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Free Probability Theory

Organised by
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Abstract. The workhop brought together leading experts, as well as promising young researchers, in areas related to recent developments in free probability theory. Some particular emphasis was on the relation of free probability with random matrix theory.

Mathematics Subject Classification (2010): 46L54.

Introduction by the Organisers

The workshop *Free Probability Theory*, organised by Alice Guionnet (Lyon), Roland Speicher (Saarbrücken), and Dan Voiculescu (Berkeley), was well attended with over 50 participants with broad geographic representation.

Free probability theory is a line of research which parallels aspects of classical probability, in a non-commutative context where tensor products are replaced by free products, and independent random variables are replaced by free random variables. It grew out from attempts to solve some longstanding problems about von Neumann algebras of free groups. In the almost 35 years since its creation, free probability has become a subject in its own right, with connections to several other parts of mathematics: operator algebras, the theory of random matrices, classical probability, the theory of large deviations, and algebraic combinatorics. Free probability also has connections with some mathematical models in theoretical physics and quantum information theory, as well as applications in statistics and wireless communications.

Free probability is certainly a very active area, with many unsolved problems ahead, as well as various recent new exciting developments. The Oberwolfach workshop brought together various mathematical backgrounds and was strong on the connections of free probability with other fields, with particular emphasis on the random matrix perspective. The diversity of the participants and the ample free time left in the programme stimulated a lot of fruitful discussions.

The programme consisted of 23 lectures of 40 minutes. Because of the various backgrounds of the participants much emphasis was put on making the lectures accessible to a broad audience; most of them provided a survey on the background as well as highlighting some recent developments in connection with free probability.

Instead of going into more detail we will let the following abstracts speak for themselves.

On behalf of all participants, the organizers would like to thank the staff and the director of the Mathematisches Forschungsinstitut Oberwolfach for providing such a stimulating and inspiring atmosphere, and for taking care of all local arrangements with extreme efficiency.

1. Special Activities

In addition to the regular talks we also scheduled three sessions of 10 minutes announcements of research results. This was mainly, but not exclusively, intended for young researchers, who could so give an idea of their work to the general audience; quite often those announcement resulted in more in depth discussions in small groups afterwards.

1.1. **List of 10 minutes research announcements.**

- Akemann: Orthogonal polynomials in the complex plane
- Augeri: Large deviations of non-commutative polynomials
- Cebron: Existence of free Stein kernels
- Cook: Limiting spectral distribution for polynomials of Ginibre matrices
- Huang: Non-intersecting random walks and quantized free convolution
- Jekel: Entropy and transport for free Gibbs laws given by convec potentials
- Lehner: Quadratic and other forms
- Levy: About Collins & Sniady's integration formula
- Maurel-Segala: Matricial cumulants, applications to limit of families of classical matrices and kernel matrices
- Nelson: Free products of finite-dimensional von Neumann algebras and free Araki-Woods factors
- Pluma: Multi-variable SYK model
- Skoufranis: Bi-free entropy
- Tarnowski: Spectral universality of the input-output Jacobians in residual neural networks
- Yin: Regularity property of noncommutative distributions: atoms and Atiyah property

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Workshop: Free Probability Theory

Table of Contents

Dan–Virgil Voiculescu
Noncommutative extreme values based on the Ando-max 7

Charles Bordenave (joint with Hubert Lacoin)
Mixing time of covered random walks 8

Frédéric Patras (joint with Kurusch Ebrahimi-Fard)
Shuffle groups in free probability 11

Takahiro Hasebe
Stable distributions in classical, free and Boolean probabilities 14

Dimitri Shlyakhtenko (joint with Sorin Popa)
Representing interpolated free group factors as group factors 15

Mireille Capitaine
Fluctuations of outliers for Hermitian polynomials in a Wigner matrix and a spiked deterministic matrix, and operator-valued subordination property 17

Mylène Maïda (joint with Alice Guionnet)
Large deviations for the largest eigenvalue of the addition of random matrices 20

Ken Dykema (joint with Fedor Sukochev, Dmitriy Zanin)
Free perturbations, mild operators and invariant subspaces 23

Gaëtan Borot (joint with Elba Garcia Failde)
Simple maps, free cumulants and topological recursion 26

Tobias Mai (joint with Marwa Banna, Roland Speicher, Moritz Weber, Sheng Yin)
Regularity properties of noncommutative distributions 31

Vadim Gorin (joint with Yi Sun)
Macroscopic fluctuations for products of random matrices 34

Moritz Weber (joint with Alexander Mang)
Unitary half-liberations 37

Philippe Biane (joint with Matthieu Josuat-Vergès)
Interval partitions and noncrossing partitions 41

Antti Knowles (joint with Yukun He, Matteo Marcozzi)
Local law and complete eigenvector delocalization for supercritical Erdős–Rényi graphs 42

Maciej A. Nowak (joint with Jacek Grela)
Extreme values versus extreme random matrices 45

Marwa Banna (joint with Guillaume Cébron)
Operator-valued matrices: the noncommutative Lindeberg method 46

Friedrich Götze (joint with Gennadii Chistyakov)
Representations of Free Commutators 48

James A. Mingo
Infinitesimal Freeness and the GOE 49

Ion Nechita, Alexandru Nica
On the number of components of random meandric systems 53

Nikhil Srivastava (joint with Shirshendu Ganguly)
Delocalization of Eigenvectors of High Girth Graphs 54

Camille Male (joint with B. Au, G. Cébron, A. Dahlqvist, F. Gabriel)
Traffic Independence implies Freeness over the Diagonal 57

Octavio Arizmendi (joint with Mauricio Salazar and Jiunchau Wang)
Berry-Esseen theorems for Boolean and Monotone convolutions 59

Serban T. Belinschi
Invariant projections for sums of random variables free over an Abelian algebra 61

Abstracts

Noncommutative extreme values based on the Ando-max

Dan–Virgil Voiculescu

The beginning of the study of extreme values in free probability is in my joint work with G. Ben Arous [1]. We had found that the appropriate notion of max to be used was the max with respect to the Ando spectral order and we found the univariate free max stable laws, a Bercovici-Pata type correspondence between free and classical max-stable laws based on coincidence of domains of attraction and the construction of free extremal projection-valued processes over a set, related to free Poisson processes.

Recently in joint work with J. Garza -Vargas [2] we found that also with respect to the non-commutative Boolean independence basic objects for extreme value theory exist, provided one restricts considerations to positive noncommutative random variables. We showed that the Boolean max-stable laws are a class of Dagum distributions and that there is also a Bercovici-Pata type correspondence with classical Frechet laws.

A third context, where such studies can be carried out, is the bi-free probability framework [6] and where I had found the max-convolution formula for bivariate cumulative distribution functions [7], which reduces the study in one of the simplest cases to a classical analysis problem. Very recently H. W. Huang and J. C. Wang [5] have carried this investigation one step further and have found the bi-free bivariate max-stable laws.

The study of non-commutative extreme values based on the spectral order for various types of independence is at an early stage and there are many open problems and things to wonder about. I would like to mention two kinds of questions that arise naturally. On one hand, since classical extreme value theory has been highly successful in applications one should wonder whether similar applications for the noncommutative extreme value results can be found (I sometimes refer jokingly to these questions, " a free dam for a free flooding in free Amsterdam ...."). Other questions are about the correspondences between classical and free. Besides Bercovici-Pata type results there are also Hasebe-Kuznetsov factorization correspondences in work of T. Hasebe, T. Simon and M.Wang [4] and it is quite intriguing whether there isn't more structure around these correspondences. Free max-stable laws coincide with the limit laws in peaks over thresholds and J. Grela and M. Nowak [3] gave an interpretation of this coincidence. It is natural to wonder whether the bi-free max-stable found by Huang and Wang have also some classical role or to try to find other ways to understand the occurrence of Dagum distributions as Boolean max-stable laws.

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Mixing time of covered random walks

Charles Bordenave

(joint work with Hubert Lacoin)

We connect the mixing properties of random walks on a finite regular graph to the strong convergence of certains Markovian matrices.

Minimal mixing time for the simple random walk. Let $3 \le d \le n-1$ be integers with nd even and let $G_n = (V_n, E_n)$ be a finite simple d-regular graph on a vertex set V_n of size $\#V_n = n$. Let $(X_t)_{t\ge 0}$ be the *simple random walk* on G_n, which is the Markov process with transition matrix, for all $x, y \in V_n$

$$P_n(x,y) = \frac{\mathbf{1}_{\{\{x,y\}\in E_n\}}}{d}.$$

The uniform measure on V_n denoted by π_n is reversible for the process. Furthermore if G_n is connected, then π_n is the unique invariant measure of P_n, G_n is not bipartite, then $P_n^t(x,\cdot)$ converges to π_n when t tends to infinity.

We are interested in estimating the time at which $P_n^t(x,\cdot)$ falls in a close neighborhood of π_n in terms of the *total variation* distance. More formally, the *total variation mixing time* associated with threshhold $\varepsilon \in (0,1)$ and initial condition $x \in V_n$, is defined by

$$T_n^{\mathrm{mix}}(x,\varepsilon) := \inf\{t \in \mathbb{N} \;\; d_n(x,t) < \varepsilon\},$$

where $d_n(x,t)$ is the *total variation* distance to equilibrium

$$d_n(x,t) := \|P_n^t(x,\cdot) - \pi_n\|_{\mathrm{TV}} = \frac{1}{2}\sum_{y\in V_n} \left|P_n^t(x,y) - \pi_n(y)\right| = \max_{A\subset V_n}\left\{P_n^t(x,A) - \pi_n(A)\right\}. \tag{1}$$

The *worst-case mixing time* is classically defined as

$$T_n^{\mathrm{mix}}(\varepsilon) = \max_{x\in V_n} T_n^{\mathrm{mix}}(x,\varepsilon).$$

The mixing properties for the random walk are intimately related to the spectrum of P_n. An illustration of this is the classical computation based on the spectral decomposition of P_n (see [8, Lemma 12.16]) allows to control the distance in function of the spectral radius of P_n projected onto the orthogonal of π_n: for all $x \in V_n$,

$$d_n(x,t) \leq \frac{\sqrt{n-1}}{2}\rho_n^t. \tag{2}$$

where

$$\rho_n := \max_{\lambda \in \mathrm{Sp}(P_n)\setminus\{1\}} |\lambda|. \tag{3}$$

This yields in particular that

$$T_n^{\mathrm{mix}}(\varepsilon) \leq \frac{1}{|\log \rho_n|}\left(\frac{1}{2}\log n - \log(2\varepsilon)\right). \tag{4}$$

The following improvement has been proved in [9] (see also [7]):

Theorem 1 ([9])**.** *Let $d \geq 3$ be an integer and let (G_n) be a sequence of connected d-regular graphs on n vertices such that their spectral radius satisfy* $\lim_{n\to\infty} \rho_n = \rho = 2\sqrt{d-1}/d$*. Then for any $\varepsilon \in (0,1)$, we have*

$$\lim_{n\to\infty} \frac{T_n^{\mathrm{mix}}(\varepsilon)}{\log n} = \frac{d}{(d-2)\log(d-1)}. \tag{5}$$

Theorem 1 is an illustration of the *cutoff phenomenon*. We refer to [5, 8] for an introduction and to [2] for an alternative characterization of cutoff.

The principal aim of this talk is to obtain a better understanding of this phenomenon via bringing the question to a larger setup.

Minimal mixing time for the anisotropic random walk. A first possible extension is to consider a random walk with biased directions. We consider an involution $* : i \mapsto i^*$ of $[d] = \{1, \dots, d\}$. We make the assumption that G_n is a *Schreier graph*: the graph may have loops or multiple edges and we assume that its adjacency matrix may be written as

$$\sum_{i=1}^{d} S_i, \tag{6}$$

where, for each i, S_i is a permutation matrix of a permutation σ_i on V_n and $\sigma_{i^*} = \sigma_i^{-1}$.

Now given a Schreier graph G_n with $\#V_n = n$ and given a probability vector $\mathbf{p} = (p_1, \dots, p_d)$ (with positive coordinates summing to one) such that

$$p_i > 0 \quad \text{and} \quad p_{i^*} = p_i, \qquad \text{for all } i \in [d], \tag{7}$$

we consider the matrix

$$P_{n,\mathbf{p}} = \sum_{i=1}^{d} p_i S_i. \tag{8}$$

Note that by construction $P_{n,\mathbf{p}}$ is a symmetric Markovian matrix. This is the transition kernel of a random walk on G_n which is called the *anisotropic random walk*. Again, π_n, the uniform measure on V_n, is reversible for this process. The spectral radius of $P_{n,\mathbf{p}}$ projected on the orthogonal of π_n is

$$\rho_{n,\mathbf{p}} := \max_{\lambda \in \mathrm{Sp}(P_{n,\mathbf{p}}) \setminus \{1\}} |\lambda|. \tag{9}$$

From what precedes, we may also define the anisotropic random walk on the infinite d-regular tree $\mathcal{T}_d$ with probability vector $\mathbf{p} = (p_1, \dots, p_d)$. We denote by $\mathcal{P}_\mathbf{p}$ its transition kernel. From [6, 4], the Alon-Bopanna lower bound for the spectral radius of $P_{n,\mathbf{p}}$ is

$$\rho_{n,\mathbf{p}} \geq (1 + o(1))\rho_\mathbf{p}, \tag{10}$$

where $\rho_\mathbf{p}$ is the spectral radius of $\mathcal{P}_\mathbf{p}$, given by the classical Akemann-Ostrand formula [1].

The mixing time of the random walk admits a minimal asymptotic value. Consider $(\mathcal{X}_t)_{t \geq 0}$ an anisotropic random walk on $\mathcal{T}_d$ with transition kernel $\mathcal{P}_\mathbf{p}$ and starting from the root of $\mathcal{T}_d$ denoted by e. The *entropy rate* $\mathfrak{h}(\mathbf{p})$ of $\mathcal{P}_\mathbf{p}$ is defined as

$$\mathfrak{h}(\mathbf{p}) := \lim_{t \to \infty} -\frac{1}{t} \sum_{x \in \mathcal{T}_d} \mathcal{P}^t_\mathbf{p}(e, x) \log \mathcal{P}^t_\mathbf{p}(e, x), \tag{11}$$

We have that for any fixed $\varepsilon \in (0, 1)$, uniformly in $x \in V_n$,

$$T^{\mathrm{mix}}_{n,\mathbf{p}}(x, 1 - \varepsilon) \geq (1 + o(1)) \frac{\log n}{\mathfrak{h}(\mathbf{p})}. \tag{12}$$

In the spirit of Theorem 1, we have the following result.

Theorem 2. *Let $d \geq 3$ be an integer, $*$ an involution on $[d]$ and let $\mathbf{p}$ be a probability vector on $[d]$ which satisfies the condition (7). Then, there exists another probability vector $\mathbf{p}'$ which satisfies the condition (7) such that the following holds. If a sequence of connected Schreier graphs G_n on n vertices as in (6) satisfies*

$$\lim_{n \to \infty} \rho_{n,\mathbf{p}'} = \rho_{\mathbf{p}'}, \tag{13}$$

then for every $\varepsilon \in (0, 1)$

$$\lim_{n \to \infty} \frac{T^{\mathrm{mix}}_{n,\mathbf{p}}(\varepsilon)}{\log n} = \frac{1}{\mathfrak{h}(\mathbf{p})}. \tag{14}$$

The condition (13) can be thought as a Ramanujan property for the anisotropic random walk with probability $\mathbf{p}'$. In some cases, this condition (13) can be relaxed to allow $n^{o(1)}$ eigenvalues outside the interval $[-\rho_{\mathbf{p}'}, \rho_{\mathbf{p}'}]$. An expression for the vector $\mathbf{p}'$ is provided in the proof. In particular we have that $\mathbf{p}' = \mathbf{p}$ if and only if $\mathbf{p}$ is the uniform vector. This result is thus a generalization of Theorem 1.

For $\mathbf{p}$ different from the uniform vector, a source of example for Theorem 2 is in [3]. Up to the involution, we consider independent permutations σ_i on $[n]$ vertices: if $i \neq i^*$, σ_i is a uniform permutation on n elements and, if $i^* = i$, we take n even and σ_i is a uniform matching on n elements (where a matching is an involution without fixed point). Then, in probability, the condition (13) is true for any probability vector $\mathbf{p}'$ which satisfies the condition (7).

In this talk, we will also discuss other extensions of this type of results which applies notably to Markov chains covered by a random walk on a non-amenable group.

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Shuffle groups in free probability

Frédéric Patras
(joint work with Kurusch Ebrahimi-Fard)

Commutative shuffle products are known to be intimately related to universal formulas for products as well as exponentials and logarithms in group theory and in the theory of free Lie algebras. Familiar examples are the Baker–Campbell–Hausdorff (BCH) formula or the analytic expression of a Lie group law in exponential coordinates in the neighbourhood of the identity [16].

Non-commutative shuffle (aka dendriform) products appear classically in several areas, including algebraic topology [8], the non-commutative representation theory of the symmetric group (e.g. through the shuffle product naturally defined on the direct sum of the symmetric group algebras [12, 15]), algebraic combinatorics (where many combinatorial Hopf algebras can be shown to carry a shuffle product) and the theory of operads (through the associated brace algebra structures) [9].

Gerstenhaber and Voronov's notion of brace operations [10] provides actually a way to understand why non-commutative shuffle products appear naturally in Voiculescu's free probability theory [18]. Let us recall that a brace algebra is a vector space V equipped with a map: $\{-;-\}$ from $V \otimes T(V)$ to V, where $T(V)$ denotes the tensor algebra with unit $\mathbf{1}$ over the vector space V, such that:

- $\forall v \in V$, $\{v;\mathbf{1}\} = v$.
- $\forall v, y_1, \ldots, y_k \in V$, $\forall w = v_1 \cdots v_n \in T(V)$:

$$\{\{v; y_1 \cdots y_k\}; w\} = \sum_{w=w_1\cdots w_{2k+1}} \{v; w_1\{y_1; w_2\}w_3 \cdots w_{2k-1}\{y_k; w_{2k}\}w_{2k+1}\}.$$

The last sum runs over all decompositions of the word $w \in T(V)$ as a concatenation product of (possibly empty) subwords. An associative product $*$ on $T(V)$ is then obtained from the brace operations as follows (see, e.g., [2] for details). Given a non-empty word $w \in T(V)$: $\forall v_1, \ldots, v_k \in V$

$$v_1 \cdots v_k * w := \sum_{w=w_1\cdots w_{2k+1}} w_1\{v_1; w_2\}w_3 \cdots w_{2k-1}\{v_k; w_{2k}\}w_{2k+1}.$$

This product is a non-commutative shuffle (aka dendriform) product, that means, it splits into two half-products ($* = \prec + \succ$):

$$v_1 \cdots v_k \prec w := \sum_{w=w_1\cdots w_{2k}} \{v_1; w_1\}w_2 \cdots w_{2k-2}\{v_k; w_{2k-1}\}w_{2k},$$

$$v_1 \cdots v_k \succ w := \sum_{\substack{w=w_1\cdots w_{2k+1}\\ w_1\neq\emptyset}} w_1\{v_1; w_2\}w_3 \cdots w_{2k-1}\{v_k; w_{2k}\}w_{2k+1}$$

that satisfy the Eilenberg–MacLane shuffle (aka dendriform) relations:

$$(x \prec y) \prec z = x \prec (y * z)$$
$$x \succ (y \prec z) = (x \succ y) \prec z$$
$$x \succ (y \succ z) = (x * y) \succ z.$$

It turns out that these shuffle structures can be defined in the context of free probability. Indeed, when suitably interpreted in terms of (dual) brace operations, Speicher's original ansatz for the definition of free cumulants [13] can be shown to give rise to such structures on $T(T(A))^*$, i.e., the dual of the double tensor algebra over a probability space (A, φ), where A is an associative algebra and φ is a linear unital map. On $T(T(A))$ this process defines in particular a graded connected non-commutative and non-cocommutative Hopf algebra structure [3, 4]. The group G of characters and the Lie algebra g of infinitesimal characters on this Hopf algebra can then be shown to be related by three exponential-type maps, $\mathcal{E}_\prec$, $\mathcal{E}_\succ$, $\mathcal{E}_*\colon g \to G$ and corresponding logarithms, $\mathcal{L}_\prec$, $\mathcal{L}_\succ$, $\mathcal{L}_*\colon G \to g$, that are naturally defined. This construction holds more generally for all bialgebras arising from brace algebra structures and leads to a theory of "shuffle groups" with the setting of free probability as a natural area of application [5].

The starting point are the moment-cumulant relations, which are naturally expressed in terms of the shuffle algebra structure on $T(T(A))^*$. Given a character $\Phi \in G$, which is induced by the linear unital map $\varphi\colon A \to \mathbb{K}$, one can show that the three logarithms applied to Φ encode the three families of free, boolean and monotone cumulants. In other words, one can compute three infinitesimal characters ρ, κ and β such that

$$\Phi = \mathcal{E}_*(\rho) = \mathcal{E}_\prec(\kappa) = \mathcal{E}_\succ(\beta).$$

For instance, applying $\Phi = \mathcal{E}_\prec(\kappa)$ to a word in $w \in T(A) \subset T(T(A))$ expresses the multivariate moment $\Phi(w)$ in terms of free cumulants. Indeed, from the algebraic computation of $\mathcal{E}_\prec(\kappa)(w)$ one recovers the free cumulant expansion of $\Phi(w)$ expressed in terms of non-crossing set partitions [14]. Analog results hold for the monotone [11] and boolean cases [17]. The combinatorial relations between these three families of cumulants [1], classically obtained by Möbius inversion techniques, can be recovered from this purely algebraic point of view [6]. For instance, the relation between free and boolean cumulants is given through

$$\kappa = \mathcal{L}_\prec \circ \mathcal{E}_\succ(\beta), \quad \beta = \mathcal{L}_\succ \circ \mathcal{E}_\prec(\kappa).$$

The existence of three exponential/logarithmic bijections between groups and Lie algebras results also in several new formal group laws together with new operations on the group and the Lie algebra, generalizing the BCH group law. For example a new notion of shuffle adjoint action particularly well fitted to the new theory: indeed, it turns out that the aforementioned relations between cumulants can also be expressed in terms of the shuffle adjoint action. For instance, one has

$$\beta = \Phi^{-1} \succ \kappa \prec \Phi.$$

Again, through the purely algebraic computation of $(\Phi^{-1} \succ \kappa \prec \Phi)(w)$ for $w \in T(A)$ one recovers the result from [1], giving the expansion of the multivariate boolean cumulant $\beta(w)$ in terms of free cumulants and irreducible non-crossing set partitions. The shuffle adjoint action gives rise to further applications, including, among others, new algebraic approaches to

- Universal products [5]
- Additive free, monotone, and boolean convolutions [5]
- Subordination products [5]
- The Bercovici–Pata correspondence [5]
- Conditionally free probability [7].

We close this short summary by noting that following [2] one can show that the shuffle adjoint action can be naturally expressed in terms of brace operations.

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Stable distributions in classical, free and Boolean probabilities

Takahiro Hasebe

In classical probability theory central limit theorem was extended to the case where random variables are iid but not necessarily of finite variance or finite mean. The possible limit distributions of affine transformations of the sums of such iid random variables form the set of stable distributions. A lot of properties of stable laws (in one dimension) are collected in [7]. Stable distributions have also been defined in free/Boolean probabilities by Bercovici-Voiculescu/Speicher-Woroudi. The main purpose of this talk is to overview known results on all these stable distributions.

Philippe Biane showed several properties of free stable distributions, including descriptions of the density, unimodality and the Zolotarev duality. Demni pointed out that the density of positive stable laws can be written in terms of a function introduced by Kanter in 1975 [4]. Hasebe and Kuznetsov proved a factorization of a classical stable r.v. into a free stable one and a power of a gamma r.v. [5] as an independent product. Arizmendi and Hasebe proved several convolution identities involving classical, free and Boolean stable laws [1]. Arizmendi and Hasebe pointed out that the exponential of some free stable r.v. has the Dykema-Haagerup distribution [2]. This is a singular value distribution of upper-triangular random matrices with independent complex Gaussian entries, but there is no interpretation or explanation about why such a limiting distribution is related to a free stable distribution.

There are also identities between stable distributions and extreme value distributions [6], the latter being classified into three types: Frechet, Weibull and Gumbel. It turns out that a classical Frechet r.v. factorizes into a free Frechet r.v. and a power of a gamma r.v. as an independent product. This gamma r.v. also appears in the stable case as mentioned above, but there is no further interpretation of this co-appearance of gamma random variables. A similar mysterious connection also exists between Boolean and classical worlds. To summarize, there are many mysterious identities, which still lack reasonable interpretations.

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Representing interpolated free group factors as group factors

Dimitri Shlyakhtenko

(joint work with Sorin Popa)

Let (M, τ) be a tracial von Neumann algebra (so that $\tau : M \to \mathbb{C}$ is a normal tracial state), and assume that M is a factor (i.e., the center of M is trivial). It is a classical result of Murray and von Neumann [MvN36, MvN43] that for any integer n and any projection p in the algebra $M_n(M)$ of $n \times n$ matrices with entries from M, the isomorphism class of $pM_n(M)p$ depends only on the value t of the non-normalized trace of p computed in $M_n(M)$. The resulting isomorphism class is denoted by M_t.

For each integer n, denote by $\mathbb{F}_n$ the free group on n generators, and let $L(\mathbb{F}_n)$ be the associated group factor. Using random matrix techniques, Voiculescu has showed in [Vo89] that

$$(L(\mathbb{F}_n))_t \cong (L(\mathbb{F}_m))_s \tag{1}$$

whenever $s, t \in (0, +\infty)$ and $n, m \in \{2, 3, \ldots, +\infty\}$ satisfy

$$t^{-2}(n-1) = s^{-2}(m-1). \tag{2}$$

Following his work, Dykema and Rădulescu introduced *interpolated free group factors* [Dyk94, Rad94, Rad92]: for each $\alpha \in (1, +\infty]$ they defined an isomorphism class $L(\mathbb{F}_\alpha)$ so as to make equations (1) and (2) true for non-integer values of n and m (thus, e.g., $L(\mathbb{F}_\alpha) = L(\mathbb{F}_2)_{1/\sqrt{\alpha-1}}$).

The interpolated free group factors are a very natural class of algebras. For example, one has the following remarkable result due to Dykema [Dyk93]: any factorial free product of amenable (in particular, finite dimensional) algebras is an interpolated free group factor.

A number of intriguing questions about interpolated free group factors remain. The most famous one is whether or not they are isomorphic for different values of α. Results of Dykema and Rădulescu [DR00] imply that there is a dichotomy: either $L(\mathbb{F}_\alpha) \not\cong L(\mathbb{F}_\beta)$ for all $\alpha \neq \beta$, or they are isomorphic for all values of α and β. Another is the extension of formulas (1)–(2) to arbitrary index subfactors: presumably, if $N \subset M \cong L(\mathbb{F}_\alpha)$ is a subfactor of index $\lambda = [M : N]$, then $N \cong L(\mathbb{F}_\beta)$ with $\beta = \lambda(\alpha - 1) + 1$. This is open even in index 2.

Another question left open is whether interpolated free group factors are actually themselves group factors: for each α, is there a group Γ_α so that $L(\Gamma_\alpha) = L(\mathbb{F}_\alpha)$? Of course, this is true for integer α. In [HV93], de la Harpe and Voiculescu conjectured that group factors associated to lattices in $PSL_2(\mathbb{R})$ are also interpolated free group factors. This conjecture is only known for certain groups admitting a free product decomposition (e.g., $L(PSL(2,\mathbb{Z})) \cong L(\mathbb{F}_{7/6})$). It is worth mentioning that by the results of [IPV10] there exist groups G with the property that for any $t \neq 1$, $L(G)_t$ is not isomorphic to the factor of any group.

Our main result is:

Theorem 1 ([PS18]). *For any $\alpha \in (1, +\infty]$, there exists a discrete group Γ_α so that $L(\Gamma_\alpha) \cong L(\mathbb{F}_\alpha)$.*

Sketch of proof. We first construct the group Γ_α. Let S_n be the permutation group on n letters, and let $S_\infty = \bigcup_n S_n$. The von Neumann algebra $R = L(S_\infty)$ is the hyperfinite II_1 factor.

Given $\alpha \in (1, +\infty]$ choose integers $k_0, k_1, \ldots$ so that

$$\alpha = 1 + \sum_{n\geq 0} \frac{k_n}{n!}.$$

Let $H_n = S_n \times \mathbb{F}_{k_n}$ and define recursively

$$G_0 = H_0, \qquad G_{n+1} = G_n *_{S_n} H_{n+1}.$$

Finally, let

$$\Gamma_\alpha = \bigcup_{n\geq 0} G_n.$$

To prove our theorem, we show that $L(\Gamma_\alpha) \cong L(\mathbb{F}_\alpha)$.

The key step in the proof is to represent $L(\Gamma_\alpha)$ as the von Neumann algebra generated by the hyperfinite II_1 factor $R = L(S_\infty)$ and a family of R-valued semicircular variables $\{X_j^{(n)} : n = 1, 2, \ldots,\ j = 1, 2, \ldots, k_n\}$. These variables are defined by requiring that they be free with amalgamation over R and that $X_j^{(n)}$ has as its covariance the completely positive map $\eta : R \to R$ given by conditional expectation onto $L(S_n) \subset R$. It turns out that $W^*(R, X_j^{(n)}) \cong L(\mathbb{F}_{1+|Q_n|^{-1}})$ [Shl99], and this allows us to conclude the proof. □

The construction of the groups G_n is inspired by the work of G. Hjorth [Hjo06] on treeable equivalence relations. He proved that any treeable ergodic equivalence relation of cost α can be generated by a free ergodic action of a certain group, whose construction is very close to that of our group Γ (Hjorth's construction is similar to ours with the groups S_n replaced by the abelian groups $(\mathbb{Z}/2\mathbb{Z})^n$). Our group Γ_α has a similar property: any treeable ergodic equivalence relation of cost α can also be generated by an action of Γ_α.

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Fluctuations of outliers for Hermitian polynomials in a Wigner matrix and a spiked deterministic matrix, and operator-valued subordination property

Mireille Capitaine

Let $W_N = (W_{ij})_{1\leq i,j\leq N}$ be a $N \times N$ Hermitian Wigner matrix such that the random variables $\{W_{ii}, \sqrt{2}\mathcal{R}W_{ij}, \sqrt{2}\mathcal{I}W_{ij}\}_{1\leq i<j}$ are independent identically distributed with law μ, μ is a centered distribution, with variance 1, and satisfies a Poincaré inequality.
Let us consider a deterministic real diagonal matrix A_N:

$$A_N = \mathrm{diag}(\theta, A_{N-1})$$

where $\theta \in \mathbb{R}$ is independent of N and A_{N-1} is a $N-1 \times N-1$ deterministic diagonal matrix. We assume that $A_{N-1} \in (M_{N-1}(\mathbb{C}), \frac{1}{N-1}\operatorname{Tr})$ converges strongly in distribution towards a non commutative random variable a in some $\mathcal{W}^*$-probability space $(\mathcal{A}, \phi)$, with ϕ faithful. Note that this implies that, for all large N, all the eigenvalues of A_{N-1} are in any small neighborhood of the spectrum of a. We assume that θ does not belong to the spectrum of a.

Fix a selfadjoint polynomial $P \in \mathbb{C} < X_1, X_2 >$. The matrix model we are interested in is

$$M_N = P\left(\frac{W_N}{\sqrt{N}}, A_N\right).$$

Denote by $\lambda_i(M_N), i = 1, \ldots, N$, its eigenvalues and by

$$\mu_{M_N} = \frac{1}{N}\sum_{i=1}^{N} \lambda_i(M_N)$$

its empirical spectral measure. According to [3, Theorem 5.4.5], we have

$$\lim_{N\to\infty} \mu_{M_N} = \mu_{P(x,a)}$$

almost surely in the weak* topology, where x is a standard semicircular non commutative random variable in $(\mathcal{A}, \phi)$ (i.e $d\mu_x = \frac{1}{2\pi}\sqrt{4-x^2}\,\mathbf{1}_{[-2,2]}(x)$), a and x are freely independent, and $\mu_{P(x,a)}$ denotes the distribution of $P(x, a)$.

The set of outliers of M_N, that is eigenvalues that move away from the rest of the spectrum, is calculated in [4] from the spiked eigenvalue θ of A_N using a so-called linearization trick and Voiculescu's matrix subordination function [9] as follows. We use the linearization procedure introduced in [1, Proposition 3]. Given a polynomial $R \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$, we call *linearization* of R any $L_R \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k\rangle$ such that $L_R := \begin{pmatrix} 0 & u \\ v & Q \end{pmatrix}$ where $m \in \mathbb{N}$, $Q \in M_{m-1}(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k\rangle$ is invertible, u is a row vector and v is a column vector, both of size $m-1$ with entries in $\mathbb{C}\langle X_1, \ldots, X_k\rangle$, the polynomial entries in Q, u and v all have degree ≤ 1, and $R = -uQ^{-1}v$. It is shown in [1] that, given a polynomial $R \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$, there exist $m \in \mathbb{N}$ and a linearization $L_R \in M_m(\mathbb{C}) \otimes \mathbb{C}\langle X_1, \ldots, X_k\rangle$. It turns out that if R is self-adjoint, L_R can be chosen to be self-adjoint.
Thus, choose a linearization L_P of P: $L_P = \gamma \otimes 1 + \alpha \otimes X_1 + \beta \otimes X_2$, α, β, γ are selfadjoint matrices in $M_m(\mathbb{C})$. In $(M_m(\mathcal{A}), \mathrm{id}_m \otimes \phi)$, $\alpha \otimes x$ is a $M_m(\mathbb{C})$-valued semicircular of variance η: $b \mapsto \alpha b \alpha$ which is free over $M_m(\mathbb{C})$ from $\beta \otimes a$ and the subordination function has the following explicit form (see [8, Chapter 9] and the end of the proof of Theorem 8.3 in [2]): for $b \in M_m(\mathbb{C}), \Im b > 0$,

$$\mathrm{id}_m \otimes \phi\left[(b \otimes 1_{\mathcal{A}} - \alpha \otimes x - \beta \otimes a)^{-1}\right] = \mathrm{id}_m \otimes \phi\left[(\omega_m(b) \otimes 1_{\mathcal{A}} - \beta \otimes a)^{-1}\right],$$

where

(1) $$\omega_m(b) = b - \alpha \mathrm{id}_m \otimes \phi \left[(b \otimes 1_{\mathcal{A}} - \alpha \otimes x - \beta \otimes a)^{-1} \right] \alpha.$$

Denote by $e_{11} \in M_m(\mathbb{C})$, the matrix such that for any $1 \leq i, j \leq m$, $(e_{11})_{ij} = \delta_{i1}\delta_{j1}$. ω_m extends as an analytic map $z \mapsto \omega_m(ze_{11} - \gamma)$ to $\mathbb{C} \setminus \mathrm{supp}(\mu_{P(x,a)})$. For any $\rho \notin \mathrm{supp}(\mu_{P(x,a)})$, define $m(\rho)$ as the multiplicity of ρ as a zero of $\det(\omega_m(\rho e_{11} - \gamma) - \theta\beta)$. [4] establishes the following.

Theorem 1. [4] *There exists $\delta_0 > 0$ such that, for any $0 < \delta \leq \delta_0$, a.s for all large N, there are exactly $m(\rho)$ eigenvalues of $P\left(\frac{W_N}{\sqrt{N}}, A_N\right)$ in $]\rho - \delta; \rho + \delta[$, counting multiplicity.*

Now, assume that there exists some real number $\rho \notin \mathrm{supp}(\mu_{P(x,a)})$ such that ρ is a zero with multiplicity one of $\det(\omega_m(\rho e_{11} - \gamma) - \theta\beta) = 0$, that is such that $m(\rho) = 1$. Therefore, for $\delta > 0$ small enough, a.s for all large N, there is exactly one eigenvalue of $P\left(\frac{W_N}{\sqrt{N}}, A_N\right)$ in $]\rho - \delta; \rho + \delta[$, say $\lambda(N, \rho)$. The following main result establishes the fluctuations of $\lambda(N, \rho)$ around a mobile point ρ_N defined through a "deterministic equivalent operator". Let a_{N-1} be a selfadjoint noncommutative random variable in $(\mathcal{A}, \phi)$ such that $\forall k \in \mathbb{N}$, $\frac{1}{N-1}\mathrm{Tr}(A_{N-1}^k) = \phi((a_{N-1}^k))$ and which is free with the semicircular variable x. Define for any $b \in M_m(\mathbb{C}), \Im b > 0$,

$$\omega_m^{(N)}(b) = b - \alpha \mathrm{id}_m \otimes \phi \left[(b \otimes 1_{\mathcal{A}} - \alpha \otimes x - \beta \otimes a_{N-1})^{-1} \right] \alpha.$$

By Hurwitz theorem, for all large N, there exists one and only one ρ_N in a small neighborhood of ρ such that $\det(\omega_m^{(N)}(\rho_N e_{11} - \gamma) - \theta\beta) = 0$.

Theorem 2. [5] *Define*

$$\mathbf{C}_m = {}^t Com(\omega_m(\rho e_{11} - \gamma) - \beta\theta),$$

$$R_\infty(\rho e_{11} - \gamma) = ((\rho e_{11} - \gamma) \otimes 1_{\mathcal{A}} - \alpha \otimes x - \beta \otimes a)^{-1},$$

$$C_\rho^{(1)} = \mathrm{Tr}_m \left(\mathbf{C}_m \left[e_{11} + \alpha \mathrm{id}_m \otimes \phi(R_\infty(\rho e_{11} - \gamma)\,(e_{11} \otimes 1_{\mathcal{A}})\, R_\infty(\rho e_{11} - \gamma))\alpha \right] \right),$$

$$C_\rho^{(2)} = \mathrm{Tr}_m \left[\mathbf{C}_m \alpha \right],$$

$$v_\rho = \left(\mathbb{E}\left(|W_{21}|^4 \right) - 2 \right) \int \left[\mathrm{Tr}_m \left(\alpha \mathbf{C}_m \alpha \left(\omega_m(\rho e_{11} - \gamma) - t\beta \right)^{-1} \right) \right]^2 d\mu_a(t)$$

$$+ \phi \left(\left[\mathrm{Tr}_m \otimes \mathrm{id}_{\mathcal{A}} \left\{ R_\infty(\rho e_{11} - \gamma)(\alpha \mathbf{C}_m \alpha) \otimes 1_{\mathcal{A}} \right\} \right]^2 \right),$$

where μ_a denotes the distribution of a and ω_m is defined by (1).
$C_\rho^{(1)}\sqrt{N}(\lambda(N, \rho) - \rho_N)$ converges in distribution to the classical convolution of the distribution of $C_\rho^{(2)} W_{11}$ and a Gaussian distribution with mean 0 and variance v_ρ.

This extends the non universality phenomenon established in [6] for additive deformations of Wigner matrices when the eigenvectors associated to the spiked eigenvalues of the deformation are localized. Using the unitarily invariance of the distribution of a G.U.E. matrix, we can readily deduce the following result.

Corollary 1. *Assume that W_N is a G.U.E. matrix. Let A_N be a deterministic Hermitian matrix such that its spectral measure μ_{A_N} weakly converges towards a compactly supported measure μ_a, θ is a spike of A_N with multiplicity one whereas the other eigenvalues of A_N converge uniformly to the compact support of μ_a. Then, for any real number $\rho \notin \operatorname{supp}(\mu_{P(x,a)})$ such that ρ is a zero with multiplicity one of $\det(\omega_m(\rho e_{11}-\gamma)-\theta\beta)=0$, $C_\rho^{(1)}\sqrt{N}(\lambda(N,\rho)-\rho_N)$ converges in distribution to a Gaussian distribution with mean 0 and variance*

$$\tilde{v}_\rho = (C_\rho^{(2)})^2 + \phi\left([\mathrm{Tr}_m \otimes \mathrm{id}_{\mathcal{A}} \{R_\infty(\rho e_{11}-\gamma)(\alpha \mathbf{C}_m \alpha)\otimes 1_{\mathcal{A}}\}]^2\right).$$

Note that [7] previously established Gaussian fluctuations for any outlier of a full rank additive deformation of a G.U.E. matrix using scalar-valued free probability theory.

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Large deviations for the largest eigenvalue of the addition of random matrices

Mylène Maïda

(joint work with Alice Guionnet)

Since the pioneering works [4] of D. Voiculescu , we know that free probability provides efficient tools to describe, at least asymptotically, the spectrum of the sum of two large Hermitian matrices in generic position from one another. More precisely, if A_N and B_N are two deterministic $N \times N$ Hermitian matrices and U_N is a unitary random matrix distributed according to the Haar measure, then, in the large N limit, A_N and $U_N B_N U_N^*$ are asymptotically free and the limiting spectral distribution of $H_N := A_N + U_N B_N U_N^*$ is given by the free convolution of the limiting spectral distributions of A_N and B_N. One can also describe the asymptotic behavior of the largest eigenvalue of H_N in terms of subordination functions (for more details, we refer to the talk of Mireille Capitaine in the same workshop). In particular, if A_N and B_N have no outliers, then the largest eigenvalue of H_N converges to the right edge of the support of the free convolution of the limiting spectral distributions of A_N and B_N. In this talk, based on the preprint [3], we investigate the large deviations of this extreme eigenvalue.

Let us now introduce some notations.

Let $(A_N)_{N\geq 1}$ and $(B_N)_{N\geq 1}$ be two sequences of deterministic real diagonal matrices, with A_N and B_N of size $N \times N$. We denote by $\lambda_1^{(A_N)} \geq \ldots \geq \lambda_N^{(A_N)}$ and $\lambda_1^{(B_N)} \geq \ldots \geq \lambda_N^{(B_N)}$ their respective eigenvalues in decreasing order, by

$$\|A_N\| := \max(|\lambda_1^{(A_N)}|, |\lambda_N^{(A_N)}|) \text{ and } \|B_N\| := \max(|\lambda_1^{(B_N)}|, |\lambda_N^{(B_N)}|)$$

their respective spectral radius and by

$$\hat{\mu}_{A_N} := \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j^{(A_N)}} \text{ and } \hat{\mu}_{B_N} := \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j^{(B_N)}}$$

their respective spectral measures.

For $\beta = 1$ or 2, we denote by m_N^{β} the Haar measure on the orthogonal group $\mathcal{O}_N$ if $\beta = 1$ and on the unitary group $\mathcal{U}_N$ if $\beta = 2$. For any $N \times N$ matrix U, we denote by $H_N(U) := A_N + UB_NU^*$ and by $\lambda_{\max}^N$ the largest eigenvalue of $H_N(U)$. Our main result is a large deviation principle for the law of $\lambda_{\max}^N$ under the Haar measure m_N^{β}. It holds under some mild assumptions, that we now detail.

For μ a compactly supported probability measure on $\mathbb{R}$, we denote by $\mathsf{r}(\mu)$ the right edge of the support of μ and by G_μ the Stieltjes transform of μ : for $\lambda \geq \mathsf{r}(\mu)$,

$$G_\mu(\lambda) := \int \frac{1}{\lambda - y}\mu(\mathrm{d}y).$$

It is decreasing on the interval $(\mathsf{r}(\mu), \infty)$. By taking the limit as λ decreases to $\mathsf{r}(\mu)$, one can also define $G_\mu(\mathsf{r}(\mu)) \in \mathbb{R}_+ \cup \infty$.

We assume the following :

- (H1) The sequences of spectral empirical measures $(\hat{\mu}_{A_N})_{N\geq 1}$ and $(\hat{\mu}_{B_N})_{N\geq 1}$ converge weakly as N grows to infinity respectively to μ_a and μ_b, compactly supported on $\mathbb{R}$. Moreover, $\sup_{N\geq 1}(\|A_N\| + \|B_N\|) < \infty$.
- (H2) The largest eigenvalues $\lambda_1^{(A_N)}$ and $\lambda_1^{(B_N)}$ converge as N grows to infinity to ρ_a and ρ_b respectively.
- (H3) The following condition holds:

$$G_{\mu_a \boxplus \mu_b}(\mathrm{r}(\mu_a \boxplus \mu_b)) \leq \min\left(G_{\mu_a}(\rho_a), G_{\mu_b}(\rho_b)\right),$$

  where $\mu_a \boxplus \mu_b$ stands for the free convolution of the two measures μ_a and μ_b.

Note that the latter condition ensures that $(H_N)_{N\geq 1}$ has no outlier, namely that $\lambda_1^{(H_N)}$ converges to $\mathrm{r}(\mu_a \boxplus \mu_b)$.

We can now state the main result:

Theorem 1. *Under Assumptions* (H1, 2, 3)*, for* $\beta = 1$ *or* 2*, the law of* $\lambda_{\max}^N$ *under* m_N^β *satisfies a large deviation principle in the scale* N *with a good rate function* I^β*, that will be defined below. It means that, for any* $x \in \mathbb{R}$*, we have the upper bound*

$$\limsup_{\delta\downarrow 0}\limsup_{N\to+\infty}\frac{1}{N}\log m_N^\beta\left(\lambda_{\max}^N \in [x-\delta, x+\delta]\right) \le -I^\beta(x),$$

and the lower bound

$$\liminf_{\delta\downarrow 0}\liminf_{N\to+\infty}\frac{1}{N}\log m_N^\beta\left(\lambda_{\max}^N \in [x-\delta, x+\delta]\right) \ge -I^\beta(x).$$

A key argument of the proof of the theorem is a tilt of the measure m_N^β by a rank one spherical integral. Similar strategies are used in the companion paper [1] to study some classes of sub-Gaussian Wigner matrices. The rank one spherical integral is defined as follows: for any $\theta \ge 0$ and M_N an Hermitian matrix of size N,

$$I_N^\beta(\theta, M_N) := \int e^{N\theta(UM_NU^*)_{11}} m_N^\beta(\mathrm{d}U) \quad \text{and} \quad J_N^\beta(\theta, M_N) := \frac{1}{N}\log I_N^\beta(\theta, M_N).$$

The rate function of our large deviation principle involves the limit of $J_N^\beta(\theta, H_N)$ as N grows to infinity, which we now describe. For $\beta = 1$ or 2, $\theta \ge 0$, μ a compactly supported probability measure and $\rho \ge \mathsf{r}(\mu)$:

$$J_\mu^\beta(\theta,\rho) := \begin{cases} \frac{\beta}{2}\int_0^{\frac{2\theta}{\beta}} R_\mu(u)\mathrm{d}u, & \text{if } 0 \le \frac{2\theta}{\beta} \le G_\mu(\rho), \\ \theta\rho - \frac{\beta}{2}\log\theta - \frac{\beta}{2}\int\log(\rho-y)\mu(\mathrm{d}y) + \frac{\beta}{2}\left(\log\frac{\beta}{2} - 1\right), & \text{if } \frac{2\theta}{\beta} > G_\mu(\rho), \end{cases}$$

where G_μ has been defined above and R_μ is the R-transform of the measure μ. For any $\theta \ge 0$ and $x \ge \mathsf{r}(\mu_a \boxplus \mu_b)$, we denote by

$$I^\beta(\theta, x) := J^\beta_{\mu_a\boxplus\mu_b}(\theta, x) - J^\beta_{\mu_a}(\theta,\rho_a) - J^\beta_{\mu_b}(\theta,\rho_b),$$

and

$$I^\beta(x) := \begin{cases} \sup_{\theta\ge 0} I^\beta(\theta, x), & \text{if } x \ge \mathsf{r}(\mu_a\boxplus\mu_b), \\ +\infty, & \text{otherwise.} \end{cases}$$

It is easy to check that I^β is indeed a good rate function. The key ingredients of the proof are the following : we first use the concentration of the spectral measure of H_N around its expectation in the scale e^{-N^2}, that is much faster than the deviations we are looking at; then, if we look at the deviations towards $x > \mathrm{r}(\mu_a \boxplus \mu_b)$, we have to tilt the measure m_N^β by a rank one spherical integral in such a way that the typical behaviour of the largest eigenvalue under the tilted measure is now x and we conclude by using the asymptotics of the spherical integral obtained in [2].

If we could get the asymptotics of any finite rank sphrerical integral, we could extend this results to the joint deviations of a finite number of extreme eigenvalues of H_N.

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Free perturbations, mild operators and invariant subspaces

Ken Dykema

(joint work with Fedor Sukochev, Dmitriy Zanin)

Let $\mathcal{M} \subseteq B(\mathcal{H})$ be a von Neumann algebra with normal, faithful tracial state τ. Let $\mathcal{S}(\mathcal{M},\tau)$ denote the $*$-algebra of (possibly unbounded) closed, densely defined operators T on $\mathcal{H}$ that are affiliated to $\mathcal{M}$. The last condition means that if $T = U|T|$ is the polar decomposition of T, then $U \in \mathcal{M}$ and every spectral projection of the form $1_{[0,r]}(|T|)$ of $|T|$ for $r \geq 0$, belongs to $\mathcal{M}$. Then we can define τ on positive operators $|T| \in \mathcal{S}(\mathcal{M},\tau)$ by setting

$$\tau(|T|) = \sup_{r>0} \tau\big(|T|1_{[0,r]}(|T|)\big)\} \in [0,+\infty].$$

When $p > 0$, we consider the usual noncommutative L_p-space

$$\mathcal{L}_p(\mathcal{M},\tau) = \{T \in \mathcal{S}(\mathcal{M},\tau) \mid \|T\|_p := \tau(|T|^p)^{1/p} < +\infty\}.$$

Recall that when $p < 1$, then $\|\cdot\|_p$ is a quasi-norm, not a norm, but that

$$d_p(S,T) := \|S-T\|_p^p$$

is a metric on it. We also consider the noncommutative space of log-integrable operators,

$$\mathcal{L}_{\log}(\mathcal{M},\tau) = \{T \in \mathcal{S}(\mathcal{M},\tau) \mid \tau(\log(1+|T|)) < +\infty\}.$$

Recall that the Brown measure (see [1]) of an operator $T \in \mathcal{M}$ is a Borel probability measure ν_T, whose support is contained in the spectrum of T, and that serves as a sort of spectral distribution measure for T. We may, alternatively, write $\nu_T^{(\mathcal{M})}$ for ν_T. Its definition uses the Fuglede–Kadison determinant (see [6]). Uffe Haagerup and Hanne Schultz showed, in [7], that there are good extensions of the notions of the Fuglede–Kadison determinant $\Delta(T) = \exp(\tau(\log(|T|))) \in [0,+\infty)$ and the Brown measure ν_T, defined for all operators $T \in \mathcal{L}_{\log}(\mathcal{M},\tau)$.

Definition 1. We say that $T \in \mathcal{L}_{\log}(\mathcal{M},\tau)$ is *mild* if

(a) $\forall\lambda \in \mathbf{C}$, $\Delta(T-\lambda) > 0$,
(b) ν_T is absolutely continuous (on $\mathbf{C}$),

(c) the function $\lambda \mapsto (\lambda - T)^{-1}$ is locally Lipscitz from $\mathbf{C}$ into $\mathcal{L}_p(\mathcal{M}, \tau)$ (endowed with the metric d_p) for some $p \in (\frac{1}{2}, 1)$.

In [8], Haagerup and Schultz proved that if X and Y are $*$-free circular elements in $\mathcal{M}$, then $Z := XY^{-1}$ belongs to $\mathcal{L}_p(\mathcal{M}, \tau)$ for all $p \in (0,1)$, and that if $T \in \mathcal{L}_{\log}(\mathcal{M}, \tau)$ is $*$-free from $\{X, Y\}$, then the free perturbation $T + Z$ is mild.

It's good to be mild:

Theorem 2 ([8])**.** *Suppose $A \in \mathcal{L}_{\log}(\mathcal{M}, \tau)$ is mild. Suppose $D \subseteq \mathbf{C}$ is a closed disk. Then there exists a projection $q = Q(A, D) \in \mathcal{M}$ such that*

(i) $Aq = qAq$,
(ii) $\nu_{Aq}^{(q\mathcal{M}q)}(\mathbf{C} \setminus D) = 0$,
(iii) $\nu_{(1-q)A}^{((1-q)\mathcal{M}(1-q))}(D) = 0$.

Main Idea of Proof. The Riemann integral $\frac{1}{2\pi i} \int_{\partial D} (\lambda - A)^{-1} \, d\lambda$ converges and yields an idempotent E, whose range projection is q, which has the desired properties. □

Haagerup and Schultz used these to prove existence of hyperinvariant subspaces for elements of $\mathcal{M}$ whose Brown measures are not concentrated at single points:

Theorem 3 ([8])**.** *Let $T \in \mathcal{M}$. Then for all Borel sets $B \subseteq \mathbf{C}$, there is a unique projection $p \in \mathcal{M}$ such that*

(i) $Tp = pTp$,
(ii) $\nu_{Tp}^{(p\mathcal{M}p)}(\mathbf{C} \setminus B) = 0$,
(iii) $\nu_{(1-p)T}^{((1-p)\mathcal{M}(1-p))}(B) = 0$.

Moreover, we have that

(iv) $\tau(p) = \nu_T(B)$

and that p is T-hyperinvariant, meaning that for all $S \in B(\mathcal{H})$ that commute with T, we have $Sp = pSp$.

Main Idea of Proof. Embed $\mathcal{M}$ in a free product $\widetilde{\mathcal{M}} = \mathcal{M} * L(\mathbf{F}_4)$, which ensures that there is a copy of $Z = XY^{-1} \in \widetilde{\mathcal{M}}$ that is $*$-free from T. In the case of a closed disk $B = D$ such that $\nu_T(\delta D) = 0$, for every $n \in \mathbf{N}$, let

$$q_n = Q(T + \frac{1}{n} Z, D) \in \widetilde{\mathcal{M}}$$

be the projection from Theorem 2 and, in an ultrapower von Neumann algebra $(\widetilde{\mathcal{M}})^\omega$, consider the projection $p = [(q_n)_{n=1}^\infty]$. Then p is T-hyperinvariant, so must actually be an element of the von Neumann algebra generated by T, in $\mathcal{M} \subseteq (\widetilde{\mathcal{M}})^\omega$, and satisfies the desired properties.

From these projections for disks D, the requisite projections for arbitrary Borel sets B can be constructed. □

Notation 4. The projection p from Theorem 3 is called the *Haagerup–Schultz projection* for T and B, and is denoted $p = P(T, B)$.

The Haagerup–Schultz projections were used to construct analogues of Schur upper triangular forms of elements of T, which yield the following:

Theorem 5 ([3]). *Let $T \in \mathcal{M}$. Then there exist $N, Q \in \mathcal{M}$ such that*

(i) $T = N + Q$,
(ii) N *is normal,*
(iii) $\nu_N = \nu_T$,
(iv) $\nu_Q = \delta_0$.

An analoguous (but somewhat weaker) result was proved for unbounded affiliated operators:

Theorem 6 ([4]). *Let $T \in \mathcal{L}_{\log}(\mathcal{M}, \tau)$. Then there exists a von Neumann algebra $\widetilde{\mathcal{M}}$ containing a copy of $\mathcal{M}$ and with a normal, faithful tracial state $\tilde{\tau}$ whose restriction to $\mathcal{M}$ is τ, and there exust $N, Q \in \mathcal{L}_{\log}(\widetilde{\mathcal{M}}, \tilde{\tau})$ such that (i)-(iv) of Theorem 5 hold.*

The proof of this theorem was analogous to the proofs of the previous sequence of results, but involved many technical difficulties. In particular, the lack of a good notion of hyperinvariant subspace for unbounded (affiliated) operators meant we could not realize N and Q in $\mathcal{M}$ itself. However, this weekeness did still allow the application described below. This application can be described as saying: every Dixmier trace is spectral. It is a very general answer to a question of Pietsch [9].

We consider a subbimodule $\widetilde{B}$ of $\mathcal{L}_{\log}(\mathcal{M}, \tau)$, namely a vector subspace closed under left- and right-multiplication by elements of $\mathcal{M}$. A *trace* (or *Dixmier trace*) on $\widetilde{B}$ is a linear functional ϕ on $\widetilde{B}$ such that $\phi(AX) = \phi(XA)$ for all $A \in \widetilde{B}$ and all $X \in \mathcal{M}$. We say that $\widetilde{B}$ is *closed under log-submajorization* if, given $T \in \widetilde{B}$ and $S \in \mathcal{S}(\mathcal{M}, \tau)$ and supposing that for all $x \in (0, 1)$ we have

$$\int_0^x \log(\mu_t(S))\, dt \leq \int_0^x \log(\mu_t(T))\, dt,$$

where $\mu_t(X)$ is the generalized singular number of $X \in \mathcal{S}(\mathcal{M}, \tau)$ (see [5]), it follows that $S \in \widetilde{B}$.

Theorem 7 ([4]). *If $\widetilde{B}$ is a subbimodule of $\mathcal{L}_{\log}(\mathcal{M}, \tau)$ and if $\widetilde{B}$ is closed under log-submajorization and if ϕ is a trace on $\widetilde{B}$, then for every $T \in \widetilde{B}$, $\phi(T)$ depends only on ν_T.*

Idea of Proof. Using the decomposition $T = N + Q$ from Theorem 6, $\widetilde{B}$ and ϕ can be extended to $\widetilde{\mathcal{B}} \subseteq \mathcal{L}_{\log}(\widetilde{\mathcal{M}}, \tilde{\tau})$ and $\tilde{\phi} : \widetilde{\mathcal{B}} \to \mathbf{C}$. We have $N, Q \in \widetilde{\mathcal{B}}$ and it suffices to show $\tilde{\phi}(Q) = 0$. This follows from results of [2]. □

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Simple maps, free cumulants and topological recursion

Gaëtan Borot

(joint work with Elba Garcia Failde)

Collins, Mingo, Speicher and Śniady [6] introduced a theory of n-th order freeness for $n \geq 2$. Using the combinatorics of non-crossing partitions on n circles, they defined n-th order free cumulants, which are additive under addition of "higher order free" elements. For instance, independent unitarily invariant matrices admitting large N limits for their cumulants are higher order free of higher non-commutative probability spaces. This theory suffers from the lack of analytic results on the relation between generating series of nth order free and nth order ordinary cumulants. We propose in [4] a different combinatorial perspective on higher order free cumulants, exploiting two standard tools in random matrix theory

- its relation with map enumeration;
- the computation of topological expansions via the Eynard-Orantin topological recursion.

which may give some insight into the analytic theory.

Let $\mathcal{H}_N$ be the space of hermitian matrices of size N, and $M \in \mathcal{H}_N$ be random with a U_N invariant distribution. The space of $U(N)$-invariant polynomial functions on $\mathcal{H}_N$ is spanned by the power sum functions, indexed by partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_\ell)$. By unitary invariance, the expectation value of any polynomial function of M (of degree L) must be a linear combination of

$$\mathbb{E}[p_\lambda(M)] = \mathbb{E}\bigg[\prod_{i=1}^{n} \mathrm{Tr}\, M^{\lambda_i}\bigg], \qquad (\text{with } |\lambda| = L) \tag{1}$$

We will be interested in another set of observables, which we call "fully simple"

$$\mathbb{E}[q_\lambda(M)] = \mathbb{E}\bigg[\prod_{i=1}^{n} \big(M_{a_1^{(i)},a_2^{(i)}} \cdots M_{a_{\lambda_i}^{(i)},a_1^{(i)}}\big)\bigg]$$

where $a_j^{(i)}$ are fixed (but arbitrary) pairwise disjoint elements of $\{1,\ldots,N\}$ (this makes sense at least for $N \geq L$). By unitary invariance we have $\mathbb{E}[q_\lambda(M)] =$

$\int_{U_N} \mathrm{d}U\, \mathbb{E}[q_\lambda(UMU^\dagger)]$ leading us to decompose, in principle, the fully simple observables on the basis of ordinary observables. Our first result is a nice combinatorial description for this "change of basis".

Theorem 1. *Let $\lambda \vdash L$ and σ be a fixed permutation in the conjugacy class λ. Let $\mathcal{T}_k^{\geq}$ (resp. $\mathcal{T}_k^{>}$) be the set of transpositions $\tau_1 = (a_1, b_1), \ldots, \tau_k = (a_k, b_k)$ such that $\max(a_i, b_i)$ increases weakly (resp. strictly) with i. We introduce the weakly monotone double Hurwitz numbers,*

$$[H_k]_{\lambda,\mu} = |\mathrm{Aut}\,\mu|^{-1} \#\big\{(\tau_1, \ldots, \tau_k) \in \mathcal{T}_k^{\geq} \;\;|\;\; \sigma \circ \tau_1 \circ \cdots \circ \tau_k \in C_\mu\big\}$$

and their strictly monotone analog $[E_k]_{\lambda,\mu}$. We have the formulas

$$\frac{\mathbb{E}[q_\lambda(M)]}{|\mathrm{Aut}\,\lambda|} = \sum_{\mu \vdash |\lambda|} N^{-|\mu|} \sum_{k \geq 0} (-N)^{-k}\, [H_k]_{\lambda,\mu}\, \mathbb{E}[p_\mu(M)]$$

$$\frac{\mathbb{E}[p_\lambda(M)]}{|\mathrm{Aut}\,\lambda|} = \sum_{\mu \vdash |\lambda|} N^{|\mu|} \sum_{k \geq 0} N^{-k}\, [E_k]_{\lambda,\mu}\, \mathbb{E}[q_\mu(M)]$$

We have at the moment two proofs of this result. The first proof is in [4] and relies on the Weingarten calculus [2] to evaluate moments of the entries of U for the Haar measure and invoke the relation between representation theory of the symmetric group and Hurwitz numbers (via Jucys-Murphy correspondence). Notice that the right-hand side of the first formula is subtraction-free. Garcia-Failde and Do, and independently Charbonnier and I, could give it a second proof via bijective combinatorics using the interpretation of the two sets of observables in terms of enumeration of maps. This interpretation and its consequences is the topic of the rest of the talk.

Now let us assume that the distribution of M is of the form

$$(2) \quad Z_N^{-1}\, \mathrm{d}M \exp\bigg(-\frac{N\,\mathrm{Tr}\,M^2}{2} + \sum_{\substack{h \geq 0,\ k \geq 1 \\ \ell_1, \ldots, \ell_k \geq 0}} \frac{N^{2-2h-k}}{k!}\, t^{(h)}_{\ell_1, \ldots, \ell_k}\, \frac{\mathrm{Tr}\,M^{\ell_1} \cdots \mathrm{Tr}\,M^{\ell_k}}{\ell_1 \cdots \ell_k} \bigg)$$

Up to our choice of dependence in N and regularity assumptions, this is the general form of a $U(N)$ invariant Gibbs measure on $\mathcal{H}_N$. Wick theorem shows that observables are weighted enumerations of "maps". The quotes mean that, unlike standard maps, faces here need not be homeomorphic to disks: they can have genus h and k boundary components of respective lengths $\ell_1, \ldots, \ell_k$, and each of them is counted with a Boltzmann weight $t^{(h)}_{\ell_1, \ldots, \ell_k}$. There is also an overall factor of N^χ where χ is the Euler characteristic. It is well-known that $\mathbb{E}[p_\lambda(M)]$ counts "maps" with n rooted labeled boundary faces $\partial_1, \ldots, \partial_n$ homeomorphic to disks and of respective lengths $\lambda_1, \ldots, \lambda_n$. It is not hard to see that $\mathbb{E}[q_\lambda(M)]$ is counting fully simple "maps".

Definition 1. *A "map" is simple if each vertex in ∂_i is incident to atmost two edges in ∂_i. It is fully simple if each vertex in ∂_i is incident to atmost two edges in $\bigcup_j \partial_j$.*

In other words, boundary faces cannot touch each other in fully simple maps: this is forbidden by the index structure in $q_\lambda(M)$ and the Wick contraction rules. Considering the classical cumulant instead of the moments

$$\kappa_n\big[\mathrm{Tr}\, M^{\lambda_1},\ldots,\mathrm{Tr}\, M^{\lambda_n}\big],\qquad \kappa_n\Big[\big(M_{a_1^{(i)}a_2^{(i)}}\cdots M_{a_{\lambda_i}^{(i)}a_1^{(i)}}\big)_{i=1}^n\Big]$$

amounts to enumerating only the connected maps (ordinary or fully simple). In fact, unitary invariance and Weingarten calculus also shows that

$$\kappa_n\Big[\big(M_{a_1^{(i)}a_2^{(i)}}\cdots M_{a_{\lambda_i}^{(i)}a_1^{(i)}}\big)_{i=1}^n\Big]=\kappa_L\Big[\big(M_{a_j^{(i)},a_{j+1}^{(i)}}\big)_{i,j}\Big] \tag{3}$$

Assuming that $\lim_{N\to\infty} N^{n-2}\,\kappa_n[p_\lambda(M)] = \kappa_n^{(0)}[p_\lambda(M)]$ exist, it enumerates planar "maps" with n boundaries. In virtue of Theorem 1 one can show that $\lim_{N\to\infty} N^{|\lambda|+n-2}\kappa_n[q_\lambda(M)] = \kappa_n^{(0)}[q_\lambda(M)]$ exist, and it enumerates planar fully simple "maps". According to [6] and (3), $\kappa_n^{(0)}[q_\lambda(M)]$ are the n-th order free cumulants of M. Our goal is to relate their respective generating series. We introduce the formal (Laurent) series

$$\begin{aligned} W_n^{(0)}(x_1,\ldots,x_n) &= \frac{1}{x}+\sum_{\ell_1,\ldots,\ell_n\geq 0}\kappa_n^{(0)}[\mathrm{Tr}\, M^{\ell_1},\ldots,\mathrm{Tr}\, M^{\ell_n}]\prod_{i=1}^n x_i^{-(\ell_i+1)}\\ X_n^{(0)}(w_1,\ldots,w_n) &= \frac{1}{w}+\sum_{\ell_1,\ldots,\ell_n\geq 1}\kappa_n^{(0)}[q_{(\ell_1,\ldots,\ell_n)}(M)]\prod_{i=1}^n w_i^{\ell_i-1}\end{aligned}$$

and the simplified notation $W(x)=W_1^{(0)}(x)$ and $X(w)=X_1^{(0)}(w)$.

Theorem 2. [6, 4] *We have* $X(W(x))=x$ *and*

$$B=\left(W_2^{(0)}(x_1,x_2)+\frac{1}{(x_1-x_2)^2}\right)\mathrm{d}x_1\mathrm{d}x_2=\left(X_2^{(0)}(w_1,w_2)+\frac{1}{(w_1-w_2)^2}\right)\mathrm{d}w_1\mathrm{d}w_2 \tag{4}$$

where we impose $w_i=W(x_i)$ *– or* $x_i=X(w_i)$.

This theorem is a difficult result in [6] starting from the combinatorics of partitioned permutations. We offer in [4] a simpler proof, by turning into functional relations the bijective decompositions of ordinary disks and cylinders shows in Figure 1: after removal of the proliferation of ordinary disks, one obtains a fully simple map.

For instance, the first picture tells us that

$$m_\ell=\sum_{\ell',m_1,\ldots,m_{\ell'}\geq 0} m_{\ell'}^*\prod_{l=1}^{\ell'} m_\ell \tag{5}$$

where m_ℓ (resp. m_ℓ^*) are the first order (free) moments. If one iterates this relation until the right-hand side only involves the m^*, one arrives to a sum over noncrossing partitions which are the well-known combinatorial objects underlying

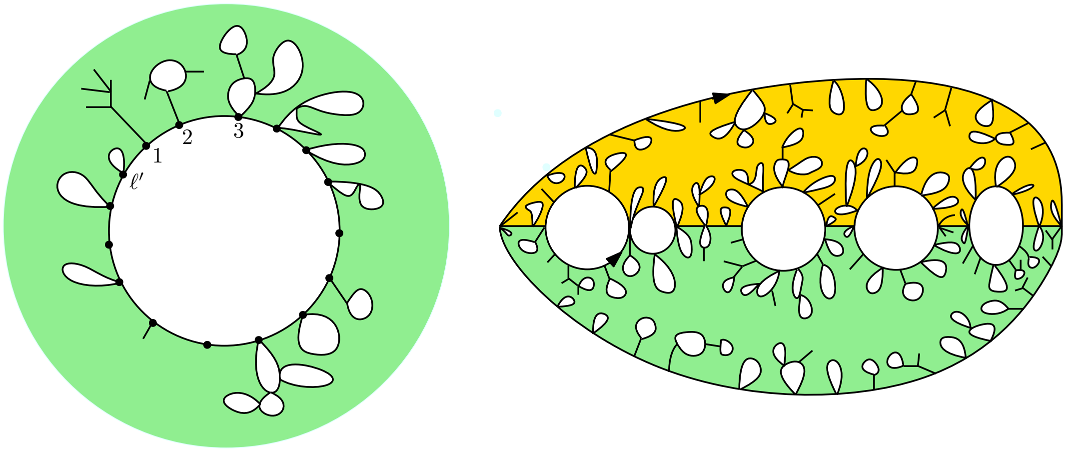

FIGURE 1. The coloured faces are the boundaries.

Voiculescu R-transform. If one make a generating series for (5) one arrives to $X(W(x)) = x$.

Let us specialize further to distributions of the form (2) with $t^{(h)}_{\ell_1,\ldots,\ell_k} = 0$ unless $(h,k) = (0,1)$ and $(0,2)$ (called "double trace models"), and assume there exists a (formal or asymptotic) expansion of the form

$$\kappa_n[p_\lambda(M)] = \sum_{g\geq 0} N^{2-2g-n}\,\kappa_n^{(g)}[p_\lambda(M)] + O(N^{-\infty})$$

and this again will imply the existence of $\kappa_n^{(g)}[q_\lambda(M)]$. We then form the generating series

$$\begin{aligned} W_{g,n}(x_1,\ldots,x_n) &= \sum_{\ell_1,\ldots,\ell_n\geq 0} \kappa_n^{(g)}[\mathrm{Tr}\,M^{\ell_1},\ldots,\mathrm{Tr}\,M^{\ell_n}]\prod_{i=1}^{n} x_i^{-(\ell_i+1)} \\ X_{g,n}(w_1,\ldots,w_n) &= \sum_{\ell_1,\ldots,\ell_n\geq 0} \kappa_n^{(g)}[q_{(\ell_1,\ldots,\ell_n)}(M)]\prod_{i=1}^{n} w_i^{\ell_i-1} \end{aligned}$$

Sufficient conditions for the existence of such an asymptotic expansion are given in [5], and then the $W_{g,n}$ can be computed by induction on $2g-2+n$ through Eynard-Orantin topological recursion [3]. In particular, $W_{g,n}$ analytically continue to meromorphic functions on $\mathcal{C}^n$ where $\mathcal{C}$ is the spectral curve of equation $w = W(x)$ – or $x = X(w)$, and it can be obtained by residue computations at the zeroes of $\mathrm{d}x$ on $\mathcal{C}$. The initial data for the topological recursion is $(\mathcal{C}, x, w, B)$. We notice that looking at fully simple maps, at least for disks and cylinders, just amount to exchanging the role of x and w while keeping the same B (4). This exchange is a "symplectic transformation", *i.e.* it preserves $|\mathrm{d}x \wedge \mathrm{d}w|$ in $\mathbb{C}^2$. In the general theory of the topological recursion, it is conjectured (although the precise form that should be given to this conjecture is still disputed) that $W_{g,0}$ is insensitive to symplectic transformations. This squares here with the fact $W_{g,0}$ enumerates maps without boundaries, so the distinction between ordinary or fully simple boundaries becomes irrelevant. This led us to conjecture

Conjecture 1. $X_n^{(g)}$ *continues analytically to* $\mathcal{C}^n$*, and is computed by the Eynard-Orantin topological recursion for the spectral curve* $(\mathcal{C}, w, x, B)$.

We stress that the enumeration of fully simple maps is encoded in the series expansion of $X_n^{(g)}$ with respect to the variables $w_i \to \infty$. For $g = 0$, this would give a direct computation of the nth order free cumulants for the double trace models. It also suggests the existence a universal theory of freeness for all genera. For the model $Z_N^{-1}\,\mathrm{d}M\,e^{-N\mathrm{Tr}(M^2/2+tM^4/4)}$ which generate quandragulations with weight $(-t)$ per quadrangle the spectral curve $\mathcal{C}$ is

$$x(z) = c(z + 1/z), \qquad w(z) = \frac{1}{cz} + \frac{tc^3}{z^3}, \qquad B(z_1, z_2) = \frac{\mathrm{d}z_1\mathrm{d}z_2}{(z_1 - z_2)^2}$$

with $c = \sqrt{\frac{1-\sqrt{1+12t}}{-6t}}$. We checked that the topological recursion for the initial data $(\mathcal{C}, w, x, B)$ produces Laurent series $X_3^{(0)}$ and $X_1^{(1)}$ with nonnegative coefficients (when $t < 0$) which are much smaller than the corresponding number of quadrangulations with ordinary boundaries. This is a non trivial fact in support of the conjecture: at least $X_n^{(g),\mathrm{TR}}$ should enumerate certain type of maps with some restriction. We also have some partial checks that $X_3^{(0),\mathrm{TR}}$ reproduces recent formulas for quadrangulated fully simple pairs of pants [1].

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Regularity properties of noncommutative distributions

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(joint work with Marwa Banna, Roland Speicher, Moritz Weber, Sheng Yin)

Noncommutative distributions are a cornerstone of free probability theory since they connect various of its facets in a unifying probabilistic language. An appropriate framework especially for applications at the interface of operator algebras and random matrix theory is that of a *tracial W^*-probability space* $(\mathcal{M}, \tau)$, i.e., a von Neumann algebra $\mathcal{M}$ that is endowed with a faithful normal tracial state τ. If, for instance, a tuple $X = (X_1, \ldots, X_n)$ consisting of selfadjoint operators $X_1, \ldots, X_n \in \mathcal{M}$ is given, then its *joint noncommutative distribution* is defined by

$$\mu_X : \mathbb{C}\langle x_1, \ldots, x_n \rangle \to \mathbb{C}, \quad x_{i_1} \cdots x_{i_k} \mapsto \tau(X_{i_1} \cdots X_{i_k}),$$

where $\mathbb{C}\langle x_1, \ldots, x_n \rangle$ denotes the $*$-algebra of *noncommutative polynomials* in the formal selfadjoint variables $x_1, \ldots, x_n$.

In recent years, much progress has been made in the understanding of how properties of $X = (X_1, \ldots, X_n)$ affect regularity properties of the associated noncommutative distribution μ_X; see, for instance, [10, 3, 8, 1, 9, 2]. But what does "regularity" actually mean here? Because a measure theoretic description of μ_X is not available in full generality, this indeed requires clarification. The strategy that one often follows is that regularity of μ_X – no matter what rigorous meaning one could give to this phrase – should be reflected in properties of all those operators $Y = f(X)$ that arise as evaluations of suitable noncommutative test functions f. Recall that if Y is selfadjoint, its noncommutative distribution can be identified with the unique Borel probability measure μ_Y on $\mathbb{R}$ that satisfies $\int_{\mathbb{R}} t^k \, d\mu_Y(t) = \tau(Y^k)$ for all integers $k \geq 0$; the latter, to which we refer as the *analytic distribution of* Y, can be studied by classical means: we may ask whether μ_Y has atoms, and if not, we can check for finer properties such as Hölder continuity of its *cumulative distribution function* $\mathcal{F}_Y$ which is defined by $\mathcal{F}_Y(t) := \mu_Y((-\infty, t])$, or absolute continuity of μ_Y with respect to the Lebesgue measure on $\mathbb{R}$.

However, even for a single operator $Y = Y^* \in \mathcal{M}$, it is far from being obvious how properties of μ_Y can be detected from Y. What one is aiming at are characterizations that are accessible to operator algebraic means. Existence of atoms, for instance, is intimately related to *zero divisors*: μ_Y has an atom at $s \in \mathbb{R}$, i.e., $\mu_Y(\{s\}) \neq 0$, if and only if we find a non-zero spectral projection p of Y such that $(Y - s)p = 0$. Moreover, if there are constants $\alpha > 1$ and $c > 0$ such that $\|(Y - s)p\|_2 \geq c\|p\|_2^\alpha$ holds for all $s \in \mathbb{R}$ and all spectral projections p of Y, then $\mathcal{F}_Y$ is Hölder continuous with exponent $\frac{2}{\alpha - 1}$; see [9, 2] and [3].

Among the classes of noncommutative test functions f which were treated in course of these investigations are

- noncommutative polynomials $P \in \mathbb{C}\langle x_1, \ldots, x_n \rangle$;
- matrices of noncommutative polynomials and *affine linear pencils* in particular, i.e., elements $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \ldots, x_n \rangle)$ that are of the form $\mathbf{P} = b_0 + b_1 x_1 + \cdots + b_n x_n$ with scalar coefficient matrices $b_0, b_1, \ldots, b_n \in M_N(\mathbb{C})$;

- *noncommutative rational functions*, i.e., elements r in the universal field of fractions $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ of the ring $\mathbb{C}\langle x_1, \ldots, x_n\rangle$, called the *free field*.

While noncommutative polynomials and matrices thereof are rather straightforward to deal with, the theory of noncommutative rational functions is significantly more involved; see [4]. Nonetheless, it leads finally to the crucial and appealing insight that $\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle$ can be understood entirely via matrices over $\mathbb{C}\langle x_1, \ldots, x_n\rangle$. It is true, for instance, that a matrix $\mathbf{Q}$ in $M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$ becomes invertible in $M_N(\mathbb{C}\langle\!\langle x_1, \ldots, x_n \rangle\!\rangle)$ if and only if $\mathbf{Q}$ is full; and the latter condition can be checked without referring to the free field: the *inner rank* $\rho(\mathbf{Q})$ of a matrix $\mathbf{Q} \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$ is defined as the least integer $k \geq 1$ for which $\mathbf{Q}$ admits a decomposition $\mathbf{Q} = \mathbf{R}_1\mathbf{R}_2$ with rectangular matrices $\mathbf{R}_1 \in M_{N\times k}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$ and $\mathbf{R}_2 \in M_{k\times N}(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$; we say that $\mathbf{Q}$ is *full* if it has full inner rank, i.e., if $\rho(\mathbf{Q}) = N$ holds.

The operators $X = (X_1, \ldots, X_n)$ that were studied align themselves roughly into two groups. In [10, 1], the case of freely independent operators $X_1, \ldots, X_n$ was considered; a more general approach using completely different techniques was developed in [3, 8, 9, 2]. The latter made contact to the groundbreaking work [11, 12, 13, 14, 15, 16, 17] of Voiculescu on analogues of entropy and Fisher information in the realm of free probability theory. Following especially the non-microstates approach [16], one can associate to the given tuple X quantities such as the *free Fisher information* $\Phi^*(X)$, the *free entropy* $\chi^*(X)$, and the *free entropy dimension* $\delta^*(X)$; a variant of the latter, denoted by $\delta^\star(X)$, was introduced in [5].

It is a common point of view – inspired primarily by results in the case $n = 1$ of a single variable – that the associated conditions $\delta^*(X) = n$ and $\delta^\star(X) = n$ rule out the atomic part in μ_X. Evidence was given to this in [3, 8, 9]; more precisely, it was shown that under the weaker condition $\delta^\star(X) = n$ the analytic distribution μ_Y of $Y = P(X)$ for any non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$ cannot have atoms. Consequently, each of the stronger conditions $\chi^*(X) > -\infty$ and $\Phi^*(X) < \infty$ is expected to entail further regularity properties of μ_X. First steps in this direction are [3, 2]; it was shown in [2] that if $\Phi^*(X) < \infty$ is assumed, then the cumulative distribution function $\mathcal{F}_Y$ of $Y = P(X)$ for any non-constant selfadjoint $P \in \mathbb{C}\langle x_1, \ldots, x_n\rangle$ is Hölder continuous with exponent $\frac{2}{3(2^d-1)}$, where $d \geq 1$ is the degree of P. Furthermore, inspired by [1], Hölder continuity of $\mathcal{F}_\mathbf{Y}$ for operators $\mathbf{Y} = \mathbf{P}(X)$ that come from certain selfadjoint affine linear pencils $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \ldots, x_n\rangle)$ was established in [9]; note that the resulting operators live in the tracial W^*-probability space $(M_N(\mathcal{M}), \frac{1}{N}\operatorname{Tr}_N \circ \tau^{(N)})$, where Tr_N denotes the trace on $M_N(\mathbb{C})$ and $\tau^{(N)}\colon M_N(\mathcal{M}) \to M_N(\mathbb{C})$ the matricial amplification of τ which is given by $\tau^{(N)}(\mathbf{Y}) := (\tau(Y_{kl}))_{k,l=1}^N$ for every $\mathbf{Y} = (Y_{kl})_{k,l=1}^N \in M_N(\mathcal{M})$.

These results have some applications in random matrix theory. It was shown in [6] that tuples $X^{(N)} = (X_1^{(N)}, \ldots, X_n^{(N)})$ of $N \times N$ hermitian random matrices which follow general Gibbs laws are described in the limit $N \to \infty$ by operators $X = (X_1, \ldots, X_n)$ that satisfy $\Phi^*(X) < \infty$. This, as proven in [2], not only implies that various "composed" random matrices $Y^{(N)} = f(X^{(N)})$ have a limiting eigenvalue distribution with Hölder continuous cumulative distribution function, but ensures also convergence in the Kolmogorov metric, in the GUE case even with an explicit rate.

Because some of the strong regularity results that were obtained in [10, 1] rely on case-specific methods, the question suggests itself how far one can still go without assuming freeness. For instance, it was proven in [10] that whenever $X_1, \dots, X_n$ are freely independent and none of the individual analytic distributions $\mu_{X_1}, \dots, \mu_{X_n}$ has atoms, then the measure of a possible atom in the analytic distribution $\mu_{\mathbf{Y}}$ of $\mathbf{Y} = \mathbf{P}(X)$, for any selfadjoint matrix $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$, can only be an integer multiple of $\frac{1}{N}$. The authors of [10] generalized for that purpose the framework of the *Strong Atiyah Conjecture* (see [7] for more details on the latter) and proved that it includes the above situation: a tuple $X = (X_1, \dots, X_n)$ of (not necessarily selfadjoint) operators in $\mathcal{M}$ is said to have the *Strong Atiyah Property* if $\operatorname{rank}(\mathbf{P}(X))$ is a nonnegative integer for every matrix $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$ of arbitrary size N; recall that $\operatorname{rank}(\mathbf{Y}) := N - (\operatorname{Tr}_N \circ \tau^{(N)})(p_{\ker(\mathbf{Y})})$ for any $\mathbf{Y} \in M_N(\mathcal{M})$, where $p_{\ker(\mathbf{Y})} \in M_N(\mathcal{M})$ is the orthogonal projection onto the kernel $\ker(\mathbf{Y})$ of $\mathbf{Y}$. It was conjectured (see [3], for instance) that these results of [10] remain true in the more general situation $\delta^\star(X) = n$. Indeed, it was shown in [9] that every tuple X with $\delta^\star(X) = n$ has the Strong Atiyah Property and that one even has $\operatorname{rank}(\mathbf{P}(X)) = \rho(\mathbf{P})$ for every matrix $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$ of arbitrary size N. It follows that under the assumption $\delta^\star(X) = n$ every operator $\mathbf{Y} = \mathbf{P}(X)$ for a selfadjoint $\mathbf{P} \in M_N(\mathbb{C}\langle x_1, \dots, x_n\rangle)$ has atoms precisely at the points in the set $\{\lambda \in \mathbb{C} \mid \mathbf{P} - \lambda \mathbf{1}_N \text{ is not full}\}$ and $\mu_{\mathbf{Y}}(\{\lambda\}) = 1 - \frac{1}{N}\rho(P - \lambda \mathbf{1}_N)$.

This has some consequences regarding evaluations of noncommutative rational functions. Let $\mathcal{A}$ be the $*$-algebra of all closed and densely defined linear operators affiliated with $\mathcal{M}$. In [9], the existence of an injective homomorphism $\operatorname{ev}_X \colon \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle \to \mathcal{A}$ extending the canonical evaluation $P \mapsto P(X)$ on $\mathbb{C}\langle x_1, \dots, x_n\rangle$ was shown; this excludes not only non-trivial rational relations among $X_1, \dots, X_n$ but also atoms in the analytic distribution of $r(X) := \operatorname{ev}_X(r)$ for every selfadjoint $r \in \mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$. This culminates in the result that if $\delta^\star(X) = n$, then the rational closure and the division closure of $\mathbb{C}\langle X_1, \dots, X_n\rangle$, the subalgebra of $\mathcal{M}$ generated by $X_1, \dots, X_n$, in $\mathcal{A}$ agree and are furthermore isomorphic to the free field; in other words, we can realize $\mathbb{C}\langle\!\langle x_1, \dots, x_n \rangle\!\rangle$ naturally inside $\mathcal{A}$, where the concrete operators $X_1, \dots, X_n$ take over the role of the formal variables $x_1, \dots, x_n$.

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Macroscopic fluctuations for products of random matrices

Vadim Gorin
(joint work with Yi Sun)

Let $Y_N^1, \ldots, Y_N^M$ be i.i.d. $N \times N$ random matrices which are right-unitarily invariant, and let $Y_N := Y_N^1 \cdots Y_N^M$ be their product. The (squared) singular values $\mu_1^N \geq \cdots \geq \mu_N^N > 0$ of Y_N occur classically in the study of ergodic theory of non-commutative random walks, and the corresponding Lyapunov exponents have a limit as $M \to \infty$ by Oseledec's multiplicative ergodic theorem. These Lyapunov exponents for various ensembles of matrices have been the object of extensive study in the dynamical systems literature, beginning with the pioneering result of Furstenberg-Kesten for matrices with positive entries. In an applied context, singular values for similar models have appeared in the study of disordered systems in statistical physics, in the study of polymers, and in the study of dynamical isometry for deep neural networks as the Jacobians of randomly initialized networks.

The goal of this work is to study the global fluctuations of the squared singular values μ^N for a variety of different distributions for Y_N^i, including rectangular Ginibre matrices (i.e., matrices with i.i.d. complex Gaussian elements), truncated Haar-random unitary matrices, and right-unitarily invariant matrices with fixed singular values. In each of these cases, we study the normalized log-spectrum

$$\lambda_i^N := \frac{1}{M} \log \mu_i^N$$

via the height function

$$\mathcal{H}_N(t) := \#\{\lambda_i^N \leq t\}.$$

We study limit shapes and fluctuations for the height function in two limit regimes, one where $N \to \infty$ with M fixed, and one where $N, M \to \infty$ simultaneously.

When M is fixed, results from free probability of Voiculescu and Nica-Speicher imply that the empirical measure $d\lambda^N$ of λ_i^N converges to a deterministic measure $d\lambda^\infty$, which implies that $\mathcal{H}_N(t)$ concentrates around a deterministic limit shape, a result we refer to as a law of large numbers. In this work we prove that the fluctuations of the height function around its mean converge to explicit Gaussian fields. We further show that these fields are log-correlated under a technical condition on the smoothness of the limit shape. Namely, this means that for polynomials f, g, we have

$$\lim_{N\to\infty} \mathrm{Cov}\left(\int (\mathcal{H}_N(t) - \mathbb{E}[\mathcal{H}_N(t)])f(t)dt, \int (\mathcal{H}_N(s) - \mathbb{E}[\mathcal{H}_N(s)])g(s)ds\right) = \int\int K(t,s)f(t)g(s)dtds$$

for an explicit covariance kernel $K(t, s)$ which satisfies

$$K(t,s) = -\frac{1}{2\pi^2} \log|t-s| + O(1)$$

for $|t - s| \to 0$.

When $M, N \to \infty$ simultaneously, the λ_i^N are the Lyapunov exponents first studied by Newman and Isopi-Newman. Again, their empirical measure converges to the deterministic measure

$$-\frac{e^{-z}}{S'_{\tilde{\mu}}(S_{\tilde{\mu}}^{-1}(e^{-z}))}\mathbf{1}_{[-\log S_{\tilde{\mu}}(-1), -\log S_{\tilde{\mu}}(0)]}dz,$$

where $S_{\tilde{\mu}}$ is the limiting S-transform of the spectral measure of $(Y_N^i)^* Y_N^i$. This yields concentration of the height function around a deterministic limit shape. In this work we prove that the rescaled fluctuations of the height function around its mean

$$M^{1/2}(\mathcal{H}_N(t) - \mathbb{E}[\mathcal{H}_N(t)])$$

again converge to explicit Gaussian fields. However, in these cases, we find that the field has a white noise component, meaning that for $|t-s| \to 0$ the covariance satisfies

$$K(t,s) = \delta(t-s) + O(1),$$

where δ denotes the Dirac delta function. In particular, as $M \to \infty$, we see a transition from a log-correlated Gaussian field to a Gaussian field with a white noise component. For this result, the relative growth rates of N and M are not important; in particular, we observe the same white noise component both for M

growing to infinity much faster than N and much slower than N. This behavior is very different from that seen in recent work of Akemann-Burda-Kieburg and Liu-Wang-Wang, where the local limit for the singular values of the products of Ginibre matrices depends on the ratio M/N.

The transition between white noise (corresponding to $M \to \infty$) and log-correlated (for finite M) statistics can be compared with a similar transition for the Dyson Brownian Motion (DBM) started from a deterministic initial condition and in which the time t is taken to be equal to M^{-1}. For DBM such transition was studied in detail for $\beta = 2$ by Duits–Johansson, for general β DBM by Huang-Landon and for a finite-temperature version of the DBM ensemble by Johansson-Lambert.

Our technique is based on the study of multivariate Bessel generating functions for log-spectral measures, which are continuous versions of the Schur generating function defined and studied by Bufetov-Gorin. Recall that for sets of variables $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$, the multivariate Bessel function is defined by

$$\mathcal{B}(a, b) := \Delta(\rho) \frac{\det(e^{a_i b_j})_{i,j=1}^N}{\Delta(a)\Delta(b)},$$

where $\rho = (N-1, \ldots, 0)$ and $\Delta(a) := \prod_{1 \le i < j \le N}(a_i - a_j)$ denotes the Vandermonde determinant. For an N-tuple $\chi_N = (\chi_{N,1} \ge \cdots \ge \chi_{N,N})$, the multivariate Bessel generating function of a measure $d\mu_N(x)$ on N-tuples $x = (x_1 \ge \cdots \ge x_N)$ with respect to χ_N is defined by

$$\phi_{\chi,N}(s) := \int \frac{\mathcal{B}(s, x)}{\mathcal{B}(\chi_N, x)} d\mu_N(x).$$

The proofs combine the extraction of the asymptotic information about the measures from their multivariate Bessel generating functions through differential operators with the asymptotic analysis of the multivariate Bessel functions through double contour integrals and determinantal representations.

The details of this talk can be found in [GS].

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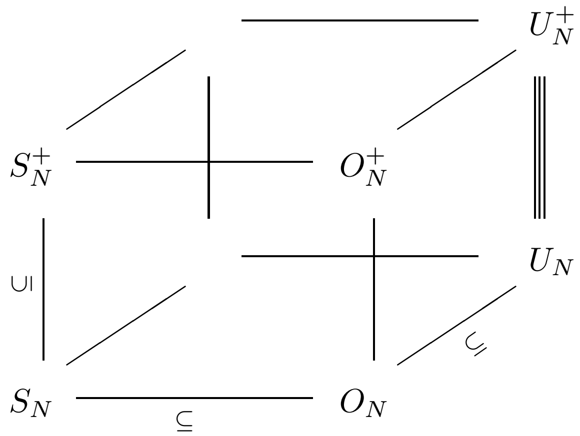
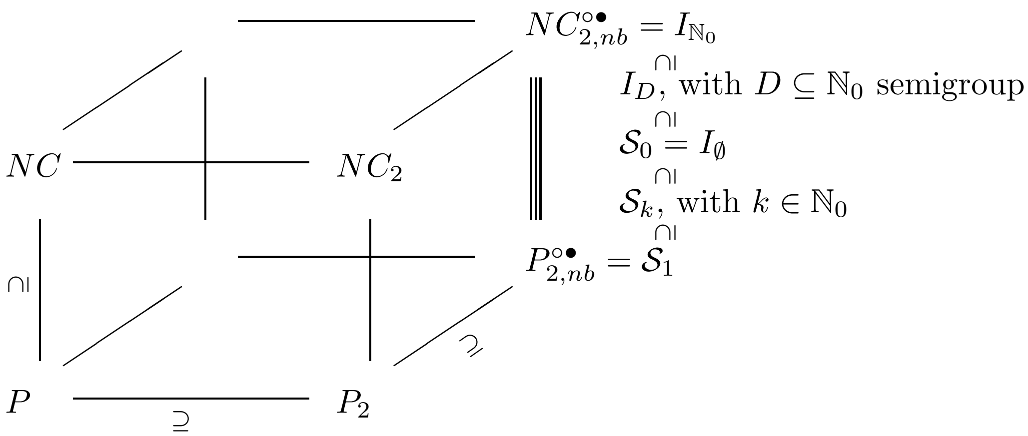
Unitary half-liberations

Moritz Weber
(joint work with Alexander Mang)

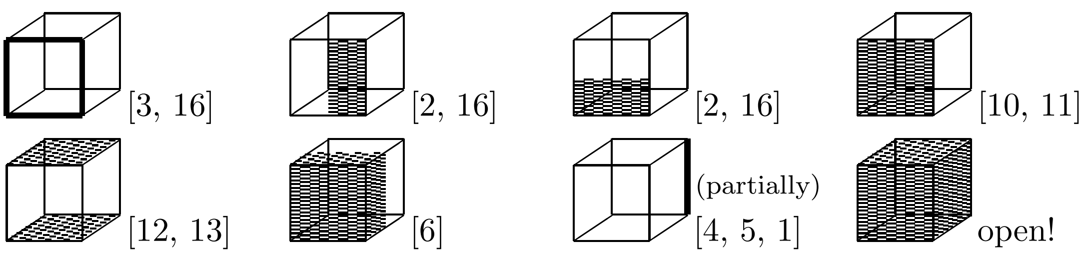
Summary We classify all unitary easy quantum groups between the unitary group U_N and the free unitary quantum group U_N^+.



In some sense, such intermediate quantum groups interpolate the classical and the free world resembling the passage from classical to free independence. On the combinatorial side, this amounts to the following complete list of categories between $NC_{2,nb}^{\circ\bullet}$ and $P_{2,nb}^{\circ\bullet}$, see [7, 8]:



Previous classification results. The following results on the classification of the above cube scheme of all easy quantum groups have been obtained before:

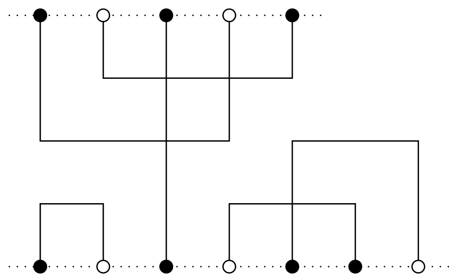


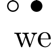
Liberation and half-liberation. In the 1980's, Woronowicz [19, 21] introduced compact quantum groups in order to provide an appropriate notion of quantum symmetry in the context of operator algebras, in particular in the noncommutative setting. In the 1990's, Sh. Wang [15] introduced a quantum version U_N^+ of the group U_N of unitary complex-valued $N \times N$-matrices. Wang's quantum group can be seen as a quantization of the unitary group which is "as free/noncommutative" as possible. It is hence also addresses as a "liberated" version of U_N. An important task is to find "half-liberated" versions of the unitary group, i.e. quantum groups G with

$$U_N \subset G \subset U_N^+.$$

Such quantum groups G can be seen as a somewhat moderate step from the classical world into the quantum world; or in other words: as an interpolation between the classical and the free world. See for instance [14, 9] for more on compact quantum groups.

Easy quantum groups. In 2009, Banica and Speicher [3] provided a powerful machine in order to produce examples of (orthogonal) compact quantum groups; their approach has been extended by Tarrago and the author [12, 13] to the unitary setting. These so called easy quantum groups are governed by the same combinatorics as in free probability theory: by partitions of finite sets. More precisely, we consider diagrams of the following form consisting in points colored either in black or in white, which are connected by some strings representing the blocks of the partition:



We consider sets $\mathcal{C}$ of partitions which are closed under horizontal and vertical concatenation as well as under swapping the upper and the lower row of points. Moreover, the partitions , , , and  are supposed to be in $\mathcal{C}$. If a set $\mathcal{C}$ of partitions satisfies these properties, we call it a category of partitions. To any partition p, we may associate a certain linear map T_p. Banica and Speicher observed that given a category of partitions $\mathcal{C}$, the linear span of all maps T_p of partitions $p \in \mathcal{C}$ forms a tensor category in Woronowicz's sense — and by Woronowicz's Tannaka-Krein Theorem [20], we obtain a compact quantum group: an easy quantum group.

The study and classification of easy quantum groups is hence equivalent to dealing with the underlying categories of partitions, which are purely combinatorial objects. See for instance [18] for an overview on easy quantum groups and [17] for links to free probability.

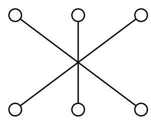
Some technical details from the work on unitary half-liberations. Recall from [15] that the free orthogonal quantum group O_N^+ is defined via the universal C^*-algebra $A_o(N)$ generated by self-adjoint elements u_{ij}, $1 \leq i, j \leq N$ and the relations $\sum_k u_{ik}u_{jk} = \sum_k u_{ki}u_{kj} = \delta_{ij}$ turning the matrix $u = (u_{ij})$ into an orthogonal one. The group O_N of orthogonal real-valued matrices however can also be seen as a quantum group, and its algebra $C(O_N)$ is the quotient of $A_o(N)$ by the relations that all u_{ij} shall commute. Conversely, one can view $A_o(N)$ as an algebra which is a "liberated" version of $C(O_N)$, in the sense that the commutativity condition on the generators is dropped. Now, in [3], Banica and Speicher defined the half-liberated orthogonal quantum group O_N^* via the quotient of $A_o(N)$ by the relations

$$abc = cba$$

where $a, b, c \in \{u_{ij}\}$. Hence, the commutativity condition is not completely dropped but rather "half-dropped". Since the algebra corresponding to O_N^* is a quotient of $A_o(N)$ and since $C(O_N)$ in turn is a quotient of the former one, we have the following chain of inclusions:

$$O_N \subset O_N^* \subset O_N^+$$

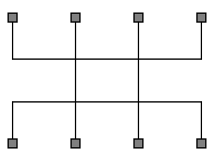
These relations $abc = cba$ are induced by the following partition:



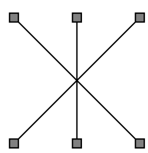
For the unitary situation, the main issue is that the generators u_{ij} are no longer self-adjoint. Therefore, one needs two colors for the points, one for the generators u_{ij} themselves and one for their adjoints. If one wants to find examples of compact quantum groups G with

$$U_N \subset G \subset U_N^+,$$

a first guess would be to color the points of the above diagram in different ways in black and white; equivalently, one would impose relations $abc = cba$ with specific choices for $a, b, c \in \{u_{ij}, u_{ij}^*\}$. This has been done in [4, 5], but this yields only two examples of such intermediate quantum groups; further ones have been found in [1]. However, in order to achieve a full classification of all such intermediate quantum groups, we proposed a change of paradigm in [7, 8] and to consider diagrams of the type



rather than



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These partitions of a "bracket type" are the key to our combinatorial classification results of all unitary easy quantum groups in between U_N and U_N^+, see [7, 8].

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Interval partitions and noncrossing partitions

Philippe Biane
(joint work with Matthieu Josuat-Vergès)

Since the work of Roland Speicher (see e.g. [2] for a recent exposition), it is known that noncrossing partitions can be used to give a combinatorial treatment of free independence. Soon after this discovery, Speicher and Woroudi [1] analogously introduced Boolean independence, which uses the lattice of interval partitions. It is known that the lattice of noncrossing partitions embeds into the symmetric group and this embedding can be given a geometric description: the permutations corresponding to noncrossing partitions are the permutation which are smaller than the full cycle $(123\dots n)$ for the *absolute order*. This absolute order can be defined smilarly for any finite Coxeter group and one can define noncrossing partitions as the set of elements of the group which are smaller than a Coxeter element. The study of these generalized noncrossing partitions has become an important branch of Coxeter group theory since the beginning of this century. We show that interval partitions can be characterized geometrically as the permutations which are smaller than $(123\dots n)$ for the *Bruhat order*. Since the Bruhat order can be defined for any finite Coxeter group, we can define again interval partitions associated with any standard Coxeter element in a finite Coxeter group. We study the relations between interval and noncrossing partitions by introducing two order relations on the Coxeter group $\sqsubset$ and $\ll$ which refine the absolute order, taking into account the Bruhat order (in the context of symmetric groups and for the set of noncrossing partitions the order $\ll$ was considered by Belinschi and Nica [3]). We show that intervals for these orders can be enumerated and are in bijection with faces of the cluster complex and we also make connections with Chapoton's F and H-triangles. A crucial property is that the two orders $\sqsubset$ and $\ll$ are related by the Kreweras inversion.

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Local law and complete eigenvector delocalization for supercritical Erdős–Rényi graphs

Antti Knowles
(joint work with Yukun He, Matteo Marcozzi)

Let $\mathcal{A} \in \{0,1\}^{N\times N}$ be the adjacency matrix of the Erdős-Rényi random graph $G(N,p)$, where $p \equiv p_N \in (0,1)$. That is, $\mathcal{A}$ is real symmetric, and its upper-triangular entries are independent Bernoulli random variables with mean p. The Erdős-Rényi graph exhibits a phase transition in its connectivity around the critical expected degree $pN = \log N$. Indeed, for fixed $\epsilon > 0$, if $pN \geq (1+\epsilon)\log N$ then $G(N,p)$ is with high probability connected, and if $pN \leq (1-\epsilon)\log N$ then $G(N,p)$ has with high probability isolated vertices. The aim of our work is to investigate the spectral and eigenvector properties of $G(N,p)$ in the supercritical regime, where $G(N,p)$ is connected with high probability.

Our main result is a *local law* for the adjacency matrix $\mathcal{A}$ in the supercritical regime $C\log N \leq pN \ll N$, where C is some universal constant. In order to describe it, it is convenient to introduce the rescaled adjacency matrix

$$A := \sqrt{\frac{1}{p(1-p)N}}\,\mathcal{A}$$

so that the typical eigenvalue spacing of A is of order N^{-1}. A local law provides control of the matrix entries $G_{ij}(z)$ of the Green function

$$G(z) := (A-z)^{-1}\,, \tag{1}$$

where $z = E + \mathrm{i}\eta$ is a spectral parameter with positive imaginary part $\eta \gg N^{-1}$ defining the spectral scale. Our main result states that that the individual entries $G_{ij}(z)$ of the Green function concentrate all the way down to the critical scale $pN = C\log N$: the quantity

$$\max_{i,j} |G_{ij}(z) - \delta_{ij} m(z)|$$

is small with high probability for all $\eta \gg N^{-1}$; here $m(z)$ is the Stieltjes transform of the semicircle law.

Such local laws have become a cornerstone of random matrix theory, ever since the seminal work [3, 4] on Wigner matrices. They serve as fundamental tools in the study of the distribution of eigenvalues and eigenvectors, as well as in establishing universality in random matrix theory.

A local law has two well-known easy consequences, one for the eigenvectors and the other for the eigenvalues of A.

(i) The first consequence is *complete eigenvector delocalization*, which states that the normalized eigenvectors $\{\mathbf{u}_i\}$ of A satisfiy with high probability

$$\max\{\|\mathbf{u}_i\|_\infty : 1 \leq i \leq N\} \leq N^{-1/2+o(1)}\,. \tag{2}$$

(ii) The second consequence is a local law for the density of states, which states that the Stieltjes transform of the empirical eigenvalue distribution

$$s(z) := \frac{1}{N} \operatorname{Tr} G(z) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{\lambda_i(A) - z}$$

satisfies $|s(z) - m(z)| = o(1)$ with high probability for all $\eta \gg N^{-1}$. Informally, this means that the semicircle law holds down to very small spectral scales.

It is a standard exercise to show that if $pN \to \infty$ then the (global) semicircle law for $s(z)$ holds, stating that $s(z) \to m(z)$ with high probability for any fixed $z \notin \mathbb{R}$. On the other hand, it is not hard to see that both consequences (i) and (ii) are wrong in the subcritical regime $pN \leq (1 - \epsilon) \log N$. Indeed, in the subcritical regime there is with high probability an isolated vertex, with an associated eigenvector localized at that vertex. Hence, the left-hand side of (2) is equal to one and (i) fails. Moreover, computing the expected number of isolated vertices in the subcritical regime $pN \ll \log N$, it is not hard to find that the number of isolated vertices is large enough to preclude (ii) because of the resulting atom at zero for the empirical spectral measure.

Thus, our assumption $pN \geq C \log N$ is optimal up to the value of the numerical constant C. A local law for the Erdős-Rényi graph was previously proved in [1, 2] under the assumption $pN \geq (\log N)^6$, and the contribution of our work is therefore to cover the very sparse range $C \log N \leq pN \leq (\log N)^6$. Our main result takes the following form.

Theorem 1. *There is a universal constant $C_* \geq 1$ such that the following holds. Let $r \geq 10$, $1 \leq q \leq N^{1/2}$, and $t > 0$. Define the fundamental error parameter*

$$\zeta \equiv \zeta(N, r, q, \eta, f) := \left(\frac{r}{q^2}\right)^{1/4} + \frac{r}{(N\eta)^{1/6}} + \frac{r}{(\log \eta + \log N) q}$$

and suppose that

$$t\zeta \leq 1\,.$$

Then

$$\mathbb{P}\Big(\max_{i,j} |G_{ij}(z) - m(z)\delta_{ij}| > t\zeta\Big) \leq N^5 \left(\frac{C_*}{t}\right)^r,$$

for $N^{-1} \leq \eta = \operatorname{Im} z \leq 1$.

Our proof strategy is based on the approach introduced in [3, 4, 5] and subsequently developed for sparse matrices in [1, 2]. Thus, we derive a self-consistent equation for the Green function G using Schur's complement formula and large deviation estimates, which is then bootstrapped in the spectral scale η to reach the smallest scale N^{-1}. The key difficulty in proving local laws for sparse matrices is that the entries are sparse random variables, and hence fluctuate much more strongly than in the Wigner case $p \asymp 1$. To that end, new large deviation estimates for sparse random vectors were developed in [1], which were however ineffective below the scale $(\log N)^6$.

The key novelty of our approach is a new family of multilinear large deviation bounds for sparse random vectors. They are optimal for very sparse vectors, and in particular allow us to reach the critical scale $pN = C \log N$. They provide bounds on multilinear functions of sparse vectors in terms of mixed ℓ^2 and ℓ^∞ norms of their coefficients. We expect them to be more generally useful in a variety of problems on sparse random graphs. To illustrate them and how they are applied, consider a sparse random vector $X \in \mathbb{R}^N$ which is a single row of the matrix $A - \mathbb{E}A$. Let (a_{ij}) be a symmetric deterministic matrix. Then, for example, we have the L^r bound

$$\Bigg\| \sum_{i \neq j} a_{ij} X_i X_j \Bigg\|_r \leq \left(\frac{4r}{1 + (\log(\psi/\gamma))_+} \vee 4 \right)^2 (\gamma \vee \psi)\,, \tag{3}$$

where

$$\gamma := \left(\max_i \frac{1}{N} \sum_j |a_{ij}|^2 \right)^{1/2}, \qquad \psi := \frac{\max_{i,j} |a_{ij}|}{pN}\,.$$

We first remark that we have to take r to be at least $\log N$. Indeed, our proof consists of an order $N^{O(1)}$ uses of such large deviation bounds. To compensate the factor $N^{O(1)}$ arising from the union bound, we therefore require bounds smaller than N^{-D} on the error probabilities for any fixed $D > 0$, which we obtain (by Chebyshev's inequality) from the large deviation bounds for $r = \log N$. The crucial feature of the bound (3) is the logarithmic factor in the denominator. Without it, there is nothing to compensate the factor $r^2 \geq (\log N)^2$ in the numerator, as $\psi \asymp (pN)^{-1} \asymp (\log N)^{-1}$ in the critical regime. Thus, the applicability of (3) hinges on the fact that ψ/γ is sufficiently large; this assumption can in fact be verified in all of our applications of (3).

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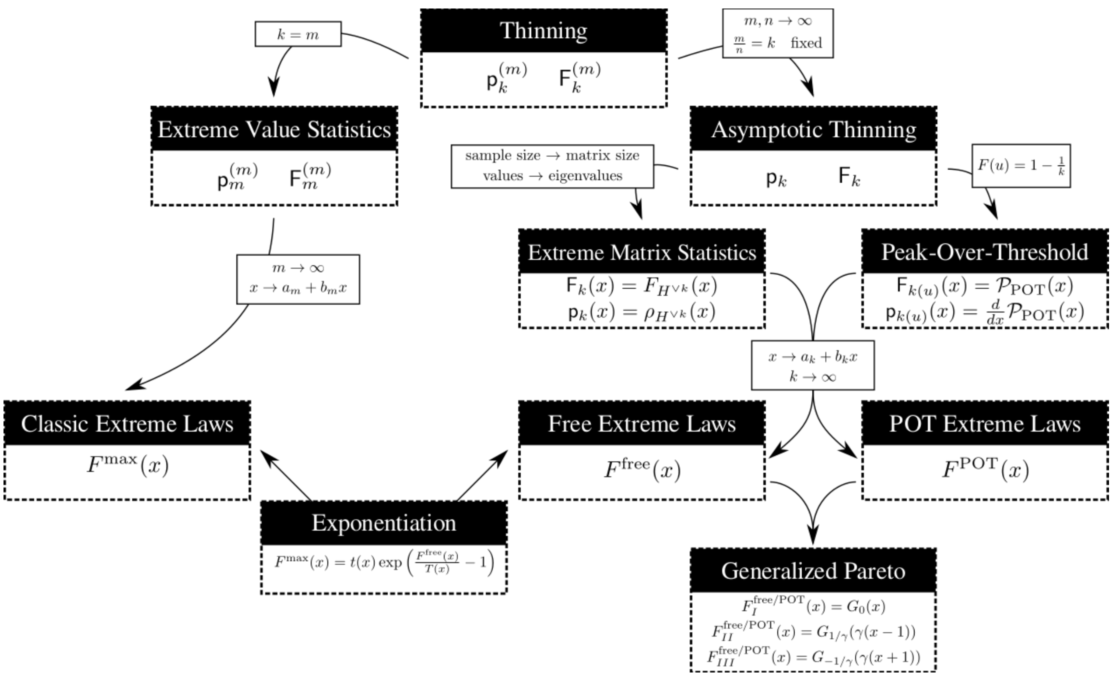
Extreme values *versus* extreme random matrices

Maciej A. Nowak
(joint work with Jacek Grela)

Using the introduced by us *thinning method* [1], we explain the link between classical Fisher-Tippett-Gnedenko classification of extreme events and their free analogue obtained by Ben Arous and Voiculescu [2] in the context of free probability calculus. In particular, we present explicit examples of large random matrix ensembles, realizing free Weibull, free Fréchet and free Gumbel limiting laws, respectively. We also explain, why these free laws are identical to Balkema-de Haan-Pickands limiting distribution for exceedances, i.e. why they have the form of generalized Pareto distributions. Finally, we derive a simple exponential relation between classical and free extreme laws. The above relations and correspondences can be visualized with the help of flow diagram:



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Operator-valued matrices: the noncommutative Lindeberg method

Marwa Banna
(joint work with Guillaume Cébron)

We consider operator-valued Wigner and Wishart matrices whose entries are free or exchangeable elements in some tracial W^*-probability space $(\mathcal{M}, \tau)$. We prove that their distribution is close to that of an operator-valued semicircular element over some subalgebra $\mathcal{B} \subset \mathcal{M}$. More precisely, we provide quantitative estimates on the associated Cauchy transforms that give in turn explicit estimates on the Kolmogorv distance for some cases. Applications to block random matrices with a Wigner or circulant structure and in independent or correlated blocks are also given.

Our approach relies on the Lindeberg method which we extend to the noncommutative setting to approximate Cauchy transforms of linear maps in noncommutating elements. In the free with amalgamation case, this gives the following:

Theorem 1. *Let $(\mathcal{M}, \tau)$ be a tracial W^*-probability space and $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{M}$. Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be two n-tuples of self-adjoint elements in $\mathcal{M}$ which are free with amalgamation over $\mathcal{B}$ and such that $\tau[x_i|\mathcal{B}] = \tau[y_i|\mathcal{B}] = 0$ and $\tau[x_i b x_i|\mathcal{B}] = \tau[y_i b y_i|\mathcal{B}]$ for any $b \in \mathcal{B}$ and $1 \leq i \leq n$. Setting*

$$\mathbf{x} = \sum_{i=1}^{n} x_i \text{ and } \mathbf{y} = \sum_{i=1}^{n} y_i$$

then, for any $z \in \mathbb{C}^+$, we have

$$\left|\tau[G_{\mathbf{x}}(z)] - \tau[G_{\mathbf{y}}(z)]\right| \leq \frac{K_\infty K_2^2}{Im(z)^4} n,$$

where $K_\infty = \max_i \left(\|x_i\|_\infty + \|y_i\|_\infty\right)$ and $K_2 = \max_i(\|x_i\|_{L^2} + \|y_i\|_{L^2})$.

Applying Theorem 1 to $N \times N$ Wigner and Wishart matrices in free entries with possibly different variances, we prove that their distribution is close to that of an operator-valued semicircular element over some subalgebra. More precisely, we provide explicit quantitative estimates for the associated Cauchy transforms which can be passed in some cases to estimates on the Kolmogorov distance.

Random block matrices are special cases of operator-valued matrices for they can be seen as matrices with entries in some noncommutative algebra. In particular, i.i.d. random matrices are exchangeable in the noncommutative space of random matrices and many random block matrix models fit nicely in the framework of matrices with exchangeable entries. We relax the freeness hypothesis in Theorem 1 and consider instead a finite family of exchangeable elements in $(\mathcal{M}, \tau)$.

Theorem 2. *Let $(\mathcal{M}, \tau)$ and $(\mathcal{N}, \varphi)$ be two tracial W^*-probability spaces. Let $(x_1, \dots, x_n)$ be an n-tuple of* exchangeable *elements in $\mathcal{M}$. Consider a family $(N_{i,k})_{1 \leq i,k \leq n}$ of independent standard Gaussian random variables and let*

$(y_1, \ldots, y_n)$ *be the n-tuple of random elements in $\mathcal{M}$ given by*

$$y_i = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} N_{i,k} \left(x_k - \frac{1}{n} \sum_{j=1}^{n} x_j \right).$$

Let $(a_1, \ldots, a_n)$ be an n-tuple of elements in $\mathcal{N}$ and set

$$\mathbf{x} = \sum_{i=1}^{n} x_i \otimes a_i + x_i^* \otimes a_i^* \quad \textit{and} \quad \mathbf{y} = \sum_{i=1}^{n} y_i \otimes a_i + y_i^* \otimes a_i^*.$$

Then, for any $z \in \mathbb{C}^+$,

$$\begin{aligned} &\left| \tau \otimes \varphi(G_{\mathbf{x}}(z)) - \mathbb{E}[\tau \otimes \varphi(G_{\mathbf{y}}(z))] \right| \\ &\leq C \cdot \left(\frac{K_1}{Im(z)^2} \|x_1\|_\infty + \frac{K_2^2}{Im(z)^3} \|x_1\|_\infty^2 \sqrt{n} + \frac{K_\infty K_2^2}{Im(z)^4} \|x_1\|_\infty^3 n + \frac{K_\infty^2 K_2^2}{Im(z)^5} \|x_1\|_\infty^4 n \right), \end{aligned}$$

where $K_\infty = \max_i \|a_i\|_\infty, K_2 = \max_i \|a_i\|_{L^2}$, $K_1 = \| \sum_{i=1}^n a_i \|_{L^1}$ and C is a universal constant.

Theorem 2 shows that sums of exchangeable operators are close in distribution to the expectation of sums in independent averaged Gaussian operators. The appearance of independent Gaussian operators and the independence structure hidden behind exchangeability might be surprising or seem unnatural but it could be explained by the fact that invariance under permutations is a commutative concept. Applying this approximation to $N \times N$ Wigner and Wishart matrices in exchangeable entries, we show that their distribution is close to that of an operator-valued semicircular element by providing quantitative estimates on the associated Cauchy transforms.

As applications to random block matrices, we consider random matrices with independent blocks and random matrices in which the blocks are themselves correlated but have i.i.d. entries. We also present an example of random block matrices where the distribution of the operator-valued semicircular element can be explicitly computed. This is the case of block matrices having a circulant block structure in i.i.d. Wigner blocks. In each of the above cases, we give quantitative estimates on the associated Cauchy transforms which lead in the latter case to quantitative estimates on the Kolmogorov distance.

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Representations of Free Commutators

Friedrich Götze
(joint work with Gennadii Chistyakov)

Let a and b be free random variables with distributions μ_a and μ_b respectively. Consider their hermitean commutator $c = i(ab - ba)$. Then μ_c can be described by combinatorial moment formulas, see [2], whereas μ_c has been characterized by five functional equations for analytic functions in $\mathbb{C} \setminus \mathbb{R}$ with a special asymptotic behavior for $|\Im z| \le |\Re z|$, $z \to \infty$ [1].

Here we provide a characterization of μ_c via multiplicative free convolutions of the symmetrization $\mu_{a_s} := \mu_a \boxplus \bar{\mu}_a$ of μ_a with $\bar{\mu}_a$, where $\bar{\mu}_a(A) := \mu_a(-A)$ for all Borel sets, with respect to the additive free convolution $\boxplus$. Similarly define μ_{b_s}. We have

Theorem 1 (Chistaykov-G. (2018)). *Let a and b be free random variables with nonzero variances. The distribution μ_{c^2} satisfies*

$$\mu_{c^2} \boxtimes \mu_0 = \mu_{a_s^2} \boxtimes \mu_{b_s^2},$$

where μ_0 is the multiplicative infinitely divisible probability measure with the density $\frac{1}{\pi} \frac{1}{\sqrt{x(4-x)}} 1_{[0,4]}(x)$.

We formulate the next results using the Σ transform of a measure μ on $\mathbb{R}_+$ related to its S transform via $\Sigma_\mu(z) = S_\mu(\frac{z}{1-z})$. The function $\Sigma_\mu(z)$ has the form $\Sigma_\mu(z) = K_\mu^{(-1)}(z)/z$, where $K_\mu(z)$ belongs to the class of the Krein functions as defined in [3].

Let a and b be **even** free random variables.

Theorem 2 (Chistyakov-G. (2018), [2]).

$$\Sigma_{\mu_{c^2}}(z) = \frac{1}{4}(2-z)\Sigma_{\mu_{a^2}}\Big(\frac{z}{2-z}\Big)\Sigma_{\mu_{b^2}}\Big(\frac{z}{2-z}\Big)$$

holds for z, where $\Sigma_{\mu_{c^2}}(z)$, $\Sigma_{\mu_{a^2}}\left(\frac{z}{2-z}\right)$ and $\Sigma_{\mu_{b^2}}\left(\frac{z}{2-z}\right)$ are defined, equivalently

$$\mu_{c^2} = \mu_0 \boxtimes \mu_{\nu_a} \boxtimes \mu_{\nu_b},$$

where the probability measures ν_a, ν_b on $[0, +\infty)$ are determined via

$$\Sigma_{\nu_a}(z) := \Sigma_{\mu_{a^2}}(K_{\kappa_0}(z)), \quad \Sigma_{\nu_b}(z) := \Sigma_{\mu_{b^2}}(K_{\kappa_0}(z))$$

with $\kappa_0 = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$ and $K_{\kappa_0}(z) = \frac{z}{2-z}$.

For general free a, b we get that if

(1) $$\mu_{a_s} = \mu_g \boxplus \mu_g \quad \text{and} \quad \mu_{b_s} = \mu_h \boxplus \mu_h$$

with even g, h, there exist probability measures ν_g, ν_h as above such that

(2) $$\mu_{c^2} = \mu_0 \boxtimes \mu_{\nu_g} \boxtimes \mu_{\nu_h}.$$

Consider **even** free random variables a and b, with distributions μ_a and μ_b.

Define $\mathcal{N}_s$ as subclass of Nevanlinna functions $f(z)$ with $f(i\mathbb{R}_+) \subset i\mathbb{R}$. In the following result we characterize in the symmetric case the distribution μ_c of the commutator c via functional equations for the Cauchy-transform $G_{\mu_c}(z)$ of μ_c in terms of subordination functions from $\mathcal{N}_s$. We have

Theorem 3 (Chistyakov-G. (2018)). *There exist unique subordination functions* $-N_1(z), -N_2(z) \in \mathcal{N}_s$

$$\frac{1}{z}N_j(z) = -\gamma_j^{-1/2} + o(1), \quad |\Re z| \le |\Im z|, \ j = 1, 2,$$

such that, for $z \in \mathbb{C}^+$, *and* $\quad Q(z) := 4z^3(1-2z)^{-2}(1-z)^{-1}$

$$N_1(z)^2 N_2(z)^2 = -\frac{1}{4}z^2 Q(N_1(z)G_{\mu_b}(N_1(z))),$$

$$\frac{1}{2}(1 + zG_{\mu_c}(z)) = N_1(z)G_{\mu_b}(N_1(z)) = N_2(z)G_{\mu_a}(N_2(z)).$$

The proofs are based on the characterization of the multiplicative free convolution in terms of Krein functions as described in [3] combined with the characterizations of μ_c in [2] and [1].

In the symmetric case Vasilchuk introduced in [1] three functional equations for analytic functions Γ_1, Γ_2 characterizing the Cauchy transform G_{μ_c} of the commutator of even free random variables a and b, see [1]. From Theorem 3 it follows now that the transforms $\frac{1+zG_{\mu_c}(z)}{2\Gamma_1(z)}$ and $\frac{1+zG_{\mu_c}(z)}{2\Gamma_2(z)}$ have to be of Nevanlinna class.

Note that using the relation (2) we can prove an analogue of Theorem 3 for probability measures μ_a and μ_b satisfying relation (1).

A remaining challenging open problem is to extend these representations to the case of commutators of general non symmetric μ_a and μ_b.

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Infinitesimal Freeness and the GOE

James A. Mingo

We present, [7], a combinatorial approach to the infinitesimal distribution of the Gaussian orthogonal ensemble (GOE). In particular we show how the infinitesimal moments are described by non-crossing pairings, but not those of type B. We demonstrate the asymptotic infinitesimal freeness of independent complex Wishart matrices and compute their infinitesimal cumulants. Using our combinatorial picture we compute the infinitesimal cumulants of the GOE and demonstrate the lack of asymptotic infinitesimal freeness of independent Gaussian orthogonal ensembles.

In this talk we consider the infinitesimal freeness of Belinschi and Shlyakhtenko [1]. Infinitesimal probability spaces have recently been used by Shlyakhtenko [10] to understand small scale perturbations in some random matrix models. Let us recall some of the connections between free probability and random matrix theory.

Let $\{A_N\}_N$ and $\{B_N\}_N$ be two self-adjoint ensembles of random matrices. By this we mean that for each integer $N \geq 1$ we have two self-adjoint matrices with random entries. The eigenvalues of A_N, $\lambda_1^{(A)} \leq \cdots \leq \lambda_N^{(A)}$, are thus random and we form a random probability measure $\mu_N^{(A)}$ with a mass of $1/N$ at each eigenvalue $\lambda_i^{(A)}$. We do the same for B_N and obtain another random measure $\mu_N^{(B)}$. For many ensembles the random measures $\mu_N^{(A)}$ and $\mu_N^{(B)}$ converge to deterministic measures, called the limit eigenvalue distributions. Two well known examples are Wigner's semi-circle law and the Marchenko-Pastur law.

A central problem in random matrix theory is to compute the limit eigenvalue distribution of $C_N = f(A_N, B_N)$ when f is a polynomial or a rational function in non-commuting variables. This would not be possible without some assumptions on the 'relative position' of A_N and B_N. By relative position we mean Voiculescu's notion of freeness or one of its extensions. We do not need freeness for finite N, but only in the large N limit; when this holds we say the ensembles are asymptotically free. When we know that A_N and B_N are asymptotically free then we can apply the analytic techniques of free probability i.e. the R and S transforms to compute the limit distribution of C_N.

The first example of asymptotic freeness was given by Voiculescu [11] where he showed that independent self-adjoint Gaussian matrices were asymptotically free. Since then there have been many generalizations and elaborations.

Infinitesimal freeness is the branch of free probability that enables us to model infinitesimal perturbations in the same way as Voiculescu's theory did for $f(A_N, B_N)$. If we start with A_N as above but now assume that B_N is a non-random fixed finite rank self-adjoint matrix, recent work of Shlyakhtenko [10] and Belinschi and Shlyakhtenko [1] shows that when A_N is complex and Gaussian then there is a universal rule for computing the effect on the outlying eigenvalues.

An infinitesimal distribution can be considered at the algebraic or the analytical level. On the algebraic level an infinitesimal distribution is a pair (μ, μ') of linear functionals on $\mathbb{C}[x]$ such that $\mu(1) = 1$ and $\mu'(1) = 0$. There are a few ways to arrive at such a pair; we shall consider the ones arising from random matrix models. Suppose $\{X_N\}_N$ is an ensemble of self-adjoint random matrices where X_N is $N \times N$ and for all k we have that the limit $\mu(x^k) := \lim_N \mathrm{E}(\mathrm{tr}(X_N^k))$ exists. Then the ensemble $\{X_N\}_N$ has a limit distribution. Suppose further that for all k we have $\mu'(x^k) := \lim_N N(\mathrm{E}(\mathrm{tr}(X_N^k)) - \mu(x^k))$ exists. Then we say that the ensemble has a *infinitesimal distribution*. This was the context of [10].

On the analytical level one can consider a pair (μ, μ') of Borel measures on $\mathbb{R}$ with μ being a probability measure and μ' a signed measure with $\mu'(\mathbb{R}) = 0$. An early examples of an infinitesimal distribution was that of the Gaussian orthogonal ensemble, given by Johansson in [5], also discussed by I. Dumitriu and A. Edelman in [3], and Ledoux in [6]. In this case μ is Wigner's semi-circle law

$$d\mu(x) = \frac{\sqrt{4-x^2}}{2\pi}\,dx \text{ on } [-2,2]$$

and μ' is the difference of the Bernoulli and the arcsine law:

$$d\mu'(x) = \frac{1}{2}\Big(\frac{\delta_{-2}+\delta_2}{2} - \frac{1}{\pi}\frac{1}{\sqrt{4-x^2}}\,dx\Big) \text{ on } [-2,2]. \tag{1}$$

Infinitesimal freeness was built on work of Biane, Goodman, and Nica [2] on freeness of type B. While this does provide a combinatorial basis for infinitesimal freeness, we show that in the orthogonal, or 'real' case, one needs to use the annular diagrams of [8]. Since there is an additional symmetry requirement (see the caption to Fig. 1), we only need the outer half of the diagram. This places infinitesimal freeness somewhere between freeness and second order freeness.

Another example of an infinitesimal distribution was given by Mingo and Nica in [8, Corollary 9.4], although is was not then described as such because the infinitesimal terminology didn't exist at the time. In [8] complex Wishart matrices were considered. In particular $X_N = \frac{1}{N}G^*G$ with G a $M\times N$ Gaussian random matrix with independent $\mathcal{N}(0,1)$ entries. When $\lim_N M/N = c$ and we get the well known Marchenko-Pastur distribution with parameter c (see [9, Ex. 2.11]). If we further assume that $c' := \lim_N(M - Nc)$ exists then there is an infinitesimal distribution with μ' given by

$$d\mu'(x) = -c'\begin{cases} \delta_0 - \frac{x+1-c}{2\pi x\sqrt{(b-x)(x-a)}}\,dx & c<1 \\ \frac{1}{2}\delta_0 - \frac{1}{2\pi\sqrt{x(4-x)}}\,dx & c=1 \\ \quad\; - \frac{x+1-c}{2\pi x\sqrt{(b-x)(x-a)}}\,dx & c>1 \end{cases}. \tag{2}$$

Note that the continuous part of μ' is supported on the interval $[a,b]$ with $a = (1-\sqrt{c})^2$ and $b = (1+\sqrt{c})^2$. We show that at a formal level we can consider μ' to be a derivative of μ. However, in [8] the distribution was given in terms of infinitesimal cumulants: $\kappa'_n = c'$ for all n, where κ'_n is an infinitesimal cumulant; the density above is obtained from the equation

$$g(z) = -r(G(z))G'(z)$$

where g and r are respectively the infinitesimal Cauchy and R-transform. The intuitive idea is to regard c' as the derivative, as $1/N \to 0$, of the shape parameter c. For a very simple case, take $c = 1$ and $c' \in \mathbb{Z}$ an integer. We let $M = N + c'$, then $M/N \to c$ and $M - cN = c'$. Earlier authors only considered the case $c' = 0$, which one can always arrange by taking (M_k, N_k) to be the k^{th} convergent in the continued fraction expansion of c.

[!h] The coefficient of the $\frac{1}{N}$ term in the expansion of $\mathrm{E}(\mathrm{tr}(X_N^n))$ in the GOE case is known to count maps on locally orientable surfaces (see [4, Thm. 1.1] and [6, §5]). What is new here is that the infinitesimal moments of the GOE are described

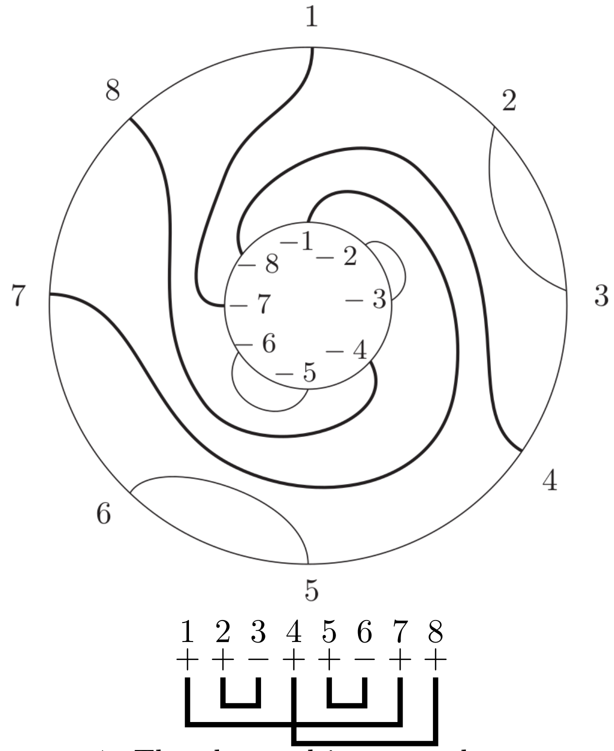

FIGURE 1. The planar objects are the non-crossing annular pairings of [8], except in this case the circles have the same orientation. Moreover we require that $(r - r)$ is never a pair and if (r, s) is a pair then $(-r, -s)$ is also a pair. These are the only conditions. The same data can be recorded and a pair (π, ϵ) where π is a crossing of [8] and ϵ is a an assignment of signs. See the lower figure, note that we have crossing diagrams.

by planar objects and thus stay within the class of the non-crossing partitions standard in free probability, but not the non-crossing partitions of type B used in [2]. We show that independent GOE's are not asymptotically infinitesimally free, nor are a GOE and a deterministic matrix. However we present a universal rule for computing mixed moments.

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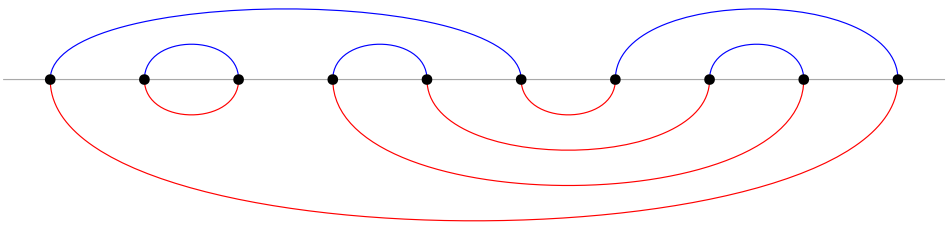
On the number of components of random meandric systems

Ion Nechita, Alexandru Nica
(joint work with M. Fukuda, and resp. I.P. Goulden, D. Puder)

A meandric system on $2n$ points is obtained by independently drawing two non-crossing pairings of the $2n$ points, above and respectively below a given horizontal line (as illustrated below).



Since the set $NC_2(2n)$ of non-crossing pairings of $2n$ points is counted by the Catalan number Cat_n, there are Cat_n^2 meandric systems on $2n$ points. Let $X_n : NC_2(2n)^2 \to \{1, \ldots, n\}$ be the random variable which gives the number of components of a random meandric system (where all meandric systems are assumed to have the same probability of $1/\mathrm{Cat}_n^2$).

In the first part of the talk, given by Alexandru Nica, we examine the expectation of the random variable X_n. We conjecture that $\lim_{n\to\infty} \mathbb{E}(X_n)/n$ exists, and in support of this conjecture we prove the bounds $\liminf_{n\to\infty} \mathbb{E}(X_n)/n \geq 0.17$ and $\limsup_{n\to\infty} \mathbb{E}(X_n)/n \leq 0.5$. Quite interestingly, our proof of the lower bound uses the derivative at time $t = 1$ for a convolution semigroup with respect to the operation $\boxplus$ of free additive convolution. This part of the talk is based on a joint work by Goulden-Nica-Puder [1].

In the second part of the talk, given by Ion Nechita, we examine the behavior of probabilities of the form $\mathbb{P}(X_n = n - r)$ for a fixed value of $r \geq 0$. We consider the generating function

$$F_r(t) = \sum_{n=r+1}^{\infty} \mathbb{P}(X_n = n - r)\mathrm{Cat}_n^2 t^n,$$

and we prove that upon doing the substitution $t = w/(1+w)^2$ one obtains an equation of the form

$$F_r(t) = \frac{w^{r+1}(1+w)}{(1-w)^{2r-1}}\widetilde{P}_r(w),$$

where $\widetilde{P}_r$ is a polynomial of degree $\leq 3r-3$. One has a precise algorithm for computing the polynomials $\widetilde{P}_r$; this allows us to derive explicit formulas for $\mathbb{P}(X_n = n-r)$ for fixed small values of r, and also the asymptotic behavior of $\mathbb{P}(X_n = n-r)$ as $n \to \infty$, for an arbitrary fixed value of r. This part of the talk is based on a joint work by Fukuda-Nechita [2].

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Delocalization of Eigenvectors of High Girth Graphs

Nikhil Srivastava

(joint work with Shirshendu Ganguly)

1. Context

Spectral graph theory studies graphs via associated linear operators such as the Laplacian and the adjacency matrix. While the extreme eigenvectors of these operators are relatively well-understood and correspond to sparse cuts and colorings, much less is known about the combinatorial meaning of the interior eigenvectors. Most of the literature about them falls into two categories:

1. **Analysis of eigenvectors of random graphs.** For example, Dekel, Lee, Linial [1] prove that any eigenvector of a dense random graph has a bounded number of nodal domains i.e., connected components where the eigenvector does not change sign. Following a sequence of results by various authors, in a recent breakthrough work Bauerschmidt, Huang, Yau [2], among various other things, show that with high probability, any 'bulk' eigenvector v of a random regular graph with n vertices and a large enough but fixed constant degree, is ℓ_∞ delocalized in the following sense:

$$||v||_\infty \leq \frac{\log^C(n)}{\sqrt{n}}||v||_2,$$

where $||\cdot||_2$, and $||\cdot||_\infty$ denote the usual ℓ_2 and ℓ_∞ norms respectively and C is a constant. For a more precise statement see Theorem 1.2 in [2]. In all of these works the randomness of the model is used heavily, and weaker notions of delocalization are also considered (see e.g. [3]).

2. A parallel story based on **asymptotic analysis of sequences of deterministic graphs**. The driving force for this is the so called Quantum Unique Ergodicity (QUE) conjecture by Rudnick and Sarnak [4]. The QUE conjecture states that on any compact negatively curved manifold all high energy eigenfunctions of the Laplacian equi-distribute. The conjecture is still widely open having been verified in only a few cases; perhaps most notably for the Hecke orthonormal basis on an arithmetic surface by Lindenstrauss [5]. Brooks-Lindenstrauss [6] initiated the study of graph-theoretic analogues of this conjecture. The analogue of negatively curved manifolds are high girth regular graphs — the girth is defined as the length of the shortest cycle in a graph. Subsequently, Anantharaman and Le-Masson [7] proved an asymptotic version of quantum ergodicity for regular expanders which converge (in the Benjamini-Schramm local topology) to the infinite $d-$regular tree.

2. New Results

We improve on the beautiful result of [6], which roughly says that if a graph does not have many short cycles, then eigenvectors cannot localize on small sets: for any eigenvector, any subset of the vertices representing a fraction of the ℓ_2^2 mass must have size n^δ for some δ depending on the fraction. Below is a statement of a special case of their theorem for graphs of large girth; it also works for graphs with few short cycles, but we do not discuss that for simplicity.
Theorem 1.(Brooks-Lindenstrauss, [6]) Suppose $G = (V, E)$ is a $(d+1)-$regular graph with adjacency matrix A and let g be the girth of G. Then for any normalized ℓ_2 eigenvector $v = (v_x)_{x \in V}$, of A and $S \subset V$ with $\|v_S\|_2^2 := \sum_{x \in S} v_x^2 \geq \epsilon$,

$$|S| \geq \Omega_d(\epsilon^2 d^{2^{-8}\epsilon^2 g}).$$

Viewed in the contrapositive, the theorem therefore says that the existence of an eigenvector of A with ϵ fraction of its mass on $k = |S|$ coordinates implies that the graph must contain a cycle of length $O(\log_d(k/\epsilon)/\epsilon^2)$. In fact, a close examination of the proof reveals that it gives an upper bound which varies between $O(\log_d(k/\epsilon)/\epsilon)$ and $O(\log_d(k/\epsilon)/\epsilon^2)$ depending on the Diophantine properties of the eigenvalue being considered.

In this paper, we contribute to the understanding of this phenomenon in two ways.

2.1. Better Delocalization.

First, we improve the above bound to $O(\log_d(k/\epsilon)/\epsilon)$ for all eigenvalues of $d+1$-regular graphs, irrespective of the number theoretic properties of the eigenvalue. The proof involves replacing the approximation-theoretic component of their proof by a simpler and more efficient method. Specifically, we prove the following
Theorem 2. Suppose G is a $(d+1)$-regular graph of girth g and v is a normalized eigenvector of the adjacency matrix of G. Then any subset S with $\|v_S\|_2^2 \geq \epsilon$ must have

$$|S| \geq \frac{d^{\epsilon g/4}\epsilon}{2d^2}.$$

The contrapositive of the above theorem implies that if there exists ϵ and k and S such that $|S| = k$ and $\|v_S\|_2^2 = \epsilon$, then

$$g \leq \frac{4\log_d(k/\epsilon) + O(1)}{\epsilon}.$$

2.2. **Sharpness of the Bound.** On the other hand, for every $d \geq 2$, sufficiently large k, and $\epsilon \in (0,1)$, we exhibit a $(d+1)-$regular graph with a localized eigenvector which has girth at least $\Omega(\log_d(k)/\epsilon)$, showing that our improved bound is sharp up to an additive $\log(1/\epsilon)$ factor in the numerator, which is negligible whenever $k = \Omega(1/\epsilon^c)$ for any c. We are able to construct such eigenvectors for a dense subset of eigenvalues in $(-2\sqrt{d}, 2\sqrt{d})$. The proof is probabilistic, and involves gluing together two trees without introducing any short cycles and while controlling their eigenvectors.

Theorem 3. For every $d \geq 2$, sufficiently large k and all $\epsilon > 0$, there is a finite $(d+1)-$regular graph G with the following properties.

(1) A_G has a normalized eigenvector v with eigenvalue $\lambda \in (-2\sqrt{d}, 2\sqrt{d})$ and

$$\|v_S\|_2^2 = \Omega_\lambda(\epsilon)$$

for a set S of size k, where the implicit constant C_λ depends on λ and is bounded away from zero on any subinterval of $(-2\sqrt{d}, 2\sqrt{d})$.

(2) G has girth at least

$$\Omega\left(\frac{\log_d(k)}{\epsilon}\right).$$

Moreover, for every fixed ϵ (or for every fixed, sufficiently large k), the set of eigenvalues attained by the above graphs is dense in $(-2\sqrt{d}, 2\sqrt{d})$.

Notice that the above theorem does not provide any bound as the eigenvalue λ approaches one of the edges $\pm 2\sqrt{d}$.

Altogether, Theorems 2 and 3 precisely quantify the delocalization properties of graphs of high girth, and establish that they are considerably weaker than those of random graphs.

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Traffic Independence implies Freeness over the Diagonal

Camille Male
(joint work with B. Au, G. Cébron, A. Dahlqvist, F. Gabriel)

Traffic probability theory was initiated in 2011 for the study of large permutation invariant random matrices [3]. A traffic is an abstract non commutative random variable, living in a space with more structure than the structure of associative algebra called *algebra over an operad*. The traffic distribution of variables is the knowledge of the *-distribution of the so-called graph polynomials in the variables. In particular the traffic distribution of a random matrix captures much more information than its *-distribution. The notion of traffic independence is a central notion in this context. In some precise sense, it encodes the different notions of non commutative independence.

Large random matrices is the motivation to introduce this setting. In the limit in large dimension N, many examples of random matrices show to be asymptotic traffic independent. This is true for random matrices with independent, not only in the Wigner regime where all entries are small but also for *sparse* matrices, such has symmetric matrices with independent entries distributed according to the Bernouilli distribution with parameter $\frac{p}{N}$ for p fixed. More generally, traffic independence is defined in order to hold for independent permutation matrices satisfying an asymptotic factorization property.

A central question raising from this conceptual picture is how traffic probability theory can be used *in practice* for the study of random matrices ? Free analysis allows to compute efficiently a numerical approximation of the density of a self-adjoint polynomial in free variables. Can we interpret traffic independence as some modification of free independence in order to use free analysis theory ? Concordant evidences from long-standing and recent results and the study of several approaches ([4, 2]) yield our research team to a first step in this approach:

Traffic Independence implies Freeness with Amalgamation over the diagonal.

In particular, the theory of free analysis with amalgamation (introduced by Voiculescu [5]) allows to use predict numerically the spectrum of polynomials of a large class of random matrices. Moreover, a weaker assumption than traffic independence is actually required to provide asymptotic freeness over the diagonal of random matrices. It follows that this phenomenon still holds after applying a variance profile to permutation invariant matrices, which recovers the original example of Shlyakhtenko [4].

Stated in terms of large matrices, our result need the two following concepts.

Definition 1. (1) *An operator-valued probability space is a triplet* $(\mathcal{A}, \mathcal{D}, \Delta)$, *where* $\mathcal{D} \subset \mathcal{A}$ *are unital* **-algebras and* $\Delta : \mathcal{A} \to \mathcal{D}$ *is a linear form such that* $\Delta(d_1 a d_2) = d_1 \Delta(a) d_2$ *for any* $a \in \mathcal{A}$, d_1, d_2 *in* $\mathcal{D}$.

(2) *A* $\mathcal{G}$*-algebra is a vector space* $\mathcal{A}$ *endowed with an* action of graph operations, *described more precisely as follow. A* K*-graph operation is a finite connected directed graph* g *with the data of an input and an output vertex. Then for each* K*-graph operation* g *is a associated is linear map*

$Z_g : \mathcal{A}^{\otimes K} \to \mathcal{A}$ satisfying natural properties (identity, associativity and equivariance [3, Definition 4.6]*).*

Any $\mathcal{G}$-algebra $\mathcal{A}$ has a structure of unital associative algebra for the product $\times$ given by linear sequence of edges, namely $Z_{\cdot\overset{1}{\leftarrow}\cdot\overset{2}{\leftarrow}\cdot}(a_1 \otimes a_2) = a_1 \times a_2$. Moreover, let g denote the graph operation consisting in a single loop. We denote the linear form $\Delta = Z_g$ and the subspace $\mathcal{D} = \Delta(\mathcal{A})$ of $\mathcal{A}$. Then $\mathcal{D}$ is an abelian subalgebra of $\mathcal{A}$ for $\times$ and the triplet $(\mathcal{A}, \mathcal{D}, \Delta)$ is an operator-valued probability space.

The space M_N of N by N complex matrices is a $\mathcal{G}$-algebra, and Δ is the projection onto diagonal matrices $\Delta(A) = \mathrm{diag}(A_{ii})_i$. Let now $\mathbf{A}_N = (A_j)_{j\in J}$ be a family of random matrices. The smallest operator-valued probability sub-space $(\mathcal{A}_N, \mathcal{D}_N, \Delta)$ of M_N containing $\mathbf{A}_N$ is described in terms of graph operation. The subspace $\mathcal{D}_N$ of diagonal matrices is generated by the $Z_g(A_{j_1} \otimes \cdots \otimes A_{j_K})$ such that g is a *well-oriented cactus* with equal input and output. Hence $\mathcal{A}_N$ is the space $\mathcal{D}_N\langle \mathbf{A}_N\rangle$ of operator-valued polynomials with coefficients in $\mathcal{D}_N$ evaluated in the $\mathbf{A}_N$.

Theorem 1. *For each $\ell = 1, \dots, L$, let $\mathbf{A}_N^{(\ell)} = (A_{N,j}^{(\ell)})_{j\in J}$ be a permutation invariant family of random matrices. Assume the families are independent and satisfy Mingo-Speicher bounds for the trace of graph polynomials in matrices (this holds if the matrices are bounded in operator norm). Then in the smallest operator-valued probability space $(\mathcal{A}_N, \mathcal{D}_N, \Delta)$ containing all the families $\mathbf{A}_N^{(\ell)}$'s, the families $\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)}$ are asymptotically free over the diagonal in the following sense. For any $n \geq 2$, any alternating sequence $\ell_1 \neq \cdots \neq \ell_n$, and for any matrices $A_i \in \mathcal{D}_N\langle \mathbf{A}_N^{(\ell_i)}\rangle$, $i = 1, \dots, n$, such that $\Delta(A_i) = 0$ can be written as a graph polynomial with bounded degree and coefficients, the diagonal matrix $\varepsilon_N = \Delta(A_1 \cdots A_n)$ tends to zero in p-Shatten norm for any $p \geq 1$. Namely for any $p \geq 1$ the expectation of*

$$\frac{1}{N}\mathrm{Tr}\big[(\varepsilon_N \varepsilon_N^*)^p\big]$$

tends to zero as N tends to infinity. Moreover, let $\big(\Gamma_{N,j}^{(\ell)}\big)_{\ell\in[L], j\in J}$ be a family of random matrices with uniformly bounded entries, independent of $(\mathbf{A}_N^{(1)}, \dots, \mathbf{A}_N^{(L)})$. Then the families $\tilde{\mathbf{A}}_N^{(1)}, \dots, \tilde{\mathbf{A}}_N^{(L)}$ are asymptotically free over the diagonal in the same sense, where

$$\tilde{\mathbf{A}}_N^{(\ell)} = \Big(A_{N,j}^{(\ell)} \circ \Gamma_{N,j}^{(\ell)}\Big)_{j\in J},$$

and $\circ$ denotes the entrywise product.

Thanks to Belinschi, Mai and Speicher fixed point algorithm [1] we can compute numerical simulations of empirical spectral distribution (e.s.d) for the sum of independent permutation invariant matrices sample from various models. The deterministic equivalents given by assuming the matrices free over the diagonal are observed to match with high precision the actual e.s.d. of the sum, even when one consider a single realization of the matrices.

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Berry-Esseen theorems for Boolean and Monotone convolutions

Octavio Arizmendi

(joint work with Mauricio Salazar and Jiunchau Wang)

The Central Limit Theorem (CLT) is possibly the most important limit theorem in probability, and it is used often in applications. A quantitative version of the CLT was proved independently by Berry [3] and Esseen [6]. The Berry-Esseen Theorem states that if μ is a probability measure vanishing mean 0, variance 1, and $\int_{\mathbb{R}} |x|^3 d\mu < \infty$, then the distance to the standard Gaussian distribution, $\mathcal{N}$, of the normalized n-fold convolution of μ is bounded as follows

$$d_{kol}(D_{\frac{1}{\sqrt{n}}}\mu^{*n}, \mathcal{N}) \leq C\frac{\int_{\mathbb{R}} |x|^3 d\mu}{\sqrt{n}},$$

where d_{kol} denotes the Kolmogorov distance between measures, $D_b\mu$ denotes the dilation of a measure μ by a factor $b > 0$, $*$ denotes the classical convolution, and C is an absolute constant.

In Non Commutative Probability, as proved by Muraki [8], there are essentially four natural notions of independence: classical, Boolean, free and monotone . For each type of independence there exists a CLT stating that the normalized sum of independent random variables with finite variance converges to the Gaussian, semicircle [10], Bernoulli [9] and arcsine [7] distributions, respectively.

For the Free Central Limit Theorem, a quantitatve version analog to Berry-Esseen Theorem was given by Kargin for the bounded case and then improved by Chistyakov and Götze [5] for measures with finite fourth moment in the following form: Let us denote $m_n(\mu) = \int x^n d\mu(x)$, if μ is a probability measure with $m_1(\mu) = 0$, $m_2(\mu) = 1$ and $m_4(\mu) < \infty$, then the distance to the standard semicircle distribution $\mathcal{S}$ satisfies

$$d_{kol}(D_{\frac{1}{\sqrt{n}}}\mu^{\boxplus n}, \mathcal{S}) \leq C'\frac{|m_3(\mu)| + |m_4(\mu)|^{1/2}}{\sqrt{n}},$$

where the symbol $\boxplus$ denotes the free convolution, and C' is an absolute constant.

In this talk we describe results regarding the quantitative versions of the monotone and Boolean Central Limit Theorems.

The first result is the main theorem from the joint work with M. Salazar [1] which gives give a qualitative description of the Boolean Central Limit Theorem in the case of measures with bounded support, and allows to obtain a bound for the Lévy distance to the Bernoulli distribution $\mathbf{b} = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$..

Theorem 1. *Let μ be a probability measure such that $m_1(\mu) = 0$, $m_2(\mu) = 1$, and $supp(\mu) \in [-K, K]$. Then the measure $\mu_n := D_{\frac{1}{\sqrt{n}}}\mu^{\uplus n}$ satisfies for $\sqrt{n} > K$ that:*

1) *$supp\,\mu_n \subset [\frac{-K}{\sqrt{n}}, \frac{K}{\sqrt{n}}] \cup \{x_1, x_2\}$, where $|(-1) - x_1| \leq \frac{K}{\sqrt{n}}$ and $|1 - x_2| \leq \frac{K}{\sqrt{n}}$.*
2) *For $p = \mu_n(\{x_1\})$, $q = \mu_n(\{x_2\})$ and $r = \mu_n([\frac{-K}{\sqrt{n}}, \frac{K}{\sqrt{n}}])$, we have that $p, q \in [\frac{1}{2} - \frac{2K}{\sqrt{n}}, \frac{1}{2} + \frac{K}{2\sqrt{n}}]$ and $r < \frac{4K}{\sqrt{n}}$.*

In particular, the Lévy distance between μ_n and $\mathbf{b}$ is bounded by

$$L(\mu_n, \mathbf{b}) \leq \frac{2K}{\sqrt{n}}.$$

The above estimate for the rate of convergence is sharp in the sense that there is an example when $L(\mu_n, \mathbf{b}) \geq \frac{c_1}{\sqrt{n}}$, for some c_1 and all $n > 0$.

The proof of the above result is based on a careful analysis of the zeros and analytic continuation to the real line of the reciprocal Cauchy transform of μ_n, defined as $F_\mu := \frac{1}{G_\mu}$, where $G_{\mu_n} = \int \frac{1}{z-t} d\mu_n$.

The second result comes from the recent joint work with M. Salazar and J.C. Wang [1] and describes the speed of convergence for the Monotone Central Limit Theorem to the arcsine distribution $\mathbf{a}$, with density

$$d\mathbf{a}(t) := \frac{1}{\pi\sqrt{2 - t^2}}\ t \in [-\sqrt{2}, \sqrt{2}].$$

Theorem 2. *Let μ be a probability measure with mean 0 and variance 1, and denote $\mu_n := D_{\frac{1}{\sqrt{n}}}\mu^{\rhd n}$. We write F_μ in its Nevanlinna integral representation*

$$F_\mu(z) = z + \int_{\mathbb{R}} \frac{1}{t - z}\, d\nu(t), \qquad z \in \mathbb{C}^+.$$

(1) *If the measure ν has the first absolute moment $c \in (0, +\infty)$, then*

$$d_{kol}(\mu_n, \mathbf{a}) \leq 71\sqrt[4]{c}\, n^{-1/8}$$

for sufficiently large n.

(2) *If the measure μ has finite sixth moment, then there exists some constant K depending on μ, such that*

$$d_{kol}(\mu_n, \mathbf{a}) \leq \frac{K}{n^{1/4}},$$

for sufficiently large n.

The rate of convergence in part (2) is sharp in the sense that there is an example when $d_{kol}(\mu_n, \mathbf{a}) \geq c_2 n^{-1/4}$, for some c_2 and all $n > 0$.

The proof of the above result is based on Bai's inequality from [4]: If two probability measures μ and ν, with cumulative distribution functions C_μ and C_ν, satisfy

$$\int_{\mathbb{R}} |C_\mu(x) - C_\nu(x)|\, dx < +\infty,$$

then

$$d_{kol}(\mu,\nu) \leq \int_{\mathbb{R}} \left| \frac{1}{F_\mu(x+iy)} - \frac{1}{F_\nu(x+iy)} \right| dx + \frac{1}{y} \sup_{r\in\mathbb{R}} \int_{|t|\leq 2\sqrt{3}y} |C_\nu(r+t) - C_\nu(t)|\, dt$$

for all $y > 0$.

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Invariant projections for sums of random variables free over an Abelian algebra

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Consider a tracial operator-valued W*-noncommutative probability space $(\mathcal{A}, E, \mathcal{L}, \tau)$ (that is, $\mathcal{A}$ is a von Neumann algebra, $\mathcal{L} \subseteq \mathcal{A}$ is a von Neumann subalgebra, $\tau\colon \mathcal{A} \to \mathbb{C}$ is a normal faithful tracial state, and $E\colon \mathcal{A} \to \mathcal{L}$ is the trace-preserving conditional expectation - see [10]). Assume that $X = X^*, Y = Y^* \in \mathcal{A}$ are two selfadjoint noncommutative random variables which are free over $\mathcal{L}$ with respect to E (in the sense of [10]). The question considered in this talk was the following: under what conditions on X, Y does there exist a projection $0 \neq p \in \mathcal{A}$ and a number $a \in \mathbb{R}$ such that $p = \ker(X + Y - a)$? A first answer to this question in the context of free probability is due to Bercovici and Voiculescu [5], in the case when $\mathcal{L} = \mathbb{C} \cdot 1$ (and in particular $E = \tau$). Their result states that $0 \neq p = \ker(X + Y - a)$ if and only if there exist $b, c \in \mathbb{R}$ and projections $q, r \in \mathcal{A}$ such that $a = b + c$, $q = \ker(X - b), r = \ker(Y - c)$, and $\tau(q + r) - 1 > 0$. In that case, $\tau(p) = \tau(q + r) - 1$. In the last couple of years there were several strong results on the *absence* of nonzero kernels for such sums [8, 7, 2]. However, no complete characterization of the existence of p in terms of properties of X and Y were known until now. Motivated by the recent article [1] of Au, Cébron, Dahlqvist, Gabriel, and Male, in which they show that, roughly speaking, independent permutation-invariant random matrices are asymptotically free over the algebra of diagonal matrices, we investigated the problem of kernels of sums of selfadjoint random variables which are free with amalgamation over an Abelian von Neumann subalgebra $\mathcal{L}$ of $\mathcal{A}$. In this case, several explicit formulae relating p and the kernels of certain affine deformations of X and Y can be obtained in terms of Voiculescu's analytic subordination functions for operator-valued distributions [9].

In order to state those results, we need the following notations. We denote by $\mathcal{L}\langle X\rangle$ the von Neumann algebra generated by $\mathcal{L}$ and X. The trace-preserving conditional expectation from $\mathcal{A}$ onto $\mathcal{L}\langle X\rangle$ is denoted by $E_{\mathcal{L}\langle X\rangle}$. In particular, $E|_{\mathcal{L}\langle X\rangle} \circ E_{\mathcal{L}\langle X\rangle} = E$. An element $\zeta \in \mathcal{A}$ has a unique decomposition $\zeta = \Re\zeta + i\Im\zeta$ in real and imaginary parts, where $\Re\zeta = \frac{\zeta+\zeta^*}{2}$ and $\Im\zeta = \frac{\zeta-\zeta^*}{2i}$ are selfadjoint. For $\zeta \in \mathcal{A}$, we write $\zeta > 0$ to signify that $\zeta = \zeta^*$ and the spectrum $\sigma(\zeta) \subset (0, +\infty)$. We also write $H^+(\mathcal{L}) = \{b \in \mathcal{L} \colon \Im b > 0\}$. It was shown by Voiculescu [9] that there exist two analytic maps $\omega_1, \omega_2 \colon H^+(\mathcal{L}) \to H^+(\mathcal{L})$ such that

$$E_{\mathcal{L}\langle X\rangle}\left[(b - X - Y)^{-1}\right] = (\omega_1(b) - X)^{-1},\ E_{\mathcal{L}\langle Y\rangle}\left[(b - X - Y)^{-1}\right] = (\omega_2(b) - X)^{-1}.$$

As a consequence of the linearizing property of the operator-valued R-transform [10], one obtains

$$E\left[(b - X - Y)^{-1}\right] = \left(\omega_1(b) + \omega_2(b) - b\right)^{-1}, \quad b \in H^+(\mathcal{L}). \tag{1}$$

We make use of the following characterization of p via Borel functional calculus:

$$p = \text{so-}\lim_{y \searrow 0} \frac{iy}{iy + a - X - Y} = \text{so-}\lim_{y \searrow 0} \frac{y^2}{y^2 + (a - X - Y)^2},$$

where so-lim stands for limit in the strong operator topology.

Using the tools, results, and methods from [4], we show that

$$\varpi_j^{\Re} := \text{so-}\lim_{y \searrow 0} \left[\frac{y}{\Im\omega_j(a + iy)}\right]^{\frac{1}{2}} \Re\omega_j(a + iy) \left[\frac{y}{\Im\omega_j(a + iy)}\right]^{\frac{1}{2}}, \quad j = 1, 2,$$

exists and is a bounded, selfadjoint operator in $\mathcal{L}$. The existence of the nonnegative operator

$$\varpi_1^{\Im} := \text{so-}\lim_{y \searrow 0} \frac{y}{\Im\omega_j(a + iy)}, \quad j = 1, 2,$$

follows quite easily from properties of classical maps between upper half-planes.

With these notations, we show that whenever $0 \neq p = \ker(X + Y - a)$, we have

$$\ker\left(\sqrt{\varpi_1^{\Im}}X\sqrt{\varpi_1^{\Im}} - \varpi_1^{\Re}\right) \ominus \ker(\varpi_1^{\Im}) \neq \{0\},$$

$$\ker\left(\sqrt{\varpi_2^{\Im}}Y\sqrt{\varpi_2^{\Im}} - \varpi_2^{\Re}\right) \ominus \ker(\varpi_2^{\Im}) \neq \{0\}.$$

Moreover,

$$\begin{aligned} (2) \qquad & E[p] + \Xi \\ & \leq E\left[\ker\left(\sqrt{\varpi_1^{\Im}}X\sqrt{\varpi_1^{\Im}} - \varpi_1^{\Re}\right) + \ker\left(\sqrt{\varpi_2^{\Im}}Y\sqrt{\varpi_2^{\Im}} - \varpi_2^{\Re}\right)\right], \end{aligned}$$

where $\Xi = \lim_{y\to 0} \frac{(\Im E[(a+iy-X-Y)^{-1}])^2}{(\Im E[(a+iy-X-Y)^{-1}])^2 + (\Re E[(a+iy-X-Y)^{-1}])^2}$ is a positive operator between $\frac{4E[p]}{4E[p]+1}$ and 1.

The fact that this result is less satisfactory than the result of Bercovici and Voiculescu from the case $\mathcal{L} = \mathbb{C} \cdot 1$ is due to the possibility that the Julia-Carathéodory derivative of the subordination functions ω_1, ω_2 at the point a may very well not exist. Indeed, a simple computation based on relation (1) shows that both ω_1, ω_2 have a Julia-Carathéodory derivative at a if and only if $E[p] > 0$. In that case, $\ker(\varpi_j^{\Im}) = \{0\}$, $j = 1, 2$, and the inequality in (2) becomes equality, with $\Xi = 1$.

Joint work in progress with H. Bercovici and W. Liu covering the case $E[p] > 0$ and $\mathcal{L}$ arbitrary, and the case $\mathcal{L}$ finite dimensional and $E[p]$ arbitrary, was also reported.

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