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Expander graphs and where to find them

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Graphs are mathematical objects composed of a collection of "dots" called vertices, some of which are joined by lines called edges. Graphs are ideal for visually representing relations between things, and mathematical properties of graphs can provide an insight into real-life phenomena. One interesting property is how connected a graph is, in the sense of how easy it is to move between the vertices along the edges. The topic dealt with here is the construction of particularly well-connected graphs, and whether or not such graphs can happily exist in worlds similar to ours.

1 Introducing graphs

In mathematics, we often invent abstract objects to study. These objects may be inspired by aspects of real life, or invented to solve a real-life problem. Sometimes, the concepts we invent have their roots in something concrete or "real", but then evolve in a way that makes this connection to the real world less visible. Take, for example, the study of roots of polynomials, which has many applications to physical problems. Once people had derived a formula to compute solutions of quadratic equations from the coefficients, one natural question to ask was whether there exist such formulas for higher-degree polynomials. This led to the invention of completely new methods and objects called "groups", which were used by the famous mathematician Galois in the early 19th century to show that no such formula can exist for polynomials of degree ≥ 5. Group theory, the study of these objects, is now a thriving branch of pure mathematics that generates its own questions and methods no longer related to its origins in polynomial equations.

Another abstract object in pure mathematics whose usefulness is rather intuitive is a *graph*. A graph is defined to be a collection of "dots" called *vertices*, some of which are joined by lines called *edges*, see Figure 1.

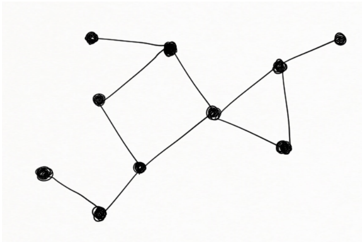

Figure 1: An example of a graph.

Graphs are good for representing relations between things. For example, drawing a vertex for each of one's friends and drawing an edge between two vertices if the people represented by these vertices are friends with each other gives a good idea of what one's social network looks like.

A graph is a particular example of a *metric space*, which is a set X endowed with a *distance function* d which tells us the distance between any two points x, y in the set. The graph distance $d(x, y)$ is defined to be the minimum number of edges in a path from x to y. Note however that there may be more than one possible shortest path between two vertices. A distance function d must satisfy the following rules:

1. The distance is always positive: $d(x, y) \geq 0$ for all x, y in X.
2. The distance is never zero, unless it's the distance between a point and itself: $d(x, y) = 0$ if and only if $x = y$.
3. The distance from x to y is the same as the distance from y to x: $d(x, y) = d(y, x)$ for all x, y in X.
4. It's quicker to go from one point to another directly, rather than via some third point: $d(x, y) \leq d(x, z) + d(z, y)$ for all x, y, z in X.

These rules are intuitive for our notion of distance in real life, and are also easy to check for the graph distance function. So we can use the distance, the induced metric space structure and associated theory to learn more about graphs and the relations they encode.

2 OK, now what?

Once we have defined potentially interesting mathematical objects, we need to see what we can do with them. Questions we can ask ourselves include:

- How restrictive are the rules that define our objects? Do we have many different objects satisfying these rules? Can we classify them?
- What kind of properties can these objects have? How are the properties related? Which properties imply others?
- How do our objects interact with other known mathematical structures?

For graphs, one property which naturally suggests itself is "connectedness", as we may hope that this will tell us about how well the things represented by the vertices are interrelated.

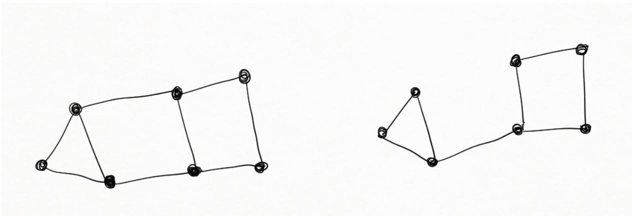

Figure 2: Which graph is better connected?

Given the two graphs in Figure 2, which one is better connected? It is intuitively clear that the graph on the left is better connected. Why is this? One answer would be that the graph on the right can be "disconnected", that is, split into two distinct pieces, by removing just one edge.

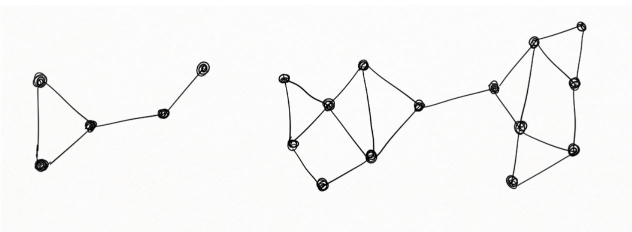

Figure 3: Which of these is better connected?

Now consider the two graphs in Figure 3. Both of these can be disconnected by removing one edge, but the parts we can disconnect from each other are much bigger for the graph on the right. Suppose now that this graph represented an electrical network: the vertices correspond to houses that are connected to each other via edges that represent the power lines that provide them with electricity. If a region is disconnected (due to failures in the power lines, which for us is represented by the removal of edges), it will be left without power. If this can happen to a very large region because of the failure of only a few power lines, it seems that the electrical network was not very well connected. With this example in mind, when we search for a good notion of connectedness, it seems reasonable to also take into account how big a region we can disconnect with how small a set of edges.

Thus, given a graph X and some subset of vertices A of X, we will look at the number of edges needed to disconnect A from the rest of the graph relative to the size of A. This set of edges will be called the boundary of A, and will be denoted by ∂A; see Figure 4 for an example.

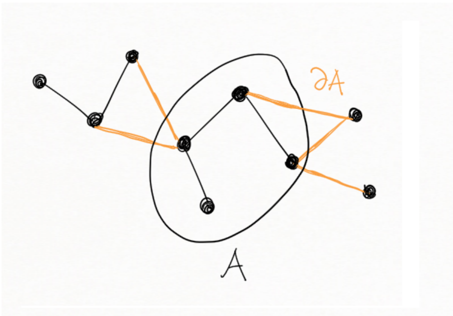

Figure 4: The boundary of a subset of vertices.

We will divide the number of edges in ∂A, which we write $|\partial A|$, by the number of vertices in A, which we write $|A|$. What we get is $\frac{|\partial A|}{|A|}$, which is just a number. We would like this quantity to be large, regardless of the subset of vertices that we choose. This leads us to the following definition. The *Cheeger constant*, denoted by $h(X)$, is defined to be

$$h(X) = \min_{A} \frac{|\partial A|}{|A|},$$

where this means that we calculate $\frac{|\partial A|}{|A|}$ for every possible subset A of vertices at most half the size of X and set $h(X)$ to be the smallest of these values. We only consider subsets at most half the size of X so that each time the graph X is split into two by removing edges, we consider the smaller part to be the subset A. This is to avoid cases such as taking the whole of X as the subset A, because this case doesn't tell us anything about the connectedness of the graph. Our interpretation of the Cheeger constant is that the bigger it is, the better connected is the graph.

Now that we have defined what is hopefully a good measure of connectedness, we can explore how it behaves for different graphs and how it is linked to other properties of graphs. For example, if the *degree* of a vertex is the number of edges connected to it, we have

$$h(X) \leq \text{ minimal degree of a vertex in } X.$$

This is because if we let the subset A be a single vertex of minimal degree, we obtain

$$\frac{|\partial A|}{|A|} = \frac{\text{min. degree of a vertex in } X}{1} = \text{ min. degree of a vertex in } X.$$

This is of course bigger than or equal to the minimum over all subsets A in X, which is the Cheeger constant $h(X)$. We have thus proved a simple upper bound on $h(X)$.

It can be calculated that for a *complete* graph with n vertices, that is, a graph where there is an edge between every pair of vertices, the Cheeger constant is given by $n/2$ if n is even and $(n+1)/2$ if n is odd. This gives us a whole family of very well-connected graphs, but at the high cost of connecting each vertex to every other vertex! Recalling the motivating example of the electrical network, connecting houses to each other via power cables in this way is definitely not a practical solution for ensuring the network is well-connected (power cables are expensive!).

3 Can it even exist?

It is useful and interesting to search for a whole family of graphs with more and more vertices and good connectedness properties. Firstly, it allows us to create a network of any size which is guaranteed to be well-connected. Secondly, it helps us understand what well-connected really means. This is because for any given graph, the Cheeger constant is a number that we know measures connectedness, but is perhaps difficult to interpret without comparing it to the Cheeger constants of other similar graphs (of the same size, say). So, given some constraint, such as a bound on the number of edges, what is the best way of connecting n vertices in order to maximize the Cheeger constant?

We can of course solve this for a given number of vertices n by simply enumerating all the possible graphs satisfying the constraint and checking their Cheeger constants. But for large n this may be difficult to compute, and moreover, boring! We would prefer to have a recipe to make bigger and bigger graphs that satisfy the constraints and that are guaranteed to have at least a certain amount of well-connectedness. Thus, what we are searching for is an infinite family of graphs (X_i) such that for every number n, we can find a graph in our family with at least n vertices, the graphs are all well-connected, and there are not too many edges.

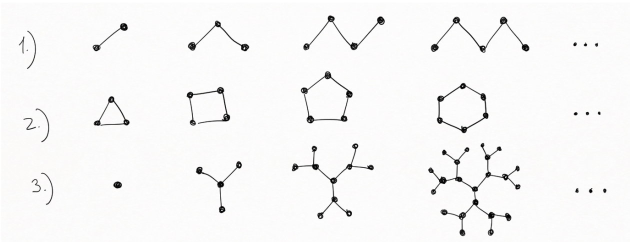

Figure 5: Some sequences of graphs.

While the first of these conditions is already formulated precisely, the others need translating into a more mathematical form. For the third assumption, a reasonable thing to try would be to impose the condition that each vertex can only be connected to at most a certain number D of other vertices, where D is independent of the size of the graph. This will help us avoid the expensive case of the family of complete graphs.

For the well-connectedness condition, we could try imposing the relatively benign-looking condition that the Cheeger constant always be bounded away from zero, that is, $h(X_i) \geq \varepsilon > 0$ for all i, where the constant ε may be small, but is strictly greater than zero and, importantly, is independent of i. This will guarantee that even if we pick a very large graph in our family, it is guaranteed to have a certain level of connectedness that is independent of its size.

Our requirements on the family of graphs (X_i) now take the following form:

- For every n, there exists an i such that $|X_i| \geq n$;
- There exists an $\varepsilon > 0$ such that $h(X_i) \geq \varepsilon$ for all i;
- There exists $D > 0$ such that the degree of all vertices in the family (X_i) is at most D.

A family (X_i) of graphs that satisfy all three of the above conditions is called an *expander*. This definition was first given by Pinsker in [14], but graphs with properties equivalent to expansion had already been studied by Kolmogorov and Barzdin in [5] in the context of neural networks. Now that we have the definition of this object with desirable properties, we would like to have an example. But where should we look? We could test some easy-to-define families of graphs, or ones which we may already have come across for other mathematical reasons.

In the examples in Figure 5, we can see that if we continue the sequences of graphs, the degree of the vertices does indeed remain bounded, but the Cheeger constant will actually get closer and closer to zero. This means the connectedness condition cannot be satisfied for these graphs. Indeed, it is not clear at first glance how to construct[1] even a single example of an expander, and one may wonder whether such examples can exist at all: after all, being well-connected and not containing too many edges may well be contradictory properties!

If mathematicians are not able to construct a certain object, the next best thing is to prove that it exists without constructing it. While non-constructive existence proofs may be difficult to use in applications, it is still interesting to know that there is some hope of constructing the object of interest in the future. Pinsker was the first to achieve this for expanders in [14]. He used techniques from probability theory to show that expanders exist long before explicit examples were given. The explicit examples came later, and in an interesting form: bringing together algebra and geometry.

4 Thanks, algebra!

Before we explain how to find a rich source of examples of expanders, we must return to groups, as mentioned in the introduction. A *group* is defined to be a set of elements, endowed with a way to compose them which satisfies certain rules. For a concrete example, let us consider the set of symmetries of an equilateral triangle. One quickly verifies that the triangle is symmetric with respect to rotation clockwise by 120°, rotation by 240° and reflection in each of the straight lines that pass through a vertex and the midpoint of the opposite side. We include in the set of symmetries the rotation by 0°, or rather, the symmetry that consists of leaving the triangle unchanged. Observe that if we compose two symmetries (that is, perform one and then another), as in Figure 6, we always obtain another symmetry. We also see that every symmetry can be "undone" by another, for instance, the rotation by 120° followed by the rotation by 240° leaves us with the original triangle.

Let us now give the formal definition of a group. Call the set of elements of our group G, and for two elements g and h in G, let us write $g * h$ for the composition of g and h. The rules that G and $*$ have to satisfy in order to form a group are:

- $g * h$ must be an element of G;

[1] By "construct" here we mean to have a recipe that is guaranteed to give us infinitely many graphs with the desired properties, even if we cannot actually construct infinitely many graphs.

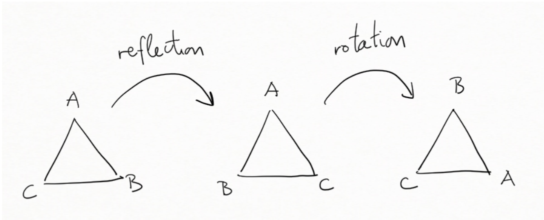

Figure 6: A reflection followed by a rotation.

- there must be an *identity* element e in G, such that $g * e = e * g = g$ for all g in G;
- for each g in G, there is an *inverse* element g^{-1} in G such that $g * g^{-1} = g^{-1} * g = e$;
- we must have $(g * h) * k = g * (h * k)$ for all g, h, k in G.

The first rule ensures that the composition of two elements is always in the group, we call this property *closure*. The second and third rules mean we must always include the "do nothing" and "undoing" elements, just as we did with the triangle example. The fourth rule, in words, means that composing g and h, and then composing the result with k gives the same thing as composing g with the composition of h and k (this is called *associativity* of the operation $*$).

We can check that taking G to be the set of symmetries of a triangle, with the composition $*$ being applying one symmetry after another, we obtain a group. Note that the order in which we compose elements matters: composing a clockwise rotation by $120°$ with a reflection, or a reflection with a clockwise rotation by $120°$, we obtain different symmetries! Actually, the symmetries of any object will form a group, making this a fundamental example in the field. Readers are encouraged to check the rules for some other examples with which they may already be familiar: the set of all integers with the operation of addition; the set of numbers $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$ with the operation of addition modulo 12 (recognisable as clock arithmetic); the set of non-zero real numbers with the operation of multiplication (can you see why we need to exclude zero?).

The study of groups is extremely rich, with many elegant results, varied techniques, and connections to other areas of mathematics to explore. One such connection is with geometry via the following construction, for which we first need to explain the notion of a *generating set* of a group.

Given a group G, we say that a subset S of its elements *generates* G if every element of G can be obtained via some composition of elements of S (we are allowed to use elements of S more than once). For example, the group of symmetries of the triangle above can be generated using only the clockwise rotations by 120° and 240°, and a reflection. See Figure 6 for an example of an element not in this set as a composition of elements in this set.

The resulting expression of an element as a composition of elements of the generating set S may not be unique, and there may be many choices of the generating set S itself. Once we have fixed one such set S, we can view our group geometrically by turning it into a graph. As the set of vertices, we take the set of elements of our group G. For the edges, we will use the set S by connecting two elements g and h whenever we can get directly from one to the other by composing with an element of S. In other words, the pairs of elements connected by edges will be those of the form $(g, g * s)$ where g is an element of G, and s is an element of S. This graph is called the *Cayley graph* of G with respect to S. Figure 7 shows an example of a Cayley graph of the group of symmetries of the triangle with the generating set given above. The generators are the elements with an edge joining them to the identity (the original triangle having the vertex at the top labeled "A", with "B" and "C" coming in clockwise succession). Note that we only draw at most one edge between the same pair of vertices, even when there could in fact be two, for simplicity.

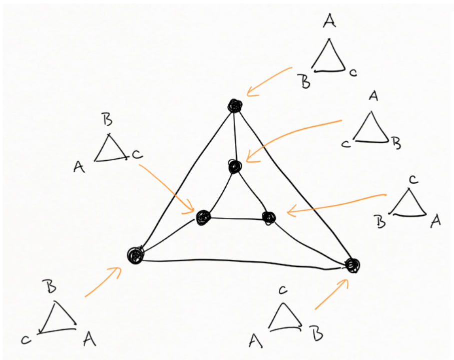

Figure 7: A Cayley graph of the group of symmetries of the triangle.

Recall from the first chapter that graphs are metric spaces – they are endowed with a notion of distance. This means that we can now view a group as a metric space, and use existing techniques in geometry in order to study groups. The field of geometric group theory is devoted to exploiting exactly these connections between algebra and geometry, and has been very successful in providing beautiful examples of the interplay between these two subjects.

Let us return to the question of the existence of expanders. In fact, imbuing groups with geometry in the above way is not only a useful tool for studying the groups themselves, but can also serve as a way to create metric spaces with desired properties. This is exactly what happened in the history of expanders, when Margulis proved in [10] that Cayley graphs of groups with certain properties can be used to construct explicit examples of expanders.

A basic but important concept from group theory, that of a *quotient*, is used here. The word “quotient” may conjure up the image of division, and indeed in some sense it is a way to divide groups by other groups. While we can divide any number by any other (non-zero) number, for groups the situation is slightly more subtle. To take the quotient of a group G by another group H, we need H to sit inside G as a *normal subgroup*. A subgroup of a group G is a subset of its elements that also forms a group (so satisfies the four rules concerning closure, identity, inverses, and associativity). As an example, consider the subset of symmetries of the triangle consisting of the rotations $\{R_{120}, R_{240}, R_{360} = Id\}$. A subgroup H of a group G is said to be *normal* in G if for any fixed g in G, the following two sets are equal:

$$\{g * h : h \in H\} = \{h * g : h \in H\}.$$

As an example, one can check that the above-mentioned subgroup of rotations of the triangle is normal. One may also wish to check, as a counterexample, that the two-element subgroup formed of the identity symmetry and any reflection is not normal.

For a normal subgroup H of G, we can form a group called the *quotient* of G by H, written as G/H. Its elements are subsets of the group G of the form $gH = \{g * h : h \text{ in } H\}$. The operation on our new quotient group is defined by $g_1 H * g_2 H = g_1 g_2 H$, and the interested reader can check that the set of all subsets of the form gH for g in G forms a group with respect to this operation. Intuitively, one can think of G as consisting of many translated copies of the subgroup H, and the quotienting process as simply shrinking each of these copies to one element, as illustrated in Figure 8.

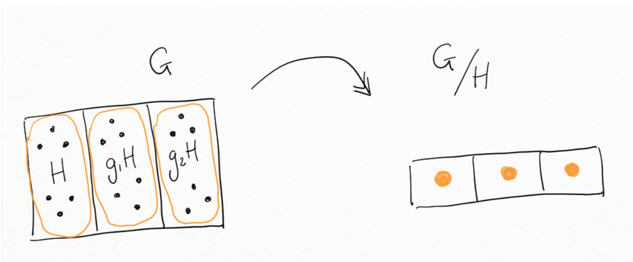

Figure 8: Quotienting by a normal subgroup H.

What Margulis proved is that if we take an infinite group that satisfies "Kazhdan's property (T)", then we can obtain an expander by looking at the Cayley graphs of an infinite sequence of finite quotients of this group. Obviously we are sweeping almost all of the details under the carpet, but these Cayley graphs do get bigger and bigger, and have bounded degree.

Let us briefly look at property (T); we shall first state it and then unpack what it means. A group G satisfies property (T) if every affine isometric action of G on a Hilbert space has a fixed point. There is a lot to unpack! Firstly, for a group to *act* on a space means that we think of the elements of the group as functions on that space that satisfy certain conditions. For example, the rotation R_{120} of the triangle can be thought of as a function on the plane $\mathbb{R}^2$ that rotates every point clockwise by 120° around the origin. A *fixed point* is a point that is left unchanged by the action. An action is *affine* if it is a linear map composed with a translation and it is *isometric* if it doesn't change the distance between points in our space:

$$d(g(x), g(y)) = d(x, y), \quad \text{for all} \quad g \in G.$$

Finally, a *Hilbert space*, named after the German mathematician David Hilbert (1862–1943), is a generalisation of the familiar Euclidean spaces. Hilbert spaces can have any number of dimensions, even infinitely many, and their defining characteristic is a geometric one: they admit an "inner product", which is a generalisation of the scalar product of two vectors that allows the measurement of distances and angles between the points of the space. Note that every Euclidean space $\mathbb{R}^n$ is a Hilbert space, in particular the plane $\mathbb{R}^2$ is an example.

It is a remarkable result that taking Cayley graphs of finite quotients of groups satisfying property (T) produces expanders – not because of any difficulty in the proof, but rather because of the idea that this is where one should look!

Since Margulis' result, several other ways of constructing expanders have appeared. For example, a purely combinatorial "zig-zag" construction was found by Reingold, Vadhan and Wigderson in [15], and asymptotically optimal expanders called Ramanujan graphs were constructed in [9] by Lubotzky, Phillips and Sarnak. One may also wish to consult [4], [6], or [2] for a more expository approach to these topics, as well as applications of expanders.

5 Expanders and their habitats

Now that we have explicit examples of expanders, we can further explore their properties. There are many aspects we could study, but here we will focus on the interaction of expanders with other spaces. To explain what we mean by this, consider the following example.

Let us take a network of stations connected by rail. We can model this as a graph, where the vertices are the stations and there is an edge between two stations when there is a direct rail line (with no intermediate stops) between them. Let us suppose for simplicity that the rail lines corresponding to edges are all of the same length.

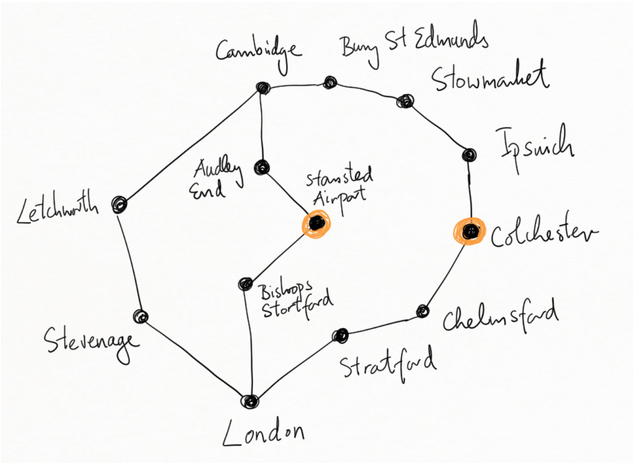

Figure 9: A graph representing a rail network.

In Figure 9, we can see that to travel by rail from Stansted Airport to Colchester, one would need to pass through London, the shortest path in our graph being Stansted Airport – Bishops Stortford – London – Stratford – Chelmsworth – Colchester. Thus, the distance in our graph between Stansted Airport and Colchester is 5.

On the other hand, looking at a map of the UK, we see that Stansted Airport is actually very close to Colchester "as the crow flies". A crow would never consider flying from Stansted Airport to Colchester via London! What we are seeing here is a difference in the metrics we can put on the set of stations: we have the metric that is induced by the rail network graph, and the metric that is induced by the location of the stations on land. This brings us to the question, for a given graph drawn on a plane, how different is the graph metric from the induced plane metric? In other words, how badly is the graph metric distorted by this embedding of the graph in the plane? Is there a way to draw it in the plane so that there is not too much difference between the metrics?

In fact, we can go ahead and ask a more general version of this question. For a given graph or sequence of graphs, in which spaces can they be embedded so that the graph metric is not too different from the metric induced on the vertices by the embedding in this space? To give a concrete meaning to "not too different", we will say that a sequence of graphs (X_i) *coarsely embeds* into a metric space Y if there exists a sequence of mappings f_i of X_i into Y and increasing, unbounded functions ρ_+ and ρ_- from the positive real numbers to the positive real numbers which control how the distances are distorted:

$$\rho_-(d_i(x,y)) \leq d_Y(f_i(x), f_i(y)) \leq \rho_+(d_i(x,y)), \text{ for all } i, \text{ and } x,y \in X_i,$$

where we have used d_i to denote the graph distance on X_i and d_Y to denote the distance on Y. This condition means that the distance in the image is bounded on both sides by (functions of) the original distance. As the name suggests, a coarse embedding is a very rough notion of inclusion of one metric space into another. Informally, it means that when we embed our graphs into the space Y, we are allowed to change some small-scale structure of the graphs, but the large-scale structure must be preserved in the image of the graphs in Y. Luckily, it is often the large-scale information which is important. For example, the Cheeger constant behaves well under coarse embeddings, so it is interesting to find out into which spaces we can coarsely embed expanders.

A first result to observe in this direction is that expanders do not coarsely embed into Hilbert spaces.[2] Intuitively, this is due to their connectivity properties: the high connectivity forces too many vertices into a small neighbourhood in the Hilbert space, which then violates the bounded degree assumption. Observe that this means, in particular, that expanders do not coarsely embed into the plane $\mathbb{R}^2$, as in our railway example.

So, if it is not possible for an expander to coarsely embed into a Hilbert space, we can try asking the same question for other types of spaces. One possibility is the family of "Banach spaces", which are a generalisation of Hilbert spaces, in that they have a distance function, but not an inner product (geometrically, this means we can measure lengths but not angles). Every Hilbert space is a Banach space, but not vice versa. A well-known class of Banach spaces are the ℓ^p spaces. For $p > 1$, an ℓ_p space is the space of finite or infinite sequences (a_i) satisfying

$$\left(\sum |a_i|^p\right)^{\frac{1}{p}} < \infty,$$

where the expression on the left is the length of (a_i) in this space. The space ℓ^2 is a Hilbert space, but this is not true for other values of p. To imagine what these spaces look like, it is instructive to draw the unit circles (the set of vectors of length 1) in such spaces of dimension 2, where the sequences consist of pairs (a_1, a_2). This is shown in Figure 10.

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[2] A proof can be found in [16] or in [13], which also serves as a good introduction to the topic of "coarse geometry", that is, geometry of groups and spaces from a large-scale point of view

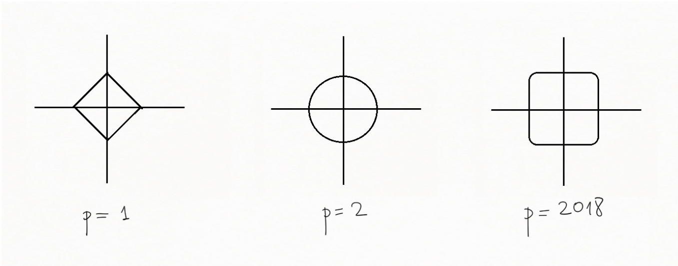

Figure 10: Unit circles for different values of p.

We can see in Figure 10 that as p gets bigger, the unit circle starts to look more and more like a square. This gives us an idea of how the metric structure changes as we explore ℓ^p for different p. The ℓ^p spaces for $p > 1$ are also "uniformly convex", which means that the midpoint of a line segment between two points on the unit circle must lie uniformly deeply in the unit disk (here we can see this is true from the picture).

It turns out that expanders cannot coarsely embed into any ℓ^p space (see [11, 16]). But what about more general Banach spaces? Do there exist expanders that can coarsely embed into some uniformly convex Banach space? Well, we don't know. This is still an important open question.

Expanders that do not coarsely embed into any uniformly convex Banach space are called *superexpanders*. While we do not know of any examples of expanders that embed into a uniformly convex Banach space, we also do not know of many examples of superexpanders – this strange dichotomy is not uncommon in mathematics, especially when the objects involved are difficult to classify or to construct. The only known examples so far have been constructed by Lafforgue in [8] using Cayley graphs of quotients of certain groups which have very particular properties related to their actions on spaces, by Mendel and Naor in [12] using "zig-zag products", and by de Laat and Vigolo in [7] using group actions on compact spaces (see also work by Fisher, Nguyen and van Limbeek in [3]).

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