Snapshots of modern mathematics
from Oberwolfach

№ 12/2019

Analogue mathematical instruments: Examples from the "theoretical dynamics" group (France, 1948–1964)

Loïc Petitgirard

Throughout the history of dynamical systems, instruments have been used to calculate and visualize (approximate) solutions of differential equations. Here we describe the approach of a group of physicists and engineers in the period 1948–1964, and we give examples of the specific (analogue) mathematical instruments they conceived and used. These examples also illustrate how their analogue culture and practices faced the advent of the digital computer, which appeared at that time as a new instrument, full of promises.

The rise of digital computers has transformed mathematical practices, both for visualization and calculation. In the 1950s, this instrument was only accessible to a few scientists around the world, primarily to the mathematicians and physicists who invented computers for their own use. At that time, a digital revolution was barely on the agenda. Here we want to address the collateral transformations in mathematical instruments, right from the beginning of the modern computer, in the domain of "dynamical systems", in a specific context.

The aim of the theory of *dynamical systems* is to describe, and if possible understand, the changes over time that occur in physical or abstract mathematical "systems", where a system could be the solar system (motion of the sun and planets), the Earth's climate, or the motion of a spring or pendulum. Any of these can be described by the mathematical notion of a dynamical system, consisting of the phase space and the dynamics. The *phase space* is the set of all possible states of the system (for instance, consisting of every position that the sun and planets of our solar system can take), and the *dynamics* is a function that transforms a point in the phase space into another. This function is understood to be a function that tells us how the states of the system evolve in time, that is, if we want to know how a particular point in the phase space changes in one unit of time, we apply the function once, for two units of time, twice, and so on. One of the first goals of dynamical systems is to understand the long-term behaviour, or *orbits* of "typical" points, where loosely speaking *typical* means that points are chosen at random from the phase space.

Among these systems are *oscillators*: devices (be they mechanical, electrical, biological, and so on) that produce oscillations. Here *oscillations* refer to a repetitive variation of a value such as the distance from the origin of a swinging pendulum. Oscillations can be continuous (such as sine waves) or discontinuous (square waves). These oscillators can be modelled by mathematical dynamical systems, often described by differential equations. Conversely, oscillators can be considered as tools to investigate those mathematical models and equations.

We focus here on a physics laboratory in Marseille (in the South of France) where a group dedicated to "theoretical dynamics" proposed new theoretical insights and created some analogue machines to serve their calculation needs. In this context, dynamics was not considered a strictly mathematical discipline: the group dealt with very practical problems (mechanical, electronic, and natural systems), requiring intense calculations, as well as highly mathematical theories (inherited from dynamical systems theory). The members of the group were essentially mathematicians, physicists, and engineers. Here we emphasize the use and the role of mathematical instruments, the invention of dedicated instruments, which were commonly analogue devices, and their evolution when facing the digital revolution.

1 "Theoretical dynamics" as defined by Théodore Vogel

Mathematical instruments have been used regularly in the long history of dynamical systems (see the discussion in [2]). Here we focus on a particular example. At the beginning of the 1950s, a small group gathered around Théodore Vogel in Marseille (CRSIM, a laboratory of the CNRS) [9]. Vogel trained as an electrotechnical engineer in the 1930s, and turned to mathematical physics after the war, obtaining his PhD in 1947 on mechanical and acoustical vibrations. He was a student of the famous French applied mathematician Joseph Peres, who had been a leader in analogue calculating devices for hydrodynamics since the 1930s. Vogel was also one of the rare (French) readers of Poincaré's work on dynamical systems. Being born in Ukraine and able to read in Russian, he also discussed the work of Poincaré with the Soviet scientists, mainly the Kiev school of nonlinear mechanics, of Nikolay Krylov, Nikolay Bogoliubov, and later Yurii Mitropolsky. Altogether, he was a hybrid of different scientific and technical cultures, which inspired his foundation and leading of the theoretical group at the CRSIM.

Vogel produced a *theory of discontinuous oscillations* in the early 1950s, which gave him an international audience in the field of nonlinear oscillations. Vogel involved a first student (Lefteri Sidériades) in 1954 and started a research program that included many ingredients: a clear empirical epistemology (that is to say, "experimental mathematics" as opposed to the very formal and abstract style of mathematics then popular in France, with the rise of the "Bourbaki" group of young French mathematicians since the 1930s), the development of specific (analogue) instruments for their mathematical investigations, and an interest in practical dynamical problems. Their instruments could be mechanical or electronic and they rendered topological or geometrical information [1] about dynamical systems, by drawing "phase portraits" that were unreachable by other means.

For dynamical systems that have two-dimensional phase spaces, that is, where the phase space is a subset of the Euclidean plane, a *phase portrait* is a geometric representation of the orbits of the points in the phase plane. Phase portraits are used to visualize the behaviour of typical points. We will give more details on phase portraits in the next section, and you can see an example in Figure 5.

Vogel's student Sidériades was an engineer in electronics, and wrote his PhD thesis on "Topological methods applied to electronics" in 1956. He used the mathematical tools of dynamical systems inherited from Poincaré and Vogel for the conception of new electronic circuits. But these electronic circuits were also machines that could, in some sense, be said to calculate their own mathematical model. Thus, they are true analogue machines for mathematical investigations.

2 Investigating solutions of dynamical systems

A good example of this analogue practice is to be found in the work of Michel Jean (a PhD student of Vogel, 1961–1965) who conceived and used a mechanical analogue of the Duffing equation. The general *Duffing equation* is a nonlinear

[1] Topology is the branch of mathematics that considers only the shape of objects, whereas geometry is concerned also with measurements such as distance and angles on these shapes. For instance, a cube and a sphere are the same from a topological point of view.

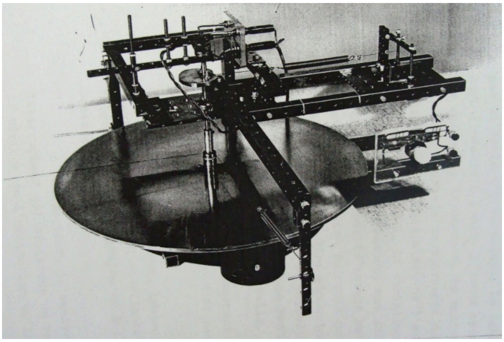

Figure 1: The mechanical oscillator.

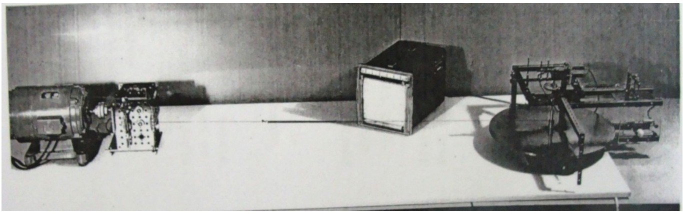

Figure 2: The mechanical oscillator with an external force.

second-order differential equation obtained by modelling a periodically forced oscillator with a nonlinear elasticity (for example the oscillations of a mass attached to a nonlinear spring with a linear damper):

$$\frac{d^2x}{dt^2} + \delta\frac{dx}{dt} + \alpha x + \beta x^3 = \gamma\cos(\omega t),$$

where δ is the amount of damping, α is the linear stiffness, β is the cubic nonlinearity parameter, and γ and ω are the amplitude and beat of the driving force. This equation models more complex behaviour than a simple harmonic oscillator. It is a low-order correction to the linear harmonic oscillator (which corresponds to the case $\delta = \beta = 0$). Depending on the values of the parameters, the behaviour of the oscillations modelled by Duffing equations can become chaotic. But before computers could tackle this equation numerically, such complex behaviours could not be grasped.

The mechanical oscillator designed by Jean is a rotating disk with damping and forcing device (see Figures 1 and 2). All the parameters are related to the geometry of the device and the properties of the different mechanical pieces (springs, torque, motor frequency, cables). Following Jean's notation, x is the

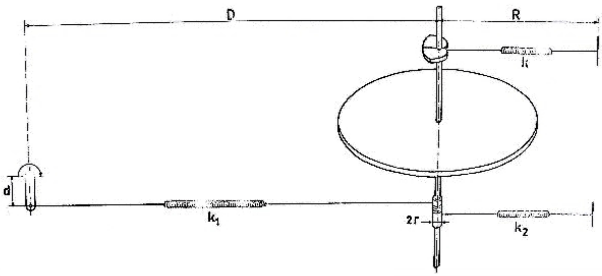

Figure 3: Outline of the geometrical aspects of the oscillator.

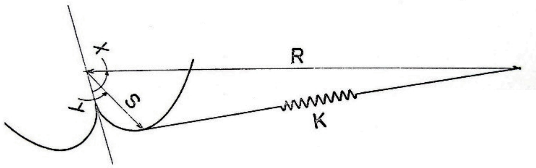

Figure 4: Focus on the torque.

angle of rotation of the disk, which changes with time t. The behaviour of the mechanical oscillator can be described by the following equation:

$$I\,\frac{d^2x}{dt^2} + 2f\,\frac{dx}{dt} + ax + bx^3 = C\cos(\omega t + \varphi).$$

In this equation, which is a Duffing equation, I stands for inertia momentum, f is the damping factor, ω is the forcing beat, and φ is the initial phase. The constants a, b, and C are determined by the geometry of the oscillator (see Figures 3 and 4):

$$C = dk_1r\ ;\quad a = 2JF_0 + (k_1 + k_2)r\ ;\quad b = 2J^2k.$$

The value F_0 is the tension of the wire (at instant $t = 0$) and k is the spring constant. The parameter J is determined by the characteristics of the machine, including the fact that the disk is subjected to a magnetic field that restrains the rotation. The parameters d, k_1, k_2, and r appear on the drawing (Figure 3).

The values of x and $\frac{dx}{dt}$ can easily be recorded and printed on paper while the machine is oscillating. It renders graphically the phase portrait of the

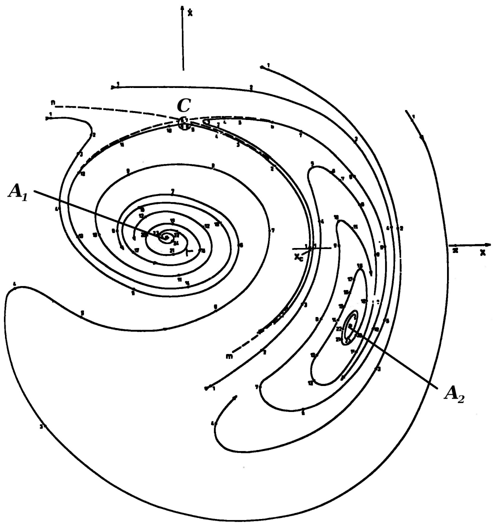

Figure 5: A phase portrait.

equation with fixed parameters. Of course, most of the parameters can be changed (external force, beat, spring stiffness, and so on) and new behaviours of another Duffing equation can be explored with the help of the machine. Although the device has limited range for each parameter, it is a convenient means of exploring oscillating solutions of the Duffing equations in this range.

In his PhD thesis [6], Jean focuses on periodic solutions of Duffing equations. This is one of the machine's strengths: it renders graphical details of the dynamics in the phase plane (the phase portrait) to give some geometrical insights regarding the dynamics and to assist in the identification of periodic solutions. An example of the measurement on the working oscillator, without any supplementary calculations, reported graphically, is shown in Figure 5. In this particular phase portrait, we see two focal points A_1 and A_2 and a saddle point C. That means, for example, that when the machine oscillates around A_1, the observed states are spiralling toward A_1, which corresponds to a periodic oscillation of the Duffing equation (A_1 and A_2 are said to be "asymptotically stable" focal points). Around the saddle point, depending on the initial condition of the oscillator, the orbit can approach or diverge from C. The behaviour of the Duffing equation is structured by its periodic solutions, that is why the device is used as a "periodic oscillations hunting system" by Jean.

3 Practice and culture of analogies

As they faced growing competition from digital computers (and we must remember that analogue instruments were, at that time, generally more efficient than digital computers), the analogue practices of the group must be emphasized, since they were based on a trade-off: there was no need for high precision results but rather quick graphical representations. When the first digital computers entered Vogel's lab (in 1963), its members quickly adopted the new techniques for use in their investigations on dynamics as a new tool with new modelling possibilities to explore different aspects of the dynamics, enhancing but not replacing the extant analogue systems. They investigated the Duffing equation digitally, with an emphasis on finding new solutions. But they also built simultaneously a new electronic device to investigate the equation analogically, to visualize the results on the oscilloscope and cross check their digital results [1, 5]. In the end, their adoption of the digital was essential for continuity in their practices and their comprehension of machines.

These works on "theoretical dynamics" remained relatively unknown until recently, but analogue calculating methods and strategies have been around since the 1930s. Some ideas of Nicolas Minorsky in the mathematics of control and his dynamical analogue calculating devices were inspired by Vannevar Bush's "differential analyzer" [8]. The meteorological (and essentially mathematical) work of Edward Lorenz later in the 1960s was, in some sense, a dialogue between mathematical theories on differential systems and digital calculations, leading to what is now called the *Lorenz attractor* in chaos theory. Another of the many examples is the rendering of chaotic attractors [2] by Otto Rössler in the 1970s [7]. The whole history of dynamical systems is comprised of mathematical theories advancing step by step with calculation and visualizing practices, developed by mathematicians but also by physicists, engineers, chemists, and biologists.

4 Analogue instruments and digital computers

The example of the "theoretical dynamics" group raises many questions regarding the use and role of analogue mathematical instruments in the 1950s and 1960s. For example: What were the different strategies regarding the choice to go on with analogue systems considering the fact that they are less precise than digital? How can these choices be related to different cultures of calculations and experimentation, given that engineers, physicists, and mathematicians did

[2] An *attractor* is a set of states in the phase space of a dynamical system whose surrounding points come closer and closer as time passes. It is called *chaotic* if it is extremely sensitive to initial conditions. In other words, for two points that are very close together, their orbits, although they remain inside the attractor, move away from each other extremely quickly.

not choose the same devices? For now, there exists no panoramic view of these diverse uses of mathematical instruments at that time.

At CRSIM, machines produce dynamics, calculate dynamics, and produce images of dynamics. They are material systems which can be easily controlled and fitted to the investigations. They are not necessarily precise, but they are fast. They are models of mathematical systems that can be experimented. The group meets, in some sense, the ideas of Vannevar Bush about such analogue instruments, that they should be used as "suggestive auxiliary to precise reasoning" [3, page 649].

Analogue instruments, analogue practices, and analogue culture were widespread and diverse. The impact of digital computers was very different from one context to another. Here we can clearly see the continuity in modelling practices from analogue to digital during the 1950s through to the 1970s, as Charles Care [4] has pointed out in other contexts. Altogether, it appears that the analogue culture was not obliterated as soon as digital techniques were available. It seems that it was not a simple matter of change from one to the other, but rather a process of hybridization and adaptation.

Image credits

Figure 1 [6, page 148].

Figure 2 [6, page 148].

Figure 3 [6, page 148].

Figure 4 [6, page 150].

Figure 5 [6, page 151].

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Loïc Petitgirard *is an associate professor at the Conservatoire national des arts et métiers, Paris – HT2S (CNAM) and Labex HASTEC (ANR-10-LABX-0085).*

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DOI
10.14760/SNAP-2019-012-EN

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ISSN 2626-1995