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Modular Forms

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Abstract. The theory of Modular Forms has been central in mathematics with a rich history and connections to many other areas of mathematics. The workshop explored recent developments and future directions with a particular focus on connections to the theory of periods.

Mathematics Subject Classification (2010): 11xx, 14xx.

Introduction by the Organizers

The workshop *Modular Forms*, organized by Jan Hendrik Bruinier (Darmstadt), Atsushi Ichino (Kyoto), Tamotsu Ikeda (Kyoto) and Özlem Imamoglu (Zürich) consisted of 14 one-hour long lectures and 8 half-hour long lectures. It covered various recent developments in the theory of modular and automorphic forms and related fields.

A particular focus was on the connection of modular forms to periods, since there have been important developments in that direction in recent years. In this context, the topics that the workshop addressed include the global Gross-Prasad conjecture and its analogs, the theory of liftings and their applications to period relations, as well special cycles on Shimura varieties and singular moduli with a view towards the Kudla program.

A period is a complex number which is usually transcendental but has important arithmetic properties, and its rigorous definition was given by Kontsevich and Zagier. For example, $\pi = 3.14159265\ldots$ is a period and appears in many formulas and conjectures in number theory, for instance in Euler's celebrated formula for special values of the Riemann zeta function. A slightly more sophisticated example is a CM period, which was extensively studied by Shimura and is defined as an integral over a 1-cycle of a holomorphic differential 1-form of an abelian variety with complex multiplication.

These periods play a central role in number theory, especially in connection with L-functions. Namely, according to the conjectures of Deligne, Beilinson-Bloch, and Bloch-Kato, special values of motivic L-functions at integral arguments should have periods as their transcendental parts and encode important arithmetic information such as ranks of Chow groups. A prototype of these conjectures is the Birch and Swinnerton-Dyer conjecture relating arithmetic invariants of an elliptic curve over $\mathbb{Q}$ to the leading term of its Hasse-Weil L-function at the central critical point. Also, periods and their connection with L-functions have incarnations in the context of modular forms, which have been developed significantly in recent years and give further insights in number theory.

Many of the lectures discussed periods associated with automorphic forms and their relations to L-functions, in particular the lectures by Claudia Alfes-Neumann, Raphaël Beuzart-Plessis, Yuanqing Cai, Tomoyoshi Ibukiyama, Hidenori Katsurada, Chao Li, Don Zagier. Special cycles on Shimura varieties and connections to periods were addressed in the talks by Kathrin Bringmann, Henri Darmon, Stephan Ehlen, Stephen Kudla, Yingkun Li, Michalis Neururer, Siddarth Sankaran. The lectures of Fabrizio Andreatta and Shunsuke Yamana dealt with p-adic L-functions and periods. Aspects of the analytic theory of automorphic forms and L-functions (mainly for higher rank groups) played an important role in the talks by Valentin Blomer, Gaëtan Chenevier, Gerard van der Geer, Paul Nelson, Aaron Pollack, and Ren-He Su.

In total, 55 researchers participated in the workshop. Out of these, 41 came from 15 countries different from Germany. Beyond the talks, the participants enjoyed ample time for discussions and collaborative research activities. The traditional hike on Wednesday afternoon led us to the Ochsenwirtshof in Schapbach. A further highlight was a piano recital on Thursday evening by Valentin Blomer.

The organizers and participants of the workshop thank the *Mathematisches Forschungsinstitut Oberwolfach* for hosting the workshop and providing such an ideal working environment.

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Workshop: Modular Forms

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Abstracts

On the dimension of spaces of Siegel modular forms for $\mathbf{Sp}_{2g}(\mathbb{Z})$

Gaëtan Chenevier
(joint work with O. Taïbi)

Let $g \geq 1$ be an integer and let $\underline{k} = (k_1, \ldots, k_g)$ be in $\mathbb{Z}^g$ with $k_1 \geq k_2 \geq \cdots \geq k_g$. In this work, we give a new and arguably "effortless" method leading to an exact formula for the dimension of the space $\mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ of vector-valued Siegel modular cuspforms of weight $\underline{k}$ for the full Siegel modular group $\mathrm{Sp}_{2g}(\mathbb{Z})$, under the assumptions $k_g > g$ and $g \leq 8$. This formula generalizes several classical results of Igusa, Tsushima ($g = 2$), Tsuyumine ($g = 3$, scalar valued case), and recover in particular the recent results of Taibi's thesis ($g \leq 7$) [Tai17].

As in [Tai17], our proof amounts to explicitly compute the geometric side $\mathrm{T}_{\mathrm{geom}}(G; \underline{k})$ of the version of Arthur's trace formula given in [Art89], in the case $G = \mathrm{Sp}_{2g}$ and for the trivial Hecke operator (the characteristic function of $G(\widehat{\mathbb{Z}})$) in weight $\underline{k}$. By induction on g, the main unknown part of this geometric side is the *elliptic part* $\mathrm{T}_{\mathrm{ell}}(G; \underline{k})$. The expression of the latter involves a number of delicate volumes (*local densities*), as well as the local orbital integrals of the characteristic functions of the $G(\mathbb{Z}_p)$ at each finite order element in $G(\mathbb{Q})$. The computation of these orbital integrals is quite difficult, and is mostly done case by case in [Tai17] up to $g = 7$, using a number of specific algorithms. In this work we proceed differently and ignore these difficulties by simply writing

$$\mathrm{T}_{\mathrm{ell}}(G; \underline{k}) = \sum_{c \in \mathrm{C}(G)} \mathrm{m}_c \ \mathrm{Trace}(c \,|\, \mathrm{V}_{\underline{k}}). \tag{1}$$

Here $\mathrm{C}(G)$ is the set of $G(\mathbb{C})$-conjugacy classes of finite order elements in $G(\mathbb{Q})$, and $V_{\underline{k}}$ is the finite dimensional algebraic representation of $G(\mathbb{C})$ with highest weight $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_g$ with $\lambda_i = k_i - (g+1)$ (we assume $k_g > g$). The set $\mathrm{C}(G)$ is finite, any class $c \in \mathrm{C}(G)$ being uniquely determined by its characteristic polynomial (a product of cyclotomic polynomials of total degree $2g$). The unknown quantity m_c is a rational number that we call the *mass* of c.

In practice, we evaluate the terms $\mathrm{Trace}(c \,|\, \mathrm{V}_{\underline{k}})$ using classical formulas of Weyl and of Koike-Terrada. Also, the full geometric side $\mathrm{T}_{\mathrm{geom}}(G; \underline{k})$ is the sum of $\mathrm{T}_{\mathrm{ell}}(G; \underline{k})$ and of the $\mathrm{T}_{\mathrm{ell}}(L; \underline{k}')$, where L runs over the proper and standard Levi subgroups of G of the form $\mathrm{Sp}_{2g'} \times \mathrm{GL}_1^a \times \mathrm{GL}_2^b$, with suitable $\underline{k}'$: see [Tai17] for the precise recipe. By induction on g, the main unknowns in $\mathrm{T}_{\mathrm{geom}}(G; \underline{k})$ are thus the masses of G. One way to state our main theorem is as follows.

Theorem: [CT19a] *Assume G is Sp_{2g} with $1 \leq g \leq 8$ or the split SO_n over $\mathbb{Z}$ with $1 \leq n \leq 17$, then the masses m_c for $c \in \mathrm{C}(G)$ are those given in* [CT19b].

Our strategy to compute the masses m_c is as follows. Arthur's trace formula writes

$$\mathrm{T}_{\mathrm{spec}}(G; \underline{k}) = \mathrm{T}_{\mathrm{geom}}(G; \underline{k}). \tag{2}$$

The spectral (left-hand) side is difficult to understand: it is the sum, over all discrete automorphic representations π of Sp_{2g} over $\mathbb{Q}$ with $\pi_p^{\mathrm{Sp}_{2g}(\mathbb{Z}_p)} \neq 0$ for each prime p, of the Euler-Poincaré characteristic of the $(\mathfrak{g}, K)$-cohomology of $\pi_\infty \otimes \mathrm{V}_{\underline{k}}^\vee$:

$$\mathrm{EP}(\pi_\infty) = \sum_{i \geq 0} (-1)^i \mathrm{H}^i(\mathfrak{g}, K; \pi_\infty \otimes \mathrm{V}_{\underline{k}}^\vee) \in \mathbb{Z}.$$

In order to study $\mathrm{T}_{\mathrm{spec}}(G; \underline{k})$ we use in a crucial way the endoscopic classification of the discrete automorphic representations of G proved by Arthur in [Art13] (see also [MW16]), as well as the description of the archimedean Arthur packets containing representations with nonzero $(\mathfrak{g}, K)$-cohomology, by Arancibia-Moeglin-Renard [AMR18] and Adams-Johnson [AJ87]. We refer to [CR15, Chap. 3 & 9], [CL19, Chap. 8] and [Tai17, Sect. 4] for the concrete form of Arthur's multiplicity formula in this context (*classification of all level* 1 *endoscopic lifts*).

Define $\mathrm{N}(\underline{k})$ as the number of selfdual cuspidal automorphic representations ϖ of GL_{2g+1} over $\mathbb{Q}$ such that ϖ_p is unramified for each prime p, and such that the eigenvalues of the infinitesimal character of ϖ_∞ (viewed as a semisimple conjugacy class in $\mathrm{M}_{2g+1}(\mathbb{C})$) are the distinct integers 0 and $\pm(k_i - i)$ for $i = 1, \ldots, g$. The aforementioned endoscopic classification allows us to write:

$$\mathrm{T}_{\mathrm{spec}}(\mathrm{Sp}_{2g}; \underline{k}) = 2^g (-1)^{\frac{g(g+1)}{2}} \mathrm{N}(\underline{k}) + \mathrm{T}_{\mathrm{endo}}(\mathrm{Sp}_{2g}; \underline{k}), \tag{3}$$

$$\dim \mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z})) = \mathrm{N}(\underline{k}) + \dim \mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\mathrm{endo}}, \tag{4}$$

where both terms $\mathrm{T}_{\mathrm{endo}}(\mathrm{Sp}_{2g}; \underline{k})$ and $\dim \mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))_{\mathrm{endo}}$ are known[1] by induction, provided we work in the induction not only with all the $\mathrm{Sp}_{2g'}$ with $g' < g$ but also with all split special orthogonal groups SO_n over $\mathbb{Q}$ with $n \leq 2g$. So in this big induction, and for a given $\underline{k}$, it is equivalent to know $\mathrm{T}_{\mathrm{spec}}(G; \underline{k})$, $\dim \mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ and $\mathrm{N}(\underline{k})$. The last key ingredient of this work is the following lemma.

Key Lemma: *For the several thousands of pairs $(g, \underline{k})$ with $\underline{k} \in \mathbb{Z}^g$ given in* [CT19b]*, we have* $\mathrm{N}(\underline{k}) = 0$.

Assuming this lemma we argue as follows. For a given g, each $\underline{k}$ with $\mathrm{N}(\underline{k}) = 0$ given by the lemma allows us to compute $\mathrm{T}_{\mathrm{spec}}(\mathrm{Sp}_{2g}; \underline{k})$ by (3) and the big induction. For such a $\underline{k}$ we thus know $\mathrm{T}_{\mathrm{geom}}(\mathrm{Sp}_{2g}; \underline{k}) = \mathrm{T}_{\mathrm{spec}}(\mathrm{Sp}_{2g}; \underline{k})$ by (2) hence $\mathrm{T}_{\mathrm{ell}}(\mathrm{Sp}_{2g}; \underline{k})$, as non elliptic geometric terms are known by induction. So for every $\underline{k}$ such that $\mathrm{N}(\underline{k}) = 0$ we obtain by (1) an explicit linear relation among the masses m_c for $c \in \mathrm{C}(G)$. It is rather easy to determine the set $\mathrm{C}(G)$. If $\sim$ denotes the equivalence relation on $\mathrm{C}(G)$ generated by $c \mapsto -c$ (we have $\mathrm{m}_c = \mathrm{m}_{-c}$), the cardinality of $\mathrm{C}(G)/\sim$ for $g = 1, 2, \ldots, 9$ is respectively

$$3, \ 12, \ 32, \ 92, \ 219, \ 530, \ 1158, \ 2521, \ 5149.$$

[1] Morally speaking, the terms with a subscript "endo" correspond to the automorphic representations (in the case $\mathrm{T}_{\mathrm{endo}}(\mathrm{Sp}_{2g}; \underline{k})$), or to the Siegel cuspidal eigenforms (in the case $\mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))$), whose associated $2g + 1$ dimensional ℓ-adic Galois representations should be reducible.

Miraculously, for each $g \leq 7$ the lemma gives us enough linear relations between the masses m_c with $c \in \mathrm{C}(\mathrm{Sp}_{2g})$ so as to invert the linear system. Of course, once the masses are known, we can then compute $\mathrm{T}_{\mathrm{geom}}(\mathrm{Sp}_{2g}; \underline{k})$, $\mathrm{N}(\underline{k})$ and $\dim \mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for each $\underline{k}$ with $k_g > g$, using (1), (2), (3) and (4). The limit of the method is the case $g = 8$, for which we do not obtain enough linear relations but can still conclude by explicitly computing some m_c with the algorithms of [Tai17]. Tables for the values $\mathrm{N}(\underline{k})$ and $\dim \mathrm{S}_{\underline{k}}(\mathrm{Sp}_{2g}(\mathbb{Z}))$ for $g \leq 8$ are given in [CT19b]. We deal with the case $G = \mathrm{SO}_n$ similarly (note we have to do it for the induction).

The last step is to explain the proof of the lemma. We use for this a method based on the Riemann-Weil explicit formula, pursuing ideas of Stark, Serre, Odlyzko, Mestre, Duke-Immamoglu, Miller and Chenevier-Lannes. The basic idea is to show that certain cuspidal automorphic representations of GL_m do not exist by showing that the explicit formula for their Rankin-Selberg L-function cannot be satisfied for well-chosen test functions: see [CT19a, Sect. 2 & 3] for the details and for new developments of this method, improving [CL19, Sect. 9.3].

An important part of the talk consisted in working out the details of this strategy in the much simpler —but illuminating— case $G = \mathrm{PGL}_2$, recovering the classical formula for $\dim \mathrm{S}_k(\mathrm{SL}_2(\mathbb{Z}))$.

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On the Kudla–Rapoport conjecture

Chao Li
(joint work with Wei Zhang)

The classical *Siegel–Weil formula* relates certain Siegel Eisenstein series with the arithmetic of quadratic forms, namely expressing special *values* of these series as theta functions — generating series of representation numbers of quadratic forms. Kudla initiated an influential program to establish the *arithmetic Siegel–Weil formula* relating certain Siegel Eisenstein series with objects in arithmetic geometry, which among others, aims to express the *central derivative* of these series as the arithmetic analogue of theta functions — generating series of arithmetic intersection numbers of n special divisors on Shimura varieties associated to $\mathrm{SO}(n-1,2)$ or $\mathrm{U}(n-1,1)$. These special divisors include Heegner points on modular or Shimura curves appearing in the Gross–Zagier formula ($n=2$), Hirzebruch–Zagier cycles on Hilbert modular surfaces and modular correspondence on the product of two modular curves in Gross–Keating and Gross–Kudla–Zagier ($n=3$).

Kudla–Rapoport made the nonarchimedean part of the conjectural arithmetic Siegel–Weil formula more precise by defining arithmetic models of the special cycles (for any n in the unitary case), now known as *Kudla–Rapoport (KR) cycles*. They formulated the *global KR conjecture* ([2, Conjecture 11.10]) for the nonsingular part of the formula. KR also explained how it would follow (at least at an unramified place) from the *local KR conjecture* ([1, Conjecture 1.3]) (see below). They further proved the conjectures in the special case when the arithmetic intersection is *non-degenerate* (i.e., of the expected dimension 0). Outside the non-degenerate case, the only known result was due to Terstiege, who proved the KR conjectures for $n=3$. Analogous results were known in the orthogonal case by the work of KR, Bruinier–Yang (non-degenerate case) and Terstiege ($n=3$).

The main result of our recent work [3] is the following theorem.

Theorem (L.-Zhang). *The Kudla–Rapoport conjectures hold for any n.*

Combining with the archimedean formulas of Liu, Garcia–Sankaran, Bruinier–Yang, we deduce cases of the arithmetic Siegel–Weil formula for unitary Shimura varieties in any dimension ([3, Theorem 1.3.2]).

The arithmetic Siegel–Weil formula (together with the doubling method) has important application to the *arithmetic inner product formula*, relating the central derivative of the standard L-function of cuspidal automorphic representations on orthogonal or unitary groups to the height pairing of certain cycles on Shimura varieties constructed from arithmetic theta liftings. It can be viewed as a higher dimensional generalization of the Gross–Zagier formula, and an arithmetic analogue of the Rallis inner product formula. Consequently, our theorem also has applications to the *Beilinson–Bloch–Kato conjectures* (which generalize the celebrated Birch and Swinnerton-Dyer conjecture) for Shimura varieties of any dimension.

To discuss our proof strategy, let us formulate the local KR conjecture more precisely. Let p be a prime. Let F_0 be a finite extension of $\mathbb{Q}_p$ with residue field $k=\mathbb{F}_q$ and a uniformizer ϖ. Let F be an unramified quadratic extension of F_0.

Let $\breve{F}$ be the completion of the maximal unramified extension of F. For any integer $n \geq 1$, the *unitary Rapoport–Zink space* $\mathcal{N} = \mathcal{N}_n$ is the formal scheme over $S = \mathrm{Spf} O_{\breve{F}}$ parameterizing hermitian formal O_F-modules of signature $(1, n-1)$ within the supersingular quasi-isogeny class. Let $\mathbb{E}$ and $\mathbb{X}$ be the framing hermitian O_F-module of signature $(1,0)$ and $(1, n-1)$ over $\bar{k}$. The space of *quasi-homomorphisms* $\mathbb{V} = \mathbb{V}_n := \mathrm{Hom}^\circ_{O_F}(\overline{\mathbb{E}}, \mathbb{X})$ carries a natural F/F_0-hermitian form, which makes $\mathbb{V}$ the unique (up to isomorphism) nondegenerate non-split F/F_0-hermitian space of dimension n. For any subset $L \subseteq \mathbb{V}$, the local *KR cycle* $\mathcal{Z}(L)$ is a closed formal subscheme of $\mathcal{N}$, over which each quasi-homomorphism $x \in L$ deforms to homomorphisms.

Let $L \subseteq \mathbb{V}$ be an O_F-lattice. We now associate to L two integers: the *arithmetic intersection number* $\mathrm{Int}(L)$ and the *derivative of the local density* $\partial\mathrm{Den}(L)$.

Let $x_1, \ldots, x_n$ be an O_F-basis of L. Define the *arithmetic intersection number*

$$\mathrm{Int}(L) := \chi(\mathcal{N}, \mathcal{O}_{\mathcal{Z}(x_1)} \otimes^{\mathbb{L}} \cdots \otimes^{\mathbb{L}} \mathcal{O}_{\mathcal{Z}(x_n)}), \tag{1}$$

where $\mathcal{O}_{\mathcal{Z}(x_i)}$ denotes the structure sheaf of the KR divisor $\mathcal{Z}(x_i)$, $\otimes^{\mathbb{L}}$ denotes the derived tensor product of coherent sheaves on $\mathcal{N}$, and χ denotes the Euler–Poincaré characteristic.

For M another hermitian O_F-lattice (of arbitrary rank), define $\mathrm{Rep} = \mathrm{Rep}_{M,L}$ to be the *scheme of integral representations of M by L*, an O_{F_0}-scheme such that for any O_{F_0}-algebra R, $\mathrm{Rep}(R) = \mathrm{Herm}(L \otimes_{O_{F_0}} R, M \otimes_{O_{F_0}} R)$, where Herm denotes the group of hermitian module homomorphisms. The *local density* of integral representations of M by L is defined to be

$$\mathrm{Den}(M, L) := \lim_{N \to +\infty} \frac{\#\mathrm{Rep}(O_{F_0}/\varpi^N)}{q^{N \cdot \dim \mathrm{Rep}_{F_0}}}.$$

Let $\langle 1 \rangle^k$ be the self-dual hermitian O_F-lattice of rank k with hermitian form given by the identity matrix $\mathbf{1}_k$. Then $\mathrm{Den}(\langle 1 \rangle^k, L)$ is a polynomial in $(-q)^{-k}$ with $\mathbb{Q}$-coefficients. Define the (normalized) *local Siegel series* of L to be the polynomial $\mathrm{Den}(X, L) \in \mathbb{Z}[X]$ such that

$$\mathrm{Den}((-q)^{-k}, L) = \frac{\mathrm{Den}(\langle 1 \rangle^{n+k}, L)}{\mathrm{Den}(\langle 1 \rangle^{n+k}, \langle 1 \rangle^n)}.$$

It satisfies a functional equation relating $X \leftrightarrow \frac{1}{X}$ and we define the *derivative of the local density*

$$\partial\mathrm{Den}(L) := -\frac{d}{dX}\bigg|_{X=1} \mathrm{Den}(X, L).$$

The local conjecture asserts an exact identity between the two integers just defined.

Theorem (Local KR). *Let $L \subseteq \mathbb{V}$ be an O_F-lattice of full rank n. Then*

$$\mathrm{Int}(L) = \partial\mathrm{Den}(L).$$

The previously known special cases of the local KR conjecture are proved via explicit computation of both the geometric and analytic sides. Explicit computation seems infeasible for the general case. Our proof instead proceeds via induction on n using the *uncertainty principle*, a standard tool from local harmonic analysis. Even for $n = 2, 3$, our proof is different from the previous proofs.

More precisely, for a fixed O_F-lattice $L^\flat \subseteq \mathbb{V} = \mathbb{V}_n$ of rank $n-1$, consider functions on $x \in \mathbb{V} \setminus L^\flat_F$,

$$\mathrm{Int}_{L^\flat}(x) := \mathrm{Int}(L^\flat + \langle x\rangle), \quad \partial\mathrm{Den}_{L^\flat}(x) := \partial\mathrm{Den}(L^\flat + \langle x\rangle).$$

Then it remains to show the equality of the two functions $\mathrm{Int}_{L^\flat} = \partial\mathrm{Den}_{L^\flat}$. Both functions vanish when x is non-integral, i.e., $\mathrm{val}(x) < 0$. Here $\mathrm{val}(x)$ denotes the valuation of the norm of x. By utilizing the inductive structure of the Rapoport–Zink spaces and local densities, it is not hard to see that if $x \perp L^\flat$ with $\mathrm{val}(x) = 0$, then

$$\mathrm{Int}_{L^\flat}(x) = \mathrm{Int}(L^\flat), \quad \partial\mathrm{Den}_{L^\flat}(x) = \partial\mathrm{Den}(L^\flat)$$

for the lattice $L^\flat \subseteq \mathbb{V}_{n-1} \cong \langle x\rangle_F^\perp$ of full rank $n-1$. By induction on n, the difference function $\phi = \mathrm{Int}_{L^\flat} - \partial\mathrm{Den}_{L^\flat}$ vanishes on $\{x \in \mathbb{V} : x \perp L^\flat, \mathrm{val}(x) \leq 0\}$. We would like to deduce that ϕ indeed vanishes identically.

The uncertainty principle asserts that if $\phi \in C_c^\infty(\mathbb{V})$ satisfies that both ϕ and its Fourier transform $\hat{\phi}$ vanish on $\{x \in \mathbb{V} : \mathrm{val}(x) \leq 0\}$, the $\phi = 0$. In other words, $\phi, \hat{\phi}$ cannot simultaneously have "small support" unless $\phi = 0$. We can then finish the proof by applying the uncertainty principle to $\phi = \mathrm{Int}_{L^\flat} - \partial\mathrm{Den}_{L^\flat}$, if we can control the support of both terms. However, both functions have singularities along the hyperplane $L^\flat_F \subseteq \mathbb{V}$, which cause trouble in computing their Fourier transforms or even in showing that $\phi \in C_c^\infty(\mathbb{V})$.

To overcome this difficulty, we isolate the singularities by decomposing

$$\mathrm{Int}_{L^\flat} = \mathrm{Int}_{L^\flat,\mathsf{H}} + \mathrm{Int}_{L^\flat,\mathsf{V}}, \quad \partial\mathrm{Den}_{L^\flat} = \partial\mathrm{Den}_{L^\flat,\mathsf{H}} + \partial\mathrm{Den}_{L^\flat,\mathsf{V}}$$

into "horizontal" and "vertical" parts. Here on the geometric side $\mathrm{Int}_{L^\flat,\mathsf{H}}$ is the contribution from the horizontal part of the KR cycles. On the analytic side we define $\partial\mathrm{Den}_{L^\flat,\mathsf{H}}$ to match with $\mathrm{Int}_{L^\flat,\mathsf{H}}$. We show the horizontal parts have logarithmic singularity along $L^\flat_F$, and vertical parts are indeed in $C_c^\infty(\mathbb{V})$. We then finish the proof by determining the Fourier transforms as

$$\widehat{\mathrm{Int}}_{L^\flat,\mathsf{V}} = -\mathrm{Int}_{L^\flat,\mathsf{V}}, \quad \widehat{\partial\mathrm{Den}}_{L^\flat,\mathsf{V}} = -\partial\mathrm{Den}_{L^\flat,\mathsf{V}}.$$

Some key ingredients of the proof include:

(1) Understand the horizontal part of KR cycles in terms of Gross's quasi-canonical liftings (using the work of Tate, Grothendieck–Messing, Breuil).
(2) Prove the Tate conjecture for certain Deligne–Lusztig varieties (using the work of Lusztig), and reduce $\mathrm{Int}_{L^\flat,\mathsf{V}}(x)$ to the intersection of Deligne–Lusztig curves and $\mathcal{Z}(x)$.
(3) Local density formula in terms of lattice theory (using the work of Cho–Yamauchi).

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Density theorems for GL(n)

Valentin Blomer

The concept of a density theorem is a familiar one from the theory of zeta functions. If $N(\sigma, T, Q)$ denotes the number of zeros with real part $\geq \sigma$ and height $\leq T$ of Dirichlet L-functions $L(s, \chi)$ for primitive Dirichlet characters χ with conductor $q \leq Q$, we have a bound

$$N(\sigma, T, Q) \ll (Q^2 T)^{c(\sigma)+\varepsilon}$$

for any $\varepsilon > 0$ and some continuous, non-increasing function $c(\sigma)$ with $c(1/2) = 1$, $c(1) = 0$, see [5, Section 10]. Of course, the Riemann hypothesis states $N(\sigma, T, Q) = 0$ for $\sigma > 1/2$, but the above bound can often serve as a good substitute for the Riemann hypothesis. Its arithmetic reformulation is the Bombieri-Vinogradov theorem which roughly states that primes $\leq x$ are equidistributed in "almost all" residue classes modulo $q \leq x^{1/2+o(1)}$.

Here we want to consider an automorphic analogue. Let us fix a place v of $\mathbb{Q}$, and for an automorphic form π on GL(n) let us denote by $\mu_\pi(v) = (\mu_\pi(v, 1), \ldots, \mu_\pi(v, n))$ its local spectral parameter (each entry viewed modulo $\frac{2\pi i}{\log p}\mathbb{Z}$ if $v = p$ is a prime). Write

$$\sigma_\pi(v) = \max_j |\Re \mu_\pi(v, j)|.$$

The representation π is tempered at v if $\sigma_\pi(v) = 0$, and the size of $\sigma_\pi(v)$ measures how far π is from being tempered at v. An example of a non-tempered representation is the trivial representation which satisfies $\sigma_{\text{triv}}(v) = (n-1)/2$ for every v. For a finite family $\mathcal{F}$ of automorphic representations for GL(n) and $\sigma \geq 0$ we define

$$N_v(\sigma, \mathcal{F}) = |\{\pi \in \mathcal{F} \mid \sigma_\pi(v) \geq \sigma\}|.$$

We have trivially $N_v(0, \mathcal{F}) = |\mathcal{F}|$, and if the trivial representation is contained in $\mathcal{F}$, we have $N_v((n-1)/2, \mathcal{F}) \geq 1$. One may hope to be able to interpolate linearly between these two extreme cases:

$$N_v(\sigma, \mathcal{F}) \ll_{v,\varepsilon} |\mathcal{F}|^{1-\frac{\sigma}{a}+\varepsilon}, \quad a = \frac{n-1}{2}, \tag{1}$$

for arbitrarily small $\varepsilon > 0$. This is precisely Sarnak's density hypothesis [9, p. 465] stated there in the context of groups G of real rank 1, the principal congruence subgroup $\Gamma(q) = \{\gamma \in G(\mathbb{Z}) \mid \gamma \equiv \text{id} \,(\text{mod}\, q)\}$ and $v = \infty$. For families of large level, Sarnak's density hypothesis has recently attracted interest in the context of lifting matrices modulo q [10] and the almost diameter of Ramanujan complexes, and for families with growing infinitesimal character in the context of Golden Gates and quantum computing [8]. In each of these cases it is not a spectral gap that is needed, but a certain kind of density result.

For the group GL(2) there exist strong density results for many automorphic families, also in number field versions and for general real rank 1 groups, e.g. [4, 2, 11]. Various results are also available for GL(3), see e.g. [2]. For higher rank groups, a very deep analysis of the Arthur-Selberg trace formular [7, 3] provides as by-products some density results for the family of Maaß forms of Laplace eigenvalue up to height T and fixed level. The value of a is however much larger than (1) for $n > 2$ (at least quadratic in n).

Here we consider the family $\mathcal{F}_I(q)$ of cuspidal automorphic representations generated by Maaß forms for the group $\Gamma_0(q) \subseteq \mathrm{SL}_n(\mathbb{Z})$ of matrices whose lowest row is congruent to $(0, \ldots, 0, *)$ modulo q for a large prime q and Laplace eigenvalue λ in a fixed interval I. If I is not too small, we have $|\mathcal{F}_I(q)| \asymp_I q^{n-1}$. For this family and any place $v \neq q$ of $\mathbb{Q}$, we go *beyond* the density hypothesis (1) and obtain a value of $a = (n-1)/4$ for this family. The Arthur-Selberg trace formula is usually not sensitive to whether the trivial representation is counted or not, but the Kuznetsov formula can be a versatile tool if no residual spectrum is involved. The following theorems are proved in [1].

Theorem 1. *Let $n \geq 3$, q a prime, v be a place of $\mathbb{Q}$ different from q, $I \subseteq [0, \infty)$ a fixed interval, $\varepsilon > 0$, and $\sigma \geq 0$. Then*

$$N_v(\sigma, \mathcal{F}_I(q)) \ll_{I,v,n,\varepsilon} q^{n-1-4\sigma+\varepsilon}.$$

Of course, by [6] we know that $N_v(\sigma, \mathcal{F}_I(q)) = 0$ for $\sigma \geq 1/2 - 1/(n^2+1)$, but for $0 < \sigma < 1/2 - 1/(n^2+1)$ we obtain a substantial power saving. For $n > 3$, Theorem 1 is completely new and it appears to be the limit of what is available by any trace formula approach, even in the case $n = 2$ nothing better is known.

The proof is based on a careful analysis of the arithmetic side of the Kuznetsov formula with a test function on the spectral side that blows up on exceptional Langlands parameters at v (and therefore increases the complexity on the arithmetic side). The key input is a detailed investigation of level q Kloosterman sums $S_{q,w}(M, N, c) = 0$ for GL(n) associated to Weyl elements $w \in W$ with entries $M, N \in \mathbb{Z}^{n-1}$ and moduli $c = (c_1, \ldots, c_{n-1})$.

Theorem 2. *Let q be a prime and let $M, N \in \mathbb{Z}^{n-1}$ with entries coprime to q. Let $n \geq 3$ and let $w \in W$. Then $S_{q,w}(M, N, (q, \ldots, q)) = 0$ unless*

$$w = \begin{pmatrix} & & 1 \\ & I_{n-2} & \\ 1 & & \end{pmatrix}$$

in which case $S_{q,w}(M, N, (q, \ldots, q)) = q^{n-2}$.

As an application of the theory developed in [1] we mention a large sieve inequality.

Theorem 3. *Let q be prime and $(\alpha(m))$ any sequence of complex numbers. Then*

$$\sum_{\pi \in \mathcal{F}_I(q)} \Big| \sum_{\substack{m \leq x \\ (m,q)=1}} \alpha(m)\lambda_\pi(m)\Big|^2 \ll_{I,n,\varepsilon} q^{n-1+\varepsilon} \sum_{\substack{m \leq x \\ (m,q)=1}} |\alpha(m)|^2$$

uniformly in $x \ll q$ for a sufficiently small implied constant (in terms of I and n).

For comparison, Venkatesh [12, Theorem 1] obtained this with $x \leq q^{1/(2n-2)}$.

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Some old and new results on singular moduli

Yingkun Li

Let $j(\tau)$ be the Klein j-variant, which has the following Fourier expansion

$$j(\tau) = q^{-1} + 744 + 196884q + \dots$$

Here $q = e^{2\pi i\tau}$ and τ is in the upper half plane $\mathbb{H}$. It induces an isomorphism from the modular curve $Y := \Gamma\backslash\mathbb{H}$ to $\mathbb{C}$, where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ via fractional linear transformation. Let $z \in Y$ be a CM point of discriminant $d < 0$, i.e. it is the image of a point in $\mathbb{H}$ satisfying a quadratic equation with integral coefficients and discriminant d. Note there are only finitely many CM points on Y with a fixed discriminant. The value $j(z)$ is called a *singular modulus*. For example,

$$j\left(\frac{1+\sqrt{-3}}{2}\right) = 0,\ j\left(\frac{1+\sqrt{-163}}{2}\right) = -2^{18}3^35^323^329^3. \tag{1}$$

The classical theory of complex multiplication tells us that a singular modulus $j(z)$ is always an algebraic integer. In fact, the number field $K(j(z))$ is the ring class field of the imaginary quadratic field $K := \mathbb{Q}(\sqrt{d})$ corresponding to the order $\mathcal{O}_d := \mathbb{Z} + \mathbb{Z}\frac{d+\sqrt{d}}{2}$. In particular, if d is fundamental, i.e. $\mathcal{O}_d$ is the ring of integers in K, then $K(j(z))$ is the Hilbert class field of K.

Motivated by effective results of André-Oort type, D. Masser asked the following question:

Question 1. *Can a singular modulus be a unit?*

Since 0 is also a singular modulus, a more general question would be

Question 2. *Can the difference of two singular moduli be a unit?*

In [1], Bilu-Habegger-Kühne gave "no" as the answer to Question 1, and raised Question 2 as a natural generalization. Recently in [7], we gave a short proof of the main result in [1] and was able to answer Question 2 as well.

One of the keys to our approach lies in the nice factorization that appears in (1), which is not a coincidence, and was studied in the seminal work of Gross and Zagier on singular moduli [3] as a prelude to the Gross-Zagier formula.

Theorem 3 ([3]). *Let $z_1, z_2 \in Y$ be CM points of discriminants d_1, d_2 respectively. Suppose d_1, d_2 are co-prime and fundamental. Then*

$$\frac{4}{w_1 w_2} \log |\mathrm{Nm}(j(z_1) - j(z_2))|^2 = -\sum_{t \in S_1} a(t), \tag{2}$$

where w_i is the number of roots of unity in $\mathbb{Q}(\sqrt{d_i})$, $F := \mathbb{Q}(\sqrt{d_1 d_2})$ is a real quadratic field with different $\mathfrak{d}_F$, $S_1 := \{t \in F : t \gg 0, \mathrm{Tr}(t) = 1\}$ and

$$a(t) := \sum_{\mathfrak{b} | (t)\mathfrak{d}_F \text{ integral ideal}} \chi(\mathfrak{b}) \log \mathrm{Nm}\mathfrak{b}$$

with χ the genus character of F corresponding to the CM extension $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})/F$.

Remark 4. *From the explicit formula of $a(t)$, one can show that $-a(t) \geq 0$, with equality for all but finitely many $t \in S_1$.*

Now, consider the function

$$G_1(\tau_1, \tau_2) := \log |j(\tau_1) - j(\tau_2)|^2 \tag{3}$$

on $Y \times Y$ with logarithmic singularity along the diagonal Y^Δ. It is harmonic with respect to the Laplacian in τ_i for both $i = 1, 2$. The factorization formula of Gross and Zagier can be rephrased as

$$G_1(Z(z_1, z_2)) + \sum_{t \in S_1} a(t) = 0, \tag{4}$$

where $Z(z_1, z_2)$ is the weighted sum of the "Galois conjugates" of the CM point $(z_1, z_2) \in Y \times Y$. This can be viewed as an equation expressing the arithmetic intersection of Y^Δ and $Z(z_1, z_2)$, which is 0, as the sum of local contributions, which are explicitly calculated.

The other key to answer Question 2 above is a result by Gross, Kohnen and Zagier, which is analogous to Theorem 3 for certain "higher Green functions" defined by

$$G_s(\tau_1,\tau_2) := -\sum_{\gamma\in\Gamma} Q_{s-1}\left(1+\frac{|\tau_1-\gamma\tau_2|}{\mathrm{Im}(\tau_1)\mathrm{Im}(\gamma\tau_2)}\right),\ \mathrm{Re}(s)>1. \tag{5}$$

Here $Q_{s-1}(t) := \int_0^\infty \left(t+\sqrt{t^2-1}\cosh v\right)^{-s} dv$ is the Legendre function of the second kind. The function $G_s(\tau_1,\tau_2)$ on $Y\times Y\backslash Y^\Delta$ is an eigenfunction of the Laplacians of τ_1 and τ_2 with eigenvalue $s(1-s)$. Furthermore, it is clear from the definition that

$$G_s(\tau_1,\tau_2) < 0$$

for any $(\tau_1,\tau_2)\in Y\times Y\backslash Y^\Delta$ and real $s>1$. One consequence of the result of Gross-Kohnen-Zagier in [4] is as follows.

Theorem 5. *Let $z_1,z_2\in Y$ be CM points of discriminants d_1,d_2 respectively. Suppose d_1,d_2 are co-prime and fundamental. Then for $k=3,5,7$,*

$$G_k(Z(z_1,z_2)) = -\sum_{t\in S_1} c_k(t)a(t), \tag{6}$$

where $c_k(t) := P_{k-1}(t-t')$ with P_{k-1} is the $(k-1)^{st}$ Legendre polynomial, and $S_1, a(t)$ are the same as in Theorem 3

Remark 6. *(1) For arbitrary odd k, similar statement still holds with G_k replaced by T_fG_k for suitable Hecke operator T_f associated to a cusp form f of weight $2k$.*
(2) With the definition of G_1 in equation (3), Theorem 5 also holds for $k=1$, which then is just Theorem 3.

Now, using the "CM value formulas" in [8] and [2], we can remove the conditions on the discriminants d_1,d_2 in Theorems 3 and 5 to obtain a slightly weaker result.

Proposition 7. *Let $z_1,z_2\in Y$ be distinct CM points of discriminants d_1,d_2. Then for $k=1,3,5,7$*

$$G_k(Z(z_1,z_2)) = -\sum_{t\in S_1} c_k(t)a(t;z_1,z_2), \tag{7}$$

where $-a(t;z_1,z_2)$ are certain non-negative real numbers, and $S_1, c_k(t)$ are the same as in Theorem 5.

Remark 8. *When $k=1$, Lauter and Viray have used the algebraic approach in [3] to obtain a similar result with more precise information concerning the coefficients $a(t;z_1,z_2)$ (see [6]). If $k=1$ and $d_1=d_2$ with milder restriction than being fundamental, then similar result can be deduced from the thesis of Hayashi [5].*

Remark 9. *The numbers $-a(t;z_1,z_2)$ can be computed for any given z_1,z_2 using the calculations in the appendix of [9], but the formula would be complicated in general. Fortunately, one can still see that $-a(t;z_1,z_2)\geq 0$ by inspecting the formulas.*

Theorem 10 (L. 2019)**.** *The difference of two singular moduli is never an algebraic unit.*

Proof. Assume that $j(z_1)$ and $j(z_2)$ are singular moduli, whose difference is a unit. Then $G_1(Z(z_1, z_2)) = \log 1 = 0$. By proposition 7, $a(t; z_1, z_2) = 0$ for all $t \in S_1$. Then $G_3(Z(z_1, z_2)) = 0$ by the same proposition, which is a contradiction since $G_3(\tau_1, \tau_2) < 0$ for any $(\tau_1, \tau_2) \in Y \times Y \backslash Y^{\Delta}$. This finishes the proof. □

Now, it is natural to ask the following question:

Question 11. *Let S be a subset of the primes. Are there finitely many differences of singular moduli that are S-units?*

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Explicit pullback formulas and a half-integral version of Harder conjecture on congruences for Siegel modular forms

Tomoyoshi Ibukiyama

Harder conjectured congruences between eigenvalues of vector valued Siegel modular forms of degree two and elliptic modular forms. On the other hand, we have given a conjecture before on bijective Shimura type correspondence between vector valued Siegel modular forms of integral weight and half-integral weight of degree two in [2] and [3]. So we can consider a half-integral version of Harder's conjecture, or equivalently Jacobi forms version. An advantage of the new versions is that it becomes a congruence between modular forms in the same space. To compare modular forms in the same space, it is often useful to use the pullback formula (Katsurada's idea). Here we give a pullback formula for Jacobi forms to describe the restriction of the image of certain vector valued automorphic differential operators on Jacobi Eisenstein series of degree 4 to diagonal blocks. Then under several conditions, we can show Harder's conjecture for the Jacobi version.

Original Harder's conjecture on congruences. *Let k, j be integers such that $j \geq 0$ is even and $k \geq 3$. Let f be a Hecke eigen elliptic cusp form of weight $2k+j-2$. Then there should exist a prime ideal $\mathfrak{l}$ dividing the algebraic part of $L_{alg}(k+j,f)$ and a vector valued Siegel cusp form $F \in S_{det^k Sym(j)}(\Gamma_2)$ of degree two such that for any prime p, the Euler p factor of $\zeta(s-k+2)\zeta(2-k-j+1)L(s,f)$ and that of the spinor L function $L(s,F,Sp)$ are congruence modulo $\mathfrak{l}$. Here $Sym(j)$ means the symmetric tensor representation of degree j.*

Conjecture on Shimura type correspondence([2], [3]) *For even integers $j \geq 0$ and odd $k \geq 3$, we should have the following isomorphisms as Hecke modules.*

$$S^+_{det^{(j+5)/2}Sym(k-3)}(\Gamma_0^{(2)}(4),\psi) \cong S_{det^k Sym(j)}(\Gamma_2),$$
$$S^+_{det^{(j+5)/2}Sym(k-3)}(\Gamma_0^{(2)}(4)) \cong S^+_{det^{(j+5)/2}Sym(k-3)}(\Gamma_0^{(2)}(4),\psi) + S_{2k+j-2}(\Gamma_1) \times S_{j+2}(\Gamma_1).$$

Here ψ means the character of $\Gamma_0^{(2)}(4)$ defined by $\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{-4}{det(d)}\right)$ and S^+ means a kind of new cusp forms called plus subspace originally defined for degree one by Kohnen. Here the second isomorphism also means that LHS should have injective lifts from $S_{2k+j-2}(\Gamma_1) \times S_{j+2}(\Gamma_1)$. Note that if k is even, the first isomorphism above is false since LHS is zero in that case. So for even k, there is no half-integral interpretation at moment.

The plus subspace of half-integral weight is bijectively interpreted as the space of Jacobi forms which are holomorphic if $j \equiv 2 \bmod 4$ and skew holomorphic if $j \equiv 0 \bmod 4$. If we assume the above Shimura type conjectural correspondence, the Harder conjecture for $j \equiv 2 \bmod 4$ is equivalent to the following conjecture. (We state it only for holomorphic Jacobi forms here, though skew holomorphic case can be similarly given.)

Holomorphic Jacobi forms version of Harder conjecture. *Assume $k \geq 3$, $j \geq 0$ and $j \equiv 2 \bmod 4$. For a Hecke eigen cusp form $f \in S_{2k+j-2}(\Gamma_1)$, take a Jacobi form ϕ_0 of degree 1 of weight $k+j/2$ of index 1 corresponding to f. Let $E(\phi_0)$ be a Klingen lift to Jacobi forms of weight $det^{(j+6)/2}Sym(k-3)$ of index 1 of degree 2. Then there should exist a holomorphic Jacobi form Φ of degree two of weight $det^{(j+6)/2}Sym(k-3)$ not coming from the lift such that for a certain prime ideal $\mathfrak{l}$ dividing $L_{alg}(k+j,f)$ and for any integral Hecke operator T, we have*

$$\lambda(T,\Phi) \equiv \lambda(T,E(\phi_0)) \equiv \bmod \mathfrak{l}.$$

where $\lambda(T,*)$ is an eigenvalue of T at $*$.

To state a pullback formula, we define automorphic differential operators. For complex domains D, Δ with $\Delta \subset D$, assume that a group H acts on Δ as biholomorphic automorphisms and that we can prolong H to the action on D equivariantly. Assume also that we have two automorphy factors J_D and J_Δ on D and Δ for H. We assume that J_D is scalar-valued and J_Δ is W-valued where W is a finite dimensional vector space over $\mathbb{C}$. We consider W-valued holomorphic partial differential operators $\mathbb{D}$ with constant coefficients on D such that the following equality holds for any holomorphic function F on D and $h \in H$.

$$Res_\Delta(\mathbb{D}(F|_{J_D}[h])) = (Res_\Delta \mathbb{D}F)|_{J_\Delta}[h],$$

where Res_Δ is the restriction to Δ. We call such operators $\mathbb{D}$ as automorphic differential operators. These operators can be regarded as intertwining operators between holomorphic discrete series. There are several standard choices of pairs (D, Δ). In our case, typically they are pairs $(H_{2n}, H_n \times H_n)$ for Siegel modular forms and $(H_{2n} \times \mathbb{C}^{2n}, (H_n \times \mathbb{C}^n) \times (H_n \times \mathbb{C}^n))$ for Jacobi forms. Also we take J_D to weight k and J_Δ to $det^k\rho \times det^k\rho$ for some irreducible representation (ρ, V_ρ) of $GL_n(\mathbb{C})$ (in both Siegel and Jacobi). When $d \geq 2n$, such operator $\mathbb{D}$ exists uniquely for any ρ for the pair $(H_{2n}, H_n \times H_n)$. In this case, by the assumption that $\mathbb{D}$ has constant coefficients, we have a certain $V_\rho \otimes V_\rho$ valued polynomial $P(T)$ in components of T where T is a $2n \times 2n$ symmetric matrix of variables such that

$$\mathbb{D} = P\left(\frac{\partial}{\partial \tau}\right), \qquad \frac{\partial}{\partial \tau} = \left(\frac{1+\delta_{ij}}{2}\frac{\partial}{\partial \tau_{ij}}\right)_{1 \leq i,j \leq 2n}, \qquad \tau = (\tau_{ij}) \in H_{2n}.$$

We write elements of $H_{2n} \times \mathbb{C}^{2n}$ as (τ, z) and $\tau = (\tau_{ij})$, $z = (z_i)$.

Lemma. *For an automorphic differential operator $\mathbb{D} = P(\frac{\partial}{\partial \tau})$ on Siegel modular forms from weight $k - 1/2$ to $det^{k-1/2}\rho \otimes det^{k-1/2}\rho$, we put*

$$\mathbb{D}_J = P\left(\left(\frac{1}{2\pi i}\frac{\partial}{\partial \tau_{ij}} - \frac{1}{4(2\pi i)^2}\frac{\partial^2}{\partial z_i \partial z_j}\right)_{1 \leq i,j \leq 2n}\right).$$

Then $\mathbb{D}_J$ is an automorphic differential operators on Jacobi forms from weight k to $det^k\rho \otimes det^k\rho$.

Now we denote by $E^k_{4,1}$ the scalar valued Jacobi Eisenstein series on $H_4 \times \mathbb{C}^4$ of degree 4 of even weight k of index 1.

Theorem *For $\rho = Sym(j)$, (τ, z), $(\zeta, w) \in H_2 \times \mathbb{C}^2$, we have*

$$\mathbb{D}_J E^k_{4,1}\begin{pmatrix} (\tau, z) & 0 \\ 0 & (\zeta, w) \end{pmatrix} = c_1 \sum_{\phi \in N(J^{cusp}_{k+j,1}(\Gamma^J_1))} Z_1(k, \phi)[\phi]^2_1(\tau, z)[\phi]^2_1(\zeta, w)$$
$$+ c_2 \sum_{\Phi \in N(J^{cusp}_{det^k Sym(j)}(\Gamma^J_2))} Z_2(k, \Phi)\Phi(\tau, z)\Phi(\zeta, w)$$

Here $N(J^{cusp}_{*,1})$ denotes an orthonormal basis of the space of Jacobi cusp forms of degree 1 and 2, $[\phi]^2_1$ is the Klingen lift of Jacobi forms from degree 1 to 2, and

$$c_1 = 2^{2-k-j}(-1)^{(k+j)/2}\frac{\pi(2k-3)_j(k-1/2)_j}{(2k+2j-3)j!(2\pi i)^j},$$
$$c_2 = 2^{6-(2k+j)}(-1)^{k+j/2}\frac{\pi^3(-1)^j(2k-3)_j(k-1/2)_j}{(2k+2j-3)(2k+j-4)(2k-5)j!(2\pi i)^j},$$

where $(x)_j$ denotes the ascending Pochhammer symbol, and

$$Z_1(k,\phi) = L(2k+j-3,f)/\zeta(2k-2)$$
$$Z_2(k,\Phi) = L_2(k-5/2,\Phi)/\zeta(2k-2)\zeta(2k-4).$$

Here L_2 is the standard L function of a Jacobi form defined by Murase and Sugano.

The case $j=0$ (i.e. without the differential operator) is nothing but Arakawa's pullback formula in [1]. Of course the same sort of explicit formulas are written also for general degrees and $\mathbb{D}_J$. Such pullback formulas have several by-products, such as algebraicity of critical values of L functions as usual, but omitted here.

Now let ϕ_0 be a Jacobi cusp form of degree one with algebraic coefficients corresponding to a cusp form $f \in S_{2k+2j-2}(\Gamma_1)$. Then the holomorphic Jacobi forms version of the Harder conjecture is proved under the following assumptions (1) to (4), denoting by $a(N,r,*)$ the Fourier coefficients.

(1) For a certain positive definite half integral matrix N and $r \in \mathbb{Z}^2$, we have $ord_{\mathfrak{l}}(a(N,r,[\phi]_1^2)) < 0$. (2) $L_{alg}(2k+j-3)a(N,r,[\phi_0]_1^2)$ is $\mathfrak{l}$ unit. (3) $\mathfrak{l}$ is prime to denominators of Fourier coefficients of $E_{4,1}^k$ and elementary coefficients of $[\phi_0]_1^2 \otimes [\phi_0]_1^2$ in the pullback formula. (All these factors can be explicitly calculated.) (4) There is no congruence between $L(s,f)$ and $L(s,g)$ modulo $\mathfrak{l}$ for any $g \in A_{2k+2j-2}(\Gamma_1)$ with $g \neq f$. (This (4) is a condition to exclude endoscopic Φ.)

The above (1) and (2) means that $\mathfrak{l}$ is a divisor of $L_{alg}(2k+j-3,f)$. For integral weight $det^k Sym(j)$, this means $\mathfrak{l}$ divides $L_{alg}(k+j,f)$ as in the original Harder conjecture. This explains the reason why such critical value appears.

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Quantum modular forms and plumbing graphs of 3-manifolds

Kathrin Bringmann

(joint work with Karl Mahlburg, Antun Milas, and Caner Nazaroglu)

I will report about modularity properties of certain q-series which arise from contour integration in work of Gukov, Pei, Putrov, and Vafa [5].

Consider a graph $G=(V,E)$ with $N \in \mathbb{N}$ vertices, $M = (m_{jk})_{1\leq j,k\leq N}$ a positive definite symmetric integral matrix associated to G such that $m_{jk} = -1$ if vertex j is connected to vertex k, and zero otherwise. Once we fix a graph G, M depends only on the labeling of vertices. To each edge $j-k$ in G we associate a rational function

$$f(w_j,w_k) := \frac{1}{\left(w_j - w_j^{-1}\right)\left(w_k - w_k^{-1}\right)}$$

and to each vertex a Laurent polynomial

$$g(w_j) := \left(w_j - w_j^{-1}\right)^2.$$

Slightly modifying the setup of Gukov, Pei, Putrov, and Vafa [5], define

$$Z(q) := \frac{q^{\frac{-3N+\mathrm{tr}(M)}{2}}}{(2\pi i)^N} \mathrm{PV} \int_{|w_j|=1} \prod_{j=1}^{N} g(w_j) \prod_{(k,\ell)\in G} f(w_k, w_\ell) \Theta_M(q; \boldsymbol{w}) \frac{dw_j}{w_j},$$

where PV means the Cauchy principal value, $\int_{|w_j|=1}$ indicates the integration $\int_{|w_1|=1} \cdots \int_{|w_N|=1}$, and the theta function is defined by

$$\Theta_M(q; \boldsymbol{w}) := \sum_{\boldsymbol{m}\in M\mathbb{Z}^N} q^{\frac{1}{2}\boldsymbol{m}^T M^{-1} \boldsymbol{m}} \prod_{j=1}^{N} w_j^{m_j}.$$

Throughout, we write $\boldsymbol{w} = (w_1, \ldots, w_N)$.

Conjecture 1 (Gukov)**.** *The function $Z(q)$ is a quantum modular form.*

To describe the meaning of this conjecture, let me first recall modular forms. In the simplest case, a *modular form f of weight $k \in \mathbb{Z}$* is a holomorphic function on the complex upper half-plane $\mathbb{H} := \{\tau = u + iv \in \mathbb{C} : v > 0\}$ that is bounded as $v \to \infty$ and satisfies

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k f(\tau) \qquad \forall \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}). \tag{1}$$

For general functions, the difference between the left-hand and right-hand side of (1) is called the *obstruction to modularity.*

If one includes a multiplier, then k may also be a half-integer. An example of a modular form of weight $\frac{1}{2}$ is the classical theta function ($q := e^{2\pi i\tau}$ throughout)

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} q^{n^2}.$$

There are also important cases in which the obstruction to modularity is not zero but explicit and "nice". The most famous examples are probably Ramanujan's mock theta functions, a list of q-series which are reminiscent of modular forms and which were introduced by Ramanujan in his last letter to Hardy. The letter contained a list of 17 examples, including the following q-hypergeometric series

$$f(q) := 1 + \sum_{n\geq 1} \frac{q^{n^2}}{(-q;q)_n^2}, \qquad \text{where} \qquad (a;q)_n := \prod_{j=0}^{n-1} \left(1 - aq^j\right) \quad \text{for } n \in \mathbb{N}_0 \cup \{\infty\}.$$

Zwegers, in his PhD thesis, viewed the mock theta functions as pieces of *harmonic Maass forms*, which transform like modular forms but instead of being meromorphic they are annihilated by the *weight k Laplace operator* $\Delta_k := -v^2(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}) + ikv(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v})$. To be more precise, adding non-holomorphic integrals of the shape (θ a weight $\frac{3}{2}$ (modular) theta function)

$$\int_{-\overline{\tau}}^{i\infty} \frac{\theta(w)}{\sqrt{-i(w+\tau)}} dw \tag{2}$$

to the mock theta functions yields harmonic Maass forms.

Another class of functions which we require are *quantum modular forms.* Following Zagier [6] these are functions on the rationals whose obstruction to modularity is "nice". Typical examples are given by *false theta functions* which, compared to the modular theta functions, are weighted by a wrong sign. For example, for $a \in \mathbb{Q}$, such a false theta function is given by

$$\sum_{n\in\mathbb{Z}+a} \operatorname{sgn}(n) q^{n^2}$$

as deleting the $\operatorname{sgn}(n)$ gives a (modular) theta function. It can be shown that (many) false theta functions are quantum modular forms. Note that Zwegers (in unpublished work) observed that false theta functions and non-holomorphic integrals like (2) asymptotically agree, when moving from the upper to the lower half-plane.

To state our first result, we restrict to manifolds coming from n-leg star graphs. An *n-leg star graph* consists of n legs joined to a central vertex. By explicitly writing $Z(q)$ in the terms of false theta functions we proved the following in [1] [1].

Theorem 2 (B.–Mahlburg–Milas)**.** *The conjecture of Gukov is true for* 3*-leg star graphs.*

Let me next turn to modularity properties of $Z(q)$ on $\mathbb{H}$. This is part of a more general framework built in [3]. Consider the false theta functions ($z = x + iy \in \mathbb{C}$)

$$\psi(z;\tau) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} \operatorname{sgn}(n) e^{2\pi i n\left(z+\frac{1}{2}\right)} q^{\frac{n^2}{2}}.$$

Now define for $w \in \mathbb{H}$ (erf denotes the error function)

$$\widehat{\psi}(z;\tau,w) := \sum_{n\in\mathbb{Z}+\frac{1}{2}} \operatorname{erf}\left(\sqrt{\pi i(\tau-w)}\left(n+\tfrac{y}{v}\right)\right) e^{2\pi i n\left(z+\frac{1}{2}\right)} q^{\frac{n^2}{2}}.$$

Since for $-\frac{v}{2} < y < \frac{v}{2}$, $\psi(z;\tau) = \lim_{w\to i\infty} \widehat{\psi}(z;\tau,w)$, $\widehat{\psi}$ may be viewed as the completion of ψ. Jointly with Nazaroglu, I showed in [3] that $\widehat{\psi}$ satisfies the following transformations which look like those of Jacobiforms.

Theorem 3 (B.–Nazaroglu)**.** *We have, with χ some multiplier*

$$\begin{aligned}\widehat{\psi}(z;\tau,w) &= -\widehat{\psi}(z+1;\tau,w) = -e^{2\pi i z} q^{\frac{1}{2}} \widehat{\psi}(z+\tau;\tau,w)\\ &= e^{-\frac{\pi i}{4}} \widehat{\psi}(z;\tau+1,w+1) = \chi\tau^{-\frac{1}{2}} e^{-\frac{\pi i z^2}{\tau}} \widehat{\psi}(z;\tau,w).\end{aligned}$$

[1]Note that in a paper [4] that appeared as a preliminary version of [1] was ready, Cheng, Chun, Ferrari, Gukov, and Harrison independently calculated $Z(q)$ for a large number of additional examples of 3-spider graphs, as well as an example of a 4-spider graph.

Corollary 4 (B.–Nazaroglu)**.** *The function $Z(q)$ has modular properties on $\mathbb{H}$.*

We next turn to to higher depth quantum modular forms. The basic idea here is that the obstruction to modularity is one depth lower than the original function. To obtain depth two objects, we consider $Z(q)$ for a family of non-Seifert plumbed 3-manifolds. The simplest plumbing graph of this kind is obtained by splicing two 3-star graphs. This yields a so-called `H`-graph with six vertices. In [2], we showed the following modularity properties of $Z(q)$ by explicitly evaluating to integral defining $Z(q)$ to obtain higher-dimensional false theta-functions.

Theorem 5 (B.–Mahlburg–Milas)**.** *For any positive definite unimodular plumbing matrix, $Z(q)$ is a quantum modular form of depth two, weight one, and quantum set $\mathbb{Q}$.*

Let me end with some open questions:

- Theorem 5 shows that the conjecture of Gukov needs to be modified to include examples where higher depth quantum modular form occur.
- As mentioned above in the evaluations of $Z(q)$ higher-dimensional false theta functions occur in some cases. However, there are no known modularity properties of these functions on the upper half-plane. Thus it would be interesting to complete higher-dimensional false theta functions.
- Finally there are hints that the false theta functions allow a picture that parallels that of mock theta functions. Thus a goal is to develop a theory of completed false theta functions analogous to the one of mock theta functions and in particular find more examples which lie in that space including Poincaré series.

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Generating series for special cycles on Shimura varieties of orthogonal type over totally real fields

Stephen S. Kudla

We reported on results of [5]. Let F be a totally real number field of degree d and let V be a quadratic space over F with signature

$$((m,2)^{d_+},(m+2,0)^{d-d_+}),\quad d_+>0.$$

To avoid issues about compactifications, we assume that V is anisotropic. Let $G = R_{F/\mathbb{Q}}(\mathrm{GSpin}(V))$ and let

$$D = \prod_{j=1}^{d_+} D_j, \qquad D_j = \{\ z \in \mathrm{Gr}_2^o(V_j) \mid (\ ,\ )\,|_z < 0\ \}$$

where $\mathrm{Gr}_2^o(V_j)$ is the space of oriented negative 2-planes in $V_j = V \otimes_{F,\sigma_j} \mathbb{R}$. For a neat compact open subgroup $K \subset G(\mathbb{A}_f)$, the associated Shimura variety

$$S_K = \mathrm{Sh}(G, D)_K = G(\mathbb{Q})\backslash D \times G(\mathbb{A}_f)/K$$

is a smooth projective variety of dimension md_+.

The construction of weighted special cycles given in [4] in the case $d_+ = 1$ carries over immediately here. Thus, for a symmetric matrix $T \in \mathrm{Sym}_n(F)$ with $\sigma_j(T) > 0$ for all j and for a weight function $\varphi \in S(V(\mathbb{A}_f)^n)^K$, we obtain an algebraic cycle $Z(T,\varphi)_K$ of codimension nd_+ in S_K. The cycles are compatible with pullback: if $K' \subset K$, then

$$\mathrm{pr} : S_{K'} \longrightarrow S_K, \qquad \mathrm{pr}^*(Z(T,\varphi)_K) = Z(T,\varphi)_{K'},$$

and hence we obtain classes

$$[Z(T,\varphi)] \ \in\ \mathrm{CH}^{nd_+}(S) := \varinjlim_{K} \mathrm{CH}^{nd_+}(S_K).$$

For semi-definite T, the classes $Z(T,\varphi)$ are defined by shifting the natural cycle of codimension $\mathrm{rank}(T)d_+$ by a power of the top Chern class $\boldsymbol{c}_S \in \mathrm{CH}^{d_+}(S_K)$ of a cotautological vector bundle $\mathcal{C}_S$ of rank d_+. The key new feature here is the the special cycles occur in codimensions nd_+ that are *multiples of* d_+. In particular, for $d_+ > 1$, there are no special divisors and there is no evident way to produce relations among special cycle of codimension nd_+ by using special cycles of codimension $(n-1)d_+$.

We obtain three main results about special cycles.

The first if the following product formula.

Theorem 1. For $T_i \in \mathrm{Sym}_{n_i}(F)$ and $\varphi_i \in S(V(\mathbb{A}_f)^{n_i})$, $i = 1, 2$.

$$[Z(T_1,\varphi_1)] \cdot [Z(T_2,\varphi_2)] = \sum_{\substack{T \in \mathrm{Sym}_{n_1+n_2}(F)_{\geq 0} \\ T = \begin{pmatrix} T_1 & * \\ {}^t* & T_2 \end{pmatrix}}} [Z(T,\varphi_1 \otimes \varphi_2)] \ \in \mathrm{CH}^{(n_1+n_2)d_+}(S).$$

In the case $d_+ = 1$, the analogous product formula for cohomology classes was proved in [4] and for classes in the Chow group in [11]. For $d_+ > 1$, the proof involves a more serious analysis of degenerate intersections using the machinery of [2]. The product formula implies the the special cycles form a subring of the Chow ring $\mathrm{CH}^\bullet(S)$.

The second results concerns the modularity of the generating series for special cycles of codimension nd_+. Recall that a conjecture of Bloch and Beilinson asserts that for a smooth projective variety S_K defined over a number field, the Abel-Jacobi map

$$\mathrm{AJ}_r : \mathrm{CH}^r(S_K)^0 \longrightarrow J_r(S_K)$$

on the space

$$\mathrm{CH}^r(S_K)^0 := \mathrm{Ker}(\mathrm{cl}_r : \mathrm{CH}^r(S_K) \longrightarrow H^{2r}(S_K)),$$

is injective, up to torsion. Here $J_r(S_K)$ is the intermediate Jacobian.

Theorem 2. Assume the Bloch-Beilinson conjecture. Then, for all n, $1 \le n \le m$, the formal q-series

$$\phi_n(\tau, \varphi, S) = \sum_{T \in \mathrm{Sym}_n(F)_{\ge 0}} [Z(T, \varphi)]\, q^T \;\in\; \mathrm{CH}^{nd_+}(S)[[q]],$$

where $\tau = (\tau_1, \dots, \tau_d) \in \mathfrak{H}_n^d$, $\varphi \in S(V(\mathbb{A}_f)^n)$, and $q^T = e(\sum_{j=1}^d \mathrm{tr}(\sigma_j(T)\tau_j))$, is a Hilbert-Siegel modular form of weight $\frac{1}{2}m + 1$. Here $\mathfrak{H}_n$ is the Siegel space of genus n.

Modularity means that if $\lambda : \mathrm{CH}^{nd_+}(S) \to \mathbb{C}$ is any linear functional, then the series

$$\phi_n(\tau, \varphi, S; \lambda) = \sum_{T \in \mathrm{Sym}_n(F)_{\ge 0}} \lambda([Z(T, \varphi)])\, q^T \;\in\; \mathbb{C}[[q]],$$

is the q-expansion of a Hilbert-Siegel modular form of weight $\frac{1}{2}m + 1$.

A key point point here is that the image of the generating series under the cycle class map is a Hilbert-Siegel modular form as a consequence of the results of [6], [7], and [8]. This was also recently remarked in [10]. Maeda [9] shows that the modularity of the generating series $\phi_n(\tau, \varphi, S)$ also follows from a combination of the Bloch-Beilinson conjecture for cl_{d_+} and Wei Zhang's inductive argument [12] under the assumption that the image of the generating series under any linear functional is absolutely convergent.

The third result is a pullback formula. Let U be a totally positive definite quadratic space over F of dimension 4ℓ and let $\tilde{V} = U \oplus V$ be the orthogonal direct sum. Let $\tilde{G} = R_{F/\mathbb{Q}}\mathrm{GSpin}(\tilde{V})$, so that there is a natural homomorphism $G \to \tilde{G}$. If $\tilde{K}$ is a neat compact open subgroup of $\tilde{G}(\mathbb{A}_f)$ and $K = \tilde{K} \cap G(\mathbb{A}_f)$, there is a morphism of Shimura varieties

$$\rho_{\tilde{K}} : S_K \longrightarrow \tilde{S}_{\tilde{K}}$$

and a ring homomorphism

$$\rho_{\tilde{K}}^* : \mathrm{CH}^\bullet(\tilde{S}_{\tilde{K}}) \longrightarrow \mathrm{CH}^\bullet(S_K).$$

Passing to the limit over $\tilde{K}$, we have

$$\rho^* : \mathrm{CH}^\bullet(\tilde{S}) \longrightarrow \mathrm{CH}^\bullet(S).$$

Theorem 3. For $\varphi \in S(V(\mathbb{A}_f)^n)$ and $\varphi_0 \in S(U(\mathbb{A}_f)^n)$,

$$\rho^*\big(\ \phi_n(\tau; \varphi^0 \otimes \varphi, \tilde{S})\ \big) = \theta(\tau, \varphi^0) \cdot \phi_n(\tau, \varphi, S),$$

where

$$\theta(\tau, \varphi^0) = \sum_{x \in U(F)^n} \varphi_0(x)\, q^{Q(x)}$$

is the Hilbert-Siegel theta function of weight 2ℓ.

This result, combined with a strong vanishing result for odd Betti numbers of $\tilde{S}_{\tilde{K}}$ and results about formal Fourier series [1], [3], is the basis for the proof of Theorem 2. Full details can be found in [5].

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On central derivatives of p-adic triple product L-functions

Shunsuke Yamana
(joint work with Ming-Lun Hsieh)

We will construct a triple product p-adic L-function and discuss its trivial or nontrivial zeros at the center of the functional equation. In the split and $+1$ sign case we will determine the trivial zeros of cyclotomic p-adic L-functions associated to three ordinary elliptic curves and identify the double or triple derivatives of the p-adic L-function with the product of the algebraic part of central L-values and suitable $\mathscr{L}$-invariants. We will also formulate the p-adic Gross-Zagier formula in the -1 sign case.

0.1. **Cyclotomic p-adic L-functions associated to three elliptic curves.** Let E_1, E_2, E_3 be rational elliptic curves of conductor N_i. We write $L(\mathbf{E}, s)$ for the degree eight motivic L-function for the triple product

$$\mathbf{V}_p^{\mathbf{E}} = \mathrm{H}^1_{\text{ét}}(E_{1/\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(E_{2/\overline{\mathbf{Q}}}, \mathbf{Q}_p) \otimes \mathrm{H}^1_{\text{ét}}(E_{3/\overline{\mathbf{Q}}}, \mathbf{Q}_p)$$

realized in the middle cohomology of the abelian variety $\mathbf{E} = E_1 \times E_2 \times E_3$ by the Künneth formula. Hence

$$L(\mathrm{H}^3_{\text{ét}}(\mathbf{E}_{/\overline{\mathbf{Q}}}, \mathbf{Q}_p), s) = L(\mathbf{E}, s) \prod_{i=1}^{3} L(E_i, s-1)^2.$$

Let $G_{\mathbf{Q}} \supset G_{\mathbf{Q}_\ell} \supset I_\ell$ be the absolute Galois group, its decomposition group at ℓ and its inertia subgroup at ℓ. We consider the central critical twist

$$V_p^{\mathbf{E}} := \mathbf{V}_p^{\mathbf{E}}(2) : G_{\mathbf{Q}} \to \mathrm{GL}_8(\mathbf{Z}_p).$$

Observe that $(V_p^{\mathbf{E}})^*(1) \simeq V_p^{\mathbf{E}}$. Fix an embedding $\iota_\infty : \overline{\mathbf{Q}} \hookrightarrow \mathbf{C}$. Let $\mathbf{Q}_\infty$ be the $\mathbf{Z}_p$-extension of $\mathbf{Q}$. The twisted triple product L-series is defined by the Euler product

$$L(\mathbf{E} \otimes \hat{\chi}, s+2) = \prod_{\ell} L_\ell(V_p^{\mathbf{E}} \otimes \chi, s)$$

for p-adic characters χ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ of finite order, where $\hat{\chi}$ is the Dirichlet character associated to $\iota_\infty \circ \chi$. If $\ell \neq p$, then

$$L_\ell(V_p^{\mathbf{E}} \otimes \chi, s) = \det(\mathbf{1}_8 - \ell^{-s} \iota_\infty(\chi(\ell)^{-1} \mathrm{Frob}_\ell | (V_p^{\mathbf{E}})^{I_\ell}))^{-1}.$$

The complete twisted triple product L-series

$$\Lambda(\mathbf{E} \otimes \hat{\chi}, s) = \Gamma_{\mathbf{C}}(s) \Gamma_{\mathbf{C}}(s-1)^3 L(\mathbf{E} \otimes \hat{\chi}, s)$$

proved to be an entire function which satisfies a simple functional equation

$$\Lambda(\mathbf{E} \otimes \hat{\chi}, s) = \varepsilon(\mathbf{E}, s) \Lambda(\mathbf{E} \otimes \hat{\chi}, 4-s)$$

by the theorem of Wiles and the integral representation discovered by Garrett [2], which was studied extensively in [8]. The global sign is given by the product of local signs $\varepsilon = \varepsilon(\mathbf{E}, 2) = -\prod_\ell \varepsilon_\ell(\mathbf{E})$.

Fix an odd prime number p at which E_1, E_2, E_3 are ordinary. We denote by $T_p(E_i) = \varprojlim_n E_i[p^n]$ the Tate module of E_i. The $G_{\mathbf{Q}_p}$-invariant subspace

$$\mathrm{Fil}^0 T_p(E_i) := T_p(E_i)^{I_p} = \mathrm{Ker}(T_p(E_i) \to T_p(E_i/\mathbb{F}_p))$$

fixed by I_p is one-dimensional, where $E_i/\mathbb{F}_p$ denotes the mod p reduction of the Neron model of E_i.

The Galois representation $V_p^{\mathbf{E}} := T_p(E_1) \otimes T_p(E_2) \otimes T_p(E_3)(-1)$ satisfies the Panchishkin condition, i.e., we define the rank four $G_{\mathbf{Q}_p}$-invariant subspace of $V_p^{\mathbf{E}}$ by

$$\begin{aligned} \mathrm{Fil}^+ V_p^{\mathbf{E}} :=& \mathrm{Fil}^0 T_p(E_1) \otimes \mathrm{Fil}^0 T_p(E_2) \otimes T_p(E_3)(-1) \\ &+ T_p(E_1) \otimes \mathrm{Fil}^0 T_p(E_2) \otimes \mathrm{Fil}^0 T_p(E_3)(-1) \\ &+ \mathrm{Fil}^0 T_p(E_1) \otimes T_p(E_2) \otimes \mathrm{Fil}^0 T_p(E_3)(-1). \end{aligned}$$

The Hodge-Tate numbers of $\mathrm{Fil}^+V_p^{\mathbf{E}}$ are all positive, while none of the Hodge-Tate numbers of $V_p^{\mathbf{E}}/\mathrm{Fil}^+V_p^{\mathbf{E}}$ is positive.

Let $\Lambda(s, E_i, \mathrm{ad})$ be the complete adjoint L-function of E_i. Define our period by $\Omega_{\mathbf{E}} = \prod_{i=1}^3 \Lambda(1, E_i, \mathrm{ad})$. When N_1, N_2, N_3 are square-free, the author and Ming-Lun Hsieh have constructed a four-variable p-adic L-function, which yields a cyclotomic p-adic L-function

$$L_p(\mathbf{E}) \in \mathbf{Z}_p[[\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})]] \otimes \mathbf{Q}_p$$

with the following interpolation property

$$L_p(\mathbf{E}, \hat{\chi}) = \frac{\Lambda(\mathbf{E} \otimes \hat{\chi}, 2)}{\Omega_{\mathbf{E}}} (\sqrt{-1})^3 \mathcal{E}_p(\mathrm{Fil}^+V_p^{\mathbf{E}} \otimes \chi)$$

for all finite-order characters $\hat{\chi}$ of $\mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q})$ in Corollary 7.9 of [6], where the modified factor at ∞ is $(\sqrt{-1})^3$ and the modified p-Euler factor is defined by

$$\mathcal{E}_p(\mathrm{Fil}^+V_p^{\mathbf{E}} \otimes \chi) = \frac{L(\mathrm{Fil}^+V_p^{\mathbf{E}} \otimes \chi, 0)}{\varepsilon(\mathrm{Fil}^+V_p^{\mathbf{E}} \otimes \chi) \cdot L((\mathrm{Fil}^+V_p^{\mathbf{E}} \otimes \chi)^\vee, 1)} \cdot \frac{1}{L_p(V_p^{\mathbf{E}} \otimes \chi, 0)}.$$

Define an analytic function $L_p(\mathbf{E}, s) := \boldsymbol{\varepsilon}_{\mathrm{cyc}}^{s-2}(L_p(\mathbf{E}))$ for $s \in \mathbf{Z}_p$, where $\boldsymbol{\varepsilon}_{\mathrm{cyc}} : \mathrm{Gal}(\mathbf{Q}_\infty/\mathbf{Q}) \to 1 + p\mathbf{Z}_p$ denotes the cyclotomic character.

0.2. **Trivial zeros.** The Euler-like factor $\mathcal{E}_p(\mathrm{Fil}^+\mathbf{V}_{\mathbf{E}}(2))$ can possibly vanish. In this case the interpolation formula forces $L_p(\mathbf{E}, 2)$ to be zero. Such a zero is called a trivial zero.

We consider the case where $L_p(\mathbf{E}, s)$ has a trivial zero at the critical value $s = 2$. We essentially only need to consider the following two cases:

- (i) all E_1, E_2 and E_3 have split multiplicative reduction at p.
- (ii) E_1 has split multiplicative reduction at p; E_2 and E_3 have good ordinary reduction at p such that $\alpha_2 = \alpha_3$, where α_i is the p-adic unit Hecke eigenvalue of E_i.

Let $\mathscr{L}_p(E_i) = \dfrac{\log_p q_{E_i}}{\mathrm{ord}_p\, q_{E_i}}$ be the $\mathscr{L}$-invariant of E_i with Tate's p-adic period q_{E_i} attached to E_i.

Theorem 1. *(1) In Case (i),* $\mathrm{ord}_{s=2} L_p(\boldsymbol{E}, s) \geq 3$, *and*

$$\left.\frac{L_p(\boldsymbol{E}, s)}{(s-2)^3}\right|_{s=2} = \mathscr{L}_p(E_1)\mathscr{L}_p(E_2)\mathscr{L}_p(E_3) \cdot \frac{L(\boldsymbol{E}, 2)}{2^4\pi^5\Omega(\boldsymbol{E})}.$$

(2) In Case (ii), $\mathrm{ord}_{s=2} L_p(\boldsymbol{E}, s) \geq 2$ *and*

$$\left.\frac{L_p(\boldsymbol{E}, s)}{(s-2)^2}\right|_{s=2} = \mathscr{L}_p(E_1)^2(-p\alpha_2^{-2})(1-\alpha_2^{-2})^2 \cdot \frac{L(\boldsymbol{E}, 2)}{2^4\pi^5\Omega(\boldsymbol{E})}.$$

In the case of a p-adic L-function $L_p(E, s)$ of an elliptic curve E over $\mathbf{Q}$ the trivial zero arises if and only if E is split multiplicative at p. An analogus formula for $L_p'(E, 1)$ was proved in [4]. Our result proves the first cases of the trivial zero conjecture where multiple trivial zeros are present and the Galois representation is not of GL(2)-type.

0.3. **Ichino's formula.** Though π_i^D is self-dual, we write $\pi_i^{D\vee}$ for its dual with future generalizations in view. Let $X = \{X_U\}_U$ denote the projective system of rational curves associated to D indexed by open compact subgroups U of $\widehat{D}^\times$.

For every place v of $\mathbf{Q}$ we define the local trilinear form $I_v : \bigotimes_{i=1}^3 \big(\pi_{i,v}^D \otimes \pi_{i,v}^{D\vee}\big) \to \mathbf{C}$ by

The global trilinear form $I : \bigotimes_{i=1}^3 \big(\pi_i^D \otimes \pi_i^{D\vee}\big) \to \mathbf{C}$ is defined to be the tensor product of the local trilinear forms I_v. This definition depends on the choice of the local invariant pairings $B_v : \bigotimes_{i=1}^3 \big(\pi_{i,v}^D \otimes \pi_{i,v}^{D\vee}\big) \to \mathbf{C}$. Normalize the local pairings by the compatibility

$$\otimes_{i=1}^3 \langle\ ,\ \rangle_i = \otimes_v B_v.$$

Here the Petersson pairing $\langle\ ,\ \rangle_i : \pi_i^D \otimes \pi_i^{D\vee} \to \mathbf{C}$ is defined by

$$\langle h_i, h_i' \rangle_i = \int_{\mathbf{A}^\times D^\times \backslash (D \otimes \mathbf{A})^\times} h_i(g) h_i'(g)\, \mathrm{d}g.$$

Define the period integral $\mathscr{P}^D : \bigotimes_{i=1}^3 \pi_i^D \to \mathbf{C}$ by

$$\mathscr{P}^D(h_1 \otimes h_2 \otimes h_3) = \int_{\mathbf{A}^\times D^\times \backslash (D \otimes \mathbf{A})^\times} h_1(g) h_2(g) h_3(g)\, \mathrm{d}g.$$

For a local reason $\mathscr{P}^{D'}$ vanishes on $\bigotimes_{i=1}^3 \pi_i^{D'}$ unless $D \simeq D'$. Ichino proved the following formula for the central critical value in [7]:

$$\mathscr{P}^D(h)\mathscr{P}^D(h') = 2^{-3}\zeta_{\mathbf{Q}}(2)^2 \frac{\Lambda(\mathbf{E}, 2)}{\prod_{i=1}^3 \Lambda(1, \pi_i, \mathrm{ad})} I(h \otimes h'),$$

where $\Lambda(s, \pi_i, \mathrm{ad})$ is the complete adjoint L-series of π_i.

0.4. **The non-trivial derivative.** From now on we assume that $\varepsilon = -1$ and $(p, N_1N_2N_3) = 1$. Then $L(\mathbf{E}, 2)$ is automatically 0. The main object of study in this case is the central derivative $L'(\mathbf{E}, 2)$ of $L(\mathbf{E}, s)$. Let D be the indefinite quaternion algebra over $\mathbf{Q}$ such that $D_\ell \not\simeq \mathrm{M}_2(\mathbf{Q}_\ell)$ if and only if $\varepsilon_\ell(\mathbf{E}) = -1$. Here we put $D_\ell = D \otimes \mathbf{Q}_\ell$ and $\widehat{D} = D \otimes \widehat{\mathbf{Q}}$. Let X_U be the (compactified) Shimura curve associated to a compact open subgroup of $\widehat{D}^\times$. We regard X_U as the codimensioin 2 cycle embedded diagonally in the threefold X_U^3. One can modify it to obtain a homologically trivial cycle, following [5]. Gross and Kudla conjectured an analogous expression for $L'(\mathbf{E}, 2)$ in terms of a height pairing of the (f_1, f_2, f_3)-isotypic component of the modified diagonal cycle.

The theorem of Wiles gives a primitive form

$$f_i = \sum_{n=1}^{\infty} \mathbf{a}(n, f_i) q^n \in S_2(\Gamma_0(N_i))$$

such that all the Fourier coefficients $\mathbf{a}(n, f_i)$ are rational integers and such that E_i is isogeneous to the elliptic curve obtained from f_i via the Eichler–Shimura construction, i.e., the Dirichlet series $\sum_{n=1}^{\infty} \mathbf{a}(n, f_i) n^{-s}$ coincides with the Hasse-Weil L-series $L(s, E_i)$. Let π_i be the automorphic representation of $\mathrm{PGL}_2(\mathbf{A})$ generated by f_i. The eigenform f_i determines an automorphic representation

$\pi_i^D \simeq \otimes'_v \pi_{i,v}^D$ of $(D \otimes \mathbf{A})^\times$ via the global correspondence of Jacquet, Langlands and Shimizu.

The projective limit X of $\{X_U\}$ is endowed with the action of $\widehat{D}^\times$. The curve X_U has a Hodge class L_U, which is the line bundle whose global sections are holomorphic modular forms of weight two. Normalize the Hodge class by $\xi_U := \frac{L_U}{\mathrm{vol}(X_U)}|\widehat{\mathbf{Z}}^\times/\mathrm{N}_{\mathbf{Q}}^D(U)|$, where

$$\mathrm{vol}(X_U) := \int_{X_U(\mathbf{C})} \frac{\mathrm{d}x\mathrm{d}y}{2\pi y^2}.$$

It is known that $\deg L_U = \mathrm{vol}(X_U)$ and that the induced action of $\widehat{D}^\times$ on the set of geometrically connected components of X_U factors through the norm map $\mathrm{N}_{\mathbf{Q}}^D : \widehat{D}^\times \to \widehat{\mathbf{Q}}^\times$. Hence the restriction of ξ_U to each geometrically connected component of X_U has degree 1.

For any abelian variety A over $\mathbf{Q}$ the space $\mathrm{Hom}^0_{\xi_U}(X_U, A)$ consists of morphisms in $\mathrm{Hom}_{\mathbf{Q}}(X_U, A) \otimes \mathbf{Q}$ which map the Hodge class ξ_U to zero in A. Since any morphism from X_U to an abelian variety factors through the Jacobian variety J_U of X_U, we also have $\mathrm{Hom}^0_{\xi_U}(X_U, A) = \mathrm{Hom}^0_{\mathbf{Q}}(J_U, A)$. We consider the $\mathbf{Q}$-vector spaces

$$\sigma_i := \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, E_i), \qquad \sigma_i^\vee := \varinjlim_U \mathrm{Hom}^0_{\xi_U}(X_U, E_i^\vee).$$

The space σ_i admits a natural action by $\widehat{D}^\times$. Actually, $\sigma_i \otimes_{\mathbf{Q}} \mathbf{C} \simeq \otimes'_q \pi_{i,q}^D$ from which $\pi_{i,q}^D$ gains the structure of a $\mathbf{Q}$-vector space.

Let $h_{i,U} : J_U \to E_i$ and $h'_{i,U} : J_U \to E_i^\vee$ be $\mathbf{Q}$-morphisms. The morphism $h'^\vee_{i,U} : E_i \to J_U$ represents the homomorphism $h'^*_{i,U} : E_i \simeq \mathrm{Pic}^0(E_i) \to \mathrm{Pic}^0(J_U)$ composed with the canonical isomorphism $\mathrm{Pic}^0(J_U) \simeq J_U$ given by the Abel-Jacobi theorem. Define a perfect $\widehat{D}^\times$-invariant pairing $\sigma_i \otimes \sigma_i^\vee \to \mathbf{Q}$.

$$B_i^\natural(h_i \otimes h'_i) = \mathrm{vol}(X_U)^{-1} h_{i,U} \circ h'^\vee_{i,U} \in \mathrm{End}^0_{\mathbf{Q}}(E_i) = \mathbf{Q}.$$

Let $B^\natural := \otimes_{i=1}^3 B_i^\natural$ and define the trilinear form

$$I^\natural = \otimes_q I_q^\natural \in \mathrm{Hom}_{\widehat{D}^\times \times \widehat{D}^\times}\Big(\bigotimes_{i=1}^3 (\sigma_i \otimes \sigma_i^\vee), \mathbf{Q}\Big)$$

by

$$I_q^\natural(h_q \otimes h'_q) = \frac{\prod_{i=1}^3 L(1, \pi_{i,q}, \mathrm{ad})}{\zeta_q(2)^2 L\big(\frac{1}{2}, \pi_{1,q} \times \pi_{2,q} \times \pi_{3,q}\big)} \int_{\mathbf{Q}_q^\times \backslash D_q^\times} B_q^\natural((\sigma_{1,q} \otimes \sigma_{2,q} \otimes \sigma_{3,q})(g_q) h_q \otimes h'_q)\, \mathrm{d}g_q.$$

For each U we let Δ_U be the diagonal cycle of X_U^3 as an element in the Chow group $\mathrm{CH}^2(X_U^3)$ of codimension 2 cycles. We obtain a homologically trivial cycle Δ_{U,ξ_U} on X_U^3 by some modification with respect to ξ_U as constructed in

[5]. The classes $\Delta^{\dagger}_{U,\xi_U} = \frac{\Delta_{U,\xi_U}}{\mathrm{vol}(X_U)}$ form a projective system and define a class $\Delta^{\dagger}_{\xi} \in \varprojlim \mathrm{CH}^2(X_U^3)^0$.

Given $h_i \in \sigma_i$ for $i = 1, 2, 3$, we get a homologically trivial class

$$h_* \Delta^{\dagger}_{\xi} \in \mathrm{CH}^2(\mathbf{E})^0, \qquad\qquad h = h_1 \times h_2 \times h_3.$$

The theory of the p-adic height pairing was developed by Néron, Zarhin, Schneider, Mazur-Tate, Perrin-Riou, Nekovář. The p-adic height pairing depends on a choice of the p-adic logarithm on the idéle class group $\mathbf{A}^{\times}/\mathbf{Q}^{\times}$ and a choice of a splitting as $\mathbf{Q}_p$-vector spaces of the Hodge filtration of the de Rham cohomology of $\mathbf{E}$ over $\mathbf{Q}_p$. We take the Iwasawa logarithm $l_{\mathbf{Q}} : \mathbf{A}^{\times}/\mathbf{Q}^{\times} \to \mathbf{Q}_p$. Since $V_p^{\mathbf{E}}$ satisfies the Panchishkin condition, we have a natural choice of the splitting obtained from $\mathrm{Fil}^+ V_p^{\mathbf{E}}$. We may therefore say that there is a canonical p-adic height pairing $\langle\ ,\ \rangle_{\mathrm{Nek}}$ on homologically trivial cycles on $\mathbf{E}$. The following conjecture is a p-adic analgoue of the conjecture first formulated by Gross-Kudla [3] and later refined by Yuan, S. W. Zhang and W. Zhang [9].

Conjecture 2.

$$\langle h_* \Delta^{\dagger}_{\xi}, h'_* \Delta^{\dagger}_{\xi} \rangle_{\mathrm{Nek}} (\sqrt{-1})^3 \mathcal{E}_p(\mathrm{Fil}^+ V_p^{\boldsymbol{E}}) = L'_p(\boldsymbol{E}, 1) I^{\natural}(h \otimes h') \left(\frac{\zeta(2)}{\pi^2} \right)^2 \cdot (\textit{power of } 2)$$

for all $h \in \bigotimes_{i=1}^{3} (\sigma_i \otimes \sigma_i^{\vee})$, *where* $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$.

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Rigid Meromorphic Cocycles

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(joint work with Alice Pozzi, Jan Vonk)

A *rigid meromorphic cocycle* is a class in $H^1(\Gamma, \mathcal{M}^\times)$, where $\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p])$ is Ihara's group and $\mathcal{M}^\times$ is the multiplicative group of non-zero rigid meromorphic functions on the Drinfeld p-adic upper half-plane $\mathcal{H}_p := \mathbb{P}_1(\mathbb{C}_p) - \mathbb{P}_1(\mathbb{Q}_p)$, endowed with its natural action of Γ by Möbius transformations.

It is a prototypical example of more general objects arising in a tentative "p-adic Borcherds theory" which was briefly alluded to at the end of the lecture.

The main motivation (so far) for studying rigid meromorphic cocycles lies in their eventual connection with explicit class field theory for real quadratic fields. More precisely, a point $\tau \in \mathcal{H}_p$ is called a *real multiplication* (RM) point if it satisfies the following equivalent properties:

(1) The field $\mathbb{Q}(\tau)$ is a real quadratic field;
(2) The stabiliser of τ in Γ is infinite.

When τ is an RM point, its stabiliser Γ_τ has rank one, and is generated up to torsion by an *automorph* $\gamma_\tau \in \Gamma$, which can be chosen consistently by fixing appropriate orientations. The *value* of the rigid meromorphic cocycle J at τ is defined to be

$$J[\tau] := J(\gamma_\tau)(\tau) \in \mathbb{C}_p^\times \cup \{0, \infty\}.$$

Although the rigid meromorphic function $J(\gamma_\tau)$ depends on the choice of a representative one-cocycle, the value of this function at τ depends only on the class of J in cohomology. The value $J[\tau]$ also depends only on the Γ-orbit of τ, i.e., $J[\gamma\tau] = J[\tau]$ for all $\gamma \in \Gamma$.

The stabiliser of the RM point τ in the matrix ring $M_2(\mathbb{Z}[1/p])$ is isomorphic to a $\mathbb{Z}[1/p]$-order, denoted $\mathcal{O}$, in the real quadratic field $F = \mathbb{Q}(\tau)$. Global class field theory gives a canonical identification

$$\mathrm{Pic}^+(\mathcal{O}) = \mathrm{Gal}(H_\tau/F),$$

where $\mathrm{Pic}^+(\mathcal{O})$ denotes the class group in the narrow sense of $\mathcal{O}$ — i.e., the Picard group of projective $\mathcal{O}$-modules equipped with an orientation at ∞. The abelian extension H_τ of F is called the *narrow ring class field* attached to τ. Together with cyclotomic fields, the narrow ring class fields generate almost the full maximal abelian extension of F.

The following conjecture was proposed in [8]:

Conjecture 1. *If J is a rigid meromorphic cocycle, then there is a finite extension H_J of $\mathbb{Q}$ — the "field of definition" of J — for which $J[\tau]$ belongs to the compositum H_J and H_τ, for all RM points τ of $\mathcal{H}_p$.*

Until recently, almost all of the evidence for Conjecture 1 has been numerical and experimental, but recent progress based on the theory of p-adic deformations of modular forms and their associated Galois representations has led to strong theoretical evidence as well, so that Conjecture 1 now appears to lie within the scope of available techniques.

The lecture focussed first on a few simple examples of rigid meromorphic cocycles and related objects:

(1) the *Dedekind-Rademacher cocycle*, an avatar of Siegel units;
(2) *elliptic modular cocycles* attached to elliptic curves of conductor p;
(3) genuine rigid meromorphic cocycles, which play the role of meromorphic modular functions like the j-function.

It concluded by attempting to place the theory of rigid meromorphic cocycles in the more general framework of automorphic forms on orthogonal groups.

Let $\mathcal{A}^\times \subset \mathcal{M}^\times$ be the multiplicative group of rigid analytic functions on $\mathcal{H}_p$. It turns out that there are no interesting genuine rigid analytic cocycles: the group $H^1(\Gamma, \mathcal{A}^\times)$ is generated by the class J_{triv} given by

$$J_{\mathrm{triv}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z) := (cz + d),$$

whose value at an RM point is the fundamental unit of the associated order. A richer class of examples is obtained by relaxing the definition and considering one-cochains that satisfy the cocycle relation up to certain multiplicative periods. The simplest example of such a "cocycle modulo periods" is the *Dedekind-Rademacher cocycle.*

To define this cocycle, we begin by noting the canonical identification

$$H^2(\Gamma, \mathbb{Q}) = H^1(\Gamma_0(p), \mathbb{Q})$$

arising from the fact that Γ is an amalgamated product of two copies of $\mathbf{SL}_2(\mathbb{Z})$ intersecting in a subgroup that is conjugate to $\Gamma_0(p)$. (This in turn follows from the transitive action of Γ on the edges of the Bruhat-Tits tree, in which the vertex and edge stabilisers are conjugate to $\mathbf{SL}_2(\mathbb{Z})$ and $\Gamma_0(p)$ respectively.) The *Dedekind-Rademacher two-cocycle* is the class $\alpha_{\mathrm{DR}} \in H^2(\Gamma, \mathbb{Z})$ that corresponds, under this identification, to the *Dedekind-Rademacher homomorphism* $\varphi_{\mathrm{DR}} \in H^1(\Gamma_0(p), \mathbb{Z})$ given by

$$\varphi_{\mathrm{DR}}(\gamma) := \frac{1}{2\pi i} \int_{z_0}^{\gamma z_0} E_2^{(p)}(z), \qquad E_2^{(p)} := \mathrm{dlog}\left(\frac{\Delta(pz)}{\Delta(z)}\right).$$

The one-cocycle φ_{DR} and the two-cocycle α_{DR} thus encode the periods of the Eisenstein series $E_2^{(p)}$ arising from the logarithmic derivative of the Siegel unit $\Delta(pz)/\Delta(z)$ on the open modular curve $Y_0(p)$.

The key fact underlying the construction of the Dedekind-Rademacher cocycle is that $p^{\alpha_{\mathrm{DR}}} \in H^2(\Gamma, \mathbb{C}_p^\times)$ *is trivialised* in the group $H^2(\Gamma, \mathcal{A}^\times)$. The following theorem from [7] refines an earlier construction from [3]:

Theorem 2. *There is a one-cochain* $J_{\mathrm{DR}} \in C^1(\Gamma, \mathcal{A}^\times)$ *satisfying*

$$\gamma_1 J_{\mathrm{DR}}(\gamma_2) \div J_{\mathrm{DR}}(\gamma_1\gamma_2) \times J_{\mathrm{DR}}(\gamma_1) = p^{\alpha_{\mathrm{DR}}(\gamma_1,\gamma_2)}, \qquad \textit{for all } \gamma_1, \gamma_2 \in \Gamma.$$

The essential triviality of $H^1(\Gamma, \mathcal{A}^\times)$ ensures that the one co-chain J_{DR} of Theorem 2 is well defined, up to powers of J_{triv} and one-coboundaries. Furthermore, it satisfies the one-cocycle relation up to powers of p. Its image in $H^1(\Gamma, \mathcal{A}^\times/p^{\mathbb{Z}})$ is therefore well-defined up to powers of J_{triv}. This image is called the *Dedekind-Rademacher cocycle*, and is also denoted by J_{DR}, by a slight abuse of notation.

Theorem 3. *Let τ be an RM point in $\mathcal{H}_p$. Up to torsion in $\mathbb{Q}_p(\tau)^\times$, the value $J_{\mathrm{DR}}[\tau]$ belongs to $\mathcal{O}_{H_\tau}[1/p]^\times$.*

This proof of this theorem emerged from a gradual series of developments over almost two decades:

1. *Relation with the Gross-Stark conjecture.* The relation

$$\log_p(\mathrm{Norm}^{\mathbb{Q}_p^2}_{\mathbb{Q}_p}(J_{\mathrm{DR}}[\tau])) = L'_p(\mathcal{C}_\tau, 0),$$

where $L_p(\mathcal{C}_\tau, s)$ is the partial p-adic L-function attached to the narrow ideal class $\mathcal{C}_\tau := [1,\tau]$ was obtained in [3]. This result shows the algebraicity properties of $J_{\mathrm{DR}}[\tau]$ asserted in Theorem 3 are satisfied by its *norm* from $(F \otimes \mathbb{Q}_p)^\times$ to $\mathbb{Q}_p^\times$, assuming Gross's p-adic analogue of the Stark conjecture on leading terms of abelian L-series at $s = 0$ [12].

2. *Proof of the Gross-Stark conjecture in rank one.* It was then shown in [4] that the Gross-Stark conjecture is true for the first derivatives of these abelian L-series, hence that $J_{\mathrm{DR}}[\tau]$ satisfies the predicted algebraicity properties, *up to torsion and elements of $(F \otimes \mathbb{Q}_p)^\times$ of norm one.* What makes Gross's p-adic analogue of the Stark conjecture more approachable than the original archimedean conjectures, which are still completely open, is the availability of the theory of p-adic deformations of modular forms and their associated p-adic Galois representations, along with the reciprocity law of global class field theory. In a sense, these ingredients are used to parlay class field theory for abelian extensions of F into explicit class field theory for F.

3. *The work of Dasgupta and Kakde.* The ambiguity by elements of norm one in $\mathbb{Q}_{p^2}^\times$ is a serious limitation of the results following from [4]. It is addressed in a remarkable series of recent works by Samit Dasgupta and Mahesh Kakde, who show that the full Theorem 3 (up to torsion) would follow from Gross's "tame refinement" of the p-adic Gross-Stark conjecture [13] and the Brumer–Stark conjecture. They are then able to *prove* these conjectures, by significantly refining and extending the techniques of [4] to the tame setting.

4. *Modular generating series.* An ongoing project with Alice Pozzi and Jan Vonk [7] aims to give an alternate, more "genuinely p-adic" proof of Theorem 3 by realising the RM values of the Dedekind-Rademacher cocycle as the fourier coefficients of a modular generating series. This approach is suggested by [6] which studies the fourier coefficients of the ordinary projection of the first derivative, with respect to the weight, of the diagonal restriction of a p-adic family of Hilbert modular Eisenstein series attached to totally odd characters of F.

The methods used to understand the RM values of J_{DR} are somewhat roundabout, relying crucially on p-adic deformations of modular forms and their associated Galois representations. These techniques do not dispel the mystery surrounding the deeper arithmetic meaning of rigid meromorphic cocycles. To underscore this point, the lecture then turned to a discussion of *elliptic cocycles*.

An elliptic rigid analytic cocycle can be attached to an elliptic curve E of conductor p, or rather to its normalised newform f of weight two, whose real and imaginary periods are encoded by two one-cocycles φ_f^+ and φ_f^- in $H^1(\Gamma_0(p), \mathbb{Z})$. Let α_f^+ and α_f^- be the corresponding two-cocycles in $H^2(\Gamma, \mathbb{Z})$. The following trivialisation result, in which α_{DR} is replaced by $\alpha_f^{\pm}$, and the prime p by the Tate period $q_E \in \mathbb{Q}_p^\times$ attached to E over $\mathbb{Q}_p$, was shown in [2] to follow from the "exceptional zero conjecture" of Mazur, Tate and Teitelbaum [17] proved by Greenberg and Stevens [14].

Theorem 4. *There are one-cochains $J_{\mathrm{f}}^+, J_f^- \in C^1(\Gamma, \mathcal{A}^\times)$ satisfying*

$$\gamma_1 J_f^{\pm}(\gamma_2) \div J_f^{\pm}(\gamma_1\gamma_2) \times J_f^{\pm}(\gamma_1) = q_E^{\alpha_f^{\pm}(\gamma_1,\gamma_2)}, \qquad \text{for all } \gamma_1, \gamma_2 \in \Gamma,$$

up to torsion in $(F \otimes \mathbb{Q}_p)^\times$.

In particular, the one-cochains J_f^+ and J_f^- satisfy the cocycle relation up to powers of q_E and up to torsion. After raising them to a suitable integer power to remove the torsion ambiguity, their images in $H^1(\Gamma, \mathcal{A}^\times/q_E^{\mathbb{Z}})$ are called the *even and odd elliptic modular cocycles* attached to f, and denoted $J_f^{\pm}$ by abuse of notation.

The RM values $J_f^{\pm}[\tau]$ are then canonical elements of $\mathbb{C}_p^\times/q_E^{\mathbb{Z}} = E(\mathbb{C}_p)$. The following conjecture, an elliptic analogue of Theorem 3, was proposed in [2].

Conjecture 5. *The RM values $J_f^+[\tau]$ and $J_f^-[\tau]$ belong to $E(H_\tau)$.*

The evidence for this conjecture so far is largely experimental [5], [11], and the speaker can discern no proof on the horizon: the tools used to handle the Dedekind-Rademacher cocycle are not available in this setting, and would perhaps need to be supplemented with a more geometric perspective on the theory of rigid cocycles.

The lecture concluded by returning to rigid meromorphic cocycles. In contrast with the fact that $H^1(\Gamma, \mathcal{A}^\times)$ is essentially trivial, the following theorem from [8] shows that rigid meromorphic cocycles exist in abundance:

Theorem 6. *Assume that $p-1$ divides 12, i.e, $p = 2, 3, 5, 7,$ or 13. For all RM points τ, there is a unique $J_\tau \in H^1(\Gamma, \mathcal{M}^\times)$ for which the rigid meromorphic functions $J_\tau(\gamma)$ have divisor supported in the Γ-orbit of τ.*

In [8], it is conjectured that the quantity

$$J_p(\tau_1, \tau_2) := J_{\tau_1}[\tau_2],$$

for pairs (τ_1, τ_2) of RM points (with co-prime discriminants, say) behave "in essentially all respects" like the differences $j(\tau_1) - j(\tau_2)$ of singular moduli studied in the work of Gross and Zagier (with τ_1 and τ_2 CM points in the Poincaré upper half-plane). The p-adic logarithm of $J_p(\tau_1, \tau_2)$ can thus be envisaged as a kind of p-adic Green's function evaluated on the pair of RM cycles attached to τ_1 and τ_2.

For example, let $\tau_1 = \varphi := (1+\sqrt{5})/2$ be the golden ratio, and $\tau_2 = 2\sqrt{2}$, which are roots of primitive binary quadratic forms of discriminants $D_1 = 5$ and $D_2 = 32$ respectively. The associated ring class fields are $H_1 = \mathbb{Q}(\sqrt{5})$ and $H_2 = \mathbb{Q}(\sqrt{2}, i)$. The pair (τ_1, τ_2) belongs to $\mathcal{H}_p \times \mathcal{H}_p$ when $p = 3$ or $p = 13$, and a computer calculation reveals that

$$\begin{aligned} J_3(\varphi, 2\sqrt{2}) &\equiv (33+56i)/(5\cdot 13) \pmod{3^{600}}, \\ J_{13}(\varphi, 2\sqrt{2}) &\equiv (1+2\sqrt{-2})/3 \pmod{13^{100}}. \end{aligned}$$

The table below lists the prime factorisations of the quantities $(D_1D_2 - t^2)/4$ when t ranges over the even integers between 0 and 12:

t	$(160-t^2)/4$	$a_3(t)$	$a_{13}(t)$
0	$2^3\cdot 5$	1	1
2	$3\cdot 13$	13	3
4	$2^2 3^2$	1	1
6	31	1	1
8	$2^3\cdot 3$	1	1
10	$3\cdot 5$	5	1
12	2^2	1	1

The two rightmost columns are obtained by picking out the terms in the second column that are divisible by 3 and 13 respectively, and taking the remaining factors. These are exactly the primes that appear in the experimentally observed factorisations of $J_3(\tau_1, \tau_2)$ and $J_{13}(\tau_1, \tau_2)$. Such patterns are reminiscent of the recipes for the factorisation of singular moduli in the theorem of Gross and Zagier [15].

The restriction on p in Theorem 6 is made to ensure that the modular curve $X_0(p)$ has genus zero, i.e., that there are no weight two cusp forms of level p. For general primes p, there is an *obstruction* to producing a rigid meromorphic cocycle with a prescribed rational RM divisor, which lies in the space of weight two cusp forms on $\Gamma_0(p)$ – or equivalently, by the Shimura correspondence, in (the Kohnen subspace of) the space $M_{3/2}(4p)$ of modular forms of weight 3/2 and level $4p$. Let $M^{!!}_{1/2}(4p)$ be the space of weakly holomorphic modular forms of weight 1/2 and level $4p$ which are regular at all the cusps except ∞ and have integer fourier coefficients at that cusp. The following theorem, which produces a systematic supply of rigid meromorphic cocycles for all primes p, is part of a work in progress [9]:

Theorem 7. *There is an injective homomorphism*

$$\mathrm{BL}^\times : M^{!!}_{1/2}(4p) \to H^1(\Gamma, \mathcal{M}^\times)$$

satisfying all the formal properties of Borcherds' multiplicative "singular theta-lift". (Notably, the RM divisor of $\mathrm{BL}^\times(g)$ *is encoded in the principal part of* g*.)*

This result suggests the following immediate generalisation, placing the theory of rigid meromorphic cocycles in the broader setting of p-arithmetic subgroups of orthogonal groups. Let V be a quadratic space over $\mathbb{Q}$ of real signature (r, s), and let $d := r + s$ be its dimension. Let $L \subset V$ be a $\mathbb{Z}[1/p]$-lattice in V, and assume for simplicity that it is equal to its dual. The orthogonal group $\Gamma := O(L)$ of this lattice is a p-arithmetic group which acts on the real symmetric space

$$X_\infty := O(V)/(O(r) \times O(s))$$

of dimension rs, as well as on the p-adic symmetric space

$$X_p := \tilde{X}_p/\mathbb{C}_p^\times, \qquad \tilde{X}_p := \{x \in V \otimes \mathbb{C}_p \text{ with } \langle x, x\rangle = 0\} - \bigcup_{\langle v,v\rangle=1} (\mathbb{Q}_p v)^\perp,$$

the union being taken over all vectors $v \in V \otimes \mathbb{Q}_p$ of norm 1. The domain X_p can be identified with the $\mathbb{C}_p$-points of a rigid analytic space over $\mathbb{Q}_p$. Let $\mathcal{M}^\times$ be the multiplicative group of non zero rigid meromorphic functions on X_p. Theorem 7 is generalised in [10] to give an injective homomorphism

$$\mathrm{BL}^\times : M^{!!}_{2-d/2}(4p) \to H^s(\Gamma, \mathcal{M}^\times)$$

with the requisite properties, most importantly, that the divisor of $\mathrm{BL}^\times(g)$ is related to a collection of "rational quadratic divisors" on X_p which can be read off from the principal part of g. The construction of $\mathrm{BL}^\times$ rests crucially on the Kudla-Millson theta kernels [16] with coefficients in the homology of the real manifold $X_\infty/O(V, \mathbb{Z})$, and on the variation of these theta-kernels in p-adic families.

In the case of signature $(3, 0)$, taking V to be the space of trace zero elements on a definite quaternion algebra B over $\mathbb{Q}$ equipped with the norm form, the image of the p-adic Borcherds lift consists of Γ-invariant meromorphic functions on $\mathcal{H}_p$ with divisor supported on CM points. The rigid analytic quotient $\Gamma\backslash\mathcal{H}_p$ is identified with the $\mathbb{C}_p$-points of a Shimura curve attached to B, thanks to the uniformisation theory of Cerednik Drinfeld. The p-adic Borcherds lift in this case leads to a new, "purely p-adic" proof of the theorem of Gross-Kohnen-Zagier asserting that a generating series made from Heegner divisors of varying discriminants is a modular form of weight 3/2.

In the case of signature $(2, 1)$, taking V to be the space of trace zero elements in the matrix ring $M_2(\mathbb{Q})$ and $L := M_2(\mathbb{Z}[1/p]) \cap V$, the image of the p-adic Borcherds lift is the "original" space of rigid meromorphic cocycles for $\Gamma := \mathbf{SL}_2(\mathbb{Z}[1/p])$ that provided the starting point for this lecture.

The case of signature $(3, 1)$ is noteworthy in light of the "accidental" isomorphisms relating the orthogonal group of this signature and the Bianchi group

$\mathbf{SL}_2(\mathcal{O}_K)$ where K is an imaginary quadratic field. That the resulting rigid meromorphic cocycles should have arithmetic significance is suggested by the work of Mak Trifkovic [19] and of Daniel Barrera and Chris Williams [1] on analytic cocycles in the Bianchi setting. Peter Scholze has proposed [Sch] that (additive, weight one) rigid analytic cocycles on the Bianchi modular group might also be relevant for understanding the conjectural correspondence between Maass forms with Laplace eigenvalue 1/4 and even two-dimensional Artin representations.

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On some theta lifts and their applications

Claudia Alfes-Neumann
(joint work with Kathrin Bringmann and Markus Schwagenscheidt)

We present recent results on extensions of the Shintani [6] and Kudla-Millson [5] theta lift to meromorphic cusp forms as well as some applications of this theory. In particular, we present finite rational formulas for the traces of cycle integrals of certain meromorphic modular forms. Moreover, we show the modularity of a completion of the generating series of the traces of reciprocal singular moduli.

Historically, theta lifts of this kind have first been considered by Shimura, Shintani, and Niwa [14, 15, 13]. They considered the generating series of traces of cycle integrals of integral weight cusp forms and showed that it is a half-integral weight modular form.

In his seminal paper [17] Zagier showed that the generating series of the (twisted) traces of singular moduli, values of the modular j-invariant at quadratic irrationalities, is a weakly holomorphic modular form of weight $1/2$ resp. $3/2$. Bruinier and Funke [10] generalized this result to arbitrary weakly holomorphic modular forms of weight 0 using a theta lift whose kernel is given by a certain theta function first considered by Kudla and Millson (accounting for the name *Kudla-Millson lift*). They obtained a generating series of the CM traces of the input function of weight $3/2$. Their result was generalized to the twisted case by the author and Ehlen in [1]. This lift was extended by Bruinier and Ono [9] and the author [2] to general negative even weight. A similar lift, using the *Millson* kernel, was considered by the author and Schwagenscheidt [3]. In particular, their work generalizes Zagier's results for weight $1/2$.

In later work, the author and Schwagenscheidt [4] as well as Bringmann, Guerzhoy and Kane [7, 8] considered extensions of the Shintani theta lift to modular forms with poles at infinity (resp. harmonic weak Maass forms). In [11], the authors consider the Shintani theta lift of differentials of the third kind making a first step towards considering meromorphic modular forms as inputs for such theta lifts.

In the following, we let $\mathcal{Q}_d$ denote the set of all integral binary quadratic forms $Q = [a, b, c]$ of discriminant $d = b^2 - 4ac$. Note that the group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{Q}_d$ with finitely many orbits if $d \neq 0$.

For $d < 0$ one is led to study CM points z_Q, i.e. the zeros of $Q(z_Q, 1)$ which lie in the upper half-plane. For $d > 0$ we obtain a geodesic

$$C_Q := \left\{ z \in \mathbb{H} : a|z|^2 + bx + c = 0 \right\} \qquad (z = x + iy).$$

By $c_Q := \Gamma_Q \backslash C_Q$ we denote the image in $\Gamma \backslash \mathbb{H}$ of the geodesic C_Q.

1. Shintani theta lifts of meromorphic cusp forms

In [6] we extend the Shintani lift to meromorphic cusp forms. We obtain the following modularity result.

Theorem 1. *For a meromorphic cusp form f of positive weight $2k$ the (twisted) Shintani theta lift of f is a real-analytic function on $\mathbb{H}$ that transforms like a modular form of weight $k+\frac{1}{2}$ for $\Gamma_0(4)$ and satisfies the Kohnen plus space condition.*

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant satisfying $(-1)^k\Delta > 0$ and let $D > 0$ be a discriminant. It turns out that the Shintani lift yields a completion of the generating function of twisted traces of regularized cycle integrals

$$\mathbf{tr}_{f,\Delta}(D) := \sum_{Q \in \mathcal{Q}_{|\Delta|D}/\Gamma} \chi_\Delta(Q) \int_{c_Q}^{\mathrm{reg}} f(z)Q(z,1)^{k-1}dz,$$

where χ_Δ is the usual genus character, and the cycle integrals have to be regularized if poles of f lie on the geodesic c_Q (for detail see Section 2.3 of [6]).

We have the following theorem.

Theorem 2. *Let f be a meromorphic cusp form of positive weight $2k$. Then the "holomorphic part" of the Fourier expansion of the Shintani theta lift of f is given by*

$$\frac{\sqrt{|\Delta|}}{2} \sum_{D>0} \mathbf{tr}_{f,\Delta}(D) e^{2\pi i D\tau}.$$

The "real-analytic part" is given by a certain linear combination of "singular" theta series.

Let $k > 0$ be even. Using the results on the Shintani lift we can then obtain rationality results for the traces of cycle integrals of the functions

$$f_{k,\mathcal{A}}(z) := \frac{|d|^{\frac{k+1}{2}}}{\pi} \sum_{Q\in\mathcal{A}} Q(z,1)^{-k}, \tag{1}$$

where $\mathcal{A} \in \mathcal{Q}_d/\Gamma$ is a fixed equivalence class of quadratic forms of discriminant $d < 0$. The poles of $f_{k,\mathcal{A}}$ lie at the CM points $z_Q \in \mathbb{H}$ for $Q \in \mathcal{A}$. Such functions were first considered by Zagier in [16]. Kohnen and Zagier [12] showed that certain simple linear combinations of the cycle integrals of such functions are rational.

Complementing these results, we obtain rational formulas for the traces

$$\mathbf{tr}_{f_{k,\mathcal{A}}}(D) := \sum_{Q\in\mathcal{Q}_D/\Gamma} \int_{c_Q} f_{k,\mathcal{A}}(z)Q(z,1)^{k-1}dz.$$

We assume that the poles of $f_{k,\mathcal{A}}$ do not lie on any of the geodesics C_Q for $Q \in \mathcal{Q}_D$. Let $z_{\mathcal{A}} := x_{\mathcal{A}} + iy_{\mathcal{A}} \in \mathbb{H}$ denote a fixed CM point z_Q for some $Q \in \mathcal{A}$.

Theorem 3. *Let F be a weakly holomorphic modular form of weight $\frac{3}{2}-k$ for $\Gamma_0(4)$ satisfying the Kohnen plus space condition. Suppose that the Fourier coefficients $a_F(-D)$ vanish for all $D > 0$ which are squares and that $a_F(-D)$ is rational for*

$D > 0$. Moreover, assume that $z_{\mathcal{A}}$ does not lie on any of the geodesics C_Q for $Q \in \mathcal{Q}_D$ for any $D > 0$ for which $a_F(-D) \neq 0$. Then the linear combinations

$$\sum_{D>0} a_F(-D)\mathbf{tr}_{f_{k,\mathcal{A}}}(D)$$

are rational.

2. Traces of reciprocal singular moduli

Complementing the results of Zagier on the modularity of the generating series of the traces of singular moduli in [17] we show the modularity of the generating series of the traces of reciprocal singular moduli

$$\mathbf{tr}_{1/j}(D) = \sum_{Q \in \mathcal{Q}_D^+/\Gamma} \frac{1/j(z_Q)}{|\overline{\Gamma}_Q|},$$

where we now consider the set of positive definite quadratic forms $\mathcal{Q}_D^+$ of discriminant $D < 0$. Notice that $1/j$ has a third order pole at $\rho = e^{\pi i/3}$, so the (CM) value of $1/j$ at this point is not defined. However, if we replace $1/j(\rho)$ by the constant term in the elliptic expansion of $1/j$ around ρ, then $\mathbf{tr}_{1/j}(D)$ is defined for every $D < 0$.

We obtain the following modularity statement.

Theorem 4. *The generating series*

$$\sum_{D \leq 0} \mathbf{tr}_{1/j}(D) q^{-D} = -\frac{1}{165888} + \frac{23}{331776}q^3 + \frac{1}{3456}q^4 - \frac{1}{3375}q^7 + \frac{1}{8000}q^8 + \dots$$

of traces of reciprocal singular moduli is a mixed mock modular form of weight $3/2$ for $\Gamma_0(4)$ (of higher depth). Its shadow is a non-zero multiple of

$$\sum_{h \pmod 3} \theta_{7/2,h}(\tau) \cdot v^4 \overline{\theta_{4,h}(\tau)}.$$

Here, we let

$$\theta_{7/2,h}(\tau) = v^{-3/2} \sum_{\substack{a \in \mathbb{Z} \\ a \equiv h \pmod 3}} H_3\left(2\sqrt{\pi v}\frac{a}{\sqrt{3}}\right) q^{a^2/3},$$

$$\theta_{4,h}(\tau) = \sum_{\substack{b,c \in \mathbb{Z} \\ b \equiv c \pmod 2 \\ b \equiv h \pmod 3}} \left(b - i\sqrt{3}c\right)^3 q^{b^2/3 + c^2},$$

with $\tau = u + iv \in \mathbb{H}$ and the Hermite polynomial $H_3(x) = 8x^3 - 12x$. Moreover, we define

$$\mathbf{tr}_{1/j}(0) = -\frac{1}{4\pi} \int_{\Gamma \backslash \mathbb{H}}^{reg} 1/j(z) \frac{dxdy}{y^2} = -\frac{1}{2^{11} \cdot 3^4} = -\frac{1}{165888}.$$

In [5] we show that a result of this kind holds for arbitrary meromorphic cusp forms of weight 0 (replacing j) using the Kudla-Millson theta lift.

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On the Kohnen plus space for Jacobi forms of half-integral weight

Ren-He Su

Let F be a totally real number field, $\mathfrak{o} = \mathfrak{d}_F$ being its ring of integers. Let $k \geq 0$, $n \geq 1$ and $r \geq 0$ be integers, where k is odd. Also, for any positive integer m, we set $L_m^{*,>0}$ to be the set of half-integral positive semi-definite matrices of size $m \times m$ with entrices in F. Then we set $\mathcal{M}$ to a matrix be in $L_{r+1}^{*,>0}$ which has the form

$$\mathcal{M} = \begin{pmatrix} \mathcal{M}_1 & L/2 \\ {}^tL/2 & 1 \end{pmatrix}$$

where $\mathcal{M}_1 \in L_r^{*,>0}$ and

$$\mathfrak{M} = 4\mathcal{M}_1 - L\,{}^tL.$$

Note that we have $\mathfrak{M} > 0$. Put $J_{k+1/2,\mathfrak{M}}^{(\mathrm{n})}$ and $J_{k+1,\mathcal{M}}^{(\mathrm{n})}$ to be the spaces of Jacob cusp forms of weight $k+1/2$ and $k+1$, matrix index $\mathfrak{M}$ and $\mathcal{M}$, respectively.

Note that any Jacobi form in $J^{(\mathrm{n})}_{k+1/2,\mathfrak{M}}$ ($J^{(\mathrm{n})}_{k+1,\mathcal{M}}$) is a holomorphic function on $(\mathfrak{h}_n \times \mathbb{C}^{n\times r})^{[F:\mathbb{Q}]}$ $((\mathfrak{h}_n \times \mathbb{C}^{n\times (r+1)})^{[F:\mathbb{Q}]})$. We want to define the Kohnen plus space of Jacobi forms of weight $k+1/2$ and index $\mathfrak{M}$, which is a subspace of $J^{(\mathrm{n})}_{k+1/2,\mathfrak{M}}$, and give the main result that it is isomorphic to $J^{(\mathrm{n})}_{k+1,\mathcal{M}}$. Now for any $\phi \in J^{(\mathrm{n})}_{k+1/2,\mathfrak{M}}$, we write its Fourier expansion as

$$\phi(\tau, z) = \sum_{M \in L_n^{*,>0}, S \in M_{n,r}(\mathfrak{o})} C(M, S) e(M\tau + S^t z).$$

Then we have the following definition of Kohnen plus space.

Definition 1. *The Kohnen plus space $J^{(\mathrm{n}),+}_{k+1/2,\mathfrak{M}}$ of Jacob cusp forms of weight $k+1/2$ and index $\mathfrak{M}$ is a subspace of $J^{(\mathrm{n})}_{k+1/2,\mathfrak{M}}$ which consists the forms $\phi(\tau, z)$ whose Fourier coefficients $C(M, S)$ vanish unless*

$$\begin{pmatrix} M & S/2 \\ {}^tS/2 & \mathfrak{M} \end{pmatrix} \equiv - \begin{pmatrix} \lambda \\ L \end{pmatrix} \begin{pmatrix} {}^t\lambda & {}^tL \end{pmatrix} (\mathrm{mod} 4L^*_{n+r})$$

for some $\lambda \in \mathfrak{o}^n$.

For $z' \in (\mathbb{C}^{n\times(r+1)})^{[F:\mathbb{Q}]}$, we write $z' = (z_1\ z_2)$, where $z_1 \in (\mathbb{C}^{n\times r})^{[F:\mathbb{Q}]}$ and $z_2 \in (\mathbb{C}^{n\times 1})^{[F:\mathbb{Q}]}$. Now for any $\psi(\tau, z') \in J^{(\mathrm{n})}_{k+1,\mathcal{M}}$, the theta decomposition of ψ is given by

$$\psi(\tau, z') = \sum_{\lambda \in \mathfrak{o}^n} \psi_\lambda(\tau, z_1) \theta_\lambda(\tau, \frac{1}{2} z_1 L + z_2).$$

Then if we put

$$\iota(\psi)(\tau, z) = \sum_{\lambda \in \mathfrak{o}^n} \psi_\lambda(4\tau, 4z),$$

this is a Jacobi cusp form in $J^{(\mathrm{n}),+}_{k+1/2,\mathfrak{M}}$. The main result is as following.

Theorem 2. *The mapping $\iota : J^{(\mathrm{n})}_{k+1,\mathcal{M}} \to J^{(\mathrm{n}),+}_{k+1/2,\mathfrak{M}}$ is an isomorphism.*

This result is an analogue of several previous results. For the classical case, that is, the case $F = \mathbb{Q}$, $n = 1$ and $r = 0$, the result was shown by Eichler and Zagier in [1]. For the case $F = \mathbb{Q}$, $n \geq 2$ and $r = 0$, the result was shown by Ibukiyama in [4]. For the case F general, $n = 1$ and $r = 0$, the result was shown by Hiraga and Ikeda in [3]. For the case $F = \mathbb{Q}$, $n \geq 1$ and $r \geq 1$, the result was shown by Hayashida in [2].

In the previous results above, the corresponding isomorphism in each case was shown to be a Hecke isomorphism. The author of this abstract is also looking forward to show that the ismorphism mentioned above is also a Hecke isomorphism.

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Doubling integrals for Brylinski-Deligne extensions of classical groups

Yuanqing Cai

In the 1980s, Piatetski-Shapiro and Rallis [8] discovered a family of Rankin-Selberg integrals for the classical groups that did not rely on Whittaker models. This is the so-called doubling method. Recently, a family of global integrals that represent the tensor product L-functions for classical groups [5] and the tensor product L-functions for covers of symplectic groups [7] was discovered. These integrals are called twisted doubling integrals and can be viewed as generalizations of the doubling method. In this note, we explain how to develop the twisted doubling integrals for all classical groups and their non-linear extensions in a more conceptual manner (see [2, 3]).

For simplicity, we will focus on the symplectic and orthogonal groups. We will describe the global integral in the linear case, and discuss some necessary modifications in the covering group case.

1. The global integral in the linear case

We start by describing the global integral in the linear case. Let F be a number field and $\mathbb{A}$ be its ring of adeles. Let $(W, \langle\ ,\ \rangle)$ be a quadratic space over F of dimension m. Let G be either $\mathrm{Sp}(W)$ or $\mathrm{SO}(W)$.

We first construct a doubled quadratic space $(W^\square, \langle\ ,\ \rangle^\square)$, where $W^\square = W_+ \oplus W_-$ and

$$\langle (x_+, x_-), (y_+, y_-)\rangle^\square = \langle x_+, y_+\rangle - \langle x_-, y_-\rangle.$$

We include the subscripts $\pm$ when it is necessary to distinguish the two copies of W. Define

$$W^\Delta = \{(x, x) \in W^\square \mid x \in W\}, \qquad W^\nabla = \{(x, -x) \in W^\square \mid x \in W\}.$$

Both are maximal totally isotropic subspaces of $W^\square$.

Fix a positive integer k. We now define a large quadratic space $(W^{\square,k}, \langle\ ,\ \rangle^{\square,k})$ where

$$(W^{\square,k}, \langle\ ,\ \rangle^{\square,k}) = (W_1^\square, \langle\ ,\ \rangle_1^\square) \oplus \cdots \oplus (W_k^\square, \langle\ ,\ \rangle_k^\square).$$

The subscripts here are included to distinguish different copies. Then

$$W^{\Delta,k} := W_1^\Delta \oplus \cdots \oplus W_k^\Delta, \qquad W^{\nabla,k} := W_1^\nabla \oplus \cdots \oplus W_k^\nabla$$

are maximal totally isotropic subspaces of $W^{\square,k}$. Let $G^{\square,k}$ be the connected component of the isometry group of $(W^{\square,k}, \langle\ ,\ \rangle^{\square,k})$.

We choose the following flag of totally isotropic subspaces:

$$0 \subset W_k^\nabla \subset W_{k-1}^\nabla \oplus W_k^\nabla \subset \cdots \subset W_2^\nabla \oplus \cdots \oplus W_k^\nabla.$$

Let $N_{\mathcal{W}}$ be the unipotent radical of the parabolic subgroup stabilizing this flag. Then

$$N_{\mathcal{W}}/[N_{\mathcal{W}}, N_{\mathcal{W}}] \simeq \mathrm{Hom}(W_{k-1}^{\nabla}, W_k^{\nabla}) \times \cdots \times \mathrm{Hom}(W_2^{\nabla}, W_3^{\nabla}) \times \mathrm{Hom}(W_1^{\square}, W_2^{\nabla}).$$

We also write this map as

$$u \mapsto (u_{k-1}, \cdots, u_1).$$

Let $\psi : F\backslash\mathbb{A} \to \mathbb{C}^{\times}$ be a nontrivial additive character. Let em be the embedding

$$\mathrm{em} : W_{1,+} \to W_1^{\square}, \qquad x_{1,+} \mapsto (x_{1,+}, 0).$$

We now define

$$\psi_{\mathcal{W}} : N_{\mathcal{W}}(F)\backslash N_{\mathcal{W}}(\mathbb{A}) \to \mathbb{C}^{\times},$$

by

$$\psi_{\mathcal{W}}(u) = \psi\left(\mathrm{tr}(u_{k-1} + \cdots + u_2) + \mathrm{tr}(u_1 \circ \mathrm{em})\right).$$

Given $(g_1, g_2) \in G \times G$, we define its action on $W^{\square,k}$ by the following formula

$$\begin{aligned}&(g_1, g_2) \cdot (x_{1,+},\ x_{1,-},\ x_{2,+},\ x_{2,-},\ \cdots,\ x_{k,+},\ x_{k,-})\\ =&(g_1x_{1,+},\ g_2x_{1,-},\ g_1x_{2,+},\ g_1x_{2,-},\ \cdots,\ g_1x_{k,+},\ g_1x_{k,-}).\end{aligned}$$

This defines a homomorphism $\iota : G\times G \to G^{\square,k}$. It is easy to check that the image $\iota(G \times G)$ lies in the stabilizer of $(N_{\mathcal{W}}, \psi_{\mathcal{W}})$.

Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_k(\mathbb{A})$. Let $\theta(m, \tau)$ be the generalized Speh representation of $\mathrm{GL}_{km}(\mathbb{A})$, constructed using residues of Eisenstein series. A key property of $\theta(m, \tau)$ is that the maximal nilpotent orbit that supports a non-zero Fourier coefficient for $\theta(m, \tau)$ is (k^m). At every local place v, the corresponding local model for $\theta(m, \tau)_v$ is unique.

Remark 1. *Here we describe two extreme cases of $\theta(m, \tau)$. When $k = 1$, then $\theta(m, \tau) = \tau \circ \det$. When $m = 1$, then $\theta(m, \tau) = \tau$.*

Let $P = P(W^{\Delta,k})$. We now consider the induced representation

$$I(s, \tau) = \mathrm{Ind}_{P(\mathbb{A})}^{G^{\square,k}(\mathbb{A})}(\theta \cdot \delta_P^s).$$

For any holomorphic section $\phi^{(s)}$ of $I(s, \tau)$, we form the associated Eisenstein series $E(\phi^{(s)})$ on $G^{\square,k}(F)\backslash G^{\square,k}(\mathbb{A})$ by

$$E(\phi^{(s)})(g) = \sum_{\gamma \in P(F)\backslash G^{\square,k}(F)} \phi^{(s)}(\gamma g).$$

The Eisenstein series converges for $\Re s \gg 0$. By the theory of Eisenstein series, it can be continued to a meromorphic function in s on all of $\mathbb{C}$ satisfying a functional equation.

Let π be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$. Let $\xi_1 \in \pi$ and $\xi_2 \in \pi^{\vee}$. We now define the following global zeta integral:

$$\begin{aligned}&Z(\xi_1 \boxtimes \xi_2, \phi^{(s)})\\ =&\int_{G(F)\backslash G(\mathbb{A})\times G(F)\backslash G(\mathbb{A})} \xi_1(g_1)\xi_2(g_2) \int_{N_{\mathcal{W}}(F)\backslash N_{\mathcal{W}}(\mathbb{A})} E(\phi^{(s)})(u \cdot \iota(g_1, g_2))\psi_{\mathcal{W}}(u)\ du\ dg_1\ dg_2.\end{aligned}$$

Remark 2. *When $k = 1$, this recovers the doubling method of Piatetski-Shapiro and Rallis.*

Let $N_{\mathcal{W}}^{\circ} = N(W^{\nabla,k}) \cap P$ and

$$f^{(s)}(g) = \int_{(P\cap N_{\mathcal{W}})(F)\backslash(P\cap N_{\mathcal{W}})(\mathbb{A})} \phi^{(s)}(ug)\psi_{\mathcal{W}}(u)\ du.$$

Now we can state the following global identity.

Theorem 3. *When $\Re s \gg 0$,*

$$Z(\xi_1 \boxtimes \xi_2, \phi^{(s)}) = \int_{G(\mathbb{A})} \mathcal{P}(\pi(g)\xi_1 \boxtimes \xi_2) \int_{N_{\mathcal{W}}^{\circ}(F)\backslash N_{\mathcal{W}}^{\circ}(\mathbb{A})} f^{(s)}(u \cdot \iota(g,e))\psi_{\mathcal{W}}(u)\ du\ dg,$$

where

$$\mathcal{P}(\xi_1 \boxtimes \xi_2) = \int_{G(F)\backslash G(\mathbb{A})} \xi_1(g)\xi_2(g)\ dg$$

is the Petersson inner product. If all the data is decomposable, then $Z(\xi_1 \boxtimes \xi_2, \phi^{(s)})$ is an Euler product.

In the symplectic case, this is proved in [5]. A proof in the conceptual setup is given in [2]. For an unramified place v, when all the data is unramified, the local version of the global integral is equal to $L(s, \pi_v \times \tau_v)$.

2. The covering group case

The twisted doubling integrals can also be developed conceptually for Brylinski-Deligne extensions of classical groups [3]. We now describe some basics of the Brylinski-Deligne covering groups [1] and discuss some necessary modifications of the twisted doubling integrals in the covering group case.

2.1. Multiplicative K_2-torsors.

Let G be a connected reductive group over a number field F. In [1], Brylinski and Deligne considered the category of multiplicative K_2-torsors on G; these are extensions of G by the sheaf K_2 of Quillen's K_2 group in the category of sheaves of groups over the big Zariski site of Spec(F):

$$1 \to K_2 \to \overline{G} \to G \to 1.$$

Brylinski and Deligne gave an elegant and functorial classification of this category in terms of enhanced root-theoretic data (or BD data), similar to the classification of split connected reductive groups by their root data.

2.2. Topological extensions.

We now assume that the base field F contains a full set of n-th roots of unity. Then at every local place v, there is a functor from the category of multiplicative K_2-torsors $\overline{G}$ on G to the category of topological central extensions:

$$1 \to \mu_n \to \overline{G}_v \to G_v = G(F_v) \to 1,$$

which glues to a central extension of the adelic group

$$1 \to \mu_n \to \overline{G}(\mathbb{A}) \to G(\mathbb{A}) \to 1.$$

The global extension is equipped with a natural splitting $G(F) \to \overline{G}(\mathbb{A})$. This naturally leads to the notion of automorphic forms on this class of groups. These topological central extensions may be considered of "algebraic origin" and can be constructed using cocycles which are essentially algebraic in nature.

In the linear case, the theory of Rankin-Selberg integrals relies heavily on the uniqueness of certain models, in particular the Whittaker model. This is no longer true for covering groups and therefore it is fundamentally difficult to find integral representations for L-functions for covering groups.

2.3. **Twisted doubling integrals.** From now on, G is either $\mathrm{Sp}(W)$ or $\mathrm{SO}(W)$. Here we briefly discuss several non-trivial issues when developing the twisted doubling integrals in the covering group setup. For covers of symplectic groups, the unfolding argument and unramified calculation were carried out in [7] using explicit 2-cocycles (see also [4] for a brief description of the method).

The first is to find suitable multiplicative K_2-torsor on $G^{\square,k}$ such that its pullback to G via $\iota : G \times G \to G^{\square,k}$ gives the desired multiplicative K_2-torsor $\overline{G}$. This is done in terms of the BD data. It is not difficult to construct a good candidate of BD data but it is a non-trivial job to verify that this candidate is a BD data for $G^{\square,k}$.

Another key ingredient is the construction of analogues of the generalized Speh representations in the covering group case. This is unknown in general. The usual construction of residues of Eisenstein series will not work nicely in the covering group setup. We refer the reader to relevant sections in [3] for more details on this matter.

Remark 4. *Another method to obtain L-functions is the constant terms of Eisenstein series on Brylinski-Deligne covering groups* [6]*. A consequence of this calculation is the meromorphic continuation of many interesting L-functions. It is an interesting question to compare these L-functions with those obtained from the twisted doubling integrals.*

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Katz type p-adic L-functions and applications

Fabrizio Andreatta
(joint work with Adrian Iovita)

In previious work I and Adrian Iovita have constructed p-adic L-functions, for $p \geq 5$, associated to a quadratic imaginary field K in which p does not split and an eigenform f, of weight $k \geq 2$, level $\Gamma_1(N)$ and nebentypus ϵ. We further assume that $N \geq 4$ is prime to p and that there exists an ideal $\mathfrak{N} \subset \mathcal{O}_K$ such that $\mathcal{O}_K/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$ (Heegner hyp.).

Such p-adic L-function interplates the algebraic part $L_{\rm alg}(f, \chi^{-1})$ of the special L-values $L\big(f, \chi^{-1}, 0\big)$ for $\chi \in \Sigma$, where Σ is the set of central critical (for f) algebraic Hecke characters. The case p split has been treated by Katz [4], for f an Eisenstein form, and by Bertolini, Darmon and Prasanna in [2], for f a cusp newform. The starting point is the following expression, for χ of ∞-type $(k+j, -j)$:

$$L_{\rm alg}\big(f, \chi^{-1}\big) := \sum_{\mathfrak{A} \in {\rm Pic}(\mathcal{O}_K)} \chi_j^{-1}(\mathfrak{A}) \delta_k^j(f)\big(\mathfrak{A} * (A, t_A, \omega_A)\big)$$

where

- $\chi_j = \chi \cdot {\rm Norm}^j$
- A is an elliptic curve with CM by $\mathcal{O}_K$,
- t_A is a generator of $A[\mathfrak{N}]$,
- ω_A is a non-zero Néron differential,
- $\mathfrak{A} * A$ is the quotient elliptic curve $A/A[\mathfrak{A}]$, $\mathfrak{A} * t_A$ is the image of t_A (taking $\mathfrak{A}$ prime to $\mathfrak{N}$), $\mathfrak{A} * \omega_A$ is the differential on $\mathfrak{A} * A$ whose pull back to A is ω_A.
- $\delta_k(f) = \frac{1}{2\pi i}\big(\frac{\partial(f)}{\partial \tau} + \frac{kf}{\tau - \overline{\tau}}\big)$ is the Shimura-Maas operator.

Notice that $\delta_k^j(f)$ is a C^∞-section of the $k+2j$-power of the Hodge bundle on $X_1(N)$. Using the trivialization of the Hodge bundle at the point (A, t_A) of $X_1(N)$ provided by $\mathfrak{A} * \omega_A$ we get that $\delta_k^j(f)|_{\mathfrak{A}*(A,t_A)}$ is a scalar, namely $\delta_k^j(f)\big(\mathfrak{A} * (A, t_A, \omega_A)\big)$, times $\big(\mathfrak{A} * \omega_A\big)^{k+2j}$. One then proves the following result, extending to the case that p is inert or ramified in K work of Katz and Bertolini-Darmon-Prasanna,

Theorem ([1]) Assume that K has odd discriminant. Then, there exists a locally analytic function $L_p\big(f, \chi^{-1}\big)$ on the p-adic completion $\widehat{\Sigma}$ of the set Σ of central critical Hecke characters whose values at $\chi \in \Sigma$ is $L_{\rm alg}\big(f, \chi^{-1}\big)$ (up to certain Euler factors at p).

In the case that f is associated to an Eisenstein series for the finite character of χ we omit f and we write $L_p\big(\chi^{-1}\big)$ for the p-adic L-function of the theorem.

We now consider an elliptic curve E, with CM by $\mathcal{O}_K$, defined over $\mathbb{Q}$ (so that K has class number 1). A classical result d Deuring states that the Hasse-Weil L-function $L(E, s+1)$ is the L-function $L(\nu_E, s)$ associated to an algebraic Hecke character of ∞-type $(1, 0)$. Let $\nu_E^* = \nu_E \circ c$ the complex conjugate character. We then have the following formula, generalizing a result of Rubin [5] fo p split in K:

Theorem Assume that $L(E, 1) = 0$ and $L'(E, 1) \neq 0$ and that p is inert in K. Then, there exists $Q \in E(\mathbb{Q})$ of infinite order such that

$$L_p\big((\nu_E^*)^{-1}\big) \equiv \frac{(\log_{\omega_E} Q)^2}{p\text{-adic period}} \mod K^*.$$

Here $\log_{\omega_E}$ is the formal p-adic logarithm on E at p defined by a Néron differential ω_E.

The proof follows closely the strategy of [3] in the case p split in K. The elliptic curve E is defined by a weight two cuspform $f = \vartheta_{\nu_E}$, the theta series associated to ν_E. We have two ingredients.

The first, that provides the right hand side of the equality, is a p-adic Gross-Zagier kind of formula, proven in [1] in the case tht p is not split. It relates the values of $L_p\big(\vartheta_{\nu_E}, \chi^{-1}\big)$, for $\chi \in \widehat{\Sigma}$ classical of ∞-type $(1, 1)$, to the p-adic logarithm of Heegner points.

The second ingredient, that provides the left hand side of the equality, uses that $L_p\big(\vartheta_{\nu_E}, \chi^{-1}\big)$ splits as the product of the p-adic L-functions associated to the two Hecke characters ν_E and ν_E^* (and χ). This is proven for classical Hecke characters using such a decomposition for classical complex L-values and then arguing by density of $\Sigma \subset \widehat{\Sigma}$.

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Eisenstein series and the cubic moment for $\mathbf{PGL_2}$

Paul D. Nelson

Motohashi [1] established a summation formula relating the fourth moment of the Riemann zeta function on the critical line to the cubic moment of central L-values of modular forms:

$$\int_{t\in\mathbb{R}} |\zeta(1/2+it)|^4 h(t)\,dt = \sum_{\varphi} \frac{L(\varphi,1/2)^3}{L^*(\varphi\times\varphi,1)}\tilde{h}(r_\varphi) + (\cdots). \tag{1}$$

Here

- h is a test function,
- φ traverses the Hecke–Maass cusp forms on $\mathrm{SL}_2(\mathbb{Z})$,
- $1/4 + r_\varphi^2$ is the eigenvalue of φ,
- $\tilde{h}$ is an explicit integral transform of h,
- $(\cdots)$ denotes analogous contributions from holomorphic forms and Eisenstein series, together with some "main terms."

Michel–Venkatesh [2] suggested a strikingly simple proof sketch of Motohashi's formula. Write $E_s^* : \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H} \to \mathbb{C}$ for the normalized Eisenstein series, whose Fourier expansion has the shape $E_s^*(x+iy) = \sum_{\pm} \xi(1\pm 2s)y^{1/2\pm s} + (\cdots)$. Working formally for now, we may evaluate the (divergent) integral

$$\int_0^\infty (E_0^*)^2(iy)\,d^\times y \tag{2}$$

in two ways. On the one hand, by Mellin–Parseval expansion and the theory of Hecke integrals, we obtain

$$\int_{t\in\mathbb{R}} \underbrace{\Big(\int_0^\infty E_0^*(iy)y^{it}\,\frac{dy}{y}\Big)}_{\sim\zeta(1/2+it)^2}\underbrace{\Big(\int_0^\infty E_0^*(iy)y^{-it}\,\frac{dy}{y}\Big)}_{\sim\zeta(1/2-it)^2}\frac{dt}{2\pi} \tag{3}$$

where $\sim$ denotes equality up to a suitable Γ-factor. On the other hand, by spectral expansion over $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H})$ and the theory of Hecke/Rankin–Selberg integrals, we obtain

$$\sum_{\varphi} \underbrace{\Big(\int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}} \varphi E_0^* E_0^*\Big)}_{\sim L(\varphi,1/2)^2}\underbrace{\int_0^\infty \varphi(iy)\,\frac{dy}{y}}_{\sim L(\varphi,1/2)} + (\cdots). \tag{4}$$

This proof sketch may be implemented rigorously. The basic ideas for doing so were suggested by Michel–Venkatesh [3, §4.5.3], using ideas of Zagier [6, 5]; we refer to [4, §11] for details. Take $s \in \mathbb{C}^3$ small and generic; eventually, let it tend to zero. On the one hand, the regularized integral

$$\int^{\mathrm{reg}} E_{s_1}^*(iy)E_{s_2}^*(iy)|y|^{s_3}\,d^\times y \tag{5}$$

may be defined as in, for instance, the integral representation

$$\zeta(s) \sim \int^{\mathrm{reg}} \theta(iy)|y|^{s/2}\, d^{\times}y$$

for the Riemann zeta function in terms of the Jacobi theta function. Such regularized integrals enjoy a modified form of Parseval's identity featuring some additional degenerate terms (see [4, §11.2]). On the other hand, for a suitable linear combination $\mathcal{E}$ of the Eisenstein series $E^*_{1/2\pm s_1 \pm s_2}$, the difference $E^*_{s_1} E^*_{s_2} - \mathcal{E}$ lies in $L^2(\Gamma\backslash\mathbb{H})$, and thus admits a spectral expansion. Carefully "commuting" this spectral expansion with the regularized y-integral leads to a formula like Motohashi's, but for some *specific* choices of weight functions h and $\tilde{h}$ (given by Γ-factors).

We obtain more general weights $h, \tilde{h}$ by working with more general Eisenstein series defined on the group quotient $\Gamma\backslash G := \mathrm{PGL}_2(\mathbb{Z})\backslash \mathrm{PGL}_2(\mathbb{R})$. For f belonging to a suitable induced representation $\mathcal{I}(s) \subseteq C^{\infty}(G)$, we define $\mathrm{Eis}^*(f, s) : \Gamma\backslash G \to \mathbb{C}$ by the formula $g \mapsto \xi(1+2s) \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} f(\gamma g)$ and then evaluate in two ways the regularized diagonal integral

$$\int_{y \in \mathbb{R}^\times/\mathbb{Z}^\times} \mathrm{Eis}^*(f_1, s_1)(a(y))\, \mathrm{Eis}^*(f_2, s_2)(a(y)) |y|^{s_3}\, d^\times y, \quad a(y) := \begin{pmatrix} y & \\ & 1 \end{pmatrix}, \tag{6}$$

on the one hand via Mellin–Parseval, and on the other hand via spectral expansion over $L^2(\Gamma\backslash G)$. The latter involves generic automorphic forms φ with Fourier expansions of the shape

$$\varphi(g) = \sum_{n \in \mathbb{Z}_{\neq 0}} W_\varphi(a(n)g) \frac{\lambda_\varphi(n)}{|n|^{1/2}} + (\cdots), \tag{7}$$

where

- W_φ denotes the Whittaker function,
- $\lambda_\varphi(n)$ denotes the normalized Hecke eigenvalue, and
- $(\cdots)$ denotes the constant term (if any).

Taking the limit as s tends to 0, we obtain a general formula of the shape: for all $f_1 \otimes f_2 \in \mathcal{I}(0) \otimes \mathcal{I}(0)$,

$$\sum_{\sigma \subseteq L^2_{\mathrm{cusp}}(\Gamma\backslash G)} \frac{L(\sigma, 1/2)^3}{L^*(\sigma \times \sigma, 1)} h_{f_1 \otimes f_2}(\sigma) = \int_t \tilde{h}_{f_1 \otimes f_2}(t) |\zeta(1/2 + it)|^4 \frac{dt}{2\pi} + (\cdots), \tag{8}$$

where $h_{f_1 \otimes f_2}, \tilde{h}_{f_1 \otimes f_2}$ are given in terms of local Hecke and Rankin–Selberg integrals by the formulas

$$h_{f_1 \otimes f_2}(\sigma) := \sum_{\varphi \in \mathcal{B}(\sigma)} (\int_{N\backslash G} \overline{W_\varphi} W_{f_1} f_2 \int_{\mathrm{diag}(\mathbb{R}^\times, 1)} W_\varphi \tag{9}$$

$$\tilde{h}_{f_1 \otimes f_2}(t) := \int_{y \in \mathbb{R}^\times} W_{f_1}(a(y))\, d^\times y \int_{y \in \mathbb{R}^\times} W_{f_2}(a(y))\, d^\times y, \tag{10}$$

and $(\cdots)$ denotes the contribution of Eisenstein σ as well as several degenerate terms. We refer to [4, §11.3] for details.

Addressing a problem raised by Michel–Venkatesh [3, §4.5.4], we show in [4, Thm 2.10] that a large class of weights $h(\sigma)$ arise as $h_f(\sigma)$ for some $f \in \mathcal{I}(0) \otimes \mathcal{I}(0)$. By "large class" we mean more precisely any $h(\sigma)$ as in the "pre–Kuznetsov formula," i.e., given for some $\phi \in C_c^\infty(G)$ by

$$h(\sigma) = \sum_{\varphi \in \mathcal{B}(\sigma)} (\int_G \overline{W_\varphi} \phi) W_\varphi(1). \tag{11}$$

Moreover, we describe $\tilde{h}$ explicitly. We obtain in this way a general "inverse" to Motohashi's formula, in the spirit of work of Ivic [7].

The above arguments are soft enough to go through adelically, over general number fields. By exploiting the nonnegativity of $L(\sigma, 1/2)$ and estimating certain weighted fourth moments, we obtain strong upper bounds for $L(\sigma, 1/2)$. For instance, [4, Thm 1.1] says that for any fixed number field F, any fixed cuspidal automorphic representation σ of $\mathrm{PGL}_2(\mathbb{A})$ and any quadratic character χ of $\mathbb{A}^\times / F^\times$,

$$L(\sigma \otimes \chi, 1/2) \ll C(\chi)^{1/3+\varepsilon}, \quad L(\chi, 1/2) \ll C(\chi)^{1/6+\varepsilon}. \tag{12}$$

These estimates generalize and refine the results of many authors, most directly those of Conrey–Iwaniec [8], who obtained similar estimates when $F = \mathbb{Q}$. We mention also the recent work of Petrow–Young [9, 10]. As a consequence, we deduce improved estimates over general number fields for

- Fourier coefficients of half-integral weight automorphic forms,
- representation numbers of ternary quadratic forms, and
- prime geodesics on $\mathrm{PSL}_2(\mathbb{Z}[i]) \backslash \mathbb{H}^3$.

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The Manin constant and the modular degree

Michalis Neururer
(joint work with Abhishek Saha, Kęstutis Česnavičius)

Let E be an elliptic curve over $\mathbb{Q}$ of conductor N and $\phi : (X_{\Gamma_0(N)})_{\mathbb{Q}} \twoheadrightarrow E$ be a modular parametrisation. The pullback of the Néron differential ω_E under ϕ equals $c_\phi \cdot \omega_f$, where ω_f is the differential associated to the normalised newform $f \in S_2(\Gamma_0(N))$ that has the same L-function as E and c_ϕ is a non-zero rational number, the *Manin constant* of ϕ. We call ϕ optimal if it has minimal degree among the parametrisations of the isogeny class of E.

Conjecture 1 (Manin)**.** *For an optimal parametrisation c_ϕ equals* 1*.*

For optimal parametrisations it is known [7, 1.2] that if $p \mid c_\phi$, then $p^2 \mid N$ (for earlier cases, see [6], [1], and [2]) and from [4, Thm. 3] that E has additive potentially ordinary reduction of Kodaira type II, III, or IV at any prime $p \geq 11$ with $p \mid c_\phi$.
For any congruence subgroup Γ between $\Gamma_1(N)$ and $\Gamma_0(N)$ let X_Γ be the coarse moduli space over $\mathbb{Z}$ associated to Γ and extend the definition of the Manin constant to parametrisations ϕ by $(X_\Gamma)_{\mathbb{Q}}$. We complement previous results on the Manin constant with the following.

Theorem 2 (Theorems 1.1 and 1.2 in [8])**.** *For an elliptic curve E over $\mathbb{Q}$ of conductor N, and for a congruence group Γ with $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$, every surjection*

$$\phi\colon (X_\Gamma)_{\mathbb{Q}} \twoheadrightarrow E \quad \textit{satisfies} \quad c_\phi \mid 6 \cdot \deg(\phi),$$

and if N is cube-free (that is, if $8 \nmid N$ and $27 \nmid N$) or if $\Gamma = \Gamma_1(N)$, then even

$$c_\phi \mid \deg(\phi);$$

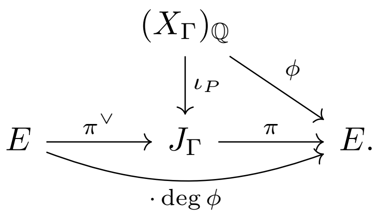
more precisely, for every prime p, we have $\mathrm{val}_p(c_\phi) \leq \mathrm{val}_p(\deg(\phi)) + \epsilon(p, N)$, where

$$\epsilon(p,N) = \begin{cases} 1 & \textit{if } p = 2 \textit{ with } \mathrm{val}_2(N) \geq 3 \textit{ and there is no } p' \mid N \textit{ with } p' \equiv 3(4), \\ 1 & \textit{if } p = 3 \textit{ with } \mathrm{val}_3(N) \geq 3 \textit{ and there is no } p' \mid N \textit{ with } p' \equiv 2(3), \\ 0 & \textit{otherwise,} \end{cases}$$

and, if for some $\Gamma \subseteq \Gamma' \subseteq \Gamma_0(N)$ the singularities of $(X_{\Gamma'})_{\mathbb{Z}_{(p)}}$ are rational (the definition is given below), then $\epsilon(p, N) = 0$.

For the proof of this theorem, let $P \in (X_\Gamma)_{\mathbb{Q}}$ be a point with $\phi(P) = 0$ and ι_P be the associated embedding of $(X_\Gamma)_{\mathbb{Q}}$ into its Jacobian J_Γ. By Jacobian functoriality ϕ factors through the Jacobian and we denote the resulting map from J_Γ to E by

π. We obtain a commutative diagram



Let $\mathcal{J}_\Gamma$ be the Néron model of J_Γ. Suppose ω_f is a differential form in the lattice of Néron differentials $H^0(\mathcal{J}_\Gamma, \Omega^1) \subseteq H^0(J_\Gamma, \Omega^1)$. Then also the pullback $(\pi^\vee)^*(\omega_f)$ would be a Néron differential in $H^0(\mathcal{E}, \Omega^1) \cong \mathbb{Z}\omega_E$ and from $\deg \phi \cdot \omega_E = (\pi \circ \pi^\vee)^* \omega_E = c_\phi (\pi^\vee)^*(\omega_f)$ we would be be able to conclude

$$c_\phi \mid \deg \phi.$$

Thus we are led to the study of the lattice $H^0(\mathcal{J}_\Gamma, \Omega^1)$. A closely related lattice is $H^0(X_\Gamma, \Omega)$, the global sections of the relative dualising sheaf of X_Γ. The map ι_P induces via pullback an isomorphism $H^0((X_\Gamma)_\mathbb{Q}, \Omega^1) \cong H^0(J_\Gamma, \Omega^1)$ by Grothendieck–Serre duality. However the corresponding map at the integral level,

$$(\star) \qquad H^0(\mathcal{J}_\Gamma, \Omega^1) \hookrightarrow H^0(X_\Gamma, \Omega)$$

is not a priori an equality in $H^0((X_\Gamma)_\mathbb{Q}, \Omega^1) \cong H^0(J_\Gamma, \Omega^1)$ and we say that X_Γ has *rational singularities*, if it is. In Proposition 4 we obtain an explicit description of $H^0(X_{\Gamma_0(N)}, \Omega)$, so for the above argument, the following question becomes important:

Question 3. *For which Γ does X_Γ have rational singularities? More generally, for which p and Γ does $(X_\Gamma)_{\mathbb{Z}_{(p)}}$ have rational singularities?*

We show that for the conclusion $\mathrm{val}_p(\phi) \leq \mathrm{val}_p(\deg(\phi))$ in Theorem 2 it is enough that $\Gamma \subseteq \Gamma'$ and $X_{\Gamma'}$ has rational singularities. So for the rest of this abstract we set $\Gamma = \Gamma_0(N)$, noting that the conclusions in Theorem 2 can be deduced from the results for $\Gamma_0(N)$. A theorem of Raynaud [9, Thm. 2] answers the question positively for $\mathrm{val}_p(N) \leq 1$ or $p \geq 5$ and $\Gamma = \Gamma_0(N)$. Using known cases of Manin's conjecture we add several new cases where $X_{\Gamma_0(N)}$ has rational singularities, among them the case $\mathrm{val}_p(N) \leq 2$.
We now turn to the study of $H^0(X_{\Gamma_0(N)}, \Omega)$. Let $f \in S_k(\Gamma_0(N))$ be a cusp form. The Fourier expansion of f at a cusp $\mathfrak{c} = \gamma\infty$ with $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ has the form

$$f|_k\gamma(z) = \sum_{n \geq 1} a_f(n;\gamma) e^{\frac{2\pi i n z}{w(\mathfrak{c})}}$$

and we define $\mathrm{val}_p(f|_\mathfrak{c}) = \inf_{n\geq 1}(\mathrm{val}_p(a_f(n;\gamma)))$. Using an integral version of the Kodaira–Spencer isomorphism together with the structure of the formal completion of $X_{\Gamma_0(N)}$ along the cusps, we obtain

Proposition 4 (Proposition 5.14 in [8])**.** *For $f \in S_2(\Gamma_0(N))$ with rational Fourier coefficients, the differential form ω_f lies in $H^0(X_{\Gamma_0(N)}, \Omega)$ if and only if for all* *primes p and all cusps* $\mathfrak{c} = A/L$ *with* $\gcd(A,N) = 1$ *and* $L|N$

$$\operatorname{val}_p(f|_{\mathfrak{c}}) \geq \begin{cases} -\operatorname{val}_p(\frac{N}{L}) & \text{if } \operatorname{val}_p(L) \in \{0, \operatorname{val}_p(N)\}, \\ -\operatorname{val}_p(\frac{N}{L}) + \frac{1}{p-1} & \text{if } 0 < \operatorname{val}_p(L) < \operatorname{val}_p(N). \end{cases}$$

If $f \in S_k(\Gamma_0(N))$ is a newform, we can associate to it an irreducible cuspidal automorphic representation π_f of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which factorises as a tensor product of $\mathrm{GL}_2(\mathbb{Q}_p)$-representations $\pi_{f,p}$. These local representations have a Whittaker model and the Fourier coefficients of f at cusps can be expressed in terms of evaluations of a special function in this model, the Whittaker newform $W_{\pi_{f,p},\psi} : \mathrm{GL}_2(\mathbb{Q}_p) \to \mathbb{C}$. These can in turn be studied with a method developed by Saha and his collaborators based on his 'basic identity' [3]. Our analysis depends heavily on the type of the representation $\pi_{f,p}$ and so I will only treat a very specific case for illustration purposes: If p is odd, $\operatorname{val}_p(N) = 2$, and $\pi_{f,p}$ is supercuspidal, then

$$\operatorname{val}_p(f|_{1/p}) \geq \operatorname{val}_p(W_{\pi_{f,p},\psi}(\left(\begin{smallmatrix} 0 & p^{-2} \\ -1 & -p^{-1} \end{smallmatrix}\right)))$$

and

$$W_{\pi_{f,p},\psi}(\left(\begin{smallmatrix} 0 & p^{-2} \\ -1 & -p^{-1} \end{smallmatrix}\right)) = \pm\tfrac{1}{p-1} + \tfrac{p^{1/2}}{p-1} \textstyle\sum_{\chi,\ \mathrm{cond}(\chi)=1} \epsilon(1/2,\chi)\epsilon(1/2,\chi^{-1}\pi),$$

where the sum is over all characters of $\mathbb{Z}_p^\times$ of conductor 1, and ϵ are the epsilon factors of the corresponding $\mathrm{GL}_1(\mathbb{Q}_p)$ or $\mathrm{GL}_2(\mathbb{Q}_p)$-representations. Viewing χ as a character of $\mathbb{Z}_p^\times/(1+p\mathbb{Z}_p) \cong \mathbb{F}_p$, we can relate both epsilon factors to Gauss sums of characters over finite fields. Using classical results on the p-adic valuations of Gauss sums we are able to show $\operatorname{val}_p(f|_{1/p}) \geq -\frac{1}{2} + \frac{1}{p-1}$. We carefully treat all possible cases for $\pi_{f,p}$ and derive general lower bounds for $\operatorname{val}_p(f|_{\mathfrak{c}})$ for any cusp $\mathfrak{c}$ (Theorem 1.3 of [8]). The proof of Theorem 2 now follows from combining our bounds with Proposition 4.

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Congruence for the Klingen-Eisenstein series and Harder's conjecture

Hidenori Katsurada

For a non-increasing sequence $\mathbf{k} = (k_1, \ldots, k_n)$ of non-negative integers we denote by $M_{\mathbf{k}}(Sp_n(\mathbb{Z}))$ and $S_{\mathbf{k}}(Sp_n(\mathbb{Z}))$ the spaces of modular forms and cusp forms of weight $\mathbf{k}$ (or, weight k, if $\mathbf{k} = (\overbrace{k, \ldots, k}^{n})$) for $Sp_n(\mathbb{Z})$, respectively. For a Hecke eigenform $F \in M_{\mathbf{k}}(Sp_n(\mathbb{Z}))$, we denote by $\mathbb{Q}(F)$ the Hecke field of F. Let $f(z)$ be a primitive form in $S_{2k+j-2}(SL_2(\mathbb{Z}))$, and let $L(s, f)$ be Hecke's L-function of f. Suppose that a 'big prime' $\mathfrak{p}$ of $\mathbb{Q}(f)$ divides the algebraic part $L_{alg}(k+j, f)$ of $L(k+j, f)$. Then, Harder [3] conjectured that there exists a Hecke eigenform F in $S_{(k+j,k)}(Sp_2(\mathbb{Z}))$ such that

$$(H_{k,j,p}) \qquad \lambda_F(T(p)) \equiv \lambda_f(T(p)) + p^{k-2} + p^{j+k-1} \bmod \mathfrak{p}'$$

for any prime number p, where for a Hecke eigenform G and a Hecke operator T, let $\lambda_G(T)$ denote the eigenvalue of the Hecke operator T on G, and $\mathfrak{p}'$ is a prime ideal of $\mathbb{Q}(f) \cdot \mathbb{Q}(F)$ lying above $\mathfrak{p}$. We call the above congruence Harder's congruence. One of main difficulties in treating this congruence arises from the fact that this is not concerning the congruence between Hecke eigenvalues of two Hecke eigenforms of the same weight. Indeed, the right-hand side of the above is not the Hecke eigenvalue of a Hecke eigenform if $j > 0$. Several attempts have been made to overcome this obstacle (cf. [4], [1]). However, as far as we know, Harder's congruence $(H_{k,j,p})$ holds for all primes p only in the case $(k, j) = (10, 4)$ (cf. [2]). In this talk we consider a conjecture concerning the congruence between two liftings of Hecke eigenforms (of integral weight) of degree two. More precisely, let k, j and n be positive even integers such that $k \geq 4$, and for the f above, let $I_n(f)$ be the Duke-Imamoglu-Ikeda lift of f to $S_{\frac{j}{2}+k+\frac{n}{2}-1}(Sp_n(\mathbb{Z}))$ with n even. For a sequence $\mathbf{k} = \Bigl(\overbrace{\frac{j}{2}+k+\frac{n}{2}-1, \ldots, \frac{j}{2}+k+\frac{n}{2}-1}^{n}, \overbrace{\frac{j}{2}+\frac{3n}{2}+1, \ldots, \frac{j}{2}+\frac{3n}{2}+1}^{n}\Bigr)$, let $[I_n(f)]^{\mathbf{k}}$ be the Klingen-Eisenstein lift of $I_n(f)$ to $M_{\mathbf{k}}(Sp_{2n}(\mathbb{Z}))$. Moreover, for a Hecke eigenform F in $S_{(k+j,k)}(Sp_2(\mathbb{Z}))$, let $A_{2n}^{(I)}(F)$ be the lift of F to $S_{\mathbf{k}}(Sp_{2n}(\mathbb{Z}))$ whose standard L-function $L(s, A_{2n}^{(I)}(F), \mathrm{St})$ is given in terms of the spin L-function $L(s, F, \mathrm{Sp})$ of F as $L(s, A_{2n}^{(I)}(F), \mathrm{St}) = \zeta(s) \prod_{i=1}^{n} L(s+n/2+j/2+k-1-i, F, \mathrm{Sp})$. The existence of this lift can be proved by Arthur's multiplicity formula (cf. [2]). Fix a $\mathbb{Z}$-module M^o of $M_{\mathbf{k}}(Sp_n(\mathbb{Z}))$ such that $M^o \otimes_{\mathbb{Z}} \mathbb{C} = M_{\mathbf{k}}(Sp_n(\mathbb{Z}))$. Let $\mathbf{L}_n$ be the Hecke ring associated to the Hecke pair $(\Gamma^{(n)}, GSp_n^+(\mathbb{Q}))$, and $\mathbf{L}_n^{\mathbf{k}}$ be its subring consisting of all elements T such that $M^o | T \subset M^o$. Then, we propose the following conjecture:

Conjecture K. *Under the above notation and the assumption, there exists a Hecke eigenform F in $S_{(k+j,k)}(Sp_2(\mathbb{Z}))$ such that*

$$\lambda_{[I_n(f)]^{\mathbf{k}}}(T) \equiv \lambda_{A_{2n}^{(I)}(F)}(T) \ \mathit{mod}\ \mathfrak{p}' \ \textit{for any}\ T \in \mathbf{L}_n^{\mathbf{k}}.$$

This conjecture derives Harder's conjecture. That is,

Theorem 1. *Let the notation and the assumption be as in Conjecture K.*

1. *Harder's congruence* $(H_{k,j,p})$ *holds for all prime* p *in the case* $j \equiv 0 \mod 4$ *if Conjecture K holds for* $n = 2$.
2. *Harder's congruence* $(H_{k,j,p})$ *holds for all primes* p *in the case* $j \equiv 2 \mod 4$ *if Conjecture K holds for* $n = 4$.

The advantage of this formulation is that one can compare the Hecke eigenvalues of two Hecke eigenforms. Indeed, by using the same argument as in Katsurada-Mizumoto [6] combined with [5], under the above assumption, we can prove that there exists a Hecke eigenform of weight $G \in M_{\mathbf{k}}(Sp_{2n}(\mathbb{Z}))$ such that G is not a constant multiple of $[I_n(f)]^{\mathbf{k}}$ and

$$\lambda_G(T) \equiv \lambda_{[I_n(f)]^{\mathbf{k}}}(T) \mod \mathfrak{p}' \text{ for any } T \in \mathbf{L}_n^{\mathbf{k}}.$$

Therefore, to prove the above conjecture, it suffices to show that G is a lift of type $A^{(I)}$. To state a main result of this talk, for a prime number p, let ζ_p be the p-th root of unity, and let $\mathcal{C}_p$ be the p-Sylow subgroup of the ideal group of $\mathbb{Q}(\zeta_p)$. For a character $\eta : \mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q}) \longrightarrow \mathbb{C}^\times$, let $\mathcal{C}_p^\eta$ denote the η-isotypical part of $\mathcal{C}_p$. Moreover for the p-adic cyclotomic character χ, we denote by ω the Teichmüller character of $\mathrm{Gal}(\mathbb{Q}(\zeta_p)/\mathbb{Q})$ corresponding to χ. Let f be a primitive form in $S_{2k+j-2}(SL_2(\mathbb{Z}))$. For two positive integers $l_1, l_2 \le 2k+j-3$ such that $l_1 - l_2$ is odd, the value $\mathbf{L}(l_1, l_2; f) := \dfrac{L(l_1, f)L(l_2, f)}{\pi^{l_1+l_2}(f, f)}$ belongs to $\mathbb{Q}(f)$. Let $f_1, \dots, f_d$ be a basis of $S_{2k+j-2}(SL_2(\mathbb{Z}))$ consisting of primitive forms with $f_1 = f$ and let $\mathfrak{D}_f$ be the ideal of $\mathbb{Q}(f)$ generated by all $\prod_{i=2}^d (\lambda_{f_i}(T(m)) - \lambda_f(T(m))$'s $(m \in \mathbb{Z}_{>0})$.

We denote by $\mathcal{H}_r(\mathbb{Z})_{>0}$ the set of positive definite half-integral symmetric matrices over $\mathbb{Z}$ of size r, and for a modular form H for $Sp_r(\mathbb{Z})$ and $A \in \mathcal{H}_r(\mathbb{Z})_{>0}$, let $a(A, H)$ denote the A-th Fourier coefficient of H. For a Hecke eigenform $F \in S_{k+j/2}(Sp_2(\mathbb{Z}))$ and a positive even integer $m \le k + j/2 - 2$, put

$$L_{alg}(m, F, \mathrm{St}) = \frac{L(m, F, \mathrm{St})}{\pi^{2(k+j/2+m)+m-3}(F,\ F)}.$$

We note that for any $A, B \in \mathcal{H}_2(\mathbb{Z})_{>0}$ $a(A, F)\overline{a(B, F)}L_{alg}(m, F, \mathrm{St})$ belongs to $\mathbb{Q}(F)$.

Theorem 2. *Let* k *and* j *be positive integers such that* $k \equiv 0 \mod 2, j \equiv 0 \mod 4$, $k \ge 4$, *and put* $\mathbf{k} = (k + j/2, k + j/2, j/2 + 4, j/2 + 4)$. *Let* f *be a primitive form in* $S_{2k+j-2}(SL_2(\mathbb{Z}))$ *and* $\mathfrak{p}$ *be a prime ideal of* $\mathbb{Q}(f)$ *and let* p *be the prime number divisible by* $\mathfrak{p}$. *Moreover let* $\rho_f : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \longrightarrow GL(V_f) \cong GL_2(K_{\widetilde{\mathfrak{p}}})$ *be the Galois representation attached to* $L(s, f)$, *where* K *is a sufficiently large number field containing* $\mathbb{Q}(f)$, *and* $\widetilde{\mathfrak{p}}$ *be a prime ideal of* K *lying above* $\mathfrak{p}$. *Suppose the following:*

(C.1) $p > 2k + 2j$.
(C.2) $\mathfrak{p}$ *divides* $L_{alg}(k + j, f)/L_{alg}(k + j/2, f)$.

(C.3) $\mathfrak{p}$ *does not divide*

$$\zeta(-1-j/2)\zeta(1-j/2)a(A_1, I_2(f))a(A, [I_2(f)]^{\mathbf{k}})L_{alg}(j/2+2, I_2(f), \mathrm{St})$$

for some $A_1 \in \mathcal{H}_2(\mathbb{Z})_{>0}$ *such that* $\mathfrak{p}$ *does not divide* $\det(2A_1)$ *and* $A \in \mathcal{H}_4(\mathbb{Z})_{>0}$.

(C.4) $\mathfrak{p}$ *does not divide* $\mathfrak{D}_f$.

(C.5) $\bar{\rho}_f$ *is absolutely irreducible.*

(C.6) $\rho_f(Gal(\bar{\mathbb{Q}}/\mathbb{Q}(\zeta_{p^\infty}))$ *contains* $SL_2(\mathbb{Z}_p)$ *(with a suitable choice of a lattice of* V_f*)*

(C.7) f *is* p*-ordinary.*

(C.8) $\#\mathcal{C}_p^{\omega^{-j}} = \#\mathcal{C}_p^{\omega^{-j-2}} = \#\mathcal{C}_p^{\omega^{-j/2}} = 1.$

(C.9) p *does note divide* $\zeta(-1-j)\zeta(1-j)\zeta(1-j/2)\zeta(-1-j/2)$.

(C.10) $\mathfrak{p}$ *divides neither* $\mathbf{L}(k+j/2, k+j+1; f)$ *nor* $\mathbf{L}(k+j-2, k+j-1; f)$.

Then there is a Hecke eigenform F_0 *in* $S_{(k+j,k)}(Sp_2(\mathbb{Z}))$ *such that*

$$\lambda_{[I_n(f)]^{\mathbf{k}}}(T) \equiv \lambda_{A_4^{(I)}(F_0)}(T) \ mod\ \mathfrak{p}' \text{ for any } T \in \mathbf{L}_n^{\mathbf{k}}.$$

Remark. The condition (C.3) can be checked though it is rather elaborate. The conditions (C.4)–(C.10) can easily be checked.

Corollary 3. *Under the same assumption as Theorem 2, Harder's congruence* $(H_{k,j,p})$ *holds for all primes* p.

Theorem 4. *Harder's congruence* $(H_{k,j,p})$ *holds for all primes* p *in the case* $(k,j) = (10,4), (14,4), (12,8), (10,12), (8,16), (6,20), (4,24)$.

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Isolation of the cuspidal spectrum and application to the Gan-Gross-Prasad conjecture for unitary groups

Raphaël Beuzart-Plessis
(joint work with Yifeng Liu, Wei Zhang, Xinwen Zhu)

In a work of 2005 [7], Venkatesh and Lindenstrauss have introduced a new method to show the existence of many (spherical) cusp forms on arithmetic locally symmetric spaces. Their strategy is to take advantage of the fact that the infinitesimal character and Hecke-eigenvalues of Eisenstein series satisfy simple relations e.g. for the standard Eisenstein series $E_{1/2+\lambda}$ on the arithmetic quotient $PGL_2(\mathbb{Z})\backslash\mathbb{H}$ of the upper half-plane, we have

$$\Delta E_{1/2+\lambda} = (\frac{1}{4} - \lambda^2)E_{1/2+\lambda},\ T_p E_{1/2+\lambda} = (p^\lambda + p^{-\lambda})E_{1/2+\lambda}$$

where Δ and T_p stand for the hyperbolic Laplacian and the Hecke operator at p respectively. Using these relations, they were able to write down explicit linear combinations of Hecke operators and Archimedean convolution operators that kill the continuous and residual spectrum. Combined with some simple (pre-)trace formula, they show that these operators are not all identically zero thus deducing the existence of cusp forms (in the example above, this gives a short proof of the existence of even Maass cusp forms). Moreover, making the argument quantitative, they were able to deduce from this construction the Weyl's law for cusp forms on congruence quotients of symmetric spaces associated to split adjoint groups over $\mathbb{Q}$. For more general applications, the Lindentrauss-Venkatesh's construction has however two flaws that we would like to get rid of:

- The operators so constructed always kill some interesting cusp forms like symmetric square lifts of Maass forms (However, these special forms are very *sparse* and so don't affect the Weyl's law);
- The construction only works for cusp forms which are spherical at the Archimedean place.

In the recent preprint [1], Y. Liu, W. Zhang, X. Zhu and I were able to go beyond the work of [7] by introducing a new way to isolate the cuspidal spectrum that doesn't have the two flaws described above. To be more specific, let G be a connected reductive group defined over $\mathbb{Q}$. A crucial ingredient for our construction is to work systematically with convolution operators which are not necessarily compactly supported at the Archimedean place but nevertheless (very) rapidly decreasing. More precisely, introduce the *Schwartz space* $\mathcal{S}(G(\mathbb{R}))$ of $G(\mathbb{R})$ as the space of C^∞ function $f : G(\mathbb{R}) \to \mathbb{C}$ such that for every polynomial differential operator D on $G(\mathbb{R})$, the function Df is bounded. This space is stable under convolution and admits a natural topology making it into a Fréchet algebra. We fix a compact-open subgroup $K = \prod_p K_p$ (i.e. a *level*) of $G(\mathbb{A}_f) = \prod' G(\mathbb{Q}_p)$ (where $\prod'$ stands for the restricted product) and S be a finite set of primes such that for $p \notin S$, K_p is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_p)$. We consider the global space of functions given by the restricted tensor product

$$\mathcal{S}(G(\mathbb{A}))_K = \mathcal{S}(G(\mathbb{R})) \otimes \bigotimes_p{}' C_c(K_p\backslash G(\mathbb{Q}_p)/K_p)$$

i.e. the space of linear combinations of functions on $G(\mathbb{A})$ of the form $f_\infty \prod_p f_p$ where $f_\infty \in \mathcal{S}(G(\mathbb{R}))$, $f_p \in C_c(K_p\backslash G(\mathbb{Q}_p)/K_p)$ and $f_p = \mathbf{1}_{K_p}$ for almost all p. Then, $\mathcal{S}(G(\mathbb{A}))_K$ will be our space of convolution operators acting on the right of $L^2(G(\mathbb{Q})\backslash G(\mathbb{A})/K)$. For $f \in \mathcal{S}(G(\mathbb{A}))_K$, we denote by $R(f)$ the associated convolution operator.

To construct the *quasi-cuspidal* operators we are looking for, we will use the action of certain *multipliers*. More precisely, the Archimedean multiplier space $\mathcal{M}_\infty(G)$ is defined as the space of continuous $\mathcal{S}(G(\mathbb{R}))$-bimodule endomorphisms of $\mathcal{S}(G(\mathbb{R}))$. This can be alternatively described as the space of invariant distributions on $G(\mathbb{R})$ which are "rapidly decreasing" in a suitable sense (more precisely: which extend by continuity to some space of "uniform moderate growth" test functions) acting on $\mathcal{S}(G(\mathbb{R}))$ by convolution. For $p \notin S$, and since $C_c(K_p\backslash G(\mathbb{Q}_p)/K_p)$ is unital and commutative by the Satake isomorphism, the analog of $\mathcal{M}_\infty(G)$ is the spherical Hecke algebra $\mathcal{M}_p(G) = C_c(K_p\backslash G(\mathbb{Q}_p)/K_p)$ acting on itself by convolution. Then, we introduce the space of S-*multipliers* as the restricted tensor product

$$\mathcal{M}^S(G) = \mathcal{M}_\infty(G) \otimes \bigotimes_{p\notin S}{}' \mathcal{M}_p(G)$$

which acts on $\mathcal{S}(G(\mathbb{A}))$ by convolution.

Before stating the main result, we need to introduce a definition. Let $\pi = \pi_\infty \otimes \bigotimes_p' \pi_p$ be an irreducible admissible representation of $G(\mathbb{A})$ with nontrivial K-invariant vectors (i.e. $\pi^K \neq 0$). We say that π is S-*CAP* if there exists an Eisenstein series E on $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ induced from a proper parabolic subgroup with the same system of Hecke-eigenvalues as π at every prime $p \notin S$. These CAP representations are the one we basically cannot isolate from the continuous (and residual) spectrum. This is partially explained by Arthur-Langlands conjecture according to which the cuspidal CAP representations are the one that should belong to the same *L-packet* as a representation in the residual spectrum. Incidentally, these are also the cuspidal representations that are expected to violate the Generalized Ramanujan Conjecture. We can now state (a weak form of) the main result of [1].

Theorem 1. *Assume that π is not S-CAP. Then, there exists a S-multiplier $\mu_\pi \in \mathcal{M}^S(G)$ such that for every $f \in \mathcal{S}(G(\mathbb{A}))_K$, we have:*

- *$R(\mu_\pi \star f)$ acts by zero on the orthogonal complement of the cuspidal subspace $L^2_{cusp}(G(\mathbb{Q})\backslash G(\mathbb{A})) \subseteq L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$;*
- *$\pi(\mu_\pi \star f) = \pi(f)$.*

The proof of the theorem roughly goes as follows. Assume for simplicity that G is split. Let $B \subset G$ be a Borel subgroup, $A \subset B$ be a maximal torus and

W be the Weyl group of A in G. Let $\widehat{A} = Hom(A, \mathbb{G}_m) \otimes \mathbb{C}^\times$ be Langlands dual torus. Then, by the Satake isomorphism, for $p \notin S$ the system of Hecke-eigenvalues of π at p can be identified with an element $\lambda_p(\pi) \in \widehat{A}/W$. Similarly, by Harish-Chandra's isomorphism, the infinitesimal character of π_∞ can be identified with an element $\lambda_\infty(\pi) \in Lie(\widehat{A})/W$. Set $\mathcal{X}^S = Lie(\widehat{A})/W \times \prod_{p \notin S} \widehat{A}/W$ and $\lambda^S(\pi) = (\lambda_\infty(\pi), (\lambda_p(\pi))_{p\notin S}) \in \mathcal{X}^S$. We define similarly $\lambda^S(E) \in \mathcal{X}^S$ when E is an Eisenstein series on $G(\mathbb{Q})\backslash G(\mathbb{A})/K$. Let $\mathcal{X}^S_{Eis} \subset \mathcal{X}^S$ be the subset consisting of $\lambda^S(E)$ for every Eisenstein series on $G(\mathbb{Q})\backslash G(\mathbb{A})/K$ induced from a proper parabolic. The space of multipliers $\mathcal{M}^S(G)$ can be seen as a space of functions on $\mathcal{X}^S$ and the theorem is essentially saying that $\mathcal{X}^S_{Eis}$ can be written as the common zero set of a family of multipliers $\mu_i \in \mathcal{M}^S(G)$, $i \in I$. To show this, we need to construct enough Archimedean multipliers (as the natural relations describing $\mathcal{X}^S_{Eis}$ are obtained by comparing each non-Archimedean component to the Archimedean one). The construction of a suitable space of Archimedean multipliers for the Schwartz space takes a significant part of [1] but we will not discuss it here (we note through that for the usual space of test functions $C_c^\infty(G(\mathbb{R}))$, the algebra of multipliers is rather small see [2, Final remark]).

Finally, we apply this construction to the Gan-Gross-Prasad conjecture for unitary groups. More precisely, let E/F be a quadratic extension of number fields, V be a Hermitian space of dimension n over E and $V' = V \oplus^\perp \langle e \rangle$ where e has norm $(e, e) = 1$. Let π and π' be cuspidal automorphic representations of $U(V)$ and $U(V')$ respectively. Consider the following linear form on $\pi \otimes \pi'$ (an example of an *automorphic period*):

$$P_{U(V)} : \varphi \otimes \varphi' \in \pi \otimes \pi' \mapsto \int_{U(V)(F)\backslash U(V)(\mathbb{A}_F)} \varphi(h)\varphi'(h)dh.$$

Let π_E, π'_E be the base-change of π, π' with respect to the extension E/F (these are automorphic representations of $GL_{n,E}$ and $GL_{n+1,E}$ respectively) and $L(s, \pi_E \times \pi'_E)$ be the associated Rankin-Selberg L-function (normalized so that the center of symmetry is at $1/2$). In [1], we prove the following result which is the "stable case" of a conjecture of Gan-Gross-Prasad [3].

Theorem 2. *Assume that π_E and π'_E are cuspidal. Then, the following assertions are equivalent:*

(1) $L(1/2, \pi_E \times \pi'_E) \neq 0$;
(2) There exist a Hermitian space W of dimension n over E and two cuspidal automorphic representations σ, σ' of $U(W)$, $U(W')$ such that $\sigma_E = \pi_E$, $\sigma'_E = \pi'_E$ (in other words σ, π on one hand and σ', π' on the other hand "belongs to the same L-packet") and $P_{U(W)} |_{\sigma\otimes\sigma'} \neq 0$.

We actually also establish at the same time a refined version of the Gan-Gross-Prasad conjecture originally due to Ichino-Ikeda [5] (see [4] for the case of unitary groups) giving a precise relation between $L(1/2, \pi_E \times \pi'_E)$ and (the square of the module of) the period $P_{U(V)}(\varphi \otimes \varphi')$ for $\varphi \in \pi$ and $\varphi' \in \pi'$. This formula is a higher rank generalization of a famous result of Waldspurger on toric periods for

$GL(2)$ [8]. These results were already obtained by W. Zhang [9], [10] under more stringent local assumptions on π, π' using a comparison of relative trace formulas proposed by Jacquet and Rallis [6]. These local assumptions originate from the use of *simple* versions of the aforementioned trace formulas: by imposing some local conditions on the test functions, the continuous and residual spectrum are automatically killed and the spectral side of the trace formulas become analytically harmless. Theorem 1 allows for such a drastic simplification without imposing local conditions on the test functions and this is essentially the main new input that allows us to establish Theorem 2.

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Modular Forms and Invariant Theory

Gerard van der Geer

(joint work with Fabien Cléry and Carel Faber)

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension g. This is an algebraic stack over $\mathbb{Z}$ and it comes with a vector bundle $\mathbb{E}$ of rank g. For every irreducible representation ρ of $\mathrm{GL}(g)$ there is a corresponding vector bundle $\mathbb{E}_\rho$ that extends to a Faltings-Chai compactification $\tilde{\mathcal{A}}_g$. Sections of $\mathbb{E}_\rho$ over $\tilde{\mathcal{A}}_g$ are Siegel modular forms of weight ρ. Over $\mathbb{C}$ these correspond to holomorphic functions $f : \mathfrak{H}_g \to W$ (with W the representation space of ρ) such that $f(\gamma(\tau)) = \rho(c\tau + d)f(\tau)$ for all $\gamma \in \Gamma_g := \mathrm{Sp}(2g, \mathbb{Z})$.

We use invariant theory to describe in principle all (vector-valued) modular forms on Γ_g for $g = 2$ and $g = 3$. The idea is that via the Torelli map $t : \mathcal{M}_g \to \mathcal{A}_g$ the moduli space of curves $\mathcal{M}_g$ is close to $\mathcal{A}_g$ for $g = 2$ and 3.

First we do $g = 2$. A curve C of genus 2 over a field of characteristic $\neq 2$ admits a description $y^2 = f(x)$ with f of degree 6 with discriminant $\neq 0$. Let V be the vector space $\langle x_1, x_2 \rangle$ and write f in homogeneous form $f = \sum_{i=0}^{6} x_1^{6-i} x_2^i$, a binary sextic. The group $\mathrm{GL}(V)$ acts and $\mathcal{M}_2$ can be written as a stack quotient

$$\mathcal{M}_2 \sim [\mathcal{X}^0/\mathrm{GL}(V)]$$

with $\mathcal{X} = \mathrm{Sym}^6(V) \otimes \det(V)^{-2}$ and $\mathcal{X}^0 \subset \mathcal{X}$ the open set corresponding to binary sextics with non-vanishing discriminant. A curve $y^2 = f$ comes with a basis of $H^0(C, \Omega^1_C)$, namely $dx/y, x\,dx/y$.

The pullback to of the Hodge bundle $\mathbb{E}$ under the Torelli morphism $t : \mathcal{M}_2 \to \mathcal{A}_2$ is the Hodge bundle on $\mathcal{M}_2$ and its pullback under the morphism $\mathcal{X}^0 \to \mathcal{M}_2$ is the equivariant bundle V. The pullback of $\det(\mathbb{E})$ to $\mathrm{Sym}^6(V)$ is $\det(V)^3$. An immediate consequence is that the pullback of a scalar-valued modular form is an invariant, that is, a polynomial in the coefficients a_i of f invariant under $\mathrm{SL}(V)$. The ring I of invariants is known by work of Clebsch and Bolza in the 19th century and generated by invariants A, B, C, D, E of degrees $2, 4, 6, 10, 15$ in the a_i.

If $\mathcal{R}_2 = \mathcal{R}_2(\mathbb{C})$ is the ring of scalar-valued Siegel modular forms of degree 2 we thus get a map $\mathcal{R}_2 \to I$. Such a map was already used by Igusa in the 1960s, who determined the structure of $\mathcal{R}_2$ in [7, 8], but he used theta functions and Thomae's formulae to relate these to cross-ratios of the zeros of binary sextics. Not every invariant corresponds to a holomorphic modular form since $t(\mathcal{M}_2) \neq \mathcal{A}_2$ and the complement is the zero locus of the modular form χ_{10}. We thus get maps

$$\mathcal{R}_2 \longrightarrow I \longrightarrow \mathcal{R}_2[1/\chi_{10}]\,.$$

We extended this in [2] to vector-valued Siegel modular forms by using covariants, that is, invariants for the action of $\mathrm{GL}(V)$ on $V \oplus \mathrm{Sym}^6(V)$. A covariant can be seen as a form in the a_i and in x_1, x_2. The most basic covariant is f, the universal binary sextic. The ring of covariants $\mathcal{C}$ was studied by Grace and Young in the early 20th century and has 26 generators.

Let $M = \oplus_{j,k} M_{j,k}(\Gamma_2)$ be the module of vector-valued modular forms over $\mathcal{R}_2$. (It is actually a ring.) We now get by pullback maps

$$M \to \mathcal{C} \xrightarrow{\nu} M_{\chi_{10}}$$

and we see that covariants map to meromorphic modular forms that become holomorphic after multiplication by a power of χ_{10}. If we apply ν to f, the most basic covariant, we get a meromorphic modular form $\chi_{6,-2}$ of weight $(6, -2)$. We can calculate this form $\chi_{6,-2}$ very explicitly and can write it as

$$\chi_{6,-2} = \sum_{i=0}^{6} \alpha_i X_1^{6-i} X_2^i$$

with α_i meromorphic on $\mathfrak{H}_2$ such that the coordinates $\chi_5 \alpha_i$ of $\chi_5 \chi_{6,-2}$ with $\chi_5^2 = \chi_{10}$ are holomorphic. Here the dummy variables X_1, X_2 are used to indicate the seven coordinates of $\chi_{6,-2}$. The main point is that the map ν is simply given by substituting α_i for a_i.

This leads to a very efficient way for calculating the Fourier series of Siegel modular forms. For example, the website [1] gives Fourier series obtained in this way, for all cases where $\dim S_{j,k}(\Gamma_2) = 1$. Using ν we showed in [5] that $\dim S_{j,2} = 0$ for $j \leq 52$, a case where the usual ways to determine dimensions did not work (until recent work of Chenevier-Taïbi, see this volume). We also determined without great effort the structure of a number of modules of modular forms with character, see [3]. As a final example, this method works also in positive characteristic and allows for example the determination of the ring $\mathcal{R}_2(\mathbb{F}_3)$, see [6].

The case $g = 3$ is analogous and treated in [4]. The Torelli map $\mathcal{M}_3 \to \mathcal{A}_3$ is a double cover (in the sense of stacks) ramified along the hyperelliptic locus. The role of χ_{10} for $g = 2$ is played here by χ_{18}, a cusp form of weight 18 that vanishes on the hyperelliptic locus. We can now look at $T_\rho = H^0(\overline{\mathcal{M}}_3, \mathbb{E}_\rho)$, the space of Teichmüller modular forms. The involution defined by the double cover $\mathcal{M}_3 \to \mathcal{A}_3$ acts and T_ρ splits as $T_\rho^+ \oplus T_\rho^-$, the $\pm$-eigenspaces. Then T_ρ^+ can be identified with a space of Siegel modular forms of weight ρ. The square root χ_9 of χ_{18}, introduced by Ichikawa, defines an odd Teichmüller modular form and $\chi_9 \, T_\rho^-$ lands in the even part.

The invariant theory of ternary quartics here plays the role that the invariant theory of binary sextics plays for $g = 2$. In fact, a non-hyperelliptic curve of genus 3 has as canonical image a plane quartic and the moduli space of these $\mathcal{M}_3^{\mathrm{nh}}$ is a quotient stack $[\mathcal{X}^0/\mathrm{GL}(V)]$, where now $V = \langle x_1, x_2, x_3 \rangle$ and $\mathcal{X}^0$ is an open part of $\mathcal{X} = \mathrm{Sym}^4(V) \otimes \det(V)^{-1}$. The role of covariants is played by so-called concomitants. The most basic example is the universal ternary quartic f.

We now get maps

$$M \to \mathcal{C} \xrightarrow{\nu} M_{\chi_9} \, ,$$

where $M = \oplus_\rho T_\rho$ is the module of vector-valued Teichmüller modular forms over the ring $\mathcal{T}_3$ of scalar-valued Teichmüller modular forms and $\mathcal{C}$ now denotes the module of concomitants over the ring of invariants of ternary quartics. The image of a concomitant is a meromorphic Teichmüller modular form that becomes holomorphic after multiplication by an appropriate power of χ_9. The universal ternary quartic f defines a Teichmüller modular form $\chi_{4,0,-1}$ with the property that $\chi_{4,0,8} = \chi_9 \chi_{4,0,-1}$ is the generator of the space of Siegel cusp forms $S_{4,0,8}(\Gamma_3)$. We can describe $\chi_{4,0,8}$ very precisely by developing the Schottky form, a Siegel modular form that generates the space of cusp forms of degree 4 and weight 8, in the normal bundle of the image of $\mathcal{A}_1 \times \mathcal{A}_3$ in $\mathcal{A}_4$. The first non-trivial term is $\Delta \otimes \chi_{4,0,8}$. This allows the determination of the Fourier series of $\chi_{4,0,8}$ up to high order. Again, the main point is that the map ν is given by substituting the coordinates α_I of $\chi_{4,0,-1}$ for the coefficients a_I of the (universal) ternary quartic $f = \sum_I a_I x^I$ in a concomitant.

The question arises which concomitants give rise to holomorphic modular forms. We can express the order of the resulting meromorphic modular form $\nu(c)$ along the hyperelliptic locus in terms of the order of the concomitant c along the locus of double conics. Again this leads to a very efficient method for calculating Fourier series of Siegel modular forms of degree 3.

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CM values of higher automorphic Green functions

Stephan Ehlen

(joint work with Jan H. Bruinier and Tonghai Yang)

In my talk, I reported on our recent progress towards a conjecture of Gross and Zagier on CM values of higher automorphic Green functions. In our preprint [1], we consider automorphic Green functions on orthogonal Shimura varieties in great generality and obtain algebraicity results on their CM values at "small" CM points.

In the following, we restrict to the classical situation of the automorphic Green function on the product of two modular curves. Throughout, let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ and consider the automorphic Green function G_s defined for $\Re(s) > 1$ by

$$G_s(z_1, z_2) = -2 \sum_{\gamma \in \Gamma} Q_{s-1}\left(1 + \frac{|z_1 - \gamma z_2|^2}{2\Im(z_1)\Im(\gamma z_2)}\right),$$

where $Q_{s-1}(t) = \int_0^\infty (t + \sqrt{t^2-1}\cosh(u))^{-s}\, du$ denotes the classical Legendre function of the second kind and z_1, z_2 are in the complex upper half-plane $\mathbb{H}$ with $z_1 \notin \Gamma z_2$. The function G_s is invariant under the action of Γ in both variables and descends to a function on $(X \times X) \setminus Z(1)$, where $X = \Gamma\backslash\mathbb{H}$ and $Z(1)$ denotes the diagonal, where it has a logarithmic singularity. It is an eigenfunction of the hyperbolic Laplacian in both variables. Moreover, G_s has a meromorphic continuation in s to the whole complex plane and satisfies a functional equation relating the values at s and $1 - s$.

Gross and Zagier considered certain linear combinations the Hecke translates

$$(1)\qquad G_s^m(z_1,z_2) = G_s(z_1,z_2) \mid T_m = -2 \sum_{\substack{\gamma\in \mathrm{Mat}_2(\mathbb{Z})\\ \det(\gamma)=m}} Q_{s-1}\left(1+\frac{|z_1-\gamma z_2|^2}{2\Im(z_1)\Im(\gamma z_2)}\right)$$

of G_s, where T_m denotes the m-th Hecke operator, acting on any of the two variables. Now we specialize the spectral parameter to $s = 1+j$ with $j \in \mathbb{Z}_{>0}$ and for a weakly holomorphic modular form $f = \sum_m c_f(m)q^m \in M^!_{-2j}$ of weight $-2j$ for Γ, we let

$$(2)\qquad G_{1+j,f}(z_1,z_2) = \sum_{m>0} c_f(-m)m^j G^m_{j+1}(z_1,z_2).$$

Finally, for a discriminant $d < 0$ we write $\mathcal{O}_d$ for the order of discriminant d in the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$, and let H_d be the corresponding ring class field.

Conjecture 1 (Gross–Zagier)**.** *Assume that $c_f(-m) \in \mathbb{Z}$ for all $m > 0$. Let z_1 be a CM point of discriminant d_1, and let z_2 be a CM point of discriminant d_2 such that $G_{j+1,f}(z_1,z_2)$ is defined at (z_1,z_2). Then there is an $\alpha \in H_{d_1}\cdot H_{d_2}$ such that*

$$(3)\qquad (d_1d_2)^{j/2}G_{j+1,f}(z_1,z_2) = \frac{w_{d_1}w_{d_2}}{4}\cdot \log|\alpha|,$$

where $w_{d_i} = \#\mathcal{O}^\times_{d_i}$.

Gross, Kohnen, and Zagier provided numerical evidence [5, Chapter V.4], [4, Chapter V.1] and considered the average of $(d_1d_2)^{j/2}G_{j+1,f}(z_1,z_2)$ over all CM points (z_1,z_2) of discriminants d_1 and d_2 and proved that it equals $\log|\beta|$ for some $\beta \in \mathbb{Q}$. Mellit proved the conjecture for $z_2 = i$ and $j = 1$ [7]. In the case that z_1 and z_2 lie in the same imaginary quadratic field Zhang [10] obtained that the conjecture holds if a certain height pairing of Heegner cycles on Kuga-Sato varieties is non-degenerate. Viazovska showed that (3) holds in this case for $\alpha \in \bar{\mathbb{Q}}$ and proved the conjecture assuming that $d_1 = d_2$ is prime [8, 9]. Another average version of the conjecture for odd j was recently shown by Li [6].

We improve these results in our paper [1]. One method we use is to only average over the CM points z_1 of discriminant d_1 and to allow z_2 to be any CM point of discriminant d_2. To make this precise, let $\mathcal{Q}_{d_1}$ denote the set of integral binary positive definite quadratic forms of discriminant $d_1 < 0$. For $Q \in \mathcal{Q}_{d_1}$ we write z_Q for the corresponding CM point in $\mathbb{H}$ and let w_Q be the order of the stabilizer Γ_Q. We then let

$$C(d_1) = \sum_{Q\in\mathcal{Q}_{d_1}/\Gamma} \frac{2}{w_Q}\cdot z_Q$$

and prove the following theorem in [1].

Theorem 2. *Let $j \in \mathbb{Z}_{>0}$. Let $d_1 < 0$ be a fundamental discriminant, and let $d_2 < 0$ be a discriminant such that d_1d_2 is not the square of an integer. If j is odd, let $k = \mathbb{Q}(\sqrt{d_1},\sqrt{d_2})$ and $H = H_{d_2}(\sqrt{d_1})$. If j is even, let $k = \mathbb{Q}(\sqrt{d_2})$ and*

$H = H_{d_2}$. If z_2 is a CM point of discriminant d_2, then there exists an algebraic number $\alpha = \alpha(f, d_1, z_2) \in H$ and an $r \in \mathbb{Z}_{>0}$ such that

$$(d_1 d_2)^{j/2} G_{j+1,f}(C(d_1), z_2^\sigma) = \frac{1}{r} \log |\alpha^\sigma|$$

for every $\sigma \in \mathrm{Gal}(H/k)$.

In fact, we also consider twists of the divisors $C(d_1)$ by genus characters, and corresponding twisted versions of the above theorem. As a corollary we obtain the following result.

Corollary 3. *Let $d_1 < 0$ be a fundamental discriminant and assume that the class group of $\mathcal{O}_{d_1}$ is trivial or has exponent 2. Let z_1 be any CM point of of discriminant d_1 and let z_2 be any CM point of discriminant $d_2 < 0$ (not necessarily fundamental), where $z_1 \neq z_2$ if $d_1 = d_2$. Then, there is an $\alpha \in H_{d_1} \cdot H_{d_2}$ and an $r \in \mathbb{Z}_{>0}$ such that*

$$(d_1 d_2)^{j/2} G_{j+1,f}(z_1, z_2) = \frac{1}{r} \log |\alpha|.$$

As mentioned above, we realize the modular curve X as an orthogonal Shimura variety and make use of the regularized theta correspondence. A key observation is that $G_s(C(d_1), z_2)$ can be obtained as the regularized theta lift of a weak Maass form of weight 1/2.

Based on this, we prove a formula for the higher Green function at a CM point in terms of a finite linear combination of the coefficients of the holomorphic part of a harmonic Maass form of weight one that maps to a weight one theta function under the ξ-operator (which is naturally associated with the CM point). We then generalize the results of [2, 3] and show that the Fourier coefficients of the holomorphic part of such a (suitably normalized) harmonic Maass form are given by logarithms of algebraic numbers in the corresponding ring class fields.

We also use our approach to prove a Gross-Kohnen-Zagier theorem for higher Heegner divisors on Kuga-Sato varieties over modular curves.

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Modular forms on exceptional groups

Aaron Pollack

Suppose that G is a reductive group over $\mathbf{Q}$, and K is a maximal compact subgroup of the real points $G(\mathbf{R})$. In case $G(\mathbf{R})/K$ has the structure of a hermitian symmetric space, one can consider those automorphic forms for G that correspond to holomorphic functions on $G(\mathbf{R})/K$. This distinguished class of automorphic functions for these groups G play a principal role in the *arithmetic* of automorphic forms and the application of automorphic forms to classical number-theoretic problems.

For example, when $G = \mathrm{Sp}_{2n}$, $K \simeq U(n)$, and the symmetric space $\mathrm{Sp}_{2n}(\mathbf{R})/K$ can be identified with Sieglel's upper half space of degree n:

$$\mathrm{Sp}_{2n}(\mathbf{R})/K \simeq \mathcal{H}_n = \{Z \in M_n(\mathbf{C}) : Z^t = Z, Im(Z) > 0\},$$

the symmetric $n \times n$ complex matrices with positive-definite imaginary part. In this case, if $\ell \geq 0$ is an integer, the classical Siegel modular forms of weight ℓ are defined to be the holomorphic functions $f : \mathcal{H}_n \to \mathbf{C}$ of moderate growth that satisfy $f(\gamma Z) = j(\gamma, Z)^\ell f(Z)$ for all γ in some congruence subgroup $\Gamma \leq \mathrm{Sp}_{2n}(\mathbf{Z})$. Here, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $n \times n$ block form, then $\gamma Z = (aZ + b)(cZ + d)^{-1}$, and $j(\gamma, Z) = \det(cZ + d)$. Of course, these Siegel modular forms have a very nice Fourier expansion:

$$f(Z) = \sum_T a_f(T) e^{2\pi i \mathrm{tr}(TZ)} \tag{1}$$

where the $a_f(T) \in \mathbf{C}$ are the Fourier coefficients and the sum is over $n \times n$ rational, symmetric matrices T which are positive semi-definite. The ability to assign the list of Fourier coefficients $\{a_f(T)\}_T$ to a Siegel modular form f is a crucial link from the Siegel modular forms to arithmetic, and very much depends on the fact that f is a holomorphic function on $\mathcal{H}_n$.

For general reductive $\mathbf{Q}$-groups G, when $G(\mathbf{R})/K$ does not have complex structure, there is no *a priori* reason to expect any distinguished or especially nice set of automorphic functions for G. For example, for the Dynkin types G_2, F_4 and E_8, there is no real form of these groups whose associated symmetric space has complex structure. Thus one can ask the following question:

Question 1: Suppose that G is reductive $\mathbf{Q}$-group whose associated symmetric space does not have complex structure. Is there any notion of "modular forms" for G, which have a Fourier expansion analogous to (1) above?

Gross and Wallach [4, 5] in fact singled out a class of groups (including split G_2 and split F_4) together with special automorphic functions on these groups that appear to deserve the moniker "modular form". These modular forms were also studied by Wallach [9], Gan–Gross–Savin [3] and Weissman [10]. The results discussed in the talk include

- A Fourier expansion for these *modular forms*, analogous to (1). This is the main theorem from [6].
- Some special examples of these modular forms on the exceptional groups E_6, E_7, E_8 [7].
- A construction of cuspidal modular forms on the exceptional group G_2 [8].

The class of groups G studied by Gross and Wallach include split G_2, forms of the exceptional groups F_4, E_6, E_7, E_8 whose real rank is four, and the groups $\mathrm{SO}(V)$ where V is a quadratic space with signature $(4,n)$ with $n \geq 3$. They defined modular forms for these groups in terms of certain infinite-dimensional unitary representations on the groups $G(\mathbf{R})$, the so-called quaternionic discrete series [4, 5]. If $K_\infty \subseteq G(\mathbf{R})$ is the maximal compact subgroup of the real points of one of these groups G, then K_∞ surjects to the compact group $\mathrm{SU}(2)/\mu_2$. Altering slightly the definition of Gross-Wallach, one can define modular forms for G as follows:

Definition 2 (See [6, 7]) Suppose $\ell \geq 1$ is an integer, and let the notation be as above. Denote by $\mathbb{V}_\ell$ the space $Sym^{2\ell}(\mathbf{C}^2)$, considered as a representation of K through the map $K \to \mathrm{SU}(2)/\mu_2$. A modular form on G of weight ℓ is an automorphic function

$$f : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbb{V}_\ell$$

satisfying the following properties:

(1) f is of moderate growth
(2) $f(gk) = k^{-1} \cdot f(g)$ for all $g \in G(\mathbf{A})$ and $k \in K_\infty$
(3) $D_\ell f \equiv 0$, for a certain specific linear differential operator D_ℓ, called the Schmid operator.

That these modular forms have nice Fourier expansions is essentially the content of the following theorem.

Theorem 3 (See [6]) Suppose $\ell \geq 1$ is an integer. For a certain index set $\{T\}$, there are completely explicit functions $\mathcal{W}_T : G(\mathbf{R}) \to \mathbb{V}_\ell$ so that if f is a modular form on G of weight ℓ, $g_f \in G(\mathbf{A}_f)$ and $g_\infty \in G(\mathbf{R})$, then

$$f_Z(g_f g_\infty) = f_0(g_f g_\infty) + \sum_T a_{f,T}(g_f)\mathcal{W}_T(g_\infty)$$

for certain $\mathbf{C}$-valued locally constant functions $a_{f,T}$ on $G(\mathbf{A}_f)$.

In the theorem, f_0 is a certain constant term of the modular form f and f_Z is a certain integral transform of f that determines the function f. The functions $\mathcal{W}_T$ should be considered the analogue of the exponential functions $e^{2\pi i \mathrm{tr}(T\bullet)}$ on the Siegel upper half space $\mathcal{H}_n$, and $a_{f,T}$ are the Fourier coefficients of f. The theorem makes explicit and extends a result of Wallach [9].

Applying Theorem 3, one can assign the Fourier coefficients $\{a_{f,T}\}_T$ to a modular form f on G. With this list of Fourier coefficients, one can then ask questions about them, such as if all the Fourier coefficients are $a_{f,T}$ valued in some small subring of $\mathbf{C}$, such as $\mathbf{Z}, \mathbf{Q}$, or $\overline{\mathbf{Q}}$. Building on work of Gan [2] and Elkies-Gross [1], the paper [7] constructs some special modular forms (in the sense of Definition 2) on the groups $E_{6,4}, E_{7,4}, E_{8,4}$, whose Fourier coefficients are rational numbers and have other special properties.

The explicit modular forms constructed in prior work [3, 10, 7] do not include cusp forms. A construction of cusp forms on the groups $\mathrm{SO}(4,4)$ and G_2 is given by the following result. Recall that one has an inclusion $G_2 \subseteq \mathrm{SO}(4,4)$.

Theorem 4(See [8]) Suppose $\ell \geq 16$ is an even integer, and f a cuspidal Siegel modular form on Sp_4 of weight ℓ.

(1) There is a cuspidal modular form $\theta(f)$ on $\mathrm{SO}(4,4)$ associated to f. If f is level one on Sp_4 and the Fourier coefficients of f are valued in a subring $R \subseteq \mathbf{C}$, then $\theta(f)$ is nonzero and has Fourier coefficients valued in R.
(2) If F is a cuspidal modular form on $\mathrm{SO}(4,4)$ of weight ℓ with Fourier coefficients valued in a ring $R \subseteq \mathrm{C}$, then the restriction $F|_{G_2}$ is a cuspidal modular form on G_2 of weight ℓ with Fourier coefficients in R.

In particular, if $\ell \geq 16$ is even, then there are nonzero cuspidal modular forms on G_2 of weight ℓ with all Fourier coefficients in $\overline{\mathbf{Q}}$.

The lift $f \mapsto \theta(f)$ of Theorem 4 is a higher rank analogue of the Saito-Kurokawa lift, considered as a lift from certain cuspidal modular forms on $\widetilde{\mathrm{SL}_2}$ to $\mathrm{SO}(2,3) = \mathrm{PGSp}_4$.

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Arithmetic special cycles and Jacobi forms

Siddarth Sankaran

1. Background

Let F be a totally real field and let V be an anisotropic quadratic space over F of dimension $p+2$ for some $p > 0$. Suppose V is of signature $((p,2),(p+2,0),\dots,(p+2,0))$;i.e. for one real embedding $\sigma_1 \colon F \to \mathbb{R}$, the space $V_1 := V \otimes_{\sigma_1,F} \mathbb{R}$ has signature $(p,2)$, while $V_i > 0$ at all other real embeddings.

Let $G = \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GSpin(V)})$ viewed as an algebraic group over $\mathbb{Q}$; given a (sufficiently small) arithmetic subgroup $\Gamma \subset G(\mathbb{R})$, we obtain a connected Shimura variety $X = X_\Gamma$, which is a projective variety defined over a number field E.

These varieties are particularly interesting because they are equipped with a family of naturally constructed E-rational "special" cycles $\{Z(T,\varphi)\}$, parametrized by matrices $T \in \mathrm{Sym}_n(F)$ and Schwartz forms $\varphi \in S(V(\mathbb{A}_f)^n)$ for $1 \le n \le p$.

In recent years, a substantial body of evidence has emerged to support a conjectural program put forward by Kudla: roughly, the idea is that when these cycles are viewed as elements of "arithmetic" Chow groups in an appropriate way, they can be identified with the Fourier coefficients of Siegel modular forms.

Let us briefly recall the theory of arithmetic Chow groups; for our purposes, we utilize the 'covariant' groups $\widehat{\mathrm{CH}}^n(\mathcal{X},\mathcal{D}_{\mathrm{cur}})$ defined in Burgos-Kramer-Kühn [5], which generalize the construction of Gillet-Soulé. Fix a ring R with $\mathcal{O}_E \subset R \subset E$ (e.g. $R = \mathcal{O}_E[1/N]$ for some N, or $R = E$) and a sufficiently nice model $\mathcal{X}/R$ of X. Then classes in $\widehat{\mathrm{CH}}^n(\mathcal{X},\mathcal{D}_{\mathrm{cur}})$ are represented by pairs $(\mathcal{Z},\mathfrak{g})$, where $\mathcal{Z}$ is a codimension n cycle on $\mathcal{X}$, and $\mathfrak{g}$ is a *Green object* for $\mathcal{Z}(\mathbb{C})$, namely a degree $2n-2$ current on the complex points $X(\mathbb{C})$ that is related to $\mathcal{Z}(\mathbb{C})$ via a certain cohomological condition (for the arithmetic Chow groups as defined by Gillet-Soulé, this condition is precisely Green's equation).

Kudla's conjecture can be formulated roughly as follows. First, for each cycle $Z(T,\varphi)$, choose a model $\mathcal{Z}(T,\varphi)$ over R and a Green object $\mathfrak{g}(T,\varphi)$, and define the class $\widehat{\mathcal{Z}}(T,\varphi) = (\mathcal{Z}(T,\varphi),\mathfrak{g}(T,\varphi)) \in \widehat{\mathrm{CH}}^n(\mathcal{X},\mathcal{D}_{\mathrm{cur}})$. Then, given judicious choices as above, the expectation is that for each $n \le p+1$, the generating series

$$\widehat{\phi}_n(\tau) = \sum_{T \in \mathrm{Sym}_n(F)} \widehat{\mathcal{Z}}(T,\varphi)\, q^T \tag{1}$$

should be the q-expansion of a Siegel modular form of genus n. For models over the full ring of integers $\mathcal{O}_E$, results of this form have been established for particular cases: for example, for full level Shimura curves over $\mathbb{Z}$ by Kudla-Rapoport-Yang [9] and for $U(p,1)$ Shimura varieties with $n = 1$ by Bruinier-Howard-Kudla-Rapoport-Yang [3]. Note that if one replaces the cycles with their images in the Chow group $\mathrm{CH}^n(X)$ of the generic fibre, the modularity of the corresponding series has been proven in [1, 4] for $F = \mathbb{Q}$, and one has conditional results in the case $F \neq \mathbb{Q}$, see e.g. [7, 10, 11]; upon passing further to cohomology, the modularity is a special case of the results of Kudla-Millson [8].

2. Results

The result discussed in the talk focuses on the case $R = E$ and $\mathcal{X} = X$, which allows us to avoid delicate issues around integral models, and therefore to consider arbitrary level structure and any value of n while retaining the data of Green currents. To this end, we use the family $\mathfrak{g}(T, v, \varphi)$ of Green currents that was constructed in previous joint work with L. Garcia [6]; these currents depend on an additional parameter $v \in \mathrm{Sym}_n(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$. We obtain classes

$$\widehat{Z}(T, v, \varphi) = (Z(T, \varphi), \mathfrak{g}(T, v, \varphi)) \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}}).$$

Theorem 1. *Let $T_2 \in \mathrm{Sym}_{n-1}(F)$. Then the generating series*

$$\widehat{\phi}_{T_2}(\tau, \varphi) := \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{Z}(T, v, \varphi)\ q^T, \tag{2}$$

with $v = \mathrm{Im}(\tau)$, is the q-expansion of a Jacobi modular form of parallel weight κ and index T_2.

In other words, this result asserts that for $\mathcal{X} = X$ and these choices of Green currents, the formal Fourier-Jacobi coefficients of the generating series $\widehat{\phi}(\tau)$ are indeed Jacobi modular forms.

Some care is required in interpreting the statement of theorem. First, we are being vague about the level; a slightly more precise formulation is that the map $\widehat{\phi}_{T_2}(\tau)\colon \varphi \mapsto \widehat{\phi}_{T_2}(\tau, \varphi)$ transforms as a vector-valued form, valued in the Weil representation acting on $S(V(\mathbb{A}_f)^n)^\vee$.

A more substantial point is that there is no obvious topology on $\widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$ with which to make sense of the convergence of a series such as (2). What is being asserted is the existence of:

1. finitely many classes $\widehat{Z}_1, \dots \widehat{Z}_r \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$;
2. finitely many Jacobi modular forms (in the usual sense) $f_1, \dots, f_r$;
3. and a Jacobi form $g(\tau)$ valued in the space of currents on X that satisfies a certain technical "uniformity" condition in τ;

such that the T'th coefficient of the Jacobi form

$$\sum_i f_i(\tau)\widehat{Z}_i + a(g(\tau))$$

coincides with $\widehat{Z}(T, \varphi, v)$. Here $a(g(\tau)) \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$ is an "archimedean class" associated to the current $g(\tau)$. Note that these forms are non-holomorphic in τ.

The first step in the proof is a decomposition $\widehat{Z}(T, v, \varphi) = \widehat{A}(T, v, \varphi) + \widehat{B}(T, v, \varphi)$, where $\widehat{B}(T, v, \varphi)$ is an archimedean class; this yields a corresponding decomposition of $\widehat{\phi}_{T_2}(\tau)$ into two generating series. The first series can be identified as a sum of products of pushforwards of generating series of divisors with standard theta functions, and the argument ultimately reduces to a modularity result due to Bruinier [2] in the case $n = 1$. The modularity of the second piece follows from an explicit calculation, with the theta series introduced by Kudla and Millson [8] playing a key role.

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Interpolated Apéry numbers, the mirror quintic, and quasiperiods of modular forms

Don Zagier

(joint work with V. Golyshev and with A. Klemm and E. Scheidegger)

The Apéry numbers $A_0 = 1,\ A_1 = 5,\ A_2 = 73, \ldots$, defined by the recursion

$$(n+1)^3 A_{n+1} = (34n^3 + 51n^2 + 27n + 5)A_n - n^3 A_{n-1}\,,$$

were used by Apéry for his famous proof in 1978 of the irrationality of $\zeta(3)$, but they also have interpretations in terms of modular forms (Beukers) and algebraic geometry (Beukers and Peters). It was predicted by V. Golyshev, and then proved, that the natural interpolation of $\{A_n\}$ to $n \in \mathbb{Q}$ (or even $n \in \mathbb{C}$) satisfies $A_{-1/2} = L(f_8, 4)$, where $f_8(\tau) = \eta(2\tau)^4\eta(4\tau)^4$ is the unique normalized cusp form of weight 4 on $\Gamma(8)$. We will explain why this is expected and why it is true. We also explain the notion of *quasiperiods* of modular forms and show that also $A_{1/2}$ (and hence A_n for all $n \in \mathbb{Z}+\frac{1}{2}$) is a linear combination of periods and quasiperiods of f_8. As another application, the periods and quasiperiods of a Hecke eigenform $f_{25} \in S_4(\Gamma_0(25))$ show up in the transition matrix between two singularities (the "MUM point" and the "conifold point") of the fundamental bases of solutions of the hypergeometric differential equation with holomorphic solution $\sum_{n=0}^{\infty} \frac{(5n)!}{n!^5} t^n$ that played the key role in the discovery of mirror symmetry. Here the question arises of a possible explicit parametrization by modular or Jacobi forms of the

Schoen quintic $X_1^5 + X_2^5 + X_3^5 + X_4^5 + X_5^5 = 5X_1X_2X_3X_4X_5$. Partial solutions of this problem are described that involve the L-series over $\mathbb{Q}$ of the elliptic curve $y^2 = x^3 + 4/5$ and a certain Picard modular forms for $U(1,2)$ over $\mathbb{Q}(\sqrt{-3})$.

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