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Singularities and Homological Aspects of Commutative Algebra

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Abstract. Commutative algebra has recently witnessed a number of spectacular developments, resulting in the resolution of long-standing problems. The new techniques and perspectives, such as methods from the theory of perfectoid spaces, are leading to an extraordinary transformation in the field. There is also remarkable progress on the study of singularities in positive characteristics, and in particular on the problem of resolution of singularities. This workshop brought together researchers driving these developments with a broader group of young researchers in commutative algebra and allied fields, with the aim of spurring new collaborations and progress.

Mathematics Subject Classification (2010): 13xx, 14xx, and 15Exx.

Introduction by the Organizers

The unifying theme of this workshop was resolutions, as in resolution of singularities in the algebraic geometry, especially in positive and mixed characteristics, and also in the sense of invariants derived from resolutions of modules over commutative rings. This workshop drew together experts that, though united by their interest in commutative algebra and singularities, do not often get an opportunity to interact. This made for a lively gathering. One of the highlights of the workshop was the presence of Heisuke Hironaka, who gave the first proof of resolution of singularities of characteristic zero algebraic varieties of arbitrary dimension in 1964. Another aspect that contributed to the success of the workshop was the participation of experts on Scholze's theory of perfectoid rings and spaces. Perfectoid techniques made a dramatic entrance in commutative algebra when Yves André used them, in 2016, to settle the Direct Summand Conjecture. André himself was present at the workshop. Other highlights of the meeting were Olivier Piltant's lecture on his monumental proof of the resolution of singularities of three-folds; Claudiu Raicu's presentation of a proof of Green's conjecture for syzygies of general canonical curves in characteristic 0 and large; and Linquan Ma's lecture on his proof of a conjecture of Stükrad and Vogel.

Resolution of singularities. As mentioned before, one focus of the workshop was on resolution of singularities, and in particular on algorithmic methods used to resolve or compute important invariants of resolution. Hironaka discussed his program to prove resolution of singularities in positive and mixed characteristic. Piltant gave an outline of his recent proof with Vincent Cossart of resolution of singularities of reduced excellent schemes of dimension three. The previous strongest result in positive characteristic was Abhyankar's 1966 theorem showing that resolutions of singularities exist for three dimensional algebraic varieties over an algebraically closed field of characteristic greater than 5.

Ana Bravo discussed recent joint work with Santiago Encinas and Beatriz Pascual-Escudero. A canonical resolution of singularities of a characteristic zero algebraic variety is determined by a generalized order function on the variety. In their work, invariants from the arc space of a variety X are used to determine the first main invariant of the generalized order function. Ana Reguera discussed joint work with Angelica Benito and Piltant. They consider the question of when the space of arcs centered in the singular locus of an algebraic variety has a finite number of irreducible components. Nash proved, using resolution of singularities, that this is true in characteristic zero. Reguera, Benito and Piltant prove there are only a finite number of components when local uniformization can be applied.

Bernd Schober explained joint work with Cossart. Given an ideal in a regular local ring R, Hironaka associated the characteristic polyhedron, which can be computed from suitable regular parameters in the formal completion of R. As a tool in resolution of singularities, it is important to find regular parameters in the original ring R which realize the characteristic polyhedron. They show that this can be achieved in some important situations. Hussein Mourtada explained his algorithm with Schober which determines if a characteristic zero hypersurface singularity has quasi-ordinary singularities. Michael Temkin reported on his joint work with Abramovich and Wlodarczyk. They prove a theorem on resolution of singularities of morphisms, which generalizes earlier work of Abramovich and Karu. After making a base change by a suitable modification, they are able to make a morphism log smooth, for dominant morphisms of integral log varieities and log DM stacks.

Applications of perfectoid techniques. André opened the proceedings with a masterful survey of his proof of the Direct Summand Conjecture, and its history. The conjecture, now a theorem, is remarkably easy to state: A regular (noetherian, commutative) ring is a direct summand in any finite extension. This statement has many equivalent formulations, which go under the rubric "The homological conjectures", and these were one of driving forces for research in commutative algebra for decades. It was proved by Hochster for rings containing fields, already by the early 1970s. The "remaining" mixed characteristic was settled by André's. His work has lead to an explosion of new results in commutative algebra—the recent work of Ma and Schwede on symbolic powers of ideals in mixed characteristic is a striking illustration of this.

Rings of differential operators, and connections with non-commutative geometry. One unexpected theme that emerged during this workshop was the theory of differential operators. While this construct, introduced by Grothendieck, has for long been of interest to commutative algebraists and algebraic geometers, the focus has been on smooth algebras and over fields of characteristic zero. This has begun to change and a number of researchers presented new advances on this subject, and coming from different points of view. Veronica Ertl spoke on her work with Miller on descent results on the sheaf cohomology of regular schemes over perfect fields of positive characteristic, with coefficients the sheafification of the rational de Rham - Witt differentials.

Holger Brenner spoke on his work with Jeffries and Núñez-Betancourt concerning the maximal rank of a free summand of P^n, the module of principal parts of degree n of an affine algebra A over a perfect field K. The A-dual of P^n consists of differential operators of degree $\leq n$, so the free rank of P^n corresponds to differential operators E that satisfy $E(f) = 1$ for some $f \in A$. Besides of intrinsic interest, the asymptotic behavior of the free rank, as n grows, provide a characteristic-free analogue of the F-signature, an important invariant in the study of singularities in positive characteristic. The F-signature is defined using the free ranks of the Frobenius modules, namely, modules A^{1/p^n}, when K is a field of positive characteristic p.

In this lecture, Anurag Singh discussed the change of rings properties the ring of differential operators. The motivation is the following question posed by K. Smith and Van den Bergh, that arose in an approach to the still-open conjecture about the simplicity of the rings of differential operators of the invariants of a linearly reductive group: Given an domain R that is flat and finitely generated over a Dedekind domain A, and a maximal ideal μ of A, is every A/μ-linear differential operator of R/μ induced by an A-linear differential operator of R? Singh explained why the coordinate ring of the Grassmannian $G(2,4)$ under the Plücker embedding furnishes a counter-example to this question. The computation, which is part of an ongoing collaboration with Jeffries, exploits a recent result of Jeffries that the derived functors of differential operators can be realised as certain local cohomology modules.

When R is a regular ring of finite type over a perfect field k, the global dimension of the ring of differential operators is finite. This was proved many years ago by Roos (in characteristic 0) and P. Smith (in positive characteristic). It came as a surprise (at least to one of the organizers) that the global dimension of the ring of differential operators is also finite when R is a normal toric algebra. This result featured in a talk by Eleonore Faber, based on her work with Muller and

K. Smith. It was derived as a byproduct of a more precise result concerning the endomorphism ring $\mathrm{End}_R(R^{1/q})$, where q is a positive integer and $R^{1/q}$ is the ring spanned by qth roots of monomials in R. Namely, that this endomorphism ring has finite global dimension for $q \gg 0$, and hence it can be viewed a non-commutative resolution of singularities of R, in the sense of Bondal and Orlov. Under certain additional constraints on R, Faber also constructs non-commutative crepant resolutions, in the sense of Van den Bergh.

Non-commutative geometry was also the topic of Vincent Gélinas' talk on his extension of the classical Bernstein, Gelfand, Gelfand (BGG) correspondence. The BGG correspondence links the derived category of an exterior algebra to the derived category of its Koszul dual symmetric algebra. Gélinas presented some of the results contained in his Ph. D. thesis wherein the exterior algebra is replaced by any Koszul Gorenstein algebra with the property that the minimal resolution of any module is eventually linear. What is remarkable about this is that Koszul dual of the algebra need not be noetherian. Using these results Gélinas has constructed counterexamples to a conjecture of Bondal, and one of Minamoto, both concerning coherence of certain algebras.

Homological aspects. Another major theme of the workshop has been the study of homological aspects of commutative algebra and related invariants. The topics discussed included Castelnuovo-Mumford regularity, syzygies, the structure of free resolutions, evolution of symbolic powers, Hilbert functions and coefficients of various types.

Castelnuovo-Mumford regularity is a universal measure of the complexity of a module and one of its incarnation is essentially homological. Central lines of investigation are the study of the behaviour of the Castelnuovo-Mumford regularity for discrete families (powers of ideals, Tor-modules etc..) and its relation with invariants of algebraic varieties. Marc Chardin presented recent achievements related to the Castelnuovo-Mumford regularity of Tor and Ext modules over non-regular rings that generalize work of Eisenbud, Huneke and Ulrich.

Raicu presented a beautiful mathematical journey that, starting from resonance varieties and the BBG correspondence and using Koszul modules, leads to a proof of Green's conjecture for the syzygies of general canonical curves of genus g in characteristic 0 and large. Matteo Varbaro discussed a remarkable homological rigidity property of algebras with straightening laws generalizing an unpublished result of Buchweitz. Steven Sam generalized Stillman's projective dimension conjecture, now theorem of Ananyan and Hochster, to cohomology tables under suitable prescriptions and, using the notion of strength, established an asymptotic version of Hartshorne's complete intersection conjecture.

Twenty years ago Ein, Lazarsfeld and K. Smith proved the following surprising result: the dn-th symbolic power of a prime ideal in a d-dimensional regular ring is contained in the n-th ordinary power. Eloísa Grifo discussed generalizations and improvements of this result to rings of finite characteristic with mild F-singularities. Hilbert coefficients of various type have been classically used to measure the singularity types and the (failure of) homological properties. Shreedevi Masuti reported on the study of the Hilbert coefficients associated to the filtration of the integral closure of the powers of m-primary ideals, the "normal" Hilbert coefficients, and their use as singularity identifiers. Ma presented a proof of a conjecture of Stükrad and Vogel asserting that for a local ring R with equidimensional completion there exist a number c such that $\ell(R/I) \leq ce(I)$ for every ideal I primary to the maximal ideal of R. The proof is a mathematical symphony with Vasconcelos' homological degree and Huneke's uniform Artin-Rees lemma as principal interpreters.

Invariants of ideals related to singularities was also the topic of Shunsuke Takagi's talk on the localization problem for finitistic test ideals in rings of positive characteristic. Brenner and Monsky constructed examples that showed that tight closure does not commute with localization, but it is an open question whether the formation of finitistic test ideals is compatible with localization. Takagi presented his results that settle this question in the affirmative for numerically $\mathbb{Q}$-Gorenstein domains over F-finite field.

Finally two talks were devoted to introduce the audience to new research directions in neighbouring fields. In the spirit of the classical results of Stanley, Martina Juhnke-Kubitzke highlighted the new frontiers of the search for inequalities that characterize discrete objects with prescribed topological and combinatorial features. Mateusz Michałek described an exciting mathematical tour that start from invariant theory and ends with the use of local cohomology to prevent injections of algebraic varieties into projective spaces.

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Workshop: Singularities and Homological Aspects of Commutative Algebra

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Abstracts

Singularities and perfectoid geometry

Yves André

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Let S be a complete (noetherian) local ring S. If S is not Cohen-Macaulay and if this creates trouble, one may adopt two different perspectives:

1) in the spirit of resolution, one may look for a *Macaulayfication of* S, i.e. a surjective birational map $Y \to \mathrm{Spec}\, S$ such that all s.o.p. (systems of parameters) in the local rings $\mathcal{O}_{Y,y}$ are regular sequences (but s.o.p. in S might not remain s.o.p. in Y); such weak resolutions have been constructed by Faltings, Kawasaki et al.

or

2) in the spirit of the homological conjectures, one may look for a (big, i.e. not necessarily noetherian) *Cohen-Macaulay* S*-algebra* T, i.e. an S-algebra in which all s.o.p. in S become regular sequences in T (but s.o.p. in T might fail to be regular sequences). If S is a finite extension of a regular local ring R, an S-algebra T is Cohen-Macaulay if and only if it is faithfully flat over R.

Theorem 1. *(big) Cohen-Macaulay algebras exist and are weakly functorial.*

This was proven by Hochster-Huneke in equal characteristic several years ago, and recently by the speaker in mixed characteristic using perfectoid techniques. This result implies many homological conjectures, such as the direct summand conjecture, the syzygy conjecture...

Here is a sketch of the construction. Let $(S, \mathfrak{m}, k)$ be a p-torsionfree complete (noetherian) local ring of char. $(0, p)$. Choose a lift $\underline{x} = (x_1, \dots, x_n)$ in S of a s.o.p. of S/p, so that S is a finite extension of $R = \Lambda[[\underline{x}]]$ (Λ = Cohen ring of k, a subring of $W(\bar{k})$), and let $g \in R$ be a discriminant of $S[\frac{1}{p}]$ over $R[\frac{1}{p}]$.

First comes the perfectoid valuation ring $V = W(\bar{k})[[p^{\frac{1}{p^\infty}}]]$. After Scholze, a p-adically complete p-torsionfree V-algebra A is called *perfectoid* if Frobenius induces an isomorphism $A/p^{\frac{1}{p}} \to A/p$.

We construct two perfectoid algebras

$A = p$-adic completion of $\cup_i W(\bar{k})[[p^{\frac{1}{p^i}}, \underline{x}^{\frac{1}{p^i}}]]$,

$A' = p$-adic completion of the p-integral closure of $A[g^{\frac{1}{p^\infty}}]$ in $A[g^{\frac{1}{p^\infty}}, \frac{1}{p}]$,

which turn out to be faitfully flat (hence Cohen-Macaulay) over R.

We then introduce the integral closure B of A' in $A' \otimes_R S[\frac{1}{pg}]$, which turns out to be "almost perfectoid" over V and "almost Cohen-Macaulay" over S (this follows from a perfectoid version of the Abhyankar lemma). Here, "almost" is taken in the sense of Almost ring theory introduced by Faltings and developed by Gabber-Ramero: a $V[g^{\frac{1}{p^\infty}}]$-module is almost zero if it is killed by the idempotent ideal $((pg)^{\frac{1}{p^\infty}})$.

In order to get rid of "almost", and get a genuine perfectoid Cohen-Macaulay S-algebra T, one may use either Hochster's modification tehcnique, or the following recent trick due to Gabber: replace B by

$T =$ the $\mathfrak{m}$-adic completion of $\mathcal{S}^{-1}(B^{\mathbb{N}}/B^{(\mathbb{N})})$, where $\mathcal{S}$ is the multiplicative set of elements $((pg)^{\epsilon_i}), \epsilon_i \in \mathbb{N}[\frac{1}{p}], \epsilon_i \to 0$. Note that the construction depends only on S and on the choice of $\underline{x}, g$.

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Using (perfectoid) Cohen-Macaulay S-algebras T, one can introduce new "closures" of ideals: $I \mapsto IT \cap S$. In this guise, Ma and Schwede developed an analog of tight closure theory - a promising tool for the study of singularities in mixed characteristic.

More surprisingly, it also serves as a bridge between singularity theory in char. 0 and char. p: for instance, Ma and Schwede show that one can detect rationality in char. 0 from F-rationality in reduction mod. p for any single p (under very mild non-degeneracy assumption). Perfectoid are hidden in the proof that the algorithm works.

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Resolution of singularities: a proof in dimension 3.

Olivier Piltant
(joint work with Vincent Cossart)

In contrast with residue characteristic zero, the resolution of singularities problem is still open in dimension higher than 3 for positive residue characteristic. It is conjectured that any quasi-excellent reduced noetherian scheme has a resolution of singularities. This extra assumption deals with the formal fibers (a condition on the local rings) and the openness of regular loci (a condition on the open covering). My talk explains some of the main ingredients of the proof of the conjecture in dimension 3: ramification of valuations, tangent cone, characteristic polyhedron and differential invariants.

Singularities of Algebras with Straightening Law

Matteo Varbaro
(joint work with Alexandru Constantinescu, Emanuela De Negri)

Let $A = \oplus_{i\in\mathbb{N}} A_i$ be a graded algebra over a field $A_0 = K$ and let $(H, \prec)$ be a finite poset set. Let $H \hookrightarrow \cup_{i>0} A_i$ be an injective function. The elements of H will be identified with their images. Given a chain $h_1 \preceq h_2 \preceq \cdots \preceq h_s$ of elements of H the corresponding product $h_1 \cdots h_s \in A$ is called *standard monomial*. One says that A is an *Algebra with Straightening Law* (ASL) on H (with respect to the given embedding H into $\cup_{i>0} A_i$) if three conditions are satisfied:

1. The elements of H generate A as a K-algebra.
2. The standard monomials are K-linearly independent.
3. For every pair h_1, h_2 of incomparable elements of H there is a relation (called the straightening law)

$$h_1 h_2 = \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{jv_j}$$

where $\lambda_j \in K \setminus \{0\}$, the $h_{j1} \cdots h_{jv_j}$ are distinct standard monomials and, assuming that $h_{j1} \preceq \cdots \preceq h_{jv_j}$, one has $h_{j1} \prec h_1$ and $h_{j1} \prec h_2$ for all j.

It then follows from the three axioms that the standard monomials form a basis of A as a K-vector space and that the straightening law gives indeed the defining equations of A as a quotient of the polynomial ring $K[H] = K[h : h \in H]$. That is, the kernel I of the canonical surjective map $K[H] \to A$ of K-algebras induced by the function $H \to \cup_{i>0} A_i$ is generated by the equations provided by the straightening law, i.e.:

$$A = K[H]/I \quad \text{with} \quad I = (h_1 h_2 - \sum_{j=1}^{u} \lambda_j h_{j1} \cdots h_{jv_j} : h_1 \not\prec h_2 \not\prec h_1).$$

Remark 1. *Equipping the polynomial ring $K[H]$ with the $\mathbb{N}$-graded structure induced by assigning to h the degree of its image in $\cup_{i>0} A_i$, then the ideal I is homogeneous.*

The ideal $J = (h_1 h_2 : h_1 \not\prec h_2 \not\prec h_1)$ of $K[H]$ defines a quotient $A_D = K[H]/J$ which is an ASL as well, called the *discrete ASL* associated to H. In [2] it is proved that A_D is the special fiber of a flat family with general fiber A. Indeed, one can obtain the same result by observing that with respect to (weighted) degrevlex associated to a total order on H that refines the given partial order $\prec$ one has $J = \text{in}(I)$. In fact ASL's can also be defined via Groebner degenerations. By this view-point, if $\mathfrak{m}$ is the maximal homogeneous ideal of $K[H]$, from the recent result of [1] the Hilbert functions of the local cohomology modules $H^i_{\mathfrak{m}}(A)$ and of $H^i_{\mathfrak{m}}(A_D)$ agree for all i. In other words, the Hilbert function of $H^i_{\mathfrak{m}}(A)$ depends only on the poset H (and not on the straightening law).

De Concini, Eisenbud and Procesi in [2, 3] expressed the feeling that many ASL's over a field of characteristic 0 should have rational singularities. In an unpublished work, Buchweitz confirmed their feeling as follows:

Theorem 2 (Buchweitz)**.** *If K has characteristic 0, A has rational singularities in its punctured spectrum, and A_D is Cohen-Macaulay (CM), then A has rational singularities.*

We can prove the same result without the CM assumption on A_D, that so becomes a consequence of A having rational singularities in its punctured spectrum. Our theorem works also in positive characteristic, by replacing rational with F-rational. In fact, our argument shows:

Theorem 3. *Let I be a homogeneous ideal of a positively graded polynomial ring S over K such that* $\text{in}(I)$ *is square-free for a degrevlex monomial order.*

(1) If K has characteristic 0 and S/I has rational singularities in its punctured spectrum, then S/I has rational singularities.
(2) If K has positive characteristic and S/I is F-rational in its punctured spectrum, then S/I is F-rational.

To prove Theorem 3 one has to show that under the assumptions S/I is CM with negative a-invariant. This suggested us the following:

Conjecture 4. *Let I be a homogeneous ideal of a standard graded polynomial ring S over K such that* $\text{in}(I)$ *is square-free for some monomial order. If I defines a nonsingular projective variety, then S/I is CM with negative a-invariant.*

Theorem 3 solves positively the conjecture for degrevlex monomial orders; otherwise, Conjecture 4 is open also for curves, for which it says: Let I be a homogeneous ideal of a standard graded polynomial ring S over K such that $\text{in}(I)$ is square-free for some monomial order. If I defines a nonsingular projective curve C, then C has genus 0.

Given a simplicial complex Δ on n vertices, we say that Δ is *Groebner-smoothable* over K if there exists a homogeneous ideal I of a standard graded polynomial ring S in n variables over K such that $\text{in}(I) = I_\Delta$ for some monomial order (where $I_\Delta \subset S$ is the Stanley-Reisner ideal of Δ) and I defines a nonsingular projective variety. By exploiting [1], Conjecture 4 is equivalent to:

Conjecture 5. *If a simplicial complex is Groebner-smoothable over K then it is Cohen-Macaulay and acyclic over K.*

In this direction we can show:

Proposition 6. *A Groebner-smoothable graph must have leaves.*

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A Fedder-type criterion over Gorenstein rings and symbolic powers

Eloísa Grifo
(joint work with Linquan Ma and Karl Schwede)

Given a radical ideal I in a regular ring R, the n-th symbolic power of I is given by

$$I^{(n)} = \bigcap_{P \in \mathrm{Min}(I)} I^n R_P \cap R.$$

While symbolic powers have good geometric properties, they can be very difficult to compute; on the other hand, ordinary powers are easily computable, but do not enjoy good geometric properties. In general, $I^n \neq I^{(n)}$. The Containment Problem tries to compare I^n and $I^{(n)}$ by asking when $I^{(a)} \subseteq I^b$.

Theorem 1 ([3, 9, 10]). *Let R be a regular ring and I be a radical ideal of big height h. For all $n \geqslant 1$, $I^{(hn)} \subseteq I^n$.*

The big height of I is the maximum height of an associated prime. In characteristic p, the result above can be improved; one has $I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q$ for all $q = p^e$.

Conjecture 2 (Harbourne, 2008). *Let R be a regular ring and I be a radical ideal of big height h. Then for all $n \geqslant 1$, $I^{(hn-h+1)} \subseteq I^n$.*

Conjecture 2 holds for nice classes such as ideals defining general points in $\mathbb{P}^2$ [7] and $\mathbb{P}^3$ [1]. In characteristic p, we have the following result:

Theorem 3 (Grifo–Huneke [6]). *Let R be a regular ring of characteristic $p > 0$ and I be a radical ideal of big height h.*

a) If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.
b) If R/I is strongly F-regular and $h \geqslant 2$, then h can be replaced by $h-1$, meaning $I^{(h(n-1)+1)} \subseteq I^n$ for all $n \geqslant 1$. In particular, when $h = 2$ we obtain $I^{(n)} = I^n$ for all $n \geqslant 1$.

When S is a ring of characteristic p, we write $F^e_*(S)$ to denote S with the the S-module given by the action of the e-th iteration F^e of the Frobenius map. An F-finite ring S is F-pure ($\equiv$ F-split) if the S-module map $F^e : S \longrightarrow F^e_*(S)$ splits.

Our goal is to improve this result to a non-regular setting. The F-purity condition is used via Fedder's Criterion, which has a counterpart for strong F-regularity in Glassbrenner's Criterion [5].

Theorem 4 (Fedder [4]). *Let $(R, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Given an ideal I in R, R/I is F-pure if and only if for all $q = p^e \gg 0$,*

$$\left(I^{[q]} : I\right) \nsubseteq \mathfrak{m}^{[q]}.$$

The main ingredients in the proof of Fedder's criterion are the following:

- If R is regular, $F^e_*(R)$ is free, and thus every $\phi \in \mathrm{Hom}_{R/I}(F^e_*(R/I), R/I)$ lifts to a map $\tilde{\phi} \in \mathrm{Hom}_R(F^e_*(R), R)$.

- If R is Gorenstein, $\mathrm{Hom}_R(F^e_*(R), R)$ is cyclic, generated by Φ_e, so that every element is of the form $\Phi_e(F^e_* r \cdot -)$ for some $r \in R$. The elements that induce maps on R/I are precisely those in $(I^{[q]} : I)$.

We will use the following generalized version of Fedder's Criterion:

Theorem 5 (G–Ma–Schwede). *Let R be an F-finite Gorenstein ring and Q be an ideal of finite projective dimension.*

(1) Every $\phi \in Hom_{R/Q}(F^e_*(R/Q), R/Q)$ lifts to a map $\tilde{\phi} \in Hom_R(F^e_*(R), R)$.
(2) If R/Q is F-pure, then $\Phi_e\left(F^e_*\left(I_e(Q) : Q\right)\right) = R$ for all e.

With this criterion, we can now obtain a generalization of Theorem 3:

Theorem 6 (Grifo–Ma–Schwede). *Let R be a Gorenstein F-finite ring and I be a radical ideal of big height h. Suppose that I has finite projective dimension.*

a) If R/I is F-pure, then $I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.
b) If R/I is strongly F-regular and $h \geqslant 2$, then h can be replaced by $h-1$, meaning $I^{(h(n-1)+1)} \subseteq I^n$ for all $n \geqslant 1$. In particular, when $h = 2$ we obtain $I^{(n)} = I^n$ for all $n \geqslant 1$.

If the projective dimension of I is infinite, take J to be the Jacobian ideal.

c) If R/I is F-pure, then $J^n I^{(hn-h+1)} \subseteq I^n$ for all $n \geqslant 1$.
d) If R/I is strongly F-regular and $h \geqslant 2$, $J^n I^{(h(n-1)+1)} \subseteq I^n$ for all $n \geqslant 1$. In particular, when $h = 2$ we obtain $J^n I^{(n)} = I^n$ for all $n \geqslant 1$.

The statements in a) and b) can fail for ideals of infinite projective dimension.

Example 7. *Consider the ideal $Q = (x, z)$ in $R = k[x, y, z]/(xy - z^a)$, where $a \geqslant 2$. This is a height 1 prime of infinite projective dimension, and R/Q is strongly F-regular. Statement b) would say $Q^{(n)} = Q^n$ for all n, but this is false. On the other hand, d) guarantees that $J^n Q^{(n)} \subseteq Q^n$. In fact, one can show that the best possible value of α such that $J^{\lceil \alpha n \rceil} Q^{(n)} \subseteq Q^n$ is $\alpha = \frac{a-1}{a}$.*

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Some extensions of the Ananyan–Hochster principle

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(joint work with Daniel Erman, Andrew Snowden)

In response to a question posed by Stillman, Ananyan and Hochster proved the following theorem [1]:

Theorem 1 (Ananyan–Hochster). *Let $d_1, \dots, d_r$ be positive integers. There exists a bound $N(d_1, \dots, d_r)$ such that any ideal in a polynomial ring over a field generated by homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$ has projective dimension at most $N(d_1, \dots, d_r)$.*

The main content is that the bound $N(d_1, \dots, d_r)$ does not depend on the number of variables in the polynomial ring. The main idea behind the proof is that once the degrees $d_1, \dots, d_r$ are fixed, polynomials behave as if they are defined in a bounded number of variables.

More precisely, define the **strength** of a homogeneous polynomial f to be the minimal k such that there is an expression $f = g_1h_1 + \cdots + g_kh_k$ where $\deg(g_i) < \deg(f)$ and $\deg(h_i) < \deg(f)$. The **collective strength** of a vector space of polynomials is the minimum strength of a nonzero homogeneous polynomial contained in it. Using this, one gets the existence of "small subalgebras":

Theorem 2 (Ananyan–Hochster). *Let $d_1, \dots, d_r$ be positive integers. There exists a bound $M(d_1, \dots, d_r)$ such that any sequence of homogeneous polynomials $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$ is contained in a subring generated by a regular sequence of length at most $M(d_1, \dots, d_r)$.*

This immediately implies the first theorem: Hilbert syzygy theorem implies that the projective dimension of $(f_1, \dots, f_r)$ over the subring is at most $M(d_1, \dots, d_r)$; by flatness, the projective dimension can be computed either over the subring or the original ring.

We extend this idea to two other settings. The first concerns Hartshorne's conjecture, which states that any smooth subvariety X of $\mathbf{P}^n$ of codimension $< n/3$ is a complete intersection. One can show using the ideas above that if X is equidimensional, then there exists a bound depending only on its codimension and degree such that if its singular locus has codimension greater than this bound, then X must be a complete intersection. The outline is as follows:

- When the codimension and degree are fixed, there is a finite list of possibilities $(d_1, \dots, d_r)$ such that the ideal of X is generated by homogeneous polynomials of these degrees [2].

- Fixing the degrees $(d_1, \dots, d_r)$ now, an elementary argument with Jacobian matrices shows that forcing the singular locus to have high codimension forces the strength of the polynomials to be high as well.
- Rearranging the generators if necessary, we may assume that the strength is weakly decreasing and that the strength of f_k is the collective strength of the span of $f_1, \dots, f_k$. If c is codimension of X, this implies that $f_1, \dots, f_c$ generates a regular sequence if the strength is sufficiently large.
- If the locus cut out by $f_1, \dots, f_c$ has multiple irreducible components, then its singular locus has codimension at most $2c$ since the intersection of two components gives singular points. Hence, if we assume that the codimension is larger than $2c$, the ideal generated by $f_1, \dots, f_c$ must be prime. However, if $k > c$, f_{c+1} is a nonzerodivisor on this complete intersection which contradicts that X has codimension exactly c. Hence X is a complete intersection defined by $f_1, \dots, f_c$.

An explicit bound can be found in [4].

Our second extension concerns the cohomology tables of coherent sheaves $\mathcal{E}$ on projective space. The bound on projective dimension in the Ananyan–Hochster theorem in fact implies that once the initial degrees are fixed, there are only a finite set of possibilities for the graded Betti numbers of the ideal. Cohomology tables are similar to graded Betti tables in many ways, and record the numbers $\dim \mathrm{H}^i(\mathcal{E}(j))$. More precisely, these dimensions are the graded Betti numbers of the associated Tate resolution of $\mathcal{E}$ (up to a re-indexing), which is a doubly-infinite minimal complex over an exterior algebra that complete encodes the data of $\mathcal{E}$. Define the kth column of a cohomology table to be the ranks of the graded components of the kth term in this complex. We prove the following analogue of finiteness of Betti tables in the polynomial ring case:

Theorem 3. *Fix the values of the kth and $(k+1)$st columns for some k. Then there are only finitely many ways to complete this data to the cohomology table of a coherent sheaf on some projective space.*

Again, the main content is that this finiteness does not depend on a fixed projective space. This follows from an analogue of the existence of small subalgebras. More precisely:

Theorem 4. *Fix the values of the kth and $(k+1)$st columns for some k. There exist bounds k_0 and n_0 such that, for any coherent sheaf $\mathcal{E}$ on $\mathbf{P}^n$ (for any n) whose cohomology table has these columns, we have:*

- *The regularity of $\mathcal{E}$ is at most k_0, and*
- *$\mathcal{E}$ is the pushforward along some linear map $\mathbf{P}^{n_0} \to \mathbf{P}^n$.*

The main input is a recent noetherianity result of Draisma [3] for polynomial functors (we just need the case of finite sums of exterior powers).

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Contact loci and Hironaka's order

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(joint work with Santiago Encinas, Beatriz Pascual-Escudero)

Constructive resolution of singularities. Hironaka's Theorem on resolution of singularities is existencial (see [8]). A *constructive resolution of singularities of an algebraic variety* X consists on giving a procedure to select a sequence of blow ups at regular centers:

$$X = X_0 \leftarrow X_1 \leftarrow \ldots \leftarrow X_n \tag{1}$$

so that $X \leftarrow X_n$ is a resolution of singularities of X. Algorithms of resolution of singularities for varieties in characteristic zero were first given in [1], [15] and [14].

The sellection of the centers in sequence (1) is made via the so called *resolution functions*. These are upper semicontinuous funtions $f_{X_i} : X_i \to (\Lambda, \geq)$ whose maximum value $\mathrm{Max}(f_{X_i})$ is achieved at a regular (closed) subscheme which defines the center of the blow up $X_i \to X_{i+1}$. The resolution functions are defined so that, first, they are constant if and only if the variety is regular and, second, so that at each step in (1), $\mathrm{Max}(f_{X_i}) > \mathrm{Max}(f_{X_{i+1}})$.

For a singular point $\xi \in X$, $f_X(\xi)$ is ussually defined as a sequence of rational numbers called *resolution invariants*. The first coordinates of $f_X(\xi)$ can be defined in terms of the Hilbert Samuel function (in the spirit of [9]) or of the multiplicity (as shown in [16]). In the latter case, the next piece of (relevant) information is what we call *Hironaka's order function in dimension* d, $\mathrm{ord}_X^{(d)}(\xi)$, where d is the dimension of X.

The definition of Hironaka's order function at a singular point requires either the use of local (étale) embeddings of X into smooth varieties, or the construction of suitable finite projections to smooth varieties (again in an étale neighborhood of a singular point of X). It is quite natural to ask whether this function can be defined in an intrinsic way. This has motivated our interest in the study of arc spaces.

Arc spaces and Nash multiplicity sequences. The work of J. Nash on the theory of arc spaces was in part motivated by Hironaka's Theorem (cf. [12]). A resolution of singularities of an algebraic variety X may not be unique; one may wonder how much information about the process of resolution can be read from the space of arcs of X, $\mathcal{L}(X)$. Arcs and singularities have been widely studied in a large number of paper by several authors.

As indicated above, we are interested in finding connections between the arc space of a singular algebraic variety and its resolution invariants. In [2] and [3], jointly with S. Encinas and B. Pascual-Escudero, we have found that $\mathrm{ord}_X^{(d)}$ can be read from the *Nash multiplicity sequences*, introduced by M. Lejeune-Jalabert in [11] and generalized afterwards by M. Hickel (see [7]). See also [13].

Suppose X is a singular variety of maximum multiplicity $m > 1$. Then, given a point $\xi \in \mathrm{Sing}(X)$ of multiplicity m, and an arc φ with center ξ, the *sequence of Nash multiplicities of* φ is a non-incressing sequence of integers,

$$m = m_0 \geq m_1 \geq m_2 \geq \dots \tag{2}$$

where $m_0 = m$ is the multiplicity at the point ξ, and the rest of the terms in the sequence can be interpreted as a *refinement of the ordinary multiplicity at* ξ *along the arc* φ.

Currently we have been interested in trying to understand the set of arcs centered at ξ whose Nash multiplicity sequences determine $\mathrm{ord}_X^{(d)}(\xi)$. In [4] we explore this question by considering the *contact loci with a singular closed point* ξ *in a variety* X, say $\mathrm{Cont}^{\geq n}(\mathfrak{m}_\xi)$ (this refers to the set of arcs that have order at least n at the maximal ideal $\mathfrak{m}_\xi$ of ξ for $n \in \mathbb{N}$; see [6], [5] and [10] where the structure of these sets is studied). Among other problems, we study the fat irreducible components of $\mathrm{Cont}^{\geq n}(\mathfrak{m}_\xi)$ and find conditions under which the Nash multiplicity sequences of their generic points determine $\mathrm{ord}_X^{(d)}(\xi)$.

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Asymptotic properties of differential operators around a singularity

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(joint work with Jack Jeffries and Luis Núñez Betancourt)

For a local algebra R over a field, we study the decomposition of the module of principal parts. A free summand of the nth module of principal parts is essentially the same as a differential operator E of order $\leq n$ with the property that the partial differential equation $E(f) = 1$ has a solution $f \in R$. Such differential operators are called *unitary operators*. The asymptotic behavior of the seize of the free part gives a measure for the singularity represented by R, which we call *differential signature*.

Motivation for this comes from the F-*signature*, which is an invariant in positive characteristic defined by looking at the asymptotic decomposition of the Frobenius. The study of the decomposition of the ring R considered as an R-module via powers of the Frobenius goes back to Kunz, with important contributions by Smith and van den Bergh, Seibert, Watanabe and Yoshida. The notion of F-signature was introduced by Huneke-Leuschke, its existence was proved by Tucker and the relation to strong F-regularity was proved by Aberbach-Leuschke. A serious problem in this theory is that in a relative arithmetic situation it is not possible to compare the Frobenius in different characteristics dierectly and there is no replacement for it in characteristic zero.

The differential signature is an attempt to give a characteristic-free variant for the F-signature. It does not give always the same numbers, but the positivity holds for the same class of singularities (at least if we assume F-pure). We show that the positivity of the differential signature implies for a standard-graded isolated Gorenstein singularity that the a-invariant is negative and that it has rational singularities. We compute this invariant for invariant rings, toric monoid rings, determinantal rings (in this case the F-signature is not known). We also compute this invariant for quadrics, where we use an algorithmic approach for the module of principal part and methods from vector bundle theory.

Witt differentials and the h-topology

Veronika Ertl
(joint work with Lance E. Miller)

The idea to use differential forms to obtain numerical invariants of algebraic varieties dates back to Picard and Lefschetz. Meanwhile sheaves of differential forms have become an important tool to study local and global properties of algebraic varieties and schemes. However it is well-known that they are not well behaved in the singular case even in characteristic 0. To get around the problems which occur here, several competing generalizations of differential forms and of the de Rham complex have been proposed.

In [3] Huber and Jörder introduce a new player using Voevodsky's h-topology [6]. It turns out that in characteristic 0 the h-sheafification of the sheaves Ω^n, $n \geq 0$, of differential forms provides a conceptual extension to singular varieties. The h-topology is useful in this context because it contains not only modifications, but also alterations, so in some sense resolutions of singularities are built in.

In characteristic $p > 0$ the h-topology encounters a number of challenges. The mein reason is that it contains inseparable morphisms. One consequence is for example that the basic Kähler differentials become zero under h-sheafification. However, it is possible to circumvent these difficulties to some extent with more subtle sheaf theoretic methods by Huber, Kebekus, and Kelly, [4].

An alternative to address these issues is to change the coefficient sheaves and so to speak lift them to characteristic 0 and to consider p-adic cohomolgy theories instead. If a variety lifts only locally, one often turns towards crystalline cohomology or one of its variants. In [5] Illusie introduced a complex of étale sheaves, called the de Rham–Witt complex, which computes crystalline cohomology on smooth schemes.

It is reasonable to expect that on local lifts, one can take advantage of the properties of the h-topology that allowed Huber and Jörder to develop their theory. The hope is to extend the program described in [3] and [4] to the de Rham–Witt complex in order to obtain an equally conceptual approach to the study of singular varieties in characteristic p.

To lay the base for this, we are especially interested in descent results for (rational) Witt differentials. An optimal result would be a cohomological descent statement analogous to [3, Cor. 6.5]. If one assumes resolution of singularities in positive characteristic we obtain indeed the following statement.

Theorem 1 (Ertl–Miller). *Let k be a perfect field of characteristic $p > 0$ and let X be a regular scheme over k. Then one has for all $i, n \geq 0$ isomorphisms*

$$H^i_{\mathrm{Zar}}(X, W\Omega^n_{\mathbb{Q}}) \cong H^i_{\mathrm{h}}(X, W\Omega^n_{\mathbb{Q},\mathrm{h}}).$$

Without the assumption of resolution of singularities such a statement is only known for Serre's Witt vector cohomology [1]. For Witt differential forms of arbitrary degree full cohomological descent without resolutions of singularities remains an unsurmountable challenge.

However, using techniques from [4] we were able to obtain the following descent result for Witt differentials of any degree without resolution of singularities.

Proposition 2 (Ertl–Miller). *Let k be a perfect field of positive characteristic p. For any regular scheme X over k, the change of topology map induces isomorphisms*

$$W\Omega^n_{\mathbb{Q}}(X) \cong W\Omega^n_{\mathbb{Q},\mathrm{h}}(X)$$

for all $n \geq 0$.

The proof relies on a construction introduced in [4] which allows one to extend a presheaf $\mathcal{F}$ on regular schemes to a presheaf $\mathcal{F}_{\mathrm{dvr}}$ on arbitrary schemes.

An important insight by Huber, Kebekus and Kelly is that this construction extends sheaves for topologies coarser than the étale topology. More precisely, if $\mathcal{F}$ is for example an étale sheaf on regular schemes, $\mathcal{F}_{\mathrm{dvr}}$ is an eh-sheaf on $\mathsf{Sch}(k)$. However, to account for the finite morphisms in the h-topology we were lead to show a similar result for the qfh-topology. Concretely, this means, that in order to show that $\mathcal{F}_{\mathrm{dvr}}$ is a qfh-sheaf, it suffices to show that $\mathcal{F}$ is a qfh-sheaf on a certain smaller category. In prticular, we obtain the following.

Proposition 3 (Ertl–Miller). *For a perfect field k of characteristic $p > 0$, the extension $W\Omega^n_{\mathbb{Q},\mathrm{dvr}}$ is a* qfh-*sheaf on* $\mathsf{Sch}(k)$.

The proof of this explicitly uses properties of Witt differentials, notably the existence of a Frobenius and a Verschiebung map. As eh- and qfh-topology generate the h-topology, this result implies that $W\Omega^n_{\mathbb{Q},\mathrm{dvr}}$ is an h-sheaf on $\mathsf{Sch}(k)$.

Another ingredient for our main theorem is the statement that if $\mathcal{F}_{\mathrm{dvr}}$ is an h-sheaf, then it has no topological torsion. In particular this is true for $W\Omega^n_{\mathbb{Q},\mathrm{dvr}}$. As a consequence the natural map $W\Omega^n_{\mathbb{Q},\mathrm{h}}(X) \to W\Omega^n_{\mathbb{Q},\mathrm{dvr}}(X)$ is injective. Our main result then follows by a diagram chase.

We can draw several expected consequences from the h-descent. This includes analogues for the rational Witt differentials of results from [3] on Kähler differentials.

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Lech's inequality and the Stückrad–Vogel conjecture

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(joint work with Patricia Klein, Pham Hung Quy, Ilya Smirnov, Yongwei Yao)

We start by reinterpreting a classical inequality of Lech [4]:

Theorem 1 (Lech)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension d. Then*

$$\inf_{\sqrt{I}=\mathfrak{m}} \{\frac{l(R/I)}{e(I)}\} \geq \frac{1}{d!e(R)}.$$

It is thus quite natural to ask whether $\sup_{\sqrt{I}=\mathfrak{m}}\{\frac{l(R/I)}{e(I)}\}$ is finite. If we let $R = k[[x, y, z]]/(xy, xz)$ and $I = (x^n, y, z)$, then it is easy to see that $l(R/I) = n$ while $e(I) = 1$. Thus the question has a negative answer in general. However, we notice that R is not equidimensional. Stückrad and Vogel [5] conjectured that this is the only obstruction for the question to have a negative answer:

Conjecture 2 (Stückrad-Vogel)**.** *Let $(R, \mathfrak{m})$ be a Noetherian local ring. Then*

$$\sup_{\sqrt{I}=\mathfrak{m}} \{\frac{l(R/I)}{e(I)}\} < \infty$$

if and only if $\widehat{R}$ is equidimensional.

In [5] they proved the "only if" direction of the conjecture. The main result in [3] settled the "if" direction of the conjecture.

Theorem 3 (Klein-Ma-Pham-Smirnov-Yao)**.** *Stückrad-Vogel's conjecture holds.*

Before we discuss the proof we point out two applications. Fix a Noetherian local ring R such that $\widehat{R}$ is equidimensional. By combing Lech's inequality and the finiteness of Stückrad-Vogel invariant, one can show that the set $\{\frac{l(R.I)}{l(R/\overline{I})}\}_{\sqrt{I}=\mathfrak{m}}$ is bounded above. Moreover, we can show that for every $\epsilon > 0$ there exists t_0 such that for all $t > t_0$ and all system of parameters $x_1, \ldots, x_d$ of R, the ratio $\frac{l(H_i(x_1^t,\ldots,x_n^t;R))}{l(H_0(x_1^t,\ldots,x_n^t;R))}$ is $< \epsilon$ for all $i > 0$. The key point is that t_0 is independent of the system of parameters $x_1, \ldots, x_d$ (so this is a strong uniform convergence result). We refer to [3] for more details.

We briefly explain the proof of the Stückrad-Vogel conjecture. Our main tool is to use Vasconcelos' homological degree [6] (see also [1]):

Definition 4 (Vasconcelos)**.** Let $(R, \mathfrak{m})$ be a complete local ring and let I be an $\mathfrak{m}$-primary ideal of R. Then for every finitely generated R-module M we define

$$hdeg(I, M) = e(I, M) + \sum_{i=0}^{\dim M-1} \binom{\dim M - 1}{i} hdeg(I, (H^i_{\mathfrak{m}}(M))^\vee).$$

Note that this is well defined recursively, because $\dim(H^i_{\mathfrak{m}}(M))^\vee < \dim M$ for every $0 \leq i \leq \dim M - 1$. This is one typical example of an "extended degree" (see [6, 1]). In particular, it satisfies the property that $hdeg(I,R) \geq l(R/I)$. We make an elementary but important observation that

$$hdeg(I,R) = \sum_{P\in\Lambda} e(I,R/P)$$

where Λ is a finite set of prime ideals of R depending only on R, but not on I.

In order to prove Stückrad-Vogel's conjecture, we can assume that R is a complete local domain by [5]. Using the properties of homological degree discussed above, it is enough to show that for every fixed prime ideal P, the set

$$\{\frac{e(I,R/P)}{e(I)}\}_{\sqrt{I}=\mathfrak{m}}$$

is bounded above. Now we pick a nonzero element $x \in P$ and pick a minimal prime Q of x contained in P (these choices depend only on R and P but not on I), and we write

$$\frac{e(I,R/P)}{e(I)} = \frac{e(I,R/P)}{e(I,R/Q)} \cdot \frac{e(I,R/Q)}{e(I,R/(x))} \cdot \frac{e(I,R/(x))}{e(I)}.$$

The first factor is bounded above independent of I by induction on $\dim R$, the second one is bounded by 1 by the associativity formula for multiplicities, and finally the last factor is bounded above independent of I by the uniform Artin-Rees number of $(x) \subseteq R$, this follows from a simple computation and using Huneke's uniform Artin-Rees lemma [2]. Putting all these together we complete the proof of Stückrad-Vogel's conjecture.

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On the Bernstein-Gel'fand-Gel'fand correspondence for absolutely Koszul Gorenstein algebras

Vincent Gélinas

1. History

Let k be a field, $S = k[x_0, \dots, x_n]$ a polynomial algebra and $\Lambda = \bigwedge^*(y_0, \dots, y_n)$ be the exterior algebra on dual variables. The classical BGG correspondence states:

Theorem (Bernstein-Gel'fand-Gel'fand '78)**.** There is an equivalence of triangulated equivalence

$$\mathrm{F} : \underline{\mathsf{grmod}}\,\Lambda \cong \mathrm{D}^b(\mathsf{coh}\,\mathbb{P}^n).$$

The algebras S and Λ are Koszul dual, so that $\mathrm{Ext}^*_S(k,k) \cong \Lambda$ and $\mathrm{Ext}^*_\Lambda(k,k) \cong S$. More generally let R be a standard graded, two-sided Noetherian, Gorenstein k-algebra (meaning $\mathsf{inj.dim}\,{}_RR < \infty$ and $\mathsf{inj.dim}\,R_R < \infty$ if R is non-commutative), and let $\mathsf{grmod}\,R$ stand for finitely presented graded R-modules. If we let

$$\mathrm{MCM}^{\mathsf{gr}}(R) := \{M \in \mathsf{grmod}\,R \mid \mathrm{Ext}^i_R(M,R) = 0 \text{ for } i > 0\}$$

be the category of finitely presented graded MCM modules, and define

$$\underline{\mathrm{MCM}}^{\mathsf{gr}}(R) := \mathrm{MCM}^{\mathsf{gr}}(R)/\{\text{projective modules}\}$$
$$\mathsf{qgr}\,R := \mathsf{grmod}\,R/\{\text{finite-length modules}\}$$

then the above becomes $\mathrm{F} : \underline{\mathrm{MCM}}^{\mathsf{gr}}(\Lambda) \cong \mathrm{D}^b(\mathsf{qgr}\,S)$. More generally:

Theorem (Buchweitz '87, [1])**.** Let R, E be Koszul dual, two-sided Noetherian Gorenstein algebras. Then there are equivalences

$$\mathrm{F}_1 : \underline{\mathrm{MCM}}^{\mathsf{gr}}(R) \cong \mathrm{D}^b(\mathsf{qgr}\,E)$$
$$\mathrm{F}_2 : \underline{\mathrm{MCM}}^{\mathsf{gr}}(E) \cong \mathrm{D}^b(\mathsf{qgr}\,R).$$

When R is commutative, this applies to complete intersections of quadrics $R = S/(\mathbf{q})$, with $\mathbf{q} = (q_1, \dots, q_c)$, in which case the Koszul dual is a graded Clifford algebra over the algebra of Gulliksen operators $E = \mathrm{Ext}^*_{S/(\mathbf{q})}(k,k) = \mathcal{C}l_{k[\chi_1,\dots,\chi_c]}(\mathbf{q})$. Conversely, having a Noetherian Yoneda algebra characterises complete intersections by Bøgvad-Halperin, and so the proof does not reach beyond this case.

2. Beyond complete intersections of quadrics

When R is either a complete intersection or its Koszul dual, one first corrects the statement to a contravariant equivalence by precomposing with a duality D

$$\widetilde{\mathrm{F}} = \mathrm{F} \circ \mathrm{D} : \underline{\mathrm{MCM}}^{\mathsf{gr}}(R)^{op} \cong \mathrm{D}^b(\mathsf{qgr}\,E)$$

so that $\widetilde{\mathrm{F}}^{-1}(\mathsf{qgr}\,E) \cong \mathcal{H}^{\mathsf{lin}}(R) \subseteq \underline{\mathrm{MCM}}^{\mathsf{gr}}(R)$ consists of *eventually linear* modules:

$$\mathcal{H}^{\mathsf{lin}}(R) := \{M \in \underline{\mathrm{MCM}}^{\mathsf{gr}}(R) \mid \mathrm{Tor}^R_i(M,k)_{i+j} = 0 \text{ for } j \neq 0 \text{ whenever } i \gg 0\}.$$

In general $\mathcal{H}^{\mathsf{lin}}(R) \subseteq \underline{\mathrm{MCM}}^{\mathsf{gr}}(R)$ need not be the heart of a bounded t-structure.

In [4], Herzog-Iyengar introduced the class of absolutely Koszul algebras in links with the rationality problem for Poincaré series of modules. These are the algebras over which minimal free resolutions are eventually dominated by their linear parts, in the sense of Eisenbud-Fløystad-Schreyer. The following is shown in [3]:

Theorem A. Let R be Koszul Gorenstein. Then $\mathcal{H}^{\mathsf{lin}}(R)$ is the heart of a bounded t-structure if and only if R is absolutely Koszul. When this holds, there is an equivalence of triangulated categories

$$\mathrm{D}^b(\mathcal{H}^{\mathsf{lin}}(R)) \xrightarrow{\cong} \underline{\mathrm{MCM}}^{\mathsf{gr}}(R).$$

Theorem B. Let R be absolutely Koszul Gorenstein and $E = \mathrm{Ext}^*_R(k, k)$. Then E is graded coherent, and there is an equivalence $\mathcal{H}^{\mathsf{lin}}(R)^{op} \xrightarrow{\cong} \mathsf{qgr}\, E$.

Theorem C (BGG Correspondence)**.** Let R be absolutely Koszul Gorenstein. Then there is an equivalence of triangulated categories

$$\underline{\mathrm{MCM}}^{\mathsf{gr}}(R)^{op} \cong \mathrm{D}^b(\mathsf{qgr}\, E)$$

restricting to $\mathcal{H}^{\mathsf{lin}}(R)^{op} \cong \mathsf{qgr}\, E$.

Theorem (Serre Duality)**.** Let R be absolutely Koszul Gorenstein with isolated singularities, and with canonical module $\omega_R \cong R(a)$. Let $\tau_\omega(-) = \Omega^a(-)(a)$ and $\nu_R = \dim R - 1 + a$. Then for any $M, N \in \mathcal{H}^{\mathsf{lin}}(R)$, we have natural isomorphisms

$$\mathrm{Ext}^i_{\mathcal{H}^{\mathsf{lin}}(R)}(M, \tau_w N) \cong \mathrm{Ext}^{\nu_R - i}_{\mathcal{H}^{\mathsf{lin}}(R)}(N, M)^*$$

and so $\mathsf{gldim}\, \mathcal{H}^{\mathsf{lin}}(R) = \nu_R < \infty$. Moreover, when $\nu_R \leq 1$ then any indecomposable MCM modules is in $\mathcal{H}^{\mathsf{lin}}(R)$ up to degree shift.

For R Koszul Gorenstein, R is absolutely Koszul if:

- R is generalised Golod (e.g. R is the image of a complete intersection under a Golod map, R has embedding codimension ≤ 4, or whenever $\nu_R \leq 1$);
- There is a map $\varphi : R \to S$ of finite flat dimension to S absolutely Koszul;
- R is the homogeneous coordinate ring (over $k = \overline{k}$) of an elliptic normal curve $E \subseteq \mathbb{P}^{d-1}$ of degree $d \geq 4$, a del Pezzo surface $S \subseteq \mathbb{P}^d$ of degree $d \geq 4$, or (in $\mathsf{char}\, k = 0$) a canonical curve $C \subseteq \mathbb{P}^{g-1}$ satisfying Petri's Theorem, a K3 surface in an appropriate embedding, or many others.

The last two points are due to Conca-Iyengar-Nguyen-Römer [2]. The second provides the following Hyperplane Section Principle:

Theorem ([2])**.** Let $X \subseteq \mathbb{P}^n$ be projective and $X \cap H$ a general hyperplane section. If the homogeneous coordinate ring $R_{X \cap H}$ is absolutely Koszul Gorenstein, then so is R_X. In this case, we have $\nu_{R_{X\cap H}} = \nu_{R_X}$.

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Noncommutative resolutions and global dimension of rings of differential operators of toric rings

Eleonore Faber
(joint work with Greg Muller, Karen E. Smith)

Let R be the coordinate ring of an affine toric variety over a field k of arbitrary characteristic. The module $R^{1/q}$ of q-th roots of R, where q is a positive integer, is then the direct sum of so-called conic modules. In this talk we are interested in homological properties of the endomorphism ring $\mathrm{End}_R(R^{1/q})$, in particular its global dimension. If k is a perfect field of prime characteristic p and $q = p^e$, then the ring of differential operators $D_k(R)$ is a direct limit of $\mathrm{End}_R(R^{1/q})$ and this description allows us to make statements about the global dimension of the non-noetherian ring $D_k(R)$.

1. A noncommutative desingularization of a commutative ring R is a certain noncommutative R-algebra of finite global dimension which can in turn be viewed as a potential analogue of a resolution of singularities of $\mathrm{Spec}(R)$. Van den Bergh introduced noncommutative crepant resolutions (=NCCRs) to interpret Bridgeland's solution to the conjecture by Bondal and Orlov on the derived invariance of flops in 2004. A *noncommutative resolution* of singularities (NCR) of a commutative noetherian ring R (or the scheme $\mathrm{Spec}(R)$) is defined to be an associative R-algebra $A = \mathrm{End}_R M$, where M is a finitely generated R-module of full support and A has finite global dimension. This notion was introduced 2015 by Dao–Iyama–Takahashi–Vial. For A to be *crepant*, that is, a NCCR, one needs additionally that M is torsion-free and A is a non-singular order.

2. Let now R be the coordinate ring of an affine normal toric variety, that is, $R = k[C \cap M]$, where $M \cong \mathbb{Z}^d$ is a lattice and C is assumed to be a full dimensional rational polyhedral cone in the vector space $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Let $R^{1/q} = \mathrm{Span}\{x^{m/q} : \frac{m}{q} \in C \cap q^{-1}M\}$. The main result of [FMS18] is the following

Theorem 3. *The global dimension of* $\mathrm{End}_R(R^{1/q})$*, for q large enough, is equal to the Krull-dimension of R. In particular,* $\mathrm{End}_R(R^{1/q})$ *is a NCR of R.*

The finiteness of the global dimension also follows from Špenko and Van den Bergh [ŠVdB17b, 1.3.6], who proved with a much more general machinery the existence of NCRs for reductive quotient singularities, albeit with less explicit bounds on the global dimension.

4. Our proof of Theorem 3 is combinatorial and constructs minimal projective resolutions of the M-graded simples of an endomorphism ring $\mathrm{End}_R(\mathbb{A})$, where $\mathbb{A}$ is a complete sum of conic modules (definitions see below).

The key observation, which dates back to Bruns–Gubeladze [BG03], is that $R^{1/q}$ is a direct sum of *conic* modules A_Δ, which can be described as follows: Let $v \in M_\mathbb{R}$, then the corresponding conic module is $A_v := \mathrm{Span}\{x^m : m \in M \cap (C + v)\}$. Conic modules are maximal Cohen–Macaulay R-modules of rank 1 and can be parametrized by *chambers of constancy* $\Delta := \{w \in M_\mathbb{R} : A_v = A_w\}$ in $M_\mathbb{R}$. Each chamber of constancy Δ is a disjoint union of open polyhedral cells. Together, ranging over all chambers of constancy, these cells define a CW decomposition of $M_\mathbb{R}$.
Since there are only finitely many isomorphism classes of conic modules, see [Bru05], we may index them by chambers of constancy (that is, $A_v = A_\Delta$ for $v \in \Delta$). For each chamber of constancy Δ, we use the combinatorics of the faces of Δ to construct a chain complex $K^\bullet_\Delta$ of conic modules. We prove an Acyclicity Lemma for these conic complexes: the complex $\mathrm{Hom}_R(A_{\Delta'}, K^\bullet_\Delta)$ is either acyclic or a resolution of the ground field k, depending on whether or not $A_\Delta \cong A_{\Delta'}$.
These results are then used as follows: we call a direct sum $\mathbb{A}$ of conic modules *complete* if every conic R-module is isomorphic to a direct summand of $\mathbb{A}$. An example of a complete conic module is $R^{1/q}$ for q large enough, see [Bru05] and also cf. [SVdB97]. The Acyclicity Lemma implies that the complex $\mathrm{Hom}_R(\mathbb{A}, K^\bullet_\Delta)$ is a finite projective resolution of a simple $\mathrm{End}_R(\mathbb{A})$-module. Using standard arguments on global dimension of noetherian rings, we show that every finitely generated $\mathrm{End}_R(\mathbb{A})$-module has finite global dimension, and thus $\mathrm{gl.dim}(\mathrm{End}_R(\mathbb{A}))$ is also finite.

5. The combinatorial structure of the conic modules allows us to show

Corollary 6. *The ring* $\mathrm{End}_R(\mathbb{A})$ *is a NCCR if and only if* $\mathrm{Spec}(R)$ *is a simplicial toric variety.*

Finally, if k is perfect of prime characteristic p, then $D_k(R) = \bigcup_{e \in \mathbb{N}} \mathrm{End}_{R^{p^e}}(R)$. The Frobenius map $F^e : R \to R^{p^e}$ induces an isomorphism $\mathrm{End}_{R^{p^e}}(R) \cong \mathrm{End}_R(R^{1/p^e})$. Combining this fact with Theorem 3 and a result about global dimension of direct limits [Ber58], if follows that in this case

$$\mathrm{gl.dim}(D_k(R)) \leq \lim_{e \in \mathbb{N}} (\mathrm{End}_{R^{p^e}}(R)) + 1 = \dim(R) + 1 \ .$$

It is not clear whether this bound is sharp.

7. Further research questions: in which case is there some (non-complete) direct sum $\mathbb{B}$ of conic modules such that $\mathrm{End}_R(\mathbb{B})$ is a NC(C)R?
More generally one can ask what conditions on a commutative ring R ensure that $D_k(R)$ has finite global dimension. And also: What conditions on R imply that the global dimension of $\mathrm{End}_R(R^{1/q})$, $q \gg 0$, is finite?
In prime characteristic, a natural candidate for $\mathrm{gl.dim}(D_k(R)) < \infty$ seem to be rings R of finite F-representation type, a class of rings including toric rings.

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Characteristic polyhedra without completion

Bernd Schober

(joint work with Vincent Cossart)

Let R be a regular local ring, $J \subset R$ be a non-zero ideal and $(u) = (u_1, \ldots, u_e)$ be a R-regular sequence. To this situation, Hironaka introduced the *characteristic polyhedron* $\Delta(J; u)$ [9], which is a useful tool in resolution of singularities providing a convex geometric viewpoint on $X = \operatorname{Spec}(R/J)$. For example, it appears in the resolution of excellent surfaces ([10], [3], [8]) and of (arithmetic) threefolds ([4], [5]), or in any dimension over fields of characteristic zero ([12]), or in the characterization of quasi-ordinary singularities via overweight deformations ([11]).

For simplicity, we restrict our attention to principal ideals $J = \langle f \rangle$, for $f \in R$ with $f \notin \langle u \rangle$. (We refer to [7] for the case of any ideals). Let $(y) = (y_1, \ldots, y_r)$ be a system of elements in R extending (u) to a regular system of parameters (r.s.p.) for R. We fix a finite expansion of f in R,

$$f = \sum_{(A,B) \in \mathbb{Z}^e_{\geq 0} \times \mathbb{Z}^r_{\geq 0}} C_{A,B} u^A y^B, \quad \text{with } C_{A,B} \in R^\times \cup \{0\}$$

(Here, we abbreviate $u^A = u_1^{A_1} \cdots u_e^{A_e}$ and $y^B = y_1^{B_1} \cdots y_r^{B_r}$). We define

$$m := m(f) := \min\{|B| = B_1 + \ldots + B_r \mid C_{0,B} \neq 0\}$$

and the symbol $\Delta(f; u; y)$ denotes the convex hull of the following subset of $\mathbb{R}^e_{\geq 0}$,

$$\left\{ \frac{A}{m - |B|} + \mathbb{R}^e_{\geq 0} \mid C_{A,B} \neq 0 \text{ and } |B| < m \right\}.$$

Using this, we can give an ad hoc definition of the characteristic polyhedron as

$$\Delta(f; u) := \bigcap_{(\widehat{y}) \,:\, (u, \widehat{y}) \text{ is r.s.p. of } \widehat{R}} \Delta(f; u; \widehat{y}) \quad \subset \mathbb{R}^e_{\geq 0}.$$

Let $\mathrm{in}_0(f) := \sum_{B:|B|=m} \overline{C_{A,B}} Y^B \in k[Y]$, where $\overline{C_{A,B}}$ denotes the image of $C_{A,B}$ in the residue field $k := R/\langle u, y\rangle$. An important hypothesis when explicitly computing $\Delta(f;u)$ is:

$$\text{There exists no proper } k\text{-submodule } T \subsetneq k[Y_1,\dots,Y_r]_1 \text{ such that } \mathrm{in}_0(f) \in k[T]. \tag{$*$}$$

Here, we consider $k[Y]$ as a standard graded ring and $k[Y]_1$ is the part homogeneous of degree 1. As a consequence of [9] Theorems (3.17) and (4.8) one gets

Theorem 1 (Hironaka). *Suppose* $(*)$ *holds. There exist* $\widehat{\varphi}_1,\dots,\widehat{\varphi}_r \in \langle u\rangle\widehat{R}$ *such that, for* $\widehat{y}_j := y_j + \widehat{\varphi}_j$, $1 \le j \le r$, *we have* $\Delta(f;u;\widehat{y}) = \Delta(f;u)$.

The main result addresses the question when it is possible to avoid passing to the completion.

Theorem 2 (Cossart-Piltant, Cossart-S.). *Suppose* $(*)$ *holds,* R *is excellent, and one of the following condition holds:*

(1) $\mathrm{char}(k) \ge \dim(X)/2 + 1$.
(2) $\dim(\mathrm{Rid}(\mathrm{in}_0(f))) = \dim(\mathrm{Dir}(\mathrm{in}_0(f)))$.
(3) $R = S[y]_{\langle u,y\rangle}$, *for* S *an excellent regular local ring, and* $f \in S[y]$ *with* $\deg_y(f) = m(f)$.

Then there exit $(z) = (z_1,\dots,z_r)$ *in* R *such that* (u,z) *is a r.s.p. for* R, $\langle z\rangle\widehat{R} = \langle\widehat{y}\rangle$, *and* $\Delta(f;u;z) = \Delta(f;u)$.

Here, $\mathrm{Rid}(\mathrm{in}_0(f))$ (resp. $\mathrm{Dir}(\mathrm{in}_0(f))$) is the *ridge* (resp. *directrix*) of the cone defined by the homogeneous polynomial $\mathrm{in}_0(f)$, which are objects encoding information on the singularity and its behavior under blowing ups, introduced and studied by Hironaka and Giraud. We refer to [1] for more details.

In [6], Cossart and Piltant proved the result for principal ideals with $r = 1$ (which implies (2)), while [7] is neither restricted to principal ideals nor to $r = 1$.

Strategy for the proof. First, we reduce to the case of an empty characteristic polyhedron. Suppose $\Delta(f;u) \neq \varnothing$. Then, there are finitely many linear forms defining half spaces in $\mathbb{R}^e_{\ge 0}$ whose intersection is $\Delta(f;u)$. Fixing one of these linear forms, say L, the number $\delta_L(f;u;y) := \min\{L(v) \mid v \in \Delta(f;u;y)\}$ measures the difference of $\Delta(f;u;y)$ and $\Delta(f;u)$ with respect to L. By passing to the initial form with respect to L, say $\mathrm{in}_L(f)$, we obtain an element in the graded ring $\mathrm{gr}_L(R)$ whose characteristic polyhedron is empty if the previous difference is non-trivial. If we can find suitable coordinates in $\mathrm{gr}_L(R)$, we can choose lifts $(y^{(1)})$ in R such that $\delta_L(f;u;y^{(1)}) > \delta_L(f;u;y)$ and one can show that the difference to $\Delta(J;u)$ strictly decreased. This procedure does not require any of the assumptions (1)–(3).

If the characteristic polyhedron is empty, the extra assumptions are essential. If (1) or (2) holds, one can deduce that the maximal Hilbert-Samuel locus in the completion is defined by $\langle\widehat{y}\rangle$. As R is excellent, the latter behaves well with respect to completions, which provides the desired elements (z). Under hypothesis (3), we can apply differential operators in the variables (y), which eventually leads to (z).

One may ask whether Theorem 2 is true for any excellent ring. This leads to

Question 3. *Let R be an excellent regular local ring with r.s.p. (u, y). Let $f \in R$ with $f \notin \langle u \rangle$ and $m = m(f)$ be as above. Suppose $(*)$ holds and $f \in \langle \widehat{y} \rangle^m$, for $\widehat{y}_j := y_j + \widehat{\varphi}_j$, with $\widehat{\varphi}_j \in \langle u \rangle \widehat{R}$ and $1 \leq j \leq r$. Do there exist $(z) = (z_1, \ldots, z_r)$ such that (u, z) is a r.s.p. for R, $\langle z \rangle \widehat{R} = \langle \widehat{y} \rangle$, and $f \in \langle z \rangle^m$?*

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Functorial semistable reduction and resolution of morphisms

Michael Temkin
(joint work with Dan Abramovich, Jarosław Włodarczyk)

1. Main results

In a joint project with Abramovich and Wlodarczyk, we construct a functorial resolution of morphisms in characteristic zero, see [2]. Already in the case of varieties, this leads to a new algorithm which is faster than the classical one and possesses better functorial properties, see [1].

1.1. **Historical background.**

1.1.1. *Classical desingularization.* Until our work there was known an essentially unique functorial (or canonical) resolution of singularities, to which we refer as the *classical algorithm.* It is based on (originally non-canonical) method of Hironaka developed further by Giraud, Bierstone-Millman, Villamayor, Wlodarczyk, Kollar, and other experts. They suggested different descriptions, of essentially the same algorithm with certain variations in combinatorial parts. In brief, the main result was

Theorem 1. *For any integral variety Z over a field k of characteristic zero there exists a modification $f\colon Z_{\rm res} \to Z$ such that $Z_{\rm res}$ is smooth. Moreover, the construction is smooth-functorial: if $Z' \to Z$ is smooth, then $Z'_{\rm res} = Z_{\rm res} \times_Z Z'$.*

The proof went by locally embedding Z into a manifold with a boundary (X, E) (i.e. X is smooth and E is an snc divisor) and principalizing the ideal $I_Z \subset \mathcal{O}_X$, i.e. finding a sequence of blow ups $g\colon (X', E') \to \cdots \to (X, E)$ such $g^{-1}(I_Z)$ is invertible with support on E'. Hironaka showed that principalization easily implies resolution, and also implies that on can resolve a closed subset $T \subsetneq Z$ to an snc divisor $T' = f^{-1}(T)$.

1.1.2. *Semistable reduction.* Kempf, Knudsen, Mumford and Saint-Donat proved the following theorem, which was the first instance of resolution of morphisms.

Theorem 2. *Let Z be an integral scheme of finite type over a trait $S = \mathrm{Spec}(R)$ of residual characteristic zero such that the generic fiber Z_η is smooth.*

(i) There exists proper $Z_{\rm res} \to Z$ with $Z_{\rm res} \to S$ log smooth and $(Z_{\rm res})_\eta = Z_\eta$.

(ii) After a finite extension of R can even make $Z_{\rm res} \to S$ semistable.

Claim (i) follows by applying Hironaka's theorem to Z and the divisor $Z \setminus Z_\eta$, and claim (ii) is then deduced by a complicated combinatorics. In general, one can not make $Z \to S$ smooth, so this is the best one might hope for. On the other side this solution is rather non-canonical, e.g. it changes when one extends R.

1.2. **Resolution of morphisms.** It turns out that the theorem of KKMS can be extended to more general morphisms and made functorial, but this requires to work within the larger category of logarithmic DM stacks with finite diagonalizable stabilizers. For simplicity, we will stick to the case of stacks of finite type over a field k of characteristic zero.

Theorem 3 ([2])**.** *To dominant morphisms $f\colon X \to S$ of integral log varieties (or log DM stacks) over k one can associate either a non-representable modification $X_{\rm res} \to X$ or a "fail output" $X_{\rm res} = \emptyset$ such that $X_{\rm res} \to S$ is log smooth and*

(i) Non-failure up to refining the base: *for any f there exists a modification $S' \to S$ such that $(X \times_S S')_{\rm res}$ is non-empty.*

(ii) Log smooth functoriality: *if $X_{\rm res}$ is non-empty and $X' \to X$ is log smooth, then $X'_{\rm res} = X_{\rm res} \times_X X'$.*

(iii) Base change functoriality: *if $X_{\rm res} \neq \emptyset$, then $(X \times_S S')_{\rm res} = X_{\rm res} \times_S S'$ for any base change $S' \to S$.*

Furthermore, generalizing the polyhedral subdivision theorem of KKMS to maps of polyhedra Adiprasito, Liu and Temkin deduced the following refinement

Theorem 4 ([3]). *After replacing S by an alteration, one can even achieve that $X_{\mathrm{res}} \to S$ is semistable.*

2. The method

2.1. **Logarithmic geometry.** Logarithmic structures are important both for classical resolution, where they are encoded by the boundary, and semistable reduction. The starting idea of our project was that in order to construct log smooth resolution of morphisms one should work log-smooth functorially. Already doing this for varieties in [1] required to modify the algorithm tremendously, and in fact the same new algorithm was extended in [2] to morphisms. We suggest:

Principle 2.1.1. If some aspects of the problem require to extend the notion of smoothness, it is preferable to run the whole algorithm in the extended setting.

Implementing it in our case suggested to work with log varieties, log smoothness, etc. In particular, resolution is reduced to principalization of ideals on log smooth (or toroidal) varieties (X, E), without the assumption that X is smooth. In addition, we replaced all basic resolution tools, such as derivation of ideals, order of ideals, hypersurface of maximal contact, etc., by their logarithmic analogs.

2.2. **Stacks.** Surprisingly for us, the log smooth functoriality forced the new principalization algorithm to perform certain weighted blow ups that produced not log smooth varieties. However, working with stacks it is possible to realize these blow ups as coarse spaces of smooth non-representable modifications, which we call Kummer blow ups. This suggested to extend our category further, in accordance with the above principle. Thus, our log smooth-functorial algorithm principalizes ideals on log smooth DM stacks even when the it starts with an ideal on a smooth variety. It is possible after that to return to log smooth or even smooth varieties by an additional modification, but the latter step can be only made smooth-functorial. Perhaps usage of stacks is unavoidable for getting a log smooth-functorial algorithm and resolution of morphisms. In the end, our algorithm operates with more complicated objects and modifications, but it is simpler and faster than its classical predecessor.

2.3. **Future works.** Our algorithm only performs weighted blow ups of a special form, and we expect that there exists a much more efficient algorithm which also works with DM stacks and performs arbitrary weighted blow ups. This is the main topic of our current research.

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Test ideals on numerically $\mathbb{Q}$-Gorenstein rings

Shunsuke Takagi

Throughout, let R be a Noetherian domain of characteristic $p > 0$, $\mathfrak{a}$ be a nonzero ideal of R and $t > 0$ be real number. We assume in addition that R is *F-finite*, that is, the Frobenius map $F : R \to R$ is a finite map. Let $\mathfrak{c}$ be an ideal of R which is a canonical module of R and $\mathfrak{c}^{(n)}$ denotes the n-th symbolic power of $\mathfrak{c}$ for each integer $n \geq 1$.

Definition 1 (Hochster-Huneke [6], Hara-Yoshida [5])**.** The *$\mathfrak{a}^t$-tight closure* $I^{*\mathfrak{a}^t}$ of an ideal $I \subseteq R$ is defined to be the ideal of R consisting of elements $x \in R$ for which there exists a nonzero element $c \in R$ such that $cx^q\mathfrak{a}^{\lceil tq \rceil} \subseteq I^{[q]}$ for all large $q = p^e$, where $\lceil tq \rceil$ is the least integer which is greater than or equal to tq and $I^{[q]}$ is the ideal generated by the q-th powers of all elements of I.

The *finitistic test ideal* $\tau_{\mathrm{fg}}(\mathfrak{a}^t)$ of $\mathfrak{a}$ with exponent t is defined by $\tau_{\mathrm{fg}}(\mathfrak{a}^t) = \bigcap_{J \subseteq R}(J : J^{\mathfrak{a}^t})$, where J runs through all ideals of R.

It is known by Brenner-Monsky [3] that tight closure does not commute with localization in general. On the other hand, the following conjecture is still open.

Conjecture 2. *The formation of finitistic test ideals commutes with localization, that is, $S^{-1}\tau_{\mathrm{fg}}(\mathfrak{a}^t) = \tau_{\mathrm{fg}}((S^{-1}\mathfrak{a})^t)$ for every multiplicative subset S of R.*

In order to state a stronger conjecture, we introduce another kind of test ideals.

Definition 3 ([6], [5])**.** Let M be a (not necessarily finitely generated) R-module. The *$\mathfrak{a}^t$-tight closure* $0_M^{*\mathfrak{a}^t}$ of the zero submodule in M is defined to be the submodule of M consisting of elements $z \in M$ for which there exists a nonzero element $c \in R$ such that $(c\mathfrak{a}^{\lceil tq \rceil})^{1/q} \otimes z = 0 \in R^{1/q} \otimes_R M$ for all large $q = p^e$.

Let $E = \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$ be the direct sum of the injective hulls of the residue fields $R/\mathfrak{m}$, where $\mathfrak{m}$ runs through all maximal ideals of R. The *big test ideal* $\tau_{\mathrm{b}}(\mathfrak{a}^t)$ of $\mathfrak{a}$ with exponent t is defined by $\tau_{\mathrm{b}}(\mathfrak{a}^t) = \mathrm{Ann}_R\, 0_E^{*\mathfrak{a}^t} \subseteq R$.

Conjecture 4. $\tau_{\mathrm{fg}}(\mathfrak{a}^t) = \tau_{\mathrm{b}}(\mathfrak{a}^t)$.

Since the formation of big test ideals commutes with localization, Conjecture 4 implies Conjecture 2. Conjecture 4 is known to be true in the following cases:

(1) ([1]) $R = \bigoplus_{n \geq 0} R_n$ is a graded ring and $\mathfrak{a}$ is a homogeneous ideal.
(2) ([7]) R is a local ring with isolated singularity and $\mathfrak{a} = R$.

(3) ([5]) R is normal and $\mathbb{Q}$-Gorenstein, that is, there exists an integer $r \geq 1$ such that the r-th symbolic power $\mathfrak{c}^{(r)}$ of $\mathfrak{c}$ is locally free.

(4) ([4]) $\mathfrak{a} = R$ and R is a normal domain such that the anti-canonical ring $\bigoplus_{n\geq 0} \mathfrak{c}^{(-n)}$ of Spec R is Noetherian, where $\mathfrak{c}^{(-n)} = \mathrm{Hom}_R(\mathfrak{c}^{(n)}, R)$.

(4) is a generalization of the case (3) when $\mathfrak{a} = R$. In this talk, we discuss another generalization of (3).

Definition 5 (cf. [2])**.** Let R be a normal domain of finite type over a field. We say that R is *numerically* $\mathbb{Q}$*-Gorenstein* if there exist a regular alteration $\pi : Y \to X = \mathrm{Spec}\, R$ and a $\mathbb{Q}$-Weil divisor on Y such that $\pi_* D = K_X$ and D is π-numerically trivial.

Remark 6. *(i)* $\mathbb{Q}$*-Gorenstein rings are numerically* $\mathbb{Q}$*-Gorenstein.*

(ii) Two-dimensional normal domains are numerically $\mathbb{Q}$*-Gorenstein.*

(iii) R *is* $\mathbb{Q}$*-Gorenstein if and only if it is numerically* $\mathbb{Q}$*-Gorenstein and the anti-canonical ring of* Spec R *is Noetherian. In particular, when* R *is a normal semigroup ring or a generic determinantal ring,* R *is* $\mathbb{Q}$*-Gorenstein if and only if it is numerically* $\mathbb{Q}$*-Gorenstein.*

The following is the main result of this talk.

Theorem 7. *Let* R *be a localization of a numerically* $\mathbb{Q}$*-Gorenstein normal domain of finite type over an* F*-finite field. Then Conjecture 4 holds.*

The proof of Theorem 7 is based on valuative techniques (Lemmas 8, 9).

Lemma 8 ([2])**.** *Let* R *be the same as in Theorem 7. Suppose that* v *is a divisorial valuation on* R *centered at* $\mathfrak{m}$*. Then* $\lim_{n\to\infty} v(\mathfrak{c}^{(n)}\mathfrak{c}^{(-n)})/n = 0$.

Lemma 9. *Let* $(R, \mathfrak{m})$ *be an* F*-finite complete local domain of characteristic* $p > 0$, $u \in R$ *and* $I \subseteq R$*. Fix a* $\mathbb{Q}$*-valued valuation* v *which is nonnegative on* R *and is positive on* $\mathfrak{m}$*. Then* $u \in I^{*\mathfrak{a}^t}$ *if and only if there exists a sequence* $\{c_e\}_{e\in\mathbb{N}}$ *of nonzero elements* $c_e \in R$ *such that* $\{v(c_e)/p^e\}_{e\in\mathbb{N}}$ *is a monotonically decreasing sequence whose limit is zero and that* $c_e x^q \mathfrak{a}^{\lceil tq \rceil} \subseteq I^{[q]}$ *for all* $q = p^e$*.*

We close this abstract with a conjecture that implies Theorem 7. Let R^+ be the absolute integral closure of R, that is, the integral closure of R in an algebraic closure of the fractional field $\mathrm{Frac}(R)$ of R. Given an ideal I of R, the *plus closure* I^+ of I is defined by $I^+ = IR^+ \cap R$. We define the ideal $P(R)$ of R by $P(R) = \bigcap_{I\subseteq R}(I : I^+)$, where I runs through all ideals of R. In general, $\tau_{\mathrm{b}}(R) \subseteq \tau_{\mathrm{fg}}(R) \subseteq P(R)$, where $\tau_{\mathrm{b}}(R)$ (resp. $\tau_{\mathrm{fg}}(R)$) is nothing but the big (resp. finitistic) test ideal $\tau_{\mathrm{b}}(R^t)$ (resp. $\tau_{\mathrm{fg}}(R^t)$) of the unit ideal R.

Conjecture 10. *Let* R *be the same as in Theorem 7. Then* $\tau_{\mathrm{b}}(R) = P(R)$*.*

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Irreducible components inside the space of arcs in positive characteristic

Ana J. Reguera
(joint work with A. Benito, O. Piltant)

In 1968, J. Nash initiated the study of the space of arcs X_∞ of a (singular) algebraic variety X with the purpose of understanding the structure of the various resolutions of singularities of X. His work [4] was done shortly after Hironaka's proof of Resolution of Singularities in characteristic zero, and it was spread by H. Hironaka and later by M. Lejeune-Jalabert.

Nash proved, using Resolution of Singularities that the space of arcs X_∞^{Sing} centered in the singular locus of X has a finite number of irreducible components. His argument, expressed in nowadays terms, is the following: let X be a variety over a field k of characteristic zero and let $\pi : Y \to X$ be a resolution of singularities. For every irreducible component E of the exceptional locus of π, the *Nash family of arcs* $N_E \subset X_\infty$ is defined to be the Zariski closure of the image of the set of arcs on Y which are centered at some point of E. Each N_E is irreducible and, moreover, N_E only depends on the divisorial valuation defined by E. Due to the properness of π, every arc in $X_\infty \setminus (\mathrm{Sing}\ X)_\infty$ which is centered at some point of $\mathrm{Sing}\ X$ lifts to Y, hence it belongs to some of the N_E's. That is, we have

$$X_\infty^{Sing} \ = \ \cup_E N_E \ \cup \ (\mathrm{Sing}\ X)_\infty. \tag{1}$$

From this one deduces that the number of irreducible components of X_∞^{Sing} is finite (see [4] or [3], [6]). It is in general not easy to deduce the decomposition of X_∞^{Sing} into its irreducible components from the one above; the Nash problem consists precisely in characterizing these irreducible components.

This Nash program extends, with some important differences, to perfect ground fields k of characteristic $p > 0$. The first difference is that Resolution of Singularities is still an open problem if $\mathrm{char}\ k = p > 0$ and $\dim X \geq 4$. Although Nash families N_E can be defined only in terms of divisorial valuations, it is not known that the indexing set in (1) can be chosen to be a finite set. In [5] we have proved that Hironaka's "Resolution of Singularities" can be substituted by the weaker notion of Zariski's generalized "Local Uniformization" along valuations in Nash's argument.

More precisely, let X be an irreducible variety over a perfect field k, and $Z \subseteq X$ be a Zariski closed subset, let X_∞^Z be the space of arcs on X centered at some point of Z. We consider valuation rings (O_v, M_v, k_v) of the field of fractions $k(X)$ of X. We say that $LU(X,\zeta)$ holds for $\zeta \in X$ an arbitrary point if for any valuation ring O_v dominating $O_{X,\zeta}$, there exists a finitely generated algebra $R = O_{X,\zeta}[f_1,\ldots,f_r] \subseteq O_v$ such that $R_{M_v\cap R}$ is a regular local ring. Then, we have:

Theorem 1 ([5], corollary 3.7). Let $X|k$ be a k-variety (not necessarily irreducible) and $Z \subseteq X$. Assume that, for every $z \in Z$ and for every irreducible subvariety $V \subseteq X$ with $z \in V$, Local Uniformization holds on V at z, i.e. $LU(V,z)$ holds. Then X_∞^Z has a finite number of irreducible components.

But it is not known whether Local Uniformization holds if char $k = p > 0$ and $\dim X \geq 4$. So, we propose the following question:
Question 1: Does X_∞^{Sing} have a finite number of irreducible components?

Another difference in extending Nash's program to positive characteristic is that, in contrast with characteristic zero, the right hand side term $(\mathrm{Sing}\ X)_\infty$ in (1) may contain some of the irreducible components of X_∞^{Sing}. Understanding these "small" components is the main purpose of our article [1]. Underlying the existence of small components when char $k = p > 0$ is the fact that Kolchin's irreducibility theorem is not valid in positive characteristic. If $X = \mathrm{Spec}R$ is an irreducible affine variety of characteristic zero, then $R_\infty := O_{X_\infty}$ is isomorphic to the differential algebra associated with R which Kolchin proved to be irreducible. But this statement does not hold in general when char $k = p > 0$; the most simple counterexample appeared in [3]. It is the irreducible surface $X := V(y^p + zx^p) \subset \mathbb{A}_k^3 = \mathrm{Spec}k[x,y,z]$. Here $\mathrm{Sing}X = V(x,y)$ is the z-axis and X_∞ has two irreducible components, $(\mathrm{Sing}X)_\infty$ and the Zariski closure of its complement in X_∞. More generally, Let k be a field of characteristic $p > 0$. For $n \geq 1$, let X_n be the $2n$-dimensional variety given by

$$\left(\cdots((y^p + z_1x_1^p)^p + z_2x_2^p)^p + \cdots\right)^p + z_nx_n^p = 0$$

in $\mathbb{A}_k^{2n+1}$. Then $(X_n)_\infty$ has exactly $n+1$ irreducible components.

In general, if X is an irreducible variety over a perfect field, we have that

$$X_\infty^\circ := \overline{X_\infty \backslash (\mathrm{Sing}X)_\infty}$$

is irreducible ([2] and [6]). Our main results in [1] are:

Theorem 2 ([1], theorem 4.4). Let $X|k$ be a k-variety, $\zeta \in X$ and $Z := \overline{\{\zeta\}} \subseteq X$. Assume that there exists $P \in X_\infty$ such that the image by the arc $h_P : \mathrm{Spec}\ \kappa(P)[[t]] \to X$ of the closed (resp. generic) point of Spec $\kappa(P)[[t]]$ is ζ (resp. is in $\mathrm{Reg}(X)$), and $\kappa(P)|k(\zeta)$ is a finite and separable field extension. Then $Z_\infty^\circ \subset X_\infty^\circ$.

Theorem 3 ([1], theorem 5.5). If $\dim O_{X,\zeta} = 1$ then the converse to theorem 2 holds.

Question 2: Given a variety X, does there exist a birational and proper morphism $Y \to X$ such that Y_∞ is irreducible?

With the help of theorem 2 and a classical result of Albanese, we have been able to give the following partial answer to question 2:

Proposition 4 ([1], proposition 6.7). Let k be an algebraically closed field of characteristic $p > 0$ and $K|k$ be a function field of dimension $d \geq 1$. If $p > d!$, there exists a projective variety $X|k$ such that $k(X) = K$ and X_∞ is irreducible.

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Can local cohomology prevent injections?

Mateusz Michałek

The following problems are our main motivation:

- construction of small sets of separating invariants [10, 5, 6];
- construction of low dimensional k-interpolating spaces [9, 11];
- construction of k-regular injections of complex varieties [3].

Let us recall the necessary definitions.

Definition 1.

- Let G be a group acting on a vector space V. A set F of invariants is *separating* if for any $v, w \in V$ we have:

$$\forall_{f\in F} f(v) = f(w) \Leftrightarrow \text{for all invariants } g \text{ we have } g(v) = g(w).$$

- A space V of functions on X is called *k-interpolating* if for any distinct points $x_1, \ldots, x_k$ and scalars $\lambda_1, \ldots, \lambda_k$ there exists such an $f \in V$ that $f(x_i) = \lambda_i$ for all $1 \leq i \leq k$.
- A map $X \to \mathbb{A}^n$ is called *k-regular* if the images of any k distinct points span a $k-1$-dimensional affine space.

The following example is not meant to be optimal, but only to demonstrate some of the connections.

Example 2. *Let G be a group of 3-rd roots of unity acting on $\mathbb{C}^2$ by coordinatewise multiplication. The ring of invariants is $\mathbb{C}[x^3, x^2y, xy^2, y^3]$. The set $\{x^3, x^2y, y^3\}$ is a separating set of invariants. The projection map $\operatorname{Spec}\mathbb{C}[x^3, x^2y, xy^2, y^3] \to \operatorname{Spec}\mathbb{C}[x^3, x^2y, y^3] \subset \mathbb{C}^3$ is 2-regular, i.e. injective, but not an embedding. The space spanned by $\{1, x^3, x^2y, y^3\}$ is 2-interpolating on $\operatorname{Spec}\mathbb{C}[x^3, x^2y, xy^2, y^3]$.*

An algebraic approach to provide solutions to all of the above problems is based on the notion of the k-th secant locus of a projective variety $X \subset \mathbb{P}^N$:

$$\sigma_k^\circ(X) = \bigcup_{x_1,\dots,x_k \in X} \langle x_1, \dots, x_k\rangle,$$

where $\langle S\rangle$ denotes the projective span of S. Trivially, a projective subspace L does not intersect $\sigma_k^\circ(X)$ if and only if dimensions of projective spans of k points of X are preserved after projection from L. In particular, for $k = 2$ we obtain a necessary and sufficient condition to obtain a (point-wise) injection. If $X' \to X \subset \mathbb{P}^N$ is (projectively) k-regular then the condition above implies that composition with projection from L remains k-regular.

One of the questions we find particularly interesting is the following.

Question 3. *Can every smooth, complex projective curve be injected to $\mathbb{P}^2$? In general, can every smooth d dimensional projective variety be injected to $\mathbb{P}^{2d}$?*

Clearly, there always exist embeddings in $\mathbb{P}^{2d+1}$. However, not every such embedding may be further projected in an injective way [13]. For recent partial results in the case of curves we refer to [1].

For nice constructions of small sets of separating invariants we refer to Dufresne-Jeffries [7] and very recent results of Görlach [8]. These allow often to find an injective map from a Segre-Veronese variety to a small projective space. However, at current stage we are not aware even if the following question has a positive answer:

Question 4. *Can every Segre-Veronese variety be injectively projected to a projective space of twice its dimension?*

In many cases we know that, if the answer is positive, then this bound is optimal. The proof relies on arguments based on local cohomology; more precisely estimation of cohomological dimension of the ideal of the separating variety. Still, this is not always the case, as there exist Segre-Veronese varieties that can be projected to even smaller spaces.

For constructions of k-regular maps we refer to a joint work with Buczyński, Januszkiewicz and Jelisiejew [4]. There, the existence of k-regular maps was related to bounds on the dimension of the so-called areole variety of the Veronese $v_r(\mathbb{P}^n)$:

$$a_k(v_r(\mathbb{P}^n, p)) = \overline{\bigcup_{\substack{\operatorname{supp} S = p, \deg S \le k \\ S \text{ smoothable}}} \langle v_r(S)\rangle},$$

for $p \in \mathbb{P}^n$. Due to classification results of small Gorenstein local schemes, the dimension bounds allow to confirm a few cases of the following general conjecture.

Conjecture 5. *There exists a continuous k-regular map* $\mathbb{C}^m \to \mathbb{C}^N$ *if and only if*

$$N \geq m(k-1)+1.$$

For upper bounds (close, but not equal to those from the conjecture), relying on methods from algebraic topology, we refer to [2]. Explicit constructions of k-regular maps and interpolating spaces, relying on Gorenstein schemes and areole varieties, can be found in [12].

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Resolution of singularities

Heisuke Hironaka

(Abstract written by Bernd Schober). Resolution of singularities has been an exciting, interesting, and important problem for many years. We explain some of the main ideas and the use of graded algebras in the recent proofs for resolution in characteristic zero. After that, we discuss new ideas for the positive characteristic case. In particular, we consider so called "differential products" which we use to modify the algebra of characteristic zero in order to obtain better insight in the singularity. Eventually, we outline a strategy for the arithmetic case using the existence of resolution of singularities in characteristic zero and positive.

The (ir)regularity of Tor and Ext

Marc Chardin
(joint work with Dipankar Ghosh, Navid Nemati)

There has been a keen interest in understanding the behavior of $\operatorname{reg}(I^n)$ as a function of n, where I is a homogeneous ideal in a polynomial ring $Q = K[X_1, \ldots, X_d]$ over a field. Following earlier results by Geramita, Gimigliano and Pitteloud, by Chandler and by Swanson, Cutkosky, Herzog and Trung and Kodiyalam independently showed that asymptotically $\operatorname{reg}(I^n)$ is a linear function of n. This behavior also has been studied for powers of more than one ideal.

One notices that $\operatorname{Tor}_1^Q(Q/I^p, Q/I^q) = I^p/I^{p+q}$ if $p \geq q$, which relates this question to more general results for finitely generated graded Q-modules M and N. The following results are known in this case

(1) If $\dim(\operatorname{Tor}_1^Q(M,N)) \leqslant 1$, then

$$\max_{0\leqslant i\leqslant d}\{\operatorname{reg}\big(\operatorname{Tor}_i^Q(M,N)\big) - i\} = \operatorname{reg}(M) + \operatorname{reg}(N).$$

This generalizes results of Sidman, Conca-Herzog, Caviglia, and Eisenbud-Huneke-Ulrich. The equality in (1) extends to the case when Q is standard graded, and M or N has finite projective dimension, replacing the right hand side by $\operatorname{reg}(M) + \operatorname{reg}(N) - \operatorname{reg}(Q)$.

(2)

$$\min_{0\leqslant i\leqslant d}\{\operatorname{indeg}(\operatorname{Ext}_Q^i(M,N)) + i\} = \operatorname{indeg}(N) - \operatorname{reg}(M)$$

where $\operatorname{indeg}(W) := \inf\{n \in \mathbb{Z} : W_n \neq 0\}$.

When working over standard graded algebras that are not regular (*i. e.* not a polynomial ring over a regular ring), one can also bound regulariy of Tor modules under the same kind of hypothesis, for instance

Theorem 1. *Suppose Q is a standard graded ring over a field, but Q is not a polynomial ring. Let M and N be finitely generated graded Q-modules, and $d := \min\{\dim(M), \dim(N)\}$. If $\dim\Big(\operatorname{Tor}_i^Q(M,N)\Big) \leqslant 1$ for all $i \geqslant i_0$, then*

$$\operatorname{reg}\Big(\operatorname{Tor}_i^Q(M,N)\Big) - i \leqslant \operatorname{reg}(M) + \operatorname{reg}(N) + \left\lfloor \frac{i+d}{2} \right\rfloor (\operatorname{reg}(Q) - 1),\ \forall i \geqslant i_0.$$

This implies that if $\operatorname{Proj}(Q)$ has isolated singularities, then the estimate above holds true for $i \geqslant \dim(Q) - 1$.

Over complete intersection ring, Ghosh and Puthenparakal controls the asymptotic behavior with respect to both a power of an ideal and the homological degree, and raised the following question,

Question 2. *For $\ell \in \{0,1\}$, do there exist $a_\ell, a'_\ell \in \mathbb{Z}_{>0}$ and $e_\ell, e'_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that*

(i) $\operatorname{reg}\Big(\operatorname{Ext}_A^{2i+\ell}(M,N)\Big) = -a_\ell \cdot i + e_\ell$ *for all $i \gg 0$?*

(ii) $\operatorname{reg}\big(\operatorname{Tor}^A_{2i+\ell}(M,N)\big) = a'_\ell \cdot i + e'_\ell$ *for all $i \gg 0$?*

In this work we are addressing this question. We prove that the answer to (i) is positive, even in a more general situation, while the answer to (ii) is negative. However, if $\dim\big(\mathrm{Tor}_i^A(M,N)\big) \leqslant 1$ for all $i \gg 0$, the second question does have a positive answer. Our main positive result on these questions is the following,

Theorem 3. *Let Q be a standard graded Noetherian algebra, $A := Q/(\mathbf{f})$, where $\mathbf{f} := f_1, \ldots, f_c$ is a homogeneous Q-regular sequence. Let M and N be finitely generated graded A-modules such that $\mathrm{Ext}^i_Q(M,N) = 0$ for all $i \gg 0$. Then,*

(i) *for $\ell \in \{0,1\}$, there exist $a_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that*
$$\mathrm{reg}\left(\mathrm{Ext}_A^{2i+\ell}(M,N)\right) = -a_\ell \cdot i + e_\ell \quad \text{for all } i \gg 0.$$

(ii) *if further Q is *local or the epimorphic image a Gorenstein ring, M has finite projective dimension over Q and*
$$\dim\big(\mathrm{Tor}_i^A(M,N)\big) \leqslant 1,\ \forall i \gg 0,$$
then, for $\ell \in \{0,1\}$, there exist $a'_\ell \in \{\deg(f_j) : 1 \leqslant j \leqslant c\}$ and $e'_\ell \in \mathbb{Z} \cup \{-\infty\}$ such that
$$\mathrm{reg}\big(\mathrm{Tor}^A_{2i+\ell}(M,N)\big) = a'_\ell \cdot i + e'_\ell,\ \forall i \gg 0.$$

On the negative side, we provide exemples showing that the behavior of the regularity of Tor modules could be very different without the assumptions as in the result above. The most striking one is maybe the following.

Example 4. *Let $Q := K[X,Y,Z,U,V,W]$ be a standard graded polynomial ring over a field K of characteristic 2. Set $A := Q/(X^2,Y^2,Z^2)$. We write $A = K[x,y,z,u,v,w]$, where x,y,z,u,v and w are the residue classes of X,Y,Z,U,V and W respectively. Set*
$$M := \mathrm{Coker}\left(\begin{bmatrix} x & y & z & 0 & 0 & 0 \\ u & v & w & x & y & z\end{bmatrix} : A(-1)^6 \longrightarrow A^2\right) \quad \text{and} \quad N := A/(x,y,z).$$
Then, for every $n \geqslant 1$, we have

(i) $\mathrm{indeg}\,(\mathrm{Ext}^n_A(M,N)) = \mathrm{reg}\,(\mathrm{Ext}^n_A(M,N)) = -n$.

(ii) $\mathrm{indeg}\,\big(\mathrm{Tor}^A_n(M,N)\big) = n$ *and* $\mathrm{reg}\,\big(\mathrm{Tor}^A_n(M,N)\big) = n + f(n)$, *where*
$$f(n) := \begin{cases} 2^{l+1}-2 & \text{if } n = 2^l - 1 \\ 2^{l+1}-1 & \text{if } 2^l \leqslant n \leqslant 2^{l+1}-2 \end{cases} \quad \text{for all integers } l \geqslant 1.$$

As a consequence, in this example,
$$\liminf_{n\to\infty} \frac{\mathrm{reg}(\mathrm{Tor}^A_n(M,N))}{n} = 2 \quad \text{and} \quad \limsup_{n\to\infty} \frac{\mathrm{reg}(\mathrm{Tor}^A_n(M,N))}{n} = 3.$$
Furthermore, for any $\alpha \in (2,3)$, by choosing any subsequence $n_\alpha(l)$ such that $|n_\alpha(l) - \lfloor 2^l/(\alpha-1)\rfloor|$ is bounded for all $l \geqslant 1$,
$$\lim_{l\to\infty} \frac{\mathrm{reg}(\mathrm{Tor}^A_{n_\alpha(l)}(M,N))}{n_\alpha(l)} = \alpha.$$
In particular, $n_\alpha(l)$ can be a sequence of even (resp. odd) integers.

Face enumeration for (balanced) manifolds

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(joint work with S. Murai, I. Novik, C. Sawaske)

At the intersection of geometry, algebra, and combinatorics is the study of the face numbers of simplicial complexes. If $f_i(\Delta)$ denotes the number of i-dimensional faces of a $(d-1)$-dimensional simplicial complex Δ, then the h-numbers $h_i(\Delta)$ are defined by $h_i(\Delta) = \sum_{j=0}^{i}(-1)^{i-j}\binom{d-i}{j-i}f_{j-1}(\Delta)$. Of the most important results in the study of face numbers of simplicial complexes, many have been elegantly phrased in the language of the h-numbers. Principal among these are the Dehn–Sommerville relations, the lower and upper bound theorems, and their culmination – the g-theorem. Our starting point is the following generalized lower bound theorem (or GLBT) conjectured by McMullen and Walkup [4] and proved by Stanley [11], and Murai and Nevo [6]:

Theorem 1. *Let P be a d-dimensional simplicial polytope. Then*

$$h_0(P) \leq h_1(P) \leq \cdots \leq h_{\lfloor \frac{d}{2} \rfloor}(P);$$

also the equality $h_{i-1}(P) = h_i(P)$ occurs for a certain $i \leq \lfloor \frac{d}{2} \rfloor$ if and only if P is $(i-1)$-stacked.

It is natural to ask to what extent these inequalities can be specialized. In particular, are there classes of simplicial polytopes whose successive h-numbers satisfy more drastic inequalities? Of recent interest have been balanced simplicial complexes (those complexes whose underlying graphs have a "minimal" coloring), introduced by Stanley in [10]. Examples of balanced simplicial complexes include barycentric subdivisions of regular CW complexes, Coxeter complexes, and Tits buildings. The following strengthening of Theorem 1 for balanced simplicial polytopes was conjectured in [3] and proved by Juhnke-Kubitzke and Murai in [1].

Theorem 2. *Let P be a d-dimensional balanced simplicial polytope. Then*

$$\frac{h_0(P)}{\binom{d}{0}} \leq \frac{h_1(P)}{\binom{d}{1}} \leq \cdots \leq \frac{h_i(P)}{\binom{d}{i}} \leq \cdots \leq \frac{h_{\lfloor d/2 \rfloor}(P)}{\binom{d}{\lfloor d/2 \rfloor}}.$$

We examine extensions of this result to more general complexes. More precisely, we study balanced $\mathbb{F}$-homology manifolds without boundary, where $\mathbb{F}$ is a field. When confining our attention to this class of simplicial complexes, the natural analog of the h-numbers turns out to be the h''-numbers (for polytopes, these are one and the same): for a $(d-1)$-dimensional complex Δ and $i < d$, $h''_i(\Delta)$ is defined by $h_i(\Delta) - \binom{d}{i}\sum_{j=1}^{i}(-1)^{i-j}\tilde{\beta}_{j-1}(\Delta)$, where $\tilde{\beta}_{j-1}(\Delta)$, $1 \leq j \leq d$, are the reduced Betti numbers computed over $\mathbb{F}$. Specifically, the manifold GLBT asserts that if Δ is a $(d-1)$-dimensional $\mathbb{F}$-homology manifold with or without boundary whose vertex links have the weak Lefschetz property, then $h''_i(\Delta, \partial\Delta) \geq h''_{i-1}(\Delta, \partial\Delta) + \binom{d}{i-1}\tilde{\beta}_{i-1}(\Delta, \partial\Delta)$ for all $i \leq \lfloor d/2 \rfloor$; see [9, eq. (9)] and [7, Theorem 1.5]. In view of this result, it seems plausible that the statement of Theorem 2 can be appropriately extended to balanced $\mathbb{F}$-homology manifolds. Indeed, the following is one of our main results [2].

Theorem 3. *Let Δ be a $(d-1)$-dimensional balanced simplicial complex. If Δ is an $\mathbb{F}$-homology manifold without boundary and $d \geq 4$, then*

$$\frac{h_2''(\Delta)}{\binom{d}{2}} \geq \frac{h_1''(\Delta)}{\binom{d}{1}} + \tilde{\beta}_1(\Delta).$$

Equivalently, $2h_2(\Delta)-(d-1)h_1(\Delta) \geq 4\binom{d}{2}(\tilde{\beta}_1(\Delta)-\tilde{\beta}_0(\Delta))$. Furthermore, if $d \geq 5$, then this inequality holds as equality if and only if each connected component of Δ is in the balanced Walkup class.

This result provides a balanced analog of [8, Theorem 5.2] (see also [5, Theorem 5.3]) and settles Conjecture 4.14 of [3] (see also [3, Remark 3.8]). It is worth mentioning that for $d-1 \geq 4$, the condition that Δ is in the balanced Walkup class is equivalent to all vertex links of Δ being stacked cross-polytopal spheres (see [3, Corollary 4.12]).

We remark that, in the general as well as in the balanced situation, there exist analog result for higher h''-numbers as well as for orientable manifolds with boundary under relatively weak additional assumptions.

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Teissier singularities: a viewpoint on quasi-ordinary singularities in positive characteristics:

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(joint work with Bernd Schober)

Let $\mathbf{K}$ be an algebraically closed field of characteristic 0. A Weierstrass polynomial $f \in \mathbf{K}[[x_1, \ldots, x_d]][z]$, satisfying $f(0) = 0$, is called quasi-ordinary if its discriminant with respect to z is a unit times a monomial in $\mathbf{K}[[x_1, \ldots, x_d]]$ (we refer to this condition as the discriminant condition). Note that we have a finite map $\{f = 0\} \longrightarrow \mathbf{K}^d$ (the projection on the first d coordinates) and that its ramification locus is the zero locus of the discriminant. If $\{f = 0\}$ is singular at the origin, we say that $0 \in \{f = 0\}$ is a quasi-ordinary singularity. Quasi-ordinary singularities appear in Jung's method of resolution of singularities in char 0: for any $f \in \mathbf{K}[[x_1, \ldots, x_d]][z]$, this method is recursive on the dimension and consists in using embedded resolution of singularities $\phi : Z \longrightarrow \mathbf{K}^d$ in dimension $d-1$, to transform the discriminant of f into a normal crossing divisor (locally a unit times a monomial). The pull back of $\{f = 0\}$ by ϕ will then have only quasi-ordinary singularities and the resolution problem is reduced to the problem of resolution of quasi-ordinary singularities and then patching these local resolutions. We first give a characterization of quasi-ordinary singularities in terms of an invariant of f, that we denote by $\kappa(f)$ (see [4]) and that we construct using a weighted version of Hironaka's characteristic polyhedron and successive embeddings of the singularity defined by f in affine spaces of higher dimensions; this invariant is inspired on one hand by resolution invariants in char 0 (and p) and on the other hand by Teissier's approach to the resolution of singularities by changing the embedding (see [6]). Note that Hironaka's characteristic polyhedron is a projection of the classical Newton polyhedron, but it has some intrinsic properties thanks to the minimizing process explained in [3]. The invariant $\kappa(f)$ is a string whose components are all in $\mathbf{Q}_+^d$, except of the last one which is either -1 or ∞. The size of $\kappa(f)$ depends on f. In [4] we prove the following theorem:

Theorem 1: *Let f be as above. The singularity $\{f = 0\}$ is quasi-ordinary with respect to the projection $\{f = 0\} \longrightarrow \boldsymbol{K}^d$ if and only if the last component of $\kappa(f)$ is ∞.*

It is worth mentioning that when $\mathbf{K} = \mathbf{C}$, we prove that the invariant $\kappa(f)$ is a complete invariant of the topological type of $(\{f = 0\}, 0) \subset (\mathbf{C}^{d+1}, 0)$.
On the one hand, while we know how to resolve quasi-ordinary singularities (in char 0), in positive characteristics, the singularities which satisfy the condition on the discriminant can be extremely wicked; e.g. hundred of pages of the proof of Cossart-Piltant of resolution in dimension 3 are dedicated to this type of singularities (see [1] for the arithmetical case). So, in positive characteristics, the reduction of the resolution of singularities problem to the singularities satisfying the discriminant condition cannot be compared with Jung's approach in characteristic 0. On the other hand, while in characteristic 0, the last component of $\kappa(f)$

being ∞ is equivalent to f being quasi-ordinary, in characteristic p, the invariant $\kappa(f)$ is still meaningful but the condition on its last component gives rise to a different condition than the one given by the discriminant. This leads us to define the following class of singularities [5]:

Definition: *Let $\boldsymbol{K}$ be an algebraically closed field of characteristic $p > 0$. Let $f \in \boldsymbol{K}[[x_1, \ldots, x_d]][z]$ satisfying $f(0) = 0$. The hypersurface singularity $(X, 0) = \{f = 0\}$ is a **Teissier singularity** if the last component of $\kappa(f)$ is ∞.*

The name "Teissier singularities" was suggested by the fact that Teissier proved that along an Abhyankar rational valuation, any hypersurface singularity can be embedded in a higher dimension affine space with a special type of equations that define an "overweight deformation" whose generic fiber is isomorphic to the singularity and whose special fiber is the toric variety associated with the graded algebra of the valuation [6]. For a Teissier singularity, all the valuations which extend rational monomial valuations on $\mathbf{K}[[x_1, \ldots, x_d]]$ to $\mathbf{K}[[x_1, \ldots, x_d]][z]/(f)$ induce the "same overweight deformation" and this property characterizes them [2]. Teissier singularities do not satisfy the discriminant condition in general, and a singularity satisfying the discriminant condition is not Teissier in general. But these singularities give a very good positive characteristics counterpart of quasi-ordinary singularities thanks to the following result:

Theorem 2: *A Teissier singularity $(X, 0)$ sits in an equisingular family χ over $\mathrm{Spec}(O_{\boldsymbol{C}_p})$ as a special fiber, and the generic fiber of χ has only quasi-ordinary singularities.*

Note that the generic fiber is defined over a field of characteristic 0. Here, equisingular means that we have a simultaneous resolution of χ. Although Teissier singularities are complicated in general, we can resolve their singularities thanks to the understanding of the neighbor quasi-ordinary singularities. Note also that any quasi-ordinary singularity in characteristic 0 gives rise to a Teissier Singularity.

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Koszul Modules

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(joint work with Marian Aprodu, Gavril Farkas, Ştefan Papadima, Jerzy Weyman)

We consider a field $\mathbf{k}$, a $\mathbf{k}$-vector space V of dimension n, and let $K \subseteq \bigwedge^2 V$ be a subspace of dimension $m \leq \binom{n}{2}$. We let $S = \mathrm{Sym}(V)$ denote the symmetric algebra on V, and define the *Koszul module* $W(V,K)$ to be the middle homology of the complex

$$K \otimes S \xrightarrow{\delta_2|_{K\otimes S}} V \otimes S \xrightarrow{\delta_1} S \tag{1}$$

where $\delta_1 : V \otimes S \longrightarrow S$ is the natural multiplication map, and

$$\delta_2 : \bigwedge^2 V \otimes S \longrightarrow V \otimes S, \quad (v \wedge v') \otimes f \xrightarrow{\delta_2} v \otimes (v'f) - v' \otimes (vf) \text{ for } v, v' \in V, f \in S,$$

is the second differential of the Koszul complex on V. We put a grading on the free modules in (1) so that K is placed in degree zero, and the maps δ_1, δ_2 are homogeneous, which implies that $W(V,K)$ is a graded S-module generated in degree zero. We note that any inclusion $K \subseteq K'$ induces a surjective homomorphism $W(V,K) \twoheadrightarrow W(V,K')$, that is, bigger subspaces K correspond to smaller Koszul modules. At the extremes we get that $K = 0$ if and only if $W(V,K) = \ker(\delta_1)$ is the module of syzygies between linear forms, and that $K = \bigwedge^2 V$ if and only if $W(V,K) = 0$. The next smallest Koszul modules are those that have finite length, for which we have the following sharp estimate on their Hilbert function [1, 2].

Theorem 1. Suppose that $n \geq 3$, and that $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) \geq n-2$. We have the equivalence

$$W_q(V,K) = 0 \text{ for } q \gg 0 \Longleftrightarrow W_q(V,K) = 0 \text{ for } q \geq n-3. \tag{2}$$

Moreover, if the equivalent statements in (2) hold then

$$\dim W_q(V,K) \leq \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2} \quad \text{for } q = 0, \cdots, n-4,$$

with equality for all q if $m = 2n-3$.

Papadima and Suciu show in [6, Lemma 2.4] that the set-theoretic support of $W(V,K)$ is given by the *resonance variety*

$$\mathcal{R}(V,K) := \left\{ a \in V^\vee \mid \text{ there exists } b \in V^\vee \text{ such that } a \wedge b \in K^\perp \setminus \{0\} \right\} \cup \{0\}, \tag{3}$$

where $K^\perp = \{\phi \in \bigwedge^2 V^\vee : \phi_{|K} = 0\}$. The equivalent statements in (2) can then be rephrased as the vanishing of the resonance $\mathcal{R}(V,K)$, which is in turn equivalent to the fact that the linear space $\mathbb{P}K^\perp$ is disjoint from the Grassmannian $\mathrm{Gr}(2, V^\vee)$ of 2-dimensional subspaces of $V^\vee$, when both are viewed as subsets of the Plücker space $\mathbb{P}\left(\bigwedge^2 V^\vee\right)$. Koszul modules have numerous incarnations, as follows.

BGG Correspondence. If we let $E = \bigwedge V^\vee$ denote the exterior algebra, think of $K^\perp$ as a space of quadrics in E, and let $A = E/\langle K^\perp \rangle$, then

$$W_q(V,K) \simeq \mathrm{Tor}^E_{q+1}(A,\mathbf{k})^\vee_{q+2} \text{ for all } q.$$

Group Cohomology. Let G be a finitely generated group, and define the pair (V,K) by letting $V = H_1(G,\mathbf{k}) = H^1(G,\mathbf{k})^\vee$, and

$$K^\perp = \ker\left(\bigwedge^2 H^1(G,\mathbf{k}) \xrightarrow{\cup} H^2(G,\mathbf{k})\right),$$

the kernel of the cup product map. We call $W(G) := W(V,K)$ the *Koszul module of G*. The Hilbert function of $W(G)$ is closely related to the *Chen ranks* of G, and Theorem 1 can be used to deduce many interesting consequences about related invariants of G (see [1]).

Vector bundles. Let $\mathcal{E}$ be a locally free sheaf of finite rank on a projective variety X and consider the exterior multiplication map

$$d_2 : \bigwedge^2 H^0(X,\mathcal{E}) \to H^0\left(X, \bigwedge^2 \mathcal{E}\right)$$

The *Koszul module of $\mathcal{E}$* is $W(X,\mathcal{E}) := W(V,K)$, where $V = H^0(X,\mathcal{E})^\vee$ and $K^\perp = \ker(d_2)$. Suppose now that $\mathcal{E}$ is globally generated and consider the associated *Lazarsfeld bundle* $M_{\mathcal{E}} = \ker\left(H^0(X,\mathcal{E}) \otimes \mathcal{O}_X \twoheadrightarrow \mathcal{E}\right)$. In analogy with a well-known result for Koszul cohomology groups, we can show that if $H^1(X,\mathcal{O}_X) = 0$ then $W_q(X,\mathcal{E})^\vee \cong H^1(X, \mathrm{Sym}^{q+1} M_{\mathcal{E}})$ for all $q \geq 0$.

$\mathfrak{sl}_2$-representation theory. Let $U = \mathrm{Span}\{1,x\}$ be a 2-dimensional vector space. There is a natural $\mathfrak{sl}$-equivariant map

$$\Psi : \bigwedge^2(\mathrm{Sym}^{n-1} U) \longrightarrow \mathrm{Sym}^{2n-4} U, \quad \Psi(x^s \wedge x^t) = (s-t)x^{s+t-1}.$$

We define $W^{(n-1)} := W(V,K)$ for $V^\vee = \mathrm{Sym}^{n-1} U$ and $K^\perp = \ker \Psi$, which is referred to as a *Weyman module* in [3, Section 3.I.B]. The map Ψ is surjective when $\mathrm{char}(\mathbf{k}) \neq 2$, so that $\dim(K) = 2n-3$, and $W^{(n-1)}$ has finite length when $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) \geq n$, so Theorem 1 applies to $W^{(n-1)}$ in the strongest form.

In [2], we use the modules $W^{(n-1)}$ to study the syzygies of the tangent developable $\mathcal{T} \subset \mathbb{P}^g$ to a rational normal curve of degree g, and complete the program put forward in [3] for the proof of Green's Conjecture in the case of general canonically embedded curves of genus g [5, Conjecture 5.6]. Writing R (resp. S) for the homogeneous coordinate ring of $\mathcal{T}$ (resp. of $\mathbb{P}^g$), and $b_{i,j}(\mathcal{T}) = \dim\left(\mathrm{Tor}^S_i(R,\mathbf{k})_{i+j}\right)$ for the *Betti numbers* of $\mathcal{T}$, we show the following.

Theorem 2. If $\mathrm{char}(\mathbf{k}) \neq 2$ then R is Gorenstein with Castelnuovo–Mumford regularity 3. Moreover,

$$b_{i,2}(\mathcal{T}) = b_{g-2-i,1}(\mathcal{T}) = \dim W^{(i+2)}_{g-3-i} \text{ for } i = 1, \cdots, g-3.$$

In particular, if $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) \geq (g+2)/2$ then

$$b_{i,2}(\mathcal{T}) = 0 \text{ for } i \leq \frac{g-3}{2}. \tag{4}$$

Using the fact that a general linear section of $\mathcal{T}$ is a canonically embedded rational g-cuspidal curve, together with the semicontinuity property of Betti numbers, we conclude that (in suitable characteristics) a general canonical curve of genus g has the same Betti numbers as $\mathcal{T}$. More precisely, we have the following.

Theorem 3. If $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) \geq (g+2)/2$ then a general curve of genus g satisfies Green's Conjecture.

In characteristic zero, this theorem has been proved using geometric methods in two landmark papers by Voisin [8, 9]. Schreyer [7] observed that Green's conjecture fails in small characteristics, for instance when $g = 7$ and $\mathrm{char}(\mathbf{k}) = 2$, or $g = 9$ and $\mathrm{char}(\mathbf{k}) = 3$. Together with Eisenbud they conjecture in [4, Conjecture 0.1] that Theorem 3 should extend to $\mathrm{char}(\mathbf{k}) \geq (g-1)/2$. However, when $\mathrm{char}(\mathbf{k}) \leq (g+1)/2$ the vanishing (4) fails, due to the fact that $\mathcal{T}$ is contained in a rational normal scroll of too large codimension. This means that (4) also fails for rational g-cuspidal curves, so they can't be used to improve on the statement of Theorem 3.

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Normal Hilbert Coefficients and Singularities

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(joint work with K. Ozeki, M. E. Rossi and H. L. Truong)

Normal Hilbert coefficients are important numerical invariants associated to an $\mathfrak{m}$-primary ideal I in an analytically unramified local ring $(R, \mathfrak{m})$. Recall that R is said to be analytically unramified if its $\mathfrak{m}$-adic completion $\widehat{R}$ is reduced. Let $\mathcal{R}(I) := \oplus_{n\geq 0} I^n t^n \subseteq R[t]$ denote the Rees algebra of I. It is well known that the integral closure of $\mathcal{R}(I)$ in $R[t]$ is $\overline{\mathcal{R}}(I) := \oplus_{n\geq 0}\overline{I^n}t^n$ where $\overline{I}$ denotes the integral closure of I. In [8] D. Rees proved that if R is analytically unramified, then $\overline{\mathcal{R}}(I)$ is a finite $\mathcal{R}(I)$-module. Hence there exist integers $\overline{\mathrm{e}}_i(I)$, called the *normal Hilbert coefficients* of I such that for all $n \gg 0$

$$\ell_R(R/\overline{I^{n+1}}) = \overline{\mathrm{e}}_0(I)\binom{n+d}{d} - \overline{\mathrm{e}}_1(I)\binom{n+d-1}{d-1} + \cdots + (-1)^d\overline{\mathrm{e}}_d(I)$$

where $d = \dim R$ and $\ell_R(N)$ denotes, for an R-module N, the length of N.

Very often the Rees algebra $\mathcal{R}(I)$ and the normalized Rees algebra $\overline{\mathcal{R}}(I)$ do not have "good" homological properties even if the ring R is "good". In fact, this is one of the main obstacles in the resolution of singularities. As a remedy one tries to find available information from other numerical invariants such as the normal Hilbert coefficients. It is a usual philosophy that if the normal Hilbert coefficients achieve "extremal" values with respect to some bounds, then the blow-up algebras have good homological properties. In this talk we are interested in the "extremal" value of $\overline{\mathrm{e}}_2(I)$.

We remark that the normal Hilbert coefficients carry important geometric information of a point corresponding to the local ring on a variety. In fact, these coefficients are useful to detect the type of singularities. For instance, let R be an excellent normal local domain of dimension two. Then R has a rational singularity (resp. minimally elliptic singularity) if and only if $\overline{\mathrm{e}}_2(I) = 0$ for every $\mathfrak{m}$-primary ideal I in R (resp. R is Gorenstein and $\max\{\overline{\mathrm{e}}_2(I) : I \text{ is } \mathfrak{m}\text{-primary}\} = 1$). Rational singularities have been investigated by J. Lipman [4] and D. Cutkosky [1] in dimension two.

Recently, T. Okuma, K.-i. Watanabe and K. Yoshida in [6] introduced a class of ideals, namely the *p_g-ideals*, which inherit nice properties of integrally closed ideals in a rational singularity. In an excellent normal local domain of dimension two $\overline{\mathrm{e}}_2(I) = 0$ characterizes the p_g-ideals [7]. We remark that $\overline{\mathrm{e}}_2(I) \geq 0$ in a normal local ring of dimension two by [2]. Thus the p_g-ideals characterize the minimal value of $\overline{\mathrm{e}}_2(I)$. However, there is a better bound on $\overline{\mathrm{e}}_2(I)$ due to S. Itoh [3] in any Cohen-Macaulay local ring of dimension $d \geq 2$.

From now onwards, let $(R, \mathfrak{m})$ be an analytically unramified Cohen-Macaulay local ring of dimension $d \geq 2$ with infinite residue field. Following Ooishi we call $\overline{g}_s(I) :=$

$\overline{\mathrm{e}}_1(I) = \overline{\mathrm{e}}_0(I) + \ell_R(R/\overline{I})$ the *normal sectional genus of* I. In [3] Itoh proved that

$$\overline{\mathrm{e}}_2(I) \geq \overline{g}_s(I) \geq \ell_R(\overline{I^2}/J\overline{I})$$

where J is a minimal reduction of I. Moreover, he showed that either of the inequality is an equality if and only if $\overline{\mathrm{r}}_J(I) \leq 2$ where

$$\overline{\mathrm{r}}_J(I) := \min\{r \geq 0 \mid \overline{I^{n+1}} = J\overline{I^n} \text{ for all } n \geq r\}$$

is the *normal reduction number* of I with respect to J. In particular, in this case the normal associated graded ring $\overline{G}(I) := \oplus_{n\geq 0}\overline{I^n}/\overline{I^{n+1}}$ is Cohen-Macaulay. In this talk we are interested in the almost minimal value of $\overline{\mathrm{e}}_2(I)$, that is, $\overline{\mathrm{e}}_2(I) = \overline{g}_s(I) + 1$. In [5] we proved that if $\overline{\mathrm{e}}_2(I) = \overline{g}_s(I)+1$ and $\overline{\mathrm{e}}_3(I) \neq 0$, then depth $\overline{G}(I) \geq d-1$. Moreover, we gave a complete structure of the Sally module in this case. We also gave an example which shows that the result is sharp. In the case $\overline{\mathrm{e}}_2(I) = \overline{g}_s(I)+1$ and $\overline{\mathrm{e}}_3(I) = 0$ in dimension 3, we proved that $\overline{G}(I^n)$ is Cohen-Macaulay for all $n \geq 2$. We remark that this case would not have been occurred in the Gorenstein case if the following conjecture by S. Itoh [3] would have been true:
Itoh's Conjecture [3]**:** Assume additionally R is Gorenstein. Then $\overline{\mathrm{e}}_3(I) = 0$ if and only if $\overline{\mathrm{r}}_J(I) \leq 2$.

The main tools that we use to prove our results are the vanishing theorem on local cohomology modules of the normalized Rees algebra [3], and the Sally module $\overline{S}_J(I) : \oplus_{n\geq 1}\overline{I^{n+1}}/J^n\overline{I}$ introduced by W. V. Vasconcelos in [9]. In particular, we study the suitable filtration $\{\overline{C}^{(i)}\}$ where $\overline{C}^{(i)} := \oplus_{n\geq i}\overline{I^{n+1}}/J^{n-i+1}\overline{I^i}$ of the Sally module that was introduced by M. Vaz Pinto in [10]. If $\overline{\mathrm{e}}_2(I) = \overline{g}_s(I) + 1$ and $\overline{\mathrm{e}}_3(I) \neq 0$, we prove that $\overline{C}^{(2)} \simeq B(-2)$ as graded B-module where $B = \mathcal{R}(J)/\mathfrak{m}\mathcal{R}(J)$. This implies, in particular, depth $\overline{G}(I) \geq d-1$ and $\overline{\mathrm{r}}_J(I) = 3$.

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Differential operators on some invariant rings

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(joint work with Jack Jeffries)

The *differential operators* on a commutative ring R are defined inductively as follows: set $\mathcal{D}^0(R) := \operatorname{Hom}_R(R, R)$, and

$$\mathcal{D}^k(R) := \{\delta \in \operatorname{Hom}_{\mathbb{Z}}(R, R) \mid \delta \circ \varphi - \varphi \circ \delta \in \mathcal{D}^{k-1}(R) \text{ for all } \varphi \in \mathcal{D}^0(R)\}.$$

It is readily verified that $\mathcal{D}(R) := \bigcup_k \mathcal{D}^k(R)$ is a subring of $\operatorname{Hom}_{\mathbb{Z}}(R, R)$.

When R is an algebra over a commutative ring A, we set $\mathcal{D}_A(R)$ to be the subring of $\mathcal{D}(R)$ consisting of differential operators that are A-linear. If R is a polynomial ring in the variables $x_1, \dots, x_n$ over A, then, by [2, Théorème 16.11.2], the ring $\mathcal{D}_A(R)$ is a free R-module, with basis

$$\frac{1}{t_1!}\frac{\partial^{t_1}}{\partial x_1^{t_1}} \cdots \frac{1}{t_n!}\frac{\partial^{t_n}}{\partial x_n^{t_n}} \qquad \text{for } (t_1, \dots, t_n) \in \mathbb{N}^n.$$

In this case, it follows that for each A-algebra B one has

$$\mathcal{D}_A(R) \otimes_A B \cong \mathcal{D}_B(R \otimes_A B).$$

The isomorphism above holds more generally whenever R is a smooth A-algebra.

If G is a group acting on R, the action extends to $\mathcal{D}(R)$ as follows:

$$g(\delta)\colon r \longmapsto g(\delta(g^{-1}(r)))$$

where $g \in G$, $\delta \in \mathcal{D}(R)$, and $r \in R$. Using $(-)^G$ to denote G-invariants, one obtains a restriction homomorphism

$$\mathcal{D}(R)^G \longrightarrow \mathcal{D}(R^G).$$

If R is an A-algebra, and G acts A-linearly on R, one similarly obtains

$$\mathcal{D}_A(R)^G \longrightarrow \mathcal{D}_A(R^G).$$

The surjectivity of this homomorphism has been studied extensively, see for example [9] and the references therein; we mention two specific results:

Theorem. (Kantor, [5]) Let G be a finite subgroup of $\operatorname{GL}_n(\mathbb{C})$, acting linearly on a polynomial ring $R := \mathbb{C}[x_1, \dots, x_n]$. Then $\mathcal{D}_{\mathbb{C}}(R)^G \longrightarrow \mathcal{D}_{\mathbb{C}}(R^G)$ is surjective if and only if the group G contains no pseudoreflections.

Theorem. (Levasseur-Stafford, [6]) Let G be either the general linear group, or the orthogonal group, or the symplectic group, acting linearly on a polynomial ring R over $\mathbb{C}$, such that R^G is, respectively, either a generic determinantal ring, or defined by the minors of a symmetric matrix of indeterminates, or by the Pfaffians of an antisymmetric matrix of indeterminates. Then $\mathcal{D}_{\mathbb{C}}(R)^G \longrightarrow \mathcal{D}_{\mathbb{C}}(R^G)$ is surjective if and only if R^G is not a regular ring.

In each of the cases covered by the theorems above, the ring $\mathcal{D}_{\mathbb{C}}(R^G)$ is a simple ring. More generally, the following has been raised as [6, Question 0.13.2], [8, Conjecture 2], and [10, Conjecture 1.1]:

Conjecture. Let G be a linearly reductive group acting linearly on a polynomial ring R over $\mathbb{C}$. Then $\mathcal{D}_{\mathbb{C}}(R^G)$ is a simple ring.

The conjecture also holds when G is a torus [7]. The analogous question in positive characteristic has an affirmative answer by [10, Theorem 1.3]; more generally:

Theorem. (Smith-Van den Bergh, [10]) Let R be a polynomial ring over a perfect field $\mathbb{F}$ of positive characteristic, and S a graded subring such that $S \subset R$ splits in the category of graded S-modules. Then $\mathcal{D}_{\mathbb{F}}(S)$ is a simple ring.

Towards an attack on the conjecture, Smith and Van den Bergh ask if *reduction modulo p* works for differential operators in the context of invariant rings: let A be a Dedekind domain and R_A a finitely generated flat A-algebra. Set

$$P^k_{R_A|A} := (R_A \otimes_A R_A)/\Delta^{k+1},$$

where Δ is the kernel of the multiplication map $R_A \otimes_A R_A \to R_A$. Suppose furthermore that each $P^k_{R_A|A}$ is a flat A-module; this can be achieved via generic flatness after inverting an element of A. For each maximal ideal μ of A, one then has an exact sequence

$$0 \to \mathcal{D}_A(R_A) \otimes_A A/\mu \to \mathcal{D}_{A/\mu}(R_A \otimes_A A/\mu) \to \mathrm{Tor}_1^A(A/\mu, R^1\mathcal{D}_A(R_A)) \to 0,$$

where $R^1\mathcal{D}_A$ is the right derived functor introduced in [10, §2.2].

Question. [10, Question 5.1.2]) Suppose A is a Dedekind domain that is a finitely generated $\mathbb{Z}$-subalgebra of $\mathbb{C}$, and R_A is a finitely generated A-algebra such that $R_A \otimes_A \mathbb{C}$ is the ring of invariants for a linearly reductive group acting linearly of a polynomial ring over $\mathbb{C}$. Then does $\mathrm{Tor}_1^A(A/\mu, R^1\mathcal{D}_A(R_A))$ vanish for all maximal ideals μ in a Zariski dense open subset of MaxSpec$A$?

The vanishing of $\mathrm{Tor}_1^A(A/\mu, R^1\mathcal{D}_A(R_A))$ implies the isomorphism

$$\mathcal{D}_A(R_A) \otimes_A A/\mu \cong \mathcal{D}_{A/\mu}(R_A \otimes_A A/\mu).$$

We establish that the answer is negative in the case of certain SL_2 invariants: Let $\mathbb{F}$ be a field. The hypersurface $\mathbb{F}[u,v,w,x,y,z]/(ux+vy+wz)$ is the homogeneous coordinate ring of the Grassmannian $G(2,4)$ under the Plücker embedding. For $\mathbb{F}$ an infinite field, this hypersurface arises as the invariant ring for an action of $\mathrm{SL}_2(\mathbb{F})$ on the polynomial ring $\mathbb{F}[X]$, where X is a 2×4 matrix of indeterminates; for characteristic-free proofs, see [1, 3]. The defining equation of the hypersurface is also the Pfaffian of an antisymmetric 4×4 matrix of indeterminates; hence this hypersurface is also the invariant ring for a symplectic group action. Set

$$R_{\mathbb{Z}} := \mathbb{Z}[u,v,w,x,y,z]/(ux+vy+wz),$$

and let $p > 0$ be an arbitrary prime integer. We prove that the map

$$\mathcal{D}_{\mathbb{Z}}(R_{\mathbb{Z}}) \otimes_{\mathbb{Z}} \mathbb{Z}/p \longrightarrow \mathcal{D}_{\mathbb{Z}/p}(R_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}/p)$$

is not surjective.

Our proof relies on a recent result of Jeffries [4] that provides an isomorphism

$$R^1\mathcal{D}_{\mathbb{Z}}(R_{\mathbb{Z}}) \cong H^6_{\Delta}(R_{\mathbb{Z}} \otimes_{\mathbb{Z}} R_{\mathbb{Z}}),$$

and on establishing that the local cohomology module $H^6_{\Delta}(R_{\mathbb{Z}} \otimes_{\mathbb{Z}} R_{\mathbb{Z}})$ has nonzero p-torsion elements for each prime integer p.

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