MATHEMATISCHES FORSCHUNGSINSTITUT OBERWOLFACH

Manifolds and Groups

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ABSTRACT. The workshop concentrated on the interplay of advances in the understanding of manifolds and geometric group theory. In particular, we discussed mapping class groups and moduli spaces of manifolds (also of high dimension) and aspects of homological stability related to them; cobordism categories and their applications; L^2-invariants, simplicial volume, and their applications; and in general, the phenomena of rigidity versus flexibility in geometry and topology.

Introduction by the Organizers

Geometric group theory and the topology of manifolds have seen significant advances and much of this based on the fertile interplay between these two areas. This workshop concentrated on the interplay of several particularly successful trends and guiding problems of this type:

- cobordism categories and their applications,
- applications of K-theory to topology,
- L^2-invariants and their relatives,
- topological aspects of volume and curvature, metrically enriched topology,
- group homology and moduli spaces.

These areas are strongly connected, progress in one area is only possible through advances and application of tools from the others. During the workshop, researchers in these fields of all levels—from promising recent graduates to the leading experts—discussed current developments and exchanged ideas. The biggest challenge (but also the biggest benefit) of this workshop was to bridge the gaps

between the different fields and mathematical languages, ranging from higher categories to Benjamini–Schramm statistics. It is our hope that seeing problems through the eyes of other fields will lead to new impulses.

The workshop programme offered 19 regular talks. The workshop also presented a one hour session on Thursday night for a *gong show* which featured five 10 minute presentations where young researchers described their work. The speakers at this gong show were Calista Bernard, Zhicheng Han, Biao Ma, Eduard Schesler, and Engelbert Peter Suchla. The meeting was attended by 50 participants from all over the world, including many early career researchers. Moreover, as usual in Oberwolfach, the breaks and evenings were filled with a variety of lively discussions in smaller groups.

The following describes the main topics and some selected recent achievements that were discussed during this workshop in more detail.

L^2-*Invariants and their relatives.* Grigori Avramidi reported on computations of mod p homology gradients for right-angled Artin groups (joint work with Okun and Schreve). Consequences of these investigations are the first construction of groups whose L^2-Betti numbers (i.e., the rational homology gradients) do *not* coincide with their mod p homology gradients as well as the first construction of counterexamples to the mod p Singer conjecture.

Dawid Kielak explained how to characterise group-theoretic virtual fibering (which is inspired by 3-manifold topology) in terms of the first L^2-Betti number.

Topological aspects of volume and curvature and metrically enriched topology. Roman Sauer (in joint work with Braun) obtained a generalisation of Gromov's main inequality between volume and simplicial volume, where the lower Ricci curvature bound is replaced by a macroscopic scalar curvature bound. The proof relies on randomised version of metric methods by Larry Guth.

K-*theory and topology.* Fabian Hebestreit and Markus Land gave a pair of linked talks on their work (with Calmès, Dotto, Harpaz, Moi, Nardin, Nikolaus, and Steimle) on algebraic cobordism categories and Grothendieck–Witt theory. The first talk explained the foundations and general structure of this theory, while the second explained how this can be used to calculate the hermitian and symplectic K-groups of the integers, in both the symmetric and quadratic situation.

Wolfgang Lück reported on extensive calculations on crystallographic groups, including their topological and algebraic K-theory as well as their L-theory. Applications were then described to the classification of C^*-algebras, to the unstable Gromov–Lawson–Rosenberg conjecture, and to the classification of topological manifolds homotopy equivalent to torus bundles over lens spaces.

Topology of moduli spaces. Manuel Krannich presented a new perspective on the relationship between pseudoisotopy theory and K-theory. Focussing on the case of an even-dimensional disc, he explained how results of Botvinnik–Perlmutter on diffeomorphism groups of handlebodies, together with Morlet's lemma of disjunction, can be used to relate the space of pseudoisotopies to algebraic K-theory of the integers in a range of degrees which is 3 times better than the classical approach.

Geometric group theory. Aspects of geometric group theory play an important role in several talks, in particular also on L^2-Betti numbers. Specific to geometric group theory was the talk of Martin Bridson which discussed rigidity aspects of the automorphism group of free groups, in particular new findings on the complex of free factors as a space with controlled topology on which this group acts, while Anna Erschler surveyed geometric group theoretical aspects of the Travelling Salesman Problem.

Workshop: Manifolds and Groups

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Abstracts

Variations of a conjecture of Singer
Grigori Avramidi

For a closed manifold M with contractible universal cover, the Singer conjecture predicts that the L^2-Betti numbers of M are concentrated in the middle dimension. In my talk I discussed what is known and unknown about this conjecture and explained why it does not have a rational analogue [Avr18]. I also explained how (following Davis, Okun, and Schreve [DO01, OS16]) to think of it as a conjecture about arbitrary discrete groups and described a variant for CAT(0) groups relating mod 2 L^2-Betti numbers (interpreted dynamically as the entropy of the group acting on homology of the boundary) to van Kampen embedding obstructions of their boundaries that fits especially well with computations in right-angled Artin groups.

Boundary Singer Conjecture. Suppose that $B\Gamma$ is a d-dimensional, finite complex and Γ is a residually finite CAT(0) or hyperbolic group. Let $\partial\Gamma$ be a CAT(0) or hyperbolic boundary for this group. If the entropy of Γ acting on $H_{d-1}^{st}(\partial\Gamma; \mathbb{F}_2)$ is positive, then $\partial\Gamma$ does not embed in S^{2d-2}.

This conjecture is the question mark in the following schematic picture.

$$
\boxed{\begin{array}{c} L^2\text{-Betti number} \\ b_d^{(2)}(B\Gamma; \mathbb{Q}) \end{array}} \xrightarrow[\text{Singer}]{} \xrightarrow[\text{Conj.}]{} \boxed{\begin{array}{c} \text{Thickening obstruction} \\ B\Gamma \not\sim M^{2d-1} \end{array}}
$$

$$
GS, L \downarrow \qquad\qquad\qquad\qquad \uparrow BKK
$$

$$
\boxed{\begin{array}{c} \text{Entropy} \\ h(\Gamma \curvearrowright \overline{H}_{d-1}^{st}(\partial\Gamma, \mathbb{F}_2)) \end{array}} \xrightarrow{?} \boxed{\begin{array}{c} \text{Embedding obstruction} \\ \partial\Gamma \not\hookrightarrow S^{2d-2} \end{array}}
$$
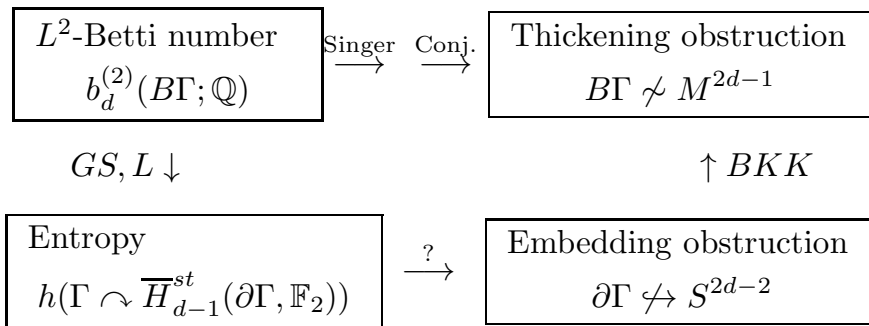
The left vertical arrow uses Lück's approximation theorem [L94] to think of L^2-Betti numbers as a limit of normalized Betti numbers of covers and compare it to the entropy (in the sense of [GS19]), which can be thought of as a limsup of normalized mod 2-Betti numbers of covers (also referred to as mod 2 L^2-Betti numbers) in this case. The right vertical arrow is work of Bestvina, Kapovitch and Kleiner [BKK02] which produces thickening obtsructions for $B\Gamma$ from embedding obstructions for the boundary $\partial\Gamma$.

The main piece of evidence for this conjecture comes from computations with right Artin groups. One can compute when the entropy is positive ([AOS20]) and when the van Kampen embedding obstruction does not vanish ([ADOS16]). The computations are quite different but the results are identical.

As mentioned before, the computation of entropy is really a computation of mod p L^2-Betti numbers of right-angled Artin groups. We compute these completely.

Theorem 1. *Let L be a flag complex, $\Gamma = A_L$ the right-angled Artin group*

$$A_L := \langle a_v, v \text{ vertex of } L \mid [a_v, a_w] = 1 \text{ if } v \text{ is adjacent to } w \rangle,$$

$\Gamma_n \trianglelefteq \Gamma$ a chain of normal finite index subgroups and k any field (e.g. $\mathbb{Q}$ or $\mathbb{F}_p$). Then

$$b_i^{(2)}(A_L; k) = \overline{b}_{i-1}(L; k),$$

where on the right we have the reduced Betti numbers of L with coefficients in k.

In particular, the mod p L^2-Betti numbers of A_L do not depend on the chain Γ_n but do depend on the characteristic of the coefficient field k. For example, if L is a flag triangulation of $\mathbb{R}P^2$ then we have

Corollary 2. $b_3^{(2)}(A_{\mathbb{R}P^2}; \mathbb{Q}) = 0$ and $b_3^{(2)}(A_{\mathbb{R}P^2}; \mathbb{F}_2) = 1$.

It follows from this that such a group $A_{\mathbb{R}P^2}$ has exponential homological torsion growth in degree two

$$\limsup_n \frac{\log |H_2(\Gamma_n; \mathbb{Z})_{\text{tors}}|}{|\Gamma/\Gamma_n|} > 0$$

answering a query of Bergeron for such a group.

In related work that was completed after the talk was given, we constructed for odd primes p closed locally CAT(0) manifolds with residually finite fundamental group and mod p L^2-Betti numbers outside the middle dimension. These closed aspherical manifolds show that the Singer conjecture on Betti number growth and Lück's conjecture on torsion homology growth are incompatible in the CAT(0) setting, so at least one of them must be wrong. (See [AOS20] for details).

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Positive scalar curvature on manifolds with odd order abelian fundamental groups

Bernhard Hanke

This talk reports on the paper [1]. The classification of closed manifolds admitting Riemannian metrics of positive scalar curvature has been a major research topic for some decades. While for simply connected manifolds of dimension at least 5 a complete classification has been achieved by Gromov-Lawson (1980) and Stolz (1992) the picture remains largely unclear even for finite fundamental groups. The following conjecture of Rosenberg (1986) addresses a specific issue in this direction.

Conjecture 1. Let M be a closed connected smooth manifold with finite fundamental group of odd order. If the universal cover of M admits a positive scalar curvature metric, then M admits a positive scalar curvature metric.

Conjecture 1 holds for $\dim M \leq 2$ and for $\dim M = 3$ it follows from the geometrisation theorem. For $\dim M = 4$ counterexamples were constructed by Hanke-Kotschick-Wehrheim (2003) using Seiberg-Witten theory. Hence this case must be excluded. Rosenberg pointed out that Conjecture 1 is false without assuming that $\pi_1(M)$ is of odd order. This and the failure of the conjecture in dimension 4 imply that the metric obtain from $\pi_1(M)$-averaging a positive scalar curvature metric on the universal cover of M is in general not of positive scalar curvature. Conjecture 1 remains open in general in dimensions larger than 4.

Definition 2. Let p be a prime. A closed oriented manifold M of dimension d is call *p-atoral* if for all $\ell \in \mathbb{N}$, $\ell \geq 1$, and cohomology classes $c_1, \ldots, c_d \in \mathrm{H}^1(M; \mathbb{Z}/p^\ell)$ we have

$$\big(c_1 \cup \cdots \cup c_d\big)([M]) = 0 \in \mathbb{Z}/p^\ell.$$

Otherwise M is called *p-toral*.

Remark 3.
- The d-torus $T^d = (S^1 \times \cdots \times S^1)^d$, $d \geq 1$, is p-toral for all p, and so are all closed manifolds which are oriented bordant, over the classifying space $B(\mathbb{Z}/p)^d$, to the canonical map $T^d = B\mathbb{Z}^d \to B(\mathbb{Z}/p)^d$.
- It may be that p-toral manifolds for odd p do not admit positive scalar curvature metrics. This would yield counterexamples to Conjecture 1.

In dimensions larger than or equal to 5 Conjecture 1 holds for p-atoral manifolds whose fundamental groups are elementary abelian p-groups (p odd). This result is due to Botvinnik-Rosenberg (2002, 2005) and to the author (2016). The following is our main result. It resolves a problem stated in the 2002 paper of Botvinnik-Rosenberg.

Theorem 4. *Let M be a closed connected smooth manifold of dimension at least 5 with odd order abelian fundamental group. Assume that M is non-spin and p-atoral for all primes p dividing the order of $\pi_1(M)$. Then M admits a Riemannian metric of positive scalar curvature.*

In spirit of other existence results for positive scalar curvature metrics on high dimensional manifolds the proof of Theorem 4 is based on the propagation of positive scalar curvature metrics along surgeries of codimension at least 3. We combine this technique with the realization of singular homology classes by manifolds with Baas-Sullivan singularities. To this end we introduce and discuss the concept of *positive scalar curvature metrics on manifolds with Baas-Sullivan singularities*. Let Ω_*^{SO} denote the oriented bordism ring and fix a family $\mathcal{Q} = (Q_{4i})_{i \geq 1}$ of closed oriented manifolds of dimension $4i$ whose bordism classes form a set of polynomial generators of $\Omega_*^{\mathrm{SO}}/\mathrm{torsion}$, and each of which is equipped with a metric of positive scalar curvature. Let $\Omega_*^{\mathrm{SO},\mathcal{Q}}(-)$ denote oriented bordism with singularities in $\mathcal{Q}$. Baas-Sullivan theory tells us that the natural transformation $\Omega_*^{\mathrm{SO},\mathcal{Q}}(-) \to \mathrm{H}_*(-;\mathbb{Z})$ is an isomorphism after inverting 2. Given a topological space X we define a subgroup $\mathrm{H}_*^{\mathcal{Q},+}(X;\mathbb{Z}) \subset \mathrm{H}_*(X;\mathbb{Z})$, called the *positive homology of X with respect to $\mathcal{Q}$*. This group is generated by singular manifolds with Baas-Sullivan singularities admitting positive scalar curvature metrics. In particular, positive homology classes need not be representable by smooth manifolds.

Theorem 5. *Let M be a closed connected oriented smooth manifold of dimension $d \geq 5$ with odd order fundamental group and which is non-spin. Let $\phi : M \to B\pi_1(M)$ be the classifying map. Then M admits a metric of positive scalar curvature, if and only if $\phi_*([M]) \in \mathrm{H}_d^{\mathcal{Q},+}(B\pi_1(M);\mathbb{Z})$.*

It remains to show that under the conditions of Theorem 4 we have $\phi_*([M]) \in \mathrm{H}_d^{\mathcal{Q},+}(B\pi_1(M);\mathbb{Z})$. For this goal we first study the positive homology $\mathrm{H}_*^{\mathcal{Q},+}(B\Gamma;\mathbb{Z})$ for finite abelian p-groups Γ. The homology of $B\Gamma$ can inductively be computed by an exact Künneth sequence (with $\alpha \geq 1$)

$$0 \to \mathrm{H}_*(B\Gamma) \otimes \mathrm{H}_*(B\mathbb{Z}/p^\alpha) \xrightarrow{\times} \mathrm{H}_*(B\Gamma \times B\mathbb{Z}/p^\alpha) \longrightarrow \mathrm{Tor}\big(\mathrm{H}_*(B\Gamma), \mathrm{H}_*(B\mathbb{Z}/p^\alpha)\big) \to 0\,.$$

The cross product can be realized by admissible products of manifolds with Baas-Sullivan singularities, and the same is true for the torsion product, which is related to a homological Toda bracket. By a variant of the well known "shrinking one factor" argument the cross product of two homology classes is positive, if one of the factors is positive. However we can in general show positivity of Toda brackets only if *both* of the factors are positive. This excludes Toda brackets involving one dimensional factors, represented by circles. Nevertheless we have the following result.

Theorem 6. *Let p be an odd prime, and let Γ be a finite abelian p-group. Then all p-atoral classes in the image of $\Omega_*^{SO}(B\Gamma) \to \mathrm{H}_*(B\Gamma;\mathbb{Z})$ are positive.*

The proof of this fact relies on the homology of abelian p-groups and the representability of certain homology classes in the classifying space of $(\mathbb{Z}/p^\alpha)^n$ by products of $\mathbb{Z}/p^\alpha$-lens spaces. Theorems 5 and 6 imply Theorem 4 by a relatively straightforward argument. We conjecture that Theorem 4 also holds for spin manifolds with vanishing α-invariants. A proof should be based on real connective K-homology instead of ordinary homology.

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Quasimorphisms on surface diffeomorphism groups

SEBASTIAN HENSEL

(joint work with Jonathan Bowden, Richard Webb)

Closed surfaces of genus $g \geq 2$ are among the most fundamental and basic objects in geometry and topology. Although surfaces are easy to describe and have been studied intensively since the early twentieth century, their homeomorphism and diffeomorphism groups show remarkable depth and complexity, and are still a rich source of interesting open problems.

One can break the study of these groups into two complementary pieces: on the one hand, there is the discrete *mapping class group* given by the path components of the group of orientation preserving diffeomorphisms:

$$\mathrm{Mcg}(S) = \pi_0(\mathrm{Diff}^+(S)),$$

and on the other one hand, there is the *identity component* $\mathrm{Diff}_0(S)$ of diffeomorphisms which are isotopic to the identity.

Until now, these two pieces have been studied with very different methods: being finitely generated, the mapping class group fits into the realm of geometric group theory, and this perspective has lead to enormous advances over the last decades. The identity component is usually studied using geometric topology, and remains much more mysterious

The philosophy behind our result is that it is possible to transfer *geometric* methods (which have proven to be successful for mapping class groups) to the realm of diffeomorphism groups.

Concretely, we proved the following:

Theorem 1 (Bowden-Hensel-Webb). *The space of unbounded quasimorphisms on* $\mathrm{Diff}_0(S)$ *is infinite-dimensional.*

Prior to this theorem, the existence of a single unbounded quasimorphism on either $\mathrm{Diff}_0(S)$ was unknown, and their existence is in fact somewhat surprising, given previous results. Namely, up to now the behaviour in higher dimensions was much better understood – and it is very different from the surface case. By classical results of Mather and Thurston, the identity component of the diffeomorphism group $\mathrm{Diff}_0(M)$ is perfect for any compact manifold. For all manifolds of dimension *not* equal to 2 or 4 these groups are even uniformly perfect (Burago-Ivanov-Polterovich, Tsuboi), meaning that any element can be written as a product of commutators of uniformly bounded length. This means in particular that these groups cannot admit any unbounded quasimorphisms.

We also note that Theorem 1 has important implications for *geometric* properties of surface homeomorphism and diffeomorphism groups. Namely, Burago-Ivanov-Polterovich asked if there are any unbounded conjugation-invariant norms on these groups. Theorem 1 then answers this in the positive and in particular the *(strong) fragmentation norm* on this group is unbounded.

To prove Theorem 1, we construct a variant of the curve graph which works in this setting. Curve graphs are a central tool in the study of mapping class groups. They were originally introduced by Harvey, but their full potential has been observed by Masur and Minsky, who showed that they are Gromov hyperbolic, and that they can be used to hierarchically encode the geometry of mapping class groups.
    Vertices of the (usual) curve graph correspond to isotopy classes of simple closed curves on the surface, with edges corresponding to disjointness (up to isotopy). By definition, the group $\mathrm{Diff}_0(S)$ will act trivially on the curve graph. We define a *fine curve graph*, whose vertices correspond to curves (as opposed to isotopy classes) and edges to actual disjointess.
    We show that the fine curve graph is hyperbolic, and in fact connect its geometry to that of "usual" curve graphs of punctured surfaces. We then construct a large class of diffeomorphisms which act as hyperbolic isometries on the fine curve graph. Using a construction by Bestvina–Fujiwara we then construct quasimorphisms which are unbounded on cyclic subgroups generated by these elements.

Fibring over the circle via group homology
Dawid Kielak

Perhaps the most naive question that one might ask when introduced to (compact, connected, oriented) 3-manifolds for the first time is: what are the examples? An extremely simple-minded way of producing many examples is to take a 2-manifold, i.e., a surface Σ, and take a product with a circle, namely $\Sigma \times \mathbb{S}^1$. A slightly more involved variation on this idea is to add a twist – one way of thinking about $\Sigma \times \mathbb{S}^1$ is that it is obtained from $\Sigma \times [0,1]$ by gluing $\Sigma \times \{0\}$ to $\Sigma \times \{1\}$ via the identity homeomorphism. One can take a different homeomorphism $f \in \mathrm{Homeo}^+(\Sigma)$ and use it instead of the identity when gluing. This way we obtain the *mapping torus* $\Sigma *_f$ of f. It is immediate that such a mapping torus is a 3-manifold. (Throughout, we will implicitly assume that Σ is a surface of genus at least 2.)

Definition 1. We say that a 3-manifold M *fibres* if and only if we have $M \cong \Sigma *_f$ for some surface Σ and its homeomorphism f.

    As we have all learned from Thurston, 3-manifolds are best looked at through their geometry. So what kind of geometry do fibring 3-manifolds carry?

Theorem 2 (Thurston [13])**.** *A 3-manifold $M \cong \Sigma *_f$ is* hyperbolic*, that is, can be endowed with a Riemannian metric of constant sectional curvature -1, if and only if f is pseudo-Anosov.*

Being pseudo-Anosov means precisely not being isotopic to a homeomorphism of finite order, or to one that permutes a finite system of disjoint closed geodesics on Σ. Among homeomorphisms of Σ, pseudo-Anosovs are generic.

Now that we know how to construct many 3-manifolds, including many hyperbolic ones, we may ask: are there any we cannot produce this way?

Theorem 3 (Agol [2]; Thurston's Virtually Fibred Conjecture). *Every hyperbolic 3-manifold M admits a finite cover which fibres.*

We will sketch the proof, in two steps.

Step 1: Let $G = \pi_1(M)$. By a result of Kahn–Marković [6], M contains sufficiently many surface subgroups to use a theorem of Bergeron–Wise [3] and conclude that G acts freely and cocompactly on a CAT(0) cube complex X. We then use Agol's result [2] and find a finite-index subgroup $H \leq G$ which acts on X specially in the sense of Haglund–Wise. Next, a paper of Haglund–Wise [5] tells us that H is a subgroup of a RAAG (right-angled Artin group), and therefore of a right-angled Coxeter group. Now Agol [1] tells us that H (and hence G) is virtually RFRS.

Step 2: We have learned that G is virtually RFRS. Let us recall the definition.

Definition 4. A finitely generated group G is *residually finite rationally solvable* (or *RFRS*) if and only if it admits a chain of subgroups

$$G = G_0 \geq G_1 \geq \cdots$$

satisfying

- for every i, the subgroup G_i is a normal subgroup of finite index of G,
- $\bigcap_i G_i = \{1\}$,
- every surjection $G_i \to G_i/G_{i+1}$ factors through $\mathbb{Z}^{n_i}$ for some n_i.

The following result connects the RFRS property back to fibring, and finishes step 2 of our proof.

Theorem 5. *Let M be an irreducible 3-manifold with non-trivial, RFRS fundamental group. Then M is virtually fibred if and only if the Euler characteristic $\chi(M)$ vanishes.*

A result of Lott–Lück [9] tells us that $\chi(M) = 0$ is equivalent to the vanishing of the first L^2-Betti number $\beta_1^{(2)}(M)$, which depends only on $G = \pi_1(M)$, and can be denoted by $\beta_1^{(2)}(G)$.

A result of Stallings [12] on the other hand says that an irreducible 3-manifold M fibres if and only if its fundamental group $G = \pi_1(M)$ *algebraically fibres*, that is, G admits an epimorphism to $\mathbb{Z}$ with a finitely generated kernel. We may thus rephrase Agol's theorem as follows.

Theorem 6. *Let G be a non-trivial RFRS group, which is the fundamental group of an irreducible 3-manifold. Then G is virtually algebraically fibred if and only if $\beta_1^{(2)}(G) = 0$.*

Phrased like this, an immediate question arises: how important is it for G to be the fundamental group of a 3-manifold? It turns out that this is not important at all.

Theorem 7 ([7])**.** *Let G be a finitely generated non-trivial RFRS group. Then G is virtually algebraically fibred if and only if $\beta_1^{(2)}(G) = 0$.*

There are two ingredients that come into the proof of the above result. The first is the Atiyah conjecture, and more specifically the fact that RFRS groups are residually {torsion-free nilpotent} (this is easy to see from the definition), that for such groups the Atiyah conjecture was proven by Schick [10], and hence that the L^2-Betti numbers of G are computed by the $\mathcal{D}(G)$-dimensions of $H_\bullet(G; \mathcal{D}(G))$, where $\mathcal{D}(G)$ is the Linnell skew-field – this last fact was shown by Linnell [8].

The second ingredient is the theorem of Sikorav [11], who showed that the kernel of an epimorphism $\phi\colon G \to \mathbb{Z}$ is finitely generated if and only if $H_1(G; \widehat{\mathbb{Z}G}^\phi) = H_1(G; \widehat{\mathbb{Z}G}^{-\phi}) = 0$, where $\widehat{\mathbb{Z}G}^\psi$ denotes the Novikov ring with respect to $\psi\colon G \to \mathbb{Z}$.

The two ingredients connect virtual fibring and L^2-Betti numbers to group homology, and in [7] a connection on the level of rings is exhibited between $\mathcal{D}(G)$ and $\widehat{\mathbb{Q}G}^\phi$. An observant reader will notice that the coefficients in the Novikov ring changed from $\mathbb{Z}$ to $\mathbb{Q}$. For homology in dimension 1 this change is inconsequential.

There is a higher dimensional version of Sikorav's theorem due to Schweizer [4], which connects the vanishing of $H_i(G; \mathbb{Q} \otimes_\mathbb{Z} \widehat{\mathbb{Z}G}^{\pm\phi})$ to the kernel of ϕ being of type $\mathtt{FP}_i(\mathbb{Q})$. In [7] however we are working with $\widehat{\mathbb{Q}G}^\phi$ rather than $\mathbb{Q} \otimes_\mathbb{Z} \widehat{\mathbb{Z}G}^\phi$, and these are not the same rings. Nevertheless, at this point one can formulate the following natural conjecture.

Conjecture 8. Let G be a RFRS group of type $\mathtt{FP}_n(\mathbb{Q})$. Then G admits a finite index subgroup H and an epimorphism $\phi\colon H \to \mathbb{Z}$ with kernel of type $\mathtt{FP}_n(\mathbb{Q})$ if and only if $\beta_i^{(2)}(G) = 0$ for every $i \leq n$.

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Pseudoisotopies of even discs, revisited

Manuel Krannich

The subject of *pseudoisotopy* or *h-cobordism theory* is the study of the homotopy type of the topological group

$$\mathrm{C}(M) = \{\phi \colon M \times [0,1] \xrightarrow{\cong} M \times [0,1] \mid \phi|_{M \times 0 \cup \partial M \times [0,1]} = \mathrm{id}\}$$

of *pseudoisotopies* of a compact smooth manifold M in the smooth topology. In my talk, I explained the following result from [5], which provides a p-local identification of this homotopy type in the case of a closed $2n$-dimensional disc D^{2n} in terms of the algebraic K-theory spectrum $K(\mathbb{Z})$ of the integers in a range up to roughly the dimension for primes p that are large with respect to the degree and the dimension.

Theorem 1. *For $n > 3$, there exists a zig-zag*

$$\mathrm{BC}(D^{2n}) \longrightarrow \cdot \longleftarrow \Omega_0^{\infty+1} K(\mathbb{Z})$$

whose maps are p-locally $\min(2n-4, 2p-4-n)$-connected for primes p.

So far, the relation of the homotopy type of spaces of pseudoisotopies with algebraic K-theory was studied via a combination of a stability result of Igusa [4] and foundational work of Waldhausen [7] and Waldhausen, Jahren, and Rognes [8]. The proof of the theorem above is independent of this approach and provides a new method to access spaces of pseudosisotopies of even-dimensional discs, which does not involve stabilising the dimension, yields a better range in many cases, and is *homological* (see [5] for an explanation). The most recent ingredient that goes into the proof of this result is Botvinnik and Perlmutter's computation of the stable homology of the moduli space of high-dimensional handlebodies [2].

Rationally and combined with a result of Randal-Williams [6] and Borel's work on the stable cohomology of arithmetic groups [1], our theorem results in the following partial computation of the rational homotopy groups of the group $\mathrm{Diff}_\partial(D^{2n+1})$ of diffeomorphisms of an odd-dimensional disc fixing the boundary pointwise.

Corollary 2. There exists an isomorphism

$$\pi_* \mathrm{BDiff}_\partial(D^{2n+1}) \otimes \mathbb{Q} \cong K_{*+1}(\mathbb{Z}) \otimes \mathbb{Q} \cong \begin{cases} \mathbb{Q} & \text{if } * \equiv 0 \pmod 4 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 0 < * < 2n-5.$$

Remark 3.

(1) In a range of degrees up to approximately $2n/3$, these groups were previously known as a result of a computation of Farrell and Hsiang [3], who combined Waldhausen's approach to pseudoisotopy theory with the study of a certain involution, neither of which the proof of the corollary requires.

(2) From work of Watanabe [9] on the value of certain characteristic classes constructed by Kontsevich on disc bundles, one can deduce that the range in the corollary is optimal up to at most three degrees.

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Scalable spaces

Fedor Manin

(joint work with Aleksandr Berdnikov)

As part of his foundational work in rational homotopy theory, Sullivan defined a notion of a *formal* space, that is, one whose rational homotopy type is a "formal consequence" of its rational cohomology algebra. For a manifold M, this means that its cohomology algebra is quasi-isomorphic (that is, related by a zig-zag of maps which are the identity on cohomology) to its algebra of differential forms. An equivalent condition turns out to be the existence of an infinite sequence of integers p and maps $M \to M$ which induce multiplication by p^n on $H^n(M;\mathbb{Q})$ [7]. A strengthened version of this notion turns out to play an important role in quantitative homotopy theory. This program, laid out by Gromov in several places in the 1990's, most notably [4], is concerned with understanding questions such as the following:

(1) How can we estimate the geometric "complexity" of the "most efficient" representative of a homotopy class of maps? How does this complexity scale with the algebraic "size" of the homotopy class?

(2) If two maps are in the same homotopy class, how does the complexity of the "most efficient" homotopy between them depend on the complexity of the maps?

Most often, complexity (or efficiency) is measured using the Lipschitz constant. In the simply connected world, the answers to these questions are closely tied to rational homotopy theory, as has already been explored in e.g. [3], [2], [6]. Our latest work [1] shows that spaces satisfying a metric version of formality (one significantly weaker than the Riemannian formulation studied by Kotschick [5] and others) exhibit particularly nice answers to these questions.

Theorem 1. *The following are equivalent for a simply connected piecewise Riemannian finite simplicial complex or compact Riemannian manifold Y:*

(i) *There is a homomorphism $H^*(Y) \to \Omega_\flat^* Y$ of differential graded algebras which sends each cohomology class to a representative of that class. (Here $\Omega_\flat^* Y$ denotes the flat forms, an algebra of not-necessarily-smooth differential forms studied by Whitney.)*

(ii) *There is a constant $C(Y)$ and infinitely many (indeed, a logarithmically dense set of) $p \in \mathbb{N}$ such that there is a $C(Y)(p+1)$-Lipschitz self-map which induces multiplication by p^n on $H^n(Y; \mathbb{R})$.*

(iii) *Y is formal, and for all finite simplicial complexes X, nullhomotopic L-Lipschitz maps $X \to Y$ have $C(X,Y)(L+1)$-Lipschitz nullhomotopies.*

(iv) *Y is formal, and for all $n < \dim Y$, homotopic L-Lipschitz maps $S^n \to Y$ have $C(X,Y)(L+1)$-Lipschitz homotopies.*

We call spaces satisfying (i)–(iv) *scalable*, after the scaling maps of (ii). It is worth noting that scalability is an invariant of rational homotopy type and that conditions (i) and (ii) can each be read as strengthenings of the equivalent conditions defining formality. Examples include symmetric spaces (where one can realize (i) by sending every class to its harmonic representative, as noted already by Sullivan) but also many others. For example, $(\mathbb{C}P^2)^{\#3} \# (\overline{\mathbb{C}P^2})^{\#3}$ is scalable while $(\mathbb{C}P^2)^{\#4}$ is not; $(S^8 \times S^8)^{\#6435}$ is scalable while $(S^8 \times S^8)^{\#6436}$ is not. The examples we have so far suggest the following line of inquiry:

Question 2. *Is scalability an $\mathbb{R}$-homotopy invariant? For that matter, is a space Y scalable if and only if there is a dga embedding $H^*(Y; \mathbb{R}) \to \bigwedge^* \mathbb{R}^N$ for some N (a finitary, purely local criterion)?*

Spaces that are formal but not scalable display some subtle properties. For example, the "distortion conjecture" of Gromov from [4] would imply in particular that if M is a punctured simply connected n-manifold and $\alpha \in \pi_{n-1}(M)$ is the homotopy class of the puncture, then $k\alpha$ has a representative with Lipschitz constant $O(k^{1/n})$—significantly more efficient than the obvious "wrap k times around the puncture", which would have Lipschitz constant $\sim k^{1/(n-1)}$. We show that

Gromov's conjecture holds for scalable spaces, while giving the following counterexample in general:

Theorem 3. *Let M be the punctured $S^2 \times (\mathbb{C}P^2)^{\#4}$. Then if $\alpha \in \pi_5(M)$ is the class of the puncture, then the growth of $\mathrm{Lip}(k\alpha)$ (the optimal Lipschitz constant of a representative of $k\alpha$) is $\omega(k^{1/6})$.*

However, this bound is tantalizingly incomplete, disproving Gromov's conjecture while giving no further information.

Question 4. *What is the true distortion of the element α above? Can one find a lower bound which grows strictly faster than $k^{1/6}$, or an upper bound lower than the obvious $k^{1/5}$?*

Finally, it is unclear how linearity of nullhomotopies plays out in non-formal spaces. We have an example of a space which is non-formal and otherwise satisfies condition (iv), but not condition (iii) of the main theorem.

Question 5. *Does condition* (iii) *imply formality?*

The answer to this, whether positive or negative, is likely to rely on highly algebraic phenomena.

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Rigidity, the complex of free factors, and the commensurator of Aut(F)

MARTIN R. BRIDSON

(joint work with Mladen Bestvina, Richard D. Wade)

There is a well established and powerful three-way analogy between mapping class groups of compact surfaces, automorphism groups of free groups, and lattices in semisimple Lie groups, particularly $\mathrm{SL}(n, \mathbb{Z})$. Classical rigidity results concerning lattices have provided inspiration for the exploration of different manifestations of rigidity among mapping class groups and automorphism groups of free groups, and the results presented in this talk contribute to this topic.

One type of rigidity result involves the identification of natural *exact models* for the groups being studied. The fundamental theorem of projective geometry provides a prototype for such theorems, and Tits' Theorem showing that all automorphisms of thick buildings arise from the underlying algebraic group extends this. In the setting of mapping class groups, Ivanov's Theorem shows that for a closed surface S of genus at least 3, the action of the mapping class group on the complex of curves $\mathcal{C}$ gives an isomorphism $\mathrm{Mod}(S) \to \mathrm{Aut}(\mathcal{C})$, where $\mathrm{Aut}(\mathcal{C})$ is the group of simplicial automorphisms. From this result, Ivanov deduced that the natural map from $\mathrm{Mod}(S)$ to its abstract commensurator is an isomorphism.

Recall that the abstract commensurator $\mathrm{Comm}(\Gamma)$ of a group Γ is the group formed by equivalence class of isomorphisms between finite-index subgroups of Γ, where two isomorphisms are equivalent if they agree on a common subgroup of finite index. A celebrated theorem of Margulis states that a lattice Γ in a semisimple Lie group is arithmetic if and only if the image of the map $\Gamma \to \mathrm{Comm}(\Gamma)$, given by conjugation, has finite index.

For $\mathrm{Aut}(F_n)$, the automorphism group of a free group of rank n, a natural analogue of the curve complex is the complex of free factors $\mathcal{F}_n$. This is the geometric realisation of the poset of free factors $A < F_n$ ordered by inclusion. There is a natural action of $\mathrm{Aut}(F_n)$ on $\mathcal{F}_n$ and of $\mathrm{Out}(F_n)$ on $[\mathcal{F}_n] = \mathcal{F}_n/\mathrm{Inn}(F_n)$.

The following theorem was proved by myself and Mladen Bestvina a few years ago and will appear on the arxiv shortly.

Theorem 1. *The natural maps* $\mathrm{Aut}(F_n) \to \mathrm{Aut}(\mathcal{F}_n)$ *and* $\mathrm{Out}(F_n) \to \mathrm{Aut}[\mathcal{F}_n]$ *are isomorphisms for* $n \geq 3$.

We had hoped that this would lead directly to a proof of commensurator rigidity for $\mathrm{Out}(F_n)$ and $\mathrm{Aut}(F_n)$, in the manner of Ivanov's theorem, but this proved difficult. By different means, Farb and Handel ($n \geq 4$) and Horbez and Wade ($n \geq 3$) proved that the natural map $\mathrm{Out}(F_n) \to \mathrm{Comm}(Out(F_n))$ is an isomorphism for $n \geq 3$.

In a recent work, Ric Wade and I classify the subgroups of $\mathrm{Aut}(F_n)$ that are direct products of r non-abelian free groups, where $r = 2n - 3$ is maximal. By combining knowledge of these subgroups with Theorem 1 and ideas from its proof we are able to prove:

Theorem 2. *For* $n \geq 3$, *the natural map* $\mathrm{Aut}(F_n) \to \mathrm{Comm}(Out(A_n))$ *is an isomorphism.*

The proof shows that $\mathrm{Aut}(F_n)$ is also the abstract commensurator of various of its natural subgroups.

Detecting second order intersections of loops with string topology

NATHALIE WAHL

(joint work with Nancy Hingston)

Let M be a closed, oriented manifold of dimension n, with $LM = \mathrm{Maps}(S^1, M)$ its free loop space. We consider the two following string topology operations on the homology of LM:

- The Chas-Sullivan product $H_*(LM \times LM) \xrightarrow{\wedge} H_{*-n}(LM)$;
- The string coproduct $H_*(LM) \xrightarrow{\vee} H_{*+1-n}(LM \times LM)$;

where the coproduct is the "extension by zero" on the constant loops of the Goresky-Hingston coproduct, as considered in [1]. In [1, Rem 3.21], it is shown that these operations do not satisfy the following Frobenius identity:

$$\vee(A \wedge B) \neq (\vee A) \wedge B) + A \wedge (\vee B).$$

The failure of this identity can be considered as a new operation

$$H_*(LM \times LM) \longrightarrow H_{*+1-2n}(LM \times LM),$$

defined as the difference between the left hand side and the right hand side in the above equation. We show that this operation detects second order intersections of loops in the following sense:

Theorem 1. *Let $A, B \in H_*(LM)$ be homology classes that admit chain representatives $\hat{A}, \hat{B}$ in smooth loops, such that for each α in the image of $\hat{A}$ and β in the image of $\hat{B}$, if $\alpha(0) = \beta(0)$, then the derivatives $\alpha'(0)$ and $\beta'(0)$ are transverse, i.e. they span a 2–dimensional subspace of $T_{\alpha(0)}M = T_{\beta(0)}M$. Then the Frobenius identity does hold for the pair (A, B), i.e.*

$$\vee(A \wedge B) = (\vee A) \wedge B) + A \wedge (\vee B).$$

This result can be reformulated as follows:

Corollary 2. *If $A, B \in H_*(LM)$ are such that $\vee(A \wedge B) \neq (\vee A) \wedge B) + A \wedge (\vee B)$, then A and B do not admit transverse representatives in the sense of the theorem.*

For example, let S^n be a sphere of odd dimension $n \geq 3$, and consider a generator $\Theta \in H_{3n-2}(LS^n) \cong \mathbb{Z}$. The class Θ can be represented in smooth loops by the class of *all circles, great and small*, where a circle is the intersection of a hyperplane with the sphere. In [1, Rem 3.21], we showed that the above identity fails for the pair (Θ, Θ). Applying our result, we can conclude that the class Θ cannot be made transverse to itself, in the sense we have described. And one can quickly check that the representative of the class Θ as all circles, great and small, is in fact far from being transverse to itself.

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The homotopy types of algebraic cobordism categories and Grothendieck-Witt theory

Fabian Hebestreit

(joint work with B. Calmès, E. Dotto, Y. Harpaz, M. Land, K. Moi, D. Nardin, T. Nikolaus, W. Steimle)

Through the work of Galatius, Madsen, Randal-Williams, Tillmann and Weiss there has been a significant increase in the understanding of the continuous group (co)homology of diffeomorphism groups (in even dimensions not 4), see for example [2, 3, 6]. In the simplest case, namely that of the manifolds

$$W_g^{2n} = (S^n \times S^n)^{\sharp g},$$

the calculations are based on equivalences

$$(1) \qquad \left(\coprod_{g \in \mathbb{N}} \mathrm{BDiff}(W_g, \mathrm{D}^{2n}) \right)^{\mathrm{grp}} \longrightarrow \Omega |\mathrm{Cob}_{2n}^{\theta_n}| \longrightarrow \Omega^\infty \mathrm{MT}\theta_n$$

The left hand term consists of classifying space of diffeomorphisms fixing a specified disc pointwise. Because of these discs the indicated collection admits a well-defined connected sum operation turning it into an E_{2n}-space (i.e. a somewhat commutative topological monoid). The superscript indicates its (homotopical) group completion. The homology of the left hand side is then related to the homologies of the individual terms $\mathrm{BDiff}(W_g, \mathrm{D}^{2n})$ by the group completion theorem of McDuff–Segal. In particular, information about the (co)homology of these classifying spaces can be obtained by studying the (co)homology of the left hand term. In the middle term Cob_d^θ is the cobordism (∞-)category associated to a d-dimensional vector bundle θ: It has as objects closed $d-1$-dimensional manifolds equipped with a bundle map $\mathrm{T}M \oplus \mathbb{R} \to \theta$ and as morphisms cobordisms of such manifolds. Its higher structure is arranged so that

$$\mathrm{Hom}_{\mathrm{Cob}_d^\theta}(M, N) \simeq \coprod_{W\,:\,M \rightsquigarrow N} \mathrm{BDiff}_\partial^\theta(W),$$

with the right hand side denoting diffeomorphisms whose differential is compatible with the θ-structure (which in general is additional structure, but let me not dwell on that point). In particular, denoting by θ_n the pullback of the universal vector bundle along the connective cover

$$\tau_{>n}(\mathrm{BO}(2n)) \longrightarrow \mathrm{BO}(2n)$$

then the forgetful map

$$\mathrm{BDiff}^{\theta_n}(W_g, \mathrm{D}^{2n}) \longrightarrow \mathrm{BDiff}(W_g, \mathrm{D}^{2n})$$

is an equivalence by obstruction theory, and we obtain maps

$$\mathrm{BDiff}(W_g, \mathrm{D}^{2n}) \longrightarrow \mathrm{Hom}_{\mathrm{Cob}_{2n}^{\theta_n}}(\emptyset, \emptyset) \longrightarrow \Omega |\mathrm{Cob}_{2n}^{\theta_n}|,$$

since every endomorphism in a category defines a loop in its realisation. As the target is a group complete monoid this map induces the left map in the original sequence (1) by the universal property of group completions.

The second map in (1) is a parametrised refinement of the Pontryagin-Thom construction, and generally $MT\theta$ is the Thom spectrum of the $-d$-dimensional virtual bundle $-\theta$. The cohomology of its infinite loop space is then accessible through the Thom isomorphism and standard methodology of algebraic topology.

In (homotopical) algebra the left hand term has an analogue: Fix a ring R, commutative for ease of presentation. Then

$$\left(\coprod_P \mathrm{BGl}(P)\right)^{\mathrm{grp}} = \mathrm{k}(R),$$

is Quillen's algebraic K-theory space, where P runs over $\mathrm{Proj}(R)$, the finitely generated projective R-modules; the left hand side is then an E_∞-space under direct sum of modules. K-theory spaces are of great interest in both geometric topology and number theory: As an illustration, on the one hand low-dimensional K-groups house invariants such as Wall's finiteness obstruction and Whitehead's torsion, and on the other the abelianised absolute Galois-group of local, global or finite fields can be extracted from a mix of their K-theory and topological cyclic homology (this recently lead to a conceptually new proof of Artin's reciprocity law in [1]). As another example the Kummer–Vandiver conjecture on class numbers of cyclotomic fields is famously equivalent to $\pi_{4k}\mathrm{k}(\mathbb{Z}) = 0$, the last remaining unknown part of $\mathrm{k}(\mathbb{Z})$.

Analogous to the algebraic K-theory one can define the Grothendieck–Witt space of R and a form parameter Λ (i.e. a choice of symmetric, quadratic, anti-symmetric, even...) by

$$\left(\coprod_{(P,q)} \mathrm{BO}(P,q)\right)^{\mathrm{grp}} = \mathrm{gw}(R,\Lambda),$$

where (P,q) runs over the unimodular Λ-forms on a finitely generated projective R-module. By work of Karoubi and Schlichting [4, 7] the spaces $\mathrm{gw}(R,\Lambda)$ are fairly well understood if 2 is a unit in R, in which case, however, much subtlety in the theory of forms is lost. They considered the hyperbolisation map

$$\mathrm{k}(R)_{\mathrm{hC}_2} \longrightarrow \mathrm{gw}(R,\Lambda),$$

essentially assigning to a projective module P the unimodular form on $P \oplus P^*$ given by evaluation. The result is that its cofibre (as E_∞-spaces) has 4-periodic homotopy groups which are given by certain flavours of Witt groups and thereby comparatively easy to compute. An extension of such a fibre sequence to general R had been variously conjectured, but no concrete suggestion for the cofibre term exists in the literature.

In the talk I explained that to a general pair (R,Λ) one can associate a Poincaré category, a concept due to Lurie [5]. It consists of a stable ∞-category $\mathcal{C}$ together

with a quadratic functor $\mathcal{Q} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{E}_\infty\text{-Grp}$ to the category of coherently commutative and associative monoids in spaces. The latter is to be thought of as associating to an object $X \in \mathcal{C}$ some flavour of hermitian forms on X. There is a non-degeneracy condition placed on $(\mathcal{C}, \mathcal{Q})$, giving in particular rise to duality self-equivalence $\mathrm{D}_\mathcal{Q} \colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}$. The underlying category associated to R is then $\mathrm{D}^{\mathrm{perf}}(R)$ consisting of perfect chain complexes over R, and the quadratic functor $\mathcal{Q}_R^\Lambda$ is the animation (or in more classical terminolgy the non-abelian derived functor) of the functor

$$\mathrm{Proj}(R)^{\mathrm{op}} \longrightarrow \mathrm{Ab}, \quad P \longmapsto \{\text{not necessarily unimodular } \Lambda\text{-forms on } P\}.$$

To the data of such a Poincaré category $(\mathcal{C}, \mathcal{Q})$ there is then associated a space of Poincaré forms in $(\mathcal{C}, \mathcal{Q})$: It consists of pairs (X, q), where $X \in \mathcal{C}$ and $q \in \mathcal{Q}(X)$, for which an associated map

$$q_\sharp \colon X \longrightarrow \mathrm{D}_\mathcal{Q}(X)$$

is an equivalence. In particular, the definitions are arranged so that any unimodular Λ-form over R gives rise to a Poincaré object in $(\mathrm{D}^{\mathrm{perf}}(R), \mathcal{Q}_R^\Lambda)$. To the data of a Poincaré category $(\mathcal{C}, \mathcal{Q})$ one can, following Lurie and Ranicki, assign an L-theory spectrum $\mathrm{L}(\mathcal{C}, \mathcal{Q})$ and furthermore, the K-theory spectrum $\mathrm{K}(\mathcal{C})$ acquires a C_2-action through the duality $\mathrm{D}_\mathcal{Q}$.

Adding to this list of invariants I constructed in the talk for a Poincaré category a cobordism $(\infty\text{-})$category $\mathrm{Cob}(\mathcal{C}, \mathcal{Q})$ and a Grothendieck–Witt spectrum $\mathrm{GW}(\mathcal{C}, \mathcal{Q})$, such that there are maps

$$(2) \qquad \mathrm{gw}(R, \Lambda) \longrightarrow \Omega|\mathrm{Cob}(\mathrm{D}^{\mathrm{perf}}(R), \mathcal{Q}_R^\Lambda)|,$$

and

$$(3) \qquad |\mathrm{Cob}(\mathcal{C}, \mathcal{Q})| \longrightarrow \Omega^{\infty-1}\mathrm{GW}(\mathcal{C}, \mathcal{Q})$$

completely analogous to those considered in (1) in the setting of diffeomorphisms. As the main result of my talk I presented the following pair of results, which put the algebraic situation into striking analogy with the geometric one sketched at the beginning:

Theorem 1 (H–Steimle). *The map (2) is an equivalence for all rings R and form parameters Λ.*

Theorem 2 (#9). *For every Poincaré category $(\mathcal{C}, \mathcal{Q})$*

    i) the map (3) is an equivalence,
    ii) there is a natural cartesian square

$$\begin{array}{ccc} \mathrm{GW}(\mathcal{C}, \mathcal{Q}) & \longrightarrow & \mathrm{L}(\mathcal{C}, \mathcal{Q}) \\ \downarrow & & \downarrow{\scriptstyle\Xi} \\ \mathrm{K}(\mathcal{C})^{hC_2} & \longrightarrow & \mathrm{K}(\mathcal{C})^{tC_2}, \end{array}$$

    and therefore
    iii) a fibre sequence

$$\mathrm{K}(\mathcal{C})_{hC_2} \longrightarrow \mathrm{GW}(\mathcal{C}, \mathcal{Q}) \longrightarrow \mathrm{L}(\mathcal{C}, \mathcal{Q}).$$

Abbreviating $\mathrm{L}(\mathrm{D}^{\mathrm{perf}}(R), \mathcal{Q}_R^\Lambda)$ to $\mathrm{L}(R, \Lambda)$ one thus obtains a fibre sequence

$$\mathrm{k}(R)_{\mathrm{hC}_2} \longrightarrow \mathrm{gw}(R, \Lambda) \longrightarrow \Omega^\infty \mathrm{L}(R, \Lambda)$$

for general R and Λ.

In a follow-up talk Markus Land presented an analysis of the terms $\mathrm{L}(R, \Lambda)$ for various choices of Λ leading, in particular, to a complete calculation of $\mathrm{gw}(\mathbb{Z}, \Lambda)$ in terms of the almost completely known algebraic K-groups of the integers. Let me here just mention, that the spectra $\mathrm{L}(R, \Lambda)$ are not generally 4-periodic if 2 is not a unit in R, and therefore not quite equivalent to any of Ranicki's classical L-theory spectra.

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Invariants of groups and metric spaces related to Travelling Salesman Problem

ANNA ERSCHLER

(joint work with Ivan Mitrofanov)

In [1] We study asymptotic invariants of metric spaces and infinite groups related to (universal) Travelling Salesman Problem (TSP).

Let (M, d) be a metric space and T be a linear order on M. For a finite subset $X \subset M$ we consider the restriction of the order T on X, and enumerate the points of X accordingly:

$$x_1 \leq_T x_2 \leq_T x_3 \leq_T \cdots \leq_T x_k$$

where $k = \#(X)$. We denote by $l_T(X)$ the length of the corresponding path

$$l_T(X) := d(x_1, x_2) + d(x_2, x_3) + \cdots + d(x_{k-1}, x_k).$$

Denote by $l_{\mathrm{opt}}(X)$ the minimal length of a path passing through all points of X, we choose this path among $k!$ paths visiting exactly once points of X.

Definition 1. For the ordered metric space $\mathrm{M} = (M, d, T)$ we define the *Ordering Ratio function*

$$\mathrm{OR}_{M,T}(k) := \sup_{X \subset M | 2 \leq \#(X) \leq k} \frac{l_T(X)}{l_{\mathrm{opt}}(X)}.$$

For an (unordered) metric space (M, d) we also define the *Ordering Ratio Function* as

$$\mathrm{OR}_M(k) := \inf_T \mathrm{OR}_{M,T}(k).$$

The ratio between the value provided by an algorithm and its actual value is called *competitive ratio* in computer science literature, so our function OR corresponds to the competitive ratio for Universal Travelling Salesman Problem, for the problem that searches the dependence on the number of points k. Recall a metric space M is *uniformly discrete*, if there exists $\delta > 0$ such that for all pairs of points $x \neq y$ the distance between x and y is at least δ. We show that for uniformly discrete spaces the asymptotic class of the Ordering Ratio function behaves good with respect to quasi-isometries. In contrast with previous works on competitive ratio of Universal Travelling Salesmans Problem, we are interested not only in this asymptotic behaviour but also in particular values of $\mathrm{OR}(k)$. We will in particular study *Travelling Salesman girth* of an order:

Definition 2. Let M be a metric space, T be an order on M. We say that the Traveling Salesman girth $\mathrm{TSgirth}_{(M,T)} = s$ if s is the smallest integer such that $\mathrm{OR}_{(M,T)}(s) < s - 1$. If such s does not exist, we say that $\mathrm{TSgirth}_M = \infty$.

The Travelling Salesman girth describes the minimal value k where for which using the given order to solve TSP has some efficiency. The word "girth" is chosen with some analogy with girth of a group G with respect to a finite generating set S, that is, the smallest length of a non-trivial loop in the Cayley graph. It is clear that the girth g of a group is related to the smallest value k such that the growth function of (G, S) is strictly smaller than the growth function of the group with the same number of generators; indeed, $k = \lceil \frac{g}{2} \rceil$. Likewise, the Travelling Salesman Girth is the smallest number of points for which $\mathrm{OR}(k)$ is strictly smaller than its maximal possible value $(k - 1)$ From definition, it is clear that if M has at least two points then $\mathrm{TSgirth}(M, T) \geq 3$ and it is easy to see that $\mathrm{TSgirth}_{(M,T)} = 3$ if M is finite. It is also not difficult to see that a bounded degree infinite graph M has the Travelling Salesman girth equal to 3 if and only if M is quasi-isometric to a ray or a line. We characterize finitely generated groups with small (≤ 4) Travelling Salesman Girth:

Theorem 3 (Theorem A). *Let G be a finitely generated group. G admits an order T with $\mathrm{TSgirth}_{(G,T)} \leq 4$ if and only if G is virtually free.*

The idea to solve a Travelling Salesman problem on a set M by ordering all its points, and then, given a finite set of M, to visit them with respect to this order, is introduced by Bartholdi and Platzman [BP82], [PB89]. Their observation was that this approach works good for subspaces of a plain lattice, significantly faster than general algorithms for finite graphs. Their argument implies a logarithmic upper bound for the function $\mathrm{OR}(k)$. There exist spaces where the function OR is even better. Namely, it is not difficult to check that $\mathrm{OR}(k)$ is bounded by constant 2 for metric trees.

We prove that this best possible situation ($\mathrm{OR}(k)$ is bounded by above by a constant) holds true for hyperbolic spaces.

Theorem 4 (Theorem B). *Let M be a δ-hyperbolic graph of bounded degree. Then there exists an order T and a constant C such that for all k*

$$\mathrm{OR}_{M,T}(k) \leq C.$$

Since by a theorem of Bonk and Schramm [BS00] any δ-hyperbolic space of bounded geometry can be quasi-isometrically imbedded to $\mathbb{H}^d$, taking in account Lemma about quasi-isometric imbeddings, the main goal in the proof of Theorem is to prove it for subsets of $\mathbb{H}^d$. We do it by choosing an appropriate tiling of this space, an appropriate tree and we study the *hierarchical order* with respect to this tree.

While in the original paper of Bartholdi and Platzman it was suggested that such efficient behaviour holds for $\mathbb{Z}^d$, as we have already mentioned, it is known that it is not the case (unless $d = 1$).

Theorem 5 (Theorem C). *If M is a metric space of finite Assouad Nagata dimension, then*

$$\mathrm{OR}_M(k) \leq C \ln k.$$

Moreover, if the Assouad Nagata dimension of M is at most m and if for all $r > 0$ the space M admits a covering, satisfying the assumption of the definition of AN dimension, with m-dimension control function at most Kr, then the positive constant C can be chosen depending only on m and K.

The worst possible case for solving Universal TSP are spaces with linear $\mathrm{OR}(k)$. An example of sequence of finite graph with linear $\mathrm{OR}(k)$ is constructed in Gorodezky et al [GKSS10], see also Bhalgat et al [BCK11] who show that a sequence of Ramanujan graphs with large of bounded diameter-by-girth ratio has this property. Since a result of Osajda allows to imbed subsequences of graphs with large girth into Cayley Graphs, combining his result with that of [GKSS10] we can conclude that there exist groups with linear $\mathrm{OR}(k)$. It can be deduced from Lusternik Schnirelman theorem that any ordering of an ε net of a sphere S^k contains snakes (zigzags) on $k + 2$ points, alternating between neighborhoods of nearly antipodal points. Combining it with the control of Ordering Ratio function for weak imbeddings of spheres we get

Theorem 6 (Theorem D). *If a space M admits a weakly imbedded sequence of arbitrarily large spheres, then for any order T on M and any k*

$$\mathrm{OR}_{M,T}(k) = k - 1$$

While we give the definition of a space to contain weakly arbitrarily large spheres and a stronger notion to contain weakly arbitrarily large cubes, we mention here that the class of such spaces includes spaces admitting uniform imbeddings of $\mathbb{Z}^d$. In particular, this condition holds for any finitely generated group G that contains $\mathbb{Z}^\infty$ as a subgroup. Further examples of spaces that weakly contain arbitrarily large cubes (and spheres) are $\mathbb{Z}^2 \wr A$, for example $\mathbb{Z}^2 \wr \mathbb{Z}/2\mathbb{Z}$ and more generally $B \wr A$, where B is an infinite group of not linear growth and A is any finite or infinite group containing at least two elements. In view of the results mentioned above we ask:

Question 7. *Let M be a metric space of infinite Assouad Nagata dimension. Is it true that the Travelling Salesman girth of M is infinite?*

Observe that if the answer to this question is positive, this would provide a positive answer to the following

Question 8 (Gap Problem for existence of orders)**.** *Let M be an uniformly discrete metric space. Is it true that either for any order T on M and all $k \geq 1$ it holds $OR_{M,T}(k) = k - 1$ or there exists an order T such that for all $k \geq 1$ it holds $OR_{M,T}(k) \leq \mathrm{Const}\ln k$?*

Given a metric space, one can formulate a stronger Gap problem, which describes behavior of all orders (rather then searches an order on the space). Our next result below solves this problem for spaces with doubling property:

Theorem 9. *Theorem E[Gap for Ordering ratio functions on spaces with doubling] Let M be a metric space with doubling and T be an order on M. Then either for all s it holds*

$$\mathrm{OR}_{T,M}(s + 1) = s$$

or there exists C (depending only on the doubling constant of M, s and ε such that $\mathrm{OR}_{M,T}(s + 1) = s - \varepsilon$) such that

$$\mathrm{OR}_{M,X}(s) \leq C \ln s$$

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Grothendieck–Witt groups of the integers

Markus Land

(joint work with B. Calmès, E. Dotto, Y. Harpaz, F. Hebestreit, K. Moi,
D. Nardin, T. Nikolaus, W. Steimle)

This talk was a follow up to a previous talk given by Fabian Hebestreit. One of
the main results explained in Hebestreit's talk was a fibre sequence of spectra

$$\text{(1)} \qquad\qquad K(\mathcal{C})_{hC_2} \longrightarrow \text{GW}(\mathcal{C}, \mathcal{Q}) \longrightarrow \text{L}(\mathcal{C}, \mathcal{Q})$$

associated to each Poincaré ∞-category $(\mathcal{C}, \mathcal{Q})$. The purpose of my talk was to
make this more explicit in an example and indicate how it leads to the calculation
of the Grothendieck–Witt groups of $\mathbb{Z}$ in its various flavours:

Theorem 1. *We have the following table of Grothendieck–Witt groups of $\mathbb{Z}$.*

n	0	1	2	3	4	5	6	7
$\text{GW}_n^s(\mathbb{Z})$	$\mathbb{Z}^2$	$(\mathbb{Z}/2)^3$	$(\mathbb{Z}/2)^2$	$\mathbb{Z}/24$	$\mathbb{Z}$	0	0	$\mathbb{Z}/240$
$\text{KSp}_n^q(\mathbb{Z})$	$\mathbb{Z} \oplus \mathbb{Z}/2$	$\mathbb{Z}/4$	$\mathbb{Z}$	$\mathbb{Z}/24$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}$	$\mathbb{Z}/240$

Let us consider a discrete ring R with an involution. Associated to this ring
we may consider its category of perfect complexes $\text{Perf(R)} = \mathcal{D}^\omega(R)$ also known
as the compact objects in the derived ∞-category of R. Concretely, objects in
this category are represented by chain complexes of finitely generated projective
R-modules which are non-zero only in a finite range of degrees. In particular, there
is a canonical functor $\text{Proj}_R^{\text{fg}} \to \text{Perf}(R)$ obtained by viewing a finitely generated
projective module as a complex concentrated in degree zero. In order to pro-
mote this category to a Poincaré ∞-category we need to equip it with a Poincaré
structure, i.e. with a non-degenerate quadratic functor. Any such gives rise to
a duality equivalence $D \colon \text{Perf}(R)^{\text{op}} \xrightarrow{\simeq} \text{Perf}(R)$. Let us now only consider those
non-degenerate quadratic functors whose associated duality is the standard one:
$D(M) = \text{Hom}_R(M, R)$. We have the following characterization of such Poincaré
structures on $\text{Perf}(R)$:

Lemma 2. Poincaré structures on $\text{Perf}(R)$ with associated duality $D(M) = \text{Hom}_R(M, R)$ are determined by an R module N and a map $N \to R^{tC_2}$ of R-
modules.

Here R^{tC_2} is an object of $\mathcal{D}(R)$ and denotes the Tate construction of the C_2-
action induced by the involution on R. The homology groups of this complex are
the Tate cohomology of C_2 acting on R. It is an R-module in a non-trivial way:
There is a map $R \to (R \otimes R)^{tC_2}$ called the Tate-diagonal [1]. Furthermore, R^{tC_2}
is an $(R \otimes R)^{tC_2}$-module as R is an $(R \otimes R)$-module since R is equipped with an
involution.

Definition 3. We call the Poincaré structure associated to the map $\tau_{\geq 0}(R^{tC_2}) \to R^{tC_2}$ the genuine Poincaré structure of the ring R. The associated quadratic
functor will be denoted by $\mathcal{Q}^g \colon \text{Perf}(R)^{\text{op}} \to \text{Sp}$.

I discussed the following characterisation of this genuine Poincaré structure:

Proposition 4. The restriction of the functor $\mathcal{Q}^g$ along the canonical inclusion $\mathrm{Proj}_R^{\mathrm{fg}} \to \mathrm{Perf}(R)$ takes a finitely generated projective module P to the (Eilenberg–Mac Lane spectrum on the) abelian group $\mathrm{Hom}_{R\otimes R}(P \otimes P, R)^{C_2}$ of symmetric bilinear forms on P. It is uniquely characterized by this property, and is the non-abelian derived functor in the sense of Dold–Puppe of $P \mapsto \mathrm{Hom}_{R\otimes R}(P \otimes P, R)^{C_2}$.

It is a theorem of Hebestreit–Steimle that $\mathrm{GW}(\mathrm{Perf}(R), \mathcal{Q}^g)$ recovers the group completion of the symmetric monoidal groupoid of non-degenerate symmetric bilinear forms on R, as indicated in Hebestreit's talk. In the following Corollary, I simply write the symbol $\mathbb{Z}$ for the Poincaré ∞-category $(\mathrm{Perf}(\mathbb{Z}), \mathcal{Q}^g)$.

Corollary 5. For $n \geq 0$ there is an isomorphism
$$\mathrm{GW}_n(\mathbb{Z})[\tfrac{1}{2}] \cong K_n(\mathbb{Z})[\tfrac{1}{2}]_{C_2} \oplus \mathrm{L}_n(\mathbb{Z})[\tfrac{1}{2}].$$

I then explained how to use algebraic surgery to prove the following theorem.

Theorem 6. *Let R be a ring with involution. Then the homotopy groups of $\mathrm{L}(\mathrm{Perf}(R), \mathcal{Q}^g)$ can be expressed as follows:*

$$\pi_k(\mathrm{L}(\mathrm{Perf}(R), \mathcal{Q}^g)) \cong \begin{cases} \mathrm{L}_k^{\mathrm{short}}(R) & \text{for } k \geq 0 \\ \mathrm{W}_{k+2}^{\mathrm{ev}}(R, -) & \text{for } k = -1, -2 \\ \mathrm{W}_k^q(R) & \text{for } k \leq -3 \end{cases}$$

Here, $\mathrm{W}_*^{\mathrm{ev}}(R, -)$ denotes the Witt groups of unimodular even, antisymmetric forms and formations, and $\mathrm{W}_*^q(R)$ denotes the Witt groups of unimodular quadratic forms and formations. The groups $\mathrm{L}_k^{\mathrm{short}}(R)$ denote the symmetric algebraic L-groups of Ranicki [2] and are such that $\mathrm{L}_k^{\mathrm{short}}(R) \cong \mathrm{W}_0^s(R)$ where the latter denotes the Witt group of unimodular symmetric forms.

The non-negative symmetric L-groups of the integers have been calculated by Ranicki, so we obtain for $n \geq 0$

$$\mathrm{L}_n^{\mathrm{short}}(\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & \text{if } n \equiv 0(4) \\ \mathbb{Z}/2 & \text{if } n \equiv 1(4) \\ 0 & \text{else.} \end{cases}$$

To then calculate the Grothendieck–Witt groups of the integers (including the 2-torsion), one considers the canonical pullback square associated to the fibre sequence (1):

$$\begin{array}{ccc} \mathrm{GW}(\mathbb{Z}) & \longrightarrow & \mathrm{L}(\mathbb{Z}) \\ \downarrow & & \downarrow \\ K(\mathbb{Z})^{hC_2} & \longrightarrow & K(\mathbb{Z})^{tC_2} \end{array}$$

Using localization and dévissage in L-theory, we can show that the right vertical map in this diagram induces a 2-adic equivalence on connective covers. Hence the same is true for the left vertical map:

Corollary 7. The map $\mathrm{GW}(\mathbb{Z}) \to K(\mathbb{Z})^{hC_2}$ induces an equivalence on connective covers.

From this, one can calculate the Grothendieck–Witt group of the integers, conditional on the Kummer–Vandiver conjecture in general, and unconditional in a range far beyond the table displayed above. This relies on the understanding of the algebraic K-theory of the integers, which by the proven Quillen–Lichtenbaum conjecture is controlled by étale cohomology. I mention that these calculations have also recently been achieved by Schlichting [3], conditional on work to appear which also proves the existence of the pullback diagram we have used.

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On the h-cobordism category

Wolfgang Steimle

(joint work with Georgios Raptis)

The h-cobordism category, denoted by $\mathrm{Cob}_d^{\sim}$, is a topological category which is (roughly) defined as follows: An object is a compact $(d-1)$-dimensional manifold M, possibly with boundary; the topology on the set of objects is defined in such a way that the space of objects becomes a classifying space for bundles of $(d-1)$-dimensional smooth manifolds. A morphism from M to N is an cobordism

$$
\begin{array}{ccccc}
M & \xrightarrow{\;\simeq\;} & W & \xleftarrow{\;\simeq\;} & N \\
\uparrow & & \uparrow & & \uparrow \\
\partial M & \xrightarrow{\;\simeq\;} & \partial_h W & \xleftarrow{\;\simeq\;} & \partial N
\end{array}
$$

between the manifolds with boundary M and N (a smooth manifold with corners at ∂M and ∂N), which is an h-cobordism in the sense that all labeled maps are homotopy equivalences. The set of morphisms is also topologized in a way that it becomes a classifying space of h-cobordism bundles. Composition is defined by glueing of cobordisms. There is a variation of this definition, denoted $\mathrm{Cob}_\theta^{\sim}$, where objects and morphisms carry generalized orientations, that is, vector bundle maps from the tangent bundle to a fixed d-dimensional vector bundle θ. Our main result relates the classifying space of this category to Waldhausen's space $H(M)$ of h-cobordisms on a given compact manifold M. A point in this space is an h-cobordism W from M to some compact manifold N, which is ∂-trivial in the sense that the "horizontal" boundary piece $\partial_h W$ of W is identified with the cylinder $M \times [0,1]$ (so, in particular, ∂M is identified with ∂N).

Theorem 1 ([1])**.** *Let* $d \geq 7$, *and let* M *be an object of* $\mathrm{Cob}_{\theta}^{\sim}$. *For any* ∂-*trivial* h-*cobordism* $(W; M, N)$ *on* M, *there is a homotopy fiber sequence*

$$\mathrm{Emb}_{\theta}^{\sim}(M, N) \to \Omega_M B\,\mathrm{Cob}_{\theta}^{\sim} \to H(M),$$

where the loop space is based at M, *and the homotopy fiber is taken over* $(W; M, N)$ *in* $H(M)$.

Here, the first space in the sequence is the space of θ-h-embeddings from M into N, where "h" refers to the requirement that the complement of the embedding be an h-cobordism, and "θ" refers to the datum of an additional identification between the θ-structures on M and N. For the specific tangential structure θ_M given by the tangent bundle of M itself, we obtain from Theorem 1 the following result:

Theorem 2 ([1])**.** *Let* $d \geq 7$ *and let* M *be a compact connected smooth* $(d-1)$-*manifold of handle dimension* k, *regarded as an object of* $\mathrm{Cob}_{\theta_M}^{\sim}$. *There is a* $(d - 2k - 2)$-*connected map*

$$\Omega_M B\,\mathrm{Cob}_{\theta_M}^{\sim} \to H(M).$$

From the stable parametrized h-cobordism theorem [3], we deduce from Theorem 2:

Corollary 3. Let M be a compact connected smooth $(d-1)$-manifold. There is a homotopy equivalence

$$\mathrm{hocolim}_n \, \Omega_{M \times D^n} B\,\mathrm{Cob}_{\theta_{M \times D^n}}^{\sim} \simeq \Omega\,\mathrm{Wh}^{\mathrm{diff}}(M).$$

Here, $\mathrm{Wh}^{\mathrm{diff}}(M)$ denotes the Diff Whitehead space of M [2], a space defined through algebraic K-theory, which only depends on the homotopy type of M.

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Asympotic of twisted Alexander polynomials and hyperbolic volume

LÉO BÉNARD

(joint work with Jérôme Dubois, Michael Heusener and Joan Porti)

We study a family of polynomial invariants of hyperbolic 3-manifolds: the twisted Alexander polynomials. Twisted Alexander polynomials of knots have been defined by Lin [4] and Wada [6]. Kitano [3] showed that they are Reidemeister torsions, generalizing Milnor's theorem on the (untwisted) Alexander polynomial [5]. Here we take the Reidemeister torsion approach to define the twisted Alexander

polynomial for oriented, cusped, hyperbolic three-manifolds of finite volume. An orientable hyperbolic three-manifold has a natural representation of its fundamental group into $\mathrm{PSL}_2(\mathbb{C})$, the hyperbolic holonomy that is unique up to conjugation. The holonomy representation lifts to $\mathrm{SL}_2(\mathbb{C})$, and a lift is unique up to multiplication with a representation into the center of $\mathrm{SL}_2(\mathbb{C})$. The corresponding twisted Alexander polynomial has been considered, among others, by Dunfield, Friedl and Jackson in [1]. Here we compose the lift of the holonomy representation with the irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ in $\mathrm{SL}_n(\mathbb{C})$, the $(n-1)$-th symmetric power, and study its asymptotic behavior. Although in the paper we consider non-compact, orientable, hyperbolic three-manifolds of finite volume in general, here we discuss only the case of a hyperbolic knot complement $S^3 \setminus K$ for simplicity. Let $\rho_n \colon \pi_1(S^3 \setminus K) \to \mathrm{SL}_n(\mathbb{C})$ be the composition of a lift of the holonomy with the $(n-1)$-th symmetric power $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_n(\mathbb{C})$. Let $\Delta_K^{\rho_n}$ denote the Alexander polynomial of K twisted by ρ_n, which equals Wada's definition for n even, but it is Wada's polynomial divided by $(t-1)$ when n is odd, so that its evaluation at $t=1$ does not vanish. The set of unit complex numbers is denoted by $\mathbb{S}^1 = \{\zeta \in \mathbb{C} \mid |\zeta| = 1\}$. The following is a particular case of the main result of this paper.

Theorem 1. *For any $\zeta \in \mathbb{S}^1$,*

$$\lim_{n \to \infty} \frac{\log |\Delta_K^{\rho_n}(\zeta)|}{n^2} = \frac{1}{4\pi} \mathrm{vol}(S^3 \setminus K)$$

uniformly on ζ.

The definition of twisted Alexander polynomial as a Reidemeister torsion requires a vanishing theorem in cohomology. Its proof mimics the classical vanishing theorem on L^2-cohomology of Matsushima–Murakami, as we explain in an Appendix. As a direct consequence of this vanishing theorem, we obtain that the twisted Alexander polynomials have no roots on the unit circle:

Theorem 2. *For any $\zeta \in \mathbb{S}^1$, $\Delta_K^{\rho_n}(\zeta) \neq 0$.*

We apply this theorem to study the dynamics of a pseudo-Anosov diffeomorphism on the variety of representations. Let Σ be a compact orientable surface, possibly with boundary and with negative Euler characteristic. For a pseudo-Anosov diffeomorphism $\phi \colon \Sigma \to \Sigma$, consider its action on the relative variety of (conjugacy classes of) representations $\phi^* \colon \mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C})) \to \mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$. The mapping torus $M(\phi)$ is a hyperbolic manifold of finite volume and its holonomy restricts to a representation of $\pi_1(\Sigma)$ in $\mathrm{SL}_2(\mathbb{C})$ whose conjugacy class is fixed by ϕ^*. In particular the conjugacy class of the composition $[\rho_n] = [\mathrm{Sym}^{n-1} \circ \mathrm{hol}|_{\pi_1(\Sigma)}]$ in $\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$ is fixed by ϕ^*. We prove:

Theorem 3. *The tangent map of ϕ^* at $[\rho_n]$ on $\mathcal{R}(\Sigma, \partial\Sigma, \mathrm{SL}_n(\mathbb{C}))$ has no eigenvalues of norm one. Namely, of ϕ^* has hyperbolic dynamics at $[\rho_n]$.*

For $n = 2$ and $\partial\Sigma = \emptyset$, this was proved by M. Kapovich in [2]. The first part of the talk will be mainly about motivations and context, and we present informally

some of our results. Then we will try to explain part of the proof of Theorem 1. The core of the proof deals with the study of the analytic torsion of some compact hyperbolic manifolds, and we will explain how they are related, and some of the techniques used to obtain the desired asymptotic.

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On the cost of Benjamini Schramm statistics with the Kazhdan property

Łukasz Grabowski

(joint work with Samuel Mellick)

This is report on an on-going joint work with S. Mellick. Recently T. Hutchcroft and G. Pete showed that the cost of any infinite Kazhdan group is 1. We generalise this result to the context of graphings. Our proof is noticbly simpler than the original proof of Hutchcroft and Pete even for graphings arising from actions of countable Kazhdan groups, in particular our arguments do not use any "hard" probability theory. The main ingredient in our approach is the analysis of the connectivity properties of partitions of the vertex space of a graphing, which are "Cheeger-optimal", i.e. minimise the amount of edges present between the parts of a partition.

We work in the context of Benjamini-Schramm statistics, which are convenient "group-like" objects roughly equivalent to "invariant random subgroups". In particular we give examples of Benjamini-Schramm statistics with the Kazhdan property which do not arise from actions of countable Kazhdan groups, by considering point processes on lattice-free Kazhdan Lie groups. Of some interest might be also a seemingly new characerisation of Kazhdan equivalence relations, as studied by M. Pichot.

Our work is partially motivated by the following "Lueck approximation type" question: Let $d \in \mathbb{N}$ with $d \geq 3$, and let M_n be a sequence of triangulated compact d-dimensional manifolds, such that the 1-skeleta of the triangulations have uniformly bounded vertex degrees and which converge to a triangulation of R^d in the sense of Benjamini-Schramm (informally speaking, this means that "M_n is a sequence of compact manifolds with growing injectivity radia"). Is it true that

$\lim \frac{\dim H_1(M_n)}{|M_n|} = 0$? Here $|M_n|$ is the number of vertices in the triangulation of M_n.

The answer is known to be positive for example when $d = 3$ and M_n is a sequence of covers of a fixed compact aspherical manifold M which converges to the universal cover of M. When $d > 3$ and M_n is a sequence of covers of a fixed compact aspherical manifold M, then the question above is equivalent (by the standard Lueck approximation theorem) to the "Singer conjecture in degree 1", i.e. the statement that the first L^2-cohomology group of the universal cover of M is trivial.

Our results imply that the question above has positive answer in the special case when the limiting Benjamini-Schramm statistics has the Kazhdan property.

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Symmetries of exotic negatively curved manifolds
MAURICIO BUSTAMANTE
(joint work with Bena Tshishiku)

Let M be a closed hyperbolic manifold of dimension n and fundamental group π. Viewed as a Riemannian manifold and in dimensions larger than 2, the Rigidity Theorem of Mostow implies that the isometries of M are captured entirely by those of its fundamental group. More precisely, the isometry group of M is isomorphic to the outer automorphism group $\mathrm{Out}(\pi)$ of π. When M is viewed only as a smooth manifold, its group of isometries is the full group of diffeomorphisms of M, but now the map

$$(1) \qquad\qquad \Psi_M : \mathrm{Diff}(M) \to \mathrm{Out}(\pi)$$

which takes a diffeomorphism to the automorphism of π that it induces, is no longer an isomorphism. Nontheless it is a split surjection by the theorem of Mostow. In my talk, I presented results about the map (1) for manifolds N homeomorphic but not diffeomorphic to M. This has to do with the quesiton of how much symmetry such exotic manifold N can have. Our results appear in [2].

Theorem 1. *Fix an n, and assume that either n is even or the order of the group Θ_n of homotopy n-spheres is not a power of 2. For all $d > 0$, there exists a closed hyperbolic manifold M and a manifold N homeomorphic but not diffeomorphic to M such that $|\mathrm{Isom}(M)| = |\mathrm{Out}(\pi)| \geq d$ and $\Psi_N : \mathrm{Diff}(N) \to \mathrm{Out}(\pi)$ is a split surjection. A section to Ψ_N can be chosen to have image isomorphic with the isometry group of some negatively curved metric on N*

Remark 2. Farrell and Jones [3] proved that Ψ_N is not onto in general.

Theorem 3. *Fix an n such that $\Theta_{n-1} \neq 0$. For all $d > 0$, there exists a closed hyperbolic manifold M, and a manifold N homeomorphic but not diffeomorphic to M such that $|\mathrm{Im}\Psi_N| \leq \frac{1}{d}|\mathrm{Out}(\pi)|$*

The proof of Theorem 1 relies on work of Belolipetsky and Lubotzky [1], where they show that for every finite group F and integer $n \geq 2$, there exist infinitely many n-dimensional hyperbolic manifolds M whose isometry group is isomorphic to F. This is used to produce hyperbolic manifolds M with isometry group isomorphic to F, and a smooth action of F on an exotic N of the form $M \# \Sigma \# \cdots \# \Sigma$, where the number of sumands $\Sigma \in \Theta_n$ satisfies certain divisibility condition with respect to $|F|$. B. Tshishiku and I showed, in addition, that among those manifolds M, there are infinitely many which are stably parallelizable and have very large injectivity radius. This guarantees that N is indeed non-diffeomorphic to M and admits a negatively curved metric with respect to which F acts by isometries. The asymmetric manifolds of Theorem 3, are of the form

$$N = M_{c,\varphi} = M \setminus S^1 \times \mathrm{int}(D^{n-1}) \cup_{id_{S^1} \times \varphi} S^1 \times D^{n-1}$$

where $[\varphi] \neq 0 \in \pi_0\mathrm{Diff}(S^{n-2}) \simeq \Theta_{n-1}$ and c is some simple closed geodesic in M with trivial normal bundle (which we identify with $S^1 \times D^{n-1}$). For this type of manifold we find an obstruction to lifting an element in $\mathrm{Out}(\pi)$ to $\mathrm{Diff}(N)$. An instance of this obstruction is as follows. Suppose that there is a cyclic group of order d acting by isometries on M. If this group is generated by an isometry α, then one can show that the smooth manifolds $M_{c,\varphi}, \ldots, M_{\alpha^{d-1}(c),\varphi}$ define *concordant* smooth structures on M. Smoothing theory tells us that concordance classes of smooth structures on M are in bijective correspondence with the set (in fact, abelian group) $[M, \mathrm{Top/O}]$ of homotopy classes of maps from M to $\mathrm{Top/O}$. This is the abelian group in which the desired obstruction lies. It turns out that the smooth structures defined by $M_{c,\varphi}, \ldots, M_{\alpha^{d-1}(c),\varphi}$ are not concordant to each other, provided M is stably parallelizable. Hence to show the second theorem one has to find hyperbolic manifolds with large finite cyclic groups in their isometry groups. For this, we use a theorem of Lubotzky [4] where they produce hyperbolic manifolds whose fundamental group surjects onto a free group.

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Stable integral simplicial volume of 3-manifolds

Marco Moraschini

(joint work with Daniel Fauser, Clara Löh and José Pedro Quintanilha)

Simplicial volume is a homotopy invariant of compact manifolds introduced by Gromov [7] and it measures the complexity of a manifold in terms of its real singular chains. Given an oriented compact connected n-manifold M (possibly with non-empty boundary) the *simplicial volume* is defined by

$$\|M, \partial M\| := \inf\left\{ \sum_{j=1}^{m} |a_j| \ \Bigg| \ \sum_{j=1}^{m} a_j \cdot \sigma_j \in C_n(M; \mathbb{R}) \text{ is a relative}\right.$$
$$\left. \text{fundamental cycle of } (M, \partial M) \right\} .$$

One of the main original aims of the investigation of simplicial volume [7] was the understanding of the relation between the topology of a manifold and its (minimal) volume. In particular, in the case of hyperbolic manifolds, one can show that the simplicial volume is proportional to the Riemannian volume [7, 11]. I report in this talk a recent work in collaboration with Fauser, Löh and Quintanilha [4], in which we investigate an approximation problem of simplicial volume. A still open problem proposed by Gromov [8, p. 232] is the following:

Question 1. *Let M be an oriented closed connected aspherical manifold. Does $\|M\| = 0$ imply the vanishing of the Euler characteristic?*

One way for studying the previous open problem is the following: If M admits enough finite coverings (i.e. if $\pi_1(M)$ is residually finite), we introduce the *stable integral simplicial volume*

$$\|M, \partial M\|_{\mathbb{Z}}^{\infty} := \inf\left\{ \frac{\|W, \partial W\|_{\mathbb{Z}}}{d} \ \Bigg| \ d \in \mathbb{N}, \ W \text{ a } d\text{-sheeted covering of } M \right\} ,$$

where the integral simplicial volume $\|W, \partial W\|_{\mathbb{Z}}$ is defined as the classical one but via $\mathbb{Z}$-singular chains instead of the real ones. One can easily check that the stable integral simplicial volume is always larger than or equal to the standard one:

$$\|M, \partial M\| \leq \|M, \partial M\|_{\mathbb{Z}}^{\infty} .$$

As for Betti numbers, ranks of fundamental groups, or logarithmic torsion of homology, one can ask which aspherical manifolds M satisfy *integral approximation* for simplicial volume, i.e. when the previous inequality is in fact an equality. Since the stable integral simplicial volume of closed manifolds provides an upper bound (up to a uniform multiplicative factor depending only on the dimension of the manifold) of the Euler characteristic [7, 5], Gromov's Question 1 can be reformulated in the following stronger way:

Question 2. *Let M be an oriented closed connected aspherical manifold with $\|M\| = 0$. Does M satisfy integral approximation for simplicial volume?*

Of course, a positive answer to Question 2 would also imply an affirmative answer to Question 1 The following classes of manifolds are already known to satisfy integral approximation for simplicial volume: closed surfaces of positive genus [7, p. 9], closed hyperbolic 3-manifolds [6, Theorem 1.7] and graph manifolds with infinite fundamental group [3] (see also the work by Fauser [1] and Frigerio, Löh, Pagliantini and Sauer [6] for other examples). In contrast, approximation fails uniformly for higher-dimensional hyperbolic manifolds [5, Theorem 2.1] and it fails for closed manifolds with non-abelian free fundamental group [6, Remark 3.9]. Our main result is the following:

Theorem 3. *Let M be an oriented closed connected aspherical 3-manifold, then*

$$\|M\| = \|M\|_{\mathbb{Z}}^{\infty} = \frac{\text{hypvol}(M)}{v_3}.$$

Here, v_3 is the volume of any regular ideal tetrahedron in $\mathbb{H}^3$, and $\text{hypvol}(M)$ denotes the sum of the volumes of the hyperbolic pieces in the JSJ decomposition of M. The equality $\|M\| = \text{hypvol}(M)/v_3$ follows from the work of Soma [10]. Moreover, with regards to Question 2, we can provide the following complete picture of the 3-dimensional case:

Proposition 4. Let M be an oriented closed connected 3-manifold with $\|M\| = 0$. Then the following are equivalent:

(1) The simplicial volume of M satisfies integral approximation.
(2) The manifold M is aspherical or M is homeomorphic to either $\mathbb{S}^2 \times \mathbb{S}^1$ or the connected sum of two copies of $\mathbb{R}P^3$.

In this talk, we present the strategy for proving Theorem 3. We only have to show that

$$\|M\|_{\mathbb{Z}}^{\infty} \leq \frac{\text{hypvol}(M)}{v_3}.$$

The main difficulties arising from the JSJ decomposition of M are the following:

- to deal with the hyperbolic pieces with toroidal boundary and
- the subadditivity with respect to glueings along tori.

We explain here the strategy to overcome the previous issues: We work with a parametrized version of the simplicial volume instead of the stable integral simplicial volume. Since in this setting we can make use of the uniform boundary condition on tori studied by Fauser and Löh [2], this allows us to avoid involved bookkeeping for restrictions and compatibility of finite coverings to the glueing tori. This leads to a nice subadditivity formula with respect to glueings along tori in terms of parametrized simplicial volume. One fundamental ingredient in the proof is that in some cases the most efficient parameter space is the profinite completion of the fundamental group. Hence, it is convenient to rewrite the stable integral simplicial volume as follows:

$$\|W, \partial W\|_{\mathbb{Z}}^{\infty} = \|W, \partial W\|_{\mathcal{F}}^{\widehat{\pi_1(W)}},$$

where $\widehat{\pi_1(W)}$ denotes the profinite completion of the fundamental group of a piece W appearing in the JSJ decomposition. Then, we also need to exploit some profinite properties of the JSJ decomposition and to keep control over the size of the boundary of the cycles appearing as representatives of the parametrised fundamental classes of its pieces. Finally, we briefly discuss how to deal with the hyperbolic pieces. To this end, we have to prove a proportionality result between the parametrised simplicial volume of the hyperbolic pieces and their Riemannian volume:

Theorem 5. *Let W be an oriented compact connected hyperbolic 3-manifold with empty or toroidal boundary and let $M := W^\circ$. Then*

$$\|W, \partial W\|_\partial^{\widehat{\pi_1(W)}} = \frac{\mathrm{vol}(M)}{v_3} \ ,$$

where the subscript ∂ denotes the boundary control mentioned before.

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Volume and macroscopic curvature

Roman Sauer

(joint work with Sabine Braun)

I discussed a generalization of the following result by Guth from closed hyperbolic manifolds to all closed Riemannian manifolds.

Guth's Volume theorem [2]
Let (M, g_{hyp}) be a d-dimensional closed hyperbolic manifold and let g be another metric on M. Suppose that

$$V_{(\widetilde{M}, \widetilde{g})}(1) \leq V_{\mathbb{H}^d}(1),$$

where $\mathbb{H}^d$ is d-dimensional hyperbolic space. Then

$$\mathrm{vol}(M, g_{hyp}) \leq \mathrm{const}(d) \cdot \mathrm{vol}(M, g).$$

To formulate the generalization one needs to replace the hyperbolic volume. We replace it by the simplicial volume. By a classical result of Gromov and Thurston the simplicial volume of a closed hyperbolic manifold is its hyperbolic volume up to dimensional constant.

Theorem

Let (M, g) be a d-dimensional closed Riemannian manifold. Suppose that

$$V_{(\widetilde{M}, \widetilde{g})}(1) \leq V_1$$

for a positive real number V_1. Then

$$\|M\| \leq \mathrm{const}(d, V_1) \cdot \mathrm{vol}(M, g).$$

This theorem also generalizes Gromov's main inequality [1]. If the fundamental group of M is residually finite, then one can modify the proof by Guth to obtain the result. The main innovation is to drop residual finiteness. The idea is that even though the fundamental group is not residually finite, one can define a good analog of the solenoidal space which is defined as the projective limit of the finite regular coverings of M. We then implement a foliated (or equivariant) version of Guth's techniques on this analog.

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Topological spines, minimal realizations and cohomology of strictly developable simple complexes of groups

NANSEN PETROSYAN

(joint work with Tomasz Prytuła)

In [1], for any finitely generated Coxeter system (W, S), Bestvina constructed an acyclic polyhedral complex $B(W, S)$ of dimension equal to $\mathrm{vcd}\, W$, on which W acts as a reflection group, properly and cocompactly. The same construction produces a contractible $B(W, S)$ with $\dim B(W, S) = \mathrm{vcd}\, W$ except possibly when $\mathrm{vcd}\, W = 2$. In fact in [2], we showed that $B(W, S)$ is equivariantly homotopy equivalent to the Davis complex Σ_W. Therefore $B(W, S)$ becomes a model for $\underline{E}W$ of minimal dimension. In [2], we derived an analogous result in the more general setting of thin strictly developable simple complexes of finite groups. One of the main motivations to construct nice models for $\underline{E}G$ comes from the Isomorphism Conjectures. Other applications include computations in group cohomology and the formulation of a generalisation from finite to infinite groups of the Atiyah-Segal Completion Theorem in topological K–theory. In recent work with Tomasz Prytuła, we are

able to extend the results in [2] from finite to infinite local groups and to more naturally occurring simple complexes of groups without the thinness assumption. Many previous methods have relied on compactly supported cohomology as a convenient tool for computations. But this restricts one to only complexes that are locally finite. To resolve this difficulty, we establish a direct link between Bredon cohomology with certain coefficients and the relative cohomology of the strata. This allows us to derive the following theorems in [3].

Theorem 1 (Petrosyan-Prytuła, [3])**.** *Suppose a group G acts properly on a* CAT(0) *polyhedral complex X with a strict fundamental domain K. Let $\mathcal{Q}$ denote the poset of cells of K ordered by the reverse inclusion (thus $|\mathcal{Q}| = K'$). Then*

$$\underline{\mathrm{cd}}(G) = \max\{n \in \mathbb{N} \mid H^n(K_C', K_{>C}') \neq 0 \text{ for some block } C \subseteq \mathcal{Q}\}.$$

Theorem 2 (Petrosyan-Prytuła, [3])**.** *Let G be a group acting chamber transitively on a building of type (W, S). Let $G(\mathcal{Q})$ be the associated simple complex of groups and let $\mathcal{F}$ be the family generated by the stabilisers. Then $D(B, G(\mathcal{Q}))$ is a realisation of the building of dimension*

$$\dim(D(B, G(\mathcal{Q}))) = \left\{ \begin{array}{lll} \mathrm{vcd}\, W & if & \mathrm{vcd}\, W \neq 2, \\ 2 \text{ or } 3 & if & \mathrm{vcd}\, W = 2. \end{array} \right.$$

and

$$\mathrm{cd}_{\mathcal{F}}\, G = \mathrm{vcd}\, W = \max\{n \in \mathbb{N} \mid H^{n-1}(K_{>J}) \neq 0 \text{ for some } J \in \mathcal{Q}\}.$$

The following are some of the remaining open questions.

Question 3. *Does the Bestvina complex support a G-invariant CAT(0) metric?*

Question 4. *Is the Bestvina complex an equivariant deformation retract of the Davis complex?*

Question 5. *When can the construction of the Bestvina complex be generalised to actions with non-compact or non-strict fundamental domains?*

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The K-and L-theory of crystallographic groups and their applications to C^*-algebras and manifolds

Wolfgang Lück

We give a report about a series of papers [1, 2, 3, 4, 5] where the K-and L-groups of group rings or C^*-algebras of certain crystallographic groups are computed. Moreover, applications to C^*-algebras and manifolds are discussed. We explain the following results. Fix a group homomorphism $\rho \colon \mathbb{Z}/m \to \operatorname{aut}(\mathbb{Z}^n)$ such that the $\mathbb{Z}/m$-action on $\mathbb{Z}^n$ is free outside the origin. Let $G = \mathbb{Z}^n \times_\rho \mathbb{Z}/m$ be the associated semi-direct product which is a crystallographic group. Let $\mathcal{M}$ be the set of conjugacy classes of maximal finite subgroups of G.

Theorem 1.

(1) *We obtain an isomorphism*
$$\omega_1 \colon K_1(C_r^*(G)) \xrightarrow{\cong} K_1(G \backslash \underline{E}G).$$

Restriction with the inclusion $k \colon \mathbb{Z}^n \to G$ induces an isomorphism
$$k^* \colon K_1(C_r^*(G)) \xrightarrow{\cong} K_1(C_r^*(\mathbb{Z}^n))^{\mathbb{Z}/m}.$$

Induction with the inclusion k yields a homomorphism
$$\overline{k_*} \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \to K_1(C_r^*(G)).$$

It fits into an exact sequence
$$0 \to \widehat{H}^{-1}(\mathbb{Z}/m, K_1(C_r^*(\mathbb{Z}^n))) \to \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_1(C_r^*(\mathbb{Z}^n)) \xrightarrow{\overline{k_*}} K_1(C_r^*(G)) \to 0.$$

In particular $\overline{k_*}$ is surjective and its kernel is annihilated by multiplication with m;

(2) *There is an exact sequence*
$$0 \to \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \xrightarrow{\bigoplus_{(M) \in \mathcal{M}} i_M} K_0(C_r^*(G)) \xrightarrow{\omega_0} K_0(G \backslash \underline{E}G) \to 0,$$

where $\widetilde{R}_{\mathbb{C}}(M)$ is the kernel of the map $R_{\mathbb{C}}(M) \to \mathbb{Z}$ sending the class $[V]$ of a complex M-representation V to $\dim_{\mathbb{C}}(\mathbb{C} \otimes_{\mathbb{C}M} V)$ and the map i_M comes from the inclusion $M \to G$ and the identification $R_{\mathbb{C}}(M) = K_0(C_r^*(M))$. We obtain a homomorphism
$$\overline{k_*} \oplus \bigoplus_{(M) \in \mathcal{M}} i_M \colon \mathbb{Z} \otimes_{\mathbb{Z}[\mathbb{Z}/m]} K_0(C_r^*(\mathbb{Z}^n)) \oplus \bigoplus_{(M) \in \mathcal{M}} \widetilde{R}_{\mathbb{C}}(M) \to K_0(C_r^*(G)).$$

It is injective. It is bijective after inverting m;

(3) *We have*
$$K_i(C_r^*(G)) \cong \mathbb{Z}^{s_i}$$

where
$$s_i = \begin{cases} \left(\sum_{(M) \in \mathcal{M}} (|M| - 1)\right) + \sum_{l \in \mathbb{Z}} \operatorname{rk}_{\mathbb{Z}}\left((\Lambda^{2l} \mathbb{Z}^n)^{\mathbb{Z}/m}\right) & \text{if } i \text{ even;} \\ \sum_{l \in \mathbb{Z}} \operatorname{rk}_{\mathbb{Z}}\left((\Lambda^{2l+1} \mathbb{Z}^n)^{\mathbb{Z}/m}\right) & \text{if } i \text{ odd;} \end{cases}$$

(4) If m is even, then $s_1 = 0$ and

$$K_1(C_r^*(G)) \cong \{0\}.$$

Theorem 2. *The Conjecture due to Adem-Ge-Petroysan-Pan is true which says that the Serre-Lyndon spectral sequence for the group cohomology associated to the extension $1 \to \mathbb{Z}^n \to G \to \mathbb{Z}/m \to 1$ collapses (in the strongest sense). It is in general not true if we drop the assumption that the $\mathbb{Z}/m$-action on $\mathbb{Z}^n$ is free outside the origin.*

Theorem 3. *The group $\mathbb{Z}^4 \times_\rho \mathbb{Z}/3$ satisfies for appropriate ρ the unstable Gromov-Lawson-Rosenberg Conjecture.*

Note that Schick proved that $\mathbb{Z}^3 \times \mathbb{Z}/4$ does not satisfy the unstable Gromov-Lawson-Rosenberg Conjecture. Suppose that $m = p$ holds for an odd a prime p.

Fix a free action of $\mathbb{Z}/p$ on a sphere S^l for an odd integer $l \geq 3$. Define a closed $(n+l)$-manifold $M := T_\rho^n \times_{\mathbb{Z}/p} S^l$.

Theorem 4. *As an abelian group we get for the topological simple structure set of M*

$$S(M) \quad \cong \quad \mathbb{Z}^{p^k(p-1)/2} \oplus \bigoplus_{i=0}^{n-1} L_{n-i}(\mathbb{Z})^{r_j},$$

where the natural number k is determined by the equality $n = k(p-1)$ and $r_j := \mathrm{rk}(\Lambda^j(\mathbb{Z}[\zeta_p]^k)^{\mathbb{Z}/p})$. Moreover, a simple homotopy equivalence $N \to M$ is homotopic to a homeomorphism if and only if certain splitting obstructions vanish and certain Rho-invariants of N and M agree.

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