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Scaling Limits in Models of Statistical Mechanics

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ABSTRACT. This conference (part of a long running series) aims to cover the interplay between probability and mathematical statistical mechanics. Specific topics addressed during the 22 talks include: Universality and critical phenomena, disordered models, Gaussian free field (GFF), random planar graphs and unimodular planar maps, reinforced random walks and non-linear σ -models, non-equilibrium dynamics. Less stress is given to topics which have running series of Oberwolfach conferences devoted to them specifically, such as random matrices or integrable models and KPZ universality class.

There were 50 participants, including 9 postdocs and graduate students, working in diverse intertwining areas of probability, statistical mechanics and mathematical physics.

Subject classification: MSC: 60,82; IMU: 10,13.

Introduction by the Organisers

This workshop was a sequel to a MFO conference, by the same organizers, which took place in 2015. More broadly, it is a sequel to MFO conferences in 2006, 2009 and 2012, organised by Ken Alexander, Marek Biskup, Remco van der Hofstad and Vladas Sidoravicius. The main focus of the conference remained on probabilistic and analytic methods of non-integrable statistical mechanics. With respect to the previous editions, greater emphasis was put on statistical mechanics models on groups and general graphs, as a lot has happened in this arena recently. The list of 50 participants reflects our attempts to maintain an optimal balance between diverse fields, leading experts and promising young researchers. Nine participants were on postdoctoral and graduate level.

In our choice of 22 talks we tried to illuminate major recent advances in the field and to expose and address at least some aspects of the works for each and every one of the participants. A more detailed account of the presentations is given below. Due to an intended intertwining of topics and themes it is hard to give an unambiguous classification.

Statistical mechanical models on groups and general graphs. Tom Hutchcroft described an improved proof of the Aizenman-Kesten-Newman arm exponent estimate, and applications to percolation on hyperbolic groups. Aran Raoufi described joint work with Duminil-Copin, Goswami, Severo and Yadin in which they showed that percolation on every graph with isoperimetric dimension at least 4 has a non-trivial phase transition. Christoph Garban described work on the inverted orbit of random walk on interval exchange transformation groups, related to the Thompson group.

Two dimensional models. The understanding of two dimensional models proceeds rapidly and we heard 5 talks on the topic.

Vincent Tassion discussed a renormalisation scheme which allows to show a quadrachotomy for the two dimensional Potts model. Béatrice de Tilière talked about massive Laplacians on isoradial graphs. Giambattista Giacomin revisited a classic paper of McCoy and Wu about analyticity of the pressure of the Ising model with columnar quenched disorder. Two more talks were on Gaussian multiplicative chaos, a model representing the scaling limit of planar random graphs: Hubert Lacoin talked about its fluctuations and Jason Miller investigated random walks and Brownian motion on it.

Statistical mechanics models with quenched disorder. We heard 3 talks on Processes in random environment: Marek Biskup analyzed degenerate dynamical random environment motivated by the Helffer-Sjöstrand representation. Bálint Tóth talked about the Lorentz gas with random obstacles, on time-scales beyond the Boltzmann-Grad limit. Quentin Berger studied directed polymers in heavy-tailed environment.

Two talks were devoted to spin glasses: David Belius discussed the TAP-Plefka equations for the spherical Sherrington-Kirkpatrick model, and Aukosh Jagannath discussed the Langevin dynamics and long equilibration time for mean-field generalised spin glasses.

Other models inspired by statistical physics. Finally, a few topics were covered by a single talk:

- Silke Rolles strengthened the connection between the vertex-reinforced jump process and the supersymmetric hyperbolic sigma model.
- Roland Bauerschmidt talked about renormalisation for hierarchical spin models and implications on the dynamical spectral gap.
- Sébastien Ott applied Ornstein-Zernike theory to study the Potts model with a defect line.

- Nicholas Crawford discussed the eigenvectors of a non-Hermitian random matrix model.
- Wendelin Werner talked about loop-soups and critical percolation in dimensions 7 and above.
- Lisa Hartung talked about the extremal set of branching Brownian motion.
- Perla Sousi talked about the capacity of the range of simple random walk in different space dimensions.
- Wioletta Ruzel investigated the scaling limit of the odometer function in sandpile models.
- Frank den Hollander talked about a population model with seed-bank and spatial structure.

Summary. The workshop was an obvious success. In particular, it helped to update the participants on the state of the art and on the important pending open problems in the fields related to their domain of research, facilitated exchange of ideas between researchers in technically disconnected areas, and it gave rise to many interesting and informative discussions. In particular, we had a lively evening session focused mostly on open problems. Some new collaborations arose, notably Hutchcroft and Pete solved a long-standing problem on the cost of Kazhdan groups (this was announced a few weeks after the conference ended).

We would like to thank the MFO personnel for the help and for the invaluable logistic support, as well as for creating a friendly and stimulating environment throughout the entire meeting.

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Workshop: Scaling Limits in Models of Statistical Mechanics**Table of Contents**

Tom Hutchcroft	
<i>The Aizenman-Kesten-Newman method revisited</i>	2541
Aran Raoufi (joint with H. Duminil-Copin, S. Goswami, F. Severo, A. Yadin)	
<i>Percolation phase transition is nontrivial for graphs with isoperimetric dimension higher than 4</i>	2543
Vincent Tassion (joint with Hugo Duminil-Copin)	
<i>Renormalization of crossing probabilities in the planar random-cluster model</i>	2545
Hubert Lacoin (joint with Rémi Rhodes, Vincent Vargas)	
<i>Large and small deviation for Gaussian multiplicative chaos, and paths integrals associated with the Mabuchi action</i>	2547
Marek Biskup (joint with Pierre-François Rodriguez)	
<i>Random walks in degenerate dynamical random environments</i>	2548
Roland Bauerschmidt (joint with Thierry Bodineau)	
<i>Spectral gap critical exponent for hierarchical spin models</i>	2551
Lisa Hartung (joint with Aser Cortines, Oren Louidor)	
<i>Quantitative estimates on extreme level sets in branching Brownian motion</i>	2553
Sébastien Ott (joint with Yvan Velenik)	
<i>Decoupling a Process with Memory, Application to FK Percolation and Potts Model with a Defect Line</i>	2555
Aukosh Jagannath	
<i>Recent progress on dynamics of mean field spin glasses</i>	2557
Quentin Berger (joint with Niccolò Torri)	
<i>Scaling limits for directed polymers in heavy-tail random environment</i> ..	2558
Wioletta M. Ruszel (joint with Leandro Chiarini, Alessandra Cipriani, Milton Jara and Rajat Hazra)	
<i>Scaling limits for the odometer of divisible sandpile models</i>	2562
Bálint Tóth (joint with Christopher Lutsko)	
<i>Invariance Principle for the Random Lorentz Gas – Beyond the Boltzmann-Grad Limit</i>	2563
Silke Rolles (joint with Franz Merkl and Pierre Tarrès)	
<i>Convergence of vertex-reinforced jump processes to an extension of the supersymmetric hyperbolic nonlinear sigma model</i>	2567

Christophe Garban	
<i>Inverted orbits of exclusion processes, diffuse-extensive-amenability and (non-?) amenability of the interval exchanges</i>	2569
Frank den Hollander (joint with Andreas Greven and Margriet Oomen)	
<i>Spatial populations with seed-bank</i>	2571
Béatrice de Tilière	
<i>The Z-Dirac and massive Laplacian operators in the Z-invariant Ising model</i>	2571
Wendelin Werner	
<i>On the GFF in dimension $d \geq 7$, loop-soups and critical percolation</i>	2572
Giambattista Giacomin (joint with Francis Comets and Rafael L. Greenblatt)	
<i>On the two dimensional Ising model with columnar disorder and the continuum limit of random 2 by 2 matrix products</i>	2573

Abstracts

The Aizenman-Kesten-Newman method revisited

TOM HUTCHCROFT

We review, build upon, and find new applications of the method of Aizenman, Kesten, and Newman [2], which was originally used to prove that Bernoulli percolation on \mathbb{Z}^d has at most one infinite cluster almost surely. Their proof also yields the following quantitative estimate for percolation on \mathbb{Z}^d . Let e be an edge of \mathbb{Z}^d and let \mathcal{A}_n be the event that e is closed and that the two endpoints of e are in distinct clusters each of which has diameter at least n . Then for every $p \in (0, 1)$ there exists a constant $C_{d,p}$ such that

$$(1) \quad \mathbf{P}_p(\mathcal{A}_n) \leq C_{d,p} \frac{\log n}{\sqrt{n}}.$$

See [5] for a discussion of and improvement to this bound. Although it is possible to adapt the Aizenman-Kesten-Newman argument to prove uniqueness on any amenable transitive graph, the bound one obtains on $\mathbf{P}_p(\mathcal{A}_n)$ becomes increasingly poor as the isoperimetry of the graph improves, and we do not obtain *any* information about percolation on *nonamenable* graphs.

Roughly speaking, their proof uses an ingenious summation-exchange argument to rewrite the probabilities of a certain two-arm event \mathcal{A} in terms of an expectation roughly of the form $\mathbb{E}[T^{-1}Z_T\mathbb{1}(T < \infty)\mathbb{1}(\mathcal{B})]$, where $(Z_n)_{n \geq 0}$ is a martingale with bounded, i.i.d. increments, T is a stopping time, and \mathcal{B} is a certain one-arm event for the percolation configuration. On the event \mathcal{B} the stopping time T must be large, and one can therefore easily bound this expectation using e.g. Doob's L^2 maximal inequality to obtain that the probability of the two-arm event is small as desired. (In fact they used large deviation techniques instead of maximal inequalities, which leads to the unnecessary logarithmic term in the above inequality.)

We show that it is in fact possible to derive a version of the Aizenman-Kesten-Newman bound that holds *universally* for all unimodular transitive graphs, and which yields improved estimates even in the case of \mathbb{Z}^d . Aside from this universality, the most significant differences between our inequality and (1) are as follows: Firstly, we work with two-arm events involving only *finite* clusters, so that in particular our inequality does not directly imply uniqueness of the infinite cluster as (1) does (although it is possible to use the same techniques to prove uniqueness in the amenable case with a little more work). Such a modification is of course necessary in order to obtain an inequality that is valid in the nonamenable setting. Secondly, rather than studying the *diameter* of clusters, we study their *volume*. This is done somewhat indirectly by introducing a *ghost field* as in [1]. This modification allows us to work directly in infinite volume rather than working in finite volume and taking limits as in [2], and also leads to stronger results since the

volume is an upper bound on the diameter. In particular, we apply the mass-transport principle to carry out the summation-exchange argument of [2] directly in infinite volume. This is where the assumption of unimodularity is required.

For each set $K \subseteq V$, we write $E(K)$ for the set of edges of G that **touch** K , i.e., have at least one endpoint in K . For each edge e of G and $n \geq 1$, let $\mathcal{S}_{e,n}$ be the event that e is closed and that the endpoints of e are in distinct, finite clusters each of which touches at least n edges.

Theorem ([7]). *Let $G = (V, E)$ be a unimodular transitive graph of degree d . Then*

$$\mathbf{P}_p(\mathcal{S}_{e,n}) \leq 82d \left[\frac{1-p}{pn} \right]^{1/2}$$

for every $e \in E$, $p \in [0, 1]$ and $n \geq 1$.

Several variations on this inequality can also be obtained. For example, if critical percolation on a unimodular transitive graph $G = (V, E)$ satisfies an inequality of the form $\mathbf{P}_{p_c}(|K_v| \geq n) \leq O(n^{-\gamma})$ for some $\gamma < 1/2$ then we obtain that

$$\mathbf{P}_p(\mathcal{S}_{e,n}) \leq O(n^{-1/2-\gamma}).$$

This inequality is of particular interest for percolation on \mathbf{Z}^d with $2 \leq d \leq 5$.

Theorem has several applications to percolation on infinite-dimensional transitive graphs. In particular, in [7] we apply to prove the following theorem.

Theorem ([7]). *For every $g > 1$ and $M < \infty$ there exist constants $C = C(g, M)$ and $\delta = \delta(g, M)$ such that for every transitive unimodular graph G with $\deg(o) \leq M$ and $\text{gr}(G) \geq g$, the bound*

$$(2) \quad \mathbf{P}_p(|E(K_o)| \geq n) \leq Cn^{-\delta}$$

holds for every $p \leq p_c$ and $n \geq 1$.

Theorem can be thought of as a quantitative form of the results of [3, 6], which established that critical percolation on any transitive graphs of exponential growth does not have any infinite clusters a.s., but did not establish any estimates on the tail of the volume of the cluster of the origin.

A corollary of Theorem is that *Schramm's locality conjecture* is true for graph sequences of uniform exponential growth. See [4] for background on this conjecture.

Theorem ([7]). *Let $(G_n)_{n \geq 1}$ be a sequence of transitive graphs converging locally to a transitive graph G , and suppose that $\liminf_{n \rightarrow \infty} \text{gr}(G_n) > 1$. Then $p_c(G_n) \rightarrow p_c(G)$ as $n \rightarrow \infty$.*

Theorem is also used in the proof of the following theorem, which was obtained in joint work with Jonathan Hermon.

Theorem ([8]). *Let $G = (V, E)$ be a connected, locally finite, transitive graph satisfying a heat kernel estimate of the form $p_n(v, v) \leq \exp[-\Omega(n^\gamma)]$ for some $\gamma > 1/2$. Then critical percolation on G has no infinite clusters almost surely.*

Theorem applies in particular to certain Cayley graphs of groups of *intermediate growth*, i.e., for which the volume of a ball $|B(v, r)|$ grows faster than any power of r but slower than any exponential of r . No such graph had previously been proven to have to infinite clusters in critical percolation.

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Percolation phase transition is nontrivial for graphs with isoperimetric dimension higher than 4

ARAN RAOUFI

(joint work with H. Duminil-Copin, S. Goswami, F. Severo, A. Yadin)

Let $G = (V, E)$ be a bounded-degree infinite graph. Percolation with parameter $p \in [0, 1]$ on G is the probability measure \mathbb{P}_p on subgraphs of G resulting from keeping each edge independently with probability p . Define

$$p_c(G) := \inf\{p \in [0, 1] : \mathbb{P}_p[\text{there exists an infinite component}] > 0\}.$$

The first step to study percolation on G is to prove the nontriviality of the phase transition, that is $p_c(G) < 1$. The fact that $p_c(\mathbb{Z}^d) < 1$ for $d \geq 2$ is among the first theorems regarding percolation theory. Benjamini and Schramm systematically started the line of research of studying percolation theory beyond \mathbb{Z}^d in their influential paper [1]. Among their first conjectures in that paper is that a transitive graph of super-linear growth have $p_c(G) < 1$. We say that a graph G has *super-linear growth* if $\limsup \frac{1}{r} |B_r(x)| = +\infty$, where $B_r(x)$ be the ball of radius r centered at x with respect to the graph distance. The following is proved in [2].

Theorem. *Let G be a quasi-transitive with super-linear growth, then $p_c(G) < 1$.*

In particular, it implies that if G is a Cayley graph of a finitely generated group without a finite index cyclic subgroup, then $p_c(G) < 1$.

Another conjecture of Benjamini and Schramm connects isoperimetric dimension of G to $p_c(G) < 1$. Define the isoperimetric dimension of G

$$(1) \quad \dim(G) := \sup \left\{ d > 0 : \inf_{S \subset V, |S| < \infty} \frac{|\partial S|}{|S|^{(d-1)/d}} > 0 \right\},$$

where ∂S is the external boundary of S . Benjamini and Schramm conjectured that if $\dim(G) > 1$, then $p_c(G) < 1$. The following theorem is proved in [2].

Theorem. *If $\dim(G) > 4$, then $p_c(G) < 1$.*

Both of the above theorems are a corollary to the following theorem proved in [2].

Theorem. *Consider a graph G with bounded degree. Assume that there exist real numbers $d > 4$ and $c > 0$ such that*

$$(H_d) \quad p_n(x, x) \leq \frac{c}{n^{d/2}} \quad \forall x \in V, \forall n \geq 1.$$

Then, there exists $p < 1$ such that for every finite set $S \subset V$,

$$(2) \quad \mathbb{P}_p(S \longleftrightarrow \infty) \geq 1 - \exp[-\frac{1}{2}\text{cap}(S)],$$

where $\text{cap}(S) := \sum_{x \in S} d(x) \mathbb{P}[X_k \notin S \forall k \geq 1 | X_0 = x]$ is the capacity of S . In particular, $p_c(G) < 1$.

The proof of Theorem starts with the existence of an infinite cluster for percolation in certain in-homogeneous random environment governed by the Gaussian free field. The Gaussian free field is the Gaussian process on the graph with mean 0 and $\text{cov}(x, y) = g(x, y)$, where g is the Green function of the simple random walk on G . Then, by proving using differential inequalities, we relate the existence of an infinite cluster in percolation in the random environment to that of percolation with a parameter $p < 1$. To prove the desired differential inequalities, we utilize a multi-scale decomposition of the GFF.

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Renormalization of crossing probabilities in the planar random-cluster model

VINCENT TASSION

(joint work with Hugo Duminil-Copin)

The study of crossing probabilities – i.e. probabilities of existence of paths crossing rectangles – has been at the heart of the theory of two-dimensional percolation since its beginning. In this talk, we present a renormalization scheme for crossing probabilities in the two-dimensional random-cluster model. The outcome of the process is a precise description of an alternative between four behaviors:

- *Subcritical*: Crossing probabilities, even with favorable boundary conditions (b.c.), converge exponentially fast to 0.
- *Supercritical*: Crossing probabilities, even with unfavorable b.c., converge exponentially fast to 1.
- *Critical discontinuous*: Crossing probabilities converge to 0 exponentially fast with unfavorable b.c. and to 1 with favorable b.c..
- *Critical continuous*: Crossing probabilities remain bounded away from 0 and 1 uniformly in the b.c..

The novelty of the approach is that the proof does not rely on self-duality, enabling it to apply in a much larger generality, including the random-cluster model on arbitrary graphs with sufficient symmetry, but also other models. Last but not least, crossing probabilities can be used to prove a number of results on the model, including speed of mixing, tails of decay of the connectivity probabilities, scaling relations, etc.

This study relies on two important (almost independent) pillars:

- The first one, called the *RSW theory*, states that lower and upper bounds on crossing probabilities of rectangles of a certain aspect ratio imply similar bounds for crossing probabilities for rectangles of other aspect ratios. The first result in this direction goes back to the seminal works of Russo [Rus78] and Seymour and Welsh [SW78] (several alternative proofs of the theorem have been obtained for Bernoulli percolation [BR06a, BR06b, BR10, Tas16]). The interest of this theorem is that it enables us to transfer lower bounds (which are usually easy to obtain) for crossing probabilities of very wide rectangles to lower bounds for crossing probabilities of very thin rectangles. By duality, it also enables one to transform upper bounds in thin rectangles into upper bounds in wide rectangles. This tool therefore simplifies greatly the study of crossing probabilities, since one can choose the aspect ratio of the rectangles under consideration freely, and therefore adapt this choice to the problem at hand.
- The second pillar is quite different in nature and has become an important facet of the theory only recently. While, in Bernoulli percolation, the crossing probability of a rectangle does not depend on the state of edges outside the rectangle, this is, of course, not the case in percolation models with dependency. As a consequence, it could be that crossing probabilities

are very different under different “boundary conditions”, and that this effect predominates the impact of the aspect ratio of rectangles. In other words, it could be that for certain boundary conditions, the crossing probabilities of very wide rectangles decay extremely fast, while the crossing probabilities of very thin rectangles go very fast to 1 for others. The RSW theory is ineffective in describing this type of phenomenology for a simple reason: the whole theory is based on a priori estimates in some rectangles to deduce estimates in others. We present a new renormalization scheme that generalizes the methods of [DST16], and allows us to describe the effect of the boundary conditions in more detail. The resulting renormalization inequalities eventually show that only the four behaviors presented at the beginning of this abstract are possible.

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Large and small deviation for Gaussian multiplicative chaos, and paths integrals associated with the Mabuchi action

HUBERT LACON

(joint work with Rémi Rhodes, Vincent Vargas)

Gaussian Multiplicative Chaos is a random measure obtained by considering the measure on a domain $D \subset \mathbb{R}^d$ formally given by

$$\mathcal{G}_\gamma dx = e^{\gamma X} dx$$

where X is a Gaussian field on D with a covariance function which displays a log-divergence in the sense that it can be written in the form $L(x, y) - \log |y - x|$ where $L(x, y)$ is a continuous function.

As X can only be defined as a distribution, $e^{\gamma X}$ has no direct interpretation, but a proper way to renormalize this measure has been developed by Kahane [1] whenever $\gamma^2 < 2d$.

The aim of this talk is to investigate the lower fluctuations properties of $\mathcal{G}_\gamma D$ and of the associated derivative chaos $\mathcal{D}_\gamma D$ formally defined by

$$\mathcal{D}_\gamma dx = \gamma X e^{\gamma X} dx.$$

In the first place we show that if one discounts the fluctuation of the averaged field $Z := \frac{1}{|D|} \int X dx$, then lower fluctuation of $\mathcal{G}_\gamma(D)$ are the same that the one given by the explicit Bouchaud-Fyodorov expression proved by Remy [2]:

$$\mathbf{P}[e^{-\gamma Z} \mathcal{G}_\gamma(D) < \varepsilon] \leq e^{-\varepsilon^{-\frac{4}{\gamma^2}}}.$$

We also prove subgaussian fluctuation bounds for the derivative chaos valid when $\gamma \in (0, \sqrt{d/2})$,

$$\mathbf{P}[\mathcal{D}_\gamma(D) < -t] \leq e^{-ct^2}.$$

We then discuss how we use these fluctuation results to define a random geometry based on the combination of the Liouville and the Mabuchi actions. The talk is based on [3].

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Random walks in degenerate dynamical random environments

MAREK BISKUP

(joint work with Pierre-François Rodriguez)

Random walks or, more precisely, Markov chains in dynamical random environments have enjoyed much attention lately. We will consider a special case of such Markov chains where the dynamics is driven by a field of random conductances indexed by, and generally varying in, space and time. The chain is confined to \mathbb{Z}^d and the transitions are only between nearest neighbors. The evolution is best described in terms of the infinitesimal generator

$$(1) \quad L_t f(x) := \sum_{y: |y-x|=1} a_t(x, y) [f(y) - f(x)],$$

where, for an edge $e = (x, y)$, the quantity $a_t(e)$ is the *conductance* of e at time t . The key assumption is that of independence of orientation of the edge,

$$(2) \quad a_t(x, y) = a_t(y, x)$$

We impose the following standard “homogeneity” requirement:

Assumption. *The conductances are non-negative and distributed according to a law \mathbb{P} which is stationary and ergodic with respect to the space-time shifts.*

Although the above setting may appear restrictive, a number of relevant examples exists. The first one of these is the random walk on *dynamical percolation*, where $a_t(e)$ takes values in $\{0, 1\}$ with (typically) $t \mapsto a_t(e)$ being either a stationary Markov chain or a renewal process with density $\mathbb{P}(a_t(e) = 1) = p \in [0, 1]$. Assuming that the evolutions at distinct edges are independent, the set of edges $\{e: a_t(e) = 1\}$ at any fixed time is distributed according to Bernoulli(p) and thus represents a sample from the bond-percolation model on \mathbb{Z}^d . Note that the random walk then effectively jumps only over the edges that are open at that moment of time. Of similar flavor are *random walks on interacting particle systems*. For instance, if $\{\eta_t(x): x \in \mathbb{Z}^d\}$ marks the evolution of an exclusion process, one can take $a_t(x, y) := \eta_t(x)\eta_t(y)$. The random walk again jumps only between pairs of occupied nearest neighbors.

A very different set of examples arises from the *Helfer-Sjöstrand random walk* representation of gradient models. These are models of scalar fields $\{\varphi_x: x \in \mathbb{Z}^d\}$

distributed according to the Gibbs measure with (formal) Hamiltonian

$$(3) \quad H(\varphi) := \frac{1}{2} \sum_{x,y: |x-y|=1} V(\varphi_x - \varphi_y),$$

where the potential $V: \mathbb{R} \rightarrow \mathbb{R}$ is an even C^2 -function that is bounded from below and growing superlinearly at infinity. The Gibbs measure is preserved by the Langevin evolution,

$$(4) \quad d\varphi_x(t) = \left(\sum_{y: |x-y|=1} V'(\varphi_y(t) - \varphi_x(t)) \right) dt + \sqrt{2} dB_x(t),$$

where $\{B_x: x \in \mathbb{Z}^d\}$ is a family of independent standard Brownian motions. Given a sample of $t, x \mapsto \varphi_x(t)$, the random walk then evolves according to (1) with

$$(5) \quad a_t(x, y) := V''(\varphi_x(t) - \varphi_y(t)).$$

The positivity requirement on the conductances translates into a convexity requirement for V . Our interest lies in the situations where V is convex, but not necessarily strictly so.

The main question to address here is under what conditions on the random environment the following concept applies:

Definition (QIP). *We say that the random walk X obeys a Quenched Invariance Principle (QIP), if the law of $t \mapsto n^{-1/2} X_{nt}$ scales, as $n \rightarrow \infty$, to a non-degenerate Brownian motion for a.e. realization of the environment.*

The natural first case to try is that of uniformly elliptic conductances; in this case, Andres [1] proved that QIP indeed holds. More recently, progress has been made solely under suitable moment assumptions. Namely, in $d \geq 2$ Andres, Chiarini, Deuschel and Slowik [2] proved the QIP assuming, apart from ergodicity of \mathbb{P} under space-time shifts, the following moment conditions

$$(6) \quad \exists p, q \in [1, \infty]: \begin{cases} \mathbb{E}[a_0(e)^p] < \infty & \text{and} & \mathbb{E}[a_0(e)^{-q}] < \infty \\ \frac{1}{p-1} + \frac{1}{q(p-1)} + \frac{1}{q} < \frac{2}{d} \end{cases}$$

As shown in Deuschel and Slowik [6], in $d = 1$ the algebraic condition is instead

$$(7) \quad \frac{1}{p-1} + \frac{1}{q(p-1)} < 1.$$

This is due to a fact that the discrete Sobolev inequality, which underpins the elliptic-regularity technique that [2] is based on, takes a different form in $d = 1$ than in $d \geq 2$.

We wish to report on two extensions of the above results. The first one of these, obtained in a joint work with Pierre-François Rodriguez [5], addresses the situations when the conductances vanish with non-zero probability. This is relevant for the aforementioned case of the random walk on dynamical percolation or the walks on interacting particle systems, but also for gradient models with

potentials V that are convex, but not necessarily strictly so. As it turns out, here the relevant control quantity turns out to be

$$(8) \quad T_e := \inf \left\{ t \geq 0 : \int_0^t ds a_s(e) \geq 1 \right\}.$$

Our main result in this case is:

Theorem (Biskup-Rodriguez [5]). *Let $d \geq 2$ and suppose that, in addition to Assumption , the conductances are confined to $[0, 1]$ and, for all edges e in \mathbb{Z}^d ,*

$$(9) \quad \exists \vartheta > 4d: \quad \mathbb{E}(T_e^\vartheta) < \infty.$$

Then the QIP holds.

Our proof of this result relies on the same elliptic regularity techniques (namely, the Moser iteration) as developed in [2]. There is, however, one important novelty: we need to work with norms that involve averaging over infinite intervals of time. This requires control of a number of intermediate steps that are not present in [2]. The work of Mourrat and Otto [7], dealing with heat-kernel estimates in such situations, has been a great inspiration in this endeavor.

The other result we wish to report concerns sharpness of the moment conditions for the QIP to hold. It has long been conjectured that just the first positive and negative moments should suffice. For static random conductances, this is known to be the case in $d = 1, 2$; see Biskup [3]. As time effectively adds one dimension to the problem, there has been a hope that a similar argument can include also dynamical random conductances in $d = 1$. This is indeed the subject of:

Theorem (Biskup [4]). *Let $d = 1$ and suppose, in addition to Assumption , that for each edge e ,*

$$(10) \quad \mathbb{E}[a_0(e)] < \infty \quad \text{and} \quad \mathbb{E}[a_0(e)^{-1}] < \infty.$$

Then the QIP holds.

The main novelty of the proofs in [4] is the use of a certain dual random walk to construct the corrector, which is the key object underlying most proofs of such results, as well as to control its growth under diffusive scaling of space and time. This random walk is just a time change of the simple symmetric random walk and it is thus much easier to analyze.

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Spectral gap critical exponent for hierarchical spin models

ROLAND BAUERSCHMIDT

(joint work with Thierry Bodineau)

I presented the results of [1], where we develop a renormalisation group approach to deriving the asymptotics of the spectral gap of the generator of Glauber type dynamics of spin systems at and near a critical point. In our approach, we derive a spectral gap inequality, or more generally a Brascamp–Lieb inequality, for the measure recursively in terms of spectral gap or Brascamp–Lieb inequalities for a sequence of renormalised measures. We apply our method to hierarchical versions of the 4-dimensional n -component $|\varphi|^4$ model at the critical point and its approach from the high temperature side, and the 2-dimensional Sine–Gordon and the Discrete Gaussian models in the rough phase (Kosterlitz–Thouless phase). For these models, we show that the spectral gap decays polynomially like the spectral gap of the dynamics of a free field (with a logarithmic correction for the $|\varphi|^4$ model), the scaling limit of these models in equilibrium.

Background. Spin systems in equilibrium have been studied by a variety of methods which led to a very complete mathematical description of the physical phenomena occurring in the different regimes of the phase diagrams. This includes in particular a good understanding of the critical phenomena in a wide range of models. Much less is known about the Glauber dynamics of spin systems. For sufficiently high temperatures, it is well understood that the dynamics relaxes exponentially fast towards the equilibrium measure. For the Ising model, the much more difficult question of fast relaxation in the entire uniqueness regime was addressed in [16, 15, 5, 13]. In the phase transition regime, at least for scalar spins, the dynamical behaviour is governed by the interface motion and the relaxation becomes much slower. In particular, the relaxation time diverges as the system size increases, but the dynamical scaling depends strongly on the choice of the boundary conditions. We refer to [14] for a review, as well as to [4, 11] for more recent results. In the vicinity of the critical point, strong correlations develop and as a consequence the dynamic evolution slows down but is no longer driven by phase separation. Even though the critical dynamical behaviour has been well investigated in physics [9], mathematical results are scarce. The only cases for which polynomial lower bounds on the relaxation or mixing times are known are the two-dimensional Ising model [12], exactly at the critical point, the Ising model on a tree [7], both without sharp exponent, and the mean-field Ising model which is fully understood [10, 6].

Results. In [1], we investigate the dynamical relaxation of hierarchical models near and at the critical point by deriving the scaling of the spectral gap in terms of the temperature (or the equivalent parameter of the model) and the system size.

Since their introduction by Dyson [8] and the pioneering work of Bleher–Sinai [3], hierarchical models have been a stepping stone to develop renormalisation group arguments. At equilibrium, sharp results on the critical behaviour of a large class of models have typically been obtained first in a hierarchical framework and then later been extended to the Euclidean lattice. For the equilibrium problem, the hierarchical framework results in a significant technical simplification, but the results and methods have turned out to be surprisingly parallel to the case of the Euclidean lattice \mathbb{Z}^d . This point of view is discussed in detail in [2], to which we also refer for an overview of results and references. Building on the results for the hierarchical set-up for the equilibrium problem, we derive recursive relations on the spectral gap after one renormalisation step. This enables us to obtain sharp asymptotic behaviour of the spectral gap for large size Sine-Gordon model in the rough phase (Kosterlitz–Thouless phase) and for the $|\varphi|^4$ model in the vicinity of the critical point. The scaling coincides in both cases with the one of the hierarchical free field dynamics (with a logarithmic corrections for the $|\varphi|^4$ model) which describes the equilibrium scaling limit of these models. Renormalisation procedures have already been used to analyze spectral gaps for Glauber dynamics, see e.g., [14], but the renormalisation scheme used in [1] is different and allows to keep sharp control from one scale to the next. In particular, for the $|\varphi|^4$ model, our result is the following one.

Theorem. Let $\gamma_N(g, \nu, n)$ be the spectral gap of the hierarchical n -component $|\varphi|^4$ model on a hierarchical cube Λ_N of dimension $d = 4$ and side length L^N . Let $L \geq L_0$, and let $g > 0$ be sufficiently small. There exists $\nu_c = \nu_c(g, n) = -C(n + 2)g + O(g^2)$ and a constant $\delta \geq 1$ (independent of n) such that for $t_0 \geq t \geq cL^{-2N}$, where t_0 is a small constant,

$$(1) \quad c_1 t (-\log t)^{-\delta(n+2)/(n+8)} \leq \gamma_N(g, \nu_c + t, n) \leq c_2 t (-\log t)^{-(n+2)/(n+8)},$$

provided that N is sufficiently large. In particular, $t \geq cL^{-2N}$ is allowed to depend on N .

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Quantitative estimates on extreme level sets in branching Brownian motion

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(joint work with Aser Cortines, Oren Louidor)

This work concerns the fine structure of extreme values of branching Brownian motion. The latter describes the motion of a particle which diffuses on the real line according to a standard Brownian motion for a time whose law is exponential with mean one and then splits into two independent child particles which repeat the same procedure starting from the last position of their parent. We denote the number of particles at time t by $n(t)$ and the particle positions by $(x_k(t))_{n(t)}$. The study of extreme values of h dates back to works of Ikeda et al. [7, 8, 9], McKean [11], Bramson [3, 4] and Lalley and Sellke [10] who derived asymptotics for the law of the maximal position of a particle at time t . Introducing the centering function

$$(1) \quad m_t := \sqrt{2}t - \frac{3}{2\sqrt{2}} \log^+ t, \quad \text{where} \quad \log^+ t := \log(t \vee 1),$$

it was shown that

$$(2) \quad \lim_{t \uparrow \infty} \mathbb{P} \left(\max_{k \leq n(t)} x_k(t) - m_t \leq x \right) = \mathbb{E} \left(e^{-CZ e^{-\sqrt{2}}} \right),$$

where C is a constant and Z is a specific random variable (the limit of the so-called derivative martingale). Other extreme values of h can be studied simultaneously by considering the extremal process:

$$(3) \quad \mathcal{E}_t := \sum_{k \leq n(t)} \delta_{x_k(t) - m_t}.$$

Asymptotics for this process were treated in the physics literature by, e.g., Brunet and Derrida [5] and more recently in the mathematical literature simultaneously by Aïdékon et al. [1] and Arguin et al. [2]. These works show that there exists a random point measure \mathcal{E} such that

$$(4) \quad \mathcal{E}_t \implies \mathcal{E} \quad \text{as } t \rightarrow \infty,$$

and \mathcal{E} is described as a Poisson cluster process. In [6] we add to this picture by providing the following more quantitative results on the extreme level sets. Let $\Sigma_t(u) = \{k \leq n(t) : x_k(t) \geq m_t - u\}$.

Theorem. $\exists \gamma$ (a constant) such that $\forall \varepsilon > 0$

$$(5) \quad \lim_{u \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P} \left(\left| \frac{\#\Sigma_t(u)}{uZe^{\sqrt{2}u}} - \gamma \right| > \varepsilon \right) = 0.$$

Next, we want to understand more on the relation between the local maxima and their recent offsprings (the so called clusters). For $0 < \alpha < 1$ define the set of recent offsprings

$$(6) \quad \mathcal{C}^{(x_k(t))} = \{x_j(t) : d(x_j(t), x_k(t)) > t - r_t\},$$

where $d(\cdot, \cdot)$ denotes the time of the most recent common ancestor (overlap) and $r_t \rightarrow \infty$ as $t \uparrow \infty$ but slowly. Set

$$(7) \quad \Sigma_t(u, \alpha) := \{y \in \Sigma_t(u) : y \in \mathcal{C}^{(x_k(t))} \text{ such that } x_k(t) > m_t - \alpha u\}$$

Then we have the following:

Theorem.

$$(8) \quad \lim_{u \rightarrow \infty} \lim_{t \rightarrow \infty} \mathbb{P} \left(\left| \frac{\#\Sigma_t(u, \alpha)}{\#\Sigma_t(u)} - \alpha \right| > \varepsilon \right) = 0.$$

Moreover, we study the tail of the distance of the two highest particles at time t . Let $h_t^{(1)}$ and $h_t^{(2)}$ denote the right-most and second right-most particle.

Theorem.

$$(9) \quad \lim_{u \rightarrow \infty} \lim_{t \rightarrow \infty} \frac{1}{u} \log \mathbb{P}(h_t^{(1)} - h_t^{(2)} > u) = -\sqrt{2}(1 + \sqrt{2}).$$

This was already conjectured by Brunet and Derrida in [5]. The results mentioned above are proven using first and second moment estimates together with a decorated random walk-like representation for the path of an extremal particle.

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Decoupling a Process with Memory, Application to FK Percolation and Potts Model with a Defect Line

SÉBASTIEN OTT

(joint work with Yvan Velenik)

Consider the FK-percolation model on \mathbb{Z}^d , $d \geq 2$, with weight $(x \geq 0, q \geq 1)$

$$\mathbb{P}(\omega) \propto x^{|\omega|} q^{\kappa(\omega)},$$

where $|\omega|$ denotes the number of edges in ω and κ is the number of connected components. It is known ([2] for $q = 1$, [3] for $q = 2$ and [9] for $q \geq 1$) that there exists a critical value x_c such that one has exponential decay of connectivities when $x < x_c$ uniformly over boundary conditions, and there exists an infinite cluster when $x > x_c$. In the former case, one can define the rate of exponential decay (or inverse correlation length) in the direction of $v \in \mathbb{R}^d$ by:

$$(1) \quad \xi^{-1}(v) = - \lim_{n \rightarrow \infty} \frac{1}{n} \log (\mathbb{P}(0 \leftrightarrow nv)).$$

As one has exponential decay of connectivity, this quantity is positive for $v \neq 0$ and classical use of the FKG inequality allows to deduce that ξ^{-1} defines a norm on \mathbb{R}^d .

In the regime $x < x_c$ a heuristic argument of Ornstein and Zernike ([11],[13], originally designed to handle correlations in gas models, it translate into this prediction through the correspondence between FK percolation and Potts model)

predicts that (\vec{e}_1 is the unit vector in the first coordinate direction)

$$(2) \quad \mathbb{P}(0 \leftrightarrow n\vec{e}_1) = \frac{C}{n^{(d-1)/2}} e^{-n\xi^{-1}(\vec{e}_1)}(1 + o_n(1)).$$

A perturbative proof of a similar statement was given in [1]. A general non-perturbative scheme to prove such asymptotics was then developed by Campanino, Ioffe and Velenik in a series of papers: [4], [5], [6] and [7]. It revolves around the proof that with high probability, a long subcritical percolation cluster can be split into a chain of irreducible pieces, each of morally constant diameter (exponential decay of the diameter). Sampling a cluster connecting 0 to $n\vec{e}_1$ can then be done by sampling chain of irreducible pieces using a kernel Ψ on the space of irreducible pieces. The OZ asymptotics then follows by the derivation of a local limit theorem for such process with memory.

Factorization of Measure. In [12], we developed an elementary way to replace the chain sampled by the kernel Ψ by a chain of iid pieces of cluster. This is done using ideas from perfect sampling, in particular the construction done in [8]. The OZ asymptotics then follows by the standard LLT for random walks in \mathbb{Z}^d . This representation of long clusters being easier to manipulate, it has potential for diverse application.

Potts with a Defect Line. The goal of [12] was to study the impact on the inverse correlation length of a modified line of coupling constants in the high temperature Potts model by extending the study of the problem made in [10] for Bernoulli percolation to FK-percolation. In terms of FK-percolation, the problem translates into the following: change the weight of a configuration to

$$\mathbb{P}_{x'}(\omega) \propto \left(\frac{x'}{x}\right)^{\omega_{\mathcal{L}}} x^{|\omega|} q^{\kappa(\omega)},$$

where $x' \geq 0$, \mathcal{L} is the first coordinate axis and $\omega_{\mathcal{L}}$ is the set of edges in ω with both endpoints on \mathcal{L} ; and study $x' \rightarrow \xi_{x'}^{-1}(\vec{e}_1)$ where $\xi_{x'}^{-1}$ is defined similarly as ξ^{-1} . Using the random walk representation of the cluster and (many!) properties of the FK-percolation measure, we were able to deduce

- There exists $x'_c(d) \geq x$ such that $\xi_{x'}^{-1}(\vec{e}_1) = \xi^{-1}(\vec{e}_1)$ when $x' \leq x'_c(d)$ and $\xi_{x'}^{-1}(\vec{e}_1) < \xi^{-1}(\vec{e}_1)$ when $x' > x'_c(d)$.
- $x'_c(2) = x'_c(3) = x$ while $x'_c(d) > x$ for $d \geq 4$.
- Precise behaviour of $x' \rightarrow \xi_{x'}^{-1}(\vec{e}_1)$ in the neighbourhood of x'_c in dimensions 2, 3.

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Recent progress on dynamics of mean field spin glasses

AUKOSH JAGANNATH

Consider a random power series in N variables, where we view N as very large. Our interest is in Markov processes that are reversible with respect to the Gibbs measure corresponding to this function. These models are called “mean-field spin glass models” in the physics literature. A central prediction in the study of spin glasses is that for any local reversible dynamics, one expects an exponential time to relaxation in the spin glass phase. We prove this prediction for a broad class of Ising spin and spherical spin glass models. We present a single framework to prove these estimates that applies equally in the discrete and manifold settings by introducing the notion of “free energy barriers”. The existence of free energy barriers will imply, by Cheeger-type estimates, that the spectral gap of the dynamics is exponentially small, and thus that the mixing is exponentially slow. We then present sufficient conditions which imply the existence of these barriers for a large class of mean field spin glass models using the notions of the “replicon eigenvalue”, the 2D Guerra—Talagrand bounds, and a quenched LDP for the overlap distribution. We then show that these sufficient conditions cover large classes of spin glass models, e.g., any homogeneous polynomial of degree $p > 3$.

We also discuss a new approach to studying spherical spin glass dynamics based on differential inequalities for one-time observables. Using this approach, we obtain an approximate phase diagram for the evolution of the energy H and its gradient under Langevin dynamics for spherical p -spin models. We then derive several consequences of this phase diagram. For example, at any temperature, uniformly over all starting points, the process must reach and remain in an absorbing region of large negative values of H and large (in norm) gradients in order 1 time.

Furthermore, if the process starts in a neighborhood of a critical point of H with negative energy, then both the gradient and energy must increase macroscopically under this evolution, even if this critical point is a saddle with index of order N . These works are joint with G. Ben Arous (NYU) and R. Gheissari (NYU).

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Scaling limits for directed polymers in heavy-tail random environment

QUENTIN BERGER

(joint work with Niccolò Torri)

The directed polymer model. We consider the so-called directed polymer model in dimension $1+1$, which has been widely studied in the physical and mathematical literature over the past decades (we refer to [7] for a general overview). Let S be a nearest-neighbor simple symmetric random walk on \mathbb{Z} , whose law is denoted by \mathbf{P} , and let $\omega = (\omega_{i,x})_{i \in \mathbb{N}, x \in \mathbb{Z}}$ be a field of i.i.d. random variables (the *environment*) with law \mathbb{P} . The *directed* random walk $(i, S_i)_{i \in \mathbb{N}_0}$ represents a polymer trajectory, and for $\beta > 0$ (the inverse temperature), we define the *polymer measure*

$$\frac{d\mathbf{P}_{n,\beta}^\omega}{d\mathbf{P}}(s) := \frac{1}{\mathbf{Z}_{n,\beta}^\omega} \exp\left(\beta \sum_{i=1}^n \omega_{i,s_i}\right),$$

where $\mathbf{Z}_{n,\beta}^\omega$ is the *partition function* of the model.

One of the main question about this model is that of the localization and super-diffusivity of paths trajectories drawn from the measure $\mathbf{P}_{n,\beta}^\omega$. The transversal exponent ξ describes the fluctuation of the end-point, that is $\mathbb{E}\mathbf{E}_{n,\beta}^\omega |S_n| \approx n^\xi$ as $n \rightarrow \infty$. If $\omega_{n,x}$ have an exponential moment, it is conjectured that one has $\xi = 2/3$ (for any $\beta > 0$). Moreover, it is expected that the point-to-point partition function, when properly centered and renormalized, converges in distribution to the GUE distribution.

The case of a heavy-tail environment. Let us consider the case where the environment distribution $\omega_{i,x}$ and has some heavy tail: there is some $\alpha > 0$ such that

$$(1) \quad \mathbb{P}(\omega_{1,1} > t) \sim t^{-\alpha} \quad \text{as } t \rightarrow +\infty.$$

Then, according to the heuristics (and terminology) of [6], three regimes should occur: (a) if $\alpha > 5$, there should be a *collective* optimization and we should have $\xi = 2/3$, as in the finite exponential moment case; (b) if $\alpha \in (2, 5)$, the optimization

strategy should be *elitist*: most of the energy should be held by a small fraction of the points visited by the path, and we should have $\xi = \frac{\alpha+1}{2\alpha-1}$; (c) if $\alpha \in (0, 2)$, the strategy is *individual*: the polymer targets few exceptional points, and we have $\xi = 1$, as shown in [3, 9].

Weak-coupling scaling limits. A recent and fruitful approach to proving universality results for the directed polymer model has been to consider its *weak-coupling limit*, that is to take β go to 0 as $n \rightarrow +\infty$. In the case where ω has an exponential moment, Alberts, Khanin and Quastel [1] show that, taking $\beta_n = \beta n^{-\gamma}$, $\gamma = 1/4$ for some fixed $\beta > 0$, the model converges to a *non-trivial* (*i.e. disordered*) continuous version of it. More precisely, letting $\lambda(s) := \log \mathbb{E}[e^{s\omega}]$,

$$\log \mathbf{Z}_{n,\beta_n}^\omega - n\lambda(\beta_n) \xrightarrow{(d)} \log \mathcal{Z}_{\sqrt{2}\beta}, \quad \text{as } n \rightarrow \infty,$$

where $\beta \mapsto \log \mathcal{Z}_{\sqrt{2}\beta}$ is the so called *cross-over process* (it interpolates, as β goes from 0 to ∞ , between Gaussian and GUE distributions, see [2]). This has been extended to the case of a heavy-tail distribution (1) by Dey and Zygouras [8]: one needs to take $\beta_n = \hat{\beta}n^{-\gamma}$ with $\gamma = 1/4$ if $\alpha > 6$ and $\gamma = 3/2\alpha$ is $\alpha \in (1/2, 6)$.

This intermediate disorder regime corresponds to the exact scaling at which disorder “kicks-in”: the effect of the random environment on path trajectories is non-trivial, but one still has a transversal fluctuation exponent $\xi = 1/2$. As suggested in [8], this is part of a larger picture: setting $\beta_n = \beta n^{-\gamma}$ for some $\beta > 0$ and some $\gamma \geq 0$, the transversal fluctuation exponent ξ should depend on α, γ in the following manner

$$(2) \quad \xi = \begin{cases} \frac{2}{3}(1-\gamma) & \text{for } \alpha \geq \frac{5-2\gamma}{1-\gamma}, \quad 0 \leq \gamma \leq \frac{1}{4}, \\ \frac{1+\alpha(1-\gamma)}{2\alpha-1} & \text{for } \alpha \leq \frac{5-2\gamma}{1-\gamma}, \quad \frac{2}{\alpha} - 1 \leq \gamma \leq \frac{3}{2\alpha}. \end{cases}$$

Outside of these regions, one should have $\xi = 1/2$ (γ large) or $\xi = 1$ (γ small). Hence, by tuning properly β_n , one should be able to observe any transversal fluctuations between \sqrt{n} and n . This is summarized in Figure 1 below.

This picture is far from being settled, and so far only the border cases where $\xi = 1$ or $\xi = 1/2$ had been proven: Dey and Zygouras [8] proved that $\xi = 1/2$ in the cases $\alpha > 6, \gamma = 1/4$ and $\alpha \in (1/2, 6), \gamma = 3/2\alpha$; Auffinger and Louidor [3] proved that $\xi = 1$ for $\alpha \in (0, 2)$ and $\gamma = \frac{2}{\alpha} - 1$. The results we present here ([4, 5]) complete the picture in the case $\alpha \in (0, 2)$.

Statement of the results. Let us define the set of rescaled paths, and their continuous “entropy” and “energy”. The rescaled paths will be in the set $\mathcal{D} := \{s : [0, 1] \rightarrow \mathbb{R} ; s \text{ is continuous and a.e. differentiable}\}$. The (continuum) entropy of a path $s \in \mathcal{D}$ is

$$(3) \quad \text{Ent}(s) = \frac{1}{2} \int_0^1 (s'(t))^2 dt,$$

which derives from the rate function of the moderate deviation of the simple random walk, *i.e.* $\mathbf{P}(S_{tn} = xn^\xi) = \exp\left(- (1 + o(1)) \frac{x^2}{2t} n^{2\xi-1}\right)$.

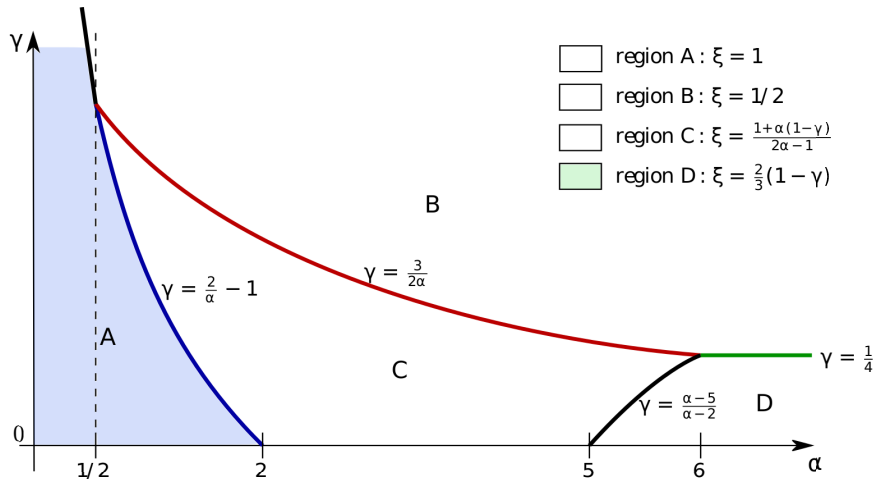


FIGURE 1. We identify four regions in the (α, γ) plane. Region **A** with $\alpha < 2$ is treated in [3] and Region **B** with $\alpha > 1/2$ in [8] and with $\alpha < 1/2$ in [4]. Region **D** is still open, and in region **C**, only the case $\alpha < 2$ has been proven, in [4].

As far as the (rescaled) disorder field is concerned, we let $\mathcal{P} := \{(w_i, t_i, x_i)\}_{i \geq 1}$ be a Poisson Point Process on $[0, 1] \times \mathbb{R} \times [0, \infty)$ of intensity measure defined as $\mu(dw dt dx) = \frac{\alpha}{2} w^{-\alpha-1} \mathbf{1}_{\{w > 0\}} dw dt dx$. For a quenched realization of \mathcal{P} , the energy of a continuous path $s \in \mathcal{D}$ is then defined by

$$(4) \quad \pi(s) = \pi^{(\mathcal{P})}(s) := \sum_{(w,t,x) \in \mathcal{P}} w \mathbf{1}_{\{s(t)=x\}},$$

which is indeed the sum of the weights “picked up” by s . Our main result is the following.

Theorem ([4, Thm. 2.4] & [5, Thm. 2.4]). *Assume that $\alpha \in (1/2, 2)$, and set $\beta_n := \beta n^{-\gamma}$ with $2/\alpha - 1 < \gamma < 3/2\alpha$ (i.e. region **A** in Figure 1). Then we have $\xi = \frac{1+(1-\gamma)\alpha}{2\alpha-1} \in (1/2, 2)$, and as $n \rightarrow +\infty$*

$$(5) \quad \frac{1}{\beta n^{2\xi-1}} (\log \mathbf{Z}_{n,\beta_n}^\omega - n\beta_n \mathbb{E}[\omega] \mathbf{1}_{\{\alpha > 1\}}) \xrightarrow{(d)} \mathcal{T} := \sup_{s \in \mathcal{D}, \text{Ent}(s) < +\infty} \{\pi(s) - \text{Ent}(s)\}.$$

We have $\mathcal{T} \in (0, +\infty)$, and the supremum is attained, by some unique continuous path s^* .

In the case $\alpha \in (0, 1/2)$, we show that no intermediate transversal fluctuations can occur.

Theorem ([3, 10] and [4, Thm. 2.10]). *Assume that $\alpha \in (0, 1/2)$, and that $\beta_n := \beta n^{-\gamma}$ with $\gamma = 2/\alpha - 1$. Then there is some $\beta_c = \beta_c(\omega) \in (0, +\infty)$ such that:*

(i) if $\beta > \beta_c$ then $\xi = 1$ and

$$\frac{1}{n} \log \mathbf{Z}_{n,\beta_n}^\omega \xrightarrow{(d)} \hat{\mathcal{T}}_\beta > 0,$$

with $\hat{\mathcal{T}}_\beta$ some energy/entropy variational problem, defined in [3] ($\hat{\mathcal{T}}_\beta = 0$ for $\beta < \beta_c$).

(ii) if $\beta < \beta_c$ then $\xi = 1/2$ and

$$\frac{1}{\beta n^{1/2-1/2\alpha}} \log \mathbf{Z}_{n,\beta_n}^\omega \xrightarrow{(d)} \mathcal{W}^{(\alpha)},$$

with $\mathcal{W}^{(\alpha)}$ some α -stable distribution, defined in [8].

Further comments. Let us simply outline here that the methods we use are very robust, and that they can be adapted to treat the case of: a higher dimension; random walks with unbounded jumps (e.g. with stretch-exponential tail); non-directed random walks... The most interesting open problem consists in completing the results in Region **C** of Figure 1, where the behavior of the system should still be driven by few large weights in the heavy-tail environment. One of the main issue comes from the fact that the variational problem in (5) becomes infinite when $\alpha > 2$, the main contribution coming from the accumulation of many small weights.

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Scaling limits for the odometer of divisible sandpile models

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(joint work with Leandro Chiarini, Alessandra Cipriani, Milton Jara and Rajat Hazra)

In this talk we want to present recent results about scaling limits of the odometer of a divisible sandpile on the torus obtained in [3, 4] and [2].

The divisible sandpile model is the continuous fixed energy counterpart of the Abelian sandpile model which was introduced by [1] as a discrete toy model displaying self-organized criticality. Self-organized critical models are characterized by a power-law behaviour of certain quantities such as 2-point functions without fine-tuning any parameter.

Consider the discrete torus $\mathbb{Z}_n^d = (\mathbb{Z}/n\mathbb{Z})^d$ and initially assign randomly to each vertex x a real number $s(x)$ drawn from a given distribution. This real number plays the role of a *mass* in case the number is positive and a *hole* otherwise. We will assume that $s(x)$ is of the form

$$(1) \quad s(x) = 1 + \sigma(x) - \frac{1}{n^d} \sum_{z \in \mathbb{Z}_n^d} \sigma(z)$$

where the random variables $(\sigma(x))_{x \in \mathbb{Z}_n^d}$ are i.i.d and satisfy one of the following conditions:

(M1) $\mathbb{E}(\sigma(x)) = 0$, $\mathbb{E}(\sigma^2(x)) = 1$

(M2) σ 's are in the domain of attraction of an α -stable distribution for $\alpha \in (0, 2]$.

At each time step, topple all vertices with mass larger than 1 by keeping 1 and redistributing the excess either

(D1) to the nearest neighbours or

(D2) to all vertices following a jump distribution of a long-range random walk with parameter $\alpha > 0$.

The configuration defined in (1) will stabilise to the all 1 configuration, it will not depend on (M1)-(M2) or (D1)-(D2).

Call the total amount of mass $u_n(x)$ emitted from each vertex $x \in \mathbb{Z}_n^d$ upon stabilisation (odometer), we can interpret the odometer function as a *random interface model* on \mathbb{Z}_n^d . Let $a_n = 4\pi^2 n^{d-4/2}$ and define a formal field on \mathbb{T}^d as

$$\Xi_n(x) := \sum_{z \in \mathbb{T}_n^d} u_n(nz) 1_{B(z, 1/2n)}(x) \quad x \in \mathbb{T}^d.$$

If the redistribution of mass happens only to the nearest neighbours (under assumption (D1)), then it was proven by [3, 5] that the odometer on the finite torus is distributed as a discrete bi-Laplacian field. Furthermore, it was proven in [3] that under the assumption of finite variance (M1), $a_n \Xi_n$ converges in distribution to a continuum bi-Laplacian field Ξ on the torus in an appropriate Sobolev space $H_{-\epsilon}(\mathbb{T}^d)$ for $\epsilon > \max\{d/4, 1 + d/2\}$ and $H_{-\epsilon}(\mathbb{T}^d)$ is the $C^\infty(\mathbb{T}^d)/\sim$ completion

under the norm

$$(f, f)_{H_a(\mathbb{T}^d)} = \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} \|\xi\|^{4a} |\widehat{f}(\xi)|^2.$$

Note that $\Xi = (-\Delta)^{-1}W$, where W is white noise, is defined as $\Xi = \{\langle \Xi, f \rangle : f \in C^\infty(\mathbb{T}^d)/\sim, \text{mean } 0\}$ such that $\langle \Xi, f \rangle \sim N(0, \|f\|_{H_{-1}}^2)$.

In [2] the authors construct the scaling limit of the odometer function to a fractional Gaussian field $(-\Delta)^{-\gamma/2}W$, $\gamma = \min\{\alpha, 2\}$ and $\alpha \in (0, \infty)$ in an appropriate Sobolev space depending on α for σ 's which have a finite second moment and under (D2). Let us stress two interesting facts. The first fact is that the case $\alpha = 1$ corresponds to the Gaussian Free Field (GFF) on the torus \mathbb{T}^d . For $d = 2$, we are constructing the GFF in the torus from a long-range random walk related to the 1-stable Lévy process, sometimes called Lévy flight. The interesting point is that the Lévy flight is not conformally invariant itself. However, the Gaussian Free Field is.

Finally, under (D1) and infinite second moments (M2), a limiting field which is not Gaussian but instead an α -stable random field is constructed in [4].

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Invariance Principle for the Random Lorentz Gas – Beyond the Boltzmann-Grad Limit

BÁLINT TÓTH

(joint work with Christopher Lutsko)

We consider the Lorentz gas with randomly placed spherical hard core scatterers in \mathbb{R}^d . That is, place spherical balls of radius r and infinite mass centred on the points of a Poisson point process of intensity ϱ in \mathbb{R}^d , and define the trajectory $t \mapsto X^{r,\varrho}(t) \in \mathbb{R}^d$ of a particle moving among these scatterers as follows:

- If the origin is covered by a scatterer then $X^{r,\varrho}(t) \equiv 0$.
- If the origin is not covered by a scatterer then $t \mapsto X^{r,\varrho}(t)$ is the trajectory of a point-like particle starting from the origin with random velocity sampled uniformly from the unit sphere S^{d-1} and flying with constant speed between successive elastic collisions on any one of the fixed, infinite mass scatterers.

Randomness is due only to the initial conditions. Otherwise, the dynamics of the moving particle is fully deterministic, governed by classical Newtonian laws. With probability 1 the trajectory $t \mapsto X^{r,\varrho}(t)$ is well defined. Due to elementary scaling and percolation arguments the probability that the moving particle is not trapped in a compact domain is positive, and arbitrarily close to 1, if ϱr^d is sufficiently small.

Establishing the diffusive scaling limit of the Lorentz particle trajectory

$$(1) \quad t \mapsto \frac{X^{r,\varrho}(Tt)}{\sqrt{T}}, \quad \text{as} \quad T \rightarrow \infty,$$

for $t \in [0, 1]$, conditioned on the event that the particle is not trapped is a major open question of (mathematical) statistical physics. The Holy Grail is a mathematically rigorous proof of invariance principle (i.e. weak convergence to Wiener process with nondegenerate variance) of the sequence of processes (1) in either one of the following two settings.

(Q) *Quenched limit*: For almost all (i.e. typical) realizations of the underlying Poisson point process, with averaging over the random initial velocity of the particle. In this case, it is expected that the variance of the limiting Wiener process is deterministic.

(AQ) *Averaged-quenched* (a.k.a. *annealed*) *limit*: Averaging over the random initial velocity of the particle *and* the random placements of the scatterers.

See the surveys [6], [2] and the monograph [7].

These questions remain open. However, our result is a non-trivial step forward in the direction of the averaged-quenched limit.

The Boltzmann-Grad limit is the following low (relative) density limit of the scatterer configuration:

$$(2) \quad r \rightarrow 0, \quad \varrho \rightarrow \infty, \quad \varrho r^{d-1} \rightarrow v_{d-1},$$

where v_{d-1} is the volume of the $(d-1)$ -dimensional unit disc/sphere. Other choices of $\lim \varrho r^{d-1} \in (0, \infty)$ are equally legitimate and would change the limit only by a time (or space) scaling factor.

It is not difficult to see that in the averaged-quenched setting and under the Boltzmann-Grad limit (2) the distribution of the first free flight length starting at any deterministic time, converges to an $EXP(1)$ and the jump in velocity after the free flight happens in a Markovian way with transition kernel

$$(3) \quad \mathbf{P}(v_{\text{out}} \in dv' \mid v_{\text{in}} = v) = \sigma(v, v') dv',$$

where dv' is the normalised (uniform) surface element on S^{d-1} and $\sigma : S^{d-1} \times S^{d-1} \rightarrow \mathbb{R}_+$ is the normalised *differential cross section* of a spherical hard core scatterer, computable as

$$(4) \quad \sigma(v, v') \sim |v - v'|^{3-d}.$$

It is intuitively compelling but far from easy to prove that under the Boltzmann-Grad limit (2)

$$(5) \quad \left\{ t \mapsto X^{r,\varrho}(t) \right\} \Rightarrow \left\{ t \mapsto Y(t) \right\},$$

where the symbol \Rightarrow stands for weak convergence (of probability measures) on the space of continuous trajectories in \mathbb{R}^d . The process $t \mapsto Y(t)$ on the right hand side is the Markovian random flight process consisting of independent free flights of $EXP(1)$ -distributed length, with Markovian velocity changes according to the transition kernel (3), (4). The limit (5), valid in any compact time interval $t \in [0, T]$, $T < \infty$, is rigorously established in [3], [4], [5] in the averaged-quenched setting and in [1] in the quenched setting. In [5] more general point processes of the scatterer positions, with sufficiently strong mixing properties are considered.

The limiting Markovian flight process $t \mapsto Y(t)$ is essentially a continuous time random walk. Therefore, taking a second, diffusive limit *after* the Boltzmann-Grad limit (5) yields indeed the invariance principle,

$$(6) \quad \left\{ t \mapsto T^{-1/2}Y(Tt) \right\} \Rightarrow \left\{ t \mapsto W(t) \right\},$$

as $T \rightarrow \infty$, where $t \mapsto W(t)$ is the Wiener process in \mathbb{R}^d of nondegenerate variance.

The natural question arises whether one could somehow interpolate between the double limit (2) and (6) taken in this order, and the plain diffusive limit (1). Our result gives a positive partial answer in dimension 3.

Theorem. *Assume $d = 3$ and let $\varrho = \varrho(r) = v_{d-1}r^{1-d}$, $T = T(r) := r^{-2+\varepsilon}$, with some $\varepsilon > 0$. Then*

$$(7) \quad \left\{ t \mapsto T^{-1/2}X^{r,\varrho}(Tt) \right\} \Rightarrow \left\{ t \mapsto W(t) \right\},$$

as $r \rightarrow 0$, in the averaged-quenched sense, where $W(t)$ is a non-degenerate d -dimensional Wiener process.

The point is that the Boltzmann-Grad limit of scatterer configuration (2) and the diffusive scaling of trajectory are done simultaneously, and not consecutively. The memory effect due to recollisions is controlled up to the time scale $T = T(r) := r^{-2+\varepsilon}$, and not beyond.

The proof is based on a coupling of the Markovian flight process $t \mapsto Y(t)$ and the averaged-quenched realisation of the Lorentz process $t \mapsto X^r(t)$, such that the maximum distance of their positions up to time T be small order of \sqrt{T} . The Lorentz process $t \mapsto X^r(t)$ is realised as an *exploration* of the environment of scatterers. That is, as time goes on, more and more information is revealed about the position of the scatterers. As long as $X^r(t)$ traverses not-yet-explored territories, it behaves just like the Markovian flight process $Y(t)$, discovering new not-yet-seen scatterers with rate 1 and scattering on them. However, unlike the Markovian flight process it has long memory, the discovered scatterers are put at their place forever and in case of return to those places, recollisions occur. Likewise, the area swept in the past by the Lorentz exploration process $X^r(t)$ – that is: a tube of radius r around its past trajectory – is recorded as a domain

where new collisions can not occur. Let the velocity of the coupled processes be $U(t) := \dot{Y}(t)$ and $V^r(t) := \dot{X}^r(t)$. The coupling is realized in such a way, that

- At the very beginning the two velocities coincide, $V^r(0) = U(0)$.
- Occasionally, with typical frequency of order r mismatches of the two velocity processes occur. These mismatches are caused by two possible effects:
 - *Recollisions* of the Lorentz exploration process with a scatterer placed in the past. This causes a collision event of $V^r(t)$ while $U(t)$ does not change.
 - Scatterings of the Markovian flight process $Y(t)$ in a moment when the Lorentz exploration process is in the explored tube, where it can not encounter a not-yet-seen new scatterers. In these moments the process $U(t)$ has a jump discontinuity, while the process $V^r(t)$ stays unchanged. We will call these events *shadowed scatterings* of the Markovian flight process.
- However, shortly after the mismatch events described above, a new jointly realised scattering event of the two processes occurs, recoupling the two velocity processes to identical values. These recouplings occur typically at an $EXP(1)$ -distributed time after the mismatches.

Summarizing: The coupled velocity processes $t \mapsto (U(t), V^r(t))$ are realized in such a way that they assume the same values except for typical time intervals of length of order 1, separated by typical intervals of lengths of order r^{-1} . Other, more complicated mismatches of the two processes occur only at time scales of order r^{-2} or longer. If all these are controlled (this is the content of the proof) then the following hold:

Up to $t < r^{-1+\varepsilon}$, with high probability there is no mismatch whatsoever between $U(t)$ and $V^r(t)$. That is, for any $\varepsilon > 0$,

$$(8) \quad \begin{aligned} & \lim_{r \rightarrow 0} \mathbf{P}(\inf\{t : V^r(t) \neq U(t)\} < r^{-1+\varepsilon}) \\ & = \lim_{r \rightarrow 0} \mathbf{P}(\inf\{t : X^r(t) \neq Y(t)\} < r^{-1+\varepsilon}) = 0, \end{aligned}$$

and therefore the invariance principle (7) follows, with $T(r) = r^{-1+\varepsilon}$, rather than $T(r) = r^{-2+\varepsilon}$. As a by-product of this argument a new and handier proof of the results of Gallavotti [3], [4] and Spohn [5] also drops out.

Going up to $t < r^{-2+\varepsilon}$, with $\varepsilon > 0$, needs more argument. The ideas exposed in the outline above lead to the following chain of bounds:

$$(9) \quad \begin{aligned} \max_{0 \leq t \leq 1} \left| \frac{X^r(Tt)}{\sqrt{T}} - \frac{Y(Tt)}{\sqrt{T}} \right| &= \frac{1}{\sqrt{T}} \max_{0 \leq t \leq 1} \left| \int_0^{Tt} (V^r(s) - U(s)) ds \right| \\ &\leq \frac{1}{\sqrt{T}} \int_0^T |V^r(s) - U(s)| ds \asymp \frac{1}{\sqrt{T}} Tr = \sqrt{T}r. \end{aligned}$$

In the \asymp step we use the arguments exposed above. Finally, choosing $T = T(r) = r^{-2+\varepsilon}$, with $\varepsilon > 0$, we obtain a tightly close coupling of the diffusively scaled

processes $t \mapsto X^r(Tt)/\sqrt{T}$ and $t \mapsto Y(Tt)/\sqrt{T}$, and hence the invariance principle (7), for this longer time scale.

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Convergence of vertex-reinforced jump processes to an extension of the supersymmetric hyperbolic nonlinear sigma model

SILKE ROLLES

(joint work with Franz Merkl and Pierre Tarrès)

Consider a finite undirected connected graph $G = (V, E)$ with weights $W_{ij} = W_{ji} > 0$ attached to the edges $e = \{i, j\} \in E$. The vertex-reinforced jump process (VRJP) $(Y_t)_{t \geq 0}$ is a stochastic process with memory in continuous time with values in V . It starts at time 0 in a fixed vertex $i_0 \in V$. Given the past of the process up to time t , it jumps to a nearest neighbor $j \in V$ of the current position Y_t at rate $\frac{1}{2}W_{Y_t j}(1 + L_j(t))$, where

$$(1) \quad L_j(t) = \int_0^t 1_{\{Y_s=j\}} ds$$

denotes the local time at j . The vertex-reinforced jump process was introduced by Werner and first studied by Davis and Volkov in [1] and [2]. Sabot and Tarrès [4] considered the time change

$$(2) \quad D(t) = \sum_{i \in V} (L_i(t)^2 - 1), \quad t \geq 0.$$

They proved that the time-changed version $Z = (Z_\sigma = Y_{D^{-1}(\sigma)})_{\sigma \geq 0}$ is a mixture of Markov jump processes. The process Z is called VRJP in exchangeable time scale. The mixing measure can be described in terms of the supersymmetric hyperbolic non-linear sigma model introduced by Zirnbauer in [5] as follows. Let $\Omega_{i_0} = \{u \in \mathbb{R}^V : u_{i_0} = 0\}$ and let \mathcal{T} denote the set of spanning trees of the graph G .

The supersymmetric hyperbolic non-linear sigma model is given by the following probability measure $\mu_{i_0}^{W,\text{susy}}$ on $(\Omega_{i_0})^2 \times \mathcal{T}$:

$$(3) \quad d\mu_{i_0}^{W,\text{susy}}(u, s, T) = \prod_{\{i,j\} \in E} \exp \left\{ -W_{ij} \left[\cosh(u_i - u_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{u_i + u_j} \right] \right\} \cdot \prod_{\{i,j\} \in T} W_{ij} e^{u_i + u_j} \prod_{j \in V \setminus \{i_0\}} \frac{e^{-u_j} du_j ds_j}{2\pi} dT,$$

where du_j, ds_j denote the Lebesgue measure on \mathbb{R} and dT is the counting measure on \mathcal{T} . The representation of VRJP as a mixture of Markov jump processes enabled many exciting developments in the field.

Consider the local times of VRJP in exchangeable time scale

$$(4) \quad l_i(\sigma) = \int_0^\sigma 1_{\{Z_\zeta=i\}} d\zeta, \quad i \in V, \sigma \geq 0.$$

Let $P_{i_0}^{W,\text{vrjp}}$ denote the law of the VRJP started at i_0 . Sabot and Tarrès [4] gave an interpretation of the u -marginal of $\mu_{i_0}^{W,\text{susy}}$ in terms of VRJP:

$$(5) \quad \mathcal{L}_{P_{i_0}^{W,\text{vrjp}}} \left(\lim_{\sigma \rightarrow \infty} \left(\frac{l_i(\sigma)}{l_{i_0}(\sigma)} \right)_{i \in V} \right) = \mathcal{L}_{\mu_{i_0}^{W,\text{susy}}} \left((e^{2u_i})_{i \in V} \right).$$

In the following, we consider two time scales $1 \ll \sigma \ll \sigma'$. We study the VRJP in exchangeable time scale on the event $A_{\sigma,\sigma'}$ that every edge has been crossed in both directions during both time intervals $[0, \sigma]$ and $[\sigma, \sigma + \sigma']$. Since the underlying graph is finite, the probability of this event tends to 1 as $\sigma \rightarrow \infty$.

Motivated by (5), we introduce new variables $v_i(\sigma)$ by

$$l_i(\sigma) = l_{i_0}(\sigma) e^{2v_i(\sigma)}, \quad i \in V.$$

The local time the process Z spends in $i \in V$ in the time interval $[\sigma, \sigma + \sigma']$ equals

$$l'_i(\sigma, \sigma') := \int_\sigma^{\sigma+\sigma'} 1_{\{Z_\zeta=i\}} d\zeta.$$

Similarly to the above, we define $u_i(\sigma, \sigma')$ by

$$(6) \quad l'_i(\sigma, \sigma') = l'_{i_0}(\sigma, \sigma') e^{2u_i(\sigma, \sigma')}, \quad i \in V.$$

For $i \in V$, consider

$$(7) \quad s_i(\sigma, \sigma') = \sqrt{l_{i_0}(\sigma)} (u_i(\sigma, \sigma') - v_i(\sigma)).$$

By (5), $\lim_{\sigma' \rightarrow \infty} u_i(\sigma, \sigma') = u_i$, where $(u_i)_{i \in V}$ is distributed according to the u -marginal of $\mu_{i_0}^{W,\text{susy}}$. Consequently, $\lim_{\sigma' \rightarrow \infty} s_i(\sigma, \sigma')$ describes the fluctuations of $l_i(\sigma)/l_{i_0}(\sigma)$ around this limit u_i on the appropriate scale.

Let $T^{\text{lastexit}}(\sigma, \sigma')$ denote the collection of directed edges taken by the jump process $(Z_\zeta)_{\sigma \leq \zeta \leq \sigma + \sigma'}$ between times σ and $\sigma + \sigma'$. This gives a spanning tree of G . The following convergence result is given in Theorems 1 and 7 in [3].

Theorem. *For the VRJP on a finite graph, the distribution of*

$$(8) \quad (s(\sigma, \sigma'), u(\sigma, \sigma'), T^{\text{lastexit}}(\sigma, \sigma'))$$

with respect to $P_{i_0}^{W, \text{vrjp}}(\cdot \cap A_{\sigma, \sigma'})$ converges weakly as $\min\{\sigma, \sigma' \sigma^{-2}\} \rightarrow \infty$ to the supersymmetric hyperbolic non-linear sigma model $\mu_{i_0}^{W, \text{susy}}$.

Moreover in [3] joint weak convergence of more quantities is shown. There, rescaled crossing numbers of the edges and the positions Z_σ and $Z_{\sigma+\sigma'}$ of the jump process are considered. All quantities are studied simultaneously for both time intervals $[0, \sigma]$ and $[\sigma, \sigma + \sigma']$ asymptotically as $\min\{\sigma, \sigma' \sigma^{-2}\} \rightarrow \infty$.

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Inverted orbits of exclusion processes, diffuse-extensive-amenability and (non-?) amenability of the interval exchanges

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The so-called **IET group** (the group of interval exchange transformations) is a group which arises naturally in dynamical systems for the study of polygonal billiards. It is not known whether it is **amenable** or not. Roughly speaking, the IET group is made of all bijections from the circle \mathbb{R}/\mathbb{Z} to itself which are càdlàg and piecewise translations. As such the IET group shares some common features with the celebrated **Thompson groups** for which the question of amenability (or not) is widely open. In the case of IET, it is believed that the group should be amenable. Recently, Juschenko, Matte Bon, Monod and de la Salle have introduced a promising new method in order to prove the amenability of the IET group. The main motivation which guided this work is to give further insights on this strategy towards amenability (or not) of IET.

Let me give some more details now. One of the (many) equivalent criteria to show that a group is amenable is the celebrated **Kesten's criterion** on the return probabilities of random walks. In the case of $G = \text{IET}$, Juschenko, Matte Bon, Monod and de la Salle introduced a new powerful criterion which is also of probabilistic nature. In order to state their criterion, it is convenient to embed finitely generated subgroups of G into the group of permutations $\mathfrak{S}(\mathbb{Z}^d)$ of the lattice \mathbb{Z}^d . Without entering into details, to any finitely generated subgroup of IET,

$\langle g_1, \dots, g_k \rangle < G$, one can find $d \geq 1$ (called the *rational rank* of $\langle g_1, \dots, g_k \rangle$) and permutations $\sigma_1, \dots, \sigma_k$ of \mathbb{Z}^d so that by the identification $g_i \mapsto \sigma_i$, the subgroup $\langle g_1, \dots, g_k \rangle$ is embedded into $\mathfrak{S}(\mathbb{Z}^d)$. In fact the embedding is even slightly better, it embeds into the so-called **wobbling group** $W(\mathbb{Z}^d) < \mathfrak{S}(\mathbb{Z}^d)$ which is made of permutations with bounded range.

Theorem (Juschenko, Matte Bon, Monod and de la Salle 2015). *The IET group G is amenable if and only if for any $d \geq 1$ and any symmetric probability measure μ supported on the set $\{\sigma_i^\pm\}_{1 \leq i \leq k}$ (where σ_i arise from IET as explained above), then the random walk $\{\tau_n\}_{n \geq 0}$ on the wobbling group $W(\mathbb{Z}^d)$ induced by μ , namely*

$$\begin{cases} \tau_0 := \text{Id} \\ \tau_n := s_n \circ s_{n-1} \circ \dots \circ s_1, \quad s_i \text{ are i.i.d. } \sim \mu \end{cases}$$

has the following property. For any $\varepsilon > 0$, one has for n sufficiently large

$$\mathbb{P}(|O_n| < \varepsilon n) \geq e^{-\varepsilon n},$$

where for each $n \geq 0$, O_n is the **inverted orbit** of the random walk $\{\tau_n\}_{n \geq 0}$, namely the random subset of \mathbb{Z}^d defined by

$$O_n = \{x \in \mathbb{Z}^d, \exists s \leq n, \tau_s(x) = 0\} = \bigcup_{0 \leq s \leq n} \tau_s^{-1}(\{0\}) \subset \mathbb{Z}^d$$

Their criterion is very powerful in the sense that it is very easy to see that if Kesten's criterion holds then their criterion holds as well, but the reverse direction is much more involved to prove. In that sense, if $G = \text{IET}$ happens to be indeed amenable, part of the difficulty has already been settled by their result.

In their work, Juschenko, Matte Bon, Monod and de la Salle asked the following question.

Question. *Is the action of the group $W(\mathbb{Z}^d)$ on \mathbb{Z}^d **extensively amenable**? Or in other words is it true that for ALL symmetric and finitely supported probability measure μ on $W(\mathbb{Z}^d)$, the random walk $\{\tau_n\}_{n \geq 0}$ induced by μ has inverted orbits $\{O_n\}_{n \geq 1}$ which satisfy the above property?*

A positive answer to this question would imply the amenability of IET. Our main contribution is to answer negatively to this question when one relaxes the hypothesis of being finitely supported in $W(\mathbb{Z}^d)$ to the hypotheses of being compactly supported in $W(\mathbb{Z}^d)$ (more precisely in $W_r(\mathbb{Z}^d)$, i.e. permutations whose range is bounded by r):

Theorem. *There exist symmetric probability measures supported on some $W_r(\mathbb{Z}^d)$, for which*

$$\mathbb{E} \left[\left(\frac{1}{2} \right)^{|O_n|} \right] \leq e^{-cn}$$

for some explicit $c > 0$.

The random walks on $W(\mathbb{Z}^d)$ used to prove this result are inspired by the **random stirring process** which was introduced by Balint Toth as a toy model for the phase transition for liquid Helium. This result gives strong support to our following conjecture.

Conjecture. *The action of $W(\mathbb{Z}^d)$ on \mathbb{Z}^d is not extensively amenable.*

Spatial populations with seed-bank

FRANK DEN HOLLANDER

(joint work with Andreas Greven and Margriet Oomen)

We consider a system of interacting Wright-Fisher diffusions with seed-bank. Individuals live in colonies and are subject to resampling and migration as long as they are *active*.

Each colony has a seed-bank into which individuals can retreat to become *dormant*, suspending their resampling and migration until they become active again. As geographic space labeling the colonies we consider \mathbb{Z}^d , $d \geq 1$. Our goal is to *classify* the long-time behaviour of the system in terms of the underlying model parameters. We want to understand the change in behaviour induced by the presence of the seed-bank, in particular, in what way the seed-bank enhances genetic diversity.

When individuals become dormant, they adopt a *random colour* that determines their wake-up time. This allow us to model wake-up times with fat tails, while preserving the Markov property of the evolution. We show that the system of continuum stochastic differential equations, describing the population in the large-colony-size limit, has a unique strong solution, converges to equilibrium, and exhibits a dichotomy of *coexistence* (= multi-type equilibrium) versus *clustering* (= mono-type equilibrium). We also establish the *finite-systems scheme* (= identify how a finite truncation of the system behaves as both the time and the truncation level tend to infinity, properly tuned together). The model has a dual, which plays a crucial role in the analysis.

We find that the seed-bank may change the long-time behaviour not only quantitatively but also qualitatively. In particular, *new universality classes* appear. For instance, both the dichotomy and the scaling in the finite-systems scheme are affected by the seed-bank when the wake-up time has infinite mean.

The Z -Dirac and massive Laplacian operators in the Z -invariant Ising model.

BÉATRICE DE TILIÈRE

We consider Baxter's Z -invariant Ising model. We prove that certain key quantities of the Ising model, *i.e.*, the partition function and probabilities of occurrence of edges in contour configurations, are explicitly expressed as a function of the Z -invariant massive Laplacian and its inverse, the massive Green function, introduced

in [1]. This establishes a deep relation between classical models of statistical mechanics: the Ising model, rooted spanning forests, random walks. In proving these results, we introduce the Z -Dirac operator and relate it to the Z -massive Green function, extending to the full Z -invariant case results proved by Kenyon at criticality [2]. Proofs consist in establishing matrix relations allowing to compare matrix inverses and also, after extra combinatorial work, determinants.

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On the GFF in dimension $d \geq 7$, loop-soups and critical percolation

WENDELIN WERNER

We report on work in progress (related to an ongoing project with Titus Lupu) about the structure of GFF in dimensions $d > 6$. Based on the representation of the discrete GFF via loop-soups on cable systems by Lupu [4], we explain how one can adapt ideas that have been developed by Michael Aizenman [1] in the context of high-dimensional percolation, in order to shed new light on the relation between the Gaussian Free Field and high-dimensional percolation, on the structure of the high-dimensional GFF and maybe also on the lace expansion itself.

For instance, one can see that typical large loop-soup clusters in a large $N \times N$ box of Z^d for $d \geq 7$ will contain no large Brownian loop, so that loop-soup percolation can be viewed in that case as a critical (somewhat spread-out) Poisson percolation model. The features that are known to hold for usual spread-out critical percolation models in $d > 6$ (or ordinary critical percolation in higher dimensions) thanks to the combination of lace expansion based results by Hara, Slade, van der Hofstad (see [3, 2] and the references therein) and Aizenman's arguments from [1], are shown to hold true here as well: When $N \rightarrow \infty$, there are $N^{d-6+o(1)}$ loop-soup clusters with diameter greater than N in this $N \times N$ box, each of which having $N^{4+o(1)}$ points.

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On the two dimensional Ising model with columnar disorder and the continuum limit of random 2 by 2 matrix products

GIAMBATTISTA GIACOMIN

(joint work with Francis Comets and Rafael L. Greenblatt)

In a series of works, in particular [2, 3], B. McCoy and T. Wu presented the solution of a model that now goes under their names: McCoy-Wu model. It is a two dimensional Ising model with *columnar disorder*, that is the interactions are nearest neighbor and the energy can be written as

$$(1) \quad H_N(\sigma) := \sum_{(i,j) \in \Lambda_N} E_2(j) \sigma_{(i,j)} \sigma_{(i,j+1)} + E_1 \sum_{(i,j) \in \Lambda_N} \sigma_{(i,j)} \sigma_{(i+1,j)},$$

with (say) $\Lambda_N = (-N, N)^2 \cap \mathbb{Z}^2$, $\sigma \in \{-1, 1\}^{\Lambda_N}$, E_1 a positive constant and $\{E_2(j)\}_{j \in \mathbb{Z}}$ a sequence of independent identically distributed positive random variables of law \mathbb{P} . Without true loss of generality let us choose for this abstract the parametrization $E_2(j) = E_2 + \Delta \eta(j)$, with E_2 a positive constant, as well as Δ and $\eta(j)$ a centered non degenerate random variable with compact support. In particular, $E_2(j) > 0$ for Δ sufficiently small. The focus is on the free energy (density)

$$(2) \quad F(\beta, \Delta) = \lim_{N \rightarrow \infty} \frac{1}{|\Lambda_N|} \mathbb{E} \log \sum_{\sigma \in \{-1, 1\}^{\Lambda_N}} \exp(\beta H_N(\sigma)),$$

with $\beta > 0$ proportional to one over the temperature. Note that if $\Delta = 0$ the model reduces to the celebrated case solved by L. Onsager in 1944. In particular Onsager found an explicit expression for $F(\beta, 0)$ and showed that $F(\cdot, 0)$ is real analytic except at one (explicit) value of β that we call $\beta_c(0)$. At this point the function fails to be C^2 : in fact $F''(\beta, 0)$ diverges (logarithmically) when β approaches $\beta_c(0)$ from both sides.

McCoy and Wu predicted that for $\Delta > 0$ small:

- (P1) $F(\cdot, \Delta)$ is real analytic except at one value of $\beta = \beta_c(\Delta)$ (explicitly given);
- (P2) at $\beta_c(\Delta)$ the free energy is C^∞ and the non-analyticity is characterized by the asymptotic behavior of the Taylor series of $F(\cdot, \Delta)$ at $\beta_c(\Delta)$.

McCoy and Wu arguments are given for a very special type of disorder, but they claim that the result should hold for *general* disorder.

In [1] we have analyzed the McCoy-Wu arguments from a mathematically rigorous perspective. We are unable at this stage to make any mathematical claim about the McCoy-Wu predictions (P1) and (P2), but we have made mathematically rigorous some of the steps of their analysis (for general disorder distributions).

In order to explain what we have done I briefly outline the steps of the McCoy-Wu argument. The first part of the argument is of an algebraic nature and leads to the remarkable formula

$$(3) \quad F(\beta, \Delta) := \frac{1}{2\pi} \int_0^\pi \mathcal{L}_{\beta, \Delta}(\theta) d\theta + \text{analytic function of } \beta,$$

where $\mathcal{L}_{\beta, \Delta}(\theta)$ is the Lyapunov exponent of the product of random matrices of the form

$$(4) \quad M_{\beta, \Delta}(\theta) := \begin{pmatrix} 1 & \frac{a}{a^2 + b^2} \\ \frac{a\lambda}{a^2 + b^2} & \frac{\lambda}{a^2 + b^2} \end{pmatrix},$$

with

$$(5) \quad a(\theta) = -2z_1 \frac{\sin(\theta)}{|1 + z_1 \exp(i\theta)|^2} \quad \text{and} \quad b(\theta) = \frac{1 - z_1^2}{|1 + z_1 \exp(i\theta)|^2},$$

where $z_1 = \tanh(\beta E_1)$, $z_2(j) = \tanh(\beta E_2(j))$ and $\lambda = \lambda(j) = z_2^2(j)$. Note that the randomness enters only via λ and that these matrices have (strictly) positive entries, except when $\sin(\theta) = 0$. Applying the results in [4] one obtains that this positivity implies that for every $c \in (0, \pi/2)$

$$(6) \quad \beta \mapsto \frac{1}{2\pi} \int_c^{\pi-c} \mathcal{L}_{\beta, \Delta}(\theta) d\theta,$$

is analytic. A different argument still relying on [4] yields analyticity also if $\pi - c$ is replaced by π . So the problem is reduced to analyzing the map

$$(7) \quad \beta \mapsto \frac{1}{2\pi} \int_0^\Delta \mathcal{L}_{\beta, \Delta}(\theta) d\theta,$$

where we have (arbitrarily, at this stage) chosen c equal to the small constant Δ .

In spite of its explicit form, the analysis of the function (7) is not an easy matter. But if we could replace $\theta \in (0, \Delta)$ with $\theta = 0$ the problem would become trivial: $M_{\beta, \Delta}(0)$ is diagonal and one computes $\mathcal{L}_{\beta, \Delta}(0)$ by applying the Law of Large Numbers:

$$(8) \quad \mathcal{L}_{\beta, \Delta}(0) = \max(0, \mathbb{E} \log(\lambda/b^2(0))).$$

Therefore $\beta \mapsto \mathcal{L}_{\beta, \Delta}(0)$ is analytic except for β such that $\mathbb{E} \log(\lambda/b^2(0)) = 0$: this defines $\beta_c(\Delta)$ and this is the way McCoy and Wu guess the value of $\beta_c(\Delta)$.

Having observed this, they pursue a finer analysis for β close to $\beta_c(\Delta)$. From a mathematical standpoint the problem splits into two subproblems:

- Showing analyticity for $\beta \neq \beta_c(\Delta)$: the analyticity argument based on the positivity of the entries fails to yield quantitative estimates on the radius of convergence of $\mathcal{L}_{\beta, \Delta}(\theta)$ around $\beta \neq \beta_c(\Delta)$. This radius goes to zero, while the result for $\theta = 0$ suggests that this may not be the case.
- The opposite should be true for $\beta = \beta_c(\Delta)$: analyticity is lacking for $\theta = 0$ and one expects that the radius of convergence goes to zero when $\theta \searrow 0$ this time, and sufficiently fast so that (7) is not analytic.

In order to substantiate their claims McCoy and Wu then make the change of variables $\beta = \beta_c(\Delta) + \alpha\Delta$, $\alpha \in \mathbb{R}$, which (since Δ is small) is a way of focusing on a neighbor of $\beta_c(\Delta)$. They claim that

$$(9) \quad \lim_{\Delta \searrow 0} \frac{1}{\Delta^2} \int_0^\Delta \mathcal{L}_{\beta_c(\Delta) + \alpha\Delta, \Delta}(\theta) d\theta = C \int_0^\eta x \frac{K_{\alpha-1}(x)}{K_\alpha(x)} dx,$$

where C and η are explicit positive constants (not depending on α) and $K_\alpha(x)$ is the modified Bessel function of 2nd kind of index $\alpha \in \mathbb{C}$ and argument $x > 0$

$$(10) \quad K_\alpha(x) := \frac{1}{2} \int_0^\infty \frac{1}{y^{1+\alpha}} \exp\left(-\frac{x}{2} \left(y + \frac{1}{y}\right)\right) dy.$$

In [1] we have proven (9) for a general class of disorder distributions. Then McCoy and Wu study the application

$$(11) \quad \alpha \mapsto \int_0^\eta x \frac{K_{\alpha-1}(x)}{K_\alpha(x)} dx,$$

and they claim that it is analytic for $\alpha \in \mathbb{R} \setminus \{0\}$ and that it is C^∞ but not analytic at $\alpha = 0$. They do so by replacing the ratio of the Bessel functions by their *leading order* expansion terms when both x and α are small: the resulting expression can be explicitly integrated and it yields a formula in terms of a polygamma function. The rest of the argument is straightforward analysis. We have not been able to justify this procedure and we have taken a different approach, based on precise asymptotic estimates on the locations on the zeros of $K_\alpha(x)$: in fact $\alpha \mapsto K_\alpha(x)$ is an entire function for every $x > 0$ and the origin of the non-analyticity is the fact that the zeros of $K_\alpha(x)$ approach the origin (from the imaginary axis) as $x \searrow 0$. This way we have been able to show analyticity of (11) out of $\alpha = 0$ and we have proven that the radius of convergence of the Taylor series at the origin $\sum_{n=0}^\infty c_n \alpha^n$ is zero: in fact $c_1 = 4\eta$, $c_{2n+1} = 0$ for every $n \in \mathbb{N}$ and the even coefficients satisfy

$$(12) \quad c_{2n} \stackrel{n \rightarrow \infty}{\sim} 4 e^{-\gamma} (-1)^{n+1} \frac{(2n-1)!}{\pi^{2n}},$$

with γ the Euler-Mascheroni constant.

In a nutshell, we have confirmed the analysis of McCoy and Wu on what arises in the limit $\Delta \searrow 0$, but (P1) and (P2) are claims for Δ (possibly extremely) small, but not zero.

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