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# POSITIVE LINE BUNDLES OVER THE IRREDUCIBLE QUANTUM FLAG MANIFOLDS

FREDY DÍAZ GARCÍA, ANDREY KRUTOV, RÉAMONN Ó BUACHALLA, PETR SOMBERG, AND KAREN R. STRUNG

Abstract. Noncommutative Kähler structures were recently introduced by the third author as a framework for studying noncommutative Kähler geometry on quantum homogeneous spaces. It was subsequently observed that the notion of a positive vector bundle directly generalises to this setting, as does the Kodaira vanishing theorem. In this paper, by restricting to covariant Kähler structures of irreducible type (those having an irreducible space of holomorphic 1-forms) we provide simple cohomological criteria for positivity, offering a means to avoid explicit curvature calculations. These general results are applied to our motivating family of examples, the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$. Building on the recently established noncommutative Borel–Weil theorem, every covariant line bundle over $\mathcal{O}_q(G/L_S)$ can be identified as positive, negative, or flat, and hence we can conclude that each Kähler structure is of Fano type. Moreover, it proves possible to extend the Borel–Weil theorem for $\mathcal{O}_q(G/L_S)$ to a direct noncommutative generalisation of the classical Bott–Borel–Weil theorem for positive line bundles.

## 1. Introduction

Positive line bundles, which is to say, line bundles whose Chern curvature is a positive definite $(1,1)$-form, play a central and ubiquitous role in modern complex geometry. Analogously, ample line bundles are fundamental objects of study in projective algebraic geometry. An ample line bundle is a line bundle $\mathcal{E}$ such that, for any coherent sheaf $\mathcal{S}$, the tensor product $\mathcal{S} \otimes \mathcal{E}^{\otimes k}$ is generated by global sections, whenever $k$

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is sufficiently large. Under the GAGA (*Géometrie Algébrique et Géométrie Analytique*) correspondence [55], positive and ample line bundles are equivalent. The existence of positive line bundles has many remarkable implications for the structure of a complex manifold. For example, the Kodaira embedding theorem says that a compact Kähler manifold is projective if and only if it admits a positive line bundle [30, § 5.3]. Positivity also has strong cohomological implications, as evidenced by the celebrated Kodaira vanishing theorem and the subsequent slew of related vanishing theorems [25]. Given the relation of positivity to these elegant structural properties, the natural impulse is to try to extend the concept to settings beyond ordinary complex geometry. This has been met with tremendous success in the study of varieties over fields of prime characteristic. Positivity, or rather in this case ampleness, has been key to understanding innate differences between these geometries, for example the failure of the Kodaira vanishing theorem in prime characteristic [51]. Another striking extension has been to the setting of noncommutative projective algebraic geometry, where ample sequences and ample pairs are by now considered foundational structures [1].

The goal of this paper is to explore the idea of positivity for the noncommutative differential geometry of quantum groups. In particular, we show that the covariant line bundles over an irreducible quantum flag manifold, endowed with their Heckenburger–Kolb calculus, are either positive, flat, or negative, **Theorems 3.3** and **4.8**. Furthermore, we are able to distinguish between these three cases using cohomological information, **Corollary 3.4**, in the form of the recently established noncommutative Akizuki–Nakano identities [48, Corollary 7.8].

Positivity in noncommutative differential geometry is a concept that has been formulated only recently in the companion paper [48]. These two papers are part of a series exploring the noncommutative complex geometry of quantum homogeneous spaces [46, 47, 48, 16, 17] based around the notion of a noncommutative Kähler structure, as introduced by the third author in [47]. In this context, the classical Koszul–Malgrange theorem [38] allows for an obvious noncommutative generalisation of the definition of a holomorphic vector bundle. As in the classical setting, every Hermitian holomorphic vector bundle has a uniquely associated Chern connection [5, Proposition 4.4]. In [48] it is observed that the definition of a positive line bundle extends directly to the noncommutative setting. Building on this observation, a corresponding Kodaira vanishing theorem can be formulated and the definition of a noncommutative Kähler structure can be refined to give the definition of a noncommutative Fano structure. The implied vanishing of cohomologies makes it possible to calculate holomorphic Euler characteristics.

Despite an abundance of structure, calculating the curvature of a line bundle in the quantum setting remains an extremely challenging task: classical tools are either not yet developed or are unavailable entirely. Any attempt at brute force calculations quickly becomes prohibitively lengthy and tedious. The complications involved can already be seen in §5, where we carry out explicit calculations for the relatively simple case of the quantum complex projective spaces, **Proposition 5.3**. Fortunately, the worst of these calculations can be avoided entirely by restricting to a particularly tractable subclass, which subsumes and generalises the quantum complex projective spaces: those covariant Kähler structures which are irreducible.

Our motivating family of examples is the irreducible, or cominiscule, quantum flag manifolds $\mathcal{O}_q(G/L_S)$, and this is the prime justification for restricting to the irreducible case. The irreducible quantum flag manifolds form a large and robust family of examples, and are a natural class to consider when attempting to extend geometric notions from the classical to the noncommutative. Indeed, it is becoming increasingly clear that the noncommutative geometry of the quantum flag manifolds is an essential key to understanding the noncommutative geometry of quantum groups in general. Here, the necessary cohomological information is provided by the forthcoming noncommutative Borel–Weil theorem [19] (see [44] for the particular case of the quantum Grassmannians). This allows us to prove in **Theorem 4.9** that every irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus, is of *Fano type* in the sense of [48, Definition 8.8]. With this information in hand, we extend the noncommutative Borel–Weil theorem to a noncommutative generalisation of the classical Bott–Borel–Weil theorem for positive line bundles, **Theorem 4.15**.

Perhaps the most significant application of the positivity results of this paper is to the proof in [17] that the appropriately twisted Dolbeault–Dirac operators of the irreducible quantum flag manifolds are Fredholm. (For the precise formulation see [17, §11.3]). These operators provide an illuminating example of how the spectrum of a noncommutative Dolbeault–Dirac operator is shaped by the geometry of the underlying de Rham complex, and a particularly satisfying demonstration of the machinery of classical complex geometry being successfully applied to the quantum world. In future work it is hoped to expand and strengthen this connection, with the search for a noncommutative GAGA equivalence serving as a motivating goal.

The paper is organised as follows. In §2 we recall necessary background material, including noncommutative Kähler structures, Hermitian and holomorphic vector bundles, and compact quantum group algebras. In particular, we recall the recently introduced notion of a compact quantum homogeneous (CQH) Kähler space [17, Definition 3.13], which details a natural set of compatibility conditions between covariant Kähler structures and compact quantum group algebras.

In §3 we develop the general theory of the paper. In particular, we introduce the notion of an irreducible CQH-Kähler space, and show that for any such space, a covariant Hermitian holomorphic line bundle is either positive, negative, or flat. We then build upon this result to show that we can distinguish between these three choices by examining the degree zero Dolbeault cohomology of the line bundle in question.

In §4 we present our motivating family of examples, the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$, their covariant line bundles $\mathcal{E}_l$, for $l \in \mathbb{Z}$, along with their Heckenberger–Kolb calculi. We recall the irreducible CQH-Kähler structure of each $\mathcal{O}_q(G/L_S)$, and the associated noncommutative generalisation of the Borel–Weil theorem. We then build upon the general theory presented in §3, and prove that, for each $k \in \mathbb{N}$, it holds that $\mathcal{E}_k > 0$, and $\mathcal{E}_{-k} < 0$. As a consequence, we observe that the Kähler structure of each $\mathcal{O}_q(G/L_S)$ is of Fano type. Finally, through an application of the results of [48], we extend the noncommutative Borel–Weil theorem to a noncommutative Bott–Borel–Weil theorem for positive line bundles.

To provide the reader with some more concrete insight, in §5 we carry out explicit curvature calculations for the positive line bundles of the quantum projective spaces, the simplest type of $A$-series irreducible quantum flag manifold. We show that the classical integer curvature $q$-deforms to quantum integer curvature.

For the reader's convenience, and to settle notation, we also include three short appendices. Appendix A Takeuchi's equivalence, the natural setting for discussing homogeneous vector bundles in the noncommutative setting. Appendix B includes some diagrammatic descriptions of the quantum flag manifolds and their canonical bundles. Finally, in Appendix C, we set notation for the relevant quantum integers used in the paper.

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## 2. Preliminaries

We recall the basic definitions and results for differential calculi, as well as complex, Hermitian, and Kähler structures and associated noncommutative vector bundles over these objects. For a more detailed introduction see [46], [47], and references therein. For an excellent presentation of classical complex and Kähler geometry see Huybrecht's book [30].

**Notation.** Throughout the paper $\mathbb{N}_0$ denotes the natural numbers, including zero, while $\mathbb{N}$ will denote the non-zero natural numbers. All algebras are assumed to be unital and defined over $\mathbb{C}$, and all unadorned tensor products are defined over $\mathbb{C}$.

2.1. **Differential Calculi.** A *differential calculus* $\big(\Omega^\bullet \simeq \bigoplus_{k\in\mathbb{N}_0}\Omega^k, \mathrm{d}\big)$ is a differential graded algebra (dg-algebra) which is generated in degree 0 as a dg-algebra, that is to say, it is generated as an algebra by the elements $a, \mathrm{d}b$, for $a, b \in \Omega^0$. We call an element $\omega \in \Omega^\bullet$ a *form*, and if $\omega \in \Omega^k$, for some $k \in \mathbb{N}$, then $\omega$ is said to be *homogeneous* of degree $|\omega| := k$. The product of two forms $\omega, \nu \in \Omega^\bullet$ is usually denoted by $\omega \wedge \nu$, unless one of the forms is of degree 0, whereupon the product is denoted by juxtaposition.

For a given algebra $B$, a *differential calculus over* $B$ is a differential calculus such that $\Omega^0 = B$. Note that for a differential calculus over $B$, each $\Omega^k$ is a $B$-bimodule. A differential calculus is said to have *total degree* $m \in \mathbb{N}_0$, if $\Omega^m \neq 0$, and $\Omega^k = 0$, for every $k > m$.

A *differential $*$-calculus* over a $*$-algebra $B$ is a differential calculus over $B$ such that the $*$-map of $B$ extends to a (necessarily unique) conjugate linear involutive map $* : \Omega^\bullet \to \Omega^\bullet$ satisfying $\mathrm{d}(\omega^*) = (\mathrm{d}\omega)^*$, and

$$\big(\omega \wedge \nu\big)^* = (-1)^{kl}\nu^* \wedge \omega^*, \qquad \text{for all } \omega \in \Omega^k,\ \nu \in \Omega^l.$$

We say that $\omega \in \Omega^\bullet$ is *closed* if $\mathrm{d}\omega = 0$, and *real* if it satisfies $\omega^* = \omega$.

2.2. **Complex Structures.** We now recall the definition of a complex structure as introduced in [35, 6]. This abstracts the properties of the de Rham complex of a classical complex manifold [30].

**Definition 2.1.** An *almost complex structure* $\Omega^{(\bullet,\bullet)}$ for a differential $*$-calculus $(\Omega^\bullet, \mathrm{d})$ is an $\mathbb{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbb{N}_0^2}\Omega^{(a,b)}$ for $\Omega^\bullet$ such that, for all $(a,b) \in \mathbb{N}_0^2$ the following hold:

(i) $\Omega^k = \bigoplus_{a+b=k}\Omega^{(a,b)}$,
(ii) $\big(\Omega^{(a,b)}\big)^* = \Omega^{(b,a)}$.

We call an element of $\Omega^{(a,b)}$ an $(a,b)$-form. For $\mathrm{proj}_{\Omega^{(a+1,b)}}$, and $\mathrm{proj}_{\Omega^{(a,b+1)}}$, the projections from $\Omega^{a+b+1}$ to $\Omega^{(a+1,b)}$, and $\Omega^{(a,b+1)}$ respectively, we denote

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}} \circ \mathrm{d}, \qquad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}} \circ \mathrm{d}.$$

A *complex structure* is an almost complex which satisfies

$$\mathrm{d}\Omega^{(a,b)} \subseteq \Omega^{(a+1,b)} \oplus \Omega^{(a,b+1)}, \qquad \text{for all } (a,b) \in \mathbb{N}_0^2. \tag{1}$$

For a complex structure, (1) implies the identities

$$\mathrm{d} = \partial + \overline{\partial}, \qquad \overline{\partial} \circ \partial = -\,\partial \circ \overline{\partial}, \qquad \partial^2 = \overline{\partial}^2 = 0.$$

In particular, $\big(\bigoplus_{(a,b)\in\mathbb{N}_0^2}\Omega^{(a,b)},\partial,\overline{\partial}\big)$ is a double complex, which we call the *Dolbeault double complex* of $\Omega^{(\bullet,\bullet)}$. Moreover, it is easily seen that both $\partial$ and $\overline{\partial}$ satisfy the graded Leibniz rule, and that

$$\partial(\omega^*) = \big(\overline{\partial}\omega\big)^*, \qquad \overline{\partial}(\omega^*) = \big(\partial\omega\big)^*, \qquad \text{for all } \omega \in \Omega^\bullet.$$

2.3. **Hermitian and Kähler Structures.** We now present the definition of an Hermitian structure, as introduced in [47, §4], as well as a Kähler structure, introduced in the same paper, [47, §7].

**Definition 2.2.** An *Hermitian structure* $(\Omega^{(\bullet,\bullet)},\sigma)$ for a differential $*$-calculus $\Omega^\bullet$ over $B$ of even total degree $2n$ is a pair consisting of a complex structure $\Omega^{(\bullet,\bullet)}$ and a central real $(1,1)$-form $\sigma$, called the *Hermitian form*, such that, with respect to the *Lefschetz operator*

$$L : \Omega^\bullet \to \Omega^\bullet, \qquad \omega \mapsto \sigma \wedge \omega,$$

isomorphisms are given by

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \qquad \text{for all } k = 0,\dots,n-1.$$

For $L$ the Lefschetz operator of an Hermitian structure, we denote

$$P^{(a,b)} := \begin{cases} \{\alpha \in \Omega^{(a,b)} \mid L^{n-a-b+1}(\alpha) = 0\}, & \text{if } a+b \leq n, \\ 0, & \text{if } a+b > n. \end{cases}$$

Moreover, we denote $P^k := \bigoplus_{a+b=k} P^{(a,b)}$, and $P^\bullet := \bigoplus_{k\in\mathbb{N}_0} P^k$. An element of $P^\bullet$ is called a *primitive form.*

An important consequence of the existence of the Lefschetz operator is the Lefschetz decomposition of the differential $*$-calculus, which we now recall below. For a proof see [47, Proposition 4.3].

**Proposition 2.3** (Lefschetz decomposition)**.** *For $L$ the Lefschetz operator of an Hermitian structure on a differential $*$-calculus $\Omega^\bullet$, a $B$-bimodule decomposition of $\Omega^k$, for all $k \in \mathbb{N}_0$, is given by*

$$\Omega^k \simeq \bigoplus_{j\geq 0} L^j\big(P^{k-2j}\big).$$

We call this the *Lefschetz decomposition* of $\Omega^\bullet$.

In classical Hermitian geometry, the Hodge map of an Hermitian metric is related to the associated Lefschetz decomposition through the well-known Weil formula (see [57, Théorème 1.2] or [30, Proposition 1.2.31]). In the noncommutative setting, we take the direct generalisation of the Weil formula for our definition of the Hodge map, and build upon this to define an Hermitian metric.

**Definition 2.4.** The *Hodge map* associated to an Hermitian structure $\big(\Omega^{(\bullet,\bullet)},\sigma\big)$ is the $B$-bimodule map $*_\sigma:\Omega^\bullet\to\Omega^\bullet$ satisfying, for any $j\in\mathbb{N}_0$,

$$*_\sigma\big(L^j(\omega)\big)=(-1)^{\frac{k(k+1)}{2}}\mathbf{i}^{a-b}\frac{j!}{(n-j-k)!}L^{n-j-k}(\omega),\qquad \omega\in P^{(a,b)}\subseteq P^{k=a+b},$$

where $\mathbf{i}=\sqrt{-1}$.

The Hodge map allows us to construct a sesquilinear map (since we will mainly deal with left modules, our convention is that such a map is conjugate-linear in the *second* variable) called the *Hermitian metric*,

$$g_\sigma:\Omega^\bullet\times\Omega^\bullet\to B,$$

which we define, for $\omega\in\Omega^k$ and $\nu\in\Omega^l$, by

$$g_\sigma(\omega,\nu):=\begin{cases} *_\sigma(\omega\wedge *_\sigma(\nu^*)), & \text{if } k=l,\\ 0, & \text{if } k\neq l.\end{cases}$$

By [17, Corollary 2.9], it holds that

$$g_\sigma(\omega,\nu)=g_\sigma(\nu,\omega)^*,\qquad \text{for all } \omega,\nu\in\Omega^\bullet.$$

Following C$^*$-algebra terminology, for a $*$-algebra $B$, the *cone of positive elements*, $B_{\geq 0}$, is defined by

$$B_{\geq 0}:=\left\{\sum_{i=1}^{l} b_i^*b_i \mid b_i\in B,\ l\in\mathbb{N}\right\}.$$

We denote the non-zero positive elements of $B$ by $B_{>0}:=B_{\geq 0}\setminus\{0\}$.

**Definition 2.5.** We say that an Hermitian structure $(\Omega^{(\bullet,\bullet)},\sigma)$ is *positive definite* if the associated metric $g_\sigma$ satisfies

$$g_\sigma(\omega,\omega)\in B_{>0},\qquad \text{for all non-zero } \omega\in\Omega^\bullet.$$

In this case we say that $\sigma$ is a *positive definite Hermitian form.*

With respect to the Hermitian metric, the Lefschetz map $L$ is adjointable, and we denote its adjoint by $L^\dagger=\Lambda$. The map $\Lambda$ can be explicitly presented as

$$\Lambda=*_\sigma^{-1}\circ L\circ *_\sigma.$$

We define the *counting operator* $H:\Omega^\bullet\to\Omega^\bullet$ by

$$H(\omega)=(k-n)\omega,\qquad \text{for } \omega\in\Omega^k.$$

Together the maps $L$, $\Lambda$, and $H$ give a representation of $\mathfrak{sl}_2$, as they satisfy the following commutation relations, see for example [17, Proposition 2.10]:

$$[H,L]=2H,\qquad [L,\Lambda]=H,\qquad [H,\Lambda]=-2\Lambda. \tag{2}$$

We finish this subsection with the definition of a Kähler structure, [47, Definition 7.1].. This is a simple strengthening of the requirements of an Hermitian structure, but as we will see below, one with profound consequences.

**Definition 2.6.** A *Kähler structure* for a differential $*$-calculus is an Hermitian structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ such that the Hermitian form $\kappa$ is closed, which is to say, $\mathrm{d}\kappa = 0$. We call such a form $\kappa$ a *Kähler form*.

2.4. **Holomorphic Vector Bundles.** Noncommutative holomorphic vector bundles have been considered in various places, for example [6], [50], and [35]. Motivated by the Serre–Swan theorem, finitely generated projective left modules are usually considered as noncommutative generalisations of vector bundles. As we now recall, one can build on this idea to define noncommutative holomorphic vector bundles via the classical Koszul–Malgrange characterisation of holomorphic bundles [38]. See [48] for a more detailed discussion.

For $\Omega^\bullet$ a differential calculus over an algebra $B$, and $\mathcal{F}$ a left $B$-module, a *connection* for $\mathcal{F}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ satisfying

$$\nabla(bf) = \mathrm{d}b \otimes f + b\nabla f, \qquad \text{for all } b \in B, f \in \mathcal{F}.$$

With respect to a choice $\Omega^{(\bullet,\bullet)}$ of complex structure on $\Omega^\bullet$, a $(0,1)$-*connection for* $\mathcal{F}$ is a connection with respect to the differential calculus $(\Omega^{(0,\bullet)}, \overline{\partial})$.

Any connection can be extended to a map $\nabla : \Omega^\bullet \otimes_B \mathcal{F} \to \Omega^\bullet \otimes_B \mathcal{F}$ uniquely defined by

$$\nabla(\omega \otimes f) = \mathrm{d}\omega \otimes f + (-1)^{|\omega|}\, \omega \wedge \nabla f, \qquad \text{for } f \in \mathcal{F},\ \omega \in \Omega^\bullet,$$

where $\omega$ is a homogeneous form, and $|\omega|$ denotes its degree.

The *curvature* of a connection is the left $B$-module map $\nabla^2 : \mathcal{F} \to \Omega^2 \otimes_B \mathcal{F}$. A connection is said to be *flat* if $\nabla^2 = 0$. Since $\nabla^2(\omega \otimes f) = \omega \wedge \nabla^2(f)$, a connection is flat if and only if the pair $(\Omega^\bullet \otimes_B \mathcal{F}, \nabla)$ is a complex.

**Definition 2.7.** For an algebra $B$, a *holomorphic vector bundle over* $B$ is a pair $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$, where $\mathcal{F}$ is a finitely generated projective left $B$-module, and the map $\overline{\partial}_{\mathcal{F}} : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F}$ is a flat $(0,1)$-connection, which we call the *holomorphic structure* for $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$.

Note that for any fixed $a \in \mathbb{N}_0$, a holomorphic vector bundle $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$ has a naturally associated complex

$$\overline{\partial}_{\mathcal{F}} : \Omega^{(a,\bullet)} \otimes_B \mathcal{F} \to \Omega^{(a,\bullet)} \otimes_B \mathcal{F}.$$

For any $b \in \mathbb{N}_0$, we denote by $H^{(a,b)}_{\overline{\partial}}(\mathcal{F})$ the $b^{\text{th}}$-cohomology group of this complex.

2.5. **Hermitian Vector Bundles.** When $B$ is a $*$-algebra, we can also generalise the classical notion of an Hermitian metric for a vector bundle, as we now recall.

**Definition 2.8.** An *Hermitian vector bundle* over a $*$-algebra $B$ is a pair $(\mathcal{F}, h_{\mathcal{F}})$, consisting of a finitely generated projective left $B$-module $\mathcal{F}$ and a sesequilinear map $h_F : \mathcal{F} \times \mathcal{F} \to B$ satisfying

(i) $h_{\mathcal{F}}(bf, g) = bh_{\mathcal{F}}(f, g)$, for all $f, g \in \mathcal{F}$ and $b \in B$,
(ii) $h_{\mathcal{F}}(f, g) = h_{\mathcal{F}}(g, f)^*$, for all $f, g \in \mathcal{F}$,
(iii) $h_{\mathcal{F}}(f, f) \in B_{>0}$, for all non-zero $f \in \mathcal{F}$.

Observe that if $(\Omega^{(\bullet,\bullet)}, \sigma)$ is an Hermitian structure for a $^*$-calculus $\Omega^\bullet$ over $B$, then $(\Omega^\bullet, g_\sigma)$ is an Hermitian vector bundle. Furthermore, if $(\mathcal{F}, h_{\mathcal{F}})$ is an Hermitian vector bundle over $B$, then a sesquilinear map

$$h_{\Omega^\bullet \otimes_B \mathcal{F}} : \Omega^\bullet \otimes_B \mathcal{F} \times \Omega^\bullet \otimes_B \mathcal{F} \to B$$

is defined by

$$h_{\Omega^\bullet \otimes_B \mathcal{F}}(\omega \otimes f, \nu \otimes g) := g_\sigma(\omega, \nu h_{\mathcal{F}}(f, g)), \qquad f, g \in \mathcal{F}, \omega, \nu \in \Omega^\bullet.$$

Since $g_\sigma(\omega b, \nu) = g_\sigma(\omega, b^*\nu)$ for all $\omega, \nu \in \Omega^\bullet$ and $b \in B$, it is straightforward to check that $(\Omega^\bullet \otimes_B \mathcal{F}, h_{\Omega^\bullet \otimes_B \mathcal{F}})$ is also an Hermitian vector bundle.

**Definition 2.9.** Let $(\mathcal{F}, h_{\mathcal{F}})$ be an Hermitian vector bundle, and consider the sesquilinear map

$$\mathfrak{h}_{\mathcal{F}} : \Omega^\bullet \otimes_B \mathcal{F} \times \Omega^\bullet \otimes_B \mathcal{F} \to \Omega^\bullet, \qquad (\omega \otimes f, \nu \otimes g) \mapsto \omega h_{\mathcal{F}}(\overline{g})(f) \wedge \nu^*.$$

A connection $\nabla \colon \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ is an *Hermitian connection* if

$$\mathrm{d}\mathfrak{h}_{\mathcal{F}}(f, g) = \mathfrak{h}_{\mathcal{F}}(\nabla(f), 1 \otimes g) + \mathfrak{h}_{\mathcal{F}}(1 \otimes f, \nabla(g)) \qquad \text{for all } f, g \in \mathcal{F}.$$

A *holomorphic Hermitian vector bundle* is a triple $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ such that $(\mathcal{F}, h_{\mathcal{F}})$ is an Hermitian vector bundle and $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$ is a holomorphic vector bundle. The following is shown in [5], see also [48].

**Lemma 2.10.** *For any Hermitian holomorphic vector bundle* $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$*, there exists a unique Hermitian connection* $\nabla : \mathcal{F} \to \Omega^1 \otimes_A \mathcal{F}$ *satisfying*

$$\overline{\partial}_{\mathcal{F}} = \big(\mathrm{proj}_{\Omega^{(0,1)}} \otimes_B \mathrm{id}\big) \circ \nabla.$$

We call $\nabla$ the *Chern connection* of $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$, and denote

$$\partial_{\mathcal{F}} := \big(\mathrm{proj}_{\Omega^{(1,0)}} \otimes_B \mathrm{id}\big) \circ \nabla.$$

We finish this subsection with the notion of positivity for a holomorphic Hermitian vector bundle. This directly generalises the classical notion of positivity, a property which is equivalent to ampleness [30, Proposition 5.3.1]. It was first introduced in [48, Definition 8.2] and requires a compatibility between Hermitian holomorphic vector bundles and Kähler structures.

**Definition 2.11.** Let $\Omega^\bullet$ be a differential calculus over a $*$-algebra $B$, and let $(\Omega^{(\bullet,\bullet)}, \kappa)$ be a Kähler structure for $\Omega^\bullet$. An Hermitian holomorphic vector bundle $(\mathcal{F}, h_\mathcal{F}, \overline{\partial}_\mathcal{F})$ is said to be *positive*, written $\mathcal{F} > 0$, if there exists $\theta \in \mathbb{R}_{>0}$, such that the Chern connection $\nabla$ of $\mathcal{F}$ satisfies

$$\nabla^2(f) = -\theta \mathbf{i}\kappa \otimes f, \qquad \text{for all } f \in \mathcal{F}.$$

Analogously, $(\mathcal{F}, h_\mathcal{F}, \overline{\partial}_\mathcal{F})$ is said to be *negative*, written $\mathcal{F} < 0$, if there exists $\theta \in \mathbb{R}_{>0}$, such that the Chern connection $\nabla$ of $\mathcal{F}$ satisfies

$$\nabla^2(f) = \theta \mathbf{i}\kappa \otimes f, \qquad \text{for all } f \in \mathcal{F}.$$

2.6. **Noncommutative Fano Structures.** In order to produce a holomorphic vector bundle from a complex structure, we recall from [46, §6.3] a refinement of the definition of a complex structure called *factorisability.* The Dolbeault double complex of every complex manifold is automatically factorisable [30, §1.2], as are the Heckenberger–Kolb calculi for all the irreducible quantum flag manifolds (see §4.3).

**Definition 2.12.** Let $\Omega^\bullet$ be a differential $*$-calculus over a $*$-algebra $B$. A complex, or almost complex, structure for $\Omega^\bullet$ is called *factorisable* if, for all $(a,b) \in \mathbb{N}_0^2$, we have $B$-bimodule isomorphisms

(i) $\wedge : \Omega^{(a,0)} \otimes_B \Omega^{(0,b)} \to \Omega^{(a,b)}, \qquad \sum_i \omega_i \otimes \nu_i \mapsto \sum_i \omega_i \wedge \nu_i,$
(ii) $\wedge : \Omega^{(0,b)} \otimes_B \Omega^{(a,0)} \to \Omega^{(a,b)}, \qquad \sum_i \omega_i \otimes \nu_i \mapsto \sum_i \omega_i \wedge \nu_i.$

An important point to note is that for any factorisable complex structure $\Omega^{(\bullet,\bullet)}$ of total degree $2n$, the pair $\big(\Omega^{(n,0)}, \overline{\partial}\big)$ is a holomorphic vector bundle. Moreover, for a *factorisable Hermitian structure*, or *factorisable Kähler structure*, which is to say an Hermitian, or Kähler, structure whose constituent complex structure is factorisable, the triple $\big(\Omega^{(n,0)}, g_\sigma, \wedge^{-1} \circ \overline{\partial}\big)$ is an Hermitian holomorphic vector bundle.

**Definition 2.13.** A *Fano structure* for a differential $*$-calculus $\Omega^\bullet$, of total degree $2n$, is a Kähler structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ such that

(i) $\Omega^{(\bullet,\bullet)}$ is a factorisable complex structure,
(ii) $\big(\Omega^{(n,0)}, g_\kappa, \overline{\partial}\big)$ is a negative holomorphic Hermitian vector bundle.

2.7. **Covariant Hermitian Structures.** Suppose that $A$ is a Hopf algebra and $B$ is a left $A$-comodule algebra. A differential calculus $\Omega^\bullet$ over $B$ is said to be *covariant* if the coaction $\Delta_L : B \to A \otimes B$ extends to a map $\Delta_L : \Omega^\bullet \to A \otimes \Omega^\bullet$ giving $\Omega^\bullet$ the structure of an $A$-comodule algebra, and such that d is a left $A$-comodule map.

A complex, or almost complex, structure for $\Omega^\bullet$ is said to be *covariant* if the $\mathbb{N}_0^2$-decomposition is a decomposition in the category of left $A$-comodules ${}^A\mathrm{Mod}$. In this case $\Omega^{(a,b)}$ is a left $A$-sub-comodule of $\Omega^\bullet$, for each $(a,b) \in \mathbb{N}_0^2$. This implies that $\partial$ and $\overline{\partial}$ are left $A$-comodule maps.

If $(\Omega^{(\bullet,\bullet)}, \sigma)$ is an Hermitian structure such that $\Omega^{(\bullet,\bullet)}$ is a covariant complex structure and $\sigma$ is left $A$-covariant, that is, $\Delta_L(\sigma) = 1 \otimes \sigma$, then we say that $(\Omega^{(\bullet,\bullet)}, \sigma)$ is a *covariant Hermitian structure*. In this case, $L$, $*_\sigma$ and $\Lambda$ are also left $A$-comodule maps. A *covariant Kähler structure* is a covariant Hermitian structure which is also a Kähler structure.

We also require the notion of covariance for vector bundles. As above, let $A$ be a Hopf algebra and $B$ an $A$-comodule algebra. Then a holomorphic vector bundle $(\mathcal{F}, \overline{\partial}_\mathcal{F})$ is *covariant* if $\mathcal{F}$ is an object ${}^A_B\mathrm{mod}_0$ (see Appendix A.1) and $\overline{\partial}_\mathcal{F} : \mathcal{F} \to \Omega^{(0,1)} \otimes_B \mathcal{F}$ is a left $A$-comodule map.

For a left $B$-module $\mathcal{F}$, denote by $\overline{\mathcal{F}}$ the *conjugate right $B$-module* of $\mathcal{F}$, as defined by the action

$$\overline{\mathcal{F}} \otimes B \to \overline{\mathcal{F}}, \qquad\qquad \overline{f} \otimes b \mapsto \overline{b^* f}.$$

Denote by ${}^\vee\mathcal{F}$ the dual module ${}^\vee\mathcal{F} := \mathrm{Hom}(\mathcal{F}, B)$, which is a right $B$-module with respect to pointwise multiplication

$$\phi\, b(f) := \phi(f)b, \qquad\qquad \phi \in {}^\vee\mathcal{F}, b \in B \text{ and } f \in \mathcal{F}.$$

A *covariant Hermitian vector bundle* is an Hermitian vector bundle $(\mathcal{F}, h_\mathcal{F})$ such that $\mathcal{F}$ is an object in ${}^A_B\mathrm{mod}_0$ and the right $B$-module map

$$\overline{\mathcal{F}} \to^\vee \mathcal{F}, \qquad f \mapsto h_\mathcal{F}(\cdot, f)$$

is a left $A$-comodule map. If $(\mathcal{F}, h_\mathcal{F}, \partial_F)$ is a covariant Hermitian holomorphic vector bundle, then the Chern connection is always a left $A$-comodule map, see [48, §7.1]. Finally, an *Hermitian*, or *holomorphic, covariant line bundle*, is a covariant Hermitian, or holomorphic, vector bundle $\mathcal{E}$ such that $\dim(\Phi(\mathcal{E})) = 1$, where $\Phi$ is the functor as given in Takeuchi's equivalence, Appendix A.1.

2.8. **CQH-Hermitian and CQH-Kähler Spaces.** Let $A$ be a Hopf algebra. For a left $A$-comodule $V$ with coaction $\Delta_L : V \to A \otimes V$, its space of matrix elements is the subcoalgebra

$$\mathcal{C}(V) := \mathrm{span}_\mathbb{C}\{(\mathrm{id} \otimes f)\Delta_L(v) \mid f \in V^*, v \in V\},$$

where $V^*$ denotes the $\mathbb{C}$-linear dual of $V$.

Recall that a Hopf algebra $A$ is *cosemisimple* if it satisfies the following three equivalent conditions:

(i) $A \cong \bigoplus_{V \in \hat{A}} \mathcal{C}(V)$, where summation is over $\hat{A}$, the set of all equivalence classes of irreducible left $A$-comodules,
(ii) the abelian category ${}^A\mathrm{Mod}$ of left $A$-comodules is semisimple,
(iii) there exists a unique linear map $\mathbf{h} : A \to \mathbb{C}$, which we call the *Haar functional*, satisfying $\mathbf{h}(1) = 1$, and

$$(\mathrm{id} \otimes \mathbf{h}) \circ \Delta(a) = \mathbf{h}(a)1, \qquad (\mathbf{h} \otimes \mathrm{id}) \circ \Delta(a) = \mathbf{h}(a)1.$$

We now give the definition of a compact quantum group algebra, the algebraic counterpart of Woronowicz's $C^*$-algebraic notion of a compact quantum group [58]. This strengthening of cosemisimplicity to the setting of Hopf $*$-algebras was introduced by Koornwinder and Dijkhuizen [20].

**Definition 2.14.** A *compact quantum group algebra*, or a *CQGA*, is a cosemisimple Hopf $*$-algebra $A$ such that $\mathbf{h}(a^*a) > 0$, for all non-zero $a \in A$.

**Definition 2.15.** Let $\pi : A \to H$ be a surjective Hopf algebra map between Hopf algebras $A$ and $H$. Then a *homogeneous right $H$-coaction* is given by the map

$$\Delta_R := (\mathrm{id} \otimes \pi) \circ \Delta : A \to A \otimes H.$$

The associated *quantum homogeneous space* is defined to be the space of coinvariant elements, $A^{\mathrm{co}(H)}$, that is,

$$A^{\mathrm{co}(H)} := \{a \in A \mid \Delta_R(a) = a \otimes 1\}.$$

A *CQGA-homogeneous space* is a quantum homogeneous space such that $A$ and $H$ are both CQGAs and $\pi : A \to H$ is a $*$-algebra map.

We now present closed integrals for Hermitian structures, abstracting the situation for a classical manifold without boundary. Note that this is a special case of an orientable differential calculus with closed integral [47, §3.2], where the Hodge map is taken as the orientation. The assumption of a closed integral is essential for establishing the codifferenial formulae presented in (4) below, as well as the noncommutative Hodge decomposition as presented in Theorem 2.18.

**Definition 2.16.** Let $(\Omega^{(\bullet,\bullet)}, \sigma)$ be an Hermitian structure of total degree $2n$, for some $n \in \mathbb{N}$. The *integral* is the linear map

$$\int := \mathbf{h} \circ *_\sigma : \Omega^{2n} \to \mathbb{C}.$$

If $\int \mathrm{d}\omega = 0$, for all $\omega \in \Omega^{2n-1}$, then the integral is said to be *closed*, and $(\Omega^{(\bullet,\bullet)}, \sigma)$ is said to be $\int$-closed.

With these definitions introduced, we are now ready to consider the definition of a CQH-Hermitian space, the main theoretic structure of this paper.

**Definition 2.17.** A *compact quantum homogeneous Hermitian space*, or simply a *CQH-Hermitian space*, is a quadruple $\mathbf{H} := \big(B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma\big)$ where

(i) $B = A^{\mathrm{co}(H)}$ is a CQGA-homogeneous space,
(ii) $\Omega^\bullet$ is a left $A$-covariant differential $*$-calculus over $B$, and an object in ${}^A_B\mathrm{mod}_0$,
(iii) $\big(\Omega^{(\bullet,\bullet)}, \sigma\big)$ is a covariant positive definite Hermitian structure for $\Omega^\bullet$ which is $\int$-closed.

We denote by $\dim(\mathbf{H})$ the total degree of the constituent differential calculus $\Omega^\bullet$, and call it the *dimension* of $\mathbf{H}$.

Over any CQH-Hermitian space $\mathbf{H} := \big(B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma\big)$, every $\mathcal{F} \in {}^A_B\mathrm{mod}_0$ admits an Hermitian structure, for more details see [17, Remark 6.4]. For a given Hermitian vector bundle $(\mathcal{F}, h_\mathcal{F}, \overline{\partial}_\mathcal{F})$ over $B$, an inner product is given by

$$\langle \cdot, \cdot \rangle_\mathcal{F} : \mathcal{F} \times \mathcal{F} \to \mathbb{C}, \qquad (f, g) \mapsto \mathbf{h}(h_\mathcal{F}(f, g)).$$

In particular, for the Hermitian vector bundle $\Omega^\bullet \otimes_B \mathcal{F}$, we have the inner product

$$\langle \cdot, \cdot \rangle_{\sigma,\mathcal{F}} : \Omega^\bullet \otimes_B \mathcal{F} \times \Omega^\bullet \otimes_B \mathcal{F} \to \mathbb{C}, \qquad (\omega \otimes f, \nu \otimes g) \mapsto \mathbf{h} \circ h_{\Omega^\bullet \otimes_B \mathcal{F}}(\omega \otimes f, \nu \otimes g). \tag{3}$$

We denote by $\partial^\dagger_\mathcal{F}$, and $\overline{\partial}^\dagger_\mathcal{F}$ the adjoint operators of $\partial_\mathcal{F}$ and $\overline{\partial}_\mathcal{F}$, respectively. That $\partial$ and $\overline{\partial}$ are adjointable with respect to this inner product is a consequence of the fact that they are covariant [48, Proposition 5.15]. We refer to any such operator as a *codifferential*. Just as in the classical case [30, §4.1], each noncommutative codifferential admits a description in terms of the Hodge map [48, Proposition 5.15]. Since such formulae will not be needed in what follows, we recall only the case where $\mathcal{F} = B$, as originally established in [47, §5.3.3]:

$$\partial^\dagger = - *_\sigma \circ \overline{\partial} \circ *_\sigma, \qquad \overline{\partial}^\dagger = - *_\sigma \circ \partial \circ *_\sigma. \tag{4}$$

The holomorphic, and anti-holomorphic, *Laplace* operators of $(\mathcal{F}, \overline{\partial}_\mathcal{F})$ are defined respectively by

$$\Delta_{\overline{\partial}_\mathcal{F}} := \overline{\partial}^\dagger_\mathcal{F} \overline{\partial}_\mathcal{F} + \overline{\partial}_\mathcal{F} \overline{\partial}^\dagger_\mathcal{F}, \qquad \Delta_{\partial_\mathcal{F}} := \partial^\dagger_\mathcal{F} \partial_\mathcal{F} + \partial_\mathcal{F} \partial^\dagger_\mathcal{F}.$$

We denote the space of *harmonic* elements by $\mathcal{H}^\bullet_{\overline{\partial}}(\mathcal{F}) := \ker(\Delta_{\overline{\partial}_\mathcal{F}})$. The following generalisation of classical Hodge decomposition was established in [48, Theorem 6.4].

**Theorem 2.18** (Twisted Hodge Decomposition)**.** *Let $(\mathcal{F}, h, \overline{\partial}_\mathcal{F})$ be an Hermitian holomorphic vector bundle over a CQH-Hermitian space $(B, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma)$. Then an orthogonal decomposition of $A$-comodules with respect to the Hermitian metric is given by*

$$\Omega^{(0,\bullet)} \otimes_B \mathcal{F} = \mathcal{H}^{(0,\bullet)}_{\overline{\partial}}(\mathcal{F}) \oplus \overline{\partial}_\mathcal{F}(\Omega^{(0,\bullet)} \otimes_B \mathcal{F}) \oplus \overline{\partial}^\dagger_\mathcal{F}(\Omega^{(0,\bullet)} \otimes_B \mathcal{F}).$$

*An isomorphism is given by the projection*

$$\mathcal{H}^{(0,\bullet)}_{\overline{\partial}}(\mathcal{F}) \to H^{(0,\bullet)}_{\overline{\partial}}(\mathcal{F}), \qquad \alpha \mapsto [\alpha].$$

**Definition 2.19.** A CQH-Kähler space $\mathbf{K} := \big(B, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa\big)$ is a CQH-Hermitian space such that $(\Omega^{(\bullet,\bullet)}, \kappa)$ is a Kähler structure.

2.9. **The Akizuki–Nakano Identity and the Kodaira Vanishing Theorem.** Suppose that $(\Omega^{(\bullet,\bullet)}, \sigma)$ is an Hermitian structure for a differential $*$-calculus over a $*$-algebra $B$, and that $\mathcal{F}$ is a left $B$-module. Define a triple of operators, acting on $\Omega^\bullet \otimes_B \mathcal{F}$, by setting

$$L_{\mathcal{F}} := L \otimes \mathrm{id}_{\mathcal{F}}, \qquad H_F := H \otimes \mathrm{id}_{\mathcal{F}}, \qquad \Lambda_{\mathcal{F}} := \Lambda \otimes \mathrm{id}_{\mathcal{F}},$$

where $L, H, \Lambda$ are respectively the Lefschetz, counting, and adjoint Lefschetz operators on $\Omega^\bullet$. It follows directly from the Lefschetz identities (2) that

$$[H_{\mathcal{F}}, L_{\mathcal{F}}] = 2L_{\mathcal{F}}, \qquad [L_{\mathcal{F}}, \Lambda_{\mathcal{F}}] = H_{\mathcal{F}}, \qquad [H_{\mathcal{F}}, \Lambda_{\mathcal{F}}] = -2\Lambda_{\mathcal{F}}.$$

For the twisted Dolbeault complex of a CQH-Kähler space, the following direct generalisation of the Kähler identities was established in [48, Theorem 7.6]. For a discussion of the classical situation, see [30, §5.3] or [18, §VII.1].

**Theorem 2.20** (Nakano identities)**.** *Let* $\mathbf{K} = (B, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa)$ *be a CQH-Kähler space, and* $(\mathcal{F}, h, \overline{\partial}_{\mathcal{F}})$ *an Hermitian holomorphic vector bundle. Denoting the Chern connection of* $\mathcal{F}$ *by* $\nabla_{\mathcal{F}} = \overline{\partial}_{\mathcal{F}} + \partial_{\mathcal{F}}$*, it holds that*

$$\begin{array}{llll} [L_{\mathcal{F}}, \partial_{\mathcal{F}}] = 0, & [L_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}}] = 0, & [\Lambda_{\mathcal{F}}, \partial^\dagger_{\mathcal{F}}] = 0, & [\Lambda_{\mathcal{F}}, \overline{\partial}^\dagger_{\mathcal{F}}] = 0, \\ [L_{\mathcal{F}}, \partial^\dagger_{\mathcal{F}}] = \mathbf{i}\overline{\partial}_{\mathcal{F}}, & [L_{\mathcal{F}}, \overline{\partial}^\dagger_{\mathcal{F}}] = -\mathbf{i}\partial_{\mathcal{F}}, & [\Lambda_{\mathcal{F}}, \partial_{\mathcal{F}}] = \mathbf{i}\overline{\partial}^\dagger_{\mathcal{F}}, & [\Lambda_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}}] = -\mathbf{i}\partial^\dagger_{\mathcal{F}}. \end{array}$$

As observed in [48, Corollary 7.8], these identities imply that the classical relationship between the Laplacians $\Delta_{\partial_{\mathcal{F}}}$ and $\Delta_{\overline{\partial}_{\mathcal{F}}}$, namely the Akizuki–Nakano identities, carries over to the noncommutative setting.

**Theorem 2.21** (Akizuki–Nakano Identity)**.** *It holds that*

$$\Delta_{\overline{\partial}_{\mathcal{F}}} = \Delta_{\partial_{\mathcal{F}}} + [\mathbf{i}\nabla^2, \Lambda_{\mathcal{F}}].$$

Note that, in the untwisted case (which is to say, the case where we do not tensor $\Omega^\bullet$ with an Hermitian vector bundle $\mathcal{F}$) the Laplacian operators coincide, and hence by the Hodge identification of harmonic forms and cohomology classes, the holomorphic and anti-holomorphic groups coincide [47, Corollary 7.7].

We finish with the noncommutative generalisation of the Kodaira vanishing theorem, originally established in [48, Theorem 8.3]. For an alternative proof, using the Akizuki–Nakano identities, see [17, Theorem 9.17].

**Theorem 2.22** (Kodaira Vanishing)**.** *Let* $\mathcal{E}_+$*, and* $\mathcal{E}_-$*, be positive, and respectively negative, line bundles over a CQH-Kähler space* $\mathbf{K} = (B, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa)$ *such that* $\dim(\mathbf{K}) = 2n$*. It holds that*

(i) $H^{(a,b)}_{\overline{\partial}}(\mathcal{E}_+) = 0,$ *for all* $a + b > n$*,*
(ii) $H^{(a,b)}_{\overline{\partial}}(\mathcal{E}_-) = 0,$ *for all* $a + b < n$*.*

## 3. Irreducible CQH-Hermitian Spaces and Positive Line Bundles

Determining positivity, or negativity, of an Hermitian holomorphic line bundle ostensibly requires one to calculate the Chern curvature explicitly. In practice, this can prove to be a very challenging technical task. This is true in the classical setting, and even more so in the noncommutative world, as can be seen in the calculations presented in §5. As we demonstrate in this section, however, self-adjointness of the Laplacian associated to a CQH-Kähler space allows us to avoid these difficulties, and to conclude positivity from the vanishing, and non-vanishing, of zeroth cohomology groups. This is most easily done if we assume irreducibility of the holomorphic, or equivalently anti-holomorphic, space of 1-forms.

3.1. **Irreducible CQH-Hermitian Spaces.** Let $\mathbf{H} = \big(B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma\big)$ be a CQH-Hermitian space. Since $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are objects in ${}^A_B\mathrm{mod}_0$, we can consider their irreducibility as objects in that category. We claim that $\Omega^{(1,0)}$ is irreducible if and only if $\Omega^{(0,1)}$ is irreducible. Indeed, for any proper non-trivial sub-object $N \subset \Omega^{(1,0)}$, let $N^*$ denote its image under the $*$-map, that is,

$$N^* := \{\omega^* \mid \omega \in N\}.$$

Then $N^* \subset \Omega^{(0,1)}$ is necessarily a proper non-trivial sub-object of $\Omega^{(0,1)}$. Clearly, the analogous argument works in the opposite direction, which proves the claim. This leads us to the next definition.

**Definition 3.1.** A CQH-Hermitian space $\mathbf{H} = \big(B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma\big)$ is said to be *irreducible* if $\Omega^{(1,0)}$, or equivalently $\Omega^{(0,1)}$, is irreducible as an object in ${}^A_B\mathrm{mod}_0$.

Irreducible Hermitian structures generalise our motivating family of examples, the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$, as presented in §4. Many of the properties of the irreducible quantum flag manifolds extend to this more general setting. In the following theorem, we present those relevant to the sequel.

**Lemma 3.2.** *Let* $\mathbf{H} = \big(B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma\big)$ *be a factorisable irreducible CQH-Hermitian space, and let* $(\mathcal{E}, h_{\mathcal{E}}, \overline{\partial}_{\mathcal{E}})$ *be a covariant Hermitian holomorphic line bundle over* $\mathbf{H}$.

(i) *The space of coinvariant* $(1,1)$*-forms is a one-dimensional space spanned by* $\sigma$*, that is,* ${}^{\mathrm{co}(A)}\Omega^{(1,1)} = \mathbb{C}\sigma$.
(ii) *The holomorphic structure* $\overline{\partial}_{\mathcal{E}}$ *of the line bundle* $\mathcal{E}$ *is unique.*
(iii) *Denoting* $2n := \dim(\mathbf{H})$*, it holds that*

$$\Lambda_{\mathcal{E}} \circ \nabla^2(e) = \lambda n \mathbf{i} e, \qquad \textit{for all } e \in \mathcal{E}.$$

(iv) *There exists a scalar* $\lambda \in \mathbb{R}$ *such that*

$$\nabla^2(e) = \lambda \mathbf{i} \sigma \otimes e, \qquad \textit{for all } e \in \mathcal{E}.$$

*Proof.* Since ${}^{H}\mathrm{mod}$ is a rigid monoidal category, $V := \Phi(\mathcal{E})$ is invertible. In particular $V \otimes V^* \simeq \mathbb{C}$, where $V^*$ denotes the dual left $H$-comodule of $V$.

By assumption $\Omega^{(\bullet,\bullet)}$ is factorisable, so we have $\Phi(\Omega^{(1,1)}) \simeq \Phi(\Omega^{(1,0)}) \otimes \Phi(\Omega^{(0,1)})$, where $\Phi$ is the functor defined in (18). Denote the decomposition of $\Phi(\Omega^{(1,1)})$ into irreducible comodules by

$$\Phi(\Omega^{(1,0)}) \otimes \Phi(\Omega^{(0,1)}) \simeq: \bigoplus_i K_i.$$

Since $\sigma$ is a left $A$-coinvariant Hermitian form, we have $[\sigma] \in {}^{\mathrm{co}(A)}\Phi(\Omega^{(1,1)})$, implying that one of the summands $K_i$ must be isomorphic to the trivial comodule. Thus $\Phi(\Omega^{(1,0)})$ and $\Phi(\Omega^{(0,1)})$ are dual. Moreover, since both $\Phi(\Omega^{(1,0)})$ and $\Phi(\Omega^{(0,1)})$ are by assumption irreducible, precisely one of the summands will be trivial. With $\mathrm{U} : \mathcal{F} \mapsto \Psi \circ \Phi(\mathcal{F})$ the unit of Takeuchi's equivalence, it is easily seen (see §10.8 [17] for example) that

$$\mathrm{U}\big({}^{\mathrm{co}(A)}\Omega^{(1,1)}\big) = 1 \otimes \big({}^{\mathrm{co}(H)}\Phi(\Omega^{(1,1)})\big) \simeq 1 \otimes \mathbb{C},$$

giving the claimed equality in (i) above.

Now we prove (ii). For a non-trivial summand $K_i$, the tensor product $K_i \otimes V$ will again be irreducible. Indeed, assume that $K_i \otimes V$ has a decomposition into a direct sum of two comodules $U_a$ and $U_b$. Then it would hold that

$$K_i \simeq K_i \otimes V \otimes V^* \simeq \big(U_a \otimes V^*\big) \oplus \big(U_b \otimes V^*\big).$$

This would contradict the irreducibility of $K_i$, and so we are forced to conclude that $K_i \otimes V$ is again irreducible. Now let us assume that $K_i \otimes V \simeq V$. Drawing again on invertibility of $V$, we see that

$$K_i \simeq K_i \otimes V \otimes V^* \simeq V \otimes V^* \simeq \mathbb{C}.$$

Since this contradicts our assumption of non-triviality of $K_i$, we are forced to conclude that $K_i \otimes V$ can never be isomorphic to $V$.

By Schur's lemma, we now have a one-dimensional space of comodule maps from $V$ to $\Phi(\Omega^{(1,1)}) \otimes V$. Explicitly, this means that all comodule maps are of the form

$$V \to \Phi(\Omega^{(1,1)}) \otimes V, \qquad\qquad v \mapsto \theta\,[\sigma] \otimes v,$$

for some $\theta \in \mathbb{C}$. Since the curvature operator is a morphism in ${}^{A}_{B}\mathrm{mod}_0$, Takeuchi's equivalence gives us the following commutative diagram:

$$\begin{array}{ccc}
\Omega^{(1,1)} \otimes_B \mathcal{E} & \xleftarrow{\quad \mathrm{U}^{-1} \quad} & A \square_H \Phi\big(\Omega^{(1,1)} \otimes_B \mathcal{E}\big) \\
\uparrow{\scriptstyle \nabla^2} & & \uparrow{\scriptstyle \Phi(\nabla^2)} \\
\mathcal{E} & \xrightarrow[\quad \mathrm{U} \quad]{} & A \square_H \Phi(\mathcal{E}),
\end{array}$$

where we have used the formula for the inverse of Takeuchi's unit, as presented in (19). Thus, for any particular element $e \in \mathcal{E}$, we see that

$$\begin{aligned}\nabla^2(e) = \mathrm{U}^{-1} \circ \Phi(\nabla^2) \circ \mathrm{U}(e) &= \mathrm{U}^{-1} \circ \Phi(\nabla^2)(e_{(-1)} \otimes [e_{(0)}]) \\ &= \theta\, \mathrm{U}^{-1}(e_{(-1)} \otimes [\sigma \wedge e_{(0)}]) \\ &= \theta e_{(-2)} S(e_{(-1)})\sigma \otimes e_{(0)} \\ &= \theta \varepsilon(e_{(-1)})\sigma \otimes e_{(0)} \\ &= \theta\sigma \otimes e.\end{aligned}$$

The operators $\Delta_{\partial_{\mathcal{E}}}$ and $\Delta_{\overline{\partial}_{\mathcal{E}}}$ are, by construction, self-adjoint operators on $\Omega^\bullet \otimes_B \mathcal{E}$. Thus any eigenvalue of $\Delta_{\partial_{\mathcal{E}}} - \Delta_{\overline{\partial}_{\mathcal{E}}}$ must be a real scalar. It now follows from the Akizuki–Nakano identity that

$$(\Delta_{\partial_{\mathcal{E}}} - \Delta_{\overline{\partial}_{\mathcal{E}}})(e) = [\mathbf{i}\nabla^2, \Lambda_{\mathcal{E}}](e) = -\mathbf{i}\Lambda_{\mathcal{E}} \circ \nabla^2(e) = -\theta\mathbf{i}\Lambda_{\mathcal{E}}(\sigma \otimes e) = -\theta\mathbf{i}\, \Lambda_{\mathcal{E}} \circ L_{\mathcal{E}}(e).$$

Recalling now the twisted Lefschetz identities, and denoting $2n := \dim(\mathbf{H})$, the above expression can be reduced to

$$-\theta\mathbf{i}\Lambda_{\mathcal{E}} \circ L_{\mathcal{E}}(e) = \theta\mathbf{i}\,[L_{\mathcal{E}}, \Lambda_{\mathcal{E}}](e) = \theta\mathbf{i}H_{\mathcal{E}}(e) = -\mathbf{i}n\theta e.$$

Thus $\theta\mathbf{i} \in \mathbb{R}$ and setting $\lambda := -\theta\mathbf{i}$ gives the equation in (iii) as claimed, establishing (ii) in the process. □

For the special case of a CQH-Kähler space, the identity in Lemma 3.2 (ii) immediately implies the following result. This serves as the principal theoretical result of the paper.

**Theorem 3.3.** *For any covariant line bundle $\mathcal{E}$ over an irreducible CQH-Kähler space, precisely one of the following three possibilities holds:*

(i) $\mathcal{E} > 0$,
(ii) $\mathcal{E}$ *is flat,*
(iii) $\mathcal{E} < 0$.

Let $\Omega^{(\bullet,\bullet)}$ be an almost complex structure. Its *opposite almost complex structure*, as considered in [44, §2.2.3], is the almost complex structure $\overline{\Omega}^{(\bullet,\bullet)}$ uniquely defined by $\overline{\Omega}^{(a,b)} := \Omega^{(b,a)}$, for all $(a,b) \in \mathbb{N}_0^2$. By [17, Lemma 8.4], if $\mathbf{H} = (B, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma)$ is a CQH-Hermitian space, then

$$\overline{\mathbf{H}} := \big(B, \Omega^\bullet, \overline{\Omega}^{(\bullet,\bullet)}, -\sigma\big)$$

is also a CQH-Hermitian space, which we call the *opposite CQH-Hermitian space.* Clearly, $\mathbf{H}$ is a CQH-Kähler space if and only if $\overline{\mathbf{H}}$ is a CQH-Kähler space, hence we also have the notion of the *opposite CQH-Kähler space.*

The general cohomological consequences of positivity presented in the Kodaira vanishing theorem now allow us to produce sufficient cohomological conditions for positivity, flatness, or negativity, of a line bundle over an irreducible CQH-Kähler space.

**Corollary 3.4.** *Let $\mathcal{E}$ be a covariant line bundle over an irreducible CQH-Kähler space* $\mathbf{K}$.

(i) *If $H^0_{\overline{\partial}}(\mathcal{E}) \neq 0$ and $H^0_{\partial}(\mathcal{E}) = 0$, then $\mathcal{E}$ is positive.*
(ii) *If $H^0_{\partial}(\mathcal{E}) = H^0_{\overline{\partial}}(\mathcal{E}) \neq 0$, then $\mathcal{E}$ is flat.*
(iii) *If $H^0_{\overline{\partial}}(\mathcal{E}) = 0$ and $H^0_{\partial}(\mathcal{E}) \neq 0$, then $\mathcal{E}$ is negative.*

*Proof.* First we show (i). By Theorem 3.3, the line bundle $\mathcal{E}$ must be positive, negative, or flat. If it were negative, then Theorem 2.22 would imply that $H^0_{\overline{\partial}}(\mathcal{E})$ is trivial. Since we are assuming that this is not the case, $\mathcal{E}$ must be flat or positive. If it were flat, then the Akizuki–Nakano identity would reduce to the equality

$$\Delta_{\overline{\partial}_{\mathcal{E}}} = \Delta_{\partial_{\mathcal{E}}}.$$

This would imply equality of harmonic forms, and hence equality of cohomologies. However, since we are assuming that $H^0_{\partial}(\mathcal{E}) = 0$, this cannot be the case. Thus we are forced to conclude that $\mathcal{E}$ is positive.

Moving now to (ii), we note that if the holomorphic and anti-holomorphic cohomologies coincide and are non-trivial, then there must exist a non-trivial

$$e \in \ker(\Delta_{\partial}) \cap \ker(\Delta_{\overline{\partial}}).$$

By the Akizuki–Nakano identity, and Lemma 3.2 (iii), this means that

$$0 = \Delta_{\overline{\partial}}(e) - \Delta_{\partial}(e) = [\mathbf{i}\nabla^2, \Lambda_{\mathcal{E}}](e) = -\Lambda_{\mathcal{E}} \circ \nabla^2(e) = -\lambda n \mathbf{i} e,$$

where $2n := \dim(\mathbf{K})$, and $\lambda$ is the unique scalar such that $\nabla^2(e) = \lambda \mathbf{i} e$. Thus we see that $\lambda$ must be zero, which is to say, $\mathcal{E}$ must be flat.

For (iii), the proof is completely analogous to that given for (i). However, in this case the argument is given in terms of the opposite CQH-Kähler space. In particular, using the assumptions that $H^0_{\overline{\partial}}(\mathcal{E}) = 0$, and $H^0_{\partial}(\mathcal{E}) \neq 0$ one eliminates the possibility that $\mathcal{E}$ is either negative or flat, and hence one can conclude that $\mathcal{E}$ is positive with respect to the opposite Kähler structure. Recalling Definition 2.11, it is easy to see that an Hermitian holomorphic vector bundle is positive if and only if it is negative with respect to the opposite CQH-Kähler space. Thus we can conclude that $\mathcal{E} < 0$. □

**Example 3.5.** We now consider an interesting family of examples, namely those irreducible CQH-Kähler spaces $\mathbf{K} = (B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa)$ of dimension 4. In particular, we bear in mind the quantum projective plane $\mathcal{O}_q(\mathbb{CP}^2)$, a member of the general family of examples discussed in §4 and §5. Note that in dimension four the Hodge map satisfies $*_\kappa^2 = \mathrm{id}$ on 2-forms. Hence $*_\kappa$ has eigenvalues 1 and $-1$. Moreover, it follows from the definition of $*_\kappa$ that $*_\kappa(\kappa) = \kappa$. Following the classical definition we define the *first Chern class* $c_1(\mathcal{E})$, of a covariant Hermitian holomorphic line bundle $\mathcal{E}$, to be $c_1(\mathcal{E}) := \mathrm{tr}\, \nabla^2$, where tr is the trace of $\nabla^2$ defined in the obvious way. By Lemma 3.2 the first Chern class $c_1(\mathcal{E})$ is proportional to $\kappa$, and so, satisfies

$$*_\kappa(c_1(\mathcal{E})) = c_1(\mathcal{E}).$$

In the classical setting such connections are called *self-dual connections*. They are of interest because they satisfy the Yang–Mills equations [3]. For a more detailed discussion for the special case of $\mathcal{O}_q(\mathbb{CP}^2)$, see the pair of papers [13, 15].

3.2. **Noncommutative Hermite–Einstein Vector Bundles.** In this subsection, which is in effect an extended remark, we observe that the classical definition of an Hermite–Einstein vector bundle extends to the setting of CQH-Kähler spaces.

Any positive, negative, or flat vector bundle $(\mathcal{F}, h, \overline{\partial}_{\mathcal{F}})$ over an irreducible CQH-Kähler space $\mathbf{K} = (B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \kappa)$ satisfies

$$\Lambda_{\mathcal{F}} \circ \nabla^2 = \gamma \mathbf{i}\, \mathrm{id}_{\mathcal{F}}, \qquad \text{for some } \gamma \in \mathbb{R}. \tag{5}$$

This is established by a verbatim extension of the argument of Lemma 3.2 (iii) from the covariant line bundle setting to the case of general Hermitian holomorphic vector bundles. What is very interesting about (5) is that it is satisfied by a far larger class of Hermitian holomorphic vector bundles than those which are simply positive, negative, or flat. Classically, this motivates the definition of an Hermite–Einstein vector bundle [30, §4.B]. As we now observe, this definition carries over directly to the noncommutative setting.

**Definition 3.6.** For a CQH-Kähler space $\mathbf{K}$, we say that an Hermitian holomorphic vector bundle $(\mathcal{F}, h, \overline{\partial}_{\mathcal{F}})$ over $\mathbf{K}$ is *Hermite–Einstein* if

$$\Lambda_{\mathcal{F}} \circ \nabla^2 = \gamma \mathbf{i}\, \mathrm{id}_{\mathcal{F}}, \qquad \text{for some } \gamma \in \mathbb{R}.$$

Hermite–Einstein vector bundles are objects of central importance in classical complex geometry. They are intimately related to the theory of Yang–Mills connections [31]. Moreover, the Donaldson–Uhlenbeck–Yau theorem relates the existence of an Hermite–Einstein metric to semi-stability of the vector bundle, see [39] for details. The investigation of how such structures and results extend to the noncommutative setting presents itself as a very interesting direction for future research.

## 4. The Heckenberger–Kolb Calculi for the Irreducible Quantum Flag Manifolds

In this section we consider our motivating family of examples: the irreducible quantum flag manifolds endowed with their Heckenberger–Kolb calculi. We assume that the reader has some familiarity with the representation theory of Lie algebras. For standard references see [29, 32, 49].

4.1. **Drinfeld–Jimbo Quantum Groups.** In this subsection we recall the necessary definitions from the theory of Drinfeld–Jimbo quantum groups. We refer the reader to [37] for further details, as well to the seminal papers [21, 33]. Let $\mathfrak{g}$ be a finite-dimensional complex semisimple Lie algebra of rank $r$. We fix a Cartan subalgebra $\mathfrak{h}$ with corresponding root system $\Delta \subseteq \mathfrak{h}^*$, where $\mathfrak{h}^*$ denotes the linear dual of $\mathfrak{h}$. With respect to a choice of simple roots $\Pi = \{\alpha_1, \dots, \alpha_r\}$, denote by $(\cdot,\cdot)$ the symmetric bilinear form induced on $\mathfrak{h}^*$ by the Killing form of $\mathfrak{g}$, normalised so that any shortest simple root $\alpha_i$ satisfies $(\alpha_i, \alpha_i) = 2$. Let $\{\varpi_1, \dots, \varpi_r\}$ denote the corresponding set of fundamental weights of $\mathfrak{g}$. The *coroot* $\alpha_i^\vee$ of a simple root $\alpha_i$ is defined by

$$\alpha_i^\vee := d_i\alpha_i = \frac{2\alpha_i}{(\alpha_i,\alpha_i)}, \qquad \text{where } d_i := \frac{2}{(\alpha_i,\alpha_i)}.$$

The Cartan matrix $A = (a_{ij})_{ij}$ of $\mathfrak{g}$ is the $(r \times r)$-matrix defined by $a_{ij} := \big(\alpha_i^\vee, \alpha_j\big)$. Let $q \in \mathbb{R}$ such that $q \notin \{-1, 0, 1\}$, and denote $q_i := q^{(\alpha_i,\alpha_i)/2}$. The *quantised enveloping algebra* $U_q(\mathfrak{g})$ is the noncommutative associative algebra generated by the elements $E_i, F_i, K_i$, and $K_i^{-1}$, for $i = 1, \dots, r$, subject to the relations

$$K_iE_j = q_i^{a_{ij}}E_jK_i, \quad K_iF_j = q_i^{-a_{ij}}F_jK_i, \quad K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$
$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

along with the *quantum Serre relations*

$$\sum\nolimits_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0, \qquad \text{for } i \neq j,$$
$$\sum\nolimits_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1 - a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0, \qquad \text{for } i \neq j;$$

where we have used the $q$-binomial coefficients defined in Appendix C. A Hopf algebra structure is defined on $U_q(\mathfrak{g})$ by

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,$$
$$S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i, \quad S(K_i) = K_i^{-1},$$
$$\varepsilon(E_i) = \varepsilon(F_i) = 0, \quad \varepsilon(K_i) = 1.$$

A Hopf $*$-algebra structure, called the *compact real form* of $U_q(\mathfrak{g})$, is defined by

$$K_i^* := K_i, \qquad E_i^* := K_iF_i, \qquad F_i^* := E_iK_i^{-1}.$$

Let $\mathcal{P}$ be the weight lattice of $\mathfrak{g}$, and $\mathcal{P}^+$ its set of dominant integral weights. For each $\mu \in \mathcal{P}^+$ there exists an irreducible finite-dimensional $U_q(\mathfrak{g})$-module $V_\mu$, uniquely defined by the existence of a vector $v_\mu \in V_\mu$, which we call a *highest weight vector*, satisfying

$$E_i \triangleright v_\mu = 0, \qquad K_i \triangleright v_\mu = q^{(\mu,\alpha_i)}v_\mu, \qquad \text{for all } i = 1, \dots, r.$$

Moreover, $v_\mu$ is unique up to scalar multiple. We call any finite direct sum of such $U_q(\mathfrak{g})$-representations a *type*-1 *representation.* In general, a vector $v \in V_\mu$ is called a *weight vector* of weight $\mathrm{wt}(v) \in \mathcal{P}$ if

$$K_i \triangleright v = q^{(\mathrm{wt}(v),\alpha_i)}v, \qquad \text{for all } i = 1,\ldots,r.$$

Finally, we note that since $U_q(\mathfrak{g})$ has an invertible antipode, we have an equivalence between ${}_{U_q(\mathfrak{g})}\mathrm{Mod}$, the category of left $U_q(\mathfrak{g})$-modules, and $\mathrm{Mod}_{U_q(\mathfrak{g})}$, the category of right $U_q(\mathfrak{g})$-modules, as induced by the antipode.

4.2. **Quantum Coordinate Algebras and Quantum Flag Manifolds.** In this subsection we recall some necessary material about quantised coordinate algebras, see [37, §6 and §7] and [52] for further details. Let $V$ be a finite-dimensional left $U_q(\mathfrak{g})$-module, $v \in V$, and $f \in V^*$, the $\mathbb{C}$-linear dual of $V$, endowed with its right $U_q(\mathfrak{g})$-module structure. An important point to note is that, with respect to the equivalence of left and right $U_q(\mathfrak{g})$-modules discussed above, the left module corresponding to $V_\mu^*$ is isomorphic to $V_{-w_0(\mu_S)}$, where $w_0$ denotes the longest element in the Weyl group of $\mathfrak{g}$.

Consider the function $c^V_{v,f} : U_q(\mathfrak{g}) \to \mathbb{C}$ defined by $c^V_{v,f}(X) := f\big(X \triangleright v\big)$. The *coordinate ring* of $V$ is the subspace

$$C(V) := \mathrm{span}_{\mathbb{C}}\big\{c^V_{f,v} \,|\, v \in V,\ f \in V^*\big\} \subseteq U_q(\mathfrak{g})^*.$$

A $U_q(\mathfrak{g})$-bimodule structure on $C(V)$ is given by

$$(Y \triangleright c^V_{f,v} \triangleleft Z)(X) := f\left((ZXY) \triangleright v\right) = c^V_{f\triangleleft Z, Y\triangleright v}(X). \tag{6}$$

Let $U_q(\mathfrak{g})^\circ$ denote the Hopf dual of $U_q(\mathfrak{g})$. It is easily checked that $C(V) \subseteq U_q(\mathfrak{g})^\circ$, and moreover that a Hopf subalgebra of $U_q(\mathfrak{g})^\circ$ is given by

$$\mathcal{O}_q(G) := \bigoplus_{\mu \in \mathcal{P}^+} C(V_\mu).$$

We call $\mathcal{O}_q(G)$ the *quantum coordinate algebra of* $G$, where $G$ is the compact, connected, simply-connected, simple Lie group having $\mathfrak{g}$ as its complexified Lie algebra.

For $\{\alpha_i\}_{i\in S}$ a subset of simple roots, consider the Hopf $*$-subalgebra

$$U_q(\mathfrak{l}_S) := \big\langle K_i, E_j, F_j \,|\, i = 1,\ldots,r; j \in S\big\rangle.$$

The Hopf $*$-algebra embedding $\iota : U_q(\mathfrak{l}_S) \hookrightarrow U_q(\mathfrak{g})$ induces the dual Hopf $*$-algebra map $\iota^\circ : U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{l}_S)^\circ$. By construction $\mathcal{O}_q(G) \subseteq U_q(\mathfrak{g})^\circ$, so we can consider the restriction map

$$\pi_S := \iota|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \to U_q(\mathfrak{l}_S)^\circ,$$

and the Hopf $*$-subalgebra $\mathcal{O}_q(L_S) := \pi_S\big(\mathcal{O}_q(G)\big) \subseteq U_q(\mathfrak{l}_S)^\circ$. The *quantum flag manifold associated* to $S$ is the CQGA-homogeneous space associated to the surjective

Hopf $*$-algebra map $\pi_S : \mathcal{O}_q(G) \to \mathcal{O}_q(L_S)$, and is denoted by

$$\mathcal{O}_q\big(G/L_S\big) := \mathcal{O}_q\big(G\big)^{\mathrm{co}(\mathcal{O}_q(L_S))}.$$

Denoting $\mu_S := \sum_{s \notin S} \varpi_s$, we choose for $V_{\mu_S}$ a weight basis $\{v_i\}_i$, with corresponding dual basis $\{f_i\}_i$. As shown in [27, Proposition 3.2], writing $N := \dim(V_{\mu_S})$, a set of generators for $\mathcal{O}_q(G/L_S)$ is given by

$$z_{ij} := c^{\mu_S}_{f_i,v_N} c^{-w_0(\mu_S)}_{v_j,f_N} \qquad \text{for } i,j = 1,\dots,N,$$

where $v_N$, and $f_N$, are the highest weight basis elements of $V_{\mu_S}$, and $V_{-w_0(\mu_S)}$, respectively, and, to ease notation, we have written

$$c^{\mu_S}_{f_i,v_N} := c^{V_{\mu_S}}_{f_i,v_N}, \qquad c^{-w_0(\mu_S)}_{v_i,f_N} := c^{V_{-w_0(\mu_S)}}_{v_i,f_N}.$$

4.3. **The Heckenberger–Kolb Calculi.** Let $S = \{1,\dots,r\} \setminus \{s\}$ where $\alpha_s$ has coefficient 1 in the expansion of the highest root of $\mathfrak{g}$. Then we say that the associated quantum flag manifold is *irreducible*. In the classical limit of $q = 1$, these homogeneous spaces reduce to the family of compact Hermitian symmetric spaces, as classified, for example, in [4]. Presented in Table 1 of Appendix B is a useful diagrammatic presentation of the set of simple roots defining the irreducible quantum flag manifolds, along with the names associated to the various series.

The irreducible quantum flag manifolds are distinguished by the existence of an essentially unique $q$-deformation of their classical de Rham complex. The existence of such a canonical deformation is one of the most important results in the study of the noncommutative geometry of quantum groups, serving as a solid base from which to investigate more general classes of quantum spaces. The following theorem is a direct consequence of results established in [27], [28], and [42]. See [17, §10] for a more detailed presentation.

**Theorem 4.1.** *Over any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists a unique finite-dimensional left $\mathcal{O}_q(G)$-covariant differential $*$-calculus*

$$\Omega^\bullet_q(G/L_S) \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}_0,$$

*which is of classical dimension, that is to say, satisfying*

$$\dim \Phi\big(\Omega^k_q(G/L_S)\big) = \binom{2M}{k}, \qquad \textit{for all } k = 0,\dots,2M,$$

*where $M$ is the complex dimension of the corresponding classical manifold, as presented in Table 2 of Appendix B.*

The calculus $\Omega^\bullet_q(G/L_S)$, which we call the *Heckenberger–Kolb calculus* of $\mathcal{O}_q(G/L_S)$, has many remarkable properties. We begin with the existence of a unique covariant complex structure, following from the results of [27], [28], and [42].

**Proposition 4.2.** *Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold, and $\Omega^\bullet_q(G/L_S)$ its Heckenberger–Kolb differential $*$-calculus. Then the following hold:*

(i) *$\Omega^\bullet_q(G/L_S)$ admits a unique left $\mathcal{O}_q(G)$-covariant almost complex structure,*
$$\Omega^\bullet_q(G/L_S) \simeq \bigoplus_{(a,b)\in\mathbb{N}_0^2} \Omega^{(a,b)} =: \Omega^{(\bullet,\bullet)},$$

(ii) *$\Omega^{(\bullet,\bullet)}$ is both integrable and factorisable,*

(iii) *$\Omega^{(1,0)}$ and $\Omega^{(0,1)}$ are irreducible as objects in ${}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}_0$.*

As observed in [47, §10.8] (using the same argument as presented in part (i) of Lemma 3.2) there exists a real left $\mathcal{O}_q(G)$-coinvariant form $\kappa \in \Omega^{(1,1)}$, and it is unique up to real scalar multiple. Moreover, by extending the representation theoretic argument given in [47, §4.4] for the case $\mathcal{O}_q(\mathbb{CP}^n)$, the form $\kappa$ is readily seen to be a closed central element of $\Omega^\bullet_q(G/L_S)$. This motivated [47, Conjecture 4.25], where it was proposed that the pair $(\Omega^{(\bullet,\bullet)}, \kappa)$ is a Kähler structure for the calculus. With suitable restrictions on the values of $q$, the conjecture was verified in [42, Theorem 5.10].

**Theorem 4.3.** *Let $\Omega^\bullet_q(G/L_S)$ be the Heckenberger–Kolb calculus of the irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$. The pair $(\Omega^{(\bullet,\bullet)}, \kappa)$ is a covariant Kähler structure for all $q \in \mathbb{R}_{>0} \setminus F$, where $F$ is a finite, possibly empty, subset of $\mathbb{R}_{>0}$. Moreover, any element of $F$ is necessarily non-transcendental.*

In [17, Lemma 10.10], positive definiteness of the Kähler structure was subsequently verified, giving us a CQH-Kähler space [17, Theorem 10.11]. Taken together with irreducibility of the holomorphic forms (as recalled in part (iii) of Proposition 4.2 above), this gives us the following theorem.

**Theorem 4.4.** *For each irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, there exists an open interval $I \subseteq \mathbb{R}_{>0}$ around $1$, such that an irreducible CQH-Kähler space is given by the quadruple*
$$\mathbf{K}_S := \big(\mathcal{O}_q(G/L_S),\, \Omega^\bullet_q(G/L_S),\, \Omega^{(\bullet,\bullet)}, \kappa\big).$$

In the rest of this section we build upon this result, using it to apply the general framework of the paper to the study of the irreducible quantum flag manifolds.

4.4. **Line Bundles over the Irreducible Quantum Flag Manifolds.** In this subsection, we recall the necessary definitions and results about noncommutative line bundles over the irreducible quantum flag manifolds. Some of the results in this section rely on the forthcoming [19], which extends the results in [44] from quantum Grassmannians (in other words, the $A$-series irreducible quantum flag manifolds). Since, at the time of writing [19] has yet to appear, we direct the interested reader to [44] where the claimed results have already been established for the quantum Grassmannians.

Classically, the algebra $\mathfrak{l}_S$ is reductive, so we have a decomposition into a direct sum $\mathfrak{l}^{\mathrm{s}}_S \oplus \mathfrak{u}_1$, comprised of a semisimple part and a commutative part, respectively. In the quantum setting, we are thus motivated to consider the Hopf subalgebra

$$U_q(\mathfrak{l}^{\mathrm{s}}_S) := \big\langle K_i, E_i, F_i \,|\, i \in S \big\rangle \subseteq U_q(\mathfrak{l}_S).$$

From the Hopf $*$-algebra embedding $\iota : U_q(\mathfrak{l}^{\mathrm{s}}_S) \hookrightarrow U_q(\mathfrak{g})$, we have the dual Hopf $*$-algebra map $\iota^\circ : U_q(\mathfrak{g})^\circ \to U_q(\mathfrak{l}^{\mathrm{s}}_S)^\circ$. By construction $\mathcal{O}_q(G) \subseteq U_q(\mathfrak{g})^\circ$, so we can consider the restriction map

$$\pi^{\mathrm{s}}_S := \iota|_{\mathcal{O}_q(G)} : \mathcal{O}_q(G) \to U_q(\mathfrak{l}^{\mathrm{s}}_S)^\circ,$$

and the Hopf $*$-subalgebra $\mathcal{O}_q(L^{\mathrm{s}}_S) := \pi^{\mathrm{s}}_S\big(\mathcal{O}_q(G)\big) \subseteq U_q(\mathfrak{l}^{\mathrm{s}}_S)^\circ$. We denote by

$$\mathcal{O}_q\big(G/L^{\mathrm{s}}_S\big) := \mathcal{O}_q\big(G\big)^{\mathrm{co}\left(\mathcal{O}_q(L^{\mathrm{s}}_S)\right)},$$

the CQGA-homogeneous space associated to the Hopf $*$-algebra map $\pi^{\mathrm{s}}_S$.

It follows directly from the defining relations of the Drinfeld–Jimbo quantum groups that the element

$$Z = K_{\det(A)\varpi_s} = K_1^{a_1} \cdot \ldots \cdot K_r^{a_r},$$

where $a_1, \ldots, a_r$ are determined by $\det(A)\varpi_s = a_1\alpha_1 + \ldots + a_r\alpha_r$, belongs to the centre of $U_q(\mathfrak{l}_S)$.

The fact that $Z$ is central implies that $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$ is closed under the right action of $Z$ defined in (6). Thus we have a well-defined $U(\mathfrak{u}_1)$-action generated by $Z$ on $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$, or equivalently an $\mathcal{O}(U_1)$-coaction. This implies an associated $\mathbb{Z}$-grading

$$O_q(G/L^{\mathrm{s}}_S) \simeq \bigoplus_{k\in\mathbb{Z}} \mathcal{E}_k.$$

Each $\mathcal{E}_k$ is clearly a bimodule over $\mathcal{E}_0 = \mathcal{O}_q(G/L_S)$. Moreover, since the action of $U(\mathfrak{u}_1)$ clearly commutes with the left $\mathcal{O}_q(G)$-coaction on $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$, each $\mathcal{E}_k$ is an $\mathcal{O}_q(G)$-sub-comodule of $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$. In [19] (see also [44]), it is shown that

$$\mathcal{E}_k \in {}^{\mathcal{O}_q(G)}_{\mathcal{O}_q(G/L_S)}\mathrm{mod}_0, \qquad\qquad \text{for all } k \in \mathbb{Z},$$

and moreover that each $\Phi(\mathcal{E}_k)$ is a one-dimensional space. From the equivalence of the category of finite-dimensional representations of $\mathfrak{l}_S$ and the category of type-1 representations of $U_q(\mathfrak{l}_S)$ it easy to deduce that $\Phi(\mathcal{E}_k)$ enumerate all one-dimensional representations of $U_q(\mathfrak{l}_S)$. Thus $\mathcal{E}_k$ classify all line bundles over $\mathcal{O}_q(G/L_S)$.

Using the general theory of principal comodule algebras, it is shown in [19] (see also [44]) that each $\mathcal{E}_k$ is projective as a left $\mathcal{O}_q(G/L_S)$-module. Thus, when $q = 1$, each $\mathcal{E}_k$ reduces to the space of sections of a line bundle over $\mathcal{O}_q(G/L_S)$.

**Example 4.5.** For the special case of quantum projective space $\mathcal{O}_q(\mathbb{CP}^n)$, the quantum homogeneous space $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$ is given by the odd-dimensional quantum sphere $\mathcal{O}_q(S^{2n-1})$, where the decomposition into line bundles is well known [43, 45].

For the case of the quantum quadrics $\mathcal{O}_q(\mathbf{Q}_n)$, the quantum homogeneous space $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$ is a $q$-deformation of the coordinate ring of $V_2(\mathbb{R}^n)$, the Stieffel manifold of orthonormal 2-frames in $\mathbb{R}^n$.

4.5. **The Borel–Weil Theorem and Positive Line Bundles.** Using the general theory of quantum principal bundles, it was shown in [44, Lemma 5.3] that each covariant line bundle over the quantum Grassmannians $\mathcal{O}_q(\mathrm{Gr}_{s,n+1})$ admits a unique covariant holomorphic structure. This result is subsequently extended to the setting of the general irreducible quantum flag manifolds in [19], as we now recall.

**Theorem 4.6.** *For every covariant line bundle $\mathcal{E}_k$ over $\mathcal{O}_q(G/L_S)$, there exists a unique covariant $(0,1)$-connection*

$$\overline{\partial}_{\mathcal{E}_k} : \mathcal{E}_k \to \Omega^{(0,1)}_q(G/L_S) \otimes_{\mathcal{O}_q(G/L_S)} \mathcal{E}_k.$$

*Moreover, $\overline{\partial}_{\mathcal{E}_k}$ is flat, which is to say, it is a holomorphic structure.*

It now follows directly from Theorem 3.3 that the covariant line bundles over the irreducible quantum flag manifolds $\mathcal{O}_q(G/L_S)$ are either positive, flat, or negative. To differentiate between these possibilities, we need some cohomological information about the bundles. This information is given by the noncommutative Borel–Weil theorem for $\mathcal{O}_q(G/L_S)$, a result to appear by the first and third authors in [19], generalising the special case of the quantum Grassmannians established in [44].

**Theorem 4.7** (Borel–Weil)**.** *For each irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, we have $U_q(\mathfrak{g})$-module isomorphisms*

(i) $H^0_{\overline{\partial}}(\mathcal{E}_k) \simeq V_{k\varpi_s}$, *for* $k \geq 0$,
(ii) $H^0_{\overline{\partial}}(\mathcal{E}_{-k}) = 0$, *for* $k > 0$.

Using this cohomological information, we can now apply Theorem 3.3 to the irreducible quantum flag manifolds.

**Theorem 4.8.** *For any irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, it holds that, for any $k \in \mathbb{N}$,*

(i) $\mathcal{E}_k > 0$,
(ii) $\mathcal{E}_{-k} < 0$,

*for choice of positive definite Kähler form $\kappa$.*

4.6. **A Fano Structure for the Irreducible Quantum Flag Manifolds.** In this subsection we show that each irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$ is of Fano type (Definition 2.13). Since we now know that a line bundle $\mathcal{E}_l$ is negative if and only if $l$ is a negative integer, this amounts to showing that $\Omega^{(M,0)} \simeq \mathcal{E}_{-k}$, for some $k > 0$. This we do by producing a general description of $k$ in terms of the Cartan matrix of $\mathfrak{g}$. In Table 2 of Appendix B, we present the explicit values of $k$ for each series of the irreducible quantum flag manifolds.

**Theorem 4.9.** *Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus $\Omega_q(G/L_s)$. For any $q \in \mathbb{R}_{>0}$, if $(\Omega^{(\bullet,\bullet)}, \kappa)$ is a Kähler structure, then it is moreover a Fano structure.*

*Proof.* Since the complex structure $\Omega^{(\bullet,\bullet)}$ is factorisable, we only need to verify condition (ii) of Definition 2.13, that is, show that $(\Omega^{(M,0)}, g_\kappa, \overline{\partial})$ is a negative line bundle, where $2M$ is the total dimension of $\Omega^\bullet_q(G/L_S)$. First, we need to identify the unique $k \in \mathbb{N}$ such that

$$\Phi\big(\Omega^{(M,0)}\big) \simeq \Phi(\mathcal{E}_{-k}) = (\Phi(\mathcal{E}_{-1}))^{\otimes k}. \tag{7}$$

To do so, we will compare the actions of the central element $Z \in U_q(\mathfrak{l}_S)$ on the $U_q(\mathfrak{l}_S)$-modules $\Phi(\mathcal{E}_{-1})$ and $\Phi(\Omega^{(M,0)})$.

Consider first $\Phi(\mathcal{E}_{-1})$, which is one-dimensional and (as observed in [19, 44]) is spanned by the element $[c^{-w_0(\varpi_s)}_{v_N,f_N}]$, where $N := \dim(V_{\mu_S})$. Explicitly, $Z$ acts as

$$[c^{-w_0(\varpi_s)}_{v_N,f_N}] \triangleleft Z = [c^{-w_0(\varpi_s)}_{v_N \triangleleft Z,f_N}] = q^{-(\varpi_s,\varpi_s)\det(A)}[c^{-w_0(\varpi_s)}_{v_N,f_N}].$$

Thus, since $\Phi(\mathcal{E}_{-k}) \simeq \Phi(\mathcal{E}_{-1})^{\otimes k}$, and $Z$ is grouplike, $Z$ must act as multiplication by the scalar $q^{-k(\varpi_s,\varpi_s)\det(A)}$.

Moving onto $\Phi\big(\Omega^{(M,0)}\big)$, we note that $\Phi(\Omega^{(1,0)})$ is irreducible as a $U_q(\mathfrak{l}_S)$-module, and hence that $Z$ acts on $\Phi(\Omega^{(1,0)})$ as multiplication by some scalar $\gamma$. Consider the subset of $J := \{1, \dots, \dim(V_{\varpi_s})\}$ given by

$$J_{(1)} := \{i \in J \mid (\varpi_s, \varpi_s - \alpha_s - \mathrm{wt}(v_i)) = 0\}.$$

As shown in [28, Proposition 3.6], a basis of $\Phi(\Omega^{(1,0)})$ is given by

$$\big\{[\partial z_{iN}] \mid \text{ for } i \in J_{(1)}\big\}.$$

Therefore, for any $i \in J_{(1)}$, we have

$$\begin{aligned}[\partial z_{iN}] \triangleleft Z &= [\partial(c^{\varpi_s}_{f_i \triangleleft Z, v_N} c^{-w_0(\varpi_s)}_{v_N \triangleleft Z, f_N})] \\ &= q^{(\varpi_s,\varpi_s-\alpha_s)\det(A)-(\varpi_s,\varpi_s)\det(A)}[\partial z_{iN}] \\ &= q^{-(\varpi_s,\alpha_s)\det(A)}[\partial z_{iN}].\end{aligned}$$

Thus we see that $\gamma = q^{-(\varpi_s,\alpha_s)\det(A)}$. It follows from [28, Proposition 2.6.2] that $\Phi(\Omega^{(M,0)})$ can be naturally considered as a submodule of $\Phi(\Omega^{(1,0)})^{\otimes M}$ (see also the remark below). From this it follows that $Z$ must act on $\Phi(\Omega^{(M,0)})$ as multiplication by $q^{-M(\varpi_s,\alpha_s)\det(A)}$.

From (7) it is now clear that $k$ is uniquely determined by the identity

$$M\,(\varpi_s, \alpha_s)\det(A) = k\,(\varpi_s, \varpi_s)\det(A).$$

Recalling that $(\varpi_s, \alpha_s) = d_s^{-1}$, and $(\varpi_s, \varpi_s) = d_s^{-1}((A^t)^{-1})_{ss} = d_s^{-1}(A^{-1})_{ss}$, where $A$ is Cartan matrix of $\mathfrak{g}$, we see that

$$k = \frac{M\,(\varpi_s, \alpha_s)}{(\varpi_s, \varpi_s)} = M/(A^{-1})_{ss}.$$

It remains to show that $k > 0$. Indeed, each entry of the matrix $(A^{-1})_{ss}$ is a positive rational number (see, for example, [49, Table 2] for an explicit presentation of the values). Thus, for every irreducible quantum flag manifold $\mathcal{O}_q(G/L_S)$, the scalar $k$, which is necessarily an integer, must be an element of $\mathbb{N}$. □

**Remark 4.10.** In the proof above, it was concluded from a calculation in [28] that $\Phi(\Omega^{(M,0)})$ can be naturally understood as a submodule of $\Phi(\Omega^{(1,0)})^{\otimes M}$. As we now observe, this fact can alternatively be concluded directly from the monoidal form of Takeuchi's equivalence, as presented in Appendix A. Indeed, since $\Omega^{(\bullet,0)}$ is a monoid object in ${}^{A}_{B}\mathrm{mod}_0$, its image under $\Phi$ is again a monoid object in ${}^{H}\mathrm{mod}$. Since $\Omega^{(\bullet,0)}$ is generated as an algebra in degree 0 and degree 1, we see that $\Phi(\Omega^{(\bullet,0)})$ is generated as an algebra in degree 0 and degree 1, giving us the surjective $U_q(\mathfrak{l}_S)$-module map

$$\Phi(\Omega^{(1,0)})^{\otimes M} \to \Phi(\Omega^{(M,0)}).$$

The category of $U_q(\mathfrak{l}_S)$-modules is semisimple, so this surjection splits, giving us an inclusion $\Phi(\Omega^{(M,0)}) \hookrightarrow \Phi(\Omega^{(1,0)})^{\otimes M}$.

4.7. **A Bott–Borel–Weil Theorem for Positive Line Bundles.** In this subsection we examine the higher cohomologies of positive line bundles, proving a direct generalisation of the classical Bott–Borel–Weil for the positive line bundles of irreducible flag manifolds [7].

4.7.1. *Holomorphic Bimodule Vector Bundles and Positivity.* We first need to recall some results and constructions from [48]. Let $\Omega^{(\bullet,\bullet)}$ be a complex structure for a differential $*$-calculus $\Omega^\bullet$ over a $*$-algebra $B$. If $\Omega^{(\bullet,\bullet)}$ is factorisable, then we have isomorphisms

$$\beta_{(a,b)} : \Omega^{(a,0)} \otimes_B \Omega^{(0,b)} \simeq \Omega^{(0,b)} \otimes_B \Omega^{(a,0)}, \qquad \text{for each } (a,b) \in \mathbb{N}_0. \tag{8}$$

Using these isomorphisms, one can construct from a factorisable complex structure a canonical type of *bimodule connection* for the space of highest degree holomorphic forms. Before we do this, let us recall the general definition of a bimodule connection.

**Definition 4.11.** Let $(\Omega^\bullet, \mathrm{d})$ be a differential calculus over an algebra $B$, and $\mathcal{F}$ a $B$-bimodule. A *bimodule connection* for $\mathcal{F}$ is a pair $(\nabla, \beta)$, where $\nabla : \mathcal{F} \to \Omega^1 \otimes_B \mathcal{F}$ is a connection and $\beta : \mathcal{F} \otimes_B \Omega^1 \to \Omega^1 \otimes_B \mathcal{F}$ is a $B$-bimodule map such that

$$\nabla(fb) = \nabla(f)b + \beta(f \otimes \mathrm{d}b), \qquad \text{for all } f \in \mathcal{F},\ b \in B.$$

Bimodule connections are important because they allow us to take tensor products. Let $(\nabla, \beta)$ be a bimodule connection for $\mathcal{F}$, and $(\nabla', \mathcal{G})$ a connection for $\mathcal{G}$, a left $B$-module. Then a connection for $\mathcal{F} \otimes_B \mathcal{G}$ is given by

$$\nabla'' := \nabla \otimes \mathrm{id}_{\mathcal{G}} + (\beta \otimes \mathrm{id}_{\mathcal{G}}) \circ (\mathrm{id}_{\mathcal{F}} \otimes \nabla') : \mathcal{F} \otimes_B \mathcal{G} \to \Omega^1 \otimes_B \mathcal{F} \otimes_B \mathcal{G}.$$

In addition to our above assumptions, let us suppose that $B$ is a $*$-algebra, that $\Omega^\bullet$ is a $*$-calculus, and let us make a choice $\Omega^{(\bullet,\bullet)}$ of complex structure for $\Omega^\bullet$.

**Definition 4.12.** A *holomorphic bimodule vector bundle* over $B$ is a triple $(\mathcal{F}, \overline{\partial}_{\mathcal{F}}, \beta)$, where $\mathcal{F}$ is $B$-bimodule, $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$ is a holomorphic vector bundle, and $(\overline{\partial}_{\mathcal{F}}, \beta)$ is a bimodule connection for $\mathcal{F}$, with respect to the differential calculus $(\Omega^{(0,\bullet)}, \overline{\partial})$.

As observed in §2.6, when $\Omega^\bullet$ is a finitely generated, projective left $B$-module, and $\Omega^{(\bullet,\bullet)}$ is factorisable, then the pair $(\Omega^{(n,0)}, \overline{\partial})$ is a holomorphic vector bundle over $B$. As we now see, the triple

$$(\Omega^{(n,0)}, \wedge^{-1} \circ \overline{\partial}, \beta := \beta_{(n,1)})$$

is a holomorphic bimodule vector bundle, where $\wedge^{-1}$ is the inverse of the map defined in Definition 2.12 (i). Note that if $B$ is a left $A$-comodule algebra, and the complex structure is also left $A$-covariant, then the holomorphic bimodule vector bundle will be covariant, in the sense that the maps $\overline{\partial}$ and $\beta$ will be comodule maps.

Here we are interested in a holomorphic bimodule vector bundle structure for $\Omega^{(0,n)}$. As explained in [48, §3], by formally dualising, one can produce a corresponding holomorphic bimodule vector bundle structure for $\Omega^{(n,0)}$. Moreover, in the covariant setting, the dual bundle will again be covariant. In practice, covariance often identifies the holomorphic structure uniquely, allowing us to take the formal dualising procedure as an existence result. Importantly, as observed in [48, §3], for any other holomorphic vector bundle $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$, the tensor product $(0,1)$-connection on $\Omega^{(0,n)} \otimes_B \mathcal{F}$ will be flat, which is to say, it will be a holomorphic structure.

With these results in hand, we are now ready to recall [48, Proposition 8.7]. See the remark below for a discussion of the original formulation of this result.

**Proposition 4.13.** *Let* $\mathbf{H} = (B = A^{\mathrm{co}(H)}, \Omega^\bullet, \Omega^{(\bullet,\bullet)}, \sigma)$ *be a* $2n$*-dimensional CQH-Hermitian space, where* $\Omega^{(\bullet,\bullet)}$ *is a factorisable complex structure, and let* $(\mathcal{F}, h_{\mathcal{F}}, \overline{\partial}_{\mathcal{F}})$ *be an Hermitian holomorphic vector bundle over* $\mathbf{H}$*. If the Hermitian holomorphic vector bundle* $\Omega^{(0,n)} \otimes_B \mathcal{F}$ *is positive, then*

$$H^{(0,k)}_{\overline{\partial}}(\mathcal{F}) = 0, \qquad \text{for all } k > 0.$$

**Remark 4.14.** Note the original formulation is presented in a more general setting, which contains the CQH-Hermitian space situation as a special case. It is stated here for CQH-Hermitian spaces for sake of convenience and brevity.

4.7.2. *The Bott–Borel–Weil Theorem.* We will now use Proposition 4.13, together with the results of §4.5 and §4.6, to establish the following noncommutative generalisation of the classical Bott–Borel–Weil theorem for the non-negative line bundles over the irreducible flag manifolds. This result extends previous partial results for line bundle cohomology over quantum projective space established in [35, 36, 14].

**Theorem 4.15** (Bott–Borel–Weil). *Let $\mathcal{O}_q(G/L_S)$ be an irreducible quantum flag manifold, endowed with its Heckenberger–Kolb calculus $\Omega^\bullet_q(G/L_S)$. Moreover, let $q \in \mathbb{R}_{>0}$ such that $(\Omega^{(\bullet,\bullet)}, \kappa)$ is a Kähler structure, and hence a Fano structure. Then we have the following $U_q(\mathfrak{g})$-module isomorphisms*

(i) $H^{(0,0)}_{\overline{\partial}}(\mathcal{E}_k) \simeq V_{k\varpi_s}$, *for all* $k \in \mathbb{N}_0$,
(ii) $H^{(0,i)}_{\overline{\partial}}(\mathcal{E}_k) = 0$, *for all* $k \in \mathbb{N}_0$, *and all* $i > 0$.

*Proof.* The isomorphisms in (i) are given by the noncommutative Borel–Weil theorem for the special case of positive line bundles. From Theorem 4.9 we know that $\Omega^{(0,M)} \simeq \mathcal{E}_j$, for some $j > 0$. Thus, as line bundles,

$$\Omega^{(0,M)} \otimes_B \mathcal{E}_k \simeq \mathcal{E}_j \otimes_B \mathcal{E}_k \simeq \mathcal{E}_{k+j}. \tag{9}$$

Since the complex structure $\Omega^{(\bullet,\bullet)}$ is factorisable, the holomorphic structure of $\Omega^{(0,M)}$ is necessarily covariant, and moreover, the holomorphic structure on $\Omega^{(0,M)} \otimes_B \mathcal{E}_k$ is covariant. Theorem 4.6 tells us that there is only one such structure on $\mathcal{E}_{k+j}$, and so, the isomorphism in (9) is an isomorphism of holomorphic vector bundles (in the obvious sense of isomorphism). Moreover, since $k + j > 0$, it follows from Theorem 4.8 that the bundle $\Omega^{(0,M)} \otimes_B \mathcal{E}_k$ is positive. Triviality of the cohomology groups in (ii) now follows from Proposition 4.13. □

## 5. The Case of Quantum Projective Space

For any complex manifold, the curvature of the Chern connection is additive over tensor products of holomorphic vector bundles. In particular, any tensor power of a positive line bundle is again positive. In the noncommutative setting tensoring two holomorphic vector bundles is more problematic. First of all, in order to define a tensor product, at least one of the bundles $(\mathcal{F}, \overline{\partial}_{\mathcal{F}})$ needs to be a bimodule. Moreover, $\overline{\partial}_{\mathcal{F}}$ needs to be a bimodule connection (in the sense of [22, 23, 24]). Even in this case, curvature does not behave additively. In particular, for a bimodule holomorphic line bundle $\mathcal{E}$, we cannot directly conclude positivity of $\mathcal{E}^{\otimes_B k}$ from positivity of $\mathcal{E}$, making the general approach of this paper all the more valuable. For the case of $\mathcal{O}_q(S^2)$ the Podleś sphere [41], and $\mathcal{O}_q(\mathbb{CP}^2)$ the quantum projective plane [15], it is known that the classical line bundle curvatures $q$-deform to quantum integer curvatures. In this section we show that this process generalises to all positive line bundles over all quantum projective spaces. This suggests that there is some type of $q$-deformed (or braided) additivity underlying these results. Understanding this process presents itself as an interesting and important future goal. Note that throughout this section $(k)_q$ denotes the quantum integer defined in Appendix C.

5.1. **General Results on Quantum Principal Bundles.** In this subsection, we recall those definitions and results from the theory of quantum principal bundles necessary for our explicit curvature calculations below.

5.1.1. *First-Order Differential Calculi.* A *first-order differential calculus* over an algebra $B$ is a pair $(\Omega^1, \mathrm{d})$, where $\Omega^1$ is a $B$-bimodule and $\mathrm{d} : B \to \Omega^1$ is a linear map for which the *Leibniz rule* holds

$$\mathrm{d}(ab) = a(\mathrm{d}b) + (\mathrm{d}a)b, \qquad a, b \in B,$$

and for which $\Omega^1$ is generated as a left $B$-module by those elements of the form $\mathrm{d}b$, for $b \in B$. The *universal first-order differential calculus* over $B$ is the pair $(\Omega^1_u(B), \mathrm{d}_u)$, where $\Omega^1_u(B)$ is the kernel of the multiplication map $m_B : B \otimes B \to B$ endowed with the obvious bimodule structure, and $\mathrm{d}_u$ is the map defined by

$$\mathrm{d}_u : B \to \Omega^1_u(B), \qquad b \mapsto 1 \otimes b - b \otimes 1.$$

By [58, Proposition 1.1], every first-order differential calculus over $B$ is of the form $(\Omega^1_u(B)/N, \mathrm{proj} \circ \mathrm{d}_u)$, where $N$ is a $B$-sub-bimodule of $\Omega^1_u(B)$, and we denote by $\mathrm{proj} : \Omega^1_u(B) \to \Omega^1_u(B)/N$ the quotient map. This gives a bijective correspondence between calculi and sub-bimodules of $\Omega^1_u(B)$. Moreover, as is well known (see for example [46, §2.5]) and is instructive to note, every first-order differential calculus admits an extension to a *maximal* differential calculus, which is to say, one from which any other extension can be obtained by quotienting.

For $A$ a Hopf algebra, and $B$ a left $A$-comodule algebra, we say that a first-order differential calculus $\Omega^1(B)$ over $B$ is *left $A$-covariant* if there exists a (necessarily unique) map $\Delta_L : \Omega^1(B) \to A \otimes \Omega^1(B)$ satisfying

$$\Delta_L(b\mathrm{d}b') = \Delta_L(b)(\mathrm{id} \otimes \mathrm{d})\Delta_L(b'), \qquad b, b' \in B.$$

For the special case of $A$ considered as a left $A$-comodule algebra over itself, we note that every covariant first-order differential calculus over $A$ is naturally an object in the category ${}^A_A\mathrm{mod}_A$.

5.1.2. *Quantum Principal Bundles and Principal Connections.* We say that a right $H$-comodule algebra $(P, \Delta_R)$ is a *Hopf–Galois extension* of $B := P^{\mathrm{co}(H)}$ if an isomorphism is given by

$$\mathrm{can} := (m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) : P \otimes_B P \to P \otimes H, \qquad r \otimes s \mapsto rs_{(1)} \otimes s_{(2)},$$

where $m_P$ denotes the multiplication in $P$. It can be shown [8, Proposition 3.6] that $P$ is a Hopf–Galois extension of $B = P^{\mathrm{co}(H)}$ if and only if an exact sequence is given by

$$0 \longrightarrow P\Omega^1_u(B)P \xrightarrow{\iota} \Omega^1_u(P) \xrightarrow{\overline{\mathrm{can}}} P \otimes H^+ \longrightarrow 0, \tag{10}$$

where $\Omega^1_u(B)$ is the restriction of $\Omega^1_u(P)$ to $B$, we denote by $\iota$ the inclusion, and $\overline{\mathrm{can}}$ is the restriction to $\Omega^1_u(P)$ of the map $(m_P \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \Delta_R) : P \otimes P \to P \otimes H$. (Note that the map's domain of definition is $P \otimes P$, rather than $P \otimes_B P$.) The following definition, due to Brzeziński and Majid [9, 10], presents sufficient criteria for the existence of a non-universal version of this sequence. A non-universal calculus on $P$ is said to be *right $H$-covariant* if the following, necessarily unique, map is well defined

$$\Delta_R : \Omega^1(P) \to \Omega^1(P) \otimes H, \qquad r\mathrm{d}s \mapsto r_{(0)}\mathrm{d}s_{(0)} \otimes r_{(1)}s_{(1)}.$$

**Definition 5.1.** Let $H$ be a Hopf algebra. A *quantum principal $H$-bundle* is a pair $(P, \Omega^1(P))$, consisting of a right $H$-comodule algebra $(P, \Delta_R)$, such that $P$ is a Hopf–Galois extension of $B = P^{\,\mathrm{co}(H)}$, together with a choice of right-$H$-covariant calculus $\Omega^1(P)$, such that for $N \subseteq \Omega^1_u(P)$ the corresponding sub-bimodule of the universal calculus, we have $\overline{\mathrm{can}}(N) = P \otimes I$, for some Ad-sub-comodule right ideal

$$I \subseteq H^+ := \ker(\varepsilon : H \to \mathbb{C}),$$

where $\mathrm{Ad} : H \to H \otimes H$ is defined by $\mathrm{Ad}(h) := h_{(2)} \otimes S(h_{(1)})h_{(3)}$.

Denoting by $\Omega^1(B)$ the restriction of $\Omega^1(P)$ to $B$, and $\Lambda^1_H := H^+/I$, the quantum principal bundle definition implies that an exact sequence is given by

$$0 \longrightarrow P\Omega^1(B)P \xrightarrow{\iota} \Omega^1(P) \xrightarrow{\overline{\mathrm{can}}} P \otimes \Lambda^1_H \longrightarrow 0, \tag{11}$$

where by abuse of notation, $\overline{\mathrm{can}}$ denotes the map induced on $\Omega^1(P)$ by identifying $\Omega^1(P)$ as a quotient of $\Omega^1_u(P)$ (for details see [26]).

A *principal connection* for a quantum principal $H$-bundle $(P, \Omega^1(P))$ is a right $H$-comodule, left $P$-module, projection $\Pi : \Omega^1(P) \to \Omega^1(P)$ satisfying

$$\ker(\Pi) = P\Omega^1(B)P.$$

The existence of a principal connection is equivalent to the existence of a left $P$-module, right $H$-comodule, splitting of the exact sequence given in (11). A principal connection $\Pi$ is called *strong* if $(\mathrm{id} - \Pi)\big(\mathrm{d}P\big) \subseteq \Omega^1(B)P$.

5.1.3. *Quantum Principal Bundles and Quantum Homogeneous Spaces.* We now restrict to the case of a homogeneous quantum principal bundle, which is to say, a quantum principal bundle whose composite $H$-comodule algebra is a quantum homogeneous space $B := A^{\mathrm{co}(H)}$, given by a surjective Hopf algebra map $\pi : A \to H$. For this special case, it is natural to restrict to calculi on $A$ which are left $A$-covariant. Any such calculus $\Omega^1(A)$ is an object in ${}^A_A\mathrm{Mod}_A$, and so, by the fundamental theorem of two-sided Hopf modules (Appendix A.2), we have the isomorphism

$$\mathrm{U} : \Omega^1(A) \simeq A \otimes F\big(\Omega^1(A)\big).$$

As a direct calculation will verify, with respect to the right $H$-coaction

$$F(\Omega^1(A)) \to F(\Omega^1(A)) \otimes H, \qquad [\omega] \mapsto [\omega_{(0)}] \otimes \pi(S(\omega_{(-1)}))\omega_{(1)},$$

the unit U of the equivalence is a right $H$-comodule map. (Here the right $H$-coaction on $A \otimes F(\Omega^1(A))$ is the usual tensor product coaction.) Thus a left $A$-covariant principal connection is equivalent to a choice of right $H$-comodule decomposition

$$F(\Omega^1(A)) \simeq F(A\Omega^1(B)A) \oplus F(A \otimes \Lambda^1_H) \simeq F(A\Omega^1(B)A) \oplus \Lambda^1_H.$$

As established in [44], for a homogeneous quantum principal bundle with cosemisimple composite Hopf algebras, all principal connections are strong.

Next, we come to the question of connections for any $\mathcal{F} \in {}^A_B\mathrm{mod}_0$. Note first that we have a natural embedding

$$j : \Omega^1(B) \otimes_B \mathcal{F} \hookrightarrow \Omega^1(B)A \,\square_H \Phi(\mathcal{F}), \qquad \omega \otimes f \mapsto \omega f_{(-1)} \otimes [f_{(0)}].$$

We claim that a strong principal connection $\Pi$ defines a connection $\nabla$ for $\mathcal{F}$ by

$$\nabla : \mathcal{F} \to \Omega^1(B) \otimes_B \mathcal{F}, \qquad f \mapsto j^{-1}\big((\mathrm{id} - \Pi)(\mathrm{d}f_{(1)}) \otimes [f_{(2)}]\big).$$

Indeed, since d and the projection $\Pi$ are both right $H$-comodule maps, their composition $(\mathrm{id} - \Pi) \circ \mathrm{d}$ is a right $H$-comodule map. Hence the image of $(\mathrm{id} - \Pi) \circ \mathrm{d}$ is contained in $j\,(\Omega^1(B) \otimes_B \mathcal{F})$, and $\nabla$ defines a connection. Moreover, if the principal connection $\Pi$ is a left $A$-comodule map, then the connection $\nabla$ is also a left $A$-comodule map.

5.2. **Line Bundles over Quantum Projective Space.** In this subsection we recall the definition of *quantum projective space*, which is to say the $A$-series irreducible quantum flag manifold $\mathcal{O}_q(\mathbb{CP}^n)$ with the defining subset of simple roots $S := \{\alpha_2, \dots, \alpha_{n+1}\}$, or equivalently the invariant subspace of the quantum Levi subalgebra

$$U_q(\mathfrak{l}_S) := \big\langle K_i, E_j, F_j \,|\, i = 1, \dots, n+1; j = 2, \dots, n+1 \big\rangle \subseteq U_q(\mathfrak{sl}_{n+1}).$$

Motivated by the classical situation, we denote $\mathcal{O}_q(U_n) := \mathcal{O}_q(L_S)$ (see [43, 45, 44]). The corresponding central element $Z$ in $U_q(\mathfrak{l}_S)$ is explicitly given by

$$Z = K_1^n K_2^{n-1} \cdot \dots \cdot K_n.$$

The quantum homogeneous space $\mathcal{O}_q(G/L^{\mathrm{s}}_S)$ reduces to the the invariant subspace of the Hopf subalgebra

$$U_q(\mathfrak{l}_S) := \big\langle K_i, E_i, F_i \,|\, i = 2, \dots, n+1 \big\rangle \subseteq U_q(\mathfrak{sl}_{n+1}).$$

In this special case, the quantum space is usually denoted by $\mathcal{O}_q(S^{2n+1})$ and called the $(2n+1)$-*dimensional quantum sphere*, as discussed in Example 4.5.

Since every finite-dimensional representation of $\mathfrak{sl}_{n+1}$ is contained in some tensor power of the first fundamental representation of $\mathfrak{sl}_{n+1}$, the matrix coefficients of $V_{\varpi_1}$ generate $\mathcal{O}_q(SU_{n+1})$ as an algebra. In particular, we can choose a weight basis $\{v_j\}_{j=1}^{n+1}$ of $V_{\varpi_1}$ such that the elements $u^i_j := c^{\varpi_1}_{f_i, v_j}$, for $i, j = 1, \dots, n+1$, coincide with the well-known FRT-presentation of $\mathcal{O}_q(SU_{n+1})$, see [52] or [37, §9] for details.

With respect to this presentation, the quantum sphere $\mathcal{O}_q(S^{2n+1})$ is generated as an algebra by the elements

$$z_i := u^i_1, \qquad \text{and} \qquad \overline{z}_i := S(u^1_i), \qquad \text{for } i = 1, \ldots, n.$$

In terms of the $\mathbb{Z}$-grading induced by the action of $Z$,

$$\mathcal{O}_q(S^{2n+1}) = \bigoplus_{k \in \mathbb{Z}} \mathcal{E}_k,$$

the elements $z_i$ have degree 1, while the elements $\bar{z}_j$ have degree $-1$. Thus, for every $k \in \mathbb{N}_0$, as objects

$$\mathcal{E}_k,\ \mathcal{E}_{-k} \in^{\mathcal{O}_q(SU_{n+1})}_{\mathcal{O}_q(\mathbb{CP}^n)} \mathrm{mod}_0,$$

the line bundle $\mathcal{E}_k$ is generated by the element $z_1^k$, while $\mathcal{E}_{-k}$ is generated by the element $\overline{z}_1^k$.

Finally, we recall that, for all $k \in \mathcal{E}_0$, the unit of Takeuchi's equivalence explicitly acts according to

$$\mathrm{U}(e) = e \otimes v_{\pm k}, \qquad \text{for all } e \in \mathcal{E}_{\pm k}, \tag{12}$$

where $v_k := [z_1^k] \in \Phi(\mathcal{E}_k)$, and $v_{-k} := [\overline{z}_1^k] \in \Phi(\mathcal{E}_{-k})$.

5.3. **A Quantum Principal Bundle Presentation of the Chern Connection of $\mathcal{O}_q(\mathbb{CP}^n)$.** In this subsection we recall the quantum principal bundle description of the Heckenberger–Kolb calculus introduced in [45]. The constituent calculus $\Omega^1_q(SU_{n+1})$ of the quantum principal bundle was originally constructed as a distinguished quotient of the standard bicovariant calculus on $\mathcal{O}_q(SU_{n+1})$ [34, 40]. Here we satisfy ourselves with presenting those properties of the calculus which are relevant to our calculations below, and refer the interested reader to [45, §4]. We stress that the calculus is far from being a natural $q$-deformation of the space of 1-forms of $SU_{n+1}$, instead it should be considered as a convenient tool for performing explicit calculations.

The calculus is left $\mathcal{O}_q(SU_{n+1})$-covariant, right $\mathcal{O}_q(U_n)$-covariant, and restricts to the Heckenberger–Kolb calculus $\Omega^1_q(\mathbb{CP}^n)$. Thus it gives us a quantum principal bundle presentation of $\Omega^1_q(\mathbb{CP}^n)$, with associated short exact sequence

$$0 \to \mathcal{O}_q(SU_{n+1})\Omega^1_q(\mathbb{CP}^n)\mathcal{O}_q(SU_{n+1}) \xrightarrow{\iota} \Omega^1_q(SU_{n+1}) \xrightarrow{\overline{\mathrm{can}}} \mathcal{O}_q(SU_{n+1}) \otimes \Lambda^1_{\mathcal{O}_q(U_n)} \to 0.$$

Since the calculus is left $\mathcal{O}_q(SU_{n+1})$-covariant, it is an $\mathcal{O}_q(SU_{n+1})$-Hopf module. A basis of $F(\Omega^1_q(SU_{n+1}))$ is given by

$$e_i^+ := [\mathrm{d}u^{i+1}_1], \qquad e^0 := [\mathrm{d}u^1_1], \qquad e_i^- := [\mathrm{d}u^1_{i+1}], \qquad \text{for } i = 1, \ldots, n.$$

Moreover, $[u^i_j] = 0$, if both $i, j \neq 1$. Let us now denote

$$\Lambda^{(1,0)} := \mathrm{span}_{\mathbb{C}}\{e_i^+ \,|\, i = 1, \ldots, n\}, \qquad \Lambda^{(0,1)} := \mathrm{span}_{\mathbb{C}}\{e_i^- \,|\, i = 1, \ldots, n\}.$$

The space $\Lambda^{(1,0)} \oplus \Lambda^{(0,1)}$ is a right $\mathcal{O}_q(SU_{n+1})$-sub-module of $\Phi_{\mathcal{O}_q(SU_{n+1})}\big(\Omega^1_q(SU_{n+1})\big)$. Explicitly, its $\mathcal{O}_q(SU_{n+1})$-sub-module structure is given by

$$e_i^{\pm} \triangleleft u^k_k = q^{\delta_{i+1,k}+\delta_{1k}-2/(n+1)} e_i^{\pm}, \qquad e_i^{\pm} \triangleleft u^k_l = 0, \quad \text{for all } k \neq l. \tag{13}$$

It is important to note that the subspace $\mathbb{C}e^0$ is *not* an $\mathcal{O}_q(SU_{n+1})$-sub-module of $F(\mathcal{O}_q(SU_{n+1}))$, nor is it even an $\mathcal{O}_q(S^{2n-1})$-sub-module. However, as shown in [45, Proposition 6.2], it *is* a sub-module over $\mathbb{C}\langle z_1\rangle$, the $*$-sub-algebra of $\mathcal{O}_q(S^{2n+1})$ generated by $z_1$.

It follows from the results of [45, §5] that

$$F\left(\mathcal{O}_q(SU_{n+1})\Omega^1_q(\mathbb{CP}^n)\mathcal{O}_q(SU_{n+1})\right) = \Lambda^{(1,0)} \oplus \Lambda^{(0,1)}.$$

Moreover, a decomposition of right $\mathcal{O}_q(U_n)$-comodules is given by

$$F\left(\Omega^1_q(SU_{n+1})\right) = \Lambda^{(1,0)} \oplus \Lambda^{(0,1)} \oplus \mathbb{C}e^0.$$

Thus we have a left $\mathcal{O}_q(SU_{n+1})$-covariant strong principal connection $\Pi$, uniquely defined by

$$F(\Pi) : F\big(\Omega^1_q(SU_{n+1})\big) \to \mathbb{C}e^0.$$

For an arbitrary covariant vector bundle $\mathcal{F}$, let us now look at the associated connection

$$\nabla : \mathcal{F} \to \Omega^1_q(\mathbb{CP}^n) \otimes_{\mathcal{O}_q(\mathbb{CP}^n)} \mathcal{F}$$

associated to $\Pi$. The linear map

$$\overline{\partial}_{\mathcal{F}} := (\mathrm{proj}_{\Omega^{(0,1)}} \otimes \mathrm{id}) \circ \nabla : \mathcal{F} \to \Omega^{(0,1)} \otimes_{\mathcal{O}_q(\mathbb{CP}^n)} \mathcal{F}$$

is a $(0,1)$-connection. Moreover, we have an analogously defined $(1,0)$-connection for $\mathcal{F}$, which we denote by $\partial_{\mathcal{F}}$. Consider next the obvious linear projections

$$\Pi^{(1,0)} : F\big(\Omega^1_q(SU_{n+1})\big) \to \Lambda^{(1,0)}, \qquad \Pi^{(0,1)} : F\big(\Omega^1_q(SU_{n+1})\big) \to \Lambda^{(0,1)}. \tag{14}$$

In terms of these operators, we have the following useful formulae:

$$\partial_{\mathcal{F}} = j^{-1} \circ ((\Pi^{(1,0)} \circ \mathrm{d}) \otimes \mathrm{id}) \circ \mathrm{U},$$
$$\overline{\partial}_{\mathcal{F}} = j^{-1} \circ ((\Pi^{(0,1)} \circ \mathrm{d}) \otimes \mathrm{id}) \circ \mathrm{U}.$$

For the special case of the covariant line bundles, it follows from the uniqueness of $(0,1)$-connections, as presented in Theorem 4.6, that $\overline{\partial}_{\mathcal{E}_k}$ is equal to the holomorphic structure of $\mathcal{E}_k$, justifying the choice of notation. We have an analogous result for the $(1,0)$-connection $\partial_{\mathcal{E}_k}$. Thus $\nabla = \partial_{\mathcal{E}_k} + \overline{\partial}_{\mathcal{E}_k}$ is equal to the Chern connection of $\mathcal{E}_k$.

5.4. **Chern Curvature of the Positive Line Bundles of $\mathcal{O}_q(\mathbb{CP}^n)$.** In this subsection we explicitly calculate the curvature of the positive line bundles over quantum projective space. We begin with the following technical lemma.

**Lemma 5.2.** *It holds that, for all $k \in \mathbb{N}$,*

$$\Pi^{(1,0)} \circ \mathrm{d}(z_1^k) = (k)_{q^{2/(n+1)}} \big(\Pi^{(1,0)} \circ \mathrm{d}(z_1)\big) z_1^{k-1}.$$

*Proof.* We will prove the formula using induction. For $k = 1$, the formula is trivially satisfied. For $k = 2$, we see that

$$\begin{aligned} \mathrm{U}\big(\Pi^{(1,0)} \circ \mathrm{d}(z_1) z_1\big) &= \mathrm{U}\big(\Pi^{(1,0)} \circ \mathrm{d}(u^1_1)\big) u^1_1 \\ &= \left(\sum_{a=2}^{n} u^1_a \otimes [\mathrm{d}(u^a_1)]\right) u^1_1 \\ &= \sum_{b=1}^{n}\sum_{a=2}^{n} u^1_a u^b_1 \otimes [\mathrm{d}(u^a_1) u^1_b]. \end{aligned}$$

Recalling the identities in (13) and the definition of $\Pi^{(1,0)}$ given in (14), we see that

$$\sum_{b=1}^{n}\sum_{a=2}^{n} u^1_a u^b_1 \otimes [\mathrm{d}(u^a_1) u^1_b] = \sum_{a=2}^{n} u^1_a u^1_1 \otimes [\mathrm{d}(u^a_1) u^1_1] = \ q^{1-\frac{2}{n+1}} \sum_{a=2}^{n} u^1_a u^1_1 \otimes [\mathrm{d}(u^a_1)].$$

The commutation relations of $\mathcal{O}_q(SU_{n+1})$ tell us that $u^1_a u^1_1 = q^{-1} u^1_1 u^1_a$ (see for example [37, §9.2] or [52, §1] for details). Thus

$$\begin{aligned} q^{1-\frac{2}{n+1}} \sum_{a=2}^{n} u^1_a u^1_1 \otimes [\mathrm{d}u^a_1] &= q^{-\frac{2}{n+1}} u^1_1 \sum_{a=2}^{n} u^1_a \otimes [\mathrm{d}u^a_1] \\ &= q^{-\frac{2}{n+1}} \mathrm{U}\left(z_1 \, \Pi^{(1,0)} \circ \mathrm{d}(z_1)\right). \end{aligned}$$

Hence we see that $z_1\big(\Pi^{(1,0)} \circ \mathrm{d}(z_1)\big) = q^{\frac{2}{n+1}} \left(\Pi^{(1,0)} \circ \mathrm{d}(z_1)\right) z_1$.

Let us now assume that the formula holds for some general $k$. By the Leibniz rule

$$\Pi^{(1,0)} \circ \mathrm{d}(z_1^{k+1}) = \Pi^{(1,0)}\big((\mathrm{d}z_1^{k-1}) z_1 + z_1^{k-1} \mathrm{d}z_1\big) \, .$$

Since $\Pi^{(1,0)}$ is a left $\mathcal{O}_q(SU_{n+1})$-module map, it must hold that

$$\Pi^{(1,0)}\big((\mathrm{d}z_1^{k-1}) z_1 + z_1^{k-1} \mathrm{d}z_1\big) = \Pi^{(1,0)}\big(\mathrm{d}z_1^{k-1} z_1\big) + z_1^{k-1} \Pi^{(1,0)}\left(\mathrm{d}z_1\right).$$

Moreover, since $\mathbb{C}e^0$ is a $\mathbb{C}\langle z_1\rangle$-sub-module of $F\big(\Omega^1_q(SU_{n+1})\big)$, the projection $\Pi^{(1,0)}$ must be a right $\mathbb{C}\langle z_1\rangle$-module map. Thus we see that

$$\Pi^{(1,0)}\big(\mathrm{d}z_1^{k-1} z_1\big) + z_1^{k-1} \Pi^{(1,0)}\left(\mathrm{d}z_1\right) = \Pi^{(1,0)}\big(\mathrm{d}z_1^{k-1}\big)\, z_1 + z_1^{k-1} \Pi^{(1,0)}\left(\mathrm{d}z_1\right).$$

Using our inductive assumption, we can reduce this expression to

$$(k-1)_{q^{2/(n+1)}} \Pi^{(1,0)}(\mathrm{d}z_1)\, z_1^{k-1} + q^{2(k-1)/(n+1)} \Pi^{(1,0)}(\mathrm{d}z_1) z_1^{k-1}.$$

By the definition of the quantum integer, this in turn reduces to

$$(k)_{q^{2/(n+1)}}\Pi^{(1,0)}(\mathrm{d}z_1)\, z_1^{k-1}.$$

The claimed formula now follows by induction. □

**Proposition 5.3.** *For any positive line bundle $\mathcal{E}_k$ over quantum projective space $\mathcal{O}_q(\mathbb{CP}^n)$, it holds that*

$$\nabla^2(e) = -(k)_{q^{-2/(n+1)}}\mathbf{i}\kappa \otimes e, \qquad \textit{for all } e \in \mathcal{E}_k,$$

*where we have chosen the unique Kähler form $\kappa$ satisfying*

$$\nabla^2(e) = -\mathbf{i}\kappa \otimes e, \qquad \textit{for all } e \in \mathcal{E}_1. \tag{15}$$

*Proof.* We are free to calculate the curvature of $\mathcal{E}_k$ by letting $\nabla^2$ act on any element of $\mathcal{E}_k$. The element $z_1^k$ presents itself as a convenient choice since, as proved in [45], it holds that $\overline{\partial}_{\mathcal{E}_k}(z_1^k) = 0$. (See [19] for the extension of this result to the general setting of irreducible quantum flag manifolds.) In particular, it holds that

$$\nabla^2(z_1^k) = (\overline{\partial}_{\mathcal{E}_k} \circ \partial_{\mathcal{E}_k} + \partial_{\mathcal{E}_k} \circ \overline{\partial}_{\mathcal{E}_k})(z_1^k) = \overline{\partial}_{\mathcal{E}_k} \circ \partial_{\mathcal{E}_k}(z_1^k).$$

Let $\alpha := q^{-2/(n+1)}$. From the quantum principal bundle presentation of $\partial_{\mathcal{E}_k}$ given in the previous subsection, together with Lemma 5.2, we see that

$$\begin{aligned}\partial_{\mathcal{E}_k}(z_1^k) &= j^{-1}\big(\Pi^{(1,0)} \circ \mathrm{d}(z_1^k) \otimes v_k\big)\\ &= (k)_\alpha\, j^{-1}\big(\big(\Pi^{(1,0)} \circ \mathrm{d}(z_1^k)\big) z_1^{k-1} \otimes v_k\big),\end{aligned}$$

where in the first identity we have used (12). We now present this expression as an element in $\Omega^{(1,0)} \otimes_B (A \Box_H \Phi(\mathcal{E}_k))$:

$$\begin{aligned}(k)_\alpha j^{-1}\big(\big(\Pi^{(1,0)} \circ \mathrm{d}(z_1)\big) z_1^{k-1} \otimes v_k\big) &= \sum_{i=1}^{n+1} (k)_\alpha j^{-1}\left(\Pi^{(1,0)} \circ \mathrm{d}(u_1^1) S(u_i^1) u_1^i z_1^{k-1} \otimes v_k\right)\\ &= \sum_{i=1}^{n+1} (k)_\alpha \partial(u_1^1 S(u_i^1)) \otimes (z_i z_1^{k-1} \otimes v_k).\end{aligned}$$

Acting on this element by $\overline{\partial}_{\mathcal{E}_k}$, and recalling that $\overline{\partial} z_i = 0$, for all $i = 1, \dots, n$, gives us the identity

$$\sum_{i=1}^{n+1} \overline{\partial}_{\mathcal{E}_k}\Big((k)_\alpha \partial(u_1^1 S(u_i^1)) \otimes (z_i z_1^{k-1} \otimes v_k)\Big) = \sum_{i=1}^{n+1} (k)_\alpha \overline{\partial}\partial(u_1^1 S(u_i^1)) \otimes (z_i z_1^{k-1} \otimes v_k).$$

Operating by $\mathrm{id} \otimes \mathrm{U}^{-1}$ produces the expression

$$(k)_\alpha \sum_{i=1}^{n+1} \overline{\partial}\partial(u_1^1 S(u_i^1)) \otimes z_i z_1^{k-1} = (k)_\alpha \nabla(u_1^1) z_1^{k-1},$$

where the right multiplication of $\nabla(u_1^1)$ by $z_1^{k-1}$ is defined with respect to the canonical embeddings of $\Omega^{(1,1)} \otimes_B \mathcal{E}_1$ and $\Omega^{(1,1)} \otimes_B \mathcal{E}_k$ into $\Omega^{(1,1)} \otimes_B \mathcal{O}_q(S^{2n-1})$. Finally, recalling that we have chosen a scaling of our Kähler form to satisfy (15), we have

$$\nabla^2(z_1^k) = (k)_\alpha(-\mathbf{i}\,\kappa \otimes z_1)z_1^{k-1} = -(k)_\alpha \mathbf{i}\kappa \otimes z_1^k,$$

which gives us the claimed identity. □

**Remark 5.4.** It is worth noting that the Chern curvature is clearly independent of any quantum principal bundle presentation of the calculus $\Omega^1_q(\mathbb{CP}^n)$. However, the quantum bundle presentation allows us to calculate the curvature in a systematic manner, and provides us with concrete insight into why the curvature undergoes a $q$-integer deformation. For those irreducible quantum flag manifolds for which a quantum bundle presentation is currently unknown, it is unclear how to perform these calculations. For the quantum Grassmannians, where a quantum principle bundle presentation is available [44], positive and negative line bundle curvature calculations will appear in subsequent work.

## Appendix A. Some Categorical Equivalences

In this appendix we present a number of categorical equivalences, all ultimately derived from Takeuchi's equivalence [56]. These equivalences play a prominent role in the paper, giving us a formal framework in which to understand covariant differential calculi as noncommutative homogeneous vector bundles.

A.1. **Takeuchi's Bimodule Equivalence.** Let $A$ and $H$ be Hopf algebras, and $B = A^{\mathrm{co}(H)}$ the quantum homogeneous space associated to a surjective Hopf algebra map $\pi : A \to H$. We define ${}^A_B\mathrm{Mod}_B$ to be the category whose objects are left $A$-comodules $\Delta_L : \mathcal{F} \to A \otimes \mathcal{F}$, endowed with a $B$-bimodule structure, such that

$$\Delta_L(bfc) = \Delta_L(b)\Delta_L(f)\Delta_L(c), \qquad \text{for all } f \in \mathcal{F}, b, c \in B, \tag{16}$$

and whose morphisms are left $A$-comodule, $B$-bimodule, maps. Let ${}^H\mathrm{Mod}$ denote the category whose objects are left $H$-comodules, and with morphisms left $H$-comodule maps.

If $\mathcal{F} \in {}^A_B\mathrm{Mod}_B$, and $B^+ := B \cap \ker(\varepsilon : A \to \mathbb{C})$, then $\mathcal{F}/(B^+\mathcal{F})$ becomes an object in ${}^H\mathrm{Mod}_B$ with the obvious right $B$-action, and left $H$-coaction given by

$$\Delta_L[f] = \pi(f_{(-1)}) \otimes [f_{(0)}], \qquad \text{for } f \in \mathcal{F}, \tag{17}$$

where $[f]$ denotes the coset of $f$ in $\mathcal{F}/(B^+\mathcal{F})$. A functor

$$\Phi : {}^A_B\mathrm{Mod}_B \to {}^H\mathrm{Mod}_B \tag{18}$$

is now defined as follows: $\Phi(\mathcal{F}) := \mathcal{F}/(B^+\mathcal{F})$, and if $g : \mathcal{F} \to \mathcal{D}$ is a morphism in ${}^A_B\mathrm{Mod}_B$, then $\Phi(g) : \Phi(\mathcal{F}) \to \Phi(\mathcal{D})$ is the map uniquely defined by $\Phi(g)[f] := [g(f)]$.

If $V \in {}^H\mathrm{Mod}_B$ with coaction $\Delta_L : V \to H \otimes V$, then the *cotensor product* of $A$ and $V$ is defined by

$$A \,\square_H V := \ker(\Delta_R \otimes \mathrm{id} - \mathrm{id} \otimes \Delta_L : A \otimes V \to A \otimes H \otimes V),$$

where $\Delta_R : A \to A \otimes H$ denotes the homogenous right $H$-coaction on $A$ (see Definition 2.15 for details). The cotensor product becomes an object in ${}^A_B\mathrm{Mod}_B$ by defining a left $B$-bimodule structure, and left $A$-comodule structure, on the first tensor factor in the obvious way, and defining a right $B$-module structure by

$$\Big(\sum a_i \otimes v_i\Big)\, b := \sum a_i b_{(1)} \otimes \big(v_i \triangleleft b_{(2)}\big),$$

for any $b \in B$, and any $\sum a_i \otimes v_i \in A \square_H V$. A functor

$$\Psi : {}^H\mathrm{Mod}_B \to {}^A_B\mathrm{Mod}_B$$

is now defined as follows: $\Psi(V) := A \square_H V$, and if $\gamma$ is a morphism in ${}^H\mathrm{Mod}_B$, then $\Psi(\gamma) := \mathrm{id} \otimes \gamma$.

For a quantum homogeneous space $B = A^{\mathrm{co}(H)}$, the algebra $A$ is said to be *faithfully flat* as a left $B$-module if the functor $A \otimes_B - : {}_B\mathrm{Mod} \to {}_{\mathbb{C}}\mathrm{Mod}$, from the category of left $B$-modules to the category of complex vector spaces, preserves and reflects exact sequences. As shown in [11, Corollary 3.4.5], for any coideal $*$-subalgebra of a CQGA faithful flatness is automatic. For example, $\mathcal{O}_q(G)$ is faithfully flat as a left module over any quantum flag manifold $\mathcal{O}_q(G/L_S)$. The following equivalence was established in [56, Theorem 1].

**Theorem A.1** (Takeuchi's Equivalence). *Let $B = A^{\mathrm{co}(H)}$ be a quantum homogeneous space such that $A$ is faithfully flat as a right $B$-module. An adjoint equivalence of categories between ${}^A_B\mathrm{Mod}_B$ and ${}^H\mathrm{Mod}_B$ is given by the functors $\Phi$ and $\Psi$ and unit, and counit, natural isomorphisms*

$$\mathrm{U} : \mathcal{F} \to \Psi \circ \Phi(\mathcal{F}), \qquad f \mapsto f_{(-1)} \otimes [f_{(0)}],$$
$$\mathrm{C} : \Phi \circ \Psi(V) \to V, \qquad \Big[\sum_i a^i \otimes v^i\Big] \mapsto \sum_i \varepsilon(a^i) v^i.$$

As observed in [46, Corollary 2.7], the inverse of the unit U of the equivalence admits a useful explicit description:

$$\mathrm{U}^{-1}\left(\sum_i f_i \otimes [g_i]\right) = \sum_i f_i S\big((g_i)_{(-1)}\big)(g_i)_{(0)}. \tag{19}$$

A.2. **The Fundamental Theorem of Two-Sided Hopf Modules.** In this subsection we consider a special case of Takeuchi's equivalence, namely the fundamental theorem of two-sided Hopf modules. (This equivalence was originally considered in [54, Theorem 5.7] using a parallel but equivalent formulation, see also [53].) For a Hopf algebra $A$, the counit $\varepsilon : A \to \mathbb{C}$ is a Hopf algebra map. The associated quantum homogeneous space is given by $A = A^{\mathrm{co}(\mathbb{C})}$, the category ${}^A_B\mathrm{Mod}_B$ specialises to

${}^A_A\mathrm{Mod}_A$, and the category ${}^H\mathrm{Mod}_B$ reduces to the category of right $A$-modules $\mathrm{Mod}_A$. In this special case we find it useful to denote the functor $\Phi$ as

$$F : {}^A_A\mathrm{Mod}_A \to \mathrm{Mod}_A, \qquad \mathcal{F} \mapsto \mathcal{F}/A^+\mathcal{F},$$

Moreover, since the cotensor product over $\mathbb{C}$ is just the usual tensor product $\otimes$, we see that the functor $\Psi$ reduces to

$$A \otimes - : \mathrm{Mod}_A \to {}^A_A\mathrm{Mod}_A, \qquad V \mapsto A \otimes V.$$

Since faithful flatness is trivially satisfied in this case, we have the following corollary of Takeuchi's equivalence.

**Theorem A.2** (Fundamental Theorem of Two-Sided Hopf Modules)**.** *An adjoint equivalence of categories between* ${}^A_A\mathrm{Mod}_A$ *and* $\mathrm{Mod}_A$ *is given by the functors* $F$ *and* $A \otimes -$*, and the unit, and counit, natural isomorphisms*

$$\mathrm{U} : \mathcal{F} \to A \otimes F(\mathcal{F}), \qquad f \mapsto f_{(-1)} \otimes [f_{(0)}],$$
$$\mathrm{C} : F(A \otimes V) \to V, \qquad [a \otimes v] \mapsto \varepsilon(a)v.$$

A.3. **Some Monoidal Equivalences.** In this subsection we recall two monoidal equivalences induced by Takeuchi's equivalence. Denote by ${}^A_B\mathrm{Mod}_0$ the full sub-category of ${}^A_B\mathrm{Mod}_B$ whose objects $\mathcal{F}$ satisfy the identity $\mathcal{F}B^+ = B^+\mathcal{F}$. Consider also the full sub-category of ${}^H\mathrm{Mod}_\mathrm{B}$ consisting of those objects endowed with the trivial right $B$-action, which is to say, those objects $V$ for which $v \triangleleft b = \varepsilon(b)v$, for all $v \in V$, and $b \in B$. This category is clearly isomorphic to ${}^H\mathrm{Mod}$, the category of left $H$-comodules, and as such, Takeuchi's equivalence induces an equivalence between ${}^A_B\mathrm{Mod}_0$ and ${}^H\mathrm{Mod}$, for details see [46, Lemma 2.8].

For $\mathcal{F}, \mathcal{D}$ two objects in ${}^A_B\mathrm{Mod}_0$, we denote by $\mathcal{F} \otimes_B \mathcal{D}$ the usual bimodule tensor product endowed with the standard left $A$-comodule structure. It is easily checked that $\mathcal{F} \otimes_B \mathcal{D}$ is again an object in ${}^A_B\mathrm{Mod}_0$, and so the tensor product $\otimes_B$ gives the category a monoidal structure. With respect to the usual tensor product of comodules in ${}^H\mathrm{Mod}$, Takeuchi's equivalence is given the structure of a monoidal equivalence (see [46, §4] for details) by the morphisms

$$\mu_{\mathcal{F},\mathcal{D}} : \Phi(\mathcal{F}) \otimes \Phi(\mathcal{D}) \to \Phi(\mathcal{F} \otimes_B \mathcal{D}), \quad [f] \otimes [d] \mapsto [f \otimes d], \qquad \text{for } \mathcal{F}, \mathcal{D} \in {}^A_B\mathrm{Mod}_0.$$

This monoidal equivalence will be tacitly assumed throughout the paper, along with the implied monoid structure on $\Phi(\mathcal{N})$, for any monoid object $\mathcal{N} \in {}^A_B\mathrm{Mod}_0$.

Consider now the category ${}^A_B\mathrm{Mod}$, whose objects are left $A$-comodules, and left $B$-modules, satisfying the obvious analogue of (16), and whose morphisms are left $A$-comodule, right $B$-module maps. We can endow any object $\mathcal{F} \in {}^A_B\mathrm{Mod}$ with a right $B$-action uniquely defined by

$$f \triangleleft b := f_{(-2)} b S(f_{(-1)}) f_{(0)}.$$

Since $e_{(-2)} b S(e_{(-1)}) e_{(0)} \in B^+\mathcal{F}$, for all $b \in B^+$, this new right module structure satisfies the defining conditions of ${}^A_B\mathrm{Mod}_0$, giving us an obvious equivalence between

${}^{A}_{B}\mathrm{Mod}$ and ${}^{A}_{B}\mathrm{Mod}_0$. In particular, we see that any left $A$-comodule, left $B$-module map between two objects in ${}^{A}_{B}\mathrm{Mod}_0$ is automatically a morphism. (We should note that the implied equivalence between ${}^{A}_{B}\mathrm{Mod}$ and ${}^{H}\mathrm{Mod}$ is the original form of Takeuchi's equivalence [56], the bimodule form presented above being an easy consequence.)

Next we examine ${}^{A}_{B}\mathrm{mod}_B$ the full sub-category of ${}^{A}_{B}\mathrm{Mod}_B$ whose objects $\mathcal{F}$ are finitely generated as left $B$-modules and ${}^{H}\mathrm{mod}_B$, the full sub-category of ${}^{H}\mathrm{Mod}_{\mathrm{B}}$ whose objects are finite-dimensional as complex vector spaces. As established in [47, Corollary 2.5] Takeuchi's equivalence induces an equivalence between these two sub-categories. We define the *dimension* of an object $\mathcal{F} \in {}^{A}_{B}\mathrm{mod}_B$ to be the vector space dimension of $\Phi(\mathcal{F})$.

Finally, consider ${}^{A}_{B}\mathrm{mod}_0$, the full sub-category of ${}^{A}_{B}\mathrm{Mod}_0$ whose objects are also objects in ${}^{A}_{B}\mathrm{mod}_B$. By the above, we see that Takeuchi's equivalence induces an equivalence between ${}^{A}_{B}\mathrm{mod}_0$ and ${}^{H}\mathrm{mod}$, the category of finite-dimensional left $H$-comodules. Both categories are easily seen to be closed under tensor products, and so both have monoidal structures. Moreover, the monoidal equivalence between ${}^{A}_{B}\mathrm{Mod}_0$ and ${}^{H}\mathrm{Mod}$ restricts to a monoidal equivalence between ${}^{A}_{B}\mathrm{mod}_0$ and ${}^{H}\mathrm{mod}$.

## Appendix B. Tables for the Irreducible Quantum Flag Manifolds

We recall the standard pictorial description of the quantum Levi subalgebras defining the irreducible quantum flag manifolds, given in terms of Dynkin diagrams. For a diagram of rank $r$ as in Table 1, let $G$ denote the corresponding compact, connected, simply-connected, simple Lie group. Each node denotes a simple root, and to the black node $\alpha_s$ we associate the set $S := \{1, \dots, n\} \backslash \{s\}$, with corresponding Levi subgroup $L_S$. The irreducible quantum flag manifold is then given by the coinvariant subspace $\mathcal{O}_q(G/L_S) \subseteq \mathcal{O}_q(G)$. Note that any automorphism of a Dynkin diagram results in an isomorphic quantum flag manifold, which is not denoted in the diagram. In particular, for the case of $D_n$ and $E_6$, colouring the second spinor node, and the first node, respectively produces an isomorphic copy of the corresponding quantum flag manifold.

We present an explicit description of the canonical line bundles of the irreducible quantum flag manifolds using the approach of Theorem 4.9. All line bundles are indexed by the negative integers, and hence, are negative in the sense of Definition 2.11. The values coincide with their classical counterparts, see for example, [32, § II.4]. This allows us to conclude in Theorem 4.9 that the Kähler structure of each irreducible quantum flag manifold is of Fano type.

**Remark B.1.** By a theorem of Atiyah, a $2m$-dimensional compact Hermitian manifold is spin if and only if its canonical line bundle $\Omega^{(m,0)}$ admits a holomorphic

TABLE 1. Irreducible Quantum Flag Manifolds: organised by series, with defining crossed node numbered according to [29, §11.4], CQGA-homogeneous space symbol and name

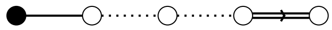
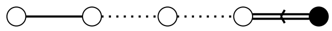
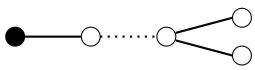
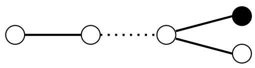
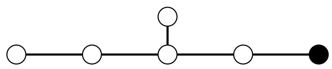
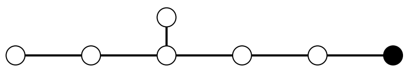
| | | | |
|---|---|---|---|
| $A_n$ |  | $\mathcal{O}_q(\mathrm{Gr}_{s,n+1})$ | quantum Grassmannian |
| $B_n$ |  | $\mathcal{O}_q(\mathbf{Q}_{2n+1})$ | odd quantum quadric |
| $C_n$ |  | $\mathcal{O}_q(\mathbf{L}_n)$ | quantum Lagrangian Grassmannian |
| $D_n$ |  | $\mathcal{O}_q(\mathbf{Q}_{2n})$ | even quantum quadric |
| $D_n$ |  | $\mathcal{O}_q(\mathbf{S}_n)$ | quantum spinor variety |
| $E_6$ |  | $\mathcal{O}_q(\mathbb{OP}^2)$ | quantum Caley plane |
| $E_7$ |  | $\mathcal{O}_q(\mathrm{F})$ | quantum Freudenthal variety |

TABLE 2. Irreducible Quantum Flag Manifolds: CQGA-homogeneous space symbol, corresponding Heckenberger–Kolb calculus complex dimension, and the space of top holomorphic forms identified as a line bundle

| $\mathcal{O}_q(G/L_S)$ | $M := \dim\big(\Omega^{(1,0)}\big)$ | Canonical line bundle $\Omega^{(M,0)}$ |
|---|---|---|
| $\mathcal{O}_q(\mathrm{Gr}_{s,n+1})$ | $s(n-s+1)$ | $\mathcal{E}_{-(n+1)}$ |
| $\mathcal{O}_q(\mathbf{Q}_{2n+1})$ | $2n-1$ | $\mathcal{E}_{-2n+1}$ |
| $\mathcal{O}_q(\mathbf{L}_n)$ | $\frac{n(n+1)}{2}$ | $\mathcal{E}_{-(n+1)}$ |
| $\mathcal{O}_q(\mathbf{Q}_{2n})$ | $2(n-1)$ | $\mathcal{E}_{-2(n-1)}$ |
| $\mathcal{O}_q(\mathbf{S}_n)$ | $\frac{n(n-1)}{2}$ | $\mathcal{E}_{-2(n-1)}$ |
| $\mathcal{O}_q(\mathbb{OP}^2)$ | 16 | $\mathcal{E}_{-12}$ |
| $\mathcal{O}_q(\mathrm{F})$ | 27 | $\mathcal{E}_{-18}$ |

square root [2, Proposition 3.2]. Thus from Table 2 we see that the classical Grassmannians $\mathrm{Gr}_{s,n+1}$, and the classical Lagrangian Grassmannians $\mathbf{L}_n$, are spin for all

$n \in 2\mathbb{N}+1$. Moreover, the even quadrics $\mathbf{Q}_{2n}$, and the spinor varieties $\mathbf{S}_n$, are spin, for all $n \in \mathbb{N}$. For the exceptional cases, both the Caley plane and the Freudenthal variety are spin. Atiyah's theorem suggests a definition for noncommutative Hermitian spin structures with a substantial ready-made family of noncommutative examples. Whether or not this will prove a fruitful idea remains to be seen, although initial investigation suggests that noncommutative Hermitian spin structures are closely related to Connes' notion of reality for spectral triples [12].

## Appendix C. Quantum Integers

Quantum integers are ubiquitous in the study of quantum groups. For this paper in particular, they arise in the defining relations of the Drinfeld–Jimbo quantum groups, and in the calculation of the curvature of the positive line bundles over quantum projective space $\mathcal{O}_q(\mathbb{CP}^n)$. In each case we use different but related formulations for quantum integers. Thus we take care here to clarify our choice of conventions.

We begin with the version of quantum integer used in the definition of the Drinfeld–Jimbo quantum groups. For $q \in \mathbb{C}$, the *quantum integer* $[m]_q$ is the complex number

$$[m]_q := q^{-m+1} + q^{-m+3} + \cdots + q^{m-3} + q^{m-1}.$$

Note that when $q \notin \{-1, 0, 1\}$, we have the identity

$$[m]_q = \frac{q^m - q^{-m}}{q - q^{-1}}.$$

We next recall the definition of the quantum binomials, which arise in the quantum Serre relations of the Drinfeld–Jimbo quantum groups. For any $n \in \mathbb{N}$, we denote

$$[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q,$$

and moreover, we denote $[0]_q! = 1$. For any non-zero $q \in \mathbb{C}$, and any $n, r \in \mathbb{N}_0$, the associated $q$-binomial coefficient is the complex number

$$\begin{bmatrix} n \\ r \end{bmatrix}_q := \frac{[n]_q!}{[r]_q!\,[n-r]_q!}.$$

By contrast, the form of quantum integer arising in curvature calculations is defined as follows: For $q \in \mathbb{C} \setminus \{1\}$, the *quantum integer* $(m)_q$ is the complex number

$$(m)_q = \frac{1 - q^m}{1 - q}.$$

When $m > 0$, we have

$$(m)_q := 1 + q + q^2 + \cdots + q^{m-1}.$$

The definition of quantum binomial also makes sense for this version of quantum integer, although we will not use it in this paper. Finally, it is instructive to note that the two conventions are related by the identity

$$[m]_q = q^{1-m}(m)_{q^2}.$$

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