Effective Stress Tensor*, Phys. Rev. 166, 1272–1279 (1968).
[8] Sergiu Klainerman, Igor Rodnianski and Jérémie Szeftel, *The bounded L^2 curvature conjecture*, Invent. Math., 202(1), 91–216, arXiv:1204.1767 (2015).

Islands of stability in anti-de Sitter spacetime

Stephen R. Green
(joint work with Alex Buchel, Luis Lehner, Steven L. Liebling, Antoine Maillard)

We study the stability of anti-de Sitter (AdS) spacetime with reflecting boundary conditions in general relativity. This system has no dissipation, as there is no black hole horizon and the boundary conditions prevent the escape of gravitational waves. Thus, linear perturbations do not decay in time, and nonlinear interactions determine the final state. It was, therefore, conjectured that AdS is unstable to black hole formation [6].

As a toy model, we restrict to spherical symmetry, and we insert a real scalar field for dynamics. We work in four spacetime dimensions. This model was studied numerically in [2], where it was shown that no matter how small the initial amplitude of the scalar field, collapse to a black hole always occured. The identified mechanism was a nonlinear transfer of energy to short wavelength modes, which are more highly peaked at the origin. The normal modes of AdS have frequency spectrum $\omega_n \propto 2n + 3$, $n = 0, 1, 2, \ldots$, so they are commensurate with each other and resonances readily transfer energy among the modes. The work [2] showed a turbulent cascade of energy to short wavelengths, and the formation of a power-law energy spectrum that put sufficient energy into small scales to form a black hole. Later work indicated, however, that different initial data (with different initial field profile) can fail to collapse for sufficiently small amplitudes [4].

The nonlinear energy transfer takes place over a time scale proportional to $1/\epsilon^2$, where ϵ is the amplitude of the scalar field. To study the dynamics in the small-amplitude limit, we therefore make use of a two timescale perturbation theory framework (TTF) [1]. This framework integrates out the rapid normal-mode oscillations, leaving a set of effective equations that describe the nonlinear interations between the modes. These nonlinear coupled ordinary differential equations for the complex mode amplitudes A_n provide access to a regime that becomes increasingly difficult to reach with traditional numerical simulations. The TTF equations conserve a Hamiltonian H, as well as the energy $E = \sum_n \omega_n^2 |A_n|^2$ and the "particle number" $N = \sum_n \omega_n |A_n|^2$ [5, 3].

In this work we study quasiperiodic (QP) equilibrium solutions to the TTF equations of the form $A_n(\tau) = \alpha_n e^{-i\beta_n \tau}$, with $\alpha_n, \beta_n \in \mathbb{R}$, and $\beta_i + \beta_j = \beta_k + \beta_l$ for $i + j = k + l$. With this ansatz, the TTF equations reduce to algebraic equations, and solutions can be found numerically with a Newton–Raphson method. Solutions extremize H for fixed E and N. QP solutions have constant-in-time energy per mode, and are characterized by a dominant mode, as well as their total energy E and particle number N. Thus, once the dominant mode is chosen, the QP solutions form 2-parameter families. The QP solutions are observed to have energy spectra with nearly-exponential decay to both sides of the dominant mode. We can find solutions up to a maximal E/N, which depends on the choice of dominant mode.

Within TTF, we perform a linear stability analysis of QP solutions, and show that perturbations are described by real-frequency modes. We argue that these QP solutions, therefore, form "islands of stability." The previously-observed non-collapsing solutions are orbits about QP solutions, and, indeed, have oscillation periods consistent with the stability analysis. The picture that emerges is that solutions fall into two classes: (1) for initial data sufficiently close to a QP solution, the solution is confined to the island of stability, and its spectrum remains nearly exponential, or (2) for initial data far from a QP solution, a power-law spectrum forms, putting far more energy in high frequencies, and leading to collapse.

Although the linear stability analysis within TTF indicates that QP solutions are stable over short time periods, the question of long-term stability remains unknown. Indeed, it is possible that over *extremely* long time scales, solutions of class (1) also collapse. This problem is reminiscent of the question of thermalization of the Fermi–Pasta–Ulam (FPU) system of nonlinearly coupled oscillators [7]. Both systems exhibit similar non-thermalization and "recurrences" of a solution close to initial data; understanding this for the FPU problem led to many developments in nonlinear dynamics. Recent work on the FPU problem, however, suggests that collapse may occur in the end [9], and it may be useful to adapt these methods to the AdS problem.

For collapsing solutions of class (2), the reason for the rapid collapse and the origin of the power law that forms remain unknown. Recent work by G. Moschidis proving that there is a one-parameter family of collapsing data in the spherically-symmetric Einstein-massless Vlasov system [8] may shed light on this problem.

References

[1] V. Balasubramanian, A. Buchel, S. R. Green, L. Lehner, and S. L. Liebling, *Holographic thermalization, stability of anti-de Sitter space, and the Fermi-Pasta-Ulam paradox*, Phys. Rev. Letters **113** (2014), 071601.

[2] P. Bizón and A. Rostworowski, *Weakly turbulent instability of anti-de Sitter spacetime*, Phys. Rev. Letters **107** (2011), 031102.

[3] A. Buchel, S. R. Green, L. Lehner, and S. L. Liebling, *Conserved quantities and dual turbulent cascades in anti-de Sitter spacetime*, Phys. Rev. D **91** (2015), 064026.

[4] A. Buchel, S. L. Liebling and L. Lehner, *Boson stars in AdS spacetime*, Phys. Rev. D **87** (2013), 123006.

[5] B. Craps, O. Evnin, and J. Vanhoof, *Renormalization, averaging, conservation laws and AdS (in)stability*, J. High Energy Phys. **01** (2015), 108.

[6] M. Dafermos and G. Holzegel, *Dynamic instability of solitons in* $4 + 1$*-dimensional gravity with negative cosmological constant*, `https://www.dpmms.cam.ac.uk/~md384/ADSinstability.pdf`, (2006).

[7] E. Fermi, J. Pasta, and S. M. Ulam, *Studies of nonlinear problems*, Los Alamos Scientific Laboratory Report No. LA-1940, (1955).
[8] G. Moschidis (2018), seminar at this workshop.
[9] M. Onorato, L. Vozella, D. Proment, and Y. V. Lvov, *Route to thermalization in the α-Fermi-Pasta-Ulam system*, Proceedings of the National Academy of Sciences **112** (2015), 4208.

Geometric Inequalities for Axially Symmetric Initial Data Sets

Ye Sle Cha

The standard picture of gravitational collapse asserts that an inequality relating the physical quantities of the initial state such as mass m and angular momentum $\mathcal{J}$ may hold. This is called the mass angular momentum inequality, as described below:

$$m \geq \sqrt{|\mathcal{J}|}. \tag{1}$$

This inequality has been intensively studied over the past decade, by Dain [6] et al. Here we consider a 3 dimensional, axially symmetric initial data set (M^3, g, k) for the Einstein equations, satisfying the dominant energy condition. Some additional assumptions may be imposed, mainly to guarantee the existence of a twist potential related to the Killing vector generating the axial symmetry, which plays an important role in the proof. Dain et al., then proved that (1) holds, if a few more assumptions are made, such that (M^3, g, k) is maximal, and that it has two ends, one strongly asymptotically flat and the other either strongly asymptotically flat or asymptotically cylindrical. Moreover, Alaee, Khuri, Kunduri recently extended the proof to show that the analogy of (1) holds if (M^4, g, k) is a 4 dimensional, bi-axisymmetric initial data set satisfying the above assumptions [1].

Naturally, one may ask how to generalize the known proof. For instance, is there a canonical way to reduce the general non-maximal case to the known maximal case? Or can we prove (1) in a similar way, if (M^3, g, k) possesses only one asymptotically flat end, and has an apparent horizon boundary?

The first question has been studied in my previous work with Khuri [3] for a 3 dimensional case, and in my upcoming work [2] for a higher dimensional case. In [3], we show that the general non-maximal case reduces to the known maximal case, if a system of elliptic equations possesses a solution with appropriate asymptotics. For a higher dimensional case, it turns out that a similar reduction argument holds [2]. Moreover, in [2], it will be shown that there exists a solution to this system of elliptic equations, if (M, g, k) is near maximal, i.e. $|Tr_g k| < \epsilon$ for $0 < \epsilon < 1$, for both cases. Even though this does not resolve the full conjecture, it lends a credence to the currently proposed strategy. Note that the above results can be extended to include the charge [2, 4].

The second question is still widely open. It is in fact a weaker version of the Penrose inequality with angular momentum

$$m^2 \geq \frac{A_{min}}{16\pi} + \frac{4\pi\mathcal{J}^2}{A_{min}}, \tag{2}$$

where A_{min} is the minimum area required to enclose one end. There has been some progress in this direction including [5], but all the known results impose more restrictive boundary conditions. Recall that Dain's proof [6] is composed of two parts : first to show that mass is bounded below by Dain's mass functional, and second to prove that this functional realizes the global minimum by the extreme Kerr data. For the manifold with boundary case, one of the main difficulties arises from the first part of the proof. A boundary integral appears, which turns out to be difficult to control. It will be very interesting to see future progress in this direction, since this problem, as well as the Penrose inequality with angular momentum, will put a cornerstone to resolve the most generalized version of the Penrose inequality.

References

[1] A. Alaee, M. Khuri, and H. Kunduri, *Proof of the mass-angular momentum inequality for bi-axisymmetric black holes with spherical topology*, Adv. Theor. Math. Phys., **20** (2016), no. 6, 1397-1441.

[2] Y.-S. Cha, *On Geometric Inequalities for Near Maximal Axially Symmetric Initial Data* (in preparation)

[3] Y.-S. Cha, and M. Khuri, *Deformations of axially symmetric initial data and the mass-angular momentum inequality*, Ann. Henri Poincaré, **16** (2015), no. 3, 841-896. arXiv:1401.3384

[4] Y.-S. Cha, and M. Khuri, *Deformations of charged axially symmetric initial data and the mass-angular momentum-charge inequality*, Ann. Henri Poincaré, **16** (2015), no. 12, 2881-2918. arXiv:1407.3621

[5] P. Chruściel, and L. Nguyen, *A lower bound for the mass of axisymmetric connected black hole data sets*, Class. Quant. Grav., **28** (2011), 125001,2011.

[6] S. Dain, *Proof of the angular momentum-mass inequality for axisymmetric black hole*, J. Differential Geom., **79** (2008), 33-67. arXiv:gr-qc/0606105

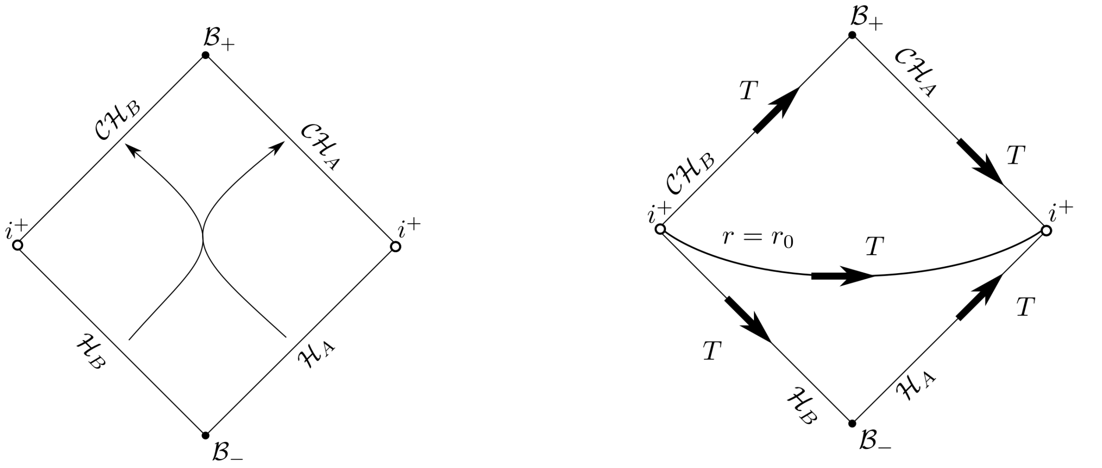
Scattering of linear waves on the interior of Reissner–Nordström black holes

Christoph Kehle
(joint work with Yakov Shlapentokh-Rothman)

There has been a lot of recent activities analyzing the *Cauchy problem* on black hole interiors. For certain physical processes, it is more natural to consider the *scattering problem* (see also [6] for scattering on the exterior of black holes). In [2] we developed a definitive scattering theory for finite energy solutions to the linear wave equation

$$\Box_g \psi = 0, \tag{1}$$

on the interior of a subextremal Reissner–Nordström black hole, from the bifurcate event horizon $\mathcal{H}$ to the bifurcate Cauchy horizon $\mathcal{CH}$ as depicted in Figure 1a. The



(A) Illustration of the scattering map.

(B) Visualization of the direction of the Killing vector field $T = \partial_t$.

FIGURE 1. Penrose diagram of the interior of the Reissner– Nordström black hole

Reissner–Nordström black holes with metric

$$g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 + r^2 g_{\mathbb{S}^2}$$

constitute a family of spacetimes, parametrized by mass M and charge $0 < |Q| < M$, which satisfy the Einstein–Maxwell system in spherical symmetry. The main result presented in the talk is *existence*, *uniqueness* and *asymptotic completeness* of finite energy scattering states. This means that for given finite energy data ψ_0 on the event horizon $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B \cup \mathcal{B}_-$, there exist unique finite energy data on the Cauchy horizon $\mathcal{CH} = \mathcal{CH}_A \cup \mathcal{CH}_B \cup \mathcal{B}_+$ arising from ψ_0 as the evolution of (1). The energy space on the event horizon is defined by imposing finiteness of the $T = \partial_t$ energy flux on $\mathcal{H}_A$ minus the T energy flux on $\mathcal{H}_B$. Since $T = \partial_t$ is past directed on $\mathcal{H}_B$ (cf. Figure 1b), the above gives a non-degenerate norm on the event horizon $\mathcal{H} = \mathcal{H}_A \cup \mathcal{H}_B \cup \mathcal{B}_-$ and similarly on the Cauchy horizon $\mathcal{CH} = \mathcal{CH}_A \cup \mathcal{CH}_B \cup \mathcal{B}_+$. More precisely, we define

$$\|\psi\|^2_{\mathcal{E}^T_{\mathcal{H}}} := \int_{\mathcal{H}_A} J^T_\mu[\psi] n^\mu_{\mathcal{H}_A} \mathrm{dvol}_{n_{\mathcal{H}_A}} - \int_{\mathcal{H}_B} J^T_\mu[\psi] n^\mu_{\mathcal{H}_B} \mathrm{dvol}_{n_{\mathcal{H}_B}},$$
$$\|\psi\|^2_{\mathcal{E}^T_{\mathcal{CH}}} := \int_{\mathcal{CH}_B} J^T_\mu[\psi] n^\mu_{\mathcal{CH}_B} \mathrm{dvol}_{n_{\mathcal{CH}_B}} - \int_{\mathcal{CH}_A} J^T_\mu[\psi] n^\mu_{\mathcal{CH}_A} \mathrm{dvol}_{n_{\mathcal{CH}_A}}.$$

Taking the completion of C^∞_c functions on the event and Cauchy horizon, respectively, defines Hilbert spaces $\mathcal{E}^T_{\mathcal{H}}$ and $\mathcal{E}^T_{\mathcal{CH}}$ on the event and Cauchy horizon, respectively. Moreover, we can define a scattering operator from data on the event horizon to the Cauchy horizon $S^T_0 : C^\infty_c(\mathcal{H}) \subset \mathcal{E}^T_{\mathcal{H}} \to \mathcal{E}^T_{\mathcal{CH}}$ by the Cauchy evolution of (1). Indeed, in [2] we have shown in the proof of the following Theorem 1 that this operator is bounded which allows us to define a scattering map $S^T : \mathcal{E}^T_{\mathcal{H}} \to \mathcal{E}^T_{\mathcal{CH}}$ by unique extension.

Theorem 1. *The scattering map* $S^T : \mathcal{E}^T_{\mathcal{H}} \to \mathcal{E}^T_{\mathcal{CH}}$ *is a Hilbert space isomorphism, i.e. a bounded and invertible linear map with bounded inverse* $B^T : \mathcal{E}^T_{\mathcal{CH}} \to \mathcal{E}^T_{\mathcal{H}}$. *In addition, the scattering map* S^T *is pseudo-unitary, meaning for* $\psi \in \mathcal{E}^T_{\mathcal{H}}$, *we have*

$$\int_{\mathcal{H}_A} |T\psi|^2 - \int_{\mathcal{H}_B} |T\psi|^2 = \int_{\mathcal{CH}_B} |TS^T\psi|^2 - \int_{\mathcal{CH}_A} |TS^T\psi|^2. \tag{2}$$

In more traditional language, this shows existence, uniqueness and asymptotic completeness of scattering states. Note that the sign-indefinite energy identity (2) does not, even a posteriori, yield boundedness of the scattering map. Indeed, in order to prove boundedness of S^T it is crucial to go to the separated picture:

It is well known that (1) on Reissner–Nordström allows separation of variables which reduces it to the radial o.d.e.

$$u'' - V_\ell u + \omega^2 u = 0$$

with potential V_ℓ, where $\omega \in \mathbb{R}$ is the time frequency and $\ell \in \mathbb{N}_0$ is the angular parameter. Indeed, most of the existing literature concerning scattering of waves in the interior of Reissner–Nordström mainly considers fixed frequency solutions [4, 5, 1]. For a purely incoming (i.e. supported only on $\mathcal{H}_A$) fixed frequency solution with parameters (ω, ℓ), we can associate transmission and reflection coefficients $\mathfrak{T}(\omega,\ell)$ and $\mathfrak{R}(\omega,\ell)$. The transmission coefficient $\mathfrak{T}(\omega,\ell)$ measures what proportion of the incoming wave is transmitted to $\mathcal{CH}_B$, whereas the reflection coefficient $\mathfrak{R}$ specifies the proportion reflected to $\mathcal{CH}_A$. An essential step to go from fixed frequency scattering to physical space scattering and to prove the boundedness of S^T is to prove uniform boundedness of $\mathfrak{T}(\omega,\ell)$ and $\mathfrak{R}(\omega,\ell)$. This is non-trivial in view of the sign-indefinite Wronskian identity $|\mathfrak{T}(\omega,\ell)|^2 - |\mathfrak{R}(\omega,\ell)|^2 = 1$. Note that this identity can be considered as the o.d.e. analog of the sign-indefinite energy identity (2). A careful o.d.e. analysis involving WKB approximations, Volterra iterations and a well chosen perturbation of special functions shows

Theorem 2. *The reflection and transmission coefficients* $\mathfrak{R}(\omega,\ell)$ *and* $\mathfrak{T}(\omega,\ell)$ *are uniformly bounded, i.e. they satisfy*

$$\sup_{\omega\in\mathbb{R}, \ell in \mathbb{N}_0} (|\mathfrak{R}(\omega,\ell)| + |\mathfrak{T}(\omega,\ell)|) \leq C(M,Q), \tag{3}$$

where $C(M,Q) > 0$ *is constant only depending on the black hole parameters* M *and* Q.

It should be remarked that in [2] we first prove Theorem 2 which eventually allows us to show Theorem 1.

The developed scattering theory has several applications. First, it can be shown that for purely incoming radiation on $\mathcal{H}_A$, there is always *non-trivial* reflection to $\mathcal{CH}_B$, see [2, Theorem 4]. Then, we can show that smooth data on $\mathcal{H}$ which are supported away from the past bifurcation sphere $\mathcal{B}_-$ and future null timelike infinity i^+ give rise to a solution which vanishes on $\mathcal{B}_+$. Moreover, we can also put the C^1 blow-up result of Chandrasekhar and Hartle [1] on rigorous footing: Indeed, we show [2, Theorem 5] that there exist smooth, compactly supported and purely incoming data on the event horizon $\mathcal{H}_A$ for which the Cauchy evolution of (1) fails to be in C^1 at the Cauchy horizon $\mathcal{CH}$.

If a cosmological constant $\Lambda \in \mathbb{R}$ is added to the Einstein–Maxwell system, we can consider the analogous (anti-) de Sitter–Reissner–Nordström family of solutions whose interiors have the same Penrose diagram as depicted in Figure 1. However, remarkably, there is no analogous T energy scattering theory for either the linear wave equation (1) or the Klein–Gordon equation with conformal mass $\Box_g\psi - \frac{2}{3}\Lambda\psi = 0$. The reason for this failure is the unboundedness of the transmission coefficient $\mathfrak{T}$ and reflection coefficients $\mathfrak{R}$ in the vanishing frequency limit $\omega \to 0$. Nevertheless, the scattering theory is crucially used in [3] for frequencies bounded away from $\omega = 0$.

References

[1] S. Chandrasekhar and J. B. Hartle, *On crossing the Cauchy horizon of a Reissner–Nordström black-hole*, Proc. Roy. Soc. London Ser. A 384.1787 (1982), 301–315.

[2] C. Kehle and Y. Shlapentokh-Rothman, *A scattering theory for linear waves on the interior of Reissner–Nordström black holes*, arXiv:1804.05438, preprint (2018).

[3] C. Kehle, *Uniform boundedness and continuity at the Cauchy horizon for linear waves on Reissner–Nordström–AdS black holes*, in preparation (2018).

[4] J. M. McNamara, *Behaviour of scalar perturbations of a Reissner–Nordström black hole inside the event horizon*, Proc. Roy. Soc. London Ser. A 364.1716 (1978), 121–134

[5] J. M. NcNamara, *Instability of black hole inner horizons*, Proc. Roy. Soc. London Ser. A 358.1695 (1978), 499–517.

[6] J. A. H. Futterman, F. A. Handler, and R. A. Matzner, *Scattering from black holes*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, (2009).

Topology of the Kerr Photon Region in the Phase Space

Sophia Jahns

(joint work with Carla Cederbaum)

We give a geometric proof that the photon region of a subcritical Kerr spacetime can be naturally viewed as a submanifold of the phase space with topology $SO(3)\times\mathbb{R}^2$.

It is well known that in the Schwarzschild spacetime with positive mass M, there is a family of photons that is *trapped* on the hypersurface $\{r = 3M\}$ (in Schwarzschild coordinates). Similar phenomena occur in other static spacetimes like Reissner–Nordström. Such a timelike hypersurface (that is, one with the property that every null geodesic once tangent to it will stay tangent) is totally umbilic [4, 9]. This geometric characterization has recently led to a number of uniqueness results for spacetimes possessing such hypersurfaces (with additional properties), see e.g. [1, 2, 3, 12, 11]. This is one of the main reasons why it seems desirable to better understand the geometry of the set of trapped light also in a non-static setting.

We therefore turn to the Kerr spacetimes as paradigmatic models of stationary, non-static spacetimes. We work with (the domain of outer communication of) subcritical Kerr spacetimes K, that is, we assume that the mass parameter M and the rotational parameter a fulfil $|a| < M$.

It is known that there is a 2-parameter family of photons who stay at a fixed coordinate radius (see e.g. [10]), and it can be verified by a direct computation using the decoupled equations of motion that these are the only photons in the Kerr exterior that are trapped in the sense that they neither approach or fall through the horizon nor escape towards infinity.

Unlike in the Schwarzschild situation, the spacetime region in a subcritical Kerr spacetime K where trapped photons are found (the *photon region*) is not a submanifold (with or without boundary) of the spacetime. Therefore, the spacetime itself seems not the right setting to investigate properties of photon regions; instead, the (co-)tangent bundle of the Kerr spacetime is a much more appropriate setting for further investigation of its geometry and topology.

With this purpose in mind, we identify a geodesic $\gamma = (t, r, \vartheta, \varphi)$ in a natural way in with the phase space point $\flat(\gamma(0), \dot{\gamma}(0)) \in T^*K$, where $\flat : TK \to T^*K$ denotes the canonical metric typechange. Similarly, the photon region P in the phase sphace of Kerr is a subset of T^*K given as

$$P := \{\flat(\gamma(0), \dot{\gamma}(0)) : \gamma \text{ is a future-directed trapped null geodesic } \} \subseteq T^*K.$$

The trapped photons in the subcritical Kerr exterior were described in [10] in terms of their energy E, angular momentum L and Carter constant $\mathfrak{Q}$ as the null geodesics that fulfil

$$\frac{L}{E} = -\frac{r^3 - 3Mr^2 + a^2 r + a^2 M}{a(r - M)},$$
$$\frac{\mathfrak{Q}}{E^2} = -\frac{r^3(r^3 - 6Mr^2 + 9M^2 r - 4a^2 M)}{a^2(r - M)^2},$$

where the radial coordinate r (in Boyer–Lindquist coordinates) ranges from $2M(1+\cos(2/3\arccos(-a/M)))$ to $2M(1+\cos(2/3\arccos(a/M)))$.

We use this characterization to show via the implicit function theorem that the photon sphere in the phase space is a smooth submanifold of codimension 3. We then proceed to determine its topology by topological and geometric arguments: working in a slice of constant time and energy $P_0 := P \cap \{t = 0, E = 1\}$, we show that P_0 is homeomorphic to $SO(3)$. In the Schwarzschild case $a = 0$, this is obvious since P_0 can be viewed as a unit sphere bundle over a sphere $\mathbb{S}^2$. In the rotating $|a| > 0$ case, we proceed as follows: We calculate the first fundamental groups of the regions of P_0 around the poles and around the equator and use the Seifert–van Kampen theorem to show that the first fundamental group of P_0 is $\mathbb{Z}_2$. By general results about the classification of closed 3-manifolds (see e.g. [7]), this allows to conclude that P_0 is homeomorphic to one of the lens spaces $L(2;0)$

or $L(2;1) \approx SO(3)$. Since these two possibilities we are left with are homotopy equivalent, we now need different machinery to distinguish between them: One can show that the projection

$$p : P_0 \ni (\vartheta, \varphi, *, *) \to (\vartheta, \varphi) \in \mathbb{S}^2$$

makes P_0 a Seifert fibered space with no exceptional fibers. On the other hand, it is known that $L(2;0)$ admits no Seifert fibration with less than 2 exceptional fibers [6]; and hence one can rule out the possibility $P_0 \approx L(2;0)$, concluding the proof that $P_0 \approx SO(3)$. Hence, the photon sphere P in T^*K has topology $SO(3) \approx SO(3) \times \mathbb{R}^2$.

It was already established in [5] that the Kerr photon region in the phase space is a submanifold. The author of [5] expresses P by means of equations whose dependence on a is smooth even in $a = 0$ (in contrast to the description of the photon region that we use, which breaks down for $a = 0$); this allows to see that the Kerr photon region in the phase space is a submanifold of the phase space which has the same topology for the rotating and the nonrotating (Schwarzschild) case. In contrast, our proof is more in a geometric spirit. It cannot directly be extended to higher dimensional Kerr spacetimes, since it makes crucial use of the classification of 3-manifolds. It has, however, the advantage that it allows to see that the Seifert fiber structure of the photon sphere in the phase space comes from a natural projection of P to a sphere $\mathbb{S}^2$. This additional structure could be useful in further investigating and using topological and geometric properties of the Kerr photon region.

References

[1] Carla Cederbaum. Uniqueness of photon spheres in static vacuum asymptotically flat spacetimes. *Commun. Anal. Geom.*, 25(2):303–320, 2017.

[2] Carla Cederbaum and Gregory J. Galloway. Uniqueness of photon spheres via positive mass rigidity. *Contemp. Math.*, 667(Complex Analysis & Dynamical Systems IV):86–99, 2015.

[3] Carla Cederbaum and Gregory J. Galloway. Uniqueness of photon spheres in electro-vacuum spacetimes. *Class. Quantum Grav.*, 33(7):075006, 2016.

[4] Clarissa-Marie Claudel, K. S. Virbhadra, and G. F. R. Ellis. The geometry of photon surfaces. *Journal of Mathematical Physics*, 42(2):818–838, 2001.

[5] Semyon Dyatlov. Asymptotics of linear waves and resonances with applications to black holes. *Communications in Mathematical Physics*, 335(1445), 2015. arXiv:1305.1723 [gr-qc].

[6] Hansjörg Geiges and Christian Lange. Seifert fibrations of lens spaces. 2016. arXiv:gr-qc/0512066.

[7] Allen Hatcher. Notes on basic 3-manifold topology, 2007. http://pi.math.cornell.edu/ hatcher/3M/3Mdownloads.html; last accessed 06/18/2018.

[8] Barrett O'Neill. *The Geometry of Kerr Black Holes.* Dover Publications, Mineola, New York, 2014.

[9] Volker Perlick. On totally umbilic submanifolds of semi-Riemannian manifolds. *Nonlinear Analysis*, 63(5-7), 2005. arXiv:gr-qc/0512066.

[10] Edward Teo. Spherical photon orbits around a Kerr black hole. *General Relativity and Gravitation*, 35(11):1909–1926, 2003.
[11] Stoytcho Yazadjiev. Uniqueness of the static spacetimes with a photon sphere in Einstein-scalar field theory. *Physical Review D*, 91(123013), 2015.
[12] Stoytcho Yazadjiev and Boian Lazov. Uniqueness of the static Einstein–Maxwell spacetimes with a photon sphere. *Class. Quantum Grav.*, 32(16):165021, 2015.

Spacetime Intrinsic Flat Convergence

Christina Sormani
(joint work with Anna Sakovich, Carlos Vega)

The intrinsic flat ($\mathcal{F}$) convergence of Riemannian manifolds was introduced by the author jointly with Stefan Wenger in [15]. The notion was first applied to General Relativity jointly with Dan Lee in [10]. Since then it has been applied in work of Lan-Hsuan Huang, Jeff Jauregui, Dan Lee, Philippe LeFloch, Anna Sakovich, and Iva Stavrov to prove the $\mathcal{F}$-stability of special cases of the Positive Mass Theorem, the Hyperbolic Positive Mass Theorem and the Penrose Inequality [9, 6, 7, 11, 12, 13].

Shing-Tung Yau has suggested that one develops a similarly useful notion of convergence for sequences of spacetimes. One might then apply it to answer the following questions:

- *What does it mean to say the universe is approximately a Friedmann–Lemaître–Robertson–Walker big bang spacetime when it has gravity wells and black holes?*
- *In what sense is one maximal development close to another if they have approximately the same initial data but the control on initial data is not strong enough to prevent gravitational collapse?*
- *In what sense is a black hole spacetime of small mass close to Minkowski space?*

First recall that stronger notions of convergence are not suited to questions where long thin gravity wells can develop. The development of such wells in sequences of Riemannian manifolds prevents smooth, Lipschitz, uniform, and Gromov–Hausdorff (GH) convergence. However, $\mathcal{F}$-convergence was designed specifically so that Ilmanen's example of a sequence of spheres with wells converges. Under $\mathcal{F}$ convergence, the wells disappear in the limit. Indeed all regions of small volume disappear and sequences of manifolds whose volume converge to 0 disappear as well. If the sequence does not disappear, the intrinsic flat limit is an *integral current space*, (X, d, T). In particular the limit space is a metric space endowed with a biLipschitz collection of charts.

Recall that Federer–Flemming developed the notion of integral currents to extend the notion of submanifolds and solve Plateau's problem. Integral currents, T, act on m forms, ω, via integration and have boundaries, $\partial T(\omega) = T(d\omega)$, and integer weighted volumes, $\mathbf{M}(T)$. Federer–Flemming define the flat distance between pairs of integral currents:

$$d_F(T_1,T_2) = \inf\{\mathbf{M}(A) + \mathbf{M}(B) : A + \partial B = T_1 - T_2\}.$$

So this is intuitively measuring the volume between the generalized submanifolds. Ambrosio–Kirchheim extended this entire theory to metric spaces in [1].

Wenger and I then defined the $\mathcal{F}$ distance:

$$d_{\mathcal{F}}((X_1,d_1,T_1),(X_2,d_2,T_2)) = \inf\left\{d_F^Z(\varphi_{1\#}T_1,\varphi_{2\#}T_2) : \varphi_i : X_i \to Z\right\}$$

where the infimum is taken over all distance preserving maps, $\varphi_i : X_i \to Z$, and over all complete metric spaces, Z, of the flat distance, d_F^Z, between the pushforwards of the integral current structures, $\varphi_{i\#}T_i$ [15]. In the same paper we proved Ilmanen's sequence of spheres with increasingly many increasingly thin wells converges to the sphere by constructing an explicit metric space Z in one dimension higher which filled in all the wells. In work with Sajjad Lakzian we proved $\mathcal{F}$ convergence for sequences of manifolds converging smoothly away from singular sets when there are volume, area, and distance bounds around those singular sets [8].

When trying to extend $\mathcal{F}$ convergence to a spacetime intrinsic flat convergence, we first encountered the difficulty that Lorentzian manifolds do not have a metric space structure and so we had no notion of distance preserving maps, φ_i such that

$$d_Z(\varphi_i(p),\varphi_i(q)) = d_{X_i}(p,q).$$

If φ_i were replaced by Riemannian isometries then the $d_{\mathcal{F}}$ would always be 0. But how would one possibly define anything but a Lorentzian isometry between spacetimes where there is no external distance structure to speak of? Lars Andersson suggested that we use a canonical time function like the cosmological time function, τ, to create a Riemannian manifold by adding twice $d\tau^2$ to the Lorentzian metric. Then define the spacetime intrinsic flat distance as the $\mathcal{F}$ distance between the two Riemannian manifolds and somehow keep track of the causal structure. After a few years, trying to deal with the singularities that could arise where τ was not smooth, this approach was abandoned.

Carlos Vega and I then decided to convert a spacetime directly into an integral current space rather than a Riemannian manifold [14]. Taking any time function, τ, on our spacetime, we defined the null distance as follows:

$$\hat{d}_\tau(p,q) = \inf_\beta \hat{L}_\tau(\beta) = \inf_\beta \sum_{i=1}^{k} |\tau(\beta(t_i)) - \tau(\beta(t_{i+1})|$$

where the inf is over all piecewise causal curves β from p to q, which are causal from $x_i = \beta(t_i)$ to $x_{i+1} = \beta(t_{i+1})$. We observed that $\hat{d}_\tau$ does not always define a metric space for arbitrary τ. For example in Minkowski space if we take $\tau = t^3$, the null distance $\hat{d}_\tau(p,q) = 0$ for all $p,q \in t^{-1}(0)$.

Carlos Vega and I proved $\hat{d}_\tau$ defines a metric space when τ is a regular cosmological time in the sense of Anderson–Howard–Galloway [3] (see also Wald–Yip [16]). The cosmological time τ at p is defined to be the supremum of the Lorentzian distance from q to p over all q in the past of p. It is "regular" if it is finite on all of M

and converges to 0 along all past inextensible curves [3]. One may envision examples like big bang spacetimes and maximal future developments from some initial data sets as possible examples of spaces with regular cosmological time functions. The great advantage of using $\hat{d}_\tau$ is that it captures causality:

$$p \text{ is in the future of } q \qquad \Longrightarrow \qquad \hat{d}_\tau(p,q) = \tau(p) - \tau(q).$$

We say that $\hat{d}_\tau$ *encodes causality* when this is an $\iff$. We prove that when $\hat{d}_\tau$ encodes causality then $\hat{d}_\tau$ is also definite and thus defines a metric space. We observed that $\hat{d}_\tau$ encodes causality in warped product spacetimes of the form $-dt^2 + f(t)^2 g_0$ where $\tau = t$. Indeed the balls in such spaces are shaped like cylinders around the lightcones. [14]

- *What spaces have regular cosmological time functions?*
- *When does $\hat{d}_\tau$ encode causality?*

Wald and Yip first introduced the cosmological time function as the maximal lifetime function in [16]. It has since been studied by Andersson–Barbot–Béguin-Zeghib [2], Cui–Jin [4], Ebrahimi [5], and many others but it must be explored further.

Carlos Vega and I are currently exploring the $\mathcal{SF}$ convergence of big bang spacetimes. Recall the classic FLRW big bang spacetimes have metrics of the form $dt^2 + f^2(t)g_0$ with $t > 0$ and $\lim_{t\to 0} f(t) = 0$. In such spaces the cosmological time, $\tau = t$, and so it is regular and in fact smooth and we can define a metric space $(X, \hat{d}_\tau)$, which encodes causality. We have proven that there is a single big bang point, p_{BB}, in the metric completion of this space, $\bar{X}$, and that cosmological time, $\tau(p) = \hat{d}_\tau(p_{BB}, p)$. We can then generalize the notion of big bang spacetime to any spacetime for which the cosmological time function defines a metric space $(X, \hat{d}_\tau)$ which encodes causality that has a big bang point, $B \in \bar{X}$ such that $\tau(p) = \hat{d}_\tau(B, p)$.

- *Which Lorentzian manifolds are generalized big bang spaces?*

The pointed $\mathcal{F}$ convergence of such spaces based at the big bang points is then well defined and one has compactness theorems with limit spaces which are integral current spaces with causal structures defined using

$$q_1 \text{ is in the future of } q_2 \qquad \iff \qquad \hat{d}_\tau(q_1, q_2) = \hat{d}_\tau(B, q_1) - \hat{d}_\tau(B, q_2).$$

Currently Anna Sakovich and I are exploring $\mathcal{SF}$ convergence for future maximal developments of initial data sets where the cosmological time function is regular and $\tau^{-1}(0)$ is the initial Cauchy surface. We are examining particular sequences of black hole spacetimes whose mass is converging to 0 to test that their metric spaces defined using the null distance do indeed converge in the intrinsic flat sense and we are formulating appropriate definitions and conjectures. It is possible that we might be able to prove a compactness theorem in this setting as well.

- *When does a future maximal development have a cosmological time function that is* 0 *on the initial Cauchy surface?*

- *Which future developments of initial data sets have null distances that encode causality?*
- *Are there other canonical time functions that are more suited to study future maximal developments of the Einstein equations?*

This research is funded in part by NSF DMS - 1309360.

References

[1] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[2] Lars Andersson, Thierry Barbot, François Béguin, and Abdelghani Zeghib. Cosmological time versus CMC time in spacetimes of constant curvature. *Asian J. Math.*, 16(1):37–87, 2012.

[3] Lars Andersson, Gregory J. Galloway, and Ralph Howard. The cosmological time function. *Classical Quantum Gravity*, 15(2):309–322, 1998.

[4] Xiaojun Cui and Liang Jin. The negative of regular cosmological time function is a viscosity solution. *J. Math. Phys.*, 55(10):102705, 9, 2014.

[5] N. Ebrahimi. Some observations on cosmological time functions. *J. Math. Phys.*, 54(5):052503, 3, 2013.

[6] Lan-Hsuan Huang, Dan A. Lee, and Christina Sormani. Intrinsic flat stability of the positive mass theorem for graphical hypersurfaces of Euclidean space. *J. Reine Angew. Math.*, 727, 2017.

[7] Jeffrey L. Jauregui. On the lower semicontinuity of the ADM mass. *Comm. Anal. Geom.*, 26(1):85–111, 2018.

[8] Sajjad Lakzian and Christina Sormani. Smooth convergence away from singular sets. *Comm. Anal. Geom.*, 21(1):39–104, 2013.

[9] Dan A. Lee and Christina Sormani. Near-equality of the Penrose inequality for rotationally symmetric Riemannian manifolds. *Ann. Henri Poincaré*, 13(7):1537–1556, 2012.

[10] Dan A. Lee and Christina Sormani. Stability of the positive mass theorem for rotationally symmetric Riemannian manifolds. *J. Reine Angew. Math.*, 686:187–220, 2014.

[11] Philippe G. LeFloch and Christina Sormani. The nonlinear stability of rotationally symmetric spaces with low regularity. *J. Funct. Anal.*, 268(7):2005–2065, 2015.

[12] Anna Sakovich and Christina Sormani. Almost rigidity of the positive mass theorem for asymptotically hyperbolic manifolds with spherical symmetry. *General Relativity and Gravitation*, 49:125, September 2017.

[13] Christina Sormani and Iva Stavrov. Geometrostatic manifolds of small ADM mass. *Preprint on arxiv:1707.03008*, 2017.

[14] Christina Sormani and Carlos Vega. Null distance on a spacetime. *Classical Quantum Gravity*, 33(8):085001, 29, 2016.

[15] Christina Sormani and Stefan Wenger. Intrinsic flat convergence of manifolds and other integral current spaces. *Journal of Differential Geometry*, 87, 2011.

[16] Robert M. Wald and Ping Yip. On the existence of simultaneous synchronous coordinates in spacetimes with spacelike singularities. *J. Math. Phys.*, 22(11):2659–2665, 1981.

The nonlinear stability of the Schwarzschild family of black holes

Martin Taylor
(joint work with M. Dafermos, G. Holzegel, I. Rodnianski)

The Schwarzschild metrics

$$g_M = -4(1-2M/r)dudv + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad M > 0, \tag{1}$$

discovered in 1915, constitute the most famous family of solutions of the vacuum Einstein equations,

$$\mathrm{Ric}[g] = 0. \tag{2}$$

Each member of the Schwarzschild family describes a static black hole, and the family arises as a subfamily of the larger stationary two parameter Kerr family of uniformly rotating black holes, $g_{a,M}$.

The most fundamental question one can ask about (1) is whether the exterior region is nonlinearly stable as a solution of (2). This talk concerned a theorem, in progress, on this question, which can be roughly formulated as follows.

Theorem 1 (Full finite codimension nonlinear asymptotic stability of the Schwarzschild family)**.** *For all characteristic initial data sufficiently close to Schwarzschild data with mass M_{init} and lying on a codimension-3 "submanifold" of the moduli space of initial data, the maximal Cauchy development $\mathcal{M}$ contains a region $\mathcal{R}$ which admits a double null gauge such that*

- (i) *$\mathcal{M}$ possesses a complete future null infinity $\mathcal{I}^+$ such that $\mathcal{R} \subset J^-(\mathcal{I}^+)$, and the future boundary of $\mathcal{R}$ in $\mathcal{M}$ is a smooth, future affine complete "event horizon" $\mathcal{H}^+$;*
- (ii) *the metric remains close to Schwarzschild in $\mathcal{R}$;*
- (iii) *the metric asymptotes, inverse polynomially, to a Schwarzschild metric with mass $M_{\mathrm{final}} \approx M_{\mathrm{init}}$ as $u \to \infty$ and $v \to \infty$.*

The proof of Theorem 1 is based on the strategy of Dafermos–Holzegel–Rodnianski [6] on the *linear stability* of Schwarzschild. An alternative proof of the linear stability of Schwarzschild, using a harmonic gauge, has also recently been obtained by Johnson [13]. See also related work of Hung–Keller–Wang [12]. Each of the above linear stability works relies heavily on methods developed in recent years to understand the dispersion of linear waves on fixed black hole backgrounds. The study of such dispersive properties began with the pioneering work of Wald [16] on the boundedness of waves on the exterior of a Schwarzschild black hole, and culminated in the recent work of Dafermos–Rodnianski–Shlapentokh-Rothman [9] on the boundedness and decay of waves on the exterior of Kerr for the full subextremal range $|a| < M$. The understanding of the phenomena of trapped null geodesics, superradiance, the celebrated redshift effect and their coupling play a crucial role in [9].

The main step of the proof of the linear stability of Schwarzschild in [6] was to prove decay results for the decoupled *Teukolsky equation*, governing the gauge-invariant part of the perturbations, via a physical space reinterpretation and novel use of transformations first introduced by Chandrasekhar [1]. This has recently been generalised to the Kerr case in [7]. See also [15]. With these more recent developments, the whole approach of the proof of Theorem 1 can in principle be generalised to Kerr.

The study of the nonlinear stability of solutions of (2) was initiated with the monumental proof of the stability of Minkowski space [4], the trivial solution of (2) (corresponding to (1) in the case $M = 0$). The proof of Theorem 1, of course, relies on the insights developed in [4] on understanding the nonlinearities of (2), in particular in the most difficult radiation zone towards null infinity $\mathcal{I}^+$ where null structure is paramount.

There are a number of nonlinear stability results for Schwarzschild under symmetry assumptions. By far the simplest case is when enough symmetry is imposed to reduce the equations to a $1+1$ dimensional system. For the case of the *Einstein-scalar field system* under spherical symmetry, the non-linear asymptotic stability of (1) follows from the more general results of [2, 8]. The vacuum equations (2) in higher space-time dimensions admit other symmetries reducing the equations to $1 + 1$, for instance the *triaxial Bianchi symmetry*. The non-linear orbital stability of the $4 + 1$-dimensional analogue of (1) under triaxial Bianchi symmetry was proven in [5] while its full asymptotic stability, in the further restricted *biaxial case*, was proven in [11].

Beyond reductions to $1 + 1$, the problem is considerably harder. Non-linear stability of (1) has been announced in the case of (2) under polarised axisymmetry (a $2 + 1$ reduction) by Klainerman–Szeftel, the first part of the proof already appearing in [14]. Note that the set of axisymmetric spacetimes constructed in the above work are included as an infinite-codimension subset of the codimension-3 set of data referred to in Theorem 1.

One should also note the recent work of Hintz–Vasy [10] on the nonlinear stability of the very slowly rotating *Kerr–de Sitter* family of black holes for the vacuum Einstein equations with a positive cosmological constant.

The restriction to finite codimension in Theorem 1 is indeed necessary to achieve point (iii) of the theorem in view of the larger Kerr family. The finite codimension of Theorem 1 is precisely equal to the dimension of *fixed mass linearised Kerr solutions*, which is 3 in our parameterisation. It is in this sense that Theorem 1 is the "full" finite codimension stability and one expects that the "submanifold" of the theorem contains *all* data close to Schwarzschild whose evolution satisfies (iii). It should be noted that the "submanifold" cannot be explicitly identified but is teleologically characterised in the proof.

The point of the restriction to perturbations which converge to a member of the Schwarzschild family (rather than a member of the Kerr family with $a \neq 0$) is so that the proof can follow that of Dafermos–Holzegel–Rodnianski [6] on the *linear stability* of Schwarzschild. In particular, the proof of Theorem 1 is based entirely in physical space.

A *double null gauge*, referred to in Theorem 1, is a coordinate system $(u, v, \theta^1, \theta^2)$ in which the metric takes the form

$$g = -4\Omega^2 du\, dv + \not{g}_{AB}(d\theta^A - b^A dv)(d\theta^B - b^B dv), \tag{3}$$

for some Ω, b, $\not{g}$. One example of a metric in double null gauge is the above form of the Schwarzschild metric (1). This is, however, far from the only way to write Schwarzschild in this form. Indeed, there exists an infinite dimensional family of diffeomorphisms of Schwarzschild preserving the double null form (3). A central difficulty in the proof of Theorem 1, on top of the issue of characterising the "submanifold" of initial data, is in determining a double null gauge of the dynamical spacetimes in which the spacetime converges to the Schwarzschild metric *in the familiar form* (1). Indeed, the double null gauge of Theorem 1 is characterised teleologically in the proof (as, in fact, is the region $\mathcal{R}$ itself). This difficulty, and such a normalisation, is present already in the work [6]. The gauge in Theorem 1, however, is more carefully normalised, both to the event horizon and to future null infinity, in order to ensure that the nonlinear error terms in the equations decay suitably. Further care needs to be taken as the location of the event horizon is not a priori known. As a bonus of the normalisation to future null infinity, the gauge in particular satisfies the *Bondi normalisation* in which the familiar laws of gravitational radiation hold, and moreover the nonlinear Christodoulou memory effect [3] can be understood.

References

[1] S. Chandrasekhar, *On the equations governing the perturbations of the Schwarzschild black hole*, Proc. Roy. Soc. A, 343(1634):289–298, 1975.

[2] D. Christodoulou, *A mathematical theory of gravitational collapse*, Commun. Math. Physics., 109(4):613–647, 1987.

[3] D. Christodoulou, *The nonlinear nature of gravitation and gravitational wave experiments*, Phys. Rev. Letters 67(12):1486–1489, 1991.

[4] D. Christodoulou and S. Klainerman, *The global nonlinear stability of the Minkowski spacetime*, volume 41 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 1993.

[5] M. Dafermos and G. Holzegel, *On the nonlinear stability of higher dimensional triaxial Bianchi-IX black holes*, Adv. Theor. Math. Phys., 10(4):503–523, 2006.

[6] M. Dafermos, G. Holzegel and I. Rodnianski, *The linear stability of the Schwarzschild solution to gravitational perturbations*, arXiv:1601.06467.

[7] M. Dafermos, G. Holzegel and I. Rodnianski, *Boundedness and decay for the Teukolsky equation on Kerr spacetimes I: the case $|a| \ll M$*, arXiv:1711.07944.

[8] M. Dafermos and I. Rodnianski, *A proof of Price's law for the collapse of self-gravitating scalar field*, Invent. Math., 162:381–457, 2005.

[9] M. Dafermos, I. Rodnianski and Y. Shlapentokh-Rothman, *Decay for solutions of the wave equation on Kerr exterior spacetimes III: The full subextremal case $|a| < M$*, Ann. of Math. (2), 183(3):787–913, 2016.

[10] P. Hintz and A. Vasy, *The global non-linear stability of the Kerr-de Sitter family of black holes*, arXiv:1606.04014.

[11] G. Holzegel, *Stability and decay rates for the five-dimensional Schwarzschild metric under biaxial perturbations*, Adv. Theor. Math. Phys., 14(5):1245–1372, 2010.

[12] P.-K. Hung, J. Keller and M.-T. Wang, *Linear stability of Schwarzschild spacetime: the Cauchy problem of metric coefficients*, arXiv:1702.02843.

[13] T. Johnson, *The linear stability of the Schwarzschild solution to gravitational perturbations in the generalised wave gauge*, arXiv:1803.04012.
[14] S. Klainerman and J. Szeftel, *Global nonlinear stability of Schwarzschild spacetime under polarized perturbations*, arXiv:1711.07597.
[15] S. Ma, *Uniform energy bound and Morawetz estimate for extreme components of spin fields in the exterior of a slowly rotating Kerr black hole II : linearized gravity*, arXiv:1708.07385.
[16] R. Wald, *Note on the stability of the Schwarzschild metric*, J. Math. Phys. 20:1056–1058, 1979.

Geometrical inequalities for isolated systems

María Eugenia Gabach-Clement

1. Introduction

Geometrical inequalities are *a priori* estimates on certain geometrical quantities defined on a physical system, that also have a clear physical interpretation.

One of the most important such geometrical inequality in General Relativity is the positivity of mass [1]. For asymptotically flat (AF) spacetimes satisfying the dominant energy condition, the ADM mass m is non negative,

$$m \geq 0 \qquad \text{(= iff Minkowski).} \tag{1}$$

If the system describes a black hole one arrives at the Penrose inequality [2], relating the mass with the horizon area A of the black hole

$$m \geq \sqrt{\frac{A}{16\pi}} \qquad \text{(= iff Schwarschild).} \tag{2}$$

The idea behind geometrical inequalities is to understand how the different physical properties of a system play with and control each other.

There is a significant difference between (1) and (2). Inequality (1) involves m, which is a global quantitiy, defined at infinity. On the other hand, (2) involves also the horizon area, which is a quasilocal quantity. Therefore, in order to prove (2) one needs to connect the bounded region (where A is defined) and infinity (where m is defined). And one needs to connect these two regions in a controlled way, so that one is able to obtain the fine relation between m and A.

In this article we will mostly refer to inequalities involving either purely global quantities or purely quasilocal quantities. We will focus on two systems: black holes and ordinary, material objects like neutron stars. We refer the reader to [3] for a recent review on the subject.

Let us start by discussing the global inequalities.

2. Global inequalities for black holes

Consider electrovacuum initial data for the Einstein equations $D := (\Sigma, g, K, \mu, j)$ where Σ is a 3-dimensional manifold, g is a Riemannian metric on Σ, K is a symmetric 2-tensor on Σ and μ, j are matter fields on Σ. Assume that D is AF, axially symmetric, maximal and that it satisfies the dominant energy condition. Some of these conditions can and have been relaxed (see [3] for details).

To describe a black hole we assume that (Σ, g) is complete with 2 ends. Both AF or one AF and one asymptotically cylindrical. Also assume that the data has total angular momentum J_∞. Then, one has [4]

$$m \geq \sqrt{J_\infty} \qquad (= \text{iff extremal Kerr}). \tag{3}$$

In order to prove (3) one first finds the bound $m \geq \mathcal{M}$, where $\mathcal{M}$ is a functional related to the energy of maps from open bounded sets of $\mathbb{R}^3$ that do not include the symmetry axis, into $\mathbb{H}^2$. Hence, minimizing m amounts finding the harmonic maps. As it turns out, the minimizer is the extreme Kerr black hole and $\mathcal{M} \geq \sqrt{J_\infty}$, giving the desired result (3).

A related inequality involves m and the total electric charge Q_∞ [5]

$$m \geq Q_\infty \qquad (= \text{iff extremal Reissner–Nordström.}) \tag{4}$$

We remark that (4) has been extended to many black holes with equality for Majumdar-Papapetrou solution (a static configuration of multiple charged black holes in equilibrium). Also, not only (4) holds for complete manifolds Σ with non trivial topology, but for manifolds with inner trapped boundary as well. These remarks lead us to the main open problems in this topic.

- *Open Problem 1:* Extend (3) to the case of Σ with trapped inner boundary
- *Open Problem 2:* Understand the multiple black hole extension of (3), partly done in [6] where the bound $m \geq \mathcal{M}_{many}$ was proven, with $\mathcal{M}_{many}$ a functional related to $\mathcal{M}$. More concretely one needs to understand
  - What the value of $\mathcal{M}_{many}$ is. Does it depend on the total J_∞ or the individual J_i and the distances between the black holes, etc.
  - What the minimizer represents. In previous results minimizers are stationary, extremal solutions, which, we know do not exist for the rotating case (at least for two black holes [7]).

3. Quasilocal inequalities for black holes

The quasilocal inequalities admit two approaches depending on the way the quasilocal black hole is described. Most of the results apply to both, stable minimal surfaces and stable marginally outer trapped surfaces (MOTS). We assume, for simplicity, that such surface S is axially symmetric and has (quasilocal) angular momentum J_S. In vacuum, one obtains [8].

$$A \geq 8\pi J_S \qquad (= \text{iff extremal Kerr}) \tag{5}$$

Inequality (5) has been generalized to include a positive cosmological constant $\Lambda \geq 0$ [9]. The inequality is saturated by extremal Kerr–dS horizon. Related to this, the main open problems are

- *Open Problem 3:* Extend (5) to explicitly include a $\Lambda \leq 0$. What role does extremal AdS play in the final estimate?
- *Open Problem 4:* Rigidity statement in the area-charge inequality.

4. Inequalities for Material Bodies

Consider initial data D containing a material body. By material body we mean an open subset Ω in Σ such that some of the matter fields are supported in Ω and that there are no horizon outside Ω. We assume the data to be AF, axially symmetric and satisfying the DEC.

The study of geometrical inequalities involving quantities that describe Ω is greatly open due to a number of difficulties, the most basic and important being

- *Q1:* Characterization of Ω. Some sort of stability may be needed.
- *Q2:* Measures of Ω that are intuitively clear, easy to compute and that can be controlled in terms of other physical-geometrical quantities.
- *Q3:* Existence of non trivial 'extremal bodies', playing the role of extremal black holes for the inequalities presented in the previous sections.

These questions are crucial in order to understand what kind of estimate we are looking for, what quantities we expect to be relevant for the description of the system and what should be the best way to represent an ordinary, material object.

Clearly, different choices produce different results and many geometrical inequalities have been obtained (see [3] for details). In the purely quasilocal setting, estimates of the form

$$J_\Omega \leq cR^2 \tag{6}$$

were proven for very specific models. Here c is a constant and R is a measure of size of Ω.

In the mixed, global-quasilocal setting there is a recent result [10] of the form

$$m \geq m_\Omega + c\frac{J^2}{\sqrt{A}R_e^2} \tag{7}$$

where m_Ω is a quasilocal mass contained in Ω, A is the area of $\partial\Omega$, c is a constant and $R_e = C_e/2\pi$ and C_e is the length of the greatest axially symmetric circle on $\partial\Omega$. There are some technical assumptions that need to be addressed in the statement of this result. As with the Penrose inequality, this result uses the IMCF to relate the object and infinity. The novel issue here is the explicit inclusion of the angular momentum.

On top of the basic questions raised above, some important open problems are

- *Open Problem 5:* Purely global inequalities.
- *Open Problem 6:* Relevance of some sort of convexity or roundedness of the object.

References

[1] Schoen, R. and Yau, S.T., Comm. Math. Phys. **65**, (1979), 45-76.
[2] G. Huisken and T. Ilmanen,J. Differential Geometry, **59**, (2001) 352-437.
[3] Dain, S. and Gabach-Clement, M.E., Living Rev. Rel., **21** (2018) 1.
[4] S. Dain, (2008) J. Differential Geometry, **79**, 33-67.
[5] Gibbons, G. W. and Hawking, S. W. and Horowitz, G.T. and Perry, M.J., Commun. Math. Phys. **88**, (1983), 295-308.
[6] Chruściel, P.T. and Li, Y. and Weinstein, G., Ann. Phys. **323**, (2008) 2591-2613.

[7] Neugebauer, G. and Hennig, J., J.Geom.Phys. **62**, (2012) 613-630.
[8] Aceña, A. and Dain S. and Gabach-Clément, M.E., Class.Quant.Grav., **28**, 105014.
[9] Gabach-Clement, M.E. and Reiris, M. and Simon, W., Class.Quant.Grav. **32** (2015) 145006.
[10] Anglada, Pablo and Gabach Clément, M. E. and Ortiz, Omar E., Class.Quant.Grav., **34** (2017) 125011.

Uniqueness and non-uniqueness results for wave equations

Jan Sbierski
(joint work with F. Eperon, H. Reall)

The well-known theorem by Choquet-Bruhat and Geroch [1] states that for given smooth initial data for the, say, vacuum Einstein equations $Ric(g) = 0$ there exists a unique (up to isometry) maximal globally hyperbolic development of the initial data. We firstly recall that the manifold on which the solution g is defined is not given a priori, but constructed at the same time as the metric g, and secondly, that in harmonic coordinates the vacuum Einstein equations take the form of a quasilinear wave equation $g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = N_{\alpha\beta}(g)(\partial g, \partial g)$, where the right hand side denotes a term that is quadratic in first derivatives of g with coefficients in g. This report, which is based on [2], investigates the question whether an analogous result holds for quasilinear wave equations that are *defined on a fixed background*. Here, we consider quasilinear wave equations of the form

$$g^{\mu\nu}(\phi, \partial\phi)\partial_\mu\partial_\nu\phi = F(\phi, \partial\phi) \ , \tag{1}$$

where $\phi \colon \mathbb{R}^{d+1} \subseteq D \to \mathbb{R}$ is a smooth function ($d \in \mathbb{N}_{\geq 1}$), $g^{\mu\nu}$ is a smooth Lorentzian-metric-valued function, and F is a smooth scalar-valued function. For the sake of simplicity let us assume in this report that we prescribe *initial data* for (1) on the hypersurface $\{t = 0\} \subseteq \mathbb{R}^{d+1}$, which then consist of smooth choices for $\phi(0, x)$ and $\partial_t\phi(0, x)$. A *globally hyperbolic development* (GHD) of this data is a smooth solution $\phi \colon \mathbb{R}^{d+1} \supseteq D \to \mathbb{R}$ of (1) such that $\{t = 0\} \subseteq D$ is a Cauchy hypersurface for D with respect to the Lorentzian metric $g(\phi, \partial\phi)$. A *maximal globally hyperbolic development* (MGHD) of the initial data is a GHD $\phi_1 \colon D_1 \to \mathbb{R}$ with the property that there does not exist any other GHD $\phi_2 \colon D_2 \to \mathbb{R}$ of the same initial data with $D_1 \subsetneq D_2$ and $\phi_2|_{D_1} = \phi_1$.

We find that in contrast to the Einstein equations, for general quasilinear wave equations defined on a fixed background there does *not* exist a unique MGHD. In particular, the time evolution of solutions to quasilinear wave equations is in general not unique. This is illustrated by means of the following counterexample: we consider the equation

$$-\big(1 + (\partial_x\phi)^2\big)\partial_t^2\phi + 2\partial_t\phi\partial_x\phi\partial_t\partial_x\phi + \big(1 - (\partial_t\phi)^2\big)\partial_x^2\phi = 0 \tag{2}$$

for a smooth function $\phi : \mathbb{R} \times \mathbb{R} \supseteq D \to \mathbb{R}$. Equation (2) is of the form (1) with

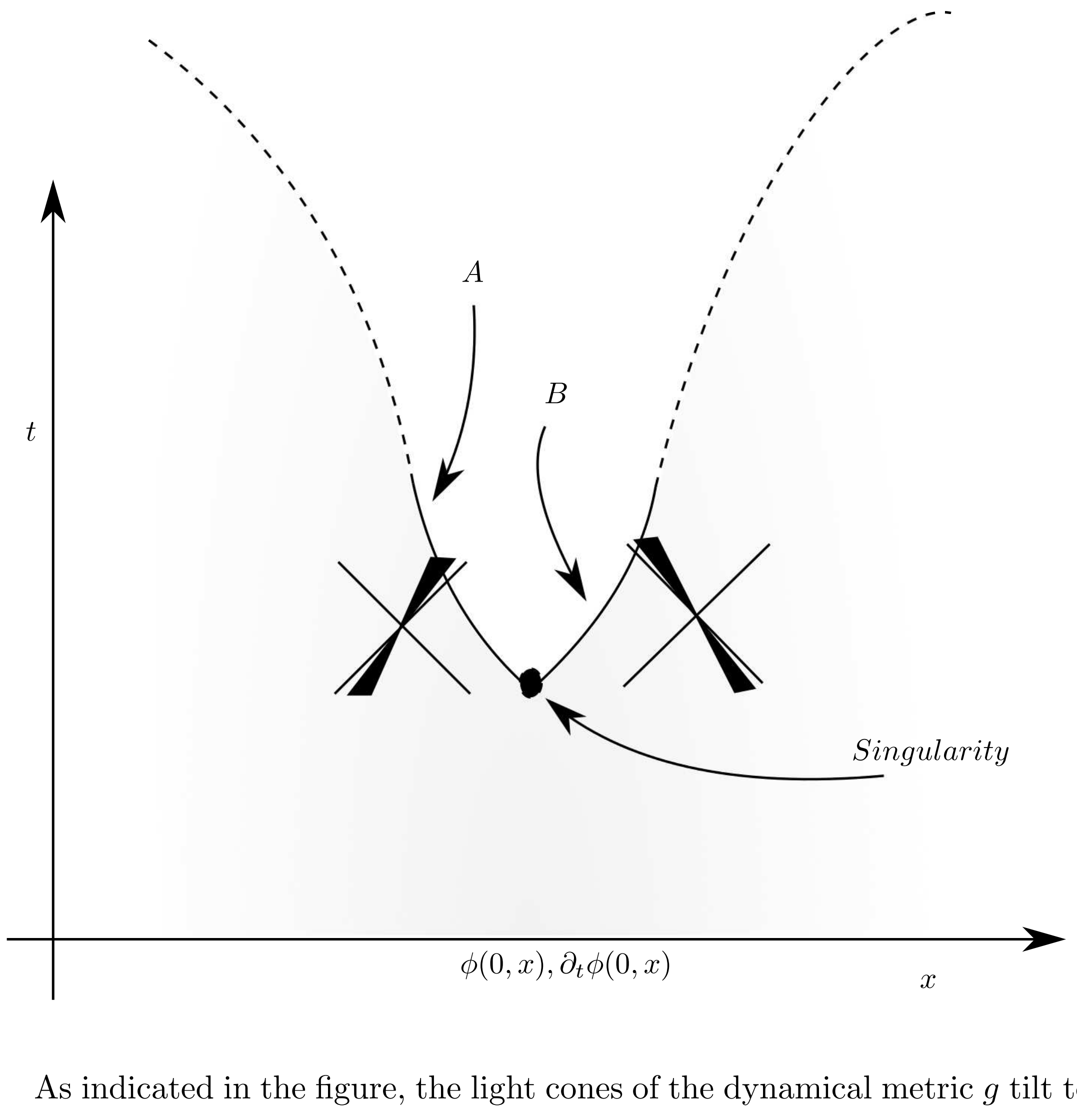
$$g^{\mu\nu} = \begin{pmatrix} -(1 + (\partial_x\phi)^2) & \partial_t\phi\partial_x\phi \\ \partial_t\phi\partial_x\phi & (1 - (\partial_t\phi)^2) \end{pmatrix} \tag{3}$$

and $F = 0$. If ϕ satisfies

$$1 + (\partial_x \phi)^2 - (\partial_t \phi)^2 > 0 \,, \tag{4}$$

then (3) is a Lorentzian metric which is moreover *subluminal*, i.e., its lightcones lie inside those of the Minkowski metric $m = -dt^2 + dx^2$. In particular, for such solutions ϕ, (2) is a quasilinear wave equation.

Equation (2) has been studied by Barbashov and Chernikov in [3], [4], who in particular show that it is exactly solvable. Using their representation formula of the general solution we construct a particular solution which satisfies (4), arises from smooth initial data on $\{t = 0\}$, and admits a unique time evolution into a globally hyperbolic region schematically depicted below.



As indicated in the figure, the light cones of the dynamical metric g tilt towards the right end of the future directed Minkowski light cone near the solid line A, which is thus spacelike with respect to g. Near the solid line B the light cones tilt to the other side, so that B is spacelike, too, with respect to g. Apart from the indicated singularity, the solution admits smooth extensions to the solid spacelike lines A and B as well as to the dashed lines which are null with respect to g. *One* *can now construct two different globally hyperbolic extensions by either extending into the region between A and B through A or through B.*

Let us emphasise that this non-uniqueness mechanism applies to smooth solutions and is thus manifestly different to a loss of uniqueness induced by a loss of regularity of the solution. We also remark that it is precisely the fixed background which prevents us from extending the solution all the way through A *as well as* all the way through B to obtain a solution which is double-valued in the region between A and B.

We conclude by mentioning one of the uniqueness criteria from [2] for solutions of quasilinear wave equations (1).

Theorem 1. *Let $\phi_i : D_i \to \mathbb{R}$, $i = 1, 2$, be two GHDs of the same initial data for a quasilinear wave equation of the form* (1)*. If $D_1 \cap D_2$ is connected, then $\phi_1 = \phi_2$ on $D_1 \cap D_2$.*

We note that in our counterexample above, the intersection of the domains of the two different globally hyperbolic extensions is disconnected. The theorem now shows that this is a necessary feature for a non-unique time evolution to arise.

Lemma 2. *Consider a quasilinear wave equation of the form* (1) *and assume that there exists a smooth vector field T such that T is timelike with respect to $g(\phi, \partial\phi)$ for all $\phi, \partial\phi$, and such that every maximal integral curve of T intersects $\{t = 0\}$ at most once. Let $\phi_i : D_i \to \mathbb{R}$ be two GHDs of the same initial data posed on $\{t = 0\}$. Then $D_1 \cap D_2$ is connected.*

The lemma holds in particular for semilinear wave equations as well as for superluminal[1] quasilinear wave equations, since one can choose $T = \partial_t$. Combined with the above theorem, this gives unique time evolution for solutions of those equations and thus also the existence of a unique MGHD (see [2]).

References

[1] Y. Choquet-Bruhat and R. Geroch, *Global Aspects of the Cauchy Problem in General Relativity*, Comm. Math. Phys. **14** (1969), 329–335.

[2] F. Eperon and H. Reall and J. Sbierski, *Predictability of subluminal and superluminal wave equations*, arXiv:1802.03020 (2018).

[3] B. Barbashov and N. Chernikov, *Solution and Quantization of a Nonlinear Two-dimensional Model for a Born-Infeld Type Field*, Sov. Phys. JETP **23**, no. 5, (1966), 861–868.

[4] B. Barbashov and N. Chernikov, *Solution of the two plane wave scattering problem in a nonlinear scalar field theory of the Born-Infeld type*,' Sov. Phys. JETP **24**, no. 2, (1967), 437–442.

[1]Here, superluminal means that the lightcones of the dynamical metric $g(\phi, \partial\phi)$ always contain the Minkowski light cones.

Horizon Stability and the Null Penrose Inequality

Henri Roesch

In the early seventies, Roger Penrose [10] conjectured that the mass contributed by a collection of black holes should be no less than $\sqrt{A/16\pi}$ where A is the total combined area of the black hole horizons. Alternatively,

$$M \geq \sqrt{\frac{A}{16\pi}} \tag{1}$$

where M represents the total (ADM,[2]) mass. One of the fundamental ingredients of Penrose's heuristic argument was the use of *weak cosmic censorship*. As a statement on the global future evolution of a system, weak cosmic censorship has direct ties to an existence theorem in general relativity. Finding a counterexample would likely indicate a failure of cosmic censorship and, conversely, a proof would provide indirect support to its validity.

The first major breakthrough on the Penrose conjecture came in 1997 with the work of Huisken and Ilmanen ([7]) when they settled a special case called the *Riemannian Penrose inequality*. Measuring the propagation of the Hawking Energy E_H (introduced by Hawking [6]), along a weakly defined inverse mean curvature flow of 2-spheres, the authors established the Penrose inequality for asymptotically flat Riemannian manifolds of non-negative scalar curvature (i.e. non-negative energy density). They verified (1) with A the area of one connected component of an outermost minimal surface. Bray lifted this restriction in 1999 ([4]), with A now the total area of *all* connected components using a conformal flow of metrics on the manifold and the positive mass theorem. Bray's argument was shown to generalize, yielding the result in dimensions $4 \leq n \leq 8$ by Bray and Lee ([5]) in 2009.

Just as the Riemannian Penrose inequality describes the result on particular space-like (hence Riemannian) slices of total ADM mass (M_{ADM}) one can formulate a variant of the Penrose conjecture on degenerate null slices or *null cones* of total *Trautman–Bondi mass* (M_{TB} [3, 14], for null cones see [8]). This is called the *Null Penrose Inequality*. In 2008, the PhD work of Sauter ([13]) in vacuum spacetimes successfully identified foliations of a null cone exhibiting non-decreasing and convergent Hawking Energy. However, comparison of this convergence with the Trautman–Bondi mass was not forth coming, except for a very special class of 'shear-free' slices. Sauter was able to use these shear-free slices to validate the conjecture. In 2015 Mars and Soria ([8]) identified the necessary asymptotics for convergence of the Hawking Energy for general null cones but—like Sauter—they faced similar difficulties in physically characterizing the Hawking Energy's convergence ([9]).

In [11], the author proposes a new quasi-local mass instead of energy. For a 2-sphere Σ with mean curvature $\vec{H}$, if Σ admits a normal null basis $\{L^-, L^+\}$

($\langle L^-, L^+\rangle = 2$) whereby $-\langle L^-, \vec{H}\rangle = 1$, then this quasi-local mass is given by:

$$m_R(\Sigma) = \frac{1}{2}\Big(\frac{1}{4\pi}\int_\Sigma \rho^{\frac{2}{3}} dA\Big)^{\frac{3}{2}} \tag{2}$$

where $\rho = \mathcal{K} - \frac{1}{4}\langle \vec{H}, \vec{H}\rangle + \nabla \cdot \tau$, $\mathcal{K}$ being the Gaussian curvature, and τ the connection 1-form associated to $\{L^-, L^+\}$. We're able to show that m_R exhibits monotonicity for a class of foliations of a null cone:

Theorem 1. *Let Ω be a null hypersurface foliated by space-like spheres $\{\Sigma_s\}$ expanding along the null flow direction $\underline{L} = \sigma L^-$ such that $|\rho(s)| > 0$. Then $m_R(s) := m_R(\Sigma_s)$ has rate of change*

$$\frac{dm_R}{ds} = \frac{(2m_R)^{\frac{1}{3}}}{8\pi}\int_{\Sigma_s}\frac{\sigma}{\rho^{\frac{1}{3}}}\Big((|\hat{\chi}^-|^2 + G(L^-, L^-))(\frac{1}{4}\langle\vec{H}, \vec{H}\rangle - \Delta\log|\rho|^{\frac{1}{3}})$$
$$+ \frac{1}{2}|\nu|^2 + G(L^-, N)\Big)dA$$

where $G = Ric - \frac{1}{2}Rg$, $\nu := \frac{2}{3}\hat{\chi}^- \cdot d\log|\rho| - \tau$, and $N := \frac{1}{9}|d\log|\rho||^2 L^- + \frac{1}{3}\nabla\log|\rho| - \frac{1}{4}L^+$.

So from the Dominant Energy Condition (DEC), Theorem 1 gives $\frac{dm_R}{ds} \geq 0$ for foliations $\{\Sigma_s\}$ satisfying $\rho > 0$ & $\frac{1}{4}\langle\vec{H}, \vec{H}\rangle \geq \frac{1}{3}\Delta\log\rho$ called *double convexity*. Surprisingly, all *asymptotically geodesic* doubly convex foliations also measure the same limiting mass, and underestimate M_{TB}. Therefore, the mass m_R overcomes the difficulty arising asymptotically from the use of the Hawking Energy.

Theorem 2. *Let Ω be a null hypersurface in a spacetime satisfying the DEC that extends to past null infinity. Given the existence of a doubly convex $\{\Sigma_s\}$ we have*

$$m_R(0) \leq \lim_{s\to\infty} m_R(\Sigma_s) =: M_R$$

(for $M_R \leq \infty$). If, in addition, Ω is past asymptotically flat with strong flux decay and $\{\Sigma_s\}$ asymptotically geodesic (see [11], *Section 4) then*

$$M_R \leq M_{TB}.$$

In case $\langle\vec{H}, \vec{H}\rangle|_{\Sigma_0} = 0$ (i.e. Σ_0 is a black hole horizon), we have the Null Penrose Inequality

$$\sqrt{\frac{|\Sigma_0|}{16\pi}} = m_R(\Sigma_0) \leq M_{TB}.$$

Furthermore, when equality holds for a strictly doubly convex foliation, we conclude that equality holds for all foliations of Ω. Moreover, any foliation of Ω agrees with some foliation of the standard null cone of the Schwarzschild spacetime with respect to the data $\{\gamma, \underline{\chi}^-, \langle\vec{H}, \vec{H}\rangle, \tau\}$.

For a proof of the Null Penrose Inequality, Theorem 2 assumes the existence of a doubly convex Σ_0 satisfying $0 = \frac{1}{4}\langle\vec{H}, \vec{H}\rangle \geq \Delta\log\rho$. Therefore, the Maximum Principle demands that $\mathcal{K} + \nabla\cdot\tau = \frac{4\pi}{|\Sigma_0|}$. We say Σ_0 is a *doubly convex MOTS* when satisfying $\langle\vec{H}, \vec{H}\rangle = 0$, $\rho_0 := \mathcal{K} + \nabla\cdot\tau = \frac{4\pi}{|\Sigma_0|}$.

In [12], we show the existence of a unique foliation by doubly convex MOTS for a class of *Weakly Isolated Horizons*, $\mathcal{H}$. Generalizing the notion of a Killing Horizon, these totally geodesic null hypersurfaces admit a null generator l whereby $D_l l = \kappa_l l$ with constant *surface gravity* κ_l.

Theorem 3. *If $\mathcal{H}$ is strictly stable, optically rigid (see [12]), and $\kappa_l > 0$, then it admits a unique foliation, $\{\Sigma_s\}$, along l satisfying $\rho_0 = \frac{4\pi}{|\Sigma|}$. Moreover, for a given Σ_s the linearization $\delta_{\psi L^- + \phi L^+}(\rho_0 - \frac{4\pi}{|\Sigma|}, \langle \vec{H}, \vec{H} \rangle)$ has bounded inverse on $\mathring{C}^{k,\alpha}(\Sigma) \times C^{l,\beta}(\Sigma)$.*

From a Lemma of S. Alexakis [1] we're able to use Theorem 3 to invoke the Implicit Function Theorem for a proof of the stability of the Null Penrose Inequality around the Schwarzschild spacetime:

Corollary 1. *Let g_λ be a smooth family of metrics satisfying the DEC off of the Schwarzschild metric g_0, then there exists $\epsilon > 0$ and a corresponding family of smooth doubly convex MOTS Σ_λ. If the past null cones $\Omega_\lambda \supset \Sigma_\lambda$ are smooth and g_λ is close to Schwarzschild (see [12]), then we have the Null Penrose Inequality.*

References

[1] S. Alexakis. The Penrose inequality on perturbations of the Schwarzschild exterior. *arXiv preprint arXiv:1506.06400*, 2015.

[2] R. Arnowitt, S. Deser, and C. W. Misner. Republication of: The dynamics of general relativity. *General Relativity and Gravitation*, 40(9):1997–2027, 2008.

[3] H. Bondi, M. Van der Burg, and A. Metzner. Gravitational waves in general relativity. vii. waves from axi-symmetric isolated systems. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences*, pages 21–52, 1962.

[4] H. L. Bray. Proof of the Riemannian Penrose inequality using the positive mass theorem. *Journal of Differential Geometry*, 59(2):177–267, 2001.

[5] H. L. Bray and D. A. Lee. On the Riemannian Penrose inequality in dimensions less than eight. *Duke Mathematical Journal*, 148(1):81–106, 2009.

[6] S. W. Hawking. Gravitational radiation in an expanding universe. *Journal of Mathematical Physics*, 9(4):598–604, 1968.

[7] G. Huisken and T. Ilmanen. The inverse mean curvature flow and the Riemannian Penrose inequality. *Journal of Differential Geometry*, 59(3):353–437, 2001.

[8] M. Mars and A. Soria. The asymptotic behaviour of the hawking energy along null asymptotically flat hypersurfaces. *Classical and Quantum Gravity*, 32(18):185020, 2015.

[9] M. Mars and A. Soria. On the Penrose inequality along null hypersurfaces. *Classical and Quantum Gravity*, 33(11):115019, 2016.

[10] R. Penrose. Gravitational collapse: The role of general relativity. Technical report, Birkbeck Coll., London, 1969.

[11] H. Roesch. Proof of a null Penrose conjecture using a new quasi-local mass. *arXiv:1609.02875 [gr-qc]*, 2016.

[12] H. Roesch. On the existence of doubly convex mots. *arXiv:1806.09675 [gr-qc]*, 2018.

[13] J. Sauter. *Foliations of null hypersurfaces and the Penrose inequality.* PhD thesis, ETH Zurich, 2008.

[14] A. Trautman. Radiation and boundary conditions in the theory of gravitation. *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.*, 6:407, 1958.

Stability Within T^2-Symmetric Expanding Spacetimes

Adam Layne
(joint work with Beverly K. Berger, James Isenberg)

We present a recently completed, nonpolarized analogue of the asymptotic characterization of T^2-symmetric Einstein flow solutions in [3]. We impose a far weaker condition, but obtain similar rates of decay for the normalized energy and associated quantities. Critical to this work have been novel numerical simulations which indicate that there is locally attractive behavior for those T^2-symmetric solutions not subject to this weakened condition. This local attractor is distinct from the local attractor in our main theorem, thereby indicating that the polarized asymptotics are on one hand stable within a larger class than merely polarised solutions, but unstable within all T^2-symmetric solutions.

Let us recall that the T^2-symmetric vacuum Einstein flows have Cauchy surfaces T^3 and two spacelike Killing vector fields. Work of [1] provides us with a global foliation

$$\begin{aligned} g =& e^{l-V+4\tau}\left(-d\tau^2 + e^{2(\rho-\tau)}d\theta^2\right) \\ &+ e^V\left[dx + Qdy + (G+QH)d\theta\right]^2 + e^{-V+2\tau}\left[dy + Hd\theta\right]^2 \end{aligned}$$

where ∂_x, ∂_y are Killing. These contain all Kasner solutions and all Gowdy solutions as subclasses, the future asymptotics of which are known.

In these coordinates, V and Q satisfy quasilinear wave equations coupled to the evolution equations for l, ρ:

$$\begin{aligned} l_\tau + \rho_\tau + 2 =& \frac{1}{2}\left[V_\tau^2 + e^{2(\tau-\rho)}V_\theta^2 + e^{2(V-\tau)}\left(Q_\tau^2 + e^{2(\tau-\rho)}Q_\theta^2\right)\right] \qquad (1) \\ \rho_\tau =& e^l. \end{aligned}$$

The structure of these two equations, and the fact that the right side of (1) has the form of an energy density for the V, Q variables, suggests that one should try to look for attractors of the flow by linearizing the ODE satisfied by

$$\int_{S^1} e^\rho \, d\theta, \quad \int_{S^1} e^{l+\rho+2\tau}\, d\theta, \quad \int_{S^1} l_\tau + \rho_\tau + 2\, d\theta.$$

This is the technique used in [3] under the assumption that the solutions are *polarised*. A T^2-symmetric Einstein flow is called *polarised* if, in the coordinates used here, $Q \equiv 0$.

Let us note that, in our coordinates, polarised Kasner solutions take the form

$$V = a\tau + b, \quad l = \left[\frac{1}{2}a^2 - 2\right]\tau + c$$

for some constants $a, b, c \in \mathbb{R}$. Non-Gowdy solutions such that l, V, Q are independent of θ are called *pseudo-homogeneous* or *PH* [4]. Due to [4], the future asymptotics of polarised PH solutions are known to be of the form

$$|V-(a\tau+b)|\to 0,\quad \left|l-\left(\left[\frac{1}{2}a^2-2\right]\tau+c\right)\right|\to 0,\quad a\in(-2,2).$$

That is, PH solutions have asymptotics of the same form as a Kasner solution, but the value of V_τ at $\tau=\infty$ is not entirely free.

In [3] it is shown that the following behavior is locally stable for polarised, non-Gowdy, non-PH T^2-symmetric solutions:

$$|V_\tau|\to 0,\quad |l_\tau|\to 0,\quad \tau\to\infty. \tag{2}$$

Note that this is distinct from the PH and Kasner cases. It was previously conjectured by Berger on the basis of numerical simulations that this behavior should be generic; all T^2-symmetric solutions (excepting sets of positive codimension) should have expanding direction asymptotics of the form (2).

We have in fact found that this behavior is not generic. There is a condition which is weaker than polarisation, but which still ensures asymptotics of the form (2). As shown in [4], the system has two conserved quantities:

$$A:=\int_{S^1}e^{\rho}\left(V_\tau-e^{2(V-\tau)}Q_\tau Q\right)d\theta$$

$$B:=\int_{S^1}e^{\rho+2(V-\tau)}Q_\tau\,d\theta.$$

Definition 1. *Let B_0 be the class of non-Gowdy, non-pseudo-homogeneous solutions for which $B=0$.*

Note that all polarised solutions are B_0 but the reverse is not true. We show, by an argument which is essentially a refinement of that used in [3], that the results of that paper extend to B_0 solutions.

We also present more sophisticated numerical simulations which provide evidence that the B_0 assumption is necessary. If one does not impose this condition, solutions seem to again have convergent first order behavior, but it is distinct from (2). In the general case it appears that $V_\tau\to\frac{1}{2}$. For more detailed asymptotics, see [2].

We thus have three conjectures supported by the numerical simulations done as part of this project.

Conjecture 2. *The local attractor in* [3] *and generalized in* [2] *is in fact a global attractor for B_0 solutions.*

There already exists in [4] a partial result in this direction.

Conjecture 3. *There is a local attractor for non-B_0 solutions satisfying*

$$\left|V_\tau-\frac{1}{2}\right|\to 0,\quad Q_\tau=O(e^{-3\tau/2}).$$

Conjecture 4. *This non-B_0 attractor is in fact a global attractor for T^2-symmetric solutions.*

References

[1] Beverly K. Berger, Piotr Chruściel, James Isenberg, and Vincent Moncrief, *Global foliations of vacuum spacetimes with T^2 isometry*, Ann. Physics, **260(1)** (1997), 117–148.

[2] Beverly K. Berger, James Isenberg, and Adam Layne, *Stability Within T^2-Symmetric Expanding Spacetimes*, forthcoming.

[3] Philippe LeFloch and Jacques Smulevici, *Future asymptotics and geodesic completeness of polarized T^2-symmetric spacetimes*, Anal. PDE, **9(2)** (2016), 363–395.

[4] Hans Ringström, *Instability of Spatially Homogeneous Solutions in the Class of T^2-Symmetric Solutions to Einstein's Vacuum Equations*, Comm. Math. Phys., **334(3)** (2015), 1299–1375.

On estimates for Bartnik's quasi-local mass for CMC Bartnik data

Armando J. Cabrera Pacheco

(joint work with Carla Cederbaum, Stephen McCormick and Pengzi Miao)

The Bartnik mass [2] is one notion of quasi-local mass, that is, a measurement of how much energy is contained in a bounded region of an initial data set. Given Bartnik data $(\Sigma \cong \mathbb{S}^2, g, H)$, where g is a metric on the surface Σ and H is a smooth function on Σ, we consider a set of suitable Riemannian 3-manifolds, called admissible extensions, and define the Bartnik mass, $\mathfrak{m}_B$, as

(1) $\mathfrak{m}_B(\Sigma, g, H) := \inf\{m_{\mathrm{ADM}}(M,\gamma) \,|\, (M,\gamma) \text{ admissible extension of } (\Sigma, g, H)\}.$

It is well-known that under certain conditions on the set of admissible extensions, the Hawking mass, $\mathfrak{m}_H$, provides a lower bound for the Bartnik mass via Huisken and Ilmanen's proof of the Riemannian Penrose inequality [6].

In [7], Mantoulidis and Schoen computed the Bartnik mass for minimal Bartnik data $(\Sigma \cong \mathbb{S}^2, g, H = 0)$. More precisely, they obtained the following theorem.

Theorem 1 (Mantoulidis and Schoen [7])**.** *If $(\Sigma \cong \mathbb{S}^2, g_o, H_o = 0)$ is a Riemannian manifold satisfying $\lambda_1(-\Delta_{g_o} + K(g_o)) > 0$, that is, the first eigenvalue of the operator $-\Delta_{g_o} + K(g_o)$, where $K(g_o)$ denotes the Gaussian curvature, is positive, then*

$$\mathfrak{m}_B(\Sigma, g_o, H_o = 0) = \mathfrak{m}_H(\Sigma, g_o, H_o = 0). \tag{2}$$

The proof consists in explicitly constructing admissible extensions for the minimal Bartnik data $(\Sigma \cong \mathbb{S}^2, g_o, H_o = 0)$, in such a way that the ADM mass of the extension is controlled and can be made arbitrarily close to the optimal value in in the Riemannian Penrose inequality, $\mathfrak{m}_H(\Sigma, g_o, H_o = 0)$. Their method can be summarized as the following two-step process.

(i) Construct a collar extension of (Σ, g_o), that is, a manifold diffeomorphic to $[0,1] \times \Sigma$ endowed with a metric such that the boundary component $\{t = 0\}$ is isometric to (Σ, g_o), minimal and outer-minimizing, and the boundary component $\{t = 1\}$ is round. Here, an essential step is to use the uniformization theorem to connect g_o with the round metric.

(ii) Smoothly glue this collar manifold to a Schwarzschild manifold of any mass $m > \mathfrak{m}_H(\Sigma, g_o, H_o = 0)$. This requires to deform the Schwarzschild manifold so it has positive scalar curvature in a small region close to the horizon.

In [5], in a joint work with Miao, the construction of Mantoulidis and Schoen was adapted to higher dimensions making use of various geometric flows in higher dimensions.

In this talk, we discuss a natural question motivated by the definition of Bartnik mass, can the Mantoulidis–Schoen construction be carried out for Bartnik data $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$ with $H_o \neq 0$? In [4], in a joint work with Cederbaum, McCormick and Miao, we prove the following theorem, which provides an upper bound for the Bartnik mass of CMC Bartnik data, that is $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$, where H_o is a positive constant.

Theorem 2 (Cabrera Pacheco, Cederbaum, McCormick and Miao [4])**.** *Given Bartnik CMC data $(\Sigma \cong \mathbb{S}^2, g_o, H_o)$, with $K(g_o) > 0$, there exist constants $0 \leq \alpha$ and $0 < \beta \leq 1$, such that if*

$$\frac{H_o^2 r_o^2}{4} < \frac{\beta}{1+\alpha},$$

where r_o denotes the area-radius of g_o, $r_o = \sqrt{\frac{|\Sigma|_{g_o}}{4\pi}}$, we have

$$\mathfrak{m}_B(\Sigma, g_o, H_o) \leq \left[1 + \left(\frac{\alpha \frac{H_o^2 r_o^2}{4}}{\beta - (1+\alpha)\frac{H_o^2 r_o^2}{4}}\right)^{\frac{1}{2}}\right] \mathfrak{m}_H(\Sigma, g_o, H_o). \tag{3}$$

The constants α and β can be regarded as a measurement of how round g_o is. The smallness condition on H_o was first used by Miao and Xie in [8]. Notice that as $H_o \to 0$, we recover Theorem 1. The proof of this theorem is obtained through a careful modification of the method in [7], making use of a collar construction in [8] to complete step (i) above.

Moreover, the construction in [7] has been further extended to the asymptotically hyperbolic setting (negative cosmological constant) in a joint work with Cederbaum and McCormick [3]. Currently, with Cederbaum and Alaee, we are carrying out the corresponding construction in the context of initial data sets for the Einstein–Maxwell equations [1].

References

[1] A. Alaee, A. J. Cabrera Pacheco, and C. Cederbaum, *Work in progress*, 2018.

[2] R. Bartnik, *A new definition of quasi-local mass*, Phys. Rev. Lett. **62** (1989), 2346.

[3] A. J. Cabrera Pacheco, C. Cederbaum, and S. McCormick, *Asymptotically hyperbolic extensions and an analogue of the Bartnik mass*, J. Geom. Phys. **132** (2018), 338-357.

[4] A. J. Cabrera Pacheco, C. Cederbaum, S. McCormick, and P. Miao, *Asymptotically flat extensions of CMC Bartnik data*, Class. Quantum Grav. **34** (2017), no. 10, 105001.

[5] A. J. Cabrera Pacheco, and P. Miao, *Higher dimensional black hole initial data with prescribed boundary metric* Math. Res. Lett., **25** (2018), no. 3, 937-956.

[6] G. Huisken, and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437.
[7] C. Mantoulidis, and R. Schoen, *On the Bartnik mass of apparent horizons*, Class. Quantum Grav. **32** (2016), no. 20, 205002, 16pp.
[8] P. Miao, and N.-Q. Xie, *On compact 3-manifolds with nonnegative scalar curvature with a CMC boundary component*, Trans. Amer. Math. Soc. **370** (2018), 5887–5906.

Structures in Gravitational Wave Memory and its EM Analog

Lydia Bieri

Gravitational waves are a common feature of interesting spacetimes in General Relativity (GR). It has been predicted that these waves change the spacetime permanently, which would show in a change of the arrangement of test masses after the wave train passed. This is the so-called memory effect. Gravitational waves were measured for the first time in 2015 by LIGO and several times since then. It is believed that the memory effect will be detected in the near future. The gravitational memory effect was found in a linear theory in 1974 by Ya. Zel'dovich and B. Polnarev [6]. Then in 1991 D. Christodoulou [4] derived within the full nonlinear theory such a memory that was much larger than expected. Together with D. Garfinkle we showed [3] that these are two different effects sourced by different events. In this talk, I will explore the memory structures as a result of different types of initial data evolving under the Einstein equations. Based on work by D. Christodoulou [4], D. Christodoulou and S. Klainerman [5] and of mine [1] I present new results [2] proving that memory produced by gravitational waves and "regular" matter-energy is of electric parity only, no magnetic parity memory can occur. Further, I discuss some structures at null infinity for spacetimes that decay very slowly to Minkowski spacetime as well as their implications for memory. Many discussions of memory followed the pioneering works. We refer to our latest papers for a detailed discussion of the history and detailed references.

References

[1] L. Bieri. *Extensions of the Stability Theorem of the Minkowski Space in General Relativity. Solutions of the Einstein Vacuum Equations.* AMS-IP. Studies in Advanced Mathematics. Cambridge. MA. (2009). Based on Ph.D. thesis of 2007.
[2] L. Bieri. *Gravitational Radiation and Memory for Asymptotically-Flat Systems.* Preprint.
[3] L. Bieri, D. Garfinkle. *Perturbative and gauge invariant treatment of gravitational wave memory.* Phys. Rev. D. 89. 084039. (2014).
[4] D. Christodoulou. *Nonlinear Nature of Gravitation and Gravitational-Wave Experiments.* Phys.Rev.Letters. **67**. (1991). no.12. 1486-1489.
[5] D. Christodoulou, S. Klainerman. *The global nonlinear stability of the Minkowski space.* Princeton Math.Series **41**. Princeton University Press. Princeton. NJ. (1993).
[6] Ya.B. Zel'dovich. A.G. Polnarev. Astron.Zh. **51**. (1974). 30. [Sov.Astron. **18**. (1974). 17.]

Reporter: Christoph Kehle

Participants

Dr. Paul Tyler Allen
Department of Mathematical Sciences
Lewis and Clark College
MSC 110
0615 S.W. Palatine Hill Road
Portland OR 97219
UNITED STATES

Dr. Hakan Andréasson
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Dr. Florian Beyer
Department of Mathematics and Statistics
University of Otago
Science III, Room 218
P.O. Box 56
Dunedin 9054
NEW ZEALAND

Prof. Dr. Lydia Bieri
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Prof. Dr. Piotr Bizon
Institute of Physics
Jagiellonian University
ul. Reymonta 4
30-059 Kraków
POLAND

Dr. Bernhard Brehm
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Dr. Armando Jose Cabrera Pacheco
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Carla Cederbaum
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Ye-Sle Cha
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Piotr T. Chrusciel
Gravitationsphysik
Universität Wien
Boltzmanngasse 5
1090 Wien
AUSTRIA

Prof. Dr. Justin Corvino
Department of Mathematics
Lafayette College
Easton, PA 18042-1781
UNITED STATES

Prof. Dr. Mihalis Dafermos
Department of Pure Mathematics and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Pau Figueras Barnera
School of Mathematical Sciences
Queen Mary
University of London
Mile End Road
London E1 4NS
UNITED KINGDOM

Prof. Dr. Helmut Friedrich
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm
GERMANY

Dr. Maria Eugenia Gabach-Clement
FAMAF
Universidad Nacional de Cordoba
Instituto de Fisica Enrique Gaviola,
IFEG, CONICET
Ciudad Universitaria
5000 Córdoba
ARGENTINA

Prof. Dr. Gregory Galloway
Department of Mathematics
and Computer Science
University of Miami
P.O. Box 248011
Coral Gables, FL 33124
UNITED STATES

Dr. Romain Gicquaud
Laboratoire de Mathématiques
Faculté des Sciences
Université de Tours
Parc de Grandmont
37200 Tours Cedex
FRANCE

Dr. Melanie Graf
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Dr. Stephen Green
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Potsdam
GERMANY

Dr. Mahir Hadzic
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Dr. Peter Hintz
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Michael Holst
Department of Mathematics
University of California, San Diego
9500 Gilman Drive
La Jolla, CA 92093-0112
UNITED STATES

Dr. Gustav Holzegel
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Gerhard Huisken
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Cécile Huneau
Centre de Mathématiques Appliquées
CML.S
École Polytechnique
Route de Saclay
91128 Palaiseau Cedex
FRANCE

Sophia Jahns
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Jeffrey Jauregui
Department of Mathematics
Union College
Bailey Hall
Schenectady, NY 12308
UNITED STATES

Christoph Kehle
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Adam Layne
Department of Mathematics
KTH Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Prof. Dr. Philippe G. LeFloch
Laboratoire Jacques-Louis Lions
Sorbonne Université
4, Place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Jonathan Luk
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Marc Mars
Facultad de Ciencias
Universidad de Salamanca
Plaza de la Merced s/n
37008 Salamanca
SPAIN

Prof. Dr. David Maxwell
Department of Mathematical Sciences
University of Alaska Fairbanks
PO Box 756660
Fairbanks, AK 99775-6660
UNITED STATES

Dr. Stephen McCormick
Matematiska Institutionen
Uppsala Universitet
Box 480
751 06 Uppsala
SWEDEN

Prof. Dr. Pengzi Miao
Department of Mathematics
University of Miami
1365 Memorial Drive
Coral Gables, FL 33146
UNITED STATES

Georgios Moschidis
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Sung-Jin Oh
School of Mathematics
Korea Institute for Advanced Study
85 Hoegi-Ro, Dongdaemun-gu
Seoul 02 455
KOREA, REPUBLIC OF

Dr. Todd Oliynyk
School of Mathematical Sciences
Monash University
Clayton, Victoria 3800
AUSTRALIA

Tim Torben Paetz
Gravitationsphysik
Universität Wien
Boltzmanngasse 5
1090 Wien
AUSTRIA

Dr. Martin Reiris-Ithurralde
Centro de Matemática
Universidad de la República
Int. 114, Oficina 3, Piso 15
CP 11400
Montevideo
URUGUAY

Prof. Dr. Hans Ringström
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Prof. Dr. Oliver Rinne
Hochschule für Technik und Wirtschaft
University of Applied Sciences
Fachbereich 4
Treskowallee 8
10318 Berlin
GERMANY

Dr. Henri Roesch
Department of Mathematics
University of California, Irvine
Irvine, CA 92697-3875
UNITED STATES

Dr. Anna Sakovich
Matematiska Institutionen
Uppsala Universitet
Box 480
751 06 Uppsala
SWEDEN

Dr. Jan J. Sbierski
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Volker Schlue
Laboratoire Jacques-Louis Lions
Université Pierre et Marie Curie
4, Place Jussieu
75005 Paris Cedex
FRANCE

Prof. Dr. Richard Schoen
Department of Mathematics
University of California, Irvine
Irvine, CA 92697-3875
UNITED STATES

Dr. Yakov Shlapentokh-Rothman
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Jacques Smulevici
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Christina Sormani
Department of Mathematics
CUNY Graduate Center
365 Fifth Avenue
New York, NY 10016
UNITED STATES

Dr. Martin Taylor
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Andras Vasy
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Prof. Dr. Robert Wald
Enrico Fermi Institute and
Department of Physics
University of Chicago
5720 S. Ellis Avenue
Chicago, IL 60637
UNITED STATES

Dr. Qian Wang
Mathematical Institute
Oxford University
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM