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Representations of Finite Groups

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Abstract. The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Meinolf Geck (Stuttgart), Radha Kessar (London) and Gabriel Navarro (Valencia). It covered a wide variety of aspects of representation theory of finite groups and its relations to other areas of mathematics.

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Introduction by the Organizers

The workshop *Representations of Finite Groups* was organised by Joseph Chuang (London), Meinolf Geck (Stuttgart), Radha Kessar (London) and Gabriel Navarro (Valencia). It covered a wide variety of aspects of representation theory of finite groups and its relations to other areas of mathematics, including Lie theory, homotopy theory, homological algebra, number theory and combinatorics. It was attended by 54 participants with broad geographical representation.

In fifteen lectures of 50 minutes each and twelve shorter contributions of 30 minutes each, speakers presented recent progress in the representation theory of finite groups and proposed new research directions. Plenty of time was available outside of the lectures for informal discussion between participants, either on continuing research cooperation or on new projects.

To give one example of a successful outcome of informal discussion, Geck solved a long-standing conjecture about the divisibility of degrees of Glauberman correspondents, using some recent results on Green functions (https://arxiv.org/pdf/1904.04586.pdf).

One of the major themes of the talks was progress on the remarkable local-global conjectures which drive the subject. Malle started the workshop with two generalisations of Brauer's height zero conjectures, one involving nonabelian defect groups and the other a pair of prime numbers. Spath explained new methods for the Alperin-McKay conjecture that lead to some verifications for special linear groups in non-defining characteristic. Three talks concerned refinements of the character counting conjectures involving Galois automorphisms: Turull on the computation of local invariants, Boltje on a reformulation in terms of matrix rings over the p-adic numbers and Schaeffer Fry on recent progress for groups of Lie Type. Sambale discussed to what extent versions of the counting conjectures can be formulated and proven for blocks of finite groups with respect to a set of primes. Semeraro spoke on the interpretation of the local-global conjectures for fusion systems. Finally, in a talk that closed the week, Hiss gave a detailed account of the Alperin weight conjecture in a particular example, providing a fitting reminder of the enduring mystery of the counting conjectures in modular representation theory.

Donovan's conjecture, and the exploration of properties of blocks of finite groups that distinguish them from finite dimensional algebras in general, was another focus of the week. Eaton surveyed recent progress on the conjecture, notably for the prime 2, and the related classification of Morita equivalence classes of blocks with a fixed defect group. An important ingredient in these results is the consideration of Picard groups of blocks; in their talks Livesey discussed the relevant Clifford theory while Eisele explained that even in a more general setting Picard groups are affine algebraic groups. Linckelmann spoke on another, possibly related, way that Lie theory enters the picture, giving a broad survey on what is known about the Lie algebra structure of the first Hochschild cohomology of a block.

Such investigations of the structure of blocks often rely on knowledge of special modules. Grodal presented a homotopy theory approach to endotrivial modules that allows classification and explicit calculation. Lassueur spoke on a variety of results on the lifting of endopermutation modules.

Several interesting new results were presented on the representation theory of symmetric groups. Bessendrodt described progress on the Kronecker problem and Saxl's Conjecture through 2-modular representations and spin characters. Giannelli spoke on certain monomial characters of symmetric groups, inspired by McKay's conjecture and by work of Navarro on p-solvable groups. Morotti spoke on the completion of the classification of representations of symmetric and alternating groups restricting to irreducible representations of subgroups. Fayers demonstrated a combinatorial formula a la Richards for defect 2 blocks of spin representations.

Lacabanne explored categorification in the the classification of unipotent characters of finite groups of Lie type. Taylor surveyed Geck's unitriangularity conjecture for decomposition matrices of such groups, including the announcement of a proof in almost all cases. Tiep presented new bounds on character degrees and character ratios of finite groups of Lie type - potential applications range from covering number problems to mixing times for random walks on groups. Turning to algebraic groups, Williamson presented a character formula for simple modules of reductive groups in terms of periodic polynomials, building on work on tilting modules presented at the previous workshop in 2015.

Several other interesting topics were covered. Margolis reported on the resolution of some open problems on units in group rings related to the Isomorphism Problem and the Zassenhaus Conjectures. Tong-Viet presented his recent results on the role of reality in character theory, in particular some interesting classification free proofs of theorems which were previously proved using the classification of finite simple groups. Symonds gave a proof of Carlson's coclass conjecture for finite p-groups (for all primes p). Benson reported on the study of invariants attached to the complexified representation ring of a finite group, viewed as a commutative Banach $*$-algebra (related to his talk in the mini workshop ID 1910c). Bouc gave a survey of the ambitious project, joint with Thevenaz, of developing a representation theory of sets.

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Abstracts

Completing the representation ring of a finite dimensional Hopf algebra, II

David Benson

This talk was given both at the mini workshop ID 1910c "Cohomology of Hopf Algebras and Tensor Categories" and at this workshop three weeks later. It breaks down naturally into two parts, so I have decided to report on the first part there and the second here.

In this writeup, I shall only discuss representation rings of finite groups, but there is an abstract definition of *representation ring* given by five axioms, described in the first part of the talk, and what is described here really only depends on those axioms. In the first part, we defined an invariant $\gamma_{\mathfrak{X}}(M)$ of kG-modules, and investigated its properties. This invariant is introduced again below, but in terms of spectral radius in a certain Banach $*$-algebra obtained by taking a suitable quotient of the completion of the representation ring.

Let G be a finite group and k a field of characteristic p. We only consider finitely generated kG-modules in this talk. The representation ring $a(G)$ has generators $[M]$ with M a kG-module, and relations $[M] + [N] = [M \oplus N]$ and $[M][N] = [M \otimes_k N]$. Additively, it is the free abelian group on the isomorphism classes $[M_i]$ of indecomposable kG-modules M_i, with i in a suitable indexing set $\mathfrak{I}$. This ring encodes information about summands of tensor products. There is an involution on $\mathfrak{I}$, $i \mapsto i^*$, given by duality: $M_{i^*} = M_i^*$.

We write $a_{\mathbb{C}}(G)$ for the complexified representation ring $a_{\mathbb{C}}(G) = \mathbb{C} \otimes_{\mathbb{Z}} a(G)$. We put a norm on $a_{\mathbb{C}}(G)$ as follows.

$$\left\| \sum_{i\in\mathfrak{I}} a_i[M_i] \right\| = \sum_{i\in\mathfrak{I}} |a_i| \dim(M_i).$$

We write $\hat{a}(G)$ for the completion of $a_{\mathbb{C}}(G)$ with respect to this norm. This is a commutative Banach $*$-algebra, where the star operation is defined as follows. If $x = \sum_i a_i[M_i]$ then $x^* = \sum_i \bar{a}_i[M_i^*]$.

A *representation ideal* $\mathfrak{X}$ of $a(G)$ is a proper subset of the indecomposable modules (or rather, of the indexing set $\mathfrak{I}$) such that if we tensor a module in $\mathfrak{X}$ with any module, all the summands of the answer are in $\mathfrak{X}$. Representation ideals are automatically closed under the star operation, so the closure of the span of the $[M_i]$ with $i \in \mathfrak{X}$ forms a Banach $*$-ideal in $\hat{a}(G)$. We may form the quotient with respect to this, and obtain a new Banach $*$-algebra $\hat{a}_{\mathfrak{X}}(G)$. The quotient norm is given by

$$\left\| \sum_{i\in\mathfrak{I}} a_i[M_i] \right\|_{\mathfrak{X}} = \sum_{i\in\mathfrak{I}} |a_i| \dim \mathsf{core}_{\mathfrak{X}}(M_i) = \sum_{i\in\mathfrak{I}\setminus\mathfrak{X}} |a_i| \dim(M_i).$$

In this expression, the $\mathfrak{X}$-core of a module M is defined by $\mathsf{core}_{\mathfrak{X}}(M) = M'$, where we write $M = M' \oplus M''$ in such a way that M'' is a direct sum of modules in $\mathfrak{X}$, and no summand of M' is in $\mathfrak{X}$.

If a is an element of a Banach algebra, the *spectrum* $\mathsf{Spec}(a)$ is the set of $\lambda \in \mathbb{C}$ such that $a - \lambda\mathbb{1}$ is not invertible. The *spectral radius* of a is $\sup_{\lambda\in\mathsf{Spec}(a)} |\lambda|$. The spectral radius formula (Gelfand 1941) says that the spectral radius of a is given by $\lim\limits_{n\to\infty} \sqrt[n]{\|a^n\|}$. Thus if M is a kG-module then the spectral radius of the image of $[M]$ in $\hat{a}_{\mathfrak{X}}(G)$ is given by

$$\gamma_{\mathfrak{X}}(M) = \lim_{n\to\infty} \sqrt[n]{\dim \mathsf{core}_{\mathfrak{X}}(M^{\otimes n})}.$$

A *species* of the representation ring $a(G)$ is defined to be a ring homomorphism $s\colon a(G) \to \mathbb{C}$. A species is continuous with respect to the norm on $a(G)$ if and only if it is *dimension bounded*, namely if and only if for all kG-modules M we have $|s([M])| \leqslant \dim(M)$. A species s descends to a $\mathbb{C}$-algebra homomorphism $s\colon \hat{a}_{\mathfrak{X}}(G) \to \mathbb{C}$ if and only if it is $\mathfrak{X}$*-core bouded*, meaning that for all kG-modules M we have

$$|s([M])| \leqslant \dim \mathsf{core}_{\mathfrak{X}}(M).$$

If a is an element of a commutative Banach algebra A then $\mathsf{Spec}(a)$ is the set of values of $s(a)$ as s runs over the $\mathbb{C}$-algebra homomorphisms $A \to \mathbb{C}$. In particular, the spectral radius of a is equal to $\sup\limits_{s\colon A\to\mathbb{C}} |s(a)|$. It follows that for a kG-module M, we have

$$\gamma_{\mathfrak{X}}(M) = \sup_{\substack{s\colon \mathfrak{a}\to\mathbb{C} \\ \mathfrak{X}\text{-core bounded}}} |s(x)|.$$

If A is a commutative Banach algebra, we give the set $\Delta(A)$ of algebra homomorphisms $s\colon A \to \mathbb{C}$ the *weak ∗-topology.* This is the coarsest topology for which the maps given by evaluation $s \mapsto s(a)$ are continuous for all $a \in A$. In the case $A = \hat{a}_{\mathfrak{X}}(G)$, we write $\Delta_{\mathfrak{X}}(G)$ for $\Delta(\hat{a}_{\mathfrak{X}}(G))$. For example, if $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ and k is a field of characteristic two, then $\Delta_{\mathsf{proj}}(G)$ is a wedge $S^1 \vee X$, where X is the one point compactification of the discrete set $\mathbb{P}^1(k) \times \mathbb{N}$.

The representation ideal $\mathfrak{X}_{\max}$ is the set of i such that $M_i \otimes M_i^*$ does not have a summand isomorphic to the trivial module. If k is algebraically closed, this is equivalent to the dimension of M_i being coprime to the characteristic of k. Every representation ideal is contained in $\mathfrak{X}_{\max}$, so it is the unique maximal one, hence the notation.

Theorem 1. *The quotient $\hat{a}_{\max}(G)$ of $\hat{a}(G)$ by the closure of the span of $\mathfrak{X}_{\max}$ is semisimple. Elements of $\hat{a}_{\max}(G)$ are separated by $\mathfrak{X}_{\max}$-core bounded species $s\colon \hat{a}_{\max}(G) \to \mathbb{C}$.*

There is another, slightly bigger, completion of $a_{\max}(G)$, which gives rise to a C^*-algebra. We define the *trace map* $\mathsf{Tr}\colon a(G) \to \mathbb{Z}$ via $\mathsf{Tr}(\sum_{i\in\mathfrak{I}} a_i[M_i]) = a_0$, where M_0 is the trivial module. Then we put an inner product on $a_{\max}(G)$ via

$\langle x, y\rangle = \mathsf{Tr}(xy^*)$, and a Hilbert norm via $|x| = \sqrt{\langle x, x\rangle}$. Thus

$$\left|\sum_{i\in\mathfrak{I}} a_i[M_i]\right| = \sqrt{\sum_{i\in\mathfrak{I}\setminus\mathfrak{X}_{\max}} n_i|a_i|^2}$$

where n_i is the multiplicity of the trivial module as a summand of $M_i \otimes M_i^*$. The completion of $a_{\max}(G)$ with respect to this norm is a Hilbert space $H(G)$. The following theorem is not as easy to prove as it looks.

Theorem 2. *For $x, y \in a_{\mathbb{C}}(G)$ we have $|xy| \leqslant \|x\|_{\max}|y|$.*

It follows from this theorem that left multiplication by elements of $a_{\mathbb{C}}(G)$ is continuous with respect to the Hilbert norm, and induces a continuous map from $\hat{a}_{\max}(G)$ to $\mathscr{L}(H)$, the C^*-algebra of continuous maps $H \to H$. This map preserves the star operation, and the closure of the image is a C^*-algebra which we denote $C^*_{\max}(G)$. As an example of an application, we have the following.

Theorem 3. *If $e \in \hat{a}_{\max}$ is an idempotent, $e \neq 0, 1$ then $0 < \mathsf{Tr}(e) < 1$.*

Proof. We have $e = e^*$ (in fact, this is true for any idempotent in a commutative C^*-algebra). $\mathsf{Tr}(e) = \mathsf{Tr}(e^*e) = \langle e, e\rangle > 0$. But $1 - e$ is also an idempotent, so $\mathsf{Tr}(1 - e) > 0$. □

It follows from this theorem that there are no non-trivial idempotents in $a_{\max}(G)$, since the traces are integers.

One intriguing question which comes out of all this is the following. Is it true in general that $\hat{a}(G)$ is a *symmetric* Banach $*$-algebra? A Banach $*$-algebra A is said to be symmetric if the spectrum of x^*x always consists of non-negative real numbers for all $x \in A$. This is equivalent to the statement that for all $s\colon A \to \mathbb{C}$ we have $s(x^*) = \overline{s(x)}$. So in our case, this would imply that for every dimension bounded species $s\colon a(G) \to \mathbb{C}$ and every kG-module M we have $s([M^*]) = \overline{s([M])}$. If this were true, it would imply that we always have

$$\gamma_{\mathfrak{X}}(M \otimes M^*) = \gamma_{\mathfrak{X}}(M)^2.$$

This is an interesting open problem.

Kronecker products and the Saxl conjecture

Christine Bessenrodt

It is a fundamental problem in representation theory to determine the decomposition of tensor products. Even for the symmetric group S_n and its complex representations this is unsolved; we report here on new contributions to a conjecture that has arisen in this context.

Let $\mathbf{S}^{\mathbb{C}}(\lambda)$ be the Specht module of S_n (over the complex numbers), to a partition λ of n, and χ^λ its irreducible character. The *Kronecker coefficients* $g(\lambda, \mu, \nu)$ are the expansion coefficients of the product

$$\chi^\lambda \cdot \chi^\mu = \sum_{\nu\vdash n} g(\lambda, \mu, \nu)\, \chi^\nu.$$

It is a wide open problem to give an efficient combinatorial description for the coefficients $g(\lambda, \mu, \nu)$, say akin to the Littlewood–Richardson rule. So far, only little is known about the Kronecker products, and mostly when the factors are very special (to hooks or 2-line partitions) or the constituents are special. Also, products with few constituents have been investigated, in particular simple products, and the classification of multiplicity-free Kronecker products was obtained recently [1]. Helpful tools are given by a recursion formula involving skew characters, and a monotonicity property of the Kronecker coefficients. Here, two approaches to proving positivity of Kronecker coefficients are described that come from unexpected directions and open up new connections.

1. Saxl's conjecture and spin representations

Inspired by their results on the square of the Steinberg character for simple groups of Lie type, Heide, Saxl, Tiep and Zalesskii [4] conjectured that for any $n \neq 2, 4, 9$ there is an irreducible S_n-character whose square contains all irreducible characters. For triangular numbers, we have more precisely:

Saxl's Conjecture. Let $\rho_k = (k, k-1, \ldots, 2, 1)$ be the staircase partition of $n = k(k+1)/2$. Then $(\chi^{\rho_k})^2$ contains all characters χ^ν, $\nu \vdash n$, as constituents, i.e., $g(\rho_k, \rho_k, \nu) > 0$ for all ν.

This conjecture has served as an important benchmark for new results and has motivated a lot of recent research (see e.g. [5, 6, 7]); notably, Ikenmeyer [5] proved that $g(\rho_k, \rho_k, \mu) > 0$ whenever μ dominates ρ_k.

Perhaps unexpectedly, results on spin character products for the double covers of the symmetric groups and of the alternating groups can be applied to obtain information on Kronecker coefficients in Saxl's square. This is due to a close link between the ordinary character and the spin character labelled by the staircase, given by multiplying the latter with the basic spin character. Using spin character values on critical conjugacy classes as a crucial tool led to the following strong new criterion for positivity.

Theorem. [2] Let μ be a partition with $\chi^\mu(\rho_k) \neq 0$ or $\chi^{h_\mu}(\rho_k) \neq 0$.
Then $g(\rho_k, \rho_k, \mu) > 0$.

The approach via spin characters gives new families of constituents in Saxl's square, in particular, all characters to double-hooks are detected [2]; this comprises several families of constituents found in earlier work, such as those to hooks, 2-line partitions and special double-hooks [5, 6, 7].

2. Kronecker positivity and decomposition numbers

There is also a surprisingly useful connection between the Kronecker problem and modular representation theory. It hinges on the fact that the staircase ρ_k is a 2-core, and thus $(\chi^{\rho_k})^2$ decomposes into characters ξ^ν to projective S_n-modules (at characteristic $p = 2$), say

$$(\chi^{\rho_k})^2 = \sum a_\nu \xi^\nu, \text{ with } a_\nu \in \mathbb{N}_0, \ \nu \text{ runs over the 2-regular partitions.}$$

Thus, for $a_\nu > 0$, the 2-decomposition number $d_{\lambda\nu} = \langle \chi^\lambda, \xi^\nu \rangle$ is a lower bound for the Kronecker coefficient $g(\rho_k, \rho_k, \lambda)$. As an immediate consequence all constituents of the projective character $\xi^{(n)}$ appear in the Saxl square, notably all irreducible characters of odd degree. A *projective strengthening of Saxl's conjecture* is the conjecture that all a_ν above are positive.

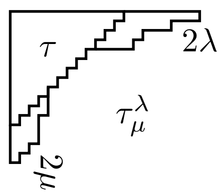
This 2-modular approach was investigated in joint work with Chris Bowman and Louise Sutton [3]. The already mentioned result by Ikenmeyer is particularly useful here. Generalising the observation on $\xi^{(n)}$ above, if λ is a 2-regular partition of n such that the 2-modular reduction of the Specht module $\mathbf{S}^{\mathbb{C}}(\lambda)$ is simple, then all constituents of ξ^λ appear in the Saxl square. As a consequence, using results from the 2-modular theory we find:

Theorem. [3] All irreducible S_n-characters of 2-height 0 are constituents of $(\chi^{\rho_k})^2$.

Unfortunately, determination of the decomposition numbers $d_{\mu\lambda}$ is a longstanding open problem in modular representation theory. But only positive lower bounds are needed here, and this allows to employ the better understood decomposition numbers for Hecke algebras of type A.

Pursuing the ideas outlined above led to study the structure of Specht modules for Hecke algebras. This work resulted in new insights both on decomposition numbers and on the Kronecker coefficients [3].

For a staircase τ and partitions λ, μ such that $\ell(\lambda) + \mu_1 \leq \ell(\tau)$, we let τ_μ^λ denote the *2-separated partition* obtained by gluing two copies of λ to the top of τ and two copies of μ to the bottom of τ.



Theorem. [3] The Specht module $\mathbf{S}^{\mathbb{C}}_{-1}(\tau_\mu^\lambda)$ of the Hecke algebra $H^{\mathbb{C}}_{-1}(n)$ is semisimple, and it decomposes as a direct sum of simples as follows:

$$\mathbf{S}^{\mathbb{C}}_{-1}(\tau_\mu^\lambda) \cong \bigoplus_\nu c(\nu^T, \lambda^T, \mu) \mathbf{D}^{\mathbb{C}}_{-1}(\tau_\emptyset^\nu),$$

where $c(\nu^T, \lambda^T, \mu)$ are Littlewood–Richardson coefficients.

In fact, even a graded version of this decomposition holds. Applying our results on the Hecke algebra decomposition numbers, we obtain the following contribution to the Saxl conjecture:

Theorem. [3] Let $w = k(k+1)/2$, $n = w(2w+1)$ and $\tau = \rho_{2w-1}$. Then for λ, μ any pair such that τ_μ^λ is a 2-separated partition, we have:

$$g(\rho_{2w}, \rho_{2w}, \tau_\mu^\lambda) \geq c(\rho_k, \lambda, \mu^T).$$

In particular, we find new constituents in Saxl's square with large multiplicities:

Corollary. [3] With notation as above, we have $g(\rho_{2w}, \rho_{2w}, \tau^{\rho(k-1)}_{(k-1,1)}) \geq k-1$.

References

[1] C. Bessenrodt, C. Bowman, *Multiplicity-free Kronecker products of characters of the symmetric groups*, Advances in Math. **322** (2017), 473–529.
[2] C. Bessenrodt, *Critical classes, Kronecker products of spin characters, and the Saxl conjecture*, Algebraic Combinatorics **1** (2018), 353–369.
[3] C. Bessenrodt, C. Bowman, and L. Sutton, *Kronecker positivity and 2-modular representation theory*, arXiv:1903.07717
[4] G. Heide, J. Saxl, P. Tiep, and A. E. Zalesski, *Conjugacy action, induced representations and the Steinberg square for simple groups of Lie type*, Proc. Lond. Math. Soc. (3) **106** (2013), 908–930.
[5] C. Ikenmeyer, *The Saxl conjecture and the dominance order*. Discrete Math. **338** (2015), 1970–1975.
[6] I. Pak, G. Panova, E. Vallejo, *Kronecker products, characters, partitions, and the tensor square conjectures.* Advances Math. **288** (2016), 702–731.
[7] I. Pak, G. Panova: *Bounds on certain classes of Kronecker and q-binomial coefficients*, J. Comb. Theory A **147** (2017), 1–17.

Alperin's weight conjecture and Wedderburn components over $\mathbb{Q}_p$

Robert Boltje
(joint work with Burkhard Külshammer)

Let p be a prime.

Definition. We call two simple $\mathbb{Q}_p$-algebras A_1 and A_2 *equivalent* if there exist positive integers m and n satisfying

$$0 \not\equiv m \equiv \pm n \mod p \quad \text{and} \quad \mathrm{Mat}_m(A_1) \cong \mathrm{Mat}_n(A_2)\,.$$

For a semisimple $\mathbb{Q}_p$-algebra C and a simple $\mathbb{Q}_p$-algebra A we denote by $[C : A]$ the number of simple factors of C which are equivalent to A.

We propose the following conjecture of block algebras of finite groups over $\mathbb{Q}_p$.

Conjecture 1. Let G be a finite group and let b be a block idempotent of $\mathbb{Z}_pG$ of positive defect. Then

$$\sum_{\sigma\in[\mathcal{P}(G)/G]} (-1)^{|\sigma|} \left[\mathrm{Mat}_{|G:G_\sigma|}(\mathbb{Q}G_\sigma b_\sigma)\right] = 0\,.$$

Here, $\mathcal{P}(G)$ denotes the set of chains $\sigma = (P_0 < P_1 < \cdots < P_n)$ of non-trivial p-subgroups of G, including the empty chain; $[\mathcal{P}(G)/G]$ denotes a set of representatives of the G-orbits of $\mathcal{P}(G)$ under the conjugation action; $|\sigma| := n$ if $\sigma = (P_0 < \cdots < P_n)$; G_σ denotes the stabilizer of σ in G; and b_σ is the sum of block idempotents of $\mathbb{Z}_pG_\sigma$ that are in Brauer correspondence to b.

Let G be a finite group, $K := \mathbb{Q}_p(\zeta)$, and $\mathcal{O} := \mathbb{Z}_p[\zeta]$, where ζ is a root of unity of order $\exp(G)$ in some algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. For any irreducible character χ of a subgroup H of G the four invariants $d(\chi)$, $r(\chi)$, $\mathbb{Q}_p(\chi)$ and $h(\chi)$ are defined as follows: Write $|H|/\chi(1) = p^{d(\chi)}r(\chi)$, with p not dividing $r(\chi)$, set $\mathbb{Q}_p(\chi) := \mathbb{Q}_p(\chi(h) \mid h \in H)$, and let $h(\chi) \in \mathbb{Q}/\mathbb{Z}$ denote the Hasse invariant of the central simple $\mathbb{Q}_p(\chi)$-algebra $\mathbb{Q}_pHe_\chi$ in the Brauer group $\mathrm{Br}(\mathbb{Q}_p(\chi)) = \mathbb{Q}/\mathbb{Z}$.

The following conjecture is a special case of a conjecture stated by Turull in [1]. In the same paper, Turull proved that the conjecture holds whenever G is a p-solvable group.

Conjecture 2. Let b be a block idempotent of $\mathcal{O}G$ of positive defect. Moreover, let d be a positive integer, $r \in \{1, \ldots, p-1\}$, $\mathbb{Q}_p \subseteq L \subseteq K$ an intermediate field, and $h \in \mathbb{Q}/\mathbb{Z}$. Then

$$\sum_{\sigma \in [\mathcal{P}(G)/G]} (-1)^{|\sigma|} \ |\mathrm{Irr}_K(G_\sigma, b_\sigma, d, r, L, h)| = 0\,.$$

Here, $\mathrm{Irr}_K(G_\sigma, b_\sigma, d, r, L, h)$ denotes the set of all irreducible characters over K of G_σ, belonging to a block occurring in the sum of blocks b_σ, and satisfying $d(\chi) = d$, $r(\chi) \equiv \pm r \mod p$, $\mathbb{Q}_p(\chi) = L$, and $h(\chi) = h$.

Our main results are the following two theorems.

Theorem A. Let b be a block idempotent of $\mathcal{O}G$ of positive defect and let $\tilde{b}$ denote the unique block idempotent of $\mathbb{Z}_pG$ with $b\tilde{b} \neq 0$. Then Conjecture 1 holds for $(G, \tilde{b})$ and all simple $\mathbb{Q}_p$-algebras A if and only if Conjecture 2 holds for (G, b) and all parameters (d, r, L, h).

Theorem B. Conjecture 1 holds for (G, b) and all simple $\mathbb{Q}_p$-algebras A, provided that b has non-trivial cyclic defect groups.

The proof of Theorem B uses the recent result by Kessar and Linckelmann in [2] that states that Rouquier's construction of splendid Rickard equivalences between blocks with cyclic defect groups and their Brauer correspondents are induced by splendid Rickard equivalences over $\mathbb{Z}_p$.

References

[1] A. Turull, *Refinements of Dade's projective conjecture for p-solvable groups*, J. Algebra **474** (2017), 424–465.

[2] R. Kessar and M. Linckelmann, *Descent of equivalences and character bijections*, in: Geometric and topological aspects of the representation theory of finite groups, 181–212, Springer Proc. Math. Stat., 242, Springer, Cham, 2018.

Simple and projective correspondence functors

Serge Bouc

(joint work with Jacques Thévenaz)

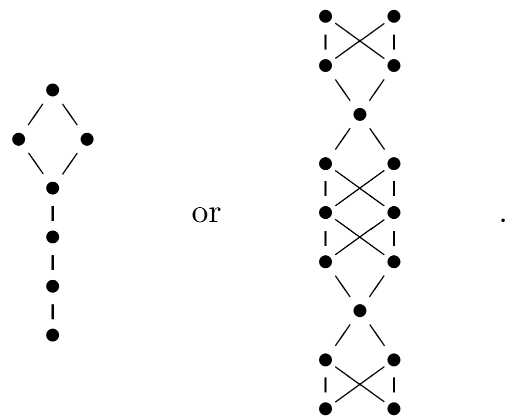
In this long term joint work with Jacques Thévenaz ([1], [3], [2], [5], [4]), we study the category of finite sets and correspondences, and its linear representations over some commutative ring k, which we call *correspondence functors* over k. The category of correspondence functors over k has various specific properties. For instance, when k is a field, we show that a finitely generated correspondence functor F over k has finite length, and this occurs if and only if the dimension of $F(X)$ is bounded by some exponential of $|X|$. Moreover such a functor is projective if and only if it is injective.

Simple correspondence functors over k are parametrized by triples (E, R, W) consisting of a finite set E, a partial order relation R on E, and a simple $k\mathrm{Aut}(E, R)$-module W. We answer the question of knowing when the simple functor $S_{E,R,W}$ is projective (or equivalently, injective):

Theorem 1. *Let k be a field and let $S_{E,R,W}$ be the simple correspondence functor parametrized by a finite set E, an order relation R on E, and a simple $k\mathrm{Aut}(E, R)$-module W. The following conditions are equivalent :*

(1) $S_{E,R,W}$ is projective.
(2) The poset (E, R) is a pole poset and W is a projective $k\mathrm{Aut}(E, R)$-module.
(3) Either (E, R) is a totally ordered poset or (E, R) is a pole poset and the characteristic of k is different from 2.

Here by a *pole poset*, we mean a finite poset obtained by stacking discrete posets of cardinality one or two, e.g.



The "official" french translation of *pole poset* is *totem.*

References

[1] S. Bouc and J. Thévenaz. Correspondence functors and finiteness conditions. *J. Algebra*, 495:150–198, 2018.
[2] S. Bouc and J. Thévenaz. The algebra of Boolean matrices, correspondence functors, and simplicity. Preprint, arXiv:1902.05422, 2019.
[3] S. Bouc and J. Thévenaz. Correspondence functors and lattices. *J. Algebra*, 518:453–518, 2019.
[4] S. Bouc and J. Thévenaz. Simple and projective correspondence functors. Preprint, arXiv:1902.09816, 2019.
[5] S. Bouc and J. Thévenaz. Tensor product of correspondence functors. Preprint, arXiv:1903.01750, 2019.

Classifying blocks of finite groups

Charles Eaton

Let $(K, \mathcal{O}, k)$ be a p-modular system where $k = \overline{\mathbb{F}}_p$. Donovan's conjecture states that, fixing a finite p-group P, there are only finitely many Morita equivalence classes amongst blocks of finite groups with defect groups $D \cong P$. The conjecture may be stated over k or $\mathcal{O}$. There are two obvious choices for a canonical choice of $\mathcal{O}$: either the ring of Witt vectors for k or the same with all $|D|$-th roots of unity attached. It seems that the former should be suitable. Donovan's conjecture defined over k was shown by Kessar to be equivalent to bounding the Cartan invariants and Morita-Frobenius number of a block in terms of P. The Morita-Frobenius number is the smallest m such that whenever B is a block of kG for a finite group G, there is a Morita equivalence between B^{σ^m} and B, where σ is the ring automorphism of kG obtained by raising coefficients of elements of G to their p-th power. An $\mathcal{O}$-Morita-Forbenius number may also be defined.

With Livesey in [5] we have shown that for P abelian, the k-Donovan conjecture reduces to checking that the Cartan invariants and the $\mathcal{O}$-Morita-Frobenius number are bounded for quasisimple groups in terms of the defect group. Farrell and Kessar in [7] bounded the Morita-Frobenius numbers for blocks of quasisimple groups, so we have the following:

Theorem. If the Cartan invariants of blocks of quasisimple groups with abelian defect groups are bounded in terms of the defect groups, then the k-Donovan conjecture holds for abelian p-groups.

With Eisele and Livesey we showed in [3] that, further:

Theorem. If the Cartan invariants of blocks of quasisimple groups with abelian defect groups are bounded in terms of the defect groups, then the $\mathcal{O}$-Donovan conjecture holds for abelian p-groups.

This involved in part showing that the $\mathcal{O}$-Donovan conjecture is equivalent to bounding the Cartan invariants and bounding the $\mathcal{O}$-Morita-Frobenius numbers.

Using work of Eaton, Kessar, Külshammer and Sambale, a consequence is:

Theorem. The $\mathcal{O}$-Donovan conjecture holds for abelan 2-groups.

In cases where Donovan's conjecture is known, we may attempt to classify Morita equivalence classes of blocks with a given defect group. Recent classifications in this direction are $\mathcal{O}$-blocks with defect groups that are abelian 2-groups of 2-rank at most three (by Eaton-Livesey [4] and Wu-Zhang-Zhou [8]), and $\mathcal{O}$-blocks with defect groups which are elementary abelian of order 16 in [2].

In applying the classification of finite simple groups to the classification of Morita equivalence classes of blocks, important steps are the comparison of blocks with those of normal subgroups when the index is prime to p or a power of p. In the case of normal subgroups of index prime to p, a main tool is the parameterisation of crossed products of an algebra by a group, as described by Külshammer. The parameterisation involves the outer automorphism group $Out(b)$ of an algebra b.

We may embed $Out(b)$ into the Picard group $Pic(b)$. The structure of Picard groups of $\mathcal{O}$ blocks has been studied recently by Boltje, Kessar and Linckelmann in [1], but limited examples were known. When b is an $\mathcal{O}$-block, in all known examples $Pic(b)$ is a finite group. With Livesey in [6] we have computed a range of examples of such Picard groups.

Progress on Donovan's conjecture and classifications is recorded in the wiki site https://wiki.manchester.ac.uk/blocks

References

[1] R. Boltje, R. Kessar, and M. Linckelmann, *On Picard groups of blocks of finite groups*, available arXiv:1805.08902

[2] C. W. Eaton, *Morita equivalence classes of blocks with elementary abelian defect groups of order 16*, arXiv 1612.03485

[3] C. W. Eaton, F. Eisele and M. Livesey, *Donovan's conjecture, blocks with abelian defect groups and discrete valuation rings*, arXiv 1809.08152

[4] C. Eaton and M. Livesey, *Classifying blocks with abelian defect groups of rank* 3 *for the prime* 2, J. Algebra **515** (2018), 1-18

[5] C. W. Eaton and M. Livesey, *Donovan's conjecture and blocks with abelian defect groups*, Proc. AMS. **147** (2019), 963-970.

[6] C. W. Eaton and M. Livesey, *Some examples of Picard groups of blocks*, arXiv 1810.10950

[7] N. Farrell and R. Kessar, *Rationality of blocks of quasi-simple finite groups*, arXiv:1805.02015v1.

[8] C. Wu, K. Zhang and Y. Zhou, *Blocks with defect group* $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^m}$, J. Algebra **510** (2018), 469–498.

Self-equivalences of blocks

Florian Eisele

(joint work with Charles W. Eaton, Michael Livesey)

Let $(K, \mathcal{O}, k)$ denote a p-modular system with $k = \bar{k}$, let G be a finite group and let $B = \mathcal{O}Gb$ be a block. We consider the group $\mathrm{Out}(B) = \mathrm{Aut}(B)/\mathrm{Inn}(B)$ of *outer automorphisms* of B, and the closely related *Picard group*

$$\mathrm{Pic}(B) = \{ \text{ invertible } B\text{-}B\text{-bimodules } \}$$

with the tensor product over B as the group operation. This group can be interpreted as the group of Morita self-equivalences of B, and it is well-known that $\mathrm{Out}(B)$ embeds into $\mathrm{Pic}(B)$ as a subgroup of finite index.

While the analogous groups for blocks defined over k are usually infinite, it was recently observed by Boltje, Kessar and Linckelmann that the outer automorphism group of a block defined over $\mathcal{O}$ is finite in all known examples. For instance, one can show that if the defect group of B is normal in G, then the group of invertible p-permutation B-B-bimodules is of finite index in $\mathrm{Pic}(B)$. Since the former is a finite group, so is the latter. It is also easy to show that if A is an $\mathcal{O}$-order which is derived equivalent to B, then $\mathrm{Pic}(A)$ is finite if and only if $\mathrm{Pic}(B)$ is. In particular, if one assumes Broué's abelian defect group conjecture over $\mathcal{O}$, then the Picard group of any block of abelian defect should be finite. However, there is, as of yet, no general explanation for the finiteness of Picard groups of blocks in general.

In this talk I will present a theorem stating that Pic(A) is an affine algebraic group over k whenever A is an $\mathcal{O}$-order in a separable K-algebra (which covers the case where A is a block). The proof involves the theory of Witt vectors, as well as variations of classical theorems of Maranda and Higman stating that lattices and $\mathcal{O}$-orders are determined up to isomorphism by their reduction modulo some large power of p, which is determined by the order of the defect group in the case of a block.

While this does not show finiteness as such, it opens up a different approach to this problem: compute the Lie algebra of this group we now know is algebraic, and compare it to the first Hochschild cohomology of A. However, whether this works (and if so, how this works) is still unclear at the moment.

I will also present a corollary of the aforementioned theorem in which the structure of an algebraic group (rather than finiteness) is sufficient to obtain the desired result. Namely, using the theorem, a well-known reduction theorem by Külshammer (see [3]) can be shown to hold over $\mathcal{O}$. This is used in [2] to reduce Donovan's conjecture for blocks of abelian defect defined over $\mathcal{O}$ to the same statement for blocks of quasi-simple groups.

References

[1] F. Eisele, *The Picard group of an order and Külshammer reduction*, preprint (2018), available at `https://arxiv.org/abs/1807.05110`

[2] C. W. Eaton, F. Eisele, M. Livesey, *Donovan's conjecture, blocks with abelian defect groups and discrete valuation rings*, preprint (2018), available at `https://arxiv.org/abs/1809.08152`

[3] B. Külshammer, *Donovan's conjecture, crossed products and algebraic group actions*, *Israel J. Math.*, 92(1-3):295–306, 1995.

Double covers of symmetric groups and Fock spaces

Matthew Fayers

This talk starts with an attempt to further our understanding of the decomposition numbers for spin representations of symmetric groups in odd characteristic; in particular, to provide an analogue of Richards's combinatorial formula [1] for the decomposition numbers for defect 2 blocks of symmetric groups. By developing the appropriate combinatorics, I have been able to find an analogue modulo a mysterious "adjustment matrix". This is intimately related to the canonical basis for the q-deformed Fock space in type $A^{(2)}_{p-1}$, via a conjecture of Leclerc–Thibon [2]. I will talk about how the mysterious adjustment can be explained using the Fock space in type $A^{(2)}_{p}$ and folding of Dynkin diagrams.

References

[1] M. Richards, *Some decomposition numbers for Hecke algebras of general linear groups*, Math. Proc. Cambridge Philos. Soc. **119** (1996), 383–402.

[2] B. Leclerc & J.–Y. Thibon, *q-deformed Fock spaces and modular representations of spin symmetric groups*, J. Physics A **30** (1997), 6163–6176.

On monomial characters and Sylow p-subgroups of Symmetric Groups

Eugenio Giannelli

In this talk we discuss the relation between the representation theory of the symmetric group S_n and that of its Sylow subgroups. The starting point is the following result, obtained in collaboration with Gabriel Navarro [2].

Theorem 1. *Let p be a prime, let $P \in \mathrm{Syl}_p(S_n)$ and let χ be an irreducible character of S_n. Then the restriction χ_P admits linear constituents.*

Theorem 1 naturally raises the following question, which is at the heart of our recent research.

Problem A: *Given $\chi \in \mathrm{Irr}(\mathfrak{S}_n)$, can we find the linear constituents of χ_P?*

Let ϕ be a linear character of $P \in \mathrm{Syl}_p(S_n)$ and let $\Omega_n(\phi)$ be the set of irreducible characters of S_n appearing as irreducible constituents of the monomial character obtained by induction of ϕ to S_n. It is clear that Problem A is equivalent to the determination of the set $\Omega_n(\phi)$, for all linear characters ϕ.

A first result, obtained in collaboration with Stacey Law [1], describes the important case where $\phi = 1_P$ is the trivial character of P.

Theorem 2. *Let $n \in \mathbb{N}$ and let $p \geqslant 3$. Let χ, χ' be the two irreducible characters of $\mathfrak{S}_n$ of degree $n-1$. Then*

$$\Omega_n(1_P) = \begin{cases} \mathrm{Irr}(\mathfrak{S}_n) \smallsetminus \{\chi, \chi'\} & \text{if } n = p^k \text{ for some } k \in \mathbb{N}, \\ \mathrm{Irr}(\mathfrak{S}_n) & \text{otherwise.} \end{cases}$$

Let $\mathcal{H}$ be the Hecke algebra corresponding to the triplet $(\mathfrak{S}_n, P, 1_P)$. A straightforward application of Theorem 3 allows us to count the number of irreducible representations of $\mathcal{H}$. The determination of the dimensions of those representations remains a mistery. In order to compute them, we would need to understand the multiplicity $\langle \chi_P, 1_P \rangle$ for all $\chi \in \mathrm{Irr}(S_n)$.

In the second part of the talk we will discuss our work on problem A, in full generality. A first key observation is the following extension to symmetric groups of a result of Navarro [3], showing that the analogous statement holds for p-solvable groups.

Theorem 3. *Let $n \in \mathbb{N}$, let p be a prime and let $\phi, \psi \in \mathrm{Lin}(P)$. Then ϕ and ψ are $N_{\mathfrak{S}_n}(P)$-conjugate if and only if $\phi^{S_n} = \psi^{S_n}$.*

We conclude by giving a precise description (for odd primes) of the sets $\Omega_n(\phi)$, for any given linear character ϕ of P. Our results show that $\Omega_n(\phi)$ contains a large proportion of irreducible characters of S_n. In particular, letting Ω_n be the intersection of the sets $\Omega_n(\phi)$, we have the following corollary.

Corollary 4. *Let p be an odd prime. Then $\frac{|\Omega_n|}{|\mathrm{Irr}(S_n)|} \to 1$, for $n \to \infty$.*

References

[1] E. Giannelli, S. Law, On permutation characters and Sylow p-subgroups of $\mathfrak{S}_n$, *J. Algebra* **506** (2018), 409–428.

[2] E. Giannelli, G. Navarro, Restricting irreducible characters to Sylow p-subgroups, *Proc. Amer. Math. Soc.* **146** (2018), no. 5, 1963–1976.

[3] G. Navarro, Linear characters of Sylow subgroups, *J. Algebra* **269** (2003), 589–598.

Understanding and classifying endotrivial modules

Jesper Grodal

In my talk I described a method for calculating endotrivial modules, using methods from homotopy theory.
The starting point was the following theorem from arXiv:1608.00499

Theorem 1. *Let G be a finite group with Sylow p-subgroup S, and k a field of characteristic p. The Green correspondence induces a bijection*

$$\{kG\text{−modules } M \text{ s.t. } M|_S \simeq k^s \oplus (kS)^t\} \longleftrightarrow k\pi_1\mathcal{O}_p^*(G)\text{−modules},$$

where $\mathcal{O}_p^*(G)$ is the orbit category with objects G/P for P a non-trivial p-subgroup and morphisms G-maps.

I then presented a number of calculations arising from this viewpoint. Taken together they enable a calculation of the group of endotrivial modules $T_k(G)$ for all finite simple groups.

On the inductive blockwise Alperin weight condition for $F_4(q)$ in characteristic 3

Gerhard Hiss
(joint work with Jianbei An, Frank Lübeck)

Britta Späth in [4] has reduced the famous Alperin weight conjecture for ℓ-blocks of finite groups to a statement called *inductive blockwise Alperin weight condition.* We present some steps on the way to verify this condition for the simple Chevalley groups $F_4(q)$ for the prime $\ell = 3$, where q is a prime power not divisible 3. Major ingredients are a classification of the semisimple conjugacy classes of $F_4(q)$ and a classification of its radical 3-subgroups (see [2, 1]). The classification of the 3-blocks and their invariants is also essential.

The groups $F_4(q)$ arise from simple, self-dual algebraic groups $\mathbf{G}$ with trivial, hence connected center. All proper Levi subgroups of $\mathbf{G}$ are of classical type.

Moreover, the outer automorphism group of $F_4(q)$ is cyclic. These facts simplify our analysis to some extent. On the other hand, 3 is a bad prime for $\mathbf{G}$, and the Sylow 3-subgroups of $F_4(q)$ are non-abelian, facts which complicate our investigation.

We indicate how the recent result by Bonnafé, Dat and Rouquier [3] can be used to reduce the problem to isolated 3-blocks. We present our main result for the principal 3-block. This exhibits a remarkably different behaviour in the two cases where 9 does or does not divide $q^2 - 1$.

References

[1] J. An and H. Dietrich, *Radical 3-subgroups of $F_4(q)$ with q even,* J. Algebra **398** (2014), 542–568.

[2] J. An and S.-C. Huang, *Radical 3-subgroups and essential 3-rank of $F_4(q)$,* J. Algebra **376** (2013), 320–340.

[3] C. Bonnafé, J.-F. Dat and R. Rouquier, *Derived categories and Deligne-Lusztig varieties II*, arXiv:1511.04714v3.

[4] B. Späth, *A reduction theorem for the blockwise Alperin weight conjecture*, J. Group Theory **16** (2013), 159–220.

Categorification of Fourier matrices for Coxeter groups with automorphism

Abel Lacabanne

In order to classify unipotent characters of a finite Chevalley or Steinberg group, Lusztig has introduced a non-abelian Fourier transform [Lus79]. It is possible to understand this transformation in terms of modular categories, which are braided fusion categories over a field with some extra assumptions. Geck and Malle [GM03] have constructed similar transform for Suzuki and Ree groups, and we give in this talk a categorical framework which enable us to recover the matrices of these Fourier transforms.

Let $\mathcal{C}$ be a modular category containing $\mathrm{Rep}(\mathbb{Z}/2\mathbb{Z})$ as a fusion subcategory. This gives a grading $\mathcal{C} = \mathcal{C}_0 \oplus \mathcal{C}_1$ and it is possible to recover matrices satisfying the same properties of the Fourier matrices, by using this grading as a crucial tool.

Moreover, there exists a generalization of these transforms to spetses, in the sense of Broué-Malle-Michel [BMM99, BMM14]. In the case of dihedral groups without automorphism, Lusztig defined an exotic Fourier transform and explains how to categorify it [Lus94]. The case of dihedral groups with automorphism as well as the Ree group of type ${}^2F_4$ fit in the categorical framework presented in this talk and are very interesting examples. We emphasize the fact that the degree 0 of the category gives the Fourier transform for the non-twisted group, whereas the degree 1 gives the Fourier transform for the twisted group.

References

[BMM99] M. Broué, G. Malle and J. Michel – *Towards spetses I*, Transform. Groups **4** (1999), no. 2-3, p. 157–218.
[BMM14] M. Broué, G. Malle and J. Michel – *Split spetses for primitive reflection groups*, Astérisque (2014), no. 359, p. vi+146.
[GM03] M. Geck and G. Malle – *Fourier transforms and Frobenius eigenvalues for finite Coxeter groups*, J. Algebra **260** (2003), no. 1, p. 162–193.
[Lus79] G. Lusztig – *Unipotent representations of a finite Chevalley group of type E_8*, Quart. J. Math. Oxford Ser. (2) **30** (1979), no. 119, p. 315–338.
[Lus94] G. Lusztig – *Exotic Fourier transform*, Duke Math. J. **73** (1994), no. 1, p. 227–241, 243–248.

On the Lifting of the Dade Group and Consequences

Caroline Lassueur
(joint work with Jacques Thévenaz)

Let p be a prime number and G be a finite group of order divisible by p. Let $\mathcal{O}$ denote a complete discrete valuation ring of characteristic 0 with residue field $k := \mathcal{O}/J(\mathcal{O})$ of positive characteristic p, and let $R \in \{k, \mathcal{O}\}$.

The Dade group and endo-permutation modules are important invariants of block theory of finite groups. For instance, they occur in the description of source algebras of blocks, or as sources of simple modules for p-soluble groups (see [Thé95, §50]). They also play an important rôle in the description of equivalences between block algebras, such as derived equivalences in the sense of Rickard or basic Morita equivalences. The final classification of endo-permutation modules was obtained by Bouc [Bou06] in 2006, but still some questions about endo-permutation modules remained open.

1. To begin with, the Dade group is defined for a p-group but its definition cannot be passed as such to arbitrary groups in general. The main reason being that transitive permutation modules over p-groups are indecomposable, but this is no longer the case over arbitrary finite groups. In [Las12, Las13], we proved that if one replaces endo-permutation modules with the so-called *strongly capped endo-p-permutation modules*, i.e. RG-modules, the endomorphism ring of which is a p-permutation module with a unique trivial direct summand, then one can define a group structure on this class of modules under the tensor product, generalising the structure of the Dade group of a p-group. The structure of this *generalised Dade group* is further investigated in [Las13].

2. In [LT18], we proved that any endo-p-permutation module is liftable from positive characteristic p to characteristic zero. Amongst finitely generated kG-modules very few classes of modules are known to be liftable to $\mathcal{O}G$-lattices. Projective kG-modules are known to lift uniquely, and more generally, so do p-permutation kG-modules. In the special case where the group G is a p-group, Alperin [Alp01] proved that endo-trivial kG-modules are liftable, and Bouc observed that so are endo-permutation kG-modules as a consequence of their classification [Bou06]. Passing to arbitrary groups, the speaker together with Malle

and Schulte [LMS16] proved that Alperin's result extends to endo-trivial modules over arbitrary groups. It is therefore legitimate to ask whether Bouc's result may be extended to arbitrary groups, as well. The class of *endo-p-permutation kG*-modules was the natural candidate for such a generalisation. We emphasise that our proof relies on a non-trivial result, namely the lifting of endo-permutation modules, which is a consequence of their classification. Moreover, there are two crucial points to our argument: the first one is the fact that reduction modulo p applied to the class of endo-p-permutation $\mathcal{O}G$-lattices preserves both indecomposability and vertices, while the second one relies on properties of the G-algebra structure of the endomorphism ring of endo-permutation RG-lattices. Finally, we note that these results also provides us with an alternative proof to the Fong/Swan-Theorem stating that simple modules for p-soluble groups are liftable.

3. Coming back to p-groups, our work in [LT18] in Part 2. above outlined that another question about the Dade group had been left open, namely: "How canonical is the lifting of endo-permutation modules?" More precisely, given a p-group P, reduction modulo p induces a group homomorphism $D_{\mathcal{O}}(P) \longrightarrow D_k(P)$ between the Dade group $D_{\mathcal{O}}(P)$ of endo-permutation $\mathcal{O}P$-lattices and the Dade group $D_k(P)$ of endo-permutation kP-modules. As aforementioned, as a consequence of Bouc's final classification [Bou06] this morphism is surjective. In [LT19] we prove that this reduction homomorphism in fact always admits a section which is again a group homomorphism. As a consequence the Dade group of endo-permutation $\mathcal{O}P$-lattices can always be express as a direct product

$$D_{\mathcal{O}}(P) \cong X_{\mathcal{O}}(P) \times D_k(P)$$

where $X_{\mathcal{O}}(P)$ is the group of one-dimensional $\mathcal{O}P$-lattices. The result is easy and well-known for $p > 3$ (see e.g. [Thé95, (29.6)]), whereas in case $p = 2$ we need to use the structure of the Dade group described by Bouc [Bou06]. However in all characteristics it is possible to lift a well-chosen set of generators of $D_k(P)$ to their unique lift with trivial determinant in order to obtain a group-theoretic section.

References

[Alp01] J. L. Alperin, *Lifting endo-trivial modules*, J. Group Theory **4** (2001), no. 1, 1–2.

[Bou06] S. Bouc, *The Dade group of a p-group*, Invent. Math. **164** (2006), 189–231.

[Las12] C. Lassueur, *Relative Projectivity and Relative Endo-trivial Modules*, Doctoral Thesis, Thèse EPFL no 5266, DOI:10.5075/epfl-thesis-5266.

[Las13] C. Lassueur, *The Dade group of a finite group*, J. Pure Appl. Algebra **217** (2013), 97–113.

[LMS16] C. Lassueur, G. Malle, E. Schulte, *Simple endotrivial modules for quasi-simple groups*, J. Reine Angew. Math. **712** (2016), 141–174.

[LT18] C. Lassueur and J. Thévenaz, *Lifting endo-p-permutation modules*, Archiv Math. **110** (2018), 205–212.

[LT19] C. Lassueur and J. Thévenaz, *On the lifting of the Dade group*, to appear in J. Group Theory (2019), DOI:10.1515/jgth-2018-0145.

[Thé95] J. Thévenaz, G-Algebras and modular representation theory. Clarendon Press, Oxford, 1995.

On the Lie algebra structure of $HH^1(B)$

Markus Linckelmann

1. Background on Hochschild cohomology

Let k be a field and A a finite-dimensional k-algebra. The *Hochschild cohomology of A* is the graded Ext-algebra $HH^*(A) = \mathrm{Ext}^*_{A\otimes_k a^{\mathrm{op}}}(A, A)$. We have $HH^0(A) = \mathrm{End}_{A\otimes_k A^{\mathrm{op}}}(A) \cong Z(A)$, and

$$HH^1(A) = \mathrm{Der}(A)/\mathrm{IDer}(A) \ ,$$

where $\mathrm{Der}(A)$ is the k-vector space of all k-linear maps $f : A \to A$ satisfying the product rule $f(ab) = f(a)b + af(b)$ for all a, $b \in A$. Any such linear map is called a *derivation* on A. For any $c \in A$, the map $[c, -]$ sending $a \in A$ to the additive commutator $[c, a] = ca - ac$ is a derivation. Any derivation of this form is called an *inner derivation* on A, and $\mathrm{IDer}(A)$ denotes the subspace of inner derivations in $\mathrm{Der}(A)$.

By results of Gerstenhaber [4], the algebra $HH^*(A)$ is graded-commutative, and carries a structure of a graded Lie algebra of degree -1, called the *Gerstenhaber bracket.* In particular, $HH^1(A)$ is a Lie algebra, with bracket induced by $[f, g] = f \circ g - g \circ f$ for any two derivations f, g on A. If $\mathrm{char}(k) = p > 0$, then $HH^1(A)$ is a restricted Lie algebra, with the p-power map induced by $f^{[p]} = f \circ f \circ \cdots \circ f$, where f is being composed p times with itself.

By a result of Tradler [19], if A is a *symmetric* k-algebra (that is, A is isomorphic to its k-dual $A^\vee = \mathrm{Hom}_k(A, k)$ as an A-A-bimodule), then there is a linear degree -1 operator

$$\Delta : HH^*(A) \to HH^{*-1}(A)$$

on $HH^*(A)$ such that $\Delta \circ \Delta = 0$ and such that for homogeneous elements ζ, τ in $HH^*(A)$ we have

$$[\zeta, \tau] = (-1)^{|\zeta|}\Delta(\zeta\tau) - (-1)^{|\zeta|}\Delta(\zeta)\tau - \zeta\Delta(\tau)$$

That is, the Gerstenhaber bracket is determined by the cup product in $HH^*(A)$ and the operator Δ, measuring how far the operator Δ is from being a derivation on $HH^*(A)$. In this way, $HH^*(A)$ has a structure of *Batalin-Vilkovisky algebra*, or BV-algebra for short. The operator Δ is called the BV operator on $HH^*(A)$.

2. Functoriality properties of $HH^*(A)$

An algebra homomorphism does not in general induce a map on Hochschild cohomology. If we consider the 2-category of symmetric algebras, with 1-morphisms the bimodules which are finitely generated projective on either side (and 2-morphisms the bimodule homomorphisms between any two such bimodules), then $HH^1(A)$ becomes functorial with respect to 1-morhisms. More precisely, by [8], given two symmetric k-algebras A, B and an A-B-bimodule M which is finitely generated projective as a left and right module, there is a graded k-linear transfer map

$$\mathrm{tr}_M : HH^*(B) \to HH^*(A)$$

which is functorial in M (that is, compatible with tensor products of bimodules). The transfer map depends on choices of bimodule isomorphisms between A, B and their duals, but it is not difficult to determine the effect of different choices on the transfer maps.

In general, tr_M preserves neither the cup product, nor the Gerstenhaber bracket or BV operator, but there is one case where this does happen. Following terminology due to Broué, an A-B-bimodule M as above is said to induce a *stable equivalence of Morita type* if $M \otimes_B M^\vee \cong A \oplus X$ for some projective A-A-bimodule X and $M^\vee \otimes_A M \cong B \oplus Y$ for some projective B-B-bimodule Y. (If $X = Y = 0$, then M and $M^\vee$ induce a Morita equivalence, and by a result of Rickard, if A and B are derived equivalent, then there exists a stable equivalence of Morita type between A and B.)

It is easy to see that then tr_M induces (up to a suitable choice of bimodule isomorphisms between A, B and their duals) an isomorphism $HH^*(B) \cong HH^*(A)$ as graded algebras, except for some adjustments in degree 0, see [8, Remark 2.13]. König, Liu, and Zhou proved in [7] that tr_M is in that case compatible with the BV operators, and hence induces an isomorphism as graded Lie algebras (and as before, with the same adjustments in degree 0). Rubio y Degrassi [15] showed moreover a compatibility with the p-power map in degree 1 on the subspaces spanned by the images of integrable derivations.

3. Hochschild cohomology of blocks of group algebras

For G a finite group, it is well-known that there is an additive decomposition

$$HH^*(kG) \cong \oplus_x \, H^*(C_G(x); k) \ ,$$

where x runs over a set of representatives of the conjugacy classes of G. The summand for $x = 1$ yields an injective graded k-algebra homomorphism

$$H^*(G; k) \to HH^*(kG) \ .$$

In general, this decomposition is not an isomorphism as graded algebras. Siegel and Witherspoon describe in [17] the cup product in $HH^*(kG)$ in terms of this decomposition and the cup products of the summands. Liu and Zhou describe in [12] explicitly the BV operator in terms of the standard Hochschild resolution. As a consequence of results of Menichi [13], the BV operator Δ on $HH^*(kG)$ preserves this additive decomposition. In recent joint work with D. Benson and R. Kessar, we describe the components of the BV operator on the summands of this decomposition (this is a special case of a more general recipe to construct degree -1 operators in Ext-algebras).

Theorem 3.1 ([2])**.** *Let G be a finite group and $z \in Z(G)$. Let (P, δ) be a projective resolution of the trivial kG-module k. Multiplication by $z - 1$ on P is a contractible chain endomorphism of P, so there is a homotopy $s : P[1] \to P$ such that $s \circ \delta + \delta \circ s$ is equal to multiplication by $z - 1$. Then* $\mathrm{Hom}_{kG}(s, k) : \mathrm{Hom}_{kG}(P, k) \to$

$\mathrm{Hom}_{kG}(P[1], k)$ is a cochain homomorphism, and the induced graded map in degree -1 in cohomology is the component $\Delta_z : H^*(G,k) \to H^{*-1}(G;k)$ of the BV operator.

In block theory, the Hochschild cohomology $HH^*(B)$ of a block B of a finite group algebra kG is one of the few global invariants of B with a direct connection to the local structure of B. The second part of the following result was first conjectured by Pakianathan and Witherspoon [14]. We assume that k is large enough and has prime characteristic p.

Theorem 3.2 ([8], [9])**.** *Let G be a finite group, B a block algebra of kG with a defect group P and fusion systen $\mathcal{F}$ on P. Then $HH^*(B)$ has a graded subalgebra $H^*(B)$ defined in terms of P and $\mathcal{F}$ (as the subalgebra of $\mathcal{F}$-stable elements in $H^*(P;k)$), such that $HH^*(B)$ is finitely generated as a module over $H^*(B)$ and such that upon taking quotients by nilpotent ideals, the inclusion $H^*(B) \to HH^*(B)$ becomes an isomorphism.*

This yields another proof of the well-known fact that the Krull dimension of $HH^*(B)$ is equal to the rank of P (by which we mean the rank of an elementary abelian subgroup of P of maximal order). This also implies that for the purpose of calculating cohomology varieties, we may as well use $HH^*(B)$ instead of $H^*(P;k)$.

The dimension of $HH^0(B) \cong Z(B)$ is equal to the number of ordinary irreducible characters of B. By a theorem of Brauer and Feit, this number is bounded in terms of a defect group P. The following result shows that the dimension of $HH^n(B)$ is bounded in terms of P for all positive integers n as well.

Theorem 3.3 ([5], [6])**.** *Let B be a block of a finite group algebra with defect group P. Then the Hilbert series $\sum_{n\geq 0} \dim_k(HH^n(B))t^n$ and the isomorphism class of P determine each other up to finitely many possibilities.*

The proof first bounds the individual dimensions of $HH^n(B)$ in terms of P and then uses Symonds' proof of Benson's regularity conjecture, stating that $\mathrm{reg}(H^*(G;k)) = 0$, to bound the degrees of generators in $HH^*(B)$ in terms of P. Donovan's conjecture for P would imply that there are only finitely many isomorphism classes of graded algebras arising as $HH^*(B)$ of some block with defect groups isomorphic to P. This would follow if we could show that the relations of a set of generators of $HH^*(B)$ involve coefficients which belong to a finite field of a size bounded in terms of P. This is still an open problem in general.

4. Degree 1 Hochschild cohomology of blocks

We keep the assumption that k is a sufficiently large field of prime characteristic p. Let B be a block of a finite group algebra kG with a nontrivial defect group P. It is not known in general whether $HH^1(B)$ is necessarily nonzero, but if it is, then $HH^1(B)$ is a restricted Lie algebra. Little is known which Lie algebras arise in this way. This question has received recently a fair amount of attention; see the papers [1], [3], [10], [11], [15], and [16], for instance. Simple Lie algebras arising from blocks with a single isomorphism class of simple modules can be described as follows.

Theorem 4.1 ([10])**.** *Let B be a block of a finite group algebra kG with a defect group P. Suppose that B has a single isomorphism class of simple modules. The following are equivalent.*

(i) *The Lie algebra $HH^1(B)$ is simple.*
(ii) *The block B is nilpotent and P is elementary abelian of order at least* 3.
(iii) *The Lie algebra $HH^1(B)$ is isomorphic to the Jacobson-Witt Lie algebra* $\mathrm{Der}(k[x_1, x_2, .., x_n]/(x_1^p, x_2^p, .., x_n^p))$ *for some positive integer n such that $p^n \geq 3$.*

We do not know whether the hypothesis on B having a single isomorphism class of simple modules is necessary for this theorem. Note that in the situation of the above theorem, none of the other simple modular Lie algebras can arise. Extending earlier results of Strametz [18] for monomial algebras, we have the following sufficient criterion for the solvability of $HH^1(A)$.

Theorem 4.2 ([11], [16])**.** *Let A be a finite-dimensional split k-algebra such that the quiver of A is a simple directed graph. Then $HH^1(A)$ is a solvable Lie algebra.*

Comprehensive results on the Lie algebra structure of $HH^1(B)$ for tame blocks B have recently been obtained in the papers [3] and [16]. A remarkable consequence of these two papers is that for tame blocks the hypothesis 'one simple module' in Theorem 4.1 is indeed not necessary. This points to the main motivation for this line of enquiry - namely that a precise knowledge of the Lie algebra structure of $HH^1(B)$ of a block B should contain significant information about numerical invariants of B. Since the Lie algebra $HH^1(B)$ is invariant under stable equivalences of Morita type, this might lead to some insight as to which numerical invariants of blocks are invariant under stable equivalences of Morita type - with the dream scenario of getting hold of special cases of the Auslander-Reiten conjecture, predicting that the number of isomorphism classes of simple B-modules is invariant under stable equivalences of Morita type.

References

[1] D. Benson, R. Kessar, and M. Linckelmann, *On blocks of defect two and one simple module, and Lie algebra structure of HH^1.* J. Pure Appl. Algebra **221** (2017), 2953–2973.

[2] D. Benson, R. Kessar, and M. Linckelmann, *On the BV structure of the Hochschild cohomology of finite group algebras.* Preprint (2018).

[3] F. Eisele and T. Raedschelders, *On solvability of the first Hochschild cohomology of a finite-dimensional algebra.* arXiv:1903.07380v1 (2019).

[4] M. Gerstenhaber, *The cohomology structure of an associative ring.* Ann. of Math. **78** (1963) 267–288.

[5] R. Kessar and M. Linckelmann, *Bounds for Hochschild cohomology of block algebras* J. Algebra **337** (2011), 318–322.

[6] R. Kessar and M. Linckelmann, *On the Hilbert series of Hochschild cohomology of block algebras* J. Algebra **371** (2012), 457–461.

[7] S. Koenig, Y. Liu, and G. Zhou, *Transfer maps in Hochschild (co)homology and applications to stable and derived invariants and to the Auslander-Reiten conjecture.* Trans. Amer. Math. Soc. **364** (2012) 195–232.
[8] M. Linckelmann, *Transfer in Hochschild cohomology of blocks of finite groups*, Algebras Representation Theory **2** (1999), 107–135.
[9] M. Linckelmann, *Hochschild and block cohomology varieties are isomorphic.* J. London Math. Soc. **81** (2010), 389–411.
[10] M. Linckelmann and L. Rubio y Degrassi, *Block algebras with HH^1 a simple Lie algebra.* Quart. J. Oxford **69** (2018), 1123–1128.
[11] M. Linckelmann and L. Rubio y Degrassi, *On the Lie algebra structure of $HH^1(A)$ of a finite-dimensional algebra.* arXiv:1903.08484 (2019).
[12] Y. Liu and G. Zhou, *The Batalin-Vilkovisky structure over the Hochschild cohomology ring of a group algebra.* J. Noncommut. Geom. **10** (2016), 811–858.
[13] L. Menichi, *Batalin–Vilkovisky algebras and cyclic cohomology of Hopf algebras*, K-Theory **32** (2004), 231–251.
[14] J. Pakianathan, S. Witherspoon, *Hochschild cohomology and Linckelmann cohomology for blocks of finite groups*, J. Pure Applied Algebra **178** (2003), 87–100.
[15] L. Rubio y Degrassi, *Invariance of the restricted p-power map on integrable derivations under stable equivalences.* J. Algebra **469** (2017), 288–301.
[16] L.Rubio y Degrassi, S. Schroll, and A. Solotar, *On the solvability of the first Hochschild cohomology space as Lie algebra.* arXiv:1903.12145 (2019).
[17] S. F. Siegel and S. J. Witherspoon, *The Hochschild cohomology ring of a group algebra*, Proc. London Math. Soc. **79** (1999), 131–157.
[18] C. Strametz, *The Lie algebra structure on the first Hochschild cohomology group of a monomial algebra.* J. Algebra Appl. **5** (3) (2006), 245–270.
[19] T. Tradler, *The Batalin–Vilkovisky algebra on Hochschild cohomology induced by infinity inner products*, Ann. Inst. Fourier (Grenoble) **58** (2008), 2351–2379.

On Picard groups of blocks with normal defect groups

Michael Livesey

Let p be a prime, $(K, \mathcal{O}, k)$ a p-modular system G be a finite group and B a block of $\mathcal{O}G$.

Definition 1.

$$\mathrm{Pic}(B) := \{M \text{ a } B\text{-}B\text{-bimodule} | M \otimes_B - \text{ induces a Morita auto-equivalence of } B\},$$

called the Picard group of B.

$$\mathcal{T}(B) := \{M \in \mathrm{Pic}(B) | M \text{ has trivial source}\},$$
$$\mathcal{L}(B) := \{M \in \mathrm{Pic}(B) | M \text{ has linear source}\},$$
$$\mathcal{E}(B) := \{M \in \mathrm{Pic}(B) | M \text{ has endopermutation source}\}.$$

Let $N \lhd G$ and b a G-stable block of $\mathcal{O}N$. When investigating the possible Morita equivalence classes of B it becomes important to understand the elements of $\mathrm{Pic}(b)$ induced by the elements of G (see [5]). So when one is studying Donovan's conjecture or attempting to classify blocks up to Morita equivalence, Picard groups of blocks are very relevant.

Theorem 1 (L). *Let G be a finite group and B a block of $\mathcal{O}G$ with normal abelian defect group D and abelian inertial quotient, then* $\mathrm{Pic}(B) = \mathcal{L}(B)$.

We note that this improves upon a result of Zhou [7, Theorem 14]. Zhou proves that if $B = \mathcal{O}(D \rtimes E)$, where D is an abelian p-group and E is an abelian p'-group then $\mathrm{Pic}(B) = \mathcal{E}(B)$. We can also compare with a result of Boltje, Linckelmann and Kessar [2, Proposition 4.3], where it is assumed in addition that $[D, E] = D$ but the result is that $\mathrm{Pic}(B) = \mathcal{T}(B)$.

Proof. By [4, Theorem A] we can assume $B = \mathcal{O}(D \rtimes E)e_\varphi$, where E is a p'-group, $Z \leq Z(E)$ is cyclic, E/Z is abelian and acts faithfully on D and ϕ is a faithful irreducible character of Z.

Now $D = D_1 \times D_2$, where $D_1 = [D, E]$ and $D_2 = C_D(E)$. Let $M \in \mathrm{Pic}(B)$ and I_M the corresponding permutation of $\mathrm{Irr}(B)$. We use perfect isometries to prove that there exists $\theta \in \mathrm{Irr}(D_2)$ such that

$$I_M(\chi \otimes 1) \mapsto \chi' \otimes \theta,$$

for all $\chi \in \mathrm{Irr}(\mathcal{O}(D_1 \rtimes E)e_\phi)$. We next note that there exists a unique subgroup $D' \leq D_1$ such that $\chi \in \mathrm{Irr}(B)$ reduces to multiple copies of the same $\varphi \in \mathrm{IBr}(B)$ if and only if $D' \leq \ker(\chi)$. Using Weiss' condition [6] this allows us to reduce to the case where B has a unique simple module.

Finally the one simple module case is dealt with by studying the basic algebra of B. This builds on work of Benson and Green [1] and Holloway and Kessar [3] where the basic algebra of $k \otimes_k B$ is calculated. □

References

[1] D. Benson, E. Green, *Non-principal blocks with one simple module*, Quart. J. Math. **55**(1) (2004), 1–11.

[2] R. Boltje, R. Kessar, and M. Linckelmann, *On Picard groups of blocks of finite groups*, J. Algebra (2019), available arXiv:1805.08902

[3] M. Holloway and R. Kessar, *Quantum complete rings and blocks with one simple module*, Quart. J. Math. **56** (2005), 209–221.

[4] B. Külshammer, *Crossed products and blocks with normal defect groups*, Comm. Algebra **13**(1) (1985), 147–168.

[5] B. Külshammer, *Donovan's conjecture, crossed products and algebraic group actions*, Israel J. Math. **92** (1995), 295–306.

[6] A. Weiss, *Rigidity of p-adic p-torsion*, Ann. Math. **127** (1988), 317–332.

[7] Y. Zhou, *Morita equivalences between some blocks for p-solvable groups*, Proc. AMS **133**(11) (2005), 3133–3142.

Around Brauer's height zero conjecture

Gunter Malle

We discussed two quite different generalisations of Brauer's height zero conjecture. The first is Robinson's conjecture from 1996 which stipulates that the defect of every irreducible character χ of a finite group G in a p-block with defect group D is bounded below by

$$p^{\mathrm{def}(\chi)} \geq |Z(D)|,$$

where $Z(D)$ denotes the centre of the defect group, with equality if and only if D is abelian. In the case when D is abelian this reduces to the proven direction of Brauer's height zero conjecture.

In joint work with Z. Feng, C. Li, Y. Liu and J. Zhang we were able to show this conjecture for all odd primes p. Our proof relies on a reduction by Murai to the case of quasi-simple groups and then a detailed investigation of the p-blocks of these groups, using the reduction theorem by Bonnafé and Rouquier and various results of Cabanes and Enguehard on blocks of finite reductive groups and their defect groups. The crucial case is the one of isolated blocks at bad primes. We also showed that Robinson's conjecture holds for all 2-blocks of quasi-simple groups of classical Lie type.

The second part of the talk concerned the following conjecture on the principal p-block $B_p(G)$ of a finite groups G:

Conjecture. *Let G be a finite group and p, q two primes. Then G has a Sylow p-subgroup commuting with a Sylow q-subgroup if and only if all characters in $B_p(G)$ have degree prime to q, and all characters in $B_q(G)$ have degree prime to p.*

Again for $p = q$ this specialises to Brauer's height zero conjecture for the principal block. In joint work with G. Navarro we proved the "only if" direction of this conjecture, and the "if" direction assuming that the Inductive Alperin–McKay Condition is satisfied for all quasi-simple groups. The proof starts off by a reduction to quasi-simple groups, and for those, the result follows by a case-by-case argument. The most difficult case of alternating groups had previously been settled in joint work with E. Giannelli and C. Vallejo.

Units in Group Rings, Characters and Blocks

Leo Margolis

(joint work with Mauricio Caicedo)

Since the study of the unit group of integral group rings $\mathbb{Z}G$ of a finite group G began with G. Higman's thesis in 1940 many conjectures have been put forward regarding the finite subgroups of units in $\mathbb{Z}G$. The strongest of those, such as the Isomorphism Problem or the Zassenhaus Conjectures, gave rice to fascinating mathematics, but turned out to be wrong in general, with counterexamples in the class of solvable groups. On the other hand the strongest possible expectations one can have concerning possible orders of units in $\mathbb{Z}G$ are known to hold for solvable groups.

Namely, call a unit normalized if its coefficients sum up to one. Then the best possible statement one can hope for regarding orders of normalized units in $\mathbb{Z}G$ is that there is a normalized unit of order n in $\mathbb{Z}G$ if and only if there is a group element of order n in G. The question if this holds for any G is known as the Spectrum Problem. The weaker form of the question which one obtains by replacing n by the product of two distinct primes is known as the Prime Graph Question. The Spectrum Problem is known to have a positive answer for solvable groups and for the Prime Graph Question also a reduction theorem, to almost simple groups, has been obtained.

I will present a result which states that if p and q are primes and the Sylow subgroup of G is cyclic of order p then $\mathbb{Z}G$ contains a normalized unit of order pq if and only if G contains an element of order pq. The main ingredient of the proof is the description of modules for blocks of defect 1 and their visualisation using Brauer trees. This directly settles the Prime Graph Question for most sporadic groups.

This is joint work with M. Caicedo.

Irreducible restrictions of representations of symmetric and alternating groups

Lucia Morotti

(joint work with Alexander Kleshchev, Pham Huu Tiep)

Let G and H be finite groups with $G < H$, F be an algebraically closed field and V be an irreducible FH-module. In general, if V is not 1-dimensional, the restriction $V{\downarrow_G}$ is reducible. There are though examples where V is not 1-dimensional and $V{\downarrow_G}$ is irreducible. The classification of such irreducible restrictions is relevant to the the Aschbacher-Scott classification of maximal subgroups of finite classical groups.

For H a symmetric or alternating group the problem of classifying irreducible restrictions has been solved by Saxl [9] in characteristic 0 and by Brundan-Kleshchev [1] and Kleshchev-Sheth [6] in characteristic ≥ 5. In characteristics 2 and 3 however only partial reduction results were known. In [3, 4, 5] we essentially complete the classification of irreducible restrictions of representations of symmetric and alternating groups in characteristics 2 and 3.

Let $p = \mathrm{char}(F)$ and $P_p(n)$ be the set of p-regular partitions. It is well known that the irreducible FS_n-modules are indexed by $P_p(n)$. For $\lambda \in P_p(n)$ let D^λ be the corresponding irreducible FS_n-module. We say that $\lambda \in P_p(n)$ is JS if $D^\lambda{\downarrow_{S_{n-1}}}$ is irreducible. Such partitions have a nice combinatorial description in terms of their parts and multiplicities. Define $P_p^A(n)$ to be the set of partitions $\lambda \in P_p(n)$ such that $D^\lambda{\downarrow_{A_n}}$ splits and let E^λ or $E^\lambda_\pm$ be the irreducible components of $D^\lambda{\downarrow_{A_n}}$. In characteristic $\neq 2$ there exists a nice description of $P_p^A(n)$ in term of the Mullineux map. Namely in this case $\lambda \in P_p^A(n)$ if and only if $\lambda = \lambda^{\mathtt{M}}$, where $\lambda^{\mathtt{M}}$ is defined by $D^{\lambda^{\mathtt{M}}} \cong D^\lambda \otimes \mathrm{sgn}$.

If $p = 2$ we say that D^λ and $E^\lambda_{(\pm)}$ are basic spin if $\lambda = (\lceil (n+1)/2\rceil, \lfloor (n-1)/2\rfloor)$. At least for symmetric groups, such modules can always be obtained by reducing modulo 2 basic spin modules of covering groups of symmetric groups.

When considering irreducible restrictions of representations of symmetric groups we in particular have the following theorem:

Theorem. [1, 5, 9] *Let $\lambda \in P_p(n)$. If $G < S_n$ and $D^\lambda{\downarrow_G}$ is irreducible, then one of the following holds:*

- *(i)* $\lambda \in \{(n), (n)^{\mathtt{M}}\}$,
- *(ii)* λ *is JS and* $G = S_{n-1}$,
- *(iii)* $\lambda \notin P_p^A(n)$ *and* $G = A_n$,
- *(iv)* $\lambda \notin P_p^A(n)$ *is JS and* $G = A_{n-1}$,
- *(v)* $\lambda \in \{(n-1,1), (n-1,1)^{\mathtt{M}}\}$ *and G is 2-transitive or* $n \equiv 0 \mod p$ *and* $G \leq S_{n-1}$ *is 2-transitive,*
- *(vi)* $p \neq 2$, $\lambda \in \{(n-2,1^2), (n-2,1^2)^{\mathtt{M}}\}$, $n = 2^m$ *and* $G = AGL_m(2)$ *or* $n = 2^m + 1 \equiv 0 \mod p$ *and* $G = AGL_m(2) \leq S_{n-1}$,
- *(vii)* $p = 2$, $\lambda = (n-1,1)$, $n \equiv 2 \mod 4$ *and* $G \leq S_{n/2} \wr S_2$,
- *(viii)* $p = 2$, D^λ *is basic spin and G is imprimitive,*
- *(ix)* $n \leq 25$.

Note that if $\lambda \in P_p(n) \setminus P_p^A(n)$ and $G < A_n$ then $E^\lambda{\downarrow_G}$ is irreducible if and only if $D^\lambda{\downarrow_G}$ is irreducible, so this case is covered by the previous theorem. For $\lambda \in P_p^A(n)$ we have the following theorem. Normal nodes of a partition $\lambda \in P_p(n)$ are certain removable nodes of λ and they can be defined combinatorially based on the sets of addable and of removable nodes of λ. The residue of a node is an element of $\mathbb{Z}/p\mathbb{Z}$.

Theorem. [5, 6, 9] *Let $\lambda \in P_p^A(n)$. If $G < A_n$ and $E^\lambda_\pm{\downarrow_G}$ irreducible, then one of the following holds:*

- *(a)* λ *is JS and* $G = A_{n-1}$, A_{n-2} *or* $A_{n-2,2}$,
- *(b)* λ *has exactly two normal nodes both of residue different from 0 and* $G = A_{n-1}$,
- *(c)* $p = 2$, $E^\lambda_\pm$ *is basic spin and G is imprimitive,*
- *(d)* $n \leq 13$.

About the reverse directions we have the following:

- Cases (i), (ii), (iii), (vi), (a) and (b): the restrictions are always irreducible. In case (i) this holds since $D^{(n)}$ is the trivial module of S_n.
- Case (iv): $D^\lambda{\downarrow_{A_{n-1}}}$ is always irreducible unless $p = 2$, $n \equiv 2 \mod 4$ and D^λ is basic spin. In particular if we are not in this case, $D^\lambda{\downarrow_{A_{n-1}}}$ is irreducible if and only if $D^\lambda{\downarrow_{S_{n-1}}}$ and $D^\lambda{\downarrow_{A_n}}$ are both irreducible.
- Case (v): it is known for which 2-transitive subgroups of S_n (or S_{n-1}) $D^{(n-1,1)}{\downarrow_G}$ is irreducible. This is mainly due to [8].

- Case (vii): an exact classification of subgroups $G \leq S_{n/2} \wr S_2$ for which $D^{(n-1,1)}{\downarrow_G}$ is irreducible can be found in [3]. Further in [5] we prove that no such subgroup is almost quasi-simple.
- Cases (viii) and (c): we cannot completely classify imprimitive subgroups to which basic spin modules restrict irreducibly. However in [3, 4, 5] we completely classify such subgroups, provided they are almost quasi-simple or maximal imprimitive.
- Cases (ix) and (d): the list of irreducible restrictions for small n can be found in [1, 5, 6, 9].

In particular in characteristic 3 the classification of irreducible restrictions of representations of symmetric and alternating groups extends that in characteristic at least 5 (at least for $n \geq 25$). In characteristic 2, for $n \geq 25$, the only differences are the following:

- Case (vi) has no equivalent. Note though that $(n-2, 1^2)$ is not 2-regular.
- Case (vii) has no corresponding case in characteristics $\neq 2$. This is however the only case where $n = ab$ and $\dim D^{(n-1,1)} = b \dim D^{(a-1,1)}$.
- Cases (viii) and (c) have no corresponding cases in larger characteristics for symmetric and alternating groups. There are however similar irreducible restrictions of basic spin modules of covering groups of symmetric and alternating groups in characteristics $\neq 2$, see [2, 7].

References

[1] J. Brundan and A.S. Kleshchev, Representations of the symmetric group which are irreducible over subgroups, *J. Reine Angew. Math.* **530** (2001), 145–190.

[2] P.B. Kleidman and D.B. Wales, The projective characters of the symmetric groups that remain irreducible on subgroups, *J. Algebra* **138** (1991), 440–478.

[3] A. Kleshchev, L. Morotti and P.H. Tiep, Irreducible restrictions of representations of symmetric groups in small characteristics: reduction theorems, *Math. Z.* to appear.

[4] A. Kleshchev, L. Morotti and P.H. Tiep, Irreducible restrictions of representations of alternating groups in small characteristics: reduction theorems, preprint.

[5] A. Kleshchev, L. Morotti and P.H. Tiep, Irreducible restrictions of representations of symmetric and alternating groups in small characteristics, preprint.

[6] A.S. Kleshchev and J. Sheth, Representations of the alternating group which are irreducible over subgroups, *Proc. London Math. Soc.* **84** (2002), 194–212.

[7] A.S. Kleshchev and P.H. Tiep, On restrictions of modular spin representations of symmetric and alternating groups. *Trans. Amer. Math. Soc.* **356** (2004), 1971–1999.

[8] B. Mortimer, The modular permutation representations of the known doubly transitive groups, *Proc. Lond. Math. Soc.* **41** (1980), 1–20.

[9] J. Saxl, Irreducible characters of the symmetric groups that remain irreducible in subgroups, *J. Algebra* **111** (1987), 210–219.

Character counting conjectures for π-separable groups

Benjamin Sambale

Many of the open conjectures in modular representation theory of finite groups are known to be true for p-solvable groups where p is the relevant prime. Richard Brauer and others have tried to replace p by a set of prime π. A convincing theory of π-blocks was eventually developed by Slattery for the family of π-separable groups. Here a finite group G is called π-separable if every composition factor of G is a π-group or a π'-group. Moreover, a π-block of G is a minimal non-empty subset $B \subseteq \mathrm{Irr}(G)$ such that B is a union of p-blocks for every $p \in \pi$. Note that $\{p\}$-separable is p-solvable and a $\{p\}$-block is a p-block. As in the original theory, let $k(B) := |B|$. Using a variant of the Fong-Reynolds Theorem, Slattery defined defect groups D of B by induction on $|G|$. In this framework it is natural to ask which of the open conjectures still hold for π-blocks. For instance, Brauer's Height Zero Conjecture and the Alperin-McKay Conjecture were proved for B above by Manz–Staszewski and Wolf respectively. In 2017, I verified Brauer's $k(B)$-Conjecture for B (stating that $k(B) \leq |D|$) which was put forward previously by Y. Liu. The proof is reduced to a non-abelian $k(GV)$-Theorem which answers a question by Pálfy and Pyber. In a second paper, I proved Brauer Problem 21 for π-blocks which states that there exists a function $f : \mathbb{N} \to \mathbb{N}$ (independent of π, B or D) such that $|D| \leq f(k(B))$. This extends the corresponding result for p-solvable groups by Külshammer. It also generalizes the classical theorem of Landau that there are only finitely many finite groups with a given class number. Finally, in a joint paper with Gabriel Navarro, we proved in 2018 a version of Alperin's Weight Conjecture for π-solvable groups. Here G is called π-solvable if the composition factors of G are π'-groups or solvable π-groups. Moreover, a π-weight of G is a pair (P, ψ) where P is a nilpotent π-subgroup of G and $\psi \in \mathrm{Irr}(\mathrm{N}_G(P)/P)$ satisfies $\psi(1)_\pi = |\mathrm{N}_G(P)/P|_\pi$. We showed that the number of conjugacy classes of π-weights of G equals the number of π-regular conjugacy classes of G. As an interesting and perhaps surprising special case one recovers Carter's theorem that every solvable group has exactly one conjugacy class of selfnormalizing nilpotent subgroups. In my talk I also proposed a (groupwise) version of Dade's conjecture which seems to hold for any π-separable group.

Equivariant Galois-McKay Bijections for the Prime 2 and Some Groups of Lie Type

Mandi A. Schaeffer Fry

Let p be a prime, G a finite group, and P a Sylow p-subgroup of G. The long-standing McKay conjecture posits that there should exist a bijection between the set $\mathrm{Irr}_{p'}(G)$ of irreducible ordinary characters of G with degree relatively prime to p and the corresponding set, $\mathrm{Irr}_{p'}(N_G(P))$, for the normalizer of P. Sometimes called the Galois-McKay conjecture, a refinement due to G. Navarro [3] says that not only should such a bijection exist, but that there should further be such a bijection which commutes with the action of a certain subgroup $\mathcal{H}$ of $\mathcal{G} := \mathrm{Gal}(\mathbb{Q}^{\mathrm{ab}}/\mathbb{Q})$.

Namely, $\mathcal{H}$ is comprised of the Galois automorphisms that map all p'-roots of unity to a given p-power of themselves.

The McKay conjecture was reduced in [1] to proving certain "inductive McKay conditions" for every simple group. In particular, the conditions require that for quasisimple groups G, there exist some $\mathrm{Aut}(G)_P$-stable $N_G(P) \leq M < G$ and bijections between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(M)$ that are $\mathrm{Aut(G)}_P$-equivariant and satisfy several other strong properties. Here at the MFO in 2014, G. Malle and B. Späth announced the proof of the ordinary McKay conjecture for the prime $p = 2$, see [2], in particular yielding the desired $\mathrm{Aut(G)}_P$-equivariant bijections in the case of groups of Lie type when $p = 2$.

Recently in [4], G. Navarro, B. Späth, and C. Vallejo have announced a reduction theorem for the Galois-McKay conjecture along the same lines, stating that the conjecture holds for all finite groups if certain "inductive Galois-McKay conditions" hold for every simple group. Here we require that the bijections between $\mathrm{Irr}_{p'}(G)$ and $\mathrm{Irr}_{p'}(M)$ from before are further $(\mathrm{Aut}(G)_P \times \mathcal{H})$-equivariant, among satisfying other strong properties.

In [6], I describe the action of $\mathcal{G}$ on the Howlett-Lehrer parameters for characters of groups with a BN pair in order to complete the proof began in [5, 7] of another conjecture of Navarro from [3], which would also be a consequence of the Galois-McKay conjecture for $p = 2$. In the present talk, I discuss how I have now extended these techniques to show that for many groups of Lie type defined in odd characteristic, the $\mathrm{Aut(G)_P}$-equivariant bijections for odd-degree characters constructed by Malle and Späth can be chosen to further be $\mathcal{H}$-equivariant, showing that they satisfy the first part of the inductive Galois-McKay conditions.

References

[1] I. Martin Isaacs, Gunter Malle, and Gabriel Navarro, *A reduction theorem for the McKay conjecture*, Invent. Math. 170(1) 33–101, 2007.

[2] Gunter Malle and Britta Späth, *Characters of odd degree*, Ann. of Math., 184(3):869–908, 2016.

[3] Gabriel Navarro, *The McKay conjecture and Galois automorphisms*, Ann. of Math. (2), 160(3), 1129–1140, 2004.

[4] Gabriel Navarro, Britta Späth, and Carolina Vallejo, *A reduction theorem for the Galois-McKay conjecture*, Preprint, 2018.

[5] A. A. Schaeffer Fry, *Odd-degree characters and self-normalizing Sylow subgroups: A reduction to simple groups*, Comm. Algebra, 44(5):1882–1904, 2016.

[6] A. A. Schaeffer Fry, *Galois automorphisms on Harish-Chandra series and Navarro's self-normalizing Sylow 2-subgroup conjecture*, Trans. Am. Math. Soc., To Appear, Published electronically: October 2, 2018

[7] A. A. Schaeffer Fry and Jay Taylor, *On self-normalising Sylow 2-subgroups in type A*, J. Lie Theory, 28(1):139–168, 2018.

“Representations” of saturated fusion systems

Jason Semeraro

Let p be a prime number. A saturated fusion system on a finite p-group S is a category whose objects are the subgroups of S where morphisms are injective homomorphisms satisfying certain axioms. A finite group G containing S as a Sylow p-subgroup provides an example, denoted $\mathcal{F}_S(G)$. The local-global counting conjectures in modular representation theory typically express an equality between a global invariant of G (such as the number of representations with a particular property) and some local integer invariant involving $\mathcal{F}_S(G)$, or some subcategory thereof. In an attempt to provide a new perspective on these conjectures we forget the group G, and consider the behaviour of the local invariants when $\mathcal{F}$ is an arbitrary saturated fusion system. Do they behave like the local-global conjectures predict they should when $\mathcal{F}$ is induced by a group? We state some conjectures, discuss recent progress and look at examples coming from algebraic topology.

The Alperin-McKay Conjecture for simple groups of type A

Britta Späth
(joint work with Julian Brough)

The McKay Conjecture and its blockwise version, the Alperin-McKay conjecture, relate the numbers of certain characters in terms of local subgroups. Let ℓ be a prime and let $\mathrm{Irr}_0(C)$ denote the set of height 0 characters in an ℓ-block C. The Alperin-McKay conjecture states the equality

$$|\mathrm{Irr}_0(B)| = |\mathrm{Irr}_0(b)|,$$

where B is an ℓ-block B of a finite group G with defect group D, and b the Brauer correspondent of B, an ℓ-block of $\mathrm{N}_G(D)$.

A reduction theorem of the Alperin-McKay conjecture from [Spä13a] shows that this conjecture follows once a stronger version of the conjecture, the so-called inductive AM condition, has been checked for all (blocks of) quasi-simple groups and all primes ℓ. It has been verified for simple groups of Lie type, when ℓ is the defining characteristic, and for alternating groups, when the prime ℓ is odd. Additionally [Mal14, SF14] have dealt with simple groups of types ${}^2\mathrm{B}_2, {}^2\mathrm{G}_2$ and ${}^2\mathrm{F}_4$, while [CS15, KS16b, KS16a] consider particular structures of the defect group of the block.

The present work is concerned with the inductive AM condition for quasi-simple groups of type A and primes ℓ different from the defining characteristic with $\ell \geq 5$. Note that the condition holds for most blocks in the defining characteristic according to [Spä13a] and [Spä13b]. In order to verify the inductive Alperin-McKay condition, we give a new criterion which will have applications to other series of simple groups. It complements the criterion given in [CS15].

Theorem 1. *Let S be a finite non-abelian simple group and ℓ a prime dividing $|S|$. Let G be the universal covering group of S, D a radical ℓ-subgroup of G and $\mathcal{B} \subseteq \mathrm{Bl}(G \mid D)$ a $\widetilde{G}_D$-stable subset with $(\widetilde{G}E)_B \leq (\widetilde{G}E)_{\mathcal{B}}$ for every $B \in \mathcal{B}$. Assume we have a semi-direct product $\widetilde{G} \rtimes E$, a $Aut(G)_{\mathcal{B},D}$-stable subgroup M with $N_G(D) \leq M \lneq G$ and a group $\widetilde{M} \leq \widetilde{G}$ with $\widetilde{M} \geq MN_{\widetilde{G}}(D)$ and $M = \widetilde{M} \cap G$ such that the following conditions hold:*

(1)
- $G = [\widetilde{G}, \widetilde{G}]$ *and E is abelian,*
- $C_{\widetilde{G} \rtimes E}(G) = Z(\widetilde{G})$ *and* $\widetilde{G}E/Z(\widetilde{G}) \cong Inn(G)Aut(G)_D$ *by the natural map,*
- *any element of* $\mathrm{Irr}_0(\mathcal{B})$ *extends to its stabiliser in* $\widetilde{G}$*,*
- *any element of* $\mathrm{Irr}_0(\mathcal{B}')$ *extends to its stabiliser in* $\widetilde{M}$*.*

(2) Let $\mathcal{B}' \subseteq Bl(M)$ be the set of all Brauer correspondents of the blocks in $\mathcal{B}$. For $\mathcal{G} := \mathrm{Irr}\left(\widetilde{G} \mid \mathrm{Irr}_0(\mathcal{B})\right)$ and $\mathcal{M} := \mathrm{Irr}\left(\widetilde{M} \mid \mathrm{Irr}_0(\mathcal{B}')\right)$ there exists an $N_{\widetilde{G}E}(D)_{\mathcal{B}}$-equivariant bijection

$$\widetilde{\Omega} : \mathcal{G} \longrightarrow \mathcal{M}$$

with

(a) $\widetilde{\Omega}\left(\mathcal{G} \cap \mathrm{Irr}(\widetilde{G} \mid \widetilde{\nu})\right) = \mathcal{M} \cap \mathrm{Irr}(\widetilde{M} \mid \widetilde{\nu})$ *for all* $\widetilde{\nu} \in \mathrm{Irr}\left(Z(\widetilde{G})\right)$*,*

(b) $bl\left(\widetilde{\Omega}(\widetilde{\chi})\right)^{\widetilde{G}} = bl(\widetilde{\chi})$ *for all* $\widetilde{\chi} \in \mathcal{G}$*, and*

(c) $\widetilde{\Omega}(\widetilde{\chi}\widetilde{\mu}) = \widetilde{\Omega}(\widetilde{\chi})Res^{\widetilde{G}}_{\widetilde{M}}(\widetilde{\mu})$ *for every* $\widetilde{\mu} \in \mathrm{Irr}(\widetilde{G} \mid 1_G)$ *and every* $\widetilde{\chi} \in \mathcal{G}$*.*

(3) For every $\widetilde{\chi} \in \mathcal{G}$ there exists some $\chi_0 \in \mathrm{Irr}(G \mid \widetilde{\chi})$ such that
- $(\widetilde{G} \rtimes E)_{\chi_0} = \widetilde{G}_{\chi_0} \rtimes E_{\chi_0}$*, and*
- χ_0 *extends to* $G \rtimes E_{\chi_0}$*.*

(4) For every $\widetilde{\psi} \in \mathcal{M}$ there exists some $\psi_0 \in \mathrm{Irr}(M \mid \widetilde{\psi})$ such that
- $O = (\widetilde{G} \cap O) \rtimes (E \cap O)$ *for* $O := G(\widetilde{G} \times E)_{D,\psi_0}$*, and*
- ψ_0 *extends to* $M(G \rtimes E)_{D,\psi_0}$*.*

(5) For any $\widetilde{G}$-orbit B in $\mathcal{B}$ the group $Out(G)_B$ is abelian.

Then the inductive AM condition holds for all ℓ-blocks in $\mathcal{B}$

This leads to the following statement, where we write $\mathrm{SL}_n(-q)$ for $\mathrm{SU}_n(q)$ and $\mathrm{GL}_n(-q)$ for $\mathrm{GU}_n(q)$.

Theorem 2. *Let ℓ be a prime, q a prime power and $\epsilon \in \{\pm 1\}$ with $\ell \nmid 3q(q-\epsilon)$, $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, $G := \mathrm{SL}_n(\epsilon q)$, B_0 an ℓ-block of G with defect group D, and B the $\mathrm{GL}_n(\epsilon q)$-orbit containing B_0. Assume that $\mathrm{PSL}_n(\epsilon q)$ is simple, G is its universal covering group and the stabilizer $\mathrm{Out}(G)_B$ is abelian.*

1. *The inductive AM condition from Definition 7.2 of* [Spä13a] *holds for B_0.*
2. *Let d be the order of q in $(\mathbb{Z}/\ell\mathbb{Z})^\times$. If D is abelian and $\mathrm{C}_{\mathbf{G}}(D)$ is a d-split Levi subgroup of $\mathbf{G}$, then the inductive BAW condition from* [Spä13b] *holds for B_0.*

In our proof a main step is to parameterize the characters of the normalizers of d-split Levi subgroups which serve as local subgroups in the inductive AM condition. Essential is to understand the Clifford theory of irreducible characters of a d-split Levi subgroup L in $\mathrm{N}_G(L)$. Furthermore, we consider the action of the stabilizer $\mathrm{Aut}(G)_{B,L}$ on the irreducible characters and verify that the corresponding inertia groups are of a particular structure.

Theorem 3. *Let* $\mathbf{G} := \mathrm{SL}_n(\overline{\mathbb{F}}_q)$, $\widetilde{\mathbf{G}} := \mathrm{GL}_n(\overline{\mathbb{F}}_q)$, $F : \widetilde{\mathbf{G}} \to \widetilde{\mathbf{G}}$ *a Frobenius endomorphism defining an* $\mathbb{F}_q$*-structure,* $\mathbf{L}$ *a d-split Levi subgroup of* $(\mathbf{G}, F)$*,* $N_0 := \mathrm{N}_{\mathbf{G}^F}(\mathbf{L})$ *and* $\widetilde{N}_0 := \mathrm{N}_{\widetilde{\mathbf{G}}^F}(\mathbf{L})$.

(1) Every $\lambda \in \mathrm{Irr}(\mathbf{L}^F)$ *extends to its inertia group in* N_0.

(2) Let $E_0 \leq \mathrm{Aut}(\widetilde{\mathbf{G}}^F)$ *be the image of* E *and let* $\psi \in \mathrm{Irr}(\mathrm{N}_0)$*. Then there exists a* $\widetilde{N}_0$*-conjugate* ψ_0 *of* ψ *such that*
  (a) $O_0 = (\widetilde{\mathbf{G}}^F \cap O_0) \rtimes (E_0 \cap O_0)$ *for* $O_0 := \mathbf{G}^F(\widetilde{\mathbf{G}}^F \rtimes E_0)_{\mathbf{L},\psi_0}$*, and*
  (b) ψ_0 *extends to* $(\mathbf{G}^F \rtimes E_0)_{\mathbf{L},\psi_0}$.

For groups of Lie type with abelian Sylow ℓ-subgroup, bijections implying the Alperin-McKay conjecture and blockwise Alperin weight were constructed in [Mal14, Theorem 2.9] assuming the first part of the above statement for analogous local subgroups. As a consequence, Alperin-McKay conjecture holds via [Mal14, Theorem 2.9 and Corollary 3.7] for special linear and unitary groups with abelian Sylow ℓ-subgroup. We then generalize Malle's approach from [Mal07], where he constructed a bijection for the inductive McKay condition. By considerations inspired by [Spä09, §10] we deduce from this the Alperin-McKay conjecture for all blocks, using results of Puig and Zhou on the so-called *inertial blocks*. Note that for $\ell \mid q$ the Alperin-McKay conjecture was proven in [Spä13a] based on earlier work by Green-Lehrer-Lusztig while for $\ell \mid (q-\epsilon)$ results of Puig in [Pui94, §5] imply the conjecture for most ℓ-blocks of $\mathrm{SL}_n(\epsilon q)$ with abelian defect.

Theorem 4. *Let* $G = \mathrm{SL}_n(\epsilon q)$*. Let* ℓ *be a prime with* $\ell \nmid 3q(q-\epsilon)$.

(1) The Alperin-McKay Conjecture holds for all ℓ*-blocks of* G.

(2) The Alperin weight Conjecture holds for all ℓ*-blocks of* G *with abelian defect.*

In a forthcoming work, we address the problem of primes dividing $q-1$, *linear primes*. In [CSFS] we prove a general statement implying the following for finite symplectic groups.

Theorem 5. *The simple groups* $PSp_{2n}(q)$ *with* q *odd satisfy the inductive Alperin-McKay condition for primes* $\ell \geq 5$ *dividing* $(q-1)$.

References

[CSFS] M. Cabanes, A. Schaeffer-Fry, and B. Späth. On the inductive Alperin-McKay conditions for linear primes *in preparation*, 2019.

[CS15] M. Cabanes and B. Späth. On the inductive Alperin-McKay condition for simple groups of type *A*. *J. Algebra*, 442:104–123, 2015.

[KS16a] S. Koshitani and B. Späth. The inductive Alperin–McKay condition for 2-blocks with cyclic defect groups. *Arch. Math. (Basel)*, 106(2):107–116, 2016.
[KS16b] S. Koshitani and B. Späth. The inductive Alperin-McKay and blockwise Alperin weight conditions for blocks with cyclic defect groups and odd primes. *J. Group Theory*, 19(5):777–813, 2016.
[Mal07] G. Malle. Height 0 characters of finite groups of Lie type. *Represent. Theory*, 11:192–220 (electronic), 2007.
[Mal14] G. Malle. On the inductive Alperin-McKay and Alperin weight conjecture for groups with abelian Sylow subgroups. *J. Algebra*, 397:190–208, 2014.
[Pui94] L. Puig. On Joanna Scopes' criterion of equivalence for blocks of symmetric groups. *Algebra Colloq.*, 1(1):25–55, 1994.
[SF14] A. Schaeffer Fry. $Sp_6(2^a)$ is "good" for the McKay, Alperin weight, and related local-global conjectures. *J. Algebra*, 401:13–47, 2014.
[Spä09] B. Späth. The McKay conjecture for exceptional groups and odd primes. *Math. Z.*, 261(3):571–595, 2009.
[Spä13a] B. Späth. A reduction theorem for the Alperin-McKay conjecture. *J. Reine Angew. Math.*, 680:153–189, 2013.
[Spä13b] B. Späth. A reduction theorem for the blockwise Alperin weight conjecture. *J. Group Theory*, 16(2):159–220, 2013.

Rank, Coclass and cohomology

Peter Symonds

The coclass classification of p-groups suggests that the p-groups of a given coclass should divide up into finitely many coclass families and the groups in a given family should have a similar structure.

This led Jon Carlson to conjecture that the p-groups of a given coclass should only have finitely many isomorphism classes of cohomology rings between them and he gave a proof for $p = 2$. Here we present a proof for all p. In fact, we prove that the p-groups of bounded sectional rank have only finitely many cohomology rings. This implies the original version because a bound on the coclass gives a bound on the rank.

The idea is to show that a group of bounded rank has a normal subgroup of bounded index that has the same cohomology as an abelian group and then to use the fact that the Castelnuove-Mumford regularity must be zero to bound the degrees and number of the generators and relations.

As a consequence, related to work of Guralnick and Tiep, we can show that if a p-group is of sectional rank r then $\dim H^i(G) \leq \binom{r(\lceil \log_2 r \rceil + 3 + e) + i - 1}{i}$, where $e = 0$ for p odd and $e = 1$ for $p = 2$.

References

[1] J.F. Carlson, *Coclass and cohomology*, J. Pure Applied Algebra **200** (2005), 251–266.
[2] P. Symonds, *Rank, coclass and cohomology*, arXiv:1902.02888.

Decomposition Matrices of Unipotent Blocks

Jay Taylor
(joint work with Olivier Brunat and Olivier Dudas)

Assume G is a finite group and $\mathrm{Irr}(G)$ is the set of complex-valued irreducible characters of G. Fix a prime $\ell > 0$ and let $\mathrm{IBr}(G)$ be the ℓ-modular Brauer characters of G, which are functions $G_{\ell'} \to \mathbb{C}$ where $G_{\ell'} \subseteq G$ is the set of elements whose order is coprime to ℓ.

If $f : G \to \mathbb{C}$ is a function then we denote by $f^0 := f|_{G_{\ell'}}$ the restriction of f to the ℓ'-elements of G. It is well known that if $\chi \in \mathrm{Irr}(G)$ then there exist integers $d_{\chi,\varphi} \geqslant 0$ such that

$$\chi^0 = \sum_{\varphi \in \mathrm{IBr}(G)} d_{\chi,\varphi}\varphi.$$

The resulting matrix $(d_{\chi,\varphi})$ is the (ℓ-)decomposition matrix of G. Obtaining information about this matrix is a central problem in the representation theory of finite groups and calculating exactly the entries $d_{\chi,\varphi}$ is an extremely challenging problem in general.

We will consider the case where $G = \mathbf{G}(k)$ is a finite reductive group and $\ell \neq p := \mathrm{char}(k)$, i.e., G is the group of k-points of a connected reductive algebraic group $\mathbf{G}$ defined over a finite field k. We will denote by $\bar{k}$ an algebraic closure of k. We then have a corresponding group $\mathbf{G}(\bar{k})$ of $\bar{k}$-points which contains G as a subgroup. We will let $\mathcal{C}_{\mathrm{u}}(\mathbf{G})$ denote the set of unipotent conjugacy classes of $\mathbf{G}(\bar{k})$.

After [7, 2, 6] we can associate to each irreducible character $\chi \in \mathrm{Irr}(G)$ a class $\mathcal{O}_\chi \in \mathcal{C}_{\mathrm{u}}(\mathbf{G})$, called the *unipotent support* of χ. It is a little delicate to define this class in general but if p is good for $\mathbf{G}$ and the centre $Z(\mathbf{G}(\bar{k}))$ is connected then it is shown in [8] that $\mathcal{O}_\chi$ is the unique unipotent class satisfying the following conditions:

- $\chi(u) \neq 0$ for some $u \in \mathcal{O}_\chi \cap G \neq \emptyset$
- if $v \in G$ is a unipotent element and $\chi(v) \neq 0$ then $v \in \overline{\mathcal{O}_\chi}$ (the Zariski closure of $\mathcal{O}_\chi$).

Example. If $1_G \in \mathrm{Irr}(G)$ is the trivial character then $\mathcal{O}_{1_G}$ is the class of regular unipotent elements and if $\mathrm{St}_G \in \mathrm{Irr}(G)$ is the Steinberg character then $\mathcal{O}_{\mathrm{St}_G}$ is the trivial unipotent class.

For finite reductive groups one has an important set of characters $\mathcal{E}(G,1) \subseteq \mathrm{Irr}(G)$, defined using ℓ-adic cohomology, known as the set of *unipotent characters*. These characters are a generic model for all the irreducible characters of G. Using the unipotent support we obtain a partition of the unipotent characters

$$\mathcal{E}(G,1) = \bigsqcup_{\mathcal{O} \in \mathcal{C}_{\mathrm{u}}(\mathbf{G})} \mathcal{E}(G,1,\mathcal{O})$$

where $\mathcal{E}(G,1,\mathcal{O}) = \{\chi \in \mathcal{E}(G,1) \mid \mathcal{O}_\chi = \mathcal{O}\}$. Note this set might be empty in general and the non-empty such sets are known as *families* of unipotent characters.

Example. Assume $G = \mathrm{Sp}_4(k)$ then $\mathcal{C}_u(\mathbf{G}) = \{\mathcal{O}_{(1^4)}, \mathcal{O}_{(2,1^2)}, \mathcal{O}_{(2^2)}, \mathcal{O}_{(4)}\}$ where each class is labelled by the sizes of the Jordan blocks in the Jordan normal form of an element under the natural representation $\mathrm{Sp}_4(\bar{k}) \to \mathrm{GL}_4(\bar{k})$. It is well known that $|\mathcal{E}(G,1)| = 6$ and the sizes of the corresponding sets $\mathcal{E}(G,1,\mathcal{O})$ are

$\mathcal{O}$	$\mathcal{O}_{(1^4)}$	$\mathcal{O}_{(2,1^2)}$	$\mathcal{O}_{(2^2)}$	$\mathcal{O}_{(4)}$
$\lvert\mathcal{E}(G,1,\mathcal{O})\rvert$	1	0	4	1

Here $\mathcal{E}(G,1,\mathcal{O}_{(1^4)}) = \{\mathrm{St}_G\}$ and $\mathcal{E}(G,1,\mathcal{O}_{(4)}) = \{1_G\}$.

On the modular side we have a corresponding subset $\mathcal{B}(G,1) \subseteq \mathrm{IBr}(G)$ of Brauer characters, which is the union of the *unipotent blocks* of G. This set is defined by a corresponding subset $\mathcal{E}_\ell(G,1) \subseteq \mathrm{Irr}(G)$ of irreducible characters, which contains the set of unipotent characters. This correspondence is such that if $\chi \in \mathcal{E}_\ell(G,1)$ and $\varphi \in \mathrm{IBr}(G)$ then $d_{\chi,\varphi} \neq 0$ implies $\varphi \in \mathcal{B}(G,1)$.

In what follows we will be interested in the following part of the decomposition matrix

$$D = (d_{\chi,\varphi} \mid \chi \in \mathcal{E}_\ell(G,1) \text{ and } \varphi \in \mathcal{B}(G,1)).$$

This matrix is, in general, not square as $|\mathcal{E}_\ell(G,1)| \geqslant |\mathcal{B}(G,1)|$. However, it has been shown by Geck–Hiß that under some mild assumptions on ℓ we have $|\mathcal{E}(G,1)| = |\mathcal{B}(G,1)|$, this holds for instance if ℓ is very good for $\mathbf{G}$. This is known to be false in general.

Let us recall that we have a natural partial order $\preceq$ on $\mathcal{C}_u(\mathbf{G})$ defined by $\mathcal{O}' \preceq \mathcal{O}$ if and only if $\mathcal{O}' \subseteq \overline{\mathcal{O}}$ (the Zariski closure). With this in hand we can state Geck's conjecture on the decomposition matrix of G. To avoid introducing more notation we will work with a stronger assumption on ℓ than is actually stated in the conjecture. We note that a weak version of this conjecture was first proposed in Geck's PhD Thesis [3]. It was then further strengthened by Geck–Hiß [5] and reached the form we state here in [4].

Geck's Unitriangularity Conjecture. Assume ℓ is a very good prime for $\mathbf{G}$. Let $\mathcal{S}_\mathbf{G} = \{\mathcal{O} \in \mathfrak{Cl}_u(\mathbf{G}) \mid \mathcal{E}(G,1,\mathcal{O}) \neq \emptyset\} = \{\mathcal{O}_1, \ldots, \mathcal{O}_r\}$ where $\mathcal{O}_r \leqslant \cdots \leqslant \mathcal{O}_1$ is a total order refining the partial order $\preceq$ on $\mathcal{S}_\mathbf{G}$. Then there is an ordering of the Brauer characters in $\mathcal{B}(G,1)$ such that

$$D = \left[\begin{array}{ccc} D_1 & 0 & 0 \\ \star & \ddots & 0 \\ \star & \star & D_r \\ \hline \star & \star & \star \end{array}\right] \begin{array}{c} \mathcal{E}(G,1,\mathcal{O}_1) \\ \vdots \\ \mathcal{E}(G,1,\mathcal{O}_r) \\ \\ \end{array}$$

where each D_i is the identity matrix with rows labelled by the irreducible characters in $\mathcal{E}(G,1,\mathcal{O}_i)$.

Example. The poset $(\mathcal{S}_\mathbf{G}, \preceq)$ contains a unique maximal element, namely the class $\mathcal{O}_{\mathrm{reg}} \in \mathcal{S}_\mathbf{G}$ of regular unipotent elements, because $\mathcal{E}(G,1,\mathcal{O}_{\mathrm{reg}}) = \{1_G\}$. In the statement of the conjecture $\mathcal{O}_1 = \mathcal{O}_{\text{reg}}$ and thus we should have 1_G^0 is an irreducible Brauer character, which it certainly is.

Similarly, the poset $(\mathcal{S}_{\mathbf{G}}, \preceq)$ contains a unique minimal element, namely the trivial class $\mathcal{O}_{\text{triv}} \in \mathcal{S}_{\mathbf{G}}$, because $\mathcal{E}(G, 1, \mathcal{O}_{\text{triv}}) = \{\text{St}_G\}$. In the statement of the conjecture $\mathcal{O}_r = \mathcal{O}_{\text{triv}}$ and St_G^0 could potentially have many irreducible constituents.

Since its inception several people have worked towards obtaining a proof of this conjecture. The conjecture was shown to be true by Dipper when $G = \text{GL}_n(k)$ and Geck when $G = \text{GU}_n(k)$. A particularly notable milestone in the life of the conjecture was achieved by Gruber–Hiß who showed the conjecture was true when G is a classical group and ℓ is a so-called linear prime for G. Together with O. Brunat and O. Dudas we have established the following.

Theorem (Brunat–Dudas–T.). Assume p is good for $\mathbf{G}$ and ℓ is very good for $\mathbf{G}$. If G has no component of type E_8 and $q \equiv 1 \pmod 4$ if G has a component of type E_7 then Geck's Unitriangularity Conjecture holds.

We are optimistic that our methods will be able to treat the cases of E_7 and E_8 and thus we hope to establish Geck's conjecture for all finite reductive groups, with appropriate assumptions on p and ℓ. As mentioned above the assumption that ℓ is very good is stronger than the assumption imposed in the original statement of the conjecture. Our result can be established with an assumption on ℓ matching that made in [4]. In fact, after work of Denoncin [1], it seems likely that some version of the unitriangularity can be established assuming only that ℓ is a good prime for $\mathbf{G}$.

References

[1] D. Denoncin, *Stable basic sets for finite special linear and unitary groups*, Adv. Math. **307** (2017), 344–368.

[2] M. Geck, *On the average values of the irreducible characters of finite groups of Lie type on geometric unipotent classes*, Doc. Math. **1** (1996), no. 15, 293–317.

[3] M. Geck, *Verallgemeinerte Gelfand–Graev Charaktere und Zerlegungszahlen endlicher Gruppen vom Lie-Typ*, Dissertation, RWTH Aachen, 1990.

[4] M. Geck, *Remarks on modular representations of finite groups of Lie type in non-defining characteristic.* Algebraic groups and quantum groups, 71–80, Contemp. Math. **565**, Amer. Math. Soc., Providence, RI, 2012.

[5] M. Geck and G. Hiss, *Modular representations of finite groups of Lie type in non-defining characteristic.* Finite reductive groups (Luminy, 1994), 195–249, Progr. Math. **141**, Birkhäuser Boston, Boston, MA, 1997.

[6] M. Geck and G. Malle, *On the existence of a unipotent support for the irreducible characters of a finite group of Lie type*, Trans. Amer. Math. Soc. **352** (2000), no. 1, 429–456.

[7] G. Lusztig, *A unipotent support for irreducible representations*, Adv. Math. **94** (1992), no. 2, 139–179.

[8] J. Taylor, *The Structure of Root Data and Smooth Regular Embeddings of Reductive Groups*, Proc. Edinb. Math. Soc. (2) **62** (2019), no. 2, 523-552.

Character bounds for finite groups of Lie type

Pham Huu Tiep

In this talk, we discuss recent results, obtained in joint work of the speaker with various collaborators, on the following problem:

Problem. Let G be a finite almost quasisimple group G and let $g \in G \smallsetminus Z(G)$. Find an explicit, and as small as possible, constant $0 < \alpha = \alpha(g) < 1$ such that $|\chi(g)| \leq \chi(1)^{\alpha}$ for all $\chi \in \mathrm{Irr}(G)$.

Even partial solutions to this problem have proved to be useful in a number of applications. The first result on this problem, in the case $G = S_n$ and $g = (m^{n/m})$ a product of disjoint cycles of the same length m, was obtained by Fomin and Lulov in [3]:

$$|\chi(g)| \leq \chi(1)^{1/m+o(1)}.$$

A full, still asymptotic, result for $G = S_n$ was later obtained by Larsen and Shalev in [8]. We may now focus on the case of finite groups of Lie type, that is, $G = \mathcal{G}^F$ for a connected reductive algebraic group $\mathcal{G}$ in characteristic $p > 0$ and a Steinberg endomorphism $F : \mathcal{G} \to \mathcal{G}$. Let r denote the rank of the semisimple subgroup $[\mathcal{G}, \mathcal{G}]$. An F-stable Levi subgroup $\mathcal{L}$ of $\mathcal{G}$ is called *split* if it is a Levi subgroup of an F-stable parabolic subgroup of $\mathcal{G}$. For any F-stable Levi subgroup $\mathcal{L}$, not a maximal torus, define

$$\alpha(\mathcal{L}^F) := \max_{1 \neq u \in \mathcal{L}^F,\ u \text{ unipotent}} \frac{\dim u^{\mathcal{L}}}{\dim u^{\mathcal{G}}}.$$

If $\mathcal{L}$ is an F-stable maximal torus, let $\alpha(\mathcal{L}^F) := 0$.

Theorem 1. [1] There exists an explicit function $f : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. Let p be a good prime for $\mathcal{G}$. Suppose that $g \in \mathcal{G}^F$ is such that $\mathbf{C}_{\mathcal{G}^F}(g) \leq \mathcal{L}^F$ for some proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$. Then

$$|\chi(g)| \leq f(r)\chi(1)^{\alpha(\mathcal{L}^F)}$$

for all $\chi \in \mathrm{Irr}(\mathcal{G}^F)$.

In our proof, f is roughly of the magnitude of $((r+1)!)^2$ (when r is not too small). However, examples show that $f(r)$ should be at least of the magnitude of $\sqrt{(r+1)!}$. On the other hand, the $\alpha(\mathcal{L}^F)$-exponent in Theorem 1 is best possible in a number of cases:

Theorem 2. [1] Let $\mathcal{G} = \mathrm{GL}_n$. There exists a constant $C(n)$ such that the following statement holds. For any $\mathcal{G}^F = \mathrm{GL}_n(q)$ with $q \geq C(n)$, and for any proper split Levi subgroup $\mathcal{L}$ of $\mathcal{G}$, there exist $g \in \mathcal{G}^F$ with $\mathbf{C}_{\mathcal{G}}(g) = \mathcal{L}$ and a unipotent irreducible character $\chi \in \mathrm{Irr}(\mathcal{G}^F)$ such that

$$|\chi(g)| \geq \frac{1}{4}\chi(1)^{\alpha(\mathcal{L}^F)}.$$

A key ingredient of the proof of Theorem 1 in [1] is to bound the *wave front set* $\mathcal{O}^*_\eta$ of any irreducible constituent of the Lusztig restriction ${}^*R^{\mathcal{G}}_{\mathcal{L}}(\chi)$ of $\chi \in \mathrm{Irr}(\mathcal{G}^F)$ by $\mathcal{O}^*_\chi$. The existence and uniqueness of $\mathcal{O}^*_\chi$ for any $\chi \in \mathrm{Irr}(\mathcal{G}^F)$ (when p is a good prime) was established by Lusztig [11] and Taylor [12], using Kawanaka's theory of generalized Gelfand-Graev representations [7].

Can one extend Theorem 1 to the case of non-split Levi subgroups? An answer to this question for groups with connected center is given in the following theorem:

Theorem 3. [13] There exists an explicit function $f : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. Let p be a good prime for $\mathcal{G}$ and let $\mathbf{Z}(\mathcal{G})$ be connected. Suppose that $\mathcal{L}$ is a proper F-stable Levi subgroup of $\mathcal{G}$ and $g \in \mathcal{L}^F$ is such that $\mathbf{C}_{\mathcal{G}}(g)^\circ \leq \mathcal{L}$. Then

$$|\chi(g)| \leq f(r)\chi(1)^{\alpha(\mathcal{L}^F)}$$

for all $\chi \in \mathrm{Irr}(\mathcal{G}^F)$.

A major part of the proof of Theorem 3 is to establish the aforementioned bounding result on $\mathcal{O}^*_\eta$ for any irreducible constituent of ${}^*R^{\mathcal{G}}_{\mathcal{L}}(\chi)$ for any proper F-stable Levi subgroup $\mathcal{L}$, by first proving its geometric analogue for parabolic induction of character sheaves. Combining the results of [1] and [13], the following asymptotically optimal character bound has been obtained in [13]:

Theorem 4. [13] There exists an explicit function $h : \mathbb{N} \to \mathbb{N}$ such that the following statement holds. Let $G := \mathrm{GL}_n(q)$ or $\mathrm{SL}_n(q)$, $n \geq 5$, and let $g \in G \smallsetminus \mathbf{Z}(G)$. Then

$$|\chi(g)| \leq h(n)\chi(1)^{(n-2)/(n-1)}$$

for all $\chi \in \mathrm{Irr}(\mathcal{G}^F)$.

In the case $\mathcal{G}$ is simple and not of type A, Theorems 1 and 4 still leave out, for instance, unipotent elements $g \in \mathcal{G}^F$. In a number of applications, however, one usually needs to bound $|\chi(g)|$ only when either $\chi(1)$ is not too large, of $|\mathbf{C}_{\mathcal{G}^F}(g)|$ is not too large. Under these conditions, various exponential character bounds for finite classical groups have been obtained in [4, 5].

Theorem 5. [4, 5] For any $\epsilon > 0$, there exists $\delta > 0$ such that the following statements hold. For any finite classical group G and for any $g \in G$ with $|\mathbf{C}_G(g)| \leq |G|^\delta$, $|\chi(g)| \leq \chi(1)^\epsilon$.

In fact, there is also an effective version of Theorem 5 for $4/5 < \epsilon < 1$ in [4, 5]. For instance, if $G = \mathrm{GL}_n(q)$ or $\mathrm{GU}_n(q)$ and $\epsilon = 8/9$, one can take $\delta = 1/12$. The proof of Theorem 5 relies on the notion of *character level*, first developed in [4] for $G = \mathrm{GL}_n(q)$ and $\mathrm{GU}_n(q)$.

Can one obtain good character bounds for all elements $g \in \mathcal{G}^F$ and for all characters $\chi \in \mathrm{Irr}(\mathcal{G}^F)$, for $\mathcal{G}$ simple not of type A? Such bounds in the case $\mathcal{G}$ is exceptional have been obtained in recent joint work of Liebeck and the speaker. Further bounds for finite classical groups have also been obtained in recent joint work of Liebeck, Shalev, and the speaker.

Exponential character bounds recently established lead to significant progress in a number of applications. We formulate one such result, which is concerned with *mixing time of random walks* on finite groups (cf. [2]), which extends the main result of [6].

Theorem 6. [13] Let $G = \mathrm{SL}_n(q)$ and let $g \in G \smallsetminus \mathbf{Z}(G)$. If q is large enough, then the mixing time of the random walk on the Cayley graph $\Gamma(G, g^G)$ is at most n.

Further applications, particularly concerning the diameter of the *McKay graph* for finite simple groups, are also discussed, see [9, 10].

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References

[1] R. Bezrukavnikov, M. W. Liebeck, A. Shalev, and Pham Huu Tiep, Character bounds for finite groups of Lie type, *Acta Math.* **221** (2018), 1–57.

[2] P. Diaconis and M. Shahshahani, Generating a random permutation with random transpositions, *Z. Wahrsch. Verw. Gebiete* **57** (1981), 159–179.

[3] S. Fomin and N. Lulov, On the number of rim hook tableaux, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **223** (1995), Teor. Predstav. Din. Sistemy, Kombin. i Algoritm. Metody. I, 219–226, 340; translation in *J. Math. Sci.* (New York) **87** (1997), 4118–4123.

[4] R. M. Guralnick, M. Larsen, and Pham Huu Tiep, Character levels and character bounds, (submitted).

[5] R. M. Guralnick, M. Larsen, and Pham Huu Tiep, Character levels and character bounds. II, (preprint).

[6] M. Hildebrand, Generating random elements in $SL_n(\mathbb{F}_q)$ by random transvections, *J. Alg. Comb.* **1** (1992), 133–150.

[7] N. Kawanaka, Generalized Gelfand-Graev representations of exceptional simple algebraic groups over a finite field. I, *Invent. Math.* **84** (1986), 575–616.

[8] M. Larsen and A. Shalev, Characters of symmetric groups: sharp bounds and applications, *Invent. Math.* **174** (2008), 645–687.

[9] M. W. Liebeck, A. Shalev, and Pham Huu Tiep, Character ratios, representation varieties and random generation of finite groups of Lie type, (submitted).

[10] M. W. Liebeck, A. Shalev, and Pham Huu Tiep, On the diameters of McKay graphs for finite simple groups, (submitted).

[11] G. Lusztig, *Characters of Reductive Groups over a Finite Field*, Annals of Math. Studies **107**, Princeton Univ. Press, Princeton, 1984.

[12] J. Taylor, Generalised Gelfand–Graev representations in small characteristics, *Nagoya Math. J.* **224** (2016), 93–167.

[13] J. Taylor and Pham Huu Tiep, Lusztig induction, unipotent support, and character bounds, (submitted).

Real conjugacy class sizes and orders of real elements

Hung P. Tong-Viet

Let G be a finite group. An element $x \in G$ is said to be *real* if x and its inverse are conjugate in G. A conjugacy class of G is real if it contains real elements. Moreover, an element $x \in G$ is said to be *strongly real* if it is inverted by an involution.

The existence or non-existence of real elements of certain orders is an important question in finite group theory. Following M. Suzuki, a finite group G is called a (C)-group if the centralizer of every involution is 2-closed, that is, having a normal Sylow 2-subgroup. It turns out that a finite group G is a (C)-group if and only if G has no real element of order $2m$ with $m > 1$ being odd; equivalently, the order of every real element of G is either a power of 2 or odd (see [2, Proposition 2.7]). These groups have been studied by Suzuki himself in [8] and recently in [2] for solvable groups. Use these results, we characterize finite (C)-groups having no real element of order 4 and deduce the following criterion for the solvability of finite groups with some restriction on the orders of real elements.

Theorem 1. ([11, Theorem A]) *Let G be a finite group and let p be a prime. If every real element of G is an involution or a p-element, then G is solvable. Moreover, if $L = \mathbf{O}^{2'}(G)$ then*

(1) *L is a 2-group and has no real element of order 4; or*
(2) *$\mathbf{O}_2(L) = 1$ and L has a cyclic Sylow 2-subgroup Q and a normal 2-complement P which is a p-group with $\mathbf{C}_P(Q) = \mathbf{C}_P(z)$, where z is the unique involution in Q.*

Our next result describes the structure of finite groups having at most three distinct real element orders.

Theorem 2. ([11, Theorem B]) *Let G be a finite group and let $L = \mathbf{O}^{2'}(G)$. If G has at most three distinct real element orders, then*

(1) *L is a 2-group and has no real element of order 8; or*
(2) *$\mathbf{O}_2(L) = 1$ and L has a cyclic Sylow 2-subgroup Q and a normal 2-complement P which is a p-group for some odd prime p with $\mathbf{C}_P(Q) = \mathbf{C}_P(z)$, where z is the unique involution in Q.*

Theorem 2 follows from Theorem 1 together with a result in [4] stating that a finite group is 2-closed if and only if it has no nontrivial real element of odd order. Note that the proof of the aforementioned result uses only Baer-Suzuki theorem. Thus the proof of Theorem 2 does not depend on the classification of finite simple groups.

It is well-known that the number of real-valued ordinary irreducible characters and the number of conjugacy classes of real elements of a finite group coincide. Hence if G has at most three real-valued ordinary irreducible characters, then G has at most three conjugacy classes of real elements. In particular, such groups must satisfy the hypothesis of Theorem 2. Therefore, Theorem 2 gives a classification-free proof of the solvability of finite groups with at most three real-valued ordinary irreducible characters (see Theorem 2.5 of [6]).

Let G be a finite group and let $A \leq S \leq G$. Recall that A is *strongly closed* in S with respect to G if whenever $a \in A, g \in G$, if $a^g \in S$, then $a^g \in A$ or equivalently $A^g \cap S \subseteq A$ for all $g \in G$. Generalizing Glauberman's Z^*-theorem, Goldschmidt [5] determines the structure of finite groups possessing an abelian strongly closed 2-subgroup. It turns out that if S is a Sylow 2-subgroup of a finite group G and $\Omega_1(S)$ is abelian, then $\Omega_1(S)$ is an elementary abelian group and hence it is strongly closed in S with respect to G. Now we can see that if a finite group G has no strongly real element of order 4, then all involutions in S commute, in particular, $\Omega_1(S)$ is abelian. Hence we can apply Goldschmidt's result to classify all finite non-abelian simple groups having no real element of order 4. Using this result, we can determine the structure of finite groups having four real-valued irreducible characters.

Theorem 3. ([11, Theorem 3.3]) *Let G be a finite group with exactly four real-valued irreducible characters. Let $L = \mathbf{O}^{2'}(G)$ and Q be a Sylow 2-subgroup of L. Then one of the following holds.*

(1) *G has a normal Sylow 2-subgroup.*
(2) *$\mathbf{O}_2(L) = 1$, Q is either cyclic or quaternion of order 8 and L has a normal 2-complement K with $\mathbf{C}_K(Q) = \mathbf{C}_K(z)$, where z is the unique involution in Q.*
(3) *G has 2-length one and Q is either homocyclic or a Suzuki 2-group.*
(4) *$G \cong \mathrm{SL}_3(2) \times K$, where K is of odd order.*

We are able to give classification-free proofs of some results in [3, 7]. (See [11]).

A classical result due to Burnside states that a finite group is of odd order if and only if the identity element is the only real element. This result has been generalized by Chillag and Mann [1], where the authors showed that if a finite group G has only one real class size or equivalently every real element lies in $\mathbf{Z}(G)$, then G is isomorphic to a direct product of a 2-group and a group of odd order. Extend this result further, we can prove the following.

Theorem 4. ([10, Theorem A]) *Let G be a finite group. If G has two real class sizes, then G is solvable.*

This confirms a conjecture due to G. Navarro, L. Sanus and P. Tiep. This result is best possible in the sense that there are non-solvable groups with exactly three real class sizes. In fact, the special linear group $\mathrm{SL}_2(q)$ of degree 2 over a finite field of size q, where $q \geq 7$ is a prime power and is congruent to -1 modulo 4, has three real class sizes, namely $1, q(q-1)$ and $q(q+1)$ but $\mathrm{SL}_2(q)$ is non-solvable.

As already noted in [7], any possible proof of Theorem 4 is complicated. Instead of giving a direct proof of this theorem, we will prove a much stronger result which implies Theorem 4. For an integer $n \geq 1$ and a prime p, the p-part of n, denoted by n_p is the largest power of p dividing n.

Theorem 5. ([10, Theorem B]) *Let G be a finite group. Suppose that all non-central real class sizes of G have the same 2-part. Then G is solvable.*

In other words, if $|x^G|_2 = 2^a$ for all non-central real elements $x \in G$, where $a \geq 0$ is a fixed integer, then G is solvable. In fact, we can say more about the structure of these groups.

Theorem 6. ([10, Theorem C]) *Let G be a finite group. Suppose that all non-central real class sizes of G have the same 2-part. Then G has 2-length one.*

Recall that a group G is said to have 2-length one if there exist normal subgroups $N \leq K \leq G$ such that N and G/K have odd order and K/N is a 2-group. Theorem 6 confirms a conjecture proposed in [9].

References

[1] D. Chillag, A. Mann, Nearly odd-order and nearly real finite groups, Comm. Algebra **26** (1998), no. 7, 2041–2064.

[2] S. Dolfi, D. Gluck, G. Navarro, On the orders of real elements of solvable groups, Israel J. Math. **210** (2015), no. 1, 1–21.

[3] S. Dolfi, G. Malle and G. Navarro, The finite groups with no real p-elements, Israel J. Math. **192** (2012), no. 2, 831–840.

[4] S. Dolfi, G. Navarro, P. H. Tiep, Primes dividing the degrees of the real characters, Math. Z. **259** (2008), no. 4, 755–774.

[5] D. M. Goldschmidt, 2-fusion in finite groups, Ann. of Math. (2) **99** (1974), 70–117.

[6] A. Moretó, G. Navarro, Groups with three real valued irreducible characters, Israel J. Math. **163** (2008), 85–92.

[7] G. Navarro, L. Sanus, P. H. Tiep, Real characters and degrees, Israel J. Math. **171** (2009), 157–173.

[8] M. Suzuki, Finite groups in which the centralizer of any element of order 2 is 2-closed, Ann. of Math. (2) **82** (1965) 191–212.

[9] H. P. Tong-Viet, Groups with some arithmetic conditions on real class sizes, Acta Math. Hungar. **140** (2013), no. 1–2, 105–116.

[10] H. P. Tong-Viet, 2-parts of real class sizes, Algebra Number Theory **12** (2018), no. 10, 2499–2514.

[11] H. P. Tong-Viet, Orders of real elements in finite groups, J. Algebra, to appear. https://doi.org/10.1016/j.jalgebra.2019.03.025

The local invariant of an irreducible character

Alexandre Turull

It is well known that associated with each irreducible character of a finite group is an element of a Brauer group. Important global/local conjectures in representation theory of finite groups have been refined over the years. In particular, these elements of the Brauer group have been incorporated in them as elements in their refinements by Turull [2]. Turull has proved that these refinements (together with many other ones) hold true in the case of p-solvable groups for the Alperin-McKay Conjecture [3] and Dade's Projective Conjecture [4].

Let p be a prime. We note that for the conjectures above the relevant base fields are finite extensions of $\mathbf{Q}_p$, the field of p-adic numbers. Let K be a finite extension of $\mathbf{Q}_p$. Then it is well known that there exists a uniquely defined isomorphism

$$inv\colon Br(K) \to \mathbf{Q}/\mathbf{Z}$$

(for example [1]).

In this talk, we show that if p-Brauer characters are defined then we automatically have a uniquely defined collection of maps inv so that, for every finite group G, we have a function

$$inv\colon Irr(G) \to \mathbf{Q}/\mathbf{Z},$$

with excellent compatibility properties.

The strengthened Alperin-McKay conjecture, and the strengthened Dade projective conjecture can be reformulated to use the local invariants given by inv

instead of the element of the Brauer group associated to each irreducible character.

We note that, for each $\chi \in Irr(G)$, the *denominator* of $inv(\chi)$ is the p-local Schur index $m_p(\chi)$ of χ. We also note that, if ψ is a Galois conjugate of χ, then $inv(\chi)$ and $inv(\psi)$ have the same *denominator* but might have different *numerator*.

This immediately suggests the problem of how to explicitly compute $inv(\chi)$. In the talk, we also describe how one can calculate $inv(\chi)$ in every case.

For the calculation of $inv(\chi)$, we describe a series of reductions that allow us to explicitly compute $inv(\chi)$. There are reductions of a small number of types that in each case relate $inv(\chi)$ to the local invariant of some irreducible character of a smaller group than G. Applying these repeatedly, we reduce the problem to the calculation of $inv(\chi)$ in the case where these reductions no longer yield smaller groups. The resulting groups can be classified into a small number of types. For groups of each of these types, we have an explicit formula that yields $inv(\chi)$.

References

[1] R. S. Pierce, "Associative algebras" Springer-Verlag, New York, Heidelberg, Berlin, 1982.

[2] A. Turull, *Strengthening the McKay Conjecture to include local fields and local Schur indices*, J. Algebra **319** (2008), 4853–4868.

[3] A. Turull, *The strengthened Alperin-McKay conjecture for p-solvable groups*, J. Algebra **394** (2013), 79–91.

[4] A. Turull, *Refinements of Dade's Projective Conjecture for p-solvable groups*, J. Algebra **474** (2017), 424–465.

A simple character formula

Geordie Williamson

(joint work with Simon Riche)

Let G denote a reductive group over an algebraically closed field of characteristic $p > 0$. Let T denote a maximal torus, $\mathscr{X}$ its character lattice, and $\mathscr{X}_+$ the dominant weights with respect to some choice of positive roots. To any $\lambda \in \mathscr{X}_+$ we can associate a simple highest weight module L_λ. The L_λ are pairwise non-isomorphic and any simple algebraic representation of G is isomoprhic to some L_λ. We would like to know how big each L_λ is, and what its character is.

For SL_2 one can do everything by hand. I'm not sure who first wrote it down. The answer for SL_3 was obtained by Mark (1939) and Braden (1967). In the 70s Jantzen discovered his sum formula [6]. The sum formula gives a complete answer for Sp_4, SL_4 and G_2.

How far does one get with Jantzen's sum formula? Careful calculations of Jantzen reduce the problem to one undetermined $a \in \{1, 2\}$ for SL_5, one undetermined $d \in \{1, 2\}$ for Sp_6 and a few undetermined quantities for $Spin_7$. (This does not mean that there is only one unknown character in each case. Jantzen shows that there are a few ambiguities in each type, but that these ambiguities are all connected via the parameters above.) Groups of rank 4 and above presumably involve many more complications!

I recalled the periodic form of Lusztig's character formula [7] from 1980. It is the statement

$$(1) \qquad [\widehat{P_A}] = \sum d_{B,\hat{A}}(1)[\widehat{L_B}]$$

where we are now working in the principal block of G_1T-modules. For a p-alcove A, we let $\widehat{L_A}$ denote the simple module of highest weight λ, where λ is the unique weight in A in the orbit under the p-dilated affine Weyl group, and $\widehat{P_A}$ denotes its projective cover. The $d_{B,A}$ are Lusztig's periodic polynomials [7]. Here $A \mapsto \hat{A}$ is the operation on p-alcoves which is uniquely determined by the following two properties: it is invariant under translations in $p\mathscr{X}$; on alcoves of the form w_0A with A in the fundamental block, it is given by $w_0A \mapsto A$.

Statement (1) implies (via Brauer-Humphreys reciprocity) character formulas for simple G_1T-modues. This in turn provides character formulas for the simple G-modules. (The key fact is that it is enough to know the character of L_λ for restricted weights, and these simple modules stay simple for G_1T.) Statement (1) is known to hold for large p above an explicit bound [2, 4]. It is also known that it is not true for many primes between h and some exponential function of h [8]. Thus it is desirable to have a feasible method of calculating these characters for small and even "medium sized" primes. For example, it would be nice if we could tell Jantzen whether $a = 1$ or 2 for SL_5!

The purpose of the lecture was to state the following new formula:

$$[\widehat{P_A}] = \sum {}^p d_{B,\tilde{A}}(1)[\widehat{L_B}]$$

This formula is valid for $p \geq 2h - 1$, where h is the Coxeter number, and has a chance to hold for all $p \geq h$ (this would be a theorem if Donkin's conjecture is true for $p \geq h$). Here the ${}^p d_{B,A}$ are periodic p-polynomials. Lusztig observed that one may express the canonical basis in the periodic module via Kazhdan-Lusztig polynomials in the spherical module [7]. Because we know that spherical p-polynomials are, this allows us to define periodic p-polynomials via Lusztig's lemma.

Although the ${}^p d_{B,A}$ are complicated, this formula probably represents the easiest way to calculate the characters of simple G-modules beyond the cases where Jantzen's sum formula provides the answer, or where Lusztig's formula is valid. For example, Jensen and Scheinmann (work in progress) were able to verify by hand that $a = 1$ in the SL_5 case above.

The proof has three main ingredients:

(1) for A in the fundamental box, we have

$$(2) \qquad \widehat{P_A} \cong (T_{\hat{A}})_{|G_1T}$$

(proved by Jantzen [5] and Donkin [3]). This is only known to be true for $p \geq 2h - 1$, and explains why we must assume $p \geq 2h - 1$ above. Donkin conjectures that (2) holds for all p. If his conjecture is true then our formula is valid for $p \geq h$.

(2) A formula for tilting characters recently established by Achar, Makisumi, Riche and the author [1].
(3) An embedding of the spherical category into the anti-spherical category, categorifying a well-known embedding.

References

[1] P. Achar, S. Makisumi, S. Riche, and G. Williamson, *Koszul duality for Kac-Moody groups and characters of tilting modules*, J. Amer. Math. Soc. **32** (2019), 261–310.

[2] H. H. Andersen, J. C. Jantzen, and W. Soergel. Representations of quantum groups at a pth root of unity and of semisimple groups in characteristic p: independence of p. *Astérisque*, (220):321, 1994.

[3] S. Donkin. On tilting modules for algebraic groups. *Math. Z.*, 212(1):39–60, 1993.

[4] P. Fiebig. An upper bound on the exceptional characteristics for Lusztig's character formula. *J. Reine Angew. Math.*, 673:1–31, 2012.

[5] J. C. Jantzen, *Darstellungen halbeinfacher Gruppen und ihrer Frobenius-Kerne*, J. Reine Angew. Math. **317** (1980), 157–199.

[6] J. C. Jantzen. *Moduln mit einem höchsten Gewicht*, volume 750 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[7] G. Lusztig. *Hecke algebras and Jantzen's generic decomposition patterns*, volume 750 of *Lecture Notes in Mathematics*, Adv. Math. **37** (1980), 121–164.

[8] G. Williamson, *Schubert calculus and torsion explosion*, J. Amer. Math. Soc. **30** (2017), 1023–1046. With an appendix by A. Kontorovich, P. McNamara and G. Williamson.

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