Snapshots of modern mathematics
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A surprising connection between quantum mechanics and shallow water waves

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We describe a connection between quantum mechanics and nonlinear wave equations and highlight a few problems at the forefront of modern research in the intersection of these areas.

1 Introduction

Quantum mechanics grew out of attempts to understand the structure of microscopic systems (likes atoms and molecules) and to explain observed phenomena in electromagnetism (such as blackbody radiation and the photoelectric effect). These phenomena could only be explained by the presence of an irreducible quantity of energy. The observed behavior of such small-scale systems is markedly different from the observed behavior of the macroscopic systems which constitute the world we observe in our daily life; for example, it is impossible to simultaneously measure a particle's position and momentum with perfect accuracy. One of the deepest insights gleaned from the modern theory of quantum mechanics is the famous notion of "wave-particle duality": particles that are sufficiently small behave like waves until observed, at which point they develop the more familiar behavior of a completely localized particle.

This duality was not a simple idea to develop. Indeed, there were a wide variety of competing theories that adopted specific wave or particle conventions

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for distinct physical phenomena on a rather *ad hoc* basis; for example, René Descartes (1596–1650) believed that light behaved like a wave in a medium, whereas Isaac Newton (1643–1727) suggested the way light reflects was best explained by a particulate model. Ultimately, two developments in the study of electromagnetism forced the quantum paradigm shift. On one hand, in the late nineteenth century, Maxwell's equations conclusively demonstrated that electromagnetic radiation was a wave generated by the motion of charged particles through an electromagnetic field. On the other hand, in the early twentieth century, experiments showed that electromagnetism propagates through a vacuum, which a wave was thought to be unable to do. In this sense, in order for physicists to understand how quantum particles traveled, it became completely fundamental to understand interactions of waves.

The study of waves dates back to humanity's earliest attempts to understand the physical world. Mathematically, we model waves using functions that are *smooth*, so small changes in the variables lead to small changes in the value of the function, and *periodic*, so the function returns to the same value after a regular interval. A basic example is the sine function, where $\sin(x) = \sin(x+2\pi)$. Here 2π is the *period*. Perhaps the first use of the mathematics of waves is due to the Babylonians, who used repeating functions to make predictions about the motions of the stars and planets. Greek mathematicians studied the properties of a vibrating string and how its physical properties influenced the sound it produced when plucked, and Hindi mathematicians in the 4th and 5th centuries developed the first trigonometric functions related to chords of the circle. Much later, Joseph Fourier (1768–1830) showed that many functions could be built by using the constructive and destructive interference of waves [3]. As such, by the twentieth century researchers had thoroughly developed the mathematics of waves, which is collectively known as harmonic analysis. Our mathematical understanding of waves has applications manifest throughout our everyday lives, from technologies built into noise-canceling headphones to the architecture of cacophonous sporting arenas; as such, it is critically important that we understand, to the best of our ability, how waves change over time.

This snapshot considers two different and seemingly unrelated models for how certain waves evolve. The first one is the Schrödinger equation. This equation gives a quantum-mechanical description of how microscopic particles interact with electrostatic potential barriers, which is the obstacle that such particles encounter when approaching the atoms that constitute matter at the microscopic scale. The second is the Korteweg–de Vries equation, which is a macroscopic model of the evolution of a cross-section of a wave in shallow water. Remarkably, these equations – designed to describe completely different phenomena – were discovered to be fundamentally linked by the mathematician Peter Lax in the late 1960s [13]. This surprising connection, which we describe below, continues to yield new mathematical insights today.

2 Quantum dynamics in one dimension

In the usual formulation of quantum mechanics, the information about the physical system under consideration is encoded in the *wave function*, $\Psi(x,t)$, which is a function of position x and time t. For simplicity, we will consider quantum dynamics in one spatial dimension (imagine a particle confined to an extremely thin wire). The standard interpretation of the wave function is that the square of its modulus, $|\Psi(x,t)|^2$, represents the likelihood of finding the quantum state near the position x at the time t. More concretely, the *probability* of finding the quantum state Ψ in the interval (a,b) at time t is given by

$$\mathrm{Prob}(a < x < b) = \int_a^b |\Psi(x,t)|^2 \, dx \tag{1}$$

Of course, the particle under consideration must be somewhere, and hence the probability of finding it in the interval $(-\infty,\infty)$ must be one. This allows us to conclude that a wave function should obey the normalization condition

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2 \, dx = 1 \text{ for every } t.$$

Then, if we wish to understand the behavior of a quantum state, we need to understand how the wave function changes as time passes. The evolution equation for a wave function corresponding to a quantum state of mass m is the *Schrödinger equation*, which reads:

$$i\hbar \frac{\partial \Psi}{\partial t}(x,t) = -\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2}(x,t) + V(x)\Psi(x,t). \tag{2}$$

This equation is a partial differential equation for the wave function Ψ. It is determined by the values of Ψ, the first-order partial derivative of Φ with respect to time t and the second-order partial derivative of Φ with respect to position x. Note that the Schrödinger equation is a complex equation, due to the presence of the imaginary number i defined by $i^2 = -1$. The wave function Ψ solution to this equation is therefore complex-valued, which is why we need to work with the square of its modulus in order to extract physically meaningful information. In the Schrödinger equation, the quantity $\hbar$ denotes the reduced Planck constant. Furthermore, $V(x)$ represents the potential energy of the quantum state Ψ at position x, which models the environment with which the particle is interacting. In analogy with classical mechanics, where systems evolve when subject to the action of external forces like gravity for example, one can think of Ψ as being acted upon by a conservative external force F, given by

$$F(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}.$$

Since the physical constants clutter the notation without changing the qualitative characteristics of solutions, we choose physical units so that $\hbar = 1$ and the mass $m = 1/2$, so our Schrödinger equation becomes

$$i\frac{\partial\Psi}{\partial t}(x,t) = -\frac{\partial^2\Psi}{\partial x^2}(x,t) + V(x)\Psi(x,t). \tag{3}$$

A *free quantum particle* is obtained by setting $V(x) \equiv 0$; this choice of V represents a quantum particle with no external forces acting upon it, hence the term "free". Then, our Schrödinger Equation (3) becomes

$$i\frac{\partial\Psi}{\partial t}(x,t) = -\frac{\partial^2\Psi}{\partial x^2}(x,t). \tag{4}$$

Based on physical intuition, we expect solutions of this equation to "propagate freely at constant speed", since there is no external force acting to slow down or speed up the particle. One might think of this as analogous to the situation in Newton's first law of motion: "An object in an inertial frame of reference either remains at rest or travels in a straight line at constant speed unless it is acted upon by a net external force."

Solutions to the free Schrödinger Equation (4) can be expressed quite elegantly in terms of the *complex exponential* function:

$$e^{ix} = \cos(x) + i\sin(x), \quad x \text{ a real number.}$$

The complex exponential function encodes the two periodic functions $\sin(x)$ and $\cos(x)$ and can be used to represent more general periodic functions. One solution of the free Schrödinger Equation (4) is

$$\Psi(x,t) = e^{i(kx-k^2t)},$$

where k is a constant. This represents a freely traveling wave, also called a *plane wave*. To see why, let us pick a value for k and focus on the real part $\cos(kx - k^2t)$. Let us choose $k = 1$ and plot $\cos(x - t)$ for $t \in \{0, 0.5, 1, 1.5\}$, as shown in Figure 1.

The observant reader will notice that the Ψ's that we have written down are not quite proper wave functions for the following reason: no matter how the variables k, x, and t are chosen,

$$|\Psi(x,t)|^2 = |e^{i(kx-k^2t)}|^2 = \cos^2(kx - k^2t) + \sin^2(kx - k^2t) = 1,$$

which means that

$$\int_{-\infty}^{\infty} |\Psi(x,t)|^2\, dx = \int_{-\infty}^{\infty} 1\, dx = \infty.$$

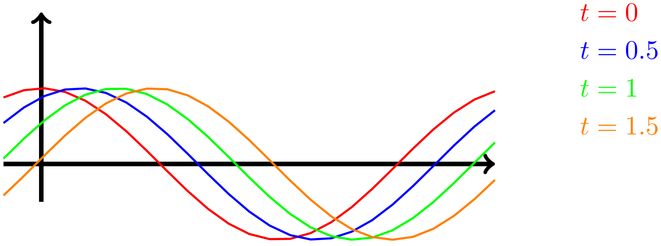

Figure 1: Traveling wave.

Despite this minor shortcoming, these plane wave solutions are exceptionally important in mathematical physics. Even though they do not represent a proper normalized wave function, plane waves are the fundamental building blocks for physical solutions to the free Schrödinger equation. In other words, wave functions can be represented in terms of such plane waves.

For the purpose of our discussion, we now need to introduce the notion of "reflectionless" potentials. Qualitatively, reflectionless potentials share many properties with a repeating periodic potential, where we recall that this means the potential satisfies

$$V(x+T) = V(x)$$

for some real $T > 0$. In other words, V is periodic if you can translate it by a nontrivial amount and land exactly where you started. The exact definition of a reflectionless potential is beyond the scope of the current article, but here is a rough description: Consider a solution $\Psi(x,t)$ of the Schrödinger equation (2). Recall that we view $|\Psi(x,t)|^2$ as a probability amplitude, giving us an estimate of how likely it is to find the wave function near x at time t. In light of this, we say that Ψ originates at $-\infty$ if for each real number a,

$$\lim_{t \to -\infty} \operatorname{Prob}(x < a) = 1,$$

where the probability is defined in (1). Similarly, we say that Ψ propagates to $+\infty$ if, for each real a,

$$\lim_{t \to +\infty} \operatorname{Prob}(x > a) = 1$$

If $V(x)$ is periodic, then any wave function $\Psi(x,t)$ that originates at $-\infty$ also propagates to $+\infty$ as time advances. Thus, solutions to the Schrödinger equation (2) with a periodic potential do not get "stuck" anywhere. The class of reflectionless potentials comprises those potentials which share this quantum dynamical property.

3 The Korteweg–de Vries equation

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped – not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the Wave of Translation.
>
> -John Scott Russell (1808–1882) [14]

John Scott Russell's "Wave of Translation", observed in Scotland's Union Canal in 1834 [14], is the first scientifically noted example of what are now known as *solitons* or *solitary waves.* Solitons are characterized as localized waves of fixed shape traveling at constant velocity. Physically speaking, solitons are very easy to generate: Imagine fixing a length of string to a wall, pulling it tight, and swiftly flicking it. The string will form a bump, approximately the height of your flick, which will travel towards the wall at a fixed speed. Mathematically, solitons are more tricky to generate, but are generally understood to be a consequence of special nonlinear differential systems[3].

One such nonlinear differential system is the Korteweg–de Vries (KdV) equation. The KdV equation describes waves in shallow water, and looks like this:

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0, \tag{5}$$

where $u(x,t)$ is the amplitude of the wave. The KdV equation gets its name from the work of Diederik Johannes Korteweg (1848–1941) and Gustav de

[3] A partial differential equation (PDE) is called *linear* when the sum of several of its solutions is also a solution. Namely, if u, $\tilde{u}$ are solutions of a linear PDE, then $u + \tilde{u}$ is also a solution of this PDE. If that is not the case, the PDE is said *nonlinear.* Notice that the Schrödinger equation is a linear PDE. In general, linear equations are much easier to solve than nonlinear ones, although they occur less frequently in applications.

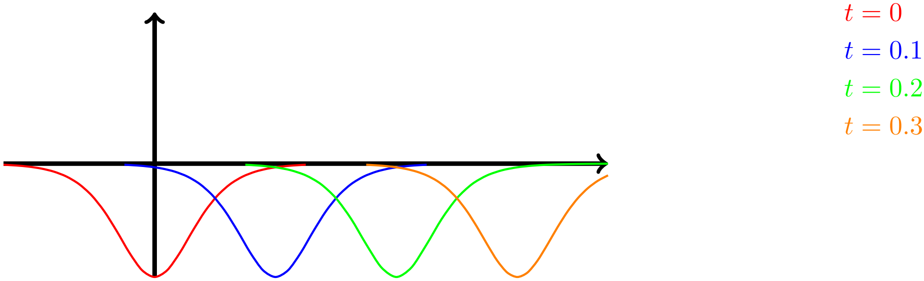

Figure 2: A moving soliton.

Vries (1868–1934) [12], though it was discovered even earlier by Joseph Valentin Boussinesq (1842–1929) [2]. The main feature that makes the KdV equation special is the existence of soliton solutions. One example of a soliton solution to Equation (5) is

$$u(x,t) = -2\left[\frac{1}{e^{\frac{1}{2}(x-t)} + e^{-\frac{1}{2}(x-t)}}\right]^2.$$

This soliton travels to the right at a speed of one unit of distance per second; see Figure 2. One of the most prominent aspects of the KdV equation is its nonlinearity. This is what makes Equation (5) so challenging and interesting to study. More specifically, the second term of Equation (5) contains a product of u and its partial derivative with respect to x, which is the source of the nonlinearity. This means that waves that solve the KdV equation do not obey the principle of superposition; that is to say, if u and $\widetilde{u}$ are solutions of Equation (5), there is no reason that $u + \widetilde{u}$ will also solve Equation (5).

However, the Schrödinger Equation (2) is a linear equation. So, how are the KdV and Schrödinger equations connected? Suppose $u(x,t)$ is a solution to the KdV equation. Then, we can imagine using the KdV equation to generate two potential energy functions for the Schrödinger equation by freezing u at two particular times; say,

$$V_0(x) = u(x,0), \quad V_1(x) = u(x,1).$$

Then, the Schrödinger equations with potentials V_0 and V_1 will share all of the same essential qualitative characteristics. For example, if V_0 is reflectionless, then V_1 shall be reflectionless as well.[4] This means that the quantum particles will behave similarly whether they propagate in a potential V_0 or V_1.

[4] These comments hold true for all fixed times t_0; we simply chose $t_0 = 0, 1$ for convenience.

More specifically, the particles have access to the same energy levels in both potentials.[5]

The mechanism that produces this relationship between Schrödinger and KdV equations is called a *Lax pair*, so called in honor of Peter Lax's seminal work on the subject [13]. The exact nature of the Lax pair is a bit beyond the scope of this article; one could describe the Lax pair by saying that it is possible to decompose the KdV equation in terms of a pair of linear differential equations in a helpful fashion. What's especially remarkable is that one of the linear differential equations in the Lax pair for the KdV equation is the Schrödinger equation! This discovery allowed mathematicians to translate knowledge about the well-studied Schrödinger equation into new facts about the KdV equation. One such application of this general technique is the *inverse scattering transform*, a method to solve some nonlinear partial differential equations, which we will not discuss here; on this topic the curious reader can see for example [9, 10].

One of the most pressing problems that remains in the study of the KdV equation is determining when a solution exists. In fact, even when a solution exists, it's hard to say that there is exactly one solution! This is an important physical consideration. After all, if a model describes a deterministic physical system, it should have a solution (something should happen) and the solution ought to be unique (exactly one thing should happen). Thus, some of the main questions that researchers would like to answer about the KdV equation are:

- For what sort of initial wave shapes is the KdV equation solvable?
- In the instances in which the KdV equation is solvable, what sorts of wave shapes are preserved by the KdV equation?
- What other kinds of quantum physics are preserved by the KdV equation?

In the 1970s, researchers focused on the case of periodic wave shapes. Many authors worked very hard to understand the KdV in this setting. For a book-length synthesis of results, see [11]. Recently, researchers have focused on these questions in the event that the initial data are almost periodic. What does it mean to be "almost periodic"? Let us recall that V is periodic (with period T) if

$$V(x+T) = V(x). \tag{6}$$

In fact, once one has a single period T, one has lots of periods! If Equation (6) holds true, then $V(x) = V(x+T) = V(x+2T) = V(x+3T)\dots$ and so on. An *almost-periodic* function is one for which this is almost true. In other words, V is almost periodic if there are many values of T for which $V(x) - V(x+T)$

[5] One of the particularities of quantum mechanics is that the energy of particles propagating in a potential can only take some specific values. We say that the energy is *quantized* in *discrete* energy levels. Other energy values cannot be attained.

is extremely small (for every x). Percy Deift has conjectured that the KdV equation is solvable when $u(x, 0)$ is almost periodic, and, moreover, that the solution remains almost periodic [7].

Utilizing the Lax pair relationship to the Schrödinger equation, reflectionless operators give us a hook into these questions! To be more specific, if the initial wave shape $u(x, 0)$ is a reflectionless potential, we know that if a solution exists, then $u(x, t_0)$ has to also be a reflectionless potential for every value of t_0. Additionally, we know that reflectionless potentials are almost periodic (under some technical restrictions that we won't discuss here). This confers several advantages:

- It narrows down the "search space." If we know that the solution should look like a bunch of reflectionless potentials stapled together, then we don't need to look at any other possible solution shapes, which narrows down the realm of possibilities to analyze.
- The reflectionless operators have a very nice way of being parameterized that allows us to change variables in the KdV equation.

Researchers have been successful with this approach in recent years, but there's still plenty of interesting work to be done. For example, it was recently discovered that there are potentials V for which the current notions of reflectionlessness are wholly inadequate [5]. Future work will need to push beyond what is currently known to understand these highly singular initial data.

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