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Mini-Workshop: Numerical Analysis for Non-Smooth PDE-Constrained Optimal Control Problems

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Abstract. This mini-workshop brought together leading experts working on various aspects of numerical analysis for optimal control problems with non-smoothness. Fifteen extended abstracts summarize the presentations at this mini-workshop.

Mathematics Subject Classification (2010): 65N30, 49J20, 49M25.

Introduction by the Organisers

The mini-workshop "Numerical Analysis for Non-Smooth PDE-Constrained Optimal Control Problems" was organized by Susanne C. Brenner (Baton Rouge), Dmitriy Leykekhman (Storrs) and Boris Vexler (Garching). This meeting was attended by 16 participants from 5 countries.

Modern real-life applications, such as optimal control of mechanical systems and identification of parameters in environmental processes, lead to optimization problems governed by systems of partial differential equations (PDEs). Finite element methods are by far the most popular choices for approximating such problems numerically. The theory of error analysis and convergence is fairly mature for smooth elliptic and parabolic problems. However, the theory of non-smooth PDE-constrained optimal control problems is far from complete. The main goal of this workshop was to bring together leading experts in the field to discuss current developments for several classes of problems with non-smoothness.

Topics discussed included problems with nonsmooth/nonlinear/novel constraints (Chrysafinos, May, Meyer, Neitzel, Rösch, Wollner), problems with nonsmooth objective functions (Leykekhman, Wachsmuth), problems with nonsmooth solutions (Apel, Pfefferer, Sung), novel discretizations (Gong, Pieper) and fast solvers (Drăgănescu, Gedicke).

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Mini-Workshop: Numerical Analysis for Non-Smooth PDE-Constrained Optimal Control Problems

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Abstracts

A priori error analysis for optimal control problems governed by VIs of the 2nd kind

Christian Meyer
(joint work with Thomas Apel, Monika Weymuth)

We consider the following optimal control problem governed by a variational inequality (VI) of the second kind:

$$\begin{cases} \min & \dfrac{1}{2}\,\|y-y_d\|^2_{L^2(\Omega)} + \dfrac{\alpha}{2}\,\|u\|^2_{L^2(\Omega)} \\ \text{s.t.} & u \in L^2(\Omega), \quad y \in H^1_0(\Omega) \\ & \displaystyle\int_\Omega \nabla y \cdot \nabla(v-y)\,dx + \|v\|_{L^1(\Omega)} - \|y\|_{L^1(\Omega)} \geq \langle u, v-y\rangle \quad \forall\, v \in H^1_0(\Omega). \end{cases}$$

Herein, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain and $y_d \in L^2(\Omega)$ and $\alpha > 0$ are given data. Due to the L^1-norm, the control-to-state map associated with the VI is in general not Gâteaux-differentiable, which is the major challenge of the problem under consideration. Based on known a priori finite element (FE) error estimate for the VI itself according to [1] and the assumption of a quadratic growth condition fulfilled by a local minimizer, one can show by standard arguments (see e.g. [2]) that the optimal control can be approximated with a convergence order that equals the square root of the order of the L^∞-error for the VI. By means of first-order necessary optimality conditions in form of strong stationarity condition as in [3] and an adaptation of the second-order analysis of [4], we construct a one-dimensional example, which exhibits minimal regularity and shows in this way that the obtained order of convergence is indeed (quasi-)optimal.

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Towards the numerical analysis of an optimal control problem with fractional constraints

Ira Neitzel
(joint work with Johannes Pfefferer, Boris Vexler)

We are concerned with the PDE-constrained optimal control problem with control $u \in L^2(\Omega)$ as well as associated state $y \in H_0^1(\Omega)$

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{1a}$$

subject to

$$-\Delta y = u, \quad \text{and} \quad (-\Delta)^s y \leq b, \tag{1b}$$

given in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^n$, $n \in \{2,3\}$, $s \in [0,1]$. Here, $-\Delta$ denotes the Laplace operator with homogeneous Dirichlet boundary conditions. We use the usual spectral definition with the help of its eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \ldots$, $\lim\limits_{k\to\infty} \lambda_k = \infty$, as well as the associated orthonormal eigenfunctions $\varphi_k \in H_0^1(\Omega)$, i.e.,

$$-\Delta y = \sum_{k=1}^{\infty} \lambda_k \left(\int_\Omega y\varphi_k dx \right) \varphi_k \quad \forall\, y \in H_0^1(\Omega)$$

and consequently

$$(-\Delta)^s y = \sum_{k=1}^{\infty} \lambda_k^s \left(\int_\Omega y\varphi_k dx \right) \varphi_k \quad \forall\, y \in H_0^1(\Omega).$$

The desired state $y_d \in L^2(\Omega)$ as well as the bound $b \in \mathbb{R}^+$ are given fixed data for problem (1).

This problem formulation includes typical control constrained model problems ($s = 1$) as well as problems with pointwise state constraints ($s = 0$). With this approach we are aiming at discussing the whole or part of the range $0 \leq s \leq 1$ in a uniform manner. To this end, we point out that by means of the transformation $z := (-\Delta)^s y$ we obtain the equivalent formulation with fractional PDE, and pointwise constraints on the state z,

$$\min_{u \in L^2(\Omega)} \frac{1}{2}\|(-\Delta)^{-s} z - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|u\|^2_{L^2(\Omega)} \tag{2a}$$

subject to

$$(-\Delta)^{1-s} z = u, \quad \text{and} \quad z \leq b; \tag{2b}$$

whereas the transformation $u = (-\Delta)^{1-s} v$ leads to a control-constrained problem with fractional PDE

$$\min_{v \in H^{2-2s}(\Omega)} \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\nu}{2}\|(-\Delta)^{1-s} v\|^2_{L^2(\Omega)} \tag{3a}$$

subject to

$$(−\Delta)^s y = v, \quad \text{and} \quad v \le b. \tag{3b}$$

For $s = 0$, i.e., enforcing pointwise state constraints in the original problem formulation (P), the latter transformation is the same as used for instance in [1], where the authors consider this approach to derive error estimates for the discretization of state-constrained problems. In the first reformulation (2), a typical way to derive first order optimality conditions would be to use typical Slater type arguments. Therefore, we have to require continuity of the state, which limits the setting for L^2-controls, for instance to $s < 1/2$ if $n = 2$. The Lagrange multipliers will be regular Borel measures.

The second reformulation (3) as control constrained problem gives rise to proving the existence of Lagrange multipliers by construction rather than relying on constraint qualifications, cf. [2] for the technique of proof for a control-constrained problem with nonfractional PDE. This, however, requires sufficient regularity of the optimal control $\bar{v}$, which is still ongoing work and seems to be promising for $1/2 \le s \le 1$.

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Discontinuous Galerkin time stepping schemes for the Allen-Cahn equation and applications to optimal control

Konstantinos Chrysafinos

Stability and error estimates for the Allen-Cahn equation are considered. The Allen-Cahn equation was introduced in [1] as the simplest phase field model, and it is a parameter dependent parabolic semi-linear PDE of the form

$$\begin{cases} u_t - \Delta u + \dfrac{1}{\epsilon^2}(u^3 - u) = f & \text{in } (0,T) \times \Omega, \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } (0,T) \times \Gamma, \\ u(0,x) = u_0 & \text{in } \Omega. \end{cases} \tag{1}$$

The principal difficulty involved, concerns the parameter $0 < \epsilon < 1$ which is very small and, typically comparable to the size of the time and space discretization parameters, τ, h respectively. Classical numerical analysis techniques based on Gronwall's type inequalities typically fail, since they introduce constants depending on quantities of $\exp(1/\epsilon)$. This problem was first circumvented in the work of [8] through the development of a suitable discrete approximation of the spectral estimate (see [2]). Various a-priori estimates were also established in the works of

[2, 3, 7], (see also references within). Fully-discrete schemes based on discontinuous Galerkin time-stepping approach combined with standard conforming finite elements in space are considered. Under minimal regularity assumptions on the given data, we prove (see [5]) that the fully-discrete solution, computed by using discontinuous Galerkin (in time) and conforming finite elements in space of arbitrary order (in time and space), denoted by u_h, satisfies the following unconditional stability estimates: $\|u_h\|_{L^2[0,T;L^2(\Omega)]} \leq C$, and $\|u_h\|_{L^\infty[0,T;L^2(\Omega)]} + \|u_h\|_{L^2[0,T;H^1(\Omega)]} \leq \frac{C}{\epsilon}$, where C denotes a constant depending on the domain Ω, the norms of $\|u_0\|_{L^2(\Omega)}$ and $\|f\|_{L^2[0,T;(H^1(\Omega))^*]}$ and the polynomial degree in time, but it is independent of τ, h, ϵ. Using the previous estimates and a compactness argument (see [10]) for DG time-stepping schemes, convergence is also established under minimal regularity assumptions. In addition, using the stability estimates, and within a neigborhood of the established convergence, we prove the following best approximation error estimate (see [5, Section 5]),

$$\|\text{error}\|_X \leq \frac{C}{\epsilon^3}(\|u\|^2_{L^\infty[0,T;H^1(\Omega)]} + \|u\|^2_{L^2[0,T;H^2(\Omega)]})\|\text{best approximation error}\|_X,$$

where $X = L^\infty[0,T;L^2(\Omega)] \cap L^2[0,T;H^1(\Omega)]$, and C denotes an algebraic constant depending only upon data, and it is independent of τ, h, ϵ for a suitable choice of τ, h, ϵ. The technique is based on the construction of a suitable auxiliary space-time projection (similar to [6]), an $L^p[0,T;L^2(\Omega)]$ estimate (see [9]), weighted estimates, and a suitable duality type argument. The above estimate is applicable in case of an optimal control problem where a tracking type functional is minimized subject to (1). The controls are of distributed type, and may satisfy point-wise control constraints. In particular, combining these estimates with the approach of [4], we establish the error estimate for the difference between the locally optimal controls and their discrete approximation when the lowest order discontinuous time-stepping scheme is used.

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Discretization error estimates for Dirichlet control problems with emphasis on graded meshes

Thomas Apel
(joint work with Mariano Mateos, Johannes Pfefferer, Arnd Rösch)

The investigation of Dirichlet control problems with L^2-regularization leads to the consideration of very weak solutions of boundary value problems, see [1]. We recall the first order optimality system and regularity results for the control, the state and the adjoint state, with emphasis on corner singularities. Furthermore, we review approximation results for discretizations of boundary value problems using graded meshes.

The main part of the talk is started with the discretization of the optimal control problem in the case that no box constraints are required for the control. We use piecewise linear finite elements on graded meshes for all variables. The error analysis is started with a general approximation result leading to the necessity to estimate three terms. The first one is a quasi-interpolation error for the control which can be treated more or less by standard means. The estimate of the second term is the most recent result: to get the optimal error order it was necessary to derive finite element error estimates in a weighted norm, see [2]. The critical detail is that we cannot use Cea's lemma since the solution is in general not in $H^1(\Omega)$. The third term is also treated by using weighted norms but here only for the interpolation error. A first numerical test shows that the overall error estimate is sharp. These results will be contained in [3].

A second test shows that with meshes with certain structural properties even better convergence orders are achievable. Therefore we discussed superconvergence meshes next. Most results from the literature concern quasi-uniform meshes. As a novel point, superconvergent graded meshes are discussed and investigated. This part of the talk is concluded by summarizing the various approximation results for the control in the unconstrained Dirichlet control problem.

At the end of the talk we summarize results for the problem with box constraints for the control. In the case of a convex domain there is not much difference to the unconstrained case. The interesting case is the non-convex domain: while the control and state variables have poles in the unconstrained case, they are flattened in the constrained case. In consequence, the solution is more regular such that a less severe mesh grading is sufficient to achieve optimal convergence, see also [3].

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Optimization of phase-field damage and fracture - and its discretization

Winnifried Wollner

(joint work with Robert Haller-Dintelmann, Hannes Meinlschmidt, Masoumeh Mohammadi, Ira Neitzel, Thomas Wick)

We consider an optimization problem governed by a regularized time-discrete phase-field damage model. As such optimization problems have received little attention in the literature, we need to provide several analytical results for the setting to allow for an analysis of the discretization error in the optimization problem. To this end, we propose a relaxation of the irreversibility of the damage variable by a penalty approach yielding the following necessary optimality conditions for the lower-level problem. Given forces q on $\Gamma_N \subset \partial\Omega \subset \mathbb{R}^2$ of finding a vector-valued displacement u^i and a scalar damage-variable ϕ^i satisfying (for $i = 1, \ldots, M$)

$$\text{(EL)} \qquad \begin{aligned} \Big(g(\phi^i)\mathbb{C}e(u^i), e(v)\Big) - (q^i, v)_{\partial_N\Omega} &= 0 \\ G_c\epsilon(\nabla\phi^i, \nabla\psi) - \frac{G_c}{\epsilon}(1-\phi^i, \psi) + (1-\kappa)(\phi^i\mathbb{C}e(u^i) : e(u^i), \psi) & \\ +\gamma([(\phi^i - \phi^{i-1})^+]^3, \psi) + \eta(\phi^i - \phi^{i-1}, \psi) &= 0 \end{aligned}$$

for all admissible test functions v, ψ where $G_c, \epsilon, \kappa, \eta, \gamma > 0$ are given parameters, $\mathbb{C}$ denotes the standard tensor of linear elasticity, $e(u)$ denotes the symmetric gradient, $g(x) = (1-\kappa)x^2 + \kappa$, and ϕ^0 is a given initial damage profile. The corresponding optimization problem, is then

$$\text{(NLP}^\gamma\text{)} \qquad \begin{aligned} &\min_{q,u,\phi} J(q,u,\phi) := \frac{1}{2}\sum_{i=1}^{M} \|u^i - u_d^i\|^2 + \frac{\alpha}{2}\sum_{i=1}^{M} \|q^i\|^2_{\partial_N\Omega} \\ &\text{s.t. } (q,u,\phi) \text{ satisfy (EL).} \end{aligned}$$

for some $\alpha > 0$ and given functions u_d^i.

Well-posedness of (EL) and existence of minimizers for the optimization problem (NLP$^\gamma$) can then be shown [1]. Thus providing the existence of at least one global minimizer $q \in L^2(\Omega)^M$, $u \in W_D^{1,p}(\Omega; \mathbb{R}^2)^M$, $\phi \in W^{1,p}(\Omega)^M$ for some, small, $p > 2$.

As additional regularity in the integrability scale does not help when deriving convergence rates of solutions, additional differentiability of the above solutions is required. To this end, we show that any solution to (EL) provides the additional regularity $u \in H^{1+s}(\Omega)$ for some, small, $s > 0$ by means of an improved regularity result for, linear, systems with mixed boundary conditions and irregular coefficients provided by [2].

Utilizing this regularity result further, [3] could show convergence rates for linear quadratic approximations of (NLP$^\gamma$) where the constants can be controlled in terms of the L^2 and $W^{1,p}$ norms of q and (u, ϕ) respectively.

Finally, we discuss the limit process $\gamma \to \infty$ in the relaxation of the irreversibility constrained as discussed in [4].

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Numerical analysis for the optimal control of a simplified mechanical damage process

Arnd Rösch

(joint work with Marita Holtmannspötter, Boris Vexler)

We discuss the optimal control of a simplified damage model. A simplified non-smooth model is given by

$$\begin{aligned} -\alpha\Delta\varphi(t) + \beta\varphi(t) &= \beta d(t) + l(t) && \text{in } \Omega, \\ \varphi(t) &= 0 && \text{on } \partial\Omega, \\ \partial_t d(t) &= \frac{1}{\delta}\max\{-\beta(d(t) - \varphi(t)) - r, 0\} && \text{a.e. in } \Omega, \\ d(0) &= d_0. \end{aligned}$$

In addition we analyze a linear version of the model

$$\begin{aligned} -\alpha\Delta\varphi(t) + \beta\varphi(t) &= \beta d(t) + l(t) && \text{in } \Omega, \\ \varphi(t) &= 0 && \text{on } \partial\Omega, \\ \partial_t d(t) &= -\frac{\beta}{\delta}(d(t) - \varphi(t)) && \text{a.e. in } \Omega, \\ d(0) &= d_0. \end{aligned}$$

In both cases the function l is considered as the control. The corresponding objective is given by

$$J(\varphi, d, l) = \frac{1}{2}\|\varphi - \varphi_d\|^2 + \frac{1}{2}\|d - d_d\|^2 + \frac{\alpha_l}{2}\|l\|^2,$$

where all norms are in $L^2(Q)$ with $Q = (0,T) \times \Omega$.
The optimal control problem is discretized by piecewise linear finite elements in space and piecewise constant finite elements in time (discontinuous Galerkin).
In the talk we present results for the optimal control problem governed by the linear model:

- existence of unique solutions for the undiscretized state equation, the semidiscrete state equation and the fully discrete equation,
- stability estimates for these different problems,
- a priori error estimates for the state equation and the optimal control problem,
- additional regularity properties of optimal solutions.

It is possible to derive similar results for the nonsmooth state equation. At the end of the talk we present several serious problems for the optimal control problem governed by the nonsmooth model.

Finite element methods for elliptic distributed optimal control problems with pointwise state constraints

Li-yeng Sung
(joint work with Susanne C. Brenner, Joscha Gedicke)

We present a general framework [3] for the convergence analysis of finite element methods for the following elliptic optimal control problem:

$$\text{Find} \quad (\bar{y}, \bar{u}) = \underset{(y,u)\in\mathbb{K}}{\operatorname{argmin}} \frac{1}{2}\Big[\|y - y_d\|^2_{L_2(\Omega)} + \beta\|u\|^2_{L_2(\Omega)}\Big],$$

where $(y, u) \in H^1_0(\Omega) \times L_2(\Omega)$ belongs to $\mathbb{K}$ if and only if

$$\begin{aligned} &\text{(i)} \quad -\Delta y = u \quad \text{in } \Omega,\\ &\text{(ii)} \qquad\quad y \le \psi \quad \text{a.e. in } \Omega. \end{aligned}$$

Here, Ω is a bounded polyhedral domain in $\mathbb{R}^d$ $(d = 2, 3)$, $y_d \in L_2(\Omega)$, $\psi \in H^3(\Omega) \cap W^{2,\infty}(\Omega)$ and $\psi > 0$ on $\partial\Omega$.

It follows from the convexity of Ω and the PDE constraint (i) that $(y, u) \in \mathbb{K}$ implies $y \in H^2(\Omega)$. The optimal control problem can therefore be reformulated as the following fourth order variational inequality:

$$\beta(D^2\bar{y}, D^2(y - \bar{y}))_{L_2(\Omega)} + (\bar{y} - y_d, y - \bar{y})_{L_2(\Omega)} \ge 0 \qquad \forall\, y \in K,$$

where D^2 is the Hessian and $K = \{y \in H^2(\Omega) \cap H^1_0(\Omega) : y \le \psi \text{ on } \Omega\}$.

Let $\mathcal{T}_h$ be a triangulation of Ω, V_h be a (C^1, nonconforming or C^0) finite element space associated with $\mathcal{T}_h$ that respects the homogeneous Dirichlet boundary condition, and $K_h = \{y_h \in V_h : y_h(p) \le \psi(p)$ at all the vertices of $\mathcal{T}_h\}$. The discrete problem is to find $\bar{y}_h \in K_h$ such that

$$\beta a_h(\bar{y}_h, y_h - \bar{y}_h) + (\bar{y}_h - y_d, y_h - \bar{y}_h)_{L_2(\Omega)} \ge 0 \qquad \forall\, y_h \in K_h,$$

where $a_h(\cdot,\cdot)$ approximates the bilinear form $(D^2\cdot, D^2\cdot)_{L_2(\Omega)}$.

The convergence analysis, which can be applied to C^1 finite element methods, classical nonconforming finite element methods and C^0 interior penalty methods, is based on the following ingredients.

Karush-Kuhn-Tucker optimality conditions:

$$\beta(\Delta\bar{y}, \Delta z)_{L_2(\Omega)} + (\bar{y} - y_d, z)_{L_2(\Omega)} = \int_\Omega z\, d\mu \qquad \forall\, z \in H^2(\Omega) \cap H_0^1(\Omega),$$

$$\mu = \text{a nonpositive finite Borel measure},$$

$$\int_\Omega (\bar{y} - \psi) d\mu = 0.$$

Regularity of $\bar{y}$:

$$\bar{y} \in H^3_{loc}(\Omega) \cap W^{2,\infty}_{loc}(\Omega) \cap H^{2+\alpha}(\Omega),$$

where $\alpha \in (0, 1]$ is determined by the geometry of Ω.

Regularity of μ:

$$\Big|\int_\Omega z\, d\mu\Big| \le C\|z\|_{H^1(\Omega)} \qquad \forall\, z \in H_0^1(\Omega).$$

Theorem. We have

$$\|\bar{y} - \bar{y}_h\|_h \le Ch^\alpha$$

for quasi-uniform meshes and

$$\|\bar{y} - \bar{y}_h\|_h \le Ch$$

for graded meshes, where $\|\cdot\|_h$ is a H^2-like norm defined according to the type of the finite element method.

We will also discuss the extension of the convergence analysis to nonconvex polyhedral domains in [1, 2], where the space $H^2(\Omega) \cap H_0^1(\Omega)$ is replaced by the space $\{y \in H_0^1(\Omega) : \Delta y \in L_2(\Omega)\}$.

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Fast solvers for a state constrained optimal control problem

Joscha Gedicke

(joint work with Susanne C. Brenner, Li-yeng Sung)

We discuss iterative solvers for an optimal control problem with pointwise state constraints. We consider a reformulation of the problem as a fourth order obstacle problem, which we discretize with the C^0 interior penalty method [1] or a P_1 finite element method with mass lumping [2].

The resulting quadratic programming problem is solved with a primal-dual active set strategy [5]. We demonstrate in numerical experiments that a nested iteration of uniformly h-refined meshes leads to mesh independent convergence in just a few outer iterations similar to the observation for second order obstacle problems made in [5].

As the primal-dual active set strategy is known to be related to a semi-smooth Newton method [4], we show empirically that a single V-cycle of a (geometric) multigrid solver as inner solver leads to convergence of the outer iteration in only a few more iterations compared to a direct inner solver.

Due to the fourth order nature of the problem, the required number of Jacobi smoothing steps in the multigrid algorithm is much larger than that for second order problems. However, we can achieve a great reduction of smoothing steps by using a Poisson smoother instead, which itself is a standard V-cycle multigrid for the Poisson problem [3]. It is an interesting observation that this remains true also in the case of the P_1 finite element method with mass lumping.

We also demonstrate that the design of the coarse levels in each inner iteration needs special care, in that we have to construct suitable coarse level active sets that are somewhat compatible with the current fine level active set approximation, cf. [6], and which are observed to be significantly different from the final (converged) active set approximations on coarser levels. Here, we chose a very simple approach of restricting a vector with entries 1 for active indices and entries 0 elsewhere to the coarser level, and then choose the coarse active set to be all indices whose entries are nonzero.

We demonstrate the efficiency of this combined nested iteration, primal-dual active set, multigrid, and Poisson smoother iterative solver in various 2d and 3d experiments. Finally we discuss the possibility to replace the multigrid inner solver by the preconditioned conjugate gradients method with the multigrid solver as preconditioner.

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Optimal control problems in non-convex domains with regularity constraint

Johannes Pfefferer
(joint work with Benedikt Berchtenbreiter, Boris Vexler)

In this talk we consider the optimal control problem:

$$(\mathcal{P}) \qquad \begin{gathered} \min_{q \in Q_{ad}} J(u,q) := \frac{1}{2}\|u-u_d\|^2_{L^2(\Omega)} + \frac{\alpha}{2}\|q\|^2_{L^2(\Omega)}, \\ \text{subject to } u \in H^1_0(\Omega): \ -\Delta u = q \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Here, $Q_{ad} \subset L^2(\Omega)$, which is further specified below, denotes the set of admissible controls, the domain Ω is a polygonally bounded subset of $\mathbb{R}^2$, u_d denotes the desired state, and $\alpha > 0$ is the regularization parameter. As specialty, the underlying domain is assumed to be non-convex. In this case, if $Q_{ad} = L^2(\Omega)$, it is well known that the solution to the Poisson equation with right hand side q, and thus the state of the optimal control problem $(\mathcal{P})$, does not belong to $H^2(\Omega)$ in general. The lack of regularity is due to the appearance of singular terms in the solution caused by the non-convex corners. However, we are interested in optimal states which nevertheless belong to $H^2(\Omega)$. As remedy, we impose a regularity constraint on the state which enforces the $H^2(\Omega)$-regularity of the state. For instance, this can be achieved by considering a non-empty, closed and convex subset of $L^2(\Omega)$ as control space Q_{ad} which only allows for $H^2(\Omega)$-regular states. Formally, this space can be defined as

$$(1) \qquad Q_{ad} := \{q = \Delta v \,:\, v \in H^2(\Omega) \cap H^1_0(\Omega)\}.$$

Alternatively, this space can be described by using the dual singular functions corresponding to the non-convex corners. Let us assume for simplicity that we have exactly one non-convex corner. Then, according to [2, Section 2.3], there is the decomposition

$$L^2(\Omega) = \{\Delta v \,:\, v \in H^2(\Omega) \cap H^1_0(\Omega)\} \overset{\perp}{\oplus} \operatorname{Span}\{p_s\},$$

which is orthogonal with respect to the $L^2(\Omega)$ inner product. The functions p_s is defined via the dual singular function corresponding to the non-convex corner. As a consequence, we can equivalently define the space Q_{ad} in (1) by

$$Q_{ad} := \left\{ q \in L^2(\Omega) \, : \, \int_\Omega q p_s = 0 \right\}. \tag{2}$$

During the talk, we discuss existence and uniqueness of the solution to the optimal control problem $(\mathcal{P})$ with control space Q_{ad} as defined in (2). Moreover, optimality conditions are presented. We also introduce several approaches to discretize the problem with finite elements, and show corresponding error estimates. Moreover, we discuss related results for the case where the control has to fulfill additional point-wise inequality constraints. The subject of this talk is inspired by the paper [1], recently submitted by Jarle Sogn and Walter Zulehner.

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Towards optimal control for compressible Navier-Stokes

Sandra May

(joint work with Xenia Kerkhoff)

To date, very little research has been done for optimal control problems that are governed by the compressible Navier-Stokes equations. The work by Collis et al. [1] is one of the few ones to name here. In this talk, we discuss this topic and present a road map towards optimal control for the compressible Navier-Stokes equations. We also hope to initiate a discussion of how to best approach optimal control problems involving hyperbolic conservation laws in general.

We start by reviewing the main properties of solutions of hyperbolic conservation laws and noting their differences to those of elliptic equations. We then discuss the pros and cons of either working with the conservation laws directly or using viscous regularization.

In the main part of the talk, we focus on the compressible Navier-Stokes equations and present methods for solving them. In one space dimension, for simplicity, they are given by

$$\begin{bmatrix} \rho \\ \rho u \\ E \end{bmatrix}_t + \begin{bmatrix} \rho u \\ \rho u^2 + p \\ (E+p)u \end{bmatrix}_x = \begin{bmatrix} 0 \\ \nu u_{xx} \\ \nu \left(\frac{u^2}{2}\right)_{xx} + \kappa \theta_{xx} \end{bmatrix},$$

with $\rho = \rho(x,t) > 0$ denoting the density, $u = u(x,t)$ the velocity, $p = p(x,t) > 0$ the pressure, and $E = \frac{p}{\gamma - 1} + \frac{1}{2}\rho u^2$ the total energy with $\gamma > 1$ being the adiabatic exponent. Finally, θ denotes the temperature, $\nu > 0$ the viscosity, and $\kappa > 0$ the heat conduction.

Typically, these equations are solved by discretizing the *conserved* variables $\mathbf{U} = [\rho, \rho u, E]^T$. In [4, 2], we developed spacetime discontinuous Galerkin (DG) schemes that are instead based on using the so called *entropy* variables as degrees of freedom. This change of variables symmetrizes the system and has a favorable effect on the structure of the diffusion term. As a consequence, we expect the approaches *discretize-then-optimize* and *optimize-then-discretize* to commute if a suitable DG discretization is used. These methods and the corresponding software serve as the starting point for the extension to an optimal control solver.

In the last part of the talk, we discuss the current status of extending the primal solver to an optimal control solver. This includes the examination of the commutative properties of our schemes for the strongly simplified model of an optimal control problem that is governed by an unsteady advection-diffusion-reaction equation following Leykekhman [3]. We conclude with numerical results for this model problem.

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Iterative hard-thresholding applied to optimal control problems with $L^0(\Omega)$ control cost

Daniel Wachsmuth

We consider optimal control problems of the type

$$\min f(u) + \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\|u\|_0, \tag{1}$$

subject to

$$u \in U_{ad} := \{v \in L^2(\Omega) : |v(x)| \le b \text{ a.e. on } \Omega\}. \tag{2}$$

Here, $\Omega \subset \mathbb{R}^n$ is an open domain. The objective functional contains the so-called L^0-norm (which is – of course – not a norm) that is defined by

$$\|u\|_0 := \text{meas}\{x : \; u(x) \neq 0\} \quad u : \Omega \to \mathbb{R} \text{ measurable.}$$

The parameters are assumed to satisfy $\alpha \ge 0$, $\beta > 0$, and $b \in [0, +\infty]$. The function f is a smooth mapping from $L^2(\Omega)$ to $\mathbb{R}$. Here, we have in mind to choose $f(u) := j(y)$, where y is the solution of a possibly nonlinear partial differential equation $E(y, u) = 0$. Let us define

$$g(u) := \frac{\alpha}{2}\|u\|^2_{L^2(\Omega)} + \beta\|u\|_0.$$

In the optimal control problem (1)–(2) the size of the support of the controls is penalized. One application of such problems are actuator location problems, where one tries to find optimal actuator locations for the controls that are small. This question was first studied in the seminal paper [1], which addresses this problem by using $\|u\|_{L^1(\Omega)}$ instead of $\|u\|_0$ in the cost functional.

Since the mapping $u \mapsto \|u\|_0$ is not weakly lower semicontinuous from $L^p(\Omega)$ to $\mathbb{R}$ for all $p \in [1,\infty)$, it is not possible to prove existence of solutions. We refer to [2] for a discussion of this issue.

In order to solve (1)–(2) numerically, we propose to use the following proximal gradient-like method. Let an iterate u_k be given. Then the next iterate u_{k+1} is determined as the solution of

$$\min_{u\in U_{ad}} f(u_k) + \nabla f(u_k)(u-u_k) + \frac{L}{2}\|u-u_k\|^2_{L^2(\Omega)} + g(u).$$

Here, the inverse L^{-1} of the positive parameter L can be interpreted as a step-size. This minimization problem can be written as the minimization of an integral functional. Its solution can be computed by a pointwise minimization of the integrands. Since the integrand is a quadratic plus L^0-norm and box constraints, an explicit solution formula is available, [3, Lemma 3.9]. Hence, the iteration is well-defined.

Let χ_k denote the characteristic function of the support of u_k. The sequence (u_k) of iterates satisfies the following properties.

Theorem ([3, Theorem 3.12])**.** Suppose $L > L_f$, where L_f is the Lipschitz constant of ∇f. Let (u_k) be a sequence of iterates generated by the algorithm above. Then it holds that:

(1) The sequences (u_k) and $(\nabla f(u_k))$ are bounded in $L^2(\Omega)$ if $\alpha > 0$ or $b < +\infty$.
(2) The sequence $(f(u_k)+g(u_k))$ is monotonically decreasing and converging.
(3) $\|u_{k+1}-u_k\|_{L^2(\Omega)} \to 0$.
(4) $\sum_{k=1}^{\infty} \|\chi_k - \chi_{k+1}\|_{L^1(\Omega)} < +\infty$.
(5) $\chi_k \to \chi$ in $L^1(\Omega)$ for some characteristic function χ.

The convergence of the sequence of characteristic functions (χ_k) implies that oscillation phenomena do not occur for the sequence (u_k). In addition, this convergence allows to prove the following result.

Theorem ([3, Theorem 3.14])**.** Let $u^* \in U_{ad}$ be a weak sequential limit point of the iterates (u_k) in $L^2(\Omega)$. Then it holds that

$$f(u^*) + g(u^*) \le \lim_{k\to\infty} (f(u_k) + g(u_k)).$$

This result is surprising, because the functional g is not weakly lower semicontinuous. For a further discussion of properties of the algorithm, we refer to [3].

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Dirichlet boundary control of Stokes equation in polygonal domain

Wei Gong

(joint work with Weiwei Hu, Mariano Mateos, John Singler, Yangwen Zhang)

Let $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ be a Lipschitz polyhedral domain with boundary $\Gamma = \partial\Omega$. We introduce the spaces

$$\begin{aligned}
\mathbf{V}^s(\Omega) &= \{\mathbf{y} \in \mathbf{H}^s(\Omega) : \ \nabla \cdot \mathbf{y} = 0, \ \ [\mathbf{y} \cdot \mathbf{n}, 1]_\Gamma = 0\}, \text{ for } s \ge 0, \\
\mathbf{V}_0^s(\Omega) &= \{\mathbf{y} \in \mathbf{H}^s(\Omega) : \ \nabla \cdot \mathbf{y} = 0, \ \mathbf{y} = 0 \text{ on } \Gamma\}, \text{ for } s > 1/2, \\
\mathbf{V}^s(\Gamma) &= \{\mathbf{u} \in \mathbf{H}^s(\Gamma) : \ \langle \mathbf{u} \cdot \mathbf{n}, 1\rangle_\Gamma = 0\}, \text{ for } 0 \le s < 3/2.
\end{aligned}$$

The spaces with negative index are defined by duality.

We are interested in the optimal control problem

$$\text{(P)} \qquad \min_{\mathbf{u} \in \mathbf{V^0(\Gamma)}} J(\mathbf{u}) = \frac{1}{2}\|\mathbf{y_u} - \mathbf{y}_d\|^2_{\mathbf{L}^2(\Omega)} + \frac{\alpha}{2}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Gamma)},$$

where $\mathbf{y_u} \in \mathbf{V}^0(\Omega)$ is the solution (in the very weak sense) of the Stokes system

$$\text{(1)} \qquad \begin{aligned} -\Delta \mathbf{y} + \nabla p = \mathbf{f} \quad \text{in } \Omega, \qquad \nabla \cdot \mathbf{y} = 0 \quad \text{in } \Omega, \\ \mathbf{y} = \mathbf{u} \quad \text{on } \Gamma, \qquad \int_\Omega p = 0. \end{aligned}$$

To introduce the definition of very weak solution of the state equation, we consider the following *compressible* Stokes problem

$$\text{(2)} \qquad \begin{aligned} -\Delta \mathbf{z} + \nabla q = \mathbf{g} \quad \text{in } \Omega, \quad \nabla \cdot \mathbf{z} = h \quad \text{in } \Omega, \\ \mathbf{z} = 0 \quad \text{on } \Gamma, \quad (q, 1)_\Omega = 0. \end{aligned}$$

Following [2, Eq. (2.2)], we define for $(\mathbf{z}, q) \in \mathbf{H}^{3/2+s}(\Omega) \times H^{1/2+s}(\Omega)$ the constant

$$\text{(3)} \qquad c(\mathbf{z}, q) = \frac{1}{|\Gamma|}\langle q - \partial_\mathbf{n}\mathbf{z} \cdot \mathbf{n}, 1\rangle_\Gamma.$$

The following definition makes sense (see, e.g., [2]):

Definition 1. Consider $0 \le s < s^* := \min\{\xi - \frac{1}{2}, \frac{1}{2}\}$ and $\mathbf{u} \in \mathbf{V}^{-s}(\Gamma)$. We say that $\mathbf{y_u} \in \mathbf{V}^0(\Omega)$, $p \in \left(H^1(\Omega)/\mathbb{R}\right)'$ is a solution in the transposition sense of (1) if

$$\text{(4)} \qquad (\mathbf{y}, \mathbf{g})_\Omega - [p, h]_\Omega = [\mathbf{u}, -\partial_\mathbf{n}\mathbf{z}_{\mathbf{g},h} + q_{\mathbf{g},h}\mathbf{n} + c(\mathbf{z}_{\mathbf{g},h}, q_{\mathbf{g},h})\mathbf{n}]_\Gamma,$$

for all $\mathbf{g} \in \mathbf{L}^2(\Omega)$ and $h \in H^1(\Omega)/\mathbb{R}$ such that $(h, 1)_\Omega = 0$, where $(\mathbf{z}_{\mathbf{g},h}, q_{\mathbf{g},h}) \in \mathbf{H}_0^1(\Omega) \times L_0^2(\Omega)$ is the solution of (2) and $c(\mathbf{z}_{\mathbf{g},h}, q_{\mathbf{g},h})$ is the constant given in (3).

Theorem 2 ([1])**.** Suppose $\mathbf{u} \in \mathbf{V}^s(\Gamma)$ for $-s^* < s < \min\{1/2+\xi, 3/2\}$, $\xi \in (0.5, 4]$ depends on the largest interior angle of Ω. Then the solution of (1) satisfies

$$\mathbf{y_u} \in \mathbf{V}^{s+1/2}(\Omega) \text{ and } p_{\mathbf{u}} \in \begin{cases} H^{s-1/2}(\Omega)/\mathbb{R} & \text{if } s \geq 1/2, \\ \left(H^{1/2-s}(\Omega)/\mathbb{R}\right)' & \text{if } s \leq 1/2. \end{cases}$$

Moreover, the mapping $\mathbf{u} \mapsto \mathbf{y_u}$ is continuous from $\mathbf{V}^s(\Gamma)$ to $\mathbf{V}^{s+1/2}(\Omega)$.

Theorem 3 ([1])**.** Let $\mathbf{u} \in \mathbf{V}^0(\Gamma)$ be the solution of problem (P). Then $\mathbf{u} \in \mathbf{V}^s(\Gamma)$ for all $0 \leq s < s^*$ and there exists $\mathbf{y} \in \mathbf{V}^{s+1/2}(\Omega)$, $p \in (H^{1/2-s}(\Omega)\backslash\mathbb{R})'$, $z \in \mathbf{V}_0^{1+r}(\Omega)$ and $q \in H^r(\Omega)\backslash\mathbb{R}$ for all $r < \min\{2, \xi\}$, satisfies

$$(\nabla \mathbf{z}, \nabla \mathbf{v}) - (q, \nabla \cdot \mathbf{v}) = (\mathbf{y} - \mathbf{y}_d, \mathbf{v}) \quad \mathbf{v} \in \mathbf{V}_0^1(\Omega), \tag{5}$$

$$(\nabla \cdot \mathbf{z}, q) = 0 \quad \forall q \in L_0^2(\Omega), \tag{6}$$

and the first order optimality condition

$$\langle \alpha \mathbf{u} + \partial_{\mathbf{n}} \mathbf{z} - q\mathbf{n}, \mu \rangle = 0 \quad \forall \mu \in \mathbf{L}_0^2(\Gamma). \tag{7}$$

Moreover, $\mathbf{u} \in \prod_{i=1}^m \mathbf{H}^{r-1/2}(\Gamma_i)$ for all $r < \min\{2, \xi\}$.

Note that the optimal control $\mathbf{u}$ for problem (P) is *discontinuous*. Let $\boldsymbol{\tau}$ be the tangential vector, we consider instead the tangential boundary control (P) with

$$\mathbf{y} = u\boldsymbol{\tau} \quad \text{on } \Gamma.$$

We have the first order optimality condition with the adjoint $(\mathbf{z}, q)$ as in (5)-(6):

$$\langle \alpha u \boldsymbol{\tau} + \partial_{\mathbf{n}} \mathbf{z}, v\boldsymbol{\tau} \rangle = 0 \quad \forall v \in L^2(\Gamma).$$

Theorem 4 ([1])**.** Suppose Ω is a convex polygon, $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $\mathbf{y}_d \in \mathbf{H}^2(\Omega)$. Let $u \in L^2(\Gamma)$ be the solution of the tangential boundary control problem. Then

$$u \in H^s(\Gamma),\ \mathbf{y} \in \mathbf{V}^{s+1/2}(\Omega),\ p \in H^{s-1/2}(\Omega)\backslash\mathbb{R},\ \mathbf{z} \in \mathbf{V}_0^{1+r}(\Omega),\ q \in H^r(\Omega)\backslash\mathbb{R}$$

for all $1/2 < s < \min\{3/2, \xi - 1/2\}$ and $1 < r < \min\{3, \xi\}$. Moreover, $u \in \prod_{i=1}^m H^{r-1/2}(\Gamma_i)$ for all $r < \min\{3, \xi\}$.

An HDG method is proposed to solve the tangential boundary control problem and optimal error estimates are derived in [1]. Specifically, for a rectangular 2D domain, $\mathbf{y}_d \in \mathbf{H}^2(\Omega)$ and $k = 1$, we obtain a priori error bounds for the velocity $\mathbf{y}$, adjoint velocity $\mathbf{z}$, their fluxes $\mathbb{L}$ and $\mathbb{G}$, pressure p and dual pressure q:

$$\|\mathbf{y} - \mathbf{y}_h\|_{0,\Omega} = O(h^{3/2-\varepsilon}), \quad \|\mathbb{L} - \mathbb{L}_h\|_{0,\Omega} = O(h^{1-\varepsilon}), \quad \|p - p_h\|_{0,\Omega} = O(h^{1-\varepsilon}),$$
$$\|\mathbf{z} - \mathbf{z}_h\|_{0,\Omega} = O(h^{3/2-\varepsilon}), \quad \|\mathbb{G} - \mathbb{G}_h\|_{0,\Omega} = O(h^{3/2-\varepsilon}),\ \|q - q_h\|_{0,\Omega} = O(h^{3/2-\varepsilon}),$$

for any $\varepsilon > 0$, and the optimal control u

$$\|u - u_h\|_{0,\Gamma} = O(h^{3/2-\varepsilon}).$$

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Optimal-order multigrid preconditioning in the semi-smooth Newton method solution of certain PDE-constrained optimization problems

Andrei Drăgănescu

(joint work with Sumaya Alzuhairy, Jyoti Saraswat, Bedřich Sousedík)

We present a multigrid preconditioning strategy for optimal control problems constrained by partial differential equations (PDEs) that exhibits a certain degree of optimality with respect to the discretization of the control space and the smoothing properties of the PDE, assuming no explicit control constraints are present in the formulation [1]. Optimality is here defined in terms of how well the preconditioner approximates the operator to be inverted; e.g., if piecewise linear controls are used for the controls, the L^2–distance between the Hessian of the cost functional and the multigrid preconditioner is $O(h^2)$, where h is the mesh-size. As a result, multigrid preconditioned conjugate gradient converges in a number of iterations that decreases with $h \downarrow 0$ at the correct rate with respect to h.

If control constraints are added, the technique is then applied to the linear systems arising in the semismooth Newton solution of the optimal control problem. As is well known, these linear systems involve principal minors of the systems arising in the unconstrained problem. The issue in the constrained case lies with the choice of the coarse spaces. It is shown in [2] that for a continous piecewise linear discretization of the controls, a natural, conforming choice of a coarse space leads to a suboptimal preconditioner that approximates the operator to $O(h^{1/2})$. On the other hand, as shown in [3], by discretizating the controls with piecewise constants and using a non-conforming coarse space for the multigrid preconditioner construction, we obtain an optimal-order preconditioner, e.g., one that approximates the operator to $O(h)$. The question of improving the $O(h)$ approximation-order for the control-constrained case has proved to be elusive for a long time. However, in our recent work, preliminary tests involving discontinuous piecewise linear discretizations of controls and adaptivity indicate that the optimal order of $O(h^2)$ is achievable for a modified semismooth Newton iteration.

Furthermore, we apply a similar strategy to a distributed optimal control problem constrained by elliptic equations with stochastic coefficients. Using a stochastic Galerkin discretization, we show not only that the multigrid preconditioner is of optimal-order $O(h^2)$, but that it is also robust with respect to the polynomial degree used to discretize the probability space and with respect to the number of terms in the truncated Karhunen-Loève expansion of the stochastic fields.

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Space-time discretization of parabolic time-optimal control problems

Konstantin Pieper

(joint work with Lucas Bonifacius, Boris Vexler)

In this talk, we present some recent results on the error analysis of time-optimal control problems in combination with a parabolic state equation. Despite the fact that parabolic time-optimal problems are a classical subject in control theory (see, e.g., [1]), their numerical analysis has so far been restricted to discretization by finite elements in space only (see, e.g., [2, 3]). For the first time, we analyze a method that also includes a discretization of the temporal variable.

The discretization concept is built upon a transformation of the time interval onto a reference interval, which yields a non-convex optimization problem with constraints on the control and the terminal state. In order to ensure the strong stability of the problem and enable error analysis, we develop new stability criteria based on a condition involving the Hamiltonian of the system [4]. These conditions can be verified explicitly if the target set is a ball in the state space around a desired state of certain structure. We derive convergence estimates for problems involving the heat equation with distributed or time-dependent parameter control, where the target sets is given as a ball in L^2. The temporal discretization is based on discontinuous Galerkin methods in time, which corresponds to a variant of the implicit Euler method in the case of piece-wise constant functions.

For the error analysis, we consider two different problem formulations: First, we consider a pure time-optimal problem, where the objective function involves only the time to reach the target set [6]. Here the solutions tend to be of bang-bang type. Optimal order error estimates for the optimal times are obtained, using only the aforementioned stability results and a mild condition on the adjoint state. This condition, which also ensures also the uniqueness and bang-bang structure of the optimal controls, is fulfilled in many cases. Under a strengthened condition on the adjoint state, rates of convergence for the control in L^1 can also be obtained. Second, we consider a problem involving additional quadratic control costs [5]. Here, the strong stability and an additional second order sufficient condition (SSC) enable optimal order error estimates for the optimal times and controls. The SSC ensures the local uniqueness of the solution, and rates of convergence for the control are obtained in L^2.

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Numerical analysis of sparse initial data identification for parabolic problems

Dmitriy Leykekhman

(joint work with Boris Vexler, Daniel Walter)

We consider a problem of identification of an unknown initial data q for a homogenous parabolic equation

$$\begin{aligned} \partial_t u - \Delta u &= 0 \quad \text{in} \quad (0,T)\times\Omega, \\ u &= 0 \quad \text{on} \quad (0,T)\times\partial\Omega, \\ u(0) &= q \quad \text{in} \quad \Omega, \end{aligned} \tag{1}$$

from a given (measured) data $u_d \approx u(T)$ of the terminal state $u(T)$ for some $T > 0$. In general, this problem is known to be exponentially ill-posed. We are interested in the situation, where the initial data we are looking for, is known to be sparse, i.e. to have a support of Lebesgue measure zero. Following the idea for measure valued formulation of sparse control problems, we seek the initial state q in the space of regular Borel measures $\mathcal{M}(\Omega)$ on the domain Ω, which is known to be isomorphic to the dual space of continuous functions $C_0(\Omega)^*$.

The corresponding optimal control we formulate as follows

$$\text{Minimize} \;\; J(q,u) = \frac{1}{2}\|u(T)-u_d\|^2_{L^2(\Omega)} + \alpha\|q\|_{\mathcal{M}(\Omega)}, \; q\in\mathcal{M}(\Omega), \text{ subject to (1)}, \tag{2}$$

where Ω is a convex polygonal/polyhedral domain in $\mathbb{R}^N$, $N = 2,3$, $I = (0,T]$ is the time interval, $u_d \in L^2(\Omega)$ is the given (desired /measured) final state, and $\alpha > 0$ is the regularization parameter.

An equivalent problem is considered in [1]. There, the initial state q is also searched for in the space $\mathcal{M}(\Omega)$. For given $\varepsilon > 0$ and $u_d \in L^2(\Omega)$ the optimal control problem in [1] is formulated as follows:

$$\text{(3)} \qquad \text{Minimize } \|q\|_{\mathcal{M}(\Omega)} \text{ subject to } \|u(T) - u_d\|_{L^2(\Omega)} \leq \varepsilon \text{ and (1).}$$

The optimal control problem (2) has a unique solution $(\bar{q}, \bar{u})$. For a numerical solution of the optimal control problem we use a discontinuous Galerkin method dG(r) of order r for temporal and linear (conforming) finite elements for spatial discretizations of the state equation (1) leading to the discrete optimal solution $(\bar{q}_{kh}, \bar{u}_{kh})$. The same type of discretization (with $r = 0$) is used in [1], where weak-star convergence $\bar{q}_{kh} \xrightharpoonup{*} \bar{q}$ in $\mathcal{M}(\Omega)$ for the control and strong convergence $\bar{u}_{kh}(T) \to \bar{u}(T)$ in $L^\infty(\Omega)$ is shown for the discretization parameters k and h tending to zero. However, no convergence rates with respect to k and h are derived in [1]. In my talk I will explain how for the general case (i.e. without any further assumptions) we can obtain an error estimate

$$\|(\bar{u} - \bar{u}_{kh})(T)\|_{L^2(\Omega)} \leq c(k^{r+\frac{1}{2}} + \ell_{kh} h),$$

where k denotes the maximal time step, h is the spatial mesh size, and ℓ_{kh} is a logarithmic term.

From the optimality system we can deduce, that the support of the optimal control (optimal initial state) $\bar{q}$ is contained in the set of maxima and minima of the adjoint state $\bar{z}(0)$. Under an additional assumption that the optimal control $\bar{q}$ consists of finitely many Dirac measures, we obtain an improved error estimate

$$\|(\bar{u} - \bar{u}_{kh})(T)\|_{L^2(\Omega)} \leq c(k^{2r+1} + \ell_{kh} h).$$

Moreover we provide a convergence result for optimal controls. Although a strong (norm) convergence of $\bar{q}_{kh}$ to $\bar{q}$ with respect to $\mathcal{M}(\Omega)$ can not be expected, we will prove convergence rates for the positions of the support points and of the corresponding coefficients. We will illustrate the theoretical results with several numerical examples.

References

[1] E. Casas, B. Vexler, and E. Zuazua, *Sparse initial data identification for parabolic PDE and its finite element approximations*, Mathematical Control and Related Fields **5** (2015), 377–399.

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