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Mini-Workshop: One-sided and Two-sided Stochastic Descriptions

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Abstract. We consider the set of discrete time stochastic processes which are dependent on their past, and the set of those that depend on both their past and their future. As long as we only allow dependence on a finite number of variables, those two sets are the same. However interesting questions appear when the dependence becomes infinite, and some of them were discussed during our mini-workshop.

Mathematics Subject Classification (2010): 60K35, 60K40, 82B26, 82B99.

Introduction by the Organizers

The mini-workshop *One-sided and Two-sided Stochastic Descriptions* was attended by 15 participants with background in different mathematical areas. The main aim of this event was to investigate the connection between the theory of stochastic processes with unbounded memory, having a one-sided stochastic description, and the theory of Gibbs states from equilibrium statistical mechanics, which have a two-sided stochastic description.

During the week, we had twelve extensive talks given by our participants. Moreover, on Thursday and Friday we also organized two broad discussion sessions involving all the participants, which allowed us to pin-point some of the most pressing and interesting open questions, as will be detailed further on.

The meeting revolved around three main topics, the first one being the most comprehensive and relating to the theoretical aspects of the two types of stochastic description, as well as the connection between them. In this sense, Sabine Jansen talked about a statistical mechanical model in dimension one at low temperature which also allowed a one-sided stochastic description and a well-defined transfer operator.

One interesting aspect about extending the connection between the two theories is to expand the known results for one-dimensional systems to results for systems in higher dimensions. However, the question remains on what a sensible definition for *one-sidedness* could be. Christof Külske reported on localized Gibbs measures and delocalized gradient Gibbs measures on trees and their coexistence. His results give indications about what one-sided description in the context of trees could mean. During the open discussion, this topic was further elaborated. One could look at e.g. Markov random fields with one-sided trivial σ-algebra, but also at continuous-time(space) processes. In the last case, one could investigate the continuity of the probability kernel appearing in the Nguyen–Georgii–Zessin formula, which seems to be analogous to the quasilocality property in the standard case.

Frédéric Paccaut focused on non-regular g-measures, and presented an existence criterion thereof in terms of the topological properties of the set of discontinuity points of the associated g-measures. Moreover, he gave a characterization of non-regular g-measures in terms of variable length Markov chains.

Stein Andreas Bethuelsen discussed in more detail the Schonmann projection as an example of renormalized Gibbs measure which is known not to be Gibbs anymore, since its specification has at least one discontinuity point. An unanswered question is whether the set of discontinuity points of the two-sided specification has measure zero. One of the results presented was that the one-sided specification does indeed satisfy this condition. Moreover, in this talk a connection to random walks in random environment was mentioned. Johannes Bäumler then proved that for a spin glass at zero temperature on a locally finite tree the uniqueness of the ground states is equivalent to the recurrence of the random walk on the tree.

The second main topic of the meeting relates to statistical mechanics of systems with long-range interactions. In this sense, Arnaud Le Ny gave a broad overview of the known results for long-range Ising models in dimension one and discussed an example of a Gibbs measure which is not a regular g-measure. Loren Coquille discussed the models in dimensions two and three and showed that in a certain regime in the two-dimensional case, non-trivial invariant states arising from Dobrushin boundary conditions do not exist. Moreover, in dimension two, a finite-volume version of the Aizenmann–Higuchi version holds for the Ising model. An interesting open question is whether this result holds also for long-range Ising models. During the discussions, Romain Durand pointed out that models for which fluctuations of the interface are known, and which satisfy the Markov or the FKG property could be good candidates for an Aizenmann–Higuchi type of result.

Aernout van Enter continued the discussion on long-range Ising models by looking at systems with random boundary conditions. Noam Berger gave an overview of known results in long-range percolation theory and emphasized similarities in proof techniques that could be applied also to long-range Ising models. As an example, the 1986 Aizenmann–Newman result about the existence of an infinite component in the critical regime in the long range percolation model, was proven to be true also for the long-range Ising model in 1988.

The third theme of the workshop refers to probabilistic cellular automata and connections to dynamical systems. In particular, Irène Marcovici discussed PCA with memory two, and gave necessary and sufficient conditions for which the invariant measure has a product form or a Markovian form. Evgeny Verbitskiy gave an extensive overview of common concepts in Statistical Mechanics and Dynamical System, like Gibbs/equilibrium states and g-measures, and their respective variational principle characterizations. Moreover, he stated a necessary and sufficient condition for a g-measure to be Gibbs, underlying the remark that for a g-measure to be Gibbs, one does not have to be in the uniqueness regime. Jeff Steif enumerated a series of open problems related to Gibbs measures, the quasi-locality of random walks in random scenery and introduced the concept of a bilaterally deterministic stochastic process, asking about the one-sided and two-sided regularity properties of its invariant states.

During the discussions, Evgeny Verbitsky mentioned it might be a good time to re-evaluate the so-called Dobrushin program for restauration of Gibbsianness and give a better classification of the weaker versions of Gibbsianness, for which the thermodynamic formalism holds.

The activities of the mini-workshop were outstandingly complemented by a small group that ran to eat the famous Black Forest Cake and a by a musical recital on Thursday evening, given by participants from all the mini-workshops. All in all, we trust that all participants benefited from the excellent scientific environment provided by the institute, the variety of talks and the ample time for informal discussions, and that this mini-workshop provided new perspectives and ideas for exciting new results.

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Mini-Workshop: One-sided and Two-sided Stochastic Descriptions

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Abstracts

Gibbs Measures for Long-Range Ising Models

Arnaud Le Ny
(joint work with Rodrigo Bissacot, Eric Ossami Endo and Aernout van Enter)

During this introductory talk given at this Oberwolfach mini-workshop, I first provided a short general description of the current state of arts and main bibliographical references concerning the DLR description of long-range polynomially decaying Ising models in dimension one (also sometimes referred as *Dyson Models*), quickly defined as follows:

Given a finite volume Λ in $\mathbb{Z}$, consider the Hamiltonian on $\Omega_\Lambda = \{-1,1\}^\Lambda$ with boundary condition $\omega_{\Lambda^c} \in \Omega_{\Lambda^c} = \{-1,1\}^{\Lambda^c}$ given by

$$H_\Lambda^\omega(\sigma_\Lambda) = -\frac{1}{2}\sum_{i,j\in\Lambda} J_{ij}\sigma_i\sigma_j - \sum_{\substack{i\in\Lambda \\ j\in\Lambda^c}} J_{ij}\sigma_i\omega_j, \tag{1}$$

where $\sigma_\Lambda = (\sigma_i)_{i\in\Lambda} \in \Omega_\Lambda$ and, for a fixed $J > 0$ and $1 < \alpha \leq 2$, the interaction $(J_{ij})_{i,j\in\mathbb{Z}}$ is defined by

$$J_{ij} = \begin{cases} J, & \text{if } |i-j| = 1, \\ \frac{1}{|i-j|^\alpha}, & \text{if } |i-j| > 1, \\ 0, & \text{if } i = j. \end{cases} \tag{2}$$

For a fixed inverse temperature $\beta > 0$, the Gibbs specification is determined by a family of probability measures $\gamma = \{\mu_\Lambda^{(\cdot)}\}_{\Lambda\subset\mathbb{Z}}$ defined by

$$\mu_\Lambda^\omega(\sigma_\Lambda) = \frac{1}{Z_\Lambda^\omega} e^{-\beta H_\Lambda^\omega(\sigma_\Lambda)} \tag{3}$$

where $\omega \in \Omega_{\Lambda^c}$ is a bounday condition, and Z_Λ^ω the usual partition function.

To proceed at infinite volume, consider $\Omega = \{-1,1\}^\mathbb{Z}$, let $\mathcal{E}$ be the Borel sigma-algebra on $\{-1,1\}$ and let $\mathcal{F} = \mathcal{E}^\mathbb{Z}$ be the product sigma-algebra on Ω. Denote by $\mathcal{M}_1(\Omega,\mathcal{F})$ the set of probability measures on the measurable space $(\Omega,\mathcal{F})$. A *Gibbs measure* μ is defined to be a probability measure on $\mathcal{M}_1(\Omega,\mathcal{F})$ whose conditional probabilities with boundary condition ω outside Λ, are of the form of Gibbsian specifications, *i.e.* the kernels μ_Λ^ω introduced in (3), and thus satisfy the *DLR equation* :

$$\mu\mu_\Lambda^{(\cdot)} = \mu, \text{ for all } \Lambda \subset \mathbb{Z}, \tag{4}$$

where $\Lambda \subset \mathbb{Z}$ means here that Λ is a finite subset of $\mathbb{Z}$.

These ferromagnetic Hamiltonians admit the same ground states as in the classical *n.n.* Ising model, namely the + (all pluses) and − (all minuses) configurations but the cost of inserting a droplet of an opposite phase of length L now depends of the decay α, thanks to the following *Landau estimate* (5), with an excess energy of subvolumic order :

$$h_L^+ := H_\Lambda(-|+) - H(+|+) = \sum_{j=-L}^{L} \sum_{k \geq L}^{\infty} \frac{1}{k^\alpha} \approx C \cdot L^{2-\alpha} \tag{5}$$

while it is dimension dependent for $n.n.$ models. Thus, for very long ranges $\alpha < 2$ ('slow decays'), the probability of occurrence of a droplet of the opposite phas eis depressed at least by

$$c \exp -\beta\zeta L^{2-\alpha}, c, \zeta > 0.$$

The analogy with the bounds going as $c \exp -\beta L^{(d-1)/d}$ for $n.n.$ Ising model on $\mathbb{Z}^d$ is evident. The decay parameter α plays thus the role of a fractional dimension that one can tune continuously to interpolate from dimension one and two or higher (see also the report of A. van Enter in this volume [10]). From this estimate, one indeed gets stability of the ground states and phase transition in dimension one for slow decays $\alpha \in (1,2)$, similarly to what happens for $2d$-$n.n.$ Ising models since the historical proof of Peierls [12].

The initial proofs of Dyson ([8], 1969; [9], 1971) used feromagnetism coupled with a comparison with a hierarchical model to solve this Kac-Thompson conjecture ([13], 1968/69), and indeed get phase transition at low temperatures, with two extremal Gibbs measures μ^- and μ^+ describing equilibrium states of opposite magnetizations. Fifteen years later, in 1982, Fröhlich and Spencer [11] introduced an implicit contour construction to provide, in the interesting borderline case $\alpha = 2$, an alternative proof, more robust. This proof has been made explicit and extended to any slow decay $\alpha \leq 2$ two extra decades after, in 2005, by Cassandro *et al.* [2] who initiated a complete geometric description of configurations in terms of triangles and contours in order to investigate more refined typical/microscopic properties of this model (Phase transition for $\alpha \in (1,2)$ in [2], 2005; Phase transition for $\alpha \in (1, \frac{3}{2})$ for the long-range RFIM in [5], 2009; Typical configurations for the long-range RFIM in [6], 2012; Interface fluctuations and cluster expansion in [4], phase separation in one dimension [3]).

In my talk, I have described briefly their results within the DLR framework and have described afterwards how they have provided a rich source of new behaviours or counterexamples. As an introduction to the workshop, I have introduced Gibbs *vs.* g-measures on the line $\mathbb{Z}$ as DLR measures possessing continuous versions of regular conditional probabilities w.r.t. the outside of finite sets (*i.e.* to the past *and* the future, for Gibbs measures) *vs.* continuous versions of conditional probabilities with respect to the past only (g-measures). As we shall learn during the workshop, although these two families of measures coincide for finite range (*cf. Global vs. local Markov properties*), it seemed since fifty years that their relation are more tight at infinite-range. As a warm-up, I described the first example of non-renormalized Gibbsian measure in dimension one (*Decimation of Dyson models for slow decays*, [7], 2017) and used a careful extension of this proof, fully described in [1], to show that for slow decays $\alpha \in (1,2)$, the plus phase μ^+ of the Dyson models, got by taking sequences of finite-volume Gibbs measures with +-boundary conditions, indeed provides an example of non-g-measures (discontinuity of conditional probabilities w.r.t. to the past) which is a Gibbs measures (continuity w.r.t. to past *and* future). The proof of this last result relies on a careful use of the description of interface fluctuations of [4], from which we get an entropic repulsion created by a droplet of minuses. We eventually recover discontinuity of the conditional probabilities for a neutral alternated boundary condition by a long-range effects of $-$ boundary condition far away in the past transmitted at the origin *via* an interface point located nearby, but in its future (while the later is absent with + boundary condition).

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On the Gibbs states of (long range) Ising models

Loren Coquille
(joint work with Yvan Velenik, Aernout van Enter, Arnaud Le Ny, Wioletta Ruszel)

In the early 1980s, in two celebrated papers, Aizenman [A80] and Higuchi [H81] independently established that all infinite-volume Gibbs measures of the two dimensional ferromagnetic nearest-neighbor Ising model at inverse temperature $\beta \geq 0$ are of the form $\alpha\mu_\beta^+ + (1-\alpha)\mu_\beta^-$, where μ_β^+ and μ_β^- are the two pure phases and $0 \leq \alpha \leq 1$. In particular, all Gibbs states are translation invariant.

The situation is more complex in dimension 3 since extremal non-translation invariant states exist at very low temperature, as it was proved by Dobrushin [D73]. The interfaces induced by so-called Dobrushin boundary conditions (+ above some hyperplane, − below), which are parallel to coordinates axes, is indeed localised, whereas it shows Brownian fluctuations in dimension 2 [GI05].

Together with Yvan Velenik, we developed a new approach to the Aizenman-Higuchi result in [CV12], with a number of advantages: (a) We obtain an optimal finite-volume, quantitative analogue (implying the classical claim); (b) the scheme of our proof seems more natural and provides a better picture of the underlying phenomenon; (c) this new approach might be applicable to systems for which the classical method fails. Our fine study of the interfaces fluctuations relies on the Ornstein-Zernike asymptotics of the two-point function below the critical temperature [CIV08]. We extended the approach to the Potts model in [CDIV14].

The presence of long-range interactions can in some cases mimic the behavior of the model in one more dimension. The case of Dyson models (long-range, dimension 1) has been discussed in the talk of Arnaud Le Ny (presence of a phase transition, fluctuation of the interface point, lack of g-measure property), see [LN19].

In a recent work with Aernout van Enter, Arnaud Le Ny and Wioletta Ruszel, we consider the two-dimensional Ising model with long-range pair interactions of the form $J_{xy} \sim |x-y|^{-\alpha}$ with $\alpha > 2$, mostly when $J_{xy \geq 0}$. We show that Dobrushin states (i.e. extremal non-translation-invariant Gibbs states selected by Dobrushin conditions which are parallel to coordinate axes) do not exist. The main ingredient of the proof is a bound on the relative entropy between a Dobrushin state and its translate, providing equivalence of translated boundary conditions [BLP79]. We also mention the existence of rigid interfaces in two long-range anisotropic contexts, following van Beijeren's approach [vB75].

Extensions of this result are possible in the direction of the Aizenman-Higuchi theorem, or concerning fluctuations of interfaces. Our aim, as a middle term project, is to prove Fröhlich and Zegarlinski's conjecture [FZ91] giving the two-point function asymptotics in the discrete Gaussian chain with long-range interactions, which is expected to be a good approximation of the 2d long-range Ising model at very low temperature.

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Transfer operator and decay of correlations for a chain of atoms at low temperature

Sabine Jansen

(joint work with Wolfgang König, Bernd Schmidt, Florian Theil)

We consider Gibbs measures for a chain of atoms in the constant pressure ensemble. The measure depends on the pair potential $v(r)$ and three parameters: the inverse temperature $\beta > 0$, the pressure $p > 0$, and a truncation parameter $m \in \mathbb{N} \cup \{\infty\}$. Nearest-neighbor and next-nearest neighbor interactions correspond to $m = 1$ and $m = 2$. The atoms have coordinates $0 = x_1 < \cdots < x_N$, with interparticle spacings $z_j = x_{j+1} - x_j$, $j = 1, \ldots, N-1$. We work with a class of potentials that includes the Lennard-Jones potential with an added hard core, given by

$$v(r) = \begin{cases} \infty, & r \le r_0, \\ r^{-12} - r^{-6}, & r > r_0, \end{cases}$$

with $r_0 > 0$ small enough. The Gibbs energy of a chain of N atoms is

$$\mathcal{E}_N^{(p)}(z_1, \dots, z_{N-1}) = \sum_{j=1}^{N} \big(v(z_j) + p z_j \big) + \sum_{k=1}^{m-1} \sum_{j=1}^{N-1-k} v(z_j + \cdots + z_{j+k}).$$

The reader more familiar with spin chains may think of spacings $z_j > 0$ as continuous spins on a one-dimensional lattice. The pressure $p > 0$ is analogous to the external field $h \in \mathbb{R}$.

We are interested in the Gibbs measures $\mu_N^{(\beta,p)}$ on $\mathbb{R}_+^{N-1}$ given by

$$\mu_N^{(\beta,p)}(A) = \frac{1}{Q_N(\beta,p)} \int_A \mathrm{e}^{-\beta \mathcal{E}_N^{(p)}(z_1,\dots,z_n)} \mathrm{d}z_1 \cdots \mathrm{d}z_{N-1}$$

with $Q_N(\beta, p)$ a normalization constant (partition function). Two limits are taken: first, the thermodynamic limit $N \to \infty$ and second, the low-temperature limit $\beta \to \infty$ at constant pressure $p > 0$.

1. Thermodynamic limit and transfer operator

To lighten notation, we suppress the p-dependence from the notation from here onwards. The spaces $\mathbb{R}_+^{\mathbb{N}}$ and $\mathbb{R}_+^{\mathbb{Z}}$ are equipped with the product topology. Infinite-volume Gibbs measures are investigated with the help of a transfer operator $\mathcal{L}_\beta$. Let

$$h(z_1, z_2, \dots) := p z_1 + \sum_{k=1}^{m} v(z_1 + \cdots + z_k)$$

be the interaction of the left-most point in a half-infinite chain with all particles to its right. The operator acts on functions $f : \mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}$ by

$$\big(\mathcal{L}_\beta f\big)(z_1, z_2, \dots) = \int_0^\infty \mathrm{e}^{-\beta h(z_0, z_1, \dots)} f(z_0, z_1, \dots) \mathrm{d}z_0.$$

The dual action on measure ν on $\mathbb{R}_+^{\mathbb{N}}$ is denoted $\mathcal{L}_\beta^* \nu$.

Lemma 1. [2] *There exist $\lambda_0(\beta) > 0$ and a probability measure ν_β on $\mathbb{R}_+^{\mathbb{N}}$ such that $\mathcal{L}_\beta^* \nu_\beta = \lambda_0(\beta) \nu_\beta$. The pair $(\lambda_0(\beta), \nu_\beta)$ is unique.*

In the limit $N \to \infty$, the measure $\mu_N^{(\beta)}$ on $\mathbb{R}_+^{N-1}$ converges in some suitable sense to the eigenmeasure ν_β. The bulk Gibbs measure is instead defined as follows. Let

$$\mathcal{W}_0\big((z_j)_{j\in\mathbb{Z}}\big) = \sum_{i=-\infty}^{0} \sum_{j=1}^{\infty} v(z_i + \cdots + z_j)$$

be the interaction between a left and right half-infinite chain. Let $\tilde{\nu}_\beta$ be the image of ν_β under the map $\mathbb{R}_+^{\mathbb{N}} \to \mathbb{R}_+^{\mathbb{Z}\cap(-\infty,0]}$, $(z_j)_{j\in\mathbb{N}} \mapsto (z'_j)_{j\leq 0}$ given by $z'_{-j} = z_{1+j}$. The measure μ_β on $\mathbb{R}_+^{\mathbb{Z}}$ with Radon-Nikodym derivative

$$\frac{\mathrm{d}\mu_\beta}{\mathrm{d}(\tilde{\nu}_\beta \otimes \nu_\beta)}\big((z_j)_{j\in\mathbb{Z}}\big) = \frac{\exp(-\beta\mathcal{W}_0((z_j)_{j\in\mathbb{Z}}))}{\tilde{\nu}_\beta \otimes \nu_\beta(\exp(-\beta\mathcal{W}_0))}$$

is translationally invariant and satisfies the DLR-conditions for the prescribed interaction. For the proofs we adapt standard arguments for finite spin spaces [4, 5] to the non-compact continuous space $\mathbb{R}_+$.

2. Path large deviations as $\beta \to \infty$

Using arguments by Gardner and Radin [1], we show that in the limit $N \to \infty$, in the bulk, the interparticle spacings $z_j^{(N)}$ of the minimizer of $\mathcal{E}_N$ converge to some lattice spacing $a = a(p) > 0$. This holds true for all $m \in \mathbb{N} \cup \{\infty\}$ and $p > 0$ not too large. In addition, the ground state energy per particle is

$$e_0 = pa + \sum_{k=1}^{m} v(ka).$$

and we have the convergence

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \lambda_0(\beta) = -e_0.$$

Define

$$\mathcal{E}_{\mathrm{surf}}\big((z_j)_{j\in\mathbb{N}}\big) := \begin{cases} \sum_{j=1}^{\infty}(h(z_j, z_{j+1}, \dots) - e_0), & \sum_{j=1}^{\infty}(z_j - a)^2 < \infty, \\ 0, & \text{else.} \end{cases}$$

A functional $\mathcal{E}_{\mathrm{bulk}}((z_j)_{j\in\mathbb{Z}})$ is defined in a similar fashion, replacing summation over $j \in \mathbb{N}$ by summation over $j \in \mathbb{Z}$.

Theorem 2. [2] *Let $p \in (0, p^*)$ and $m \in \mathbb{N} \cup \{\infty\}$. As $\beta \to \infty$, the families $(\mu_\beta)_{\beta>0}$ and $(\nu_\beta)_{\beta>0}$ satisfy the large deviation principle with speed β and respective rate functions $\mathcal{E}_{\mathrm{bulk}}$ and $\mathcal{E}_{\mathrm{surf}} - \min \mathcal{E}_{\mathrm{surf}}$. The rate functions are good (lower semicontinuous with compact level sets, product topology). The unique minimizer of $\mathcal{E}_{\mathrm{bulk}}$ is the constant sequence $z_j \equiv a$.*

A key step in the proof is a fixed point equation for the rate function for $(\nu_\beta)_{\beta>1}$

$$I(z_1, z_2, \dots) = \big(h(z_1, z_2, \dots) - e_0)\big) + I(z_2, z_3, \dots) \tag{1}$$

which is deduced from $\mathcal{L}_\beta^* \nu_\beta = \lambda_0(\beta)\nu_\beta$. The proof shares some similarities with path large deviations for Markov processes, the fixed point equation (1) plays a role analogous to Hamilton-Jacobi-Bellman equations from control theory.

3. Decay of correlations

Pollicott's method of conditional expectations [3] yields an a priori bound on the decay of correlations in the bulk Gibbs measure: For $f, g : \mathbb{R}_+^{\mathbb{Z}} \to \mathbb{R}$ bounded, write $f_0((z_j)_{j\in\mathbb{Z}}) := f(z_0)$, $g_n((z_j)_{j\in\mathbb{Z}}) := g(z_n)$, then as $n \to \infty$,

$$|\mu_\beta(f_0 g_n) - \mu_\beta(f_0)\mu_\beta(g_n)| \leq \begin{cases} C\exp(c\beta)\, ||f||_\infty\, ||g||_\infty / n^{(s-2)(1-\varepsilon)}, & m = \infty, \\ C\exp(c\beta)(1-\exp(-c\beta))^n ||f||_\infty\, ||g||_\infty, & m < \infty \end{cases}$$

with $s = 6$ for the Lennard-Jones potential, $\varepsilon > 0$ small enough, and $C, c > 0$ some constants. This a priori bound uses very few properties of the potential, the most important one being the decay $v(r) = O(r^{-s})$ as $r \to \infty$.

For the Lennard-Jones potential, all spacings z_j of every energy minimizer lie in some interval $[z_{\min}, z_{\max}]$ and the total energy U_N has a Hessian $\frac{\partial^2}{\partial z_i \partial z_j} E_N$ that is diagonally dominant in $[z_{\min}, z_{\max}]^{N-1}$, uniformly in N.

For finite m, a finite-range substitute for the transfer operator $\mathcal{L}_\beta$ is an integral operator K_β in $L^2(\mathbb{R}^{m-1})$. A thorough spectral analysis of K_β in the limit $\beta \to \infty$ reveals that K_β can be approximated by a Gaussian operator and the spectral gap stays bounded as $\beta \to \infty$.

For infinite m, we expect that the decay of correlations is polynomial and comparable to that of a Gaussian measure whose covariance kernel is the inverse of the Hessian of the matrix, but for the full Gibbs measure μ_β this is so far unproven.

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Coexistence of Gibbs measures and gradient Gibbs measures on trees

Christof Külske

(joint work with Florian Henning)

Gradient models of statistical mechanics have a Hamiltonian of the type

$$H = \sum_{x\sim y} V(\sigma_x - \sigma_y)$$

where spin variables σ_x, indexed by the vertices x of an underlying supporting graph take values in a local spin space which can be one of the spaces $\mathbb{R}, \mathbb{R}^k, \mathbb{Z}, \mathbb{Z}^k$.

In the present work we are dealing with the case where the supporting graph is equal to an infinite regular tree with $d+1$ nearest neighbors, and the local spin space is the integer lattice $\mathbb{Z}^k$.

We are interested in the Gibbs measures of such a model for local specifications which are defined via a *transfer operator* which is given by the function $Q(i) = e^{-\beta V(i)}$ where the potential V is an even function on $\mathbb{Z}^k$ and $V(0) = 0$. These local specifications provide a Markovian multi-sided description of the model. Two important concrete examples are the SOS-model for which $Q(i) = e^{-\beta|i|}$ and the log-potential model for which $Q(i) = (1+|i|)^{-\beta}$, and $k = 1$.

Our first result concerns the **existence of localized Gibbs measures** under minimal decay assumptions on the transfer operator [3]. This is expressed as follows. We put $\gamma = \|Q\|_{\frac{d+1}{2},\mathbb{Z}^k}$ and $\delta = \|Q\|_{d+1,\mathbb{Z}^k\setminus\{0\}}$ and define a *good set* G_d of pairs (γ,δ) for which there exists an $\epsilon > 0$ such that $\delta + \gamma\epsilon^d \leq \epsilon$ and $2d\gamma\epsilon^{d-1} + 2d\delta\epsilon^d < 1$. Then our theorem states that for interactions inside this good set there exists a family of tree-automorphism invariant μ_i which are related via shift in the local state space. These measures μ_i are *localized* at spin value i of the local state space. For this we also have quantitative upper and lower localization bounds on $\mu_i(\sigma_v \neq i)$ where σ_v is the spin variable at vertex v.

The proof is based on the construction of solutions to Zachary's boundary law equation for normalizable boundary laws [1]. To any such solution corresponds a Gibbs measure in infinite volume which is also a tree-indexed Markov chain. The correspondence is via Zachary's formula. It gives a multi-sided description of the finite-volume marginals of the Gibbs measure which is derived from the specification with additional terms for the spins at boundary sites involving the particular boundary law. The normalizability requirement puts us to the space of positive functions on the local state space with finite $d+1$-norm. Restricting to small ϵ-balls of solutions concentrated at i we construct fixed points for a non-linear operator T. In this, the first inequality in the definition of the good set expresses the invariance of the ball and the second expresses the contractivity of T.

The second result of [3] concerns the **existence of delocalized gradient Gibbs measures** under summability of the transfer operator Q, for which we restrict to the local state space $\mathbb{Z}$. The boundary law equation may also posses q-height-periodic solutions λ_q, in which case it becomes an equation for $q-1$ real numbers. As such solutions can never be normalizable in Zachary's sense they do not correspond to Gibbs measures. It is nevertheless possible to assign a *gradient Gibbs measure* on the set of increments to λ_q via a two-layer sampling procedure. In this procedure a first layer measure is constructed for a $\mathbb{Z}_q$-valued discrete rotation symmetric clock model. It is a tree-indexed Markov chain Gibbs measure depending on λ_q for an effective interaction Q_q, which is build from the original interaction Q. From this so-called fuzzy measure the gradient Gibbs measure is obtained via sampling of the full increments in $\mathbb{Z}$ conditional on the $\mathbb{Z}_q$-valued fuzzy increments with weights derived from Q, independently over the edges of the tree.

In a previous paper [2] already an analysis of certain families of solutions for the SOS model for periods $q = 3, 4$ was given. As a main new general existence result for such gradient measures we prove the following: For interactions Q with norms in the very same good set G_d which appeared for the Gibbs measures there exist tree-automorphism invariant gradient Gibbs measures for large enough periods $q \geq q_0(Q, d)$. (This holds under a further summability assumption which can be easily checked for our concrete models.) The proof uses a variant of the above contraction method devised for the Gibbs measures, but now applied to the local state space $\mathbb{Z}_q$. This allows to relate both types of existence questions via the limit of a large period q.

We can see that both types of measures really are fundamentally different by looking at the distribution of a sum of increments W_n taken along a path of length n on the tree. For the localized Gibbs measure we have convergence to a non-degenerate probability distribution. On the contrary for the gradient Gibbs measures W_n is a random walk in a q-periodic random environment from which one sees that we have vague convergence to zero (hence delocalization).

In our last part we discuss the SOS model and the log potential model quantitatively and present an analysis in the limit of large degrees of the tree for general potentials.

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Non regular g-measures and Variable Length Memory Chains

Frédéric Paccaut

(joint work with Peggy Cénac, Brigitte Chauvin, Ricardo F. Ferreira, Sandro Gallo, Nicolas Pouyanne)

Non regular g-measures. A natural way to generalize Markov chains is to prescribe the conditional probabilities with respect to the whole past (instead of the last symbol). These kind of processes are called chains of infinite order, or g-chains, where the function g prescribes the conditional probabilities. A stationary measure of a g-chain is called a g-measure. These are measures on $X = A^{-(\mathbb{N}\setminus\{0\})}$ (we will only consider finite alphabets A) which can be naturally extended on $A^{\mathbb{Z}}$ thanks to stationarity. When the dependence of the conditional probabilities (the function g) on the past is continuous, the existence of a g-measure is straightforward. One can use an abstract fixed point theorem for the dual of the transfer operator (where $x \in X$ and xa denotes the concatenation of x and a)

$$\mathbf{L}f(x) = \sum_{a \in A} g(xa)f(xa)$$

which acts in this case on the space of continuous functions (X is endowed with the product of discrete topologies).

If g is no longer continuous, let $\mathcal{D}$ be the set of discontinuity points of g. What conditions on the size/shape of $\mathcal{D}$ can be put for the existence of a g-measure ? The first result was given in [4].

Theorem 1. *If $\inf_X g > 0$ and the topological pressure of $\mathcal{D}$ is strictly negative then there exists a g-measure μ with $\mu(\mathcal{D}) = 0$,*

where the topological pressure mixes the way the orbit of $\mathcal{D}$ accumulates and the value of g on this orbit (see [4] for a precise definition). A more general result is obtained in [3] by using the following procedure. Let $T : A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ be the left shift mapping. Let $x \in X$, define the measure μ^x on $A^{\mathbb{Z}}$ to be equal to the Dirac measure δ_x on X and such that, for all $a_0^{n-1} \in A^n$,

$$\mu^x(a_0^{n-1}) = g(xa_o)g(xa_0a_1)\dots g(xa_0\dots a_{n-1}).$$

By Kolmogorov extension theorem, μ^x is uniquely defined. Now, by compactness, the sequence of Cesaro sums $\left(\frac{1}{k}\sum_{i=0}^{k-1} \mu^x \circ T^{-i}\right)_{k \geq 1}$ has a limit point $\tilde{\mu}^x$. The result is the following.

Theorem 2. *If there exists $x \in X$ and $\tilde{\mu}^x$ such that $\tilde{\mu}^x(\mathcal{D}) = 0$, then $\tilde{\mu}^x$ is a g-measure.*

As corollaries, here are three cases in which, for every $x \in X$, $\tilde{\mu}^x(\mathcal{D}) = 0$ and consequently $\tilde{\mu}^x$ is a g-measure.

(1) $\inf_X g > 0$ and $\mathcal{D}$ is countable,
(2) the topological pressure of $\mathcal{D}$ is strictly negative (without any extra positivity assumption on g),
(3) $\inf_X g > 0$ and there exists a finite string v which does not appear in $\mathcal{D}$ ($\mathcal{D}$ is called *v-free*).

In [3]are also given conditions to ensure uniqueness and the weak Bernoulli property.

Variable length memory chains. Let us now adopt another point of view on the g-chains and put more specific assumptions on the funtion g under which necessary and sufficient conditions of existence and uniqueness are obtained in [2]. First we will reverse the direction of time (past on the right, future on the left) to avoid the use of reversed words. Second, we will assume that the relevant pasts needed to emit the following symbol (called *contexts* in the sequel) are countable. These contexts (possibly infinite) are stored as the leaves of a saturated rooted tree, the *context tree* $\mathcal{T}$. To every context c is associated a probability distribution q_c on A. For every $\alpha \in A$, $q_c(\alpha)$ is the probability to emit the symbol α knowing that the relevant past is c. The couple $(\mathcal{T}, q)$ is called a probabilised context tree. The associated variable length memory chain (VLMC) is a g-chain where the function g is piecewise constant on the cylinders based on the contexts. A *stationary* measure for the VLMC is a g-measure.

For any right infinite or finite external (to the tree) word w, denote by $c(w)$ the only prefix of w which is a context.

Here are combinatorial objects, the LIS and the α-LIS, that are key ingredients to state the results. If w is a non-empty finite word, w can be uniquely written as

$$w = \beta_1\beta_2 \dots \beta_{p_w}\alpha_w s_w,$$

where

- $p_w \geq 0$ and $\beta_i \in A$, for all $i \in \{1, \dots, p_w\}$,
- $\alpha_w \in A$,
- s_w is the longest internal (for the tree $\mathcal{T}$) strict suffix of w.

The Longest Internal Suffix s_w is abbreviated as the *LIS* of w; the non-internal suffix $\alpha_w s_w$ is called the *alpha-LIS* of w. Let π be a stationary measure for the VLMC. If a finite word w writes $w = \alpha v$ where $\alpha \in A$ and v is a non internal word, then $\pi(w) = q_{c(v)}(\alpha)\pi(v)$. Iterating this formula with the alpha-LIS leads to the following: for any finite non-empty word w, write $w = v\alpha_w s_w$ where v is a finite word and $\alpha_w s_w$ is the α-LIS of w,

$$\pi(w) = \operatorname{casc}(w)\pi(\alpha_w s_w),$$

where $\operatorname{casc}(w)$, the *cascade* of w, is defined as

$$\operatorname{casc}(w) = \prod_{1\leq k\leq p_w} q_{c(\beta_{k+1}\dots\beta_{p_w}\alpha_w s_w)}(\beta_k).$$

Elementary arguments on measures show that any stationary probability measure is determined by its value on the cylinders based on alpha-LIS of contexts. Denote by S the set of all alpha-LIS of finite contexts. This set is at most countable. It turns out that, whenever π is stationary, all the $\pi(\alpha s)$, for $\alpha s \in S$ are related by the linear system

$$\pi(\alpha s) = \sum_{\beta t \in S} \pi(\beta t) Q_{\beta t, \alpha s},$$

where the square matrix $Q = (Q_{\alpha s, \beta t})_{(\alpha s, \beta t)\in S^2}$ is defined by

$$Q_{\alpha s, \beta t} = \sum_{\substack{c \in C^f \\ c = t\cdots \\ c = \cdots[\alpha s]}} \operatorname{casc}(\beta c).$$

In this formula, C^f denotes the set of finite contexts, the notation $c = \cdots[\alpha s]$ means that αs is the alpha-LIS of c, while $c = t\cdots$ means that t is a prefix of c. In otherwords, $(\pi(\alpha s))_{\alpha s \in S}$ is a left-fixed vector of the matrix Q. The study of the matrix Q indexed by the alpha-LIS of contexts is a key tool to characterize a stationary measure for the VLMC.

A tree is said stable when it is stable by the shift. In other words, $\forall \alpha \in A$, for any finite word w, if $\alpha w \in \mathcal{T}$ then $w \in \mathcal{T}$. In the stable case, the crux of the matter is that the matrix Q is always stochastic and can be interpreted as the transition matrix of some Markov chain on the successive alpha-LIS that appear in the chain. Indeed, any stable VLMC induces a semi-Markov chain (Z_n) with values in the set S of the context alpha-LIS. This induced semi-Markov chain brings out some renewal times that clarify a lot the structure of the chain. As an example, the following (very simple) NSC is obtained.

Theorem 3. *Assume that $\mathcal{T}$ is stable, S is a finite set and $q_c(\alpha) > 0$ for any context c and for any $\alpha \in A$. Then there exists a unique stationary measure for the VLMC if and only if*

$$\forall \alpha s \in S, \quad \sum_{c \in C^f,\ c = \cdots [\alpha s]} \operatorname{casc}(c) < +\infty$$

Finally, it is worth pointing out that any stationary measure π satisfies $\pi(\mathcal{D}) = 0$.

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One-sided and Two-sided Stochastic Descriptions of the Schonmann Projection

Stein Andreas Bethuelsen

(joint work with Diana Conache)

In 1989 Schonmann [10] proved that the projection of the plus-phase of the two-dimensional Ising model onto a one-dimensional line may not be a Gibbs measure. In the years following, this result was slightly improved in [6] and [7]. In particular, they concluded that *the Schonmann projection* does not possess a continuous two-sided stochastic description in terms of a Gibbsian specification in the phase transition regime. After many years of continued research which have revealed further properties of this measure, the question whether or not the Schonmann projection possesses a continuous two-sided stochastic description in an almost sure sense remains open.

Motivated by recent developments on one-sided vs. two-sided stochastic descriptions, in [3] we considered the Schonmann projection from a one-sided point of view. One of the main results of [3] is that the Schonmann projection possesses a one-sided stochastic description in terms of a g-function which is continuous almost surely. Although our proof method does not suffice for us to conclude that it is continuous everywhere, in contrast to the two-sided point of view, we in fact conjecture that this is indeed the case.

The apparent difference between the two-sided and one-sided description of the Schonmann projection is also seen by a study of its mixing properties. A key lemma in [10] is that, for $\beta > \beta_c$, and for each $n \in \mathbb{N}$, there exists $N(n) \geq n$ such that

$$\lim_{n\to\infty} \mu^+_{\beta,0}(\cdot \mid \eta = -1 \text{ on } [(-N(n), -n) \cup (n, N(n)] \times \{0\}) = \mu^-_{\beta,0}(\cdot). \tag{1}$$

Here $\mu^+_{\beta,0}$ denotes the plus-phase of the two-dimensional Ising model and $\mu^-_{\beta,0}(\cdot)$ denotes the minus-phase. Both of them are measures on $\{-1,+1\}^{\mathbb{Z}^2}$. Equation (1) implies that the Schonmann projection depends strongly on the two-sided conditioning. On the contrary, as proven in [3], this is no longer the case for the one-sided conditioning, for which it is shown that

$$\lim_{n\to\infty} \mu^+_{\beta,0}(\cdot \mid \eta = -1 \text{ on } (-\infty, -n) \times \{0\}) = \mu^+_{\beta,0}(\cdot). \tag{2}$$

(In fact, a slightly stronger mixing property is proven [3]). This result can be interpreted in the following way: a piece of wall adsorbing the $-$ phase generates no longitudinal wetting.

An open question motivated by the above one-sided considerations is whether there is a finitary coding of the Schonmann projection. It is known that any regular g-measure (i.e. possessing a one-sided continuous stochastic description) which is conditionally positive associated (or strong FKG) has a finitary coding if and only if the measure is unique, see [8]. Note that the mixing property in (2) implies by use of the strong FKG property that the Schonmann projection is a unique g-measure. For comparison, and as shown by [2], interestingly there is no finitary coding of the entire plus-phase of the Ising model on $\mathbb{Z}^2$.

It is also natural to consider the corresponding questions in higher dimensions. In fact, in [7] it was proven that the projection of the Ising model on $\mathbb{Z}^d$ onto a $(d-1)$-dimensional layer has no continuous Gibbsian (spatial) stochastic description. Much of the theory in [3] can also be extended to higher dimensions for which the corresponding notion of one-sided stochastic description is that of partially ordered models, see e.g. [5]. The proof of (2) combines a general result for positively correlated measures by Liggett and Steif [9] with a comparison of certain large deviation estimates for the plus and minus phase version of the Schonmann projection and the Aizenmann-Higuchi theorem for the Ising model on $\mathbb{Z}^2$. Of these three only the latter does not extend to dimension 3 or higher. Particularly, one can show that, for any $d \geq 3$,

$$\lim_{n\to\infty} \mu^+_{\beta,0}(\cdot \mid \eta = -1 \text{ on } (-\infty, -n) \times \mathbb{Z}^{d-2} \times \{0\}) \neq \mu^-_{\beta,0}(\cdot), \tag{3}$$

where $\mu^\pm_{\beta,0}$ denotes the plus-phase and minus-phase of the Ising model on $\mathbb{Z}^d$, respectively. Further, the left side of (3) is an extremal measure consistent with the Ising specification on $\mathbb{Z}^d$. Following the analysis in [3], the remaining hurdle is thus to show that the left side of (3) cannot be any extremal measure other than the plus-phase. Due to the presence of Dobrushin states in dimension 3 and higher there are potentially many candidates one needs to exclude.

In the talk, the study of the Schonmann projection from a one-sided point of view was also motivated by recent progress for random walks in random environment (RWRE). For such models, often very strong mixing properties (e.g. cone-mixing) are required to prove limiting properties of the random walk (see [4] and [1]). It can be shown that considerably weaker mixing property suffices (work in progress, jointly with Florian Völlering). For the plus phase of the Ising model on $\mathbb{Z}^2$ this improved mixing condition translates into requiring that $\sum_{n\geq 1} a_n < \infty$, where

$$a_n = \left| \mu^{+}_{\beta,0}(\eta(0,0) = +1 \mid \eta = -1 \text{ on } (-\infty,-n) \times \{\gamma_n\}) - \mu^{-}_{\beta,0}(\eta(0,0) = +1) \right|, \tag{4}$$

and where $(\gamma_n)_{\geq 0}$ is any possible random walk path on $\mathbb{Z}$ started at $\gamma_0 = 0$. A key technical property of the Schonmann projection used in most of the papers mentioned above is that it is translation invariant. The mixing condition in (4) motivates also the study of projections (of e.g. the Ising model) of random fields onto one-dimensional and directed subsets which are not translation invariant.

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On the relation between the Gibbs measures in Statistical Mechanics and g-measures in Dynamical Systems

Evgeny Verbitskiy

In my overview talk I discussed various definitions of 'Gibbs' measures in Statistical Mechanics and Dynamical Systems. The basic question we address is: What is the relation between the Gibbs and g-measures.

The first rigorous definition of Gibbs states for multidimensional lattice systems in Statistical Mechanics is due to Dobrushin [1] and Lanford & Ruelle [2] in the late sixties.

We will be primarily interested in translation invariant Gibbs measures in dimension 1. We need to define the so-called interactions: $\Phi = \{\Phi_\Lambda : \Lambda \Subset \mathbb{Z}\}$ – collection of functions indexed by finite subsets of $\mathbb{Z}$, such that the functions $\Phi_\Lambda : X \to \mathbb{R}$ depend only on the values of x in Λ, translation invariant: $\Phi_{\Lambda+n}(x) = \Phi_\Lambda(T^n x)$, $T : X \to X$ is a left shift, and is *uniformly absolutely convergent* (UAC):

$$|||\Phi||| = \sum_{0 \ni \Lambda \Subset \mathbb{Z}^d} ||\Phi_V(\cdot)||_\infty < \infty.$$

and let $H_\Lambda = -\sum_{V \cap \Lambda \neq \varnothing} \Phi_V(x)$.

Definition 1. A translation invariant measure μ on $X = A^{\mathbb{Z}}$ is called Gibbs for interaction Φ, if for all finite subsets $\Lambda \Subset \mathbb{Z}^d$, one has

$$\mu(x_\Lambda | x_{\Lambda^c}) = \frac{\exp(-H_\Lambda(x))}{\sum_{a_\Lambda \in A^\Lambda} \exp(-H_\Lambda(a_\Lambda x_{\Lambda^c}))} =: \gamma_\Lambda(x_\Lambda | x_{\Lambda^c}) \quad \mu\text{-a.s.}$$

The collection $\gamma = \{\gamma_\Lambda : \Lambda \Subset \mathbb{Z}\}$ is called a Gibbsian specification for interaction Φ. The class of translation invariant Gibbs measures can equivalently be defined as follows.

Definition 2. Denote by $\Gamma(X)$ the class of continuous functions $\gamma_0 : X \to (0,1)$ that are normalised:

$$\sum_{\boldsymbol{a} \in A} \gamma_0(\ldots, x_{-2}, x_{-1}, a, x_1, x_2, \ldots) = 1,$$

for all $x = (\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots) \in X$. A translation-invariant measure μ is called Gibbs for $\gamma_0 \in \Gamma(X)$ if

$$\mu(x_0 | x_{-\infty}^{-1}, x_1^\infty) = \gamma_0(\ldots, x_{-1}, x_0, x_1, \ldots) = \gamma(x) \quad \mu\text{-a.s.}$$

Note that one can uniquely reconstruct the whole translation invariant specification γ (Definition 1) from γ_0 (Definition 2).

Sinai [3] in early 1970's was first to introduce Gibbs formalism in Dynamical Systems. Again for simplicity we will consider the symbolic setup $X = A^{\mathbb{Z}}$, and $T : X \to X$ will denote the left shift.

Suppose μ_0 a normalized measure T-invariant on X. For each function h that is essentially bounded above and below we consider the expression

$$\Xi_{m,n}\left(h|\mu_0\right) = \int_X \exp\left(\sum_{k=-n}^{m} h(T^k x)\right) d\mu_0(x), \quad m > 0, n > 0.$$

Form the sequence of normalized measures $\mu_{m,n}(h)$ absolutely continuous with respect to μ_0, for which the Radon-Nikodym derivative is given by

$$\frac{d\mu_{m,n}}{d\mu_0} = \frac{\exp \sum_{k=-n}^{m} h\left(T^k x\right)}{\Xi_{m,n}\left(h|\mu_0\right)}.$$

Definition 3. [3] A measure that is a limit point, in the sense of weak convergence, of the sequence of measures $\mu_{m,n}(h)$ is called a Gibbs measure constructed from μ_0 and h.

Bowen [4] has introduced another convenient class of measures, which, by analogy, he also called Gibbs measures. However, Bowen warned that this class of measures is different from the class of measures introduced by Dobrushin-Lanford-Ruelle.

Definition 5. A translation invariant measure μ on $X_+ = A^{\mathbb{Z}_+}$ is called **Bowen Gibbs** for a continuous potential $f : X_+ \to \mathbb{R}$ if there exist constants $c > 1$ and $P \in \mathbb{R}$, such that for all $x \in X_+$ and every $n \in \mathbb{N}$ one has

$$\frac{1}{c} \leq \frac{\mu\big(\{x' \in X_+ : x'_0 = x_0, \ldots, x\_n-1 = x_{n-1}\}\big)}{\exp\Big(\sum_{k=0}^{n-1} f(T^k x) - nP\Big)} \leq c.$$

Finally, we turn to the last class of measures: the so-called g-measures introduced by Keane [5, 6]. Let

$$G(A^{\mathbb{Z}_+}) = \left\{ g : A^{\mathbb{Z}_+} \to [0,1] : \ g \ \textbf{continuous,} \sum_{z \in T^{-1}(x)} g(z) = 1 \text{ for each } x \in X_+ \right\}.$$

Definition 6. A translation invariant measure μ on $X_+ = A^{\mathbb{Z}_+}$ is a g-measure if

$$\mu(x_0|x_1, x_2 = g(x) \quad \mu - a.s.$$

We now turn to the overview of known results on the relation between the various classes of measures.

(1) Definitions 1 and 2 are equivalent: the classes of Gibbs measures coincide. Indeed, any UAC translation invariant potential gives rise to a positive continuous specification. In the opposite direction, however, by the celebrated results of Kozlov & Sullivan [7, 8], for a positive continuous specification, one can find an UAC interaction, which is not necessarily translation invariant, or a translation invariant interaction which is not necessarily UAC. It was shown recently [9] that the result can not be improved.

(2) A translation invariant measure μ on $\Omega = A^{\mathbb{Z}}$ or $\Omega = A^{\mathbb{Z}_+}$ is called *equilibrium state* for a continuous potential (observable) $f : \Omega \to \mathbb{R}$ if

$$h(\mu) + \int_\Omega f d\mu = \sup_{\lambda \in \mathcal{M}^1_T(\Omega)} \Big(h(\lambda) + \int_\Omega f d\lambda \Big),$$

where $h(\mu)$ is the Kolmogorov-Sinai entropy of μ, and the supremum is taken over all translation invariant measures on Ω. Gibbs (Def. 1) and Bowen Gibbs measures are equilibrium states for $f_\Phi = -\sum_{V \ni 0} \frac{1}{|V|} \Phi_V(\cdot)$ and f, respectively. Moreover, for an translation invariant UAC Φ, equilibrium states for f_Φ are Gibbs states for Φ as well.

(3) Sinai in his original work showed that any Hölder continuous function h is *co-homologous* to a one sided Hölder continuous function $h^+ : X_+ \to \mathbb{R}$, i.e., there exists a continuous function $u : X \to \mathbb{R}$ such that $h(x) = h^+(\pi x) + u(x) - u(Tx)$, where $\pi : X \to X_+$ is the natural projection. In fact his result immediately implies that the corresponding Gibbs measures are Gibbs in the sense of Definitions 1 and 3, as well as, are g-measures.

(4) Walters [10] extended Sinai's result to the class of functions with summable variation. In particular, he extended the validity of Ruelle-Perron-Frobenius theorem to this class functions. Suppose $f : X_+ \to \mathbb{R}$ is a continuous function, and consider the transfer operator $\mathcal{L}_f$ acting on continuous functions on X_+ as follows

$$\mathcal{L}_f h(x) = \sum_{y \in T^{-1}x} e^{f(y)} h(y) = \sum_{a \in A} e^{f(ax)} h(ax).$$

It is relatively straightforward to show that the dual operator $\mathcal{L}_f^*$ acting on measures, has an eigenmeasure for the maximal eigenvalue $\lambda > 0$: $\mathcal{L}_\phi^* \nu = \lambda \nu$. Furthermore, if the operator $\mathcal{L}_f$ also has a positive continuous eigenfunction h, $\mathcal{L}_f h = \lambda h$. Without loss of generality, we may assume that $\int h d\nu = 1$. Then $\mu = h \cdot \nu$ is a translation invariant measure, which is also an equilibrium state for ϕ. Moreover, μ is Note that in this case μ is a g-measure for

$$g(x) = \frac{e^{\phi(x)} \cdot h(x)}{\lambda \cdot h(Tx)}.$$

The natural question is whether for an equilibrium state μ for potential f, existence of a positive continuous eigenfunction is equivalent to μ being a g-measures?

(5) The question whether Gibbs measures are g and vice versa has been studied in Statistical Mechanics as well, see e.g., [11, 12]. Recently, examples of measures which are g, but not Gibbs [13], and which are Gibbs, but not g [14], have been found. The necessary and sufficient conditions for g-measures to be Gibbs have been given in [15]. The problem of deciding whether a Gibbs measure is g seems to be much more difficult.

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Survey on long-range percolation

Noam Berger

The aim of this talk is to give a survey of some of the main results and some interesting open problems in the field of long-range percolation. Long-Range percolation was first introduced by Schulman [7] in 1983.

This is the following percolation model: Let $\big(p_j : j \in \mathbb{Z}^d \setminus \{0\}\big)$ be a vector satisfying that $p_j \in (0,1)$ and that $p_j = p_{-j}$ for all j. Then we consider the percolation model where there is an open edge between x and y with probability p_{x-y}, and the different edges are all independent. In this talk we mostly focus on the one dimensional case, and the main question that we ask is whether an infinite cluster exists.

It turns out that the interesting regime is when $p_j \approx \beta\|j\|^{-s}$ for some β and s. It is very easy to be convinced that for $s \leq 1$, a.s. an infinite component exists. Schulman [7] showed that if $s > 2$, then a.s. an infinite component does not exist. Therefore, the first interesting question is what happens when $1 < s \leq 2$. This was answered by Newman and Schulman in 1986 in [6].

Theorem 1 (Newman and Schulman, 1986)**.** (1) *if* $\sum p_k \leq 1$ *then a.s. there exists no infinite cluster.*
(2) *if* $1 < s < 2$ *then there exists* $p'_1 < 1$ *s.t. if we replace* p_1 *(and* p_{-1}*) by* p'_1*, then a.s. there exists an infinite cluster.*
(3) *if* $s = 2$ *and* $\beta > 1$ *then there exists* $p'_1 < 1$ *s.t. if we replace* p_1 *(and* p_{-1}*) by* p'_1*, then a.s. there exists an infinite cluster.*

Thus this theorem insures the existence of a phase transition. Two questions that immediately come to mind are the case $s = 2, \beta \leq 1$ and the behaviour at the critical value.

Some answers are as follows. The first two statements below were proven by Aizenman and Newman in 1986 in [1].

The case $s = 2, \beta \leq 1$ is dealt with in the following theorem.

Theorem 2 (Aizenman and Newman, 1986)**.** *if* $s = 2$ *and* $\beta \leq 1$*, then a.s. there is no infinite component.*

The critical case behaves differently according to the value of s. The case $s = 2$ is proven in [1].

Theorem 3 (Aizenman and Newman, 1986)**.** *if* $s = 2$ *and* $\beta > 1$*, then a.s. there is an infinite component at criticality.*

The case $s < 2$ behaves quite differently, as shown in [3, 4]

Theorem 4. *if* $1 < s < 2$ *then a.s. there is no infinite component at criticality.*

Once we understand this, we may start asking more subtle questions about the critical behaviour. One natural question would be whether long-range percolation satisfies the triangle condition (see e.g. [2] for the formulation of this condition). For $s < 4/3$, this was answered by Heydenreich, van der Hofstad and Sakai in 2008 in [5].

Theorem 5 (Heydenreich, van der Hofstad and Sakai, 2008)**.** *if* $1 < s < 4/3$ *then the triangle condition is satisfied.*

We finish the talk with two open problems of great interest.

Problem 1. *Is it true that the triangle condition is not satisfied when* $s > 4/3$*? what happens when* $s = 4/3$*?*

Problem 2. *Are the following two statements true:*

(1) *in* $s = 2$ *in the critical case, the infinite cluster is two-ended.*
(2) *in* $s = 2$ *in the supercritical case, the infinite cluster has infinitely many ends.*

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Dyson models with random boundary conditions

A.C.D. van Enter

(joint work with E.O. Endo, A. Le Ny)

I discussed the low-temperature behaviour of Dyson models (long-range Ising models in one dimension) in the presence of random boundary conditions. As for typical boundary conditions Chaotic Size Dependence occurs, that is, the thermodynamic limit of the states does not exist but the sequence of states moves between various possible limit points, it makes sense to study distributional limits, the so-called metastates. The Dyson model is known to have a phase transition for decay parameters α between 1 and 2. We have shown that the metastate changes character at $\alpha = \frac{3}{2}$, between being supported on two pure Gibbs measures to being supported on mixtures thereof.

1. Summary of talk

Dyson models, long-range Ising models with polynomial decay, whose formal Hamiltonian is of the form $-\beta H = \sum_{i,j \in Z} |i-j|^{-\alpha} \sigma_i \sigma_j$ where $\sigma_i = \pm 1$, and $1 < \alpha \leq 2$, have been studied since Dyson in his fundamental work [9] more than 50 years ago proved the existence of a phase transition for $1 < \alpha < 2$, thereby establishing a conjecture due to Kac and Thompson [20].

Since then a number of different proofs and further results on the model have been found. For alternative proofs of the phase transition, see e.g. [17, 16, 6, 19].

Other, more recent, results, beyond the existence of the phase transition, include the absence of interface states [5, 15], the mesoscopic scale of interface fluctuations when the Dyson model is subjected to Dobrushin boundary conditions [7], the critical behaviour [18], behaviour under decimation[13], the fact (of especial interest for the workshop) that at low temperature the Gibbs measures fail to be g-measures [4], metastable behaviour [12], the behaviour of the Dyson model in random[1, 8] or decaying [3] fields, etc. Some of these issues have been reviewed recently in [2, 10].

During my talk I reported on work in progress with E. Endo and A. Le Ny; we study the situation when the Dyson model is subjected to random boundary conditions.For a short description of our results, see also [11].

In that situation the finite-volume measures do not converge, but Chaotic Size Dependence [22] occurs; that is, there exist multiple limit points among the set of infinite-volume Gibbs measures (all this in the weak topology). Although there is no weak convergence, there can exist convergence in distribution. The limit objects, which are distributions on infinite-volume Gibbs measures, have been introduced by Aizenman and Wehr [1], and independently by Newman and Stein [21], and carry the name "metastates". If their support consists of more than a single point, they are called "dispersed". They are expected to play an important role in the theory of quenched disordered systems, in particular for the description of spin glasses.

Our Dyson models with random boundary conditions constitute a class of toy models of quenched disordered systems. Our main observation is that a random negative half-line, interacting with the positive half-line via Dyson couplings, has almost surely a *finite* interaction energy $W(\eta, \sigma) = \sum_{i \leq 0, j>0} |i - j^{-\alpha} \eta_i \sigma_j$ under the condition $\alpha > \frac{3}{2}$, when the right half-line is either in the all plus configuration $\sigma_j = +$, for all $j > 0$, or when one considers the configuration σ_{max} which realises the maximum interaction between the two half-lines. The (absolute) value of $W(\eta, \sigma)$ for those σ is η-almost surely infinite otherwise, when $\alpha \leq \frac{3}{2}$.

To indicate why this is so, let η_i for $i \leq 0$ be random, independently chosen from the symmetric Bernoulli distribution, and let $W_i = \sum_{j>0} \eta_i |i-j|^{-\alpha} \sigma_j$. Then the expectation of W is a sum of expectations, $EW = \sum EW_i$=0, and its variance, due to the independence of the η_i, is a sum of the individual variances: $EW^2 = \sum_{i \leq 0} E(W_i)^2 = \sum_{i \leq 0} O(|i|^{2-2\alpha})$, which is finite when $\alpha > \frac{3}{2}$.
As the weights λ in the mixtures of Gibbs measures μ^+ and μ^- are given by $\lambda = \frac{\exp W}{\exp W + \exp -W}$, the situations where the W's are finite are situations where the mixed Gibbs measures $\mu = \lambda \mu^+ + (1-\lambda)\mu^-$ occur, as infinite-volume limits. So if W is finite almost surely, then the integral over the weights of the non-trivial mixtures in the dispersed metastate equals one; in the opposite case, however, the only limit points occurring with positive density are the pure plus and the pure minus state, μ^+ and $\mu-$, similarly to what happens in short-range models in higher dimensions[14], and the metastate is again dispersed, but now it is supported on only two points.

Note that if we consider the maximal energy of a positive site j, interacting with the negative random half-line configuration η, it decays as $|j|^{\frac{1}{2}-\alpha}$, which again leads to a finite sum when $\alpha > \frac{3}{2}$. As the energies from sites on the positive half-line are strongly correlated when the sites are close enough (they use the same random η after all), the behaviour of the interaction energy of the maximal configuration or the absolute value of the interaction energy of the all-plus configuration interacting with the random negative half-line is similar.

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Probabilistic cellular automata with memory two: invariant laws and multidirectional reversibility

Irène Marcovici
(joint work with Jérôme Casse)

Probabilistic cellular automata (PCA) are a class of random discrete dynamical systems. They can be seen both as the synchronous counterparts of finite-range interacting particle systems, and as a generalization of deterministic cellular automata: time is discrete and at each time step, all the cells are updated independently in a random fashion, according to a distribution depending only on the states of a finite number of their neighbours [1, 4].

In this work, we focus on a family of one-dimensional probabilistic cellular automata with *memory two* (or *order two*): the value of a given cell at time $t+1$ is drawn according to a distribution which is a function of the states of its two nearest neighbours at time t, and of its own state at time $t-1$ (see Fig. 1 for an illustration). The space-time diagrams describing the evolution of the states can thus be represented on a two-dimensional grid.

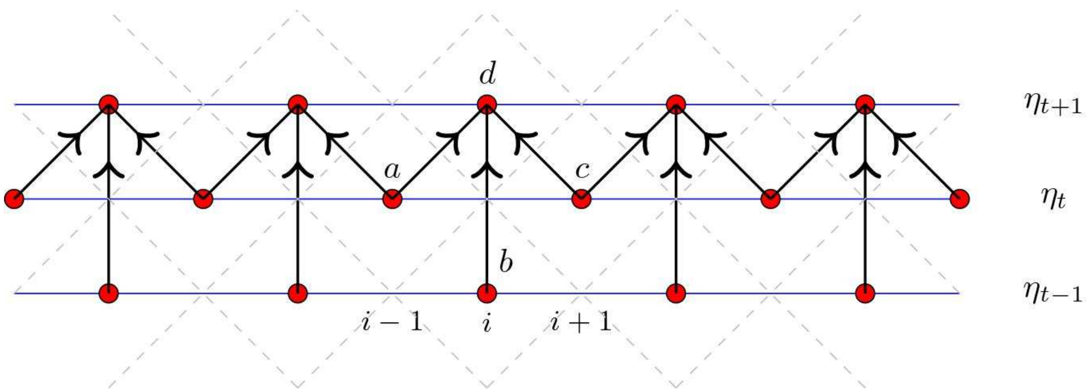

Figure 1. Illustration of the way the configuration η_{t+1} is obtained from η_t and η_{t-1}. The value $\eta_{t+1}(i)$ is equal to d with probability $T(a,b,c;d)$, and conditionally on η_t and η_{t-1}, the values $(\eta_{t+1}(i))_{i\in\mathbb{Z}_{t+1}}$ are independent.

We study the invariant measures of these PCA with memory two. In particular, we give necessary and sufficient conditions for which the invariant measure has a product form or a Markovian form, and we prove an ergodicity result holding in that context. We also show that when the parameters of the PCA satisfy some conditions, the stationary space-time diagram presents some multidirectional (quasi)-reversibility property: the random field has the same distribution as if we had iterated a PCA with memory two in another direction (the same PCA in the reversible case, or another PCA in the quasi-reversible case). Stationary space-time diagrams of PCA are known to be Gibbs random fields [2, 3]. The family of PCA that we describe thus provide examples of Gibbs fields with i.i.d. lines in many directions and nice combinatorial and geometric properties (see Fig. 2 for an illustration).

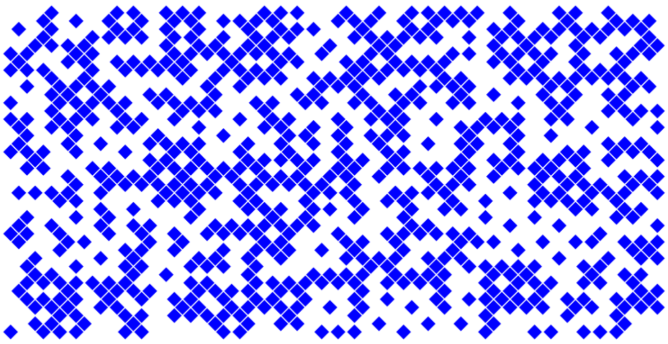

FIGURE 2. Example of PCA presenting a multidirectional reversibiliy: its stationary space-time diagram has the same distribution as if we had iterated the same PCA in any other of the four cardinal directions. In addition, any straight line drawn along the space-time diagram is made of i.i.d. random variables. The transition kernel of this PCA is given by: $T(a, b, c; \cdot) = p\,\delta_{a+b+c \mod 2} + (1-p)\,\delta_{a+b+c+1 \mod 2}$ (with here, $p = 0.2$).

PCA with memory two naturally arise in the study of different models coming from statistical physics. We review from a PCA approach some results on the 8-vertex model, and we also show that our methods allow to find new results for an extension of the classical TASEP model. As another original result, we describe some families of PCA for which the invariant measure can be explicitly computed, although it does not have a simple product or Markovian form.

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Some results and open questions concerning Gibbs states, quasilocality and bilateral determinism

Jeffrey Steif
(joint work with Sébastien Blachère, Robert M. Burton, Frank den Hollander)

I list below various results interspersed with questions.

1. Strong coupling for unique Gibbs states

Question A: If a specification on Z^d has a unique Gibbs state (among all translation invariant and nontranslation invariant measures), is it true that for all $\epsilon > 0$, there is an N such that for all $n \geq N$, any two boundary conditions outside of the n-box yield measures inside the box which are within ϵ in the Vaserstein distance meaning that they can be coupled so that the expected number of differences is at most ϵ. (This question can be asked for finite range or infinite range systems and will hold under the Dobrushin uniqueness condition.) There is a variant of this question for extremal Gibbs states where one asks the above to hold for most (with respect to the measure) pairs of boundary conditions.

2. Phase transition for quasilocality for random walk in random scenery

For random walk in random scenery in 1 dimension, where the walk stays put with probability ϵ, moves right with probability $p(1-\epsilon)$ and moves left with probability $(1-p)(1-\epsilon)$, there is a phase transition in the quasilocality behavior. The following is proved in [1]. For (p, ϵ) close to $(1, 0)$, one has quasilocality (all configurations are good) while for $p \in (1/2, 4/5)$, one does not have quasilocality in the very strong sense that all configurations are bad (essential discontinuities for the conditional probabilities).

Question B: Determine the phase diagram for quasilocality, presumably showing quasilocality when $p > 4/5$ for all ϵ and nonquasilocality when $p < 4/5$ for all ϵ.

3. Tail sigma fields, quasilocality and bilateral determinism

While there are easy examples of 1 dimensional stationary processes whose left tail sigma algebra is trivial and its right tail sigma algebra is everything, the single spin state spaces for these are necessarily infinite. There is a theorem stating that the left tail and right tail are the same (up to sets of measure zero) for stationary processes with a finite state space. However, the proof requires entropy theory.

Question C: Prove the above theorem using "only probability theory" without using entropy theory.

There are number of people who have proved the existence of stationary processes with a finite state space for which the left (and hence the right) tail is trivial but the 2-sided tail is everything. These are called bilaterally deterministic processes. The first example is due to Gurevich and then there have been examples by Ornstein and Weiss, followed by further examples (using parity checks) by Burton, Denker and Smorodinsky. The last construction was modified in [2] in order to obtain a number of interesting stationary processes which have different 1-sided and 2-sided behavior with respect to various properties. (The above reference also introduces a definition of weak Bernoulli (absolute regularity) for random fields.)

Question D: Do the examples given in [2] (or variants thereof) have interesting quasilocality properties such as having different 1-sided and 2-sided behavior with respect to quasilocality?

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Uniqueness and Non-Uniqueness for Spin-Glass Ground States on Trees

Johannes Bäumler

The Hamiltonian of the Edwards-Anderson (EA) Spin Glass Model is defined as

$$H(\sigma) = -\sum_{x \sim y} J_{xy} \sigma_x \sigma_y$$

where the sum is over all nearest neighbour pairs of vertices of a graph $G = (V, E)$, the spins σ_x are $\{-1, +1\}$-valued and the J_{xy}-s are independent and identically distributed random variables which are absolutely continuous with respect to the Lebesgue measure. In particular the couplings $(J_e)_{e \in E}$ can also attain negative values. In this talk we consider the set of Ground States for this Hamiltonian, which are precisely the elements of $\{-1, +1\}^V$ that satisfy

$$\sum_{\{x,y\} \in \partial B} J_{xy} \sigma_x \sigma_y \geq 0 \quad \forall \text{ finite } B \subset V \ . \tag{1}$$

We call an edge $e = \{x, y\}$ *satisfied* if $J_{xy} \sigma_x \sigma_y \geq 0$. Whenever the graph contains loops it is possible that not all edges can be satisfied at the same time. This phenomenon is called frustration. When the underlying graph is a tree there are no loops and hence all edges can be satisfied. This absence of frustration simplifies the analysis of ground states a lot. There are many results for spin glasses on trees, see for example [4, 5, 6, 7]. For the number of ground states of spin glasses there are rigorous results on the half-plane [1] and on trees [2], on which we focus in this talk.

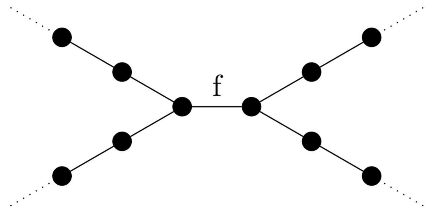

Figure 1

1. Ground states for infinite trees

From now on we fix a vertex $0 \in V$ and call it the root of the tree. For trees there are two *natural ground states*, the ones where $J_{xy}\sigma_x\sigma_y \geq 0$ for every edge $\{x, y\} \in E$. However, there might be much more than these two. We show that the cardinality of the set of ground states of a tree

$$\mathcal{G}(J) = \left\{\sigma \in \{-1, +1\}^V : \sigma \text{ is a ground state}\right\}$$

is related to the return properties of the Simple Random Walk on the tree. More precisely, the following theorem holds.

Theorem. *Let $T = (V, E)$ be a tree and suppose $(J_e)_{e\in E}$ are i.i.d. with $\mathbb{P}(J_e \in (-\epsilon, \epsilon)) = \theta(\epsilon)$. Then the following are equivalent:*
i) : The natural ground states are the only ground states almost surely.
ii) : MaxFlow$(0 \to \infty, \langle|J_e|\rangle) = 0$ almost surely.
iii) : The Simple Random Walk on the tree is recurrent.
Furthermore, $|\mathcal{G}(J)| = \infty$ almost surely in the case on non-uniqueness.

Our main tool in proving the equivalence of *ii*) and *iii*) in the above theorem is Theorem 3.1 in [3], which relates the conductance of the network to the maximal flow when the couplings have exponential distribution. From here on, one can relate the Maximal Flow in the network with exponentially distributed couplings to the Maximal Flow in the tree with any coupling distribution satisfying $\mathbb{P}(J_e \in (-\epsilon, \epsilon)) = \theta(\epsilon)$. Distributions satisfying $\mathbb{P}(J_e \in (-\epsilon, \epsilon)) = \theta(\epsilon)$ are also called *distributions of linear growth*.

Furthermore, we discuss the set of ground states for the tree depicted in Figure 1, where we have one edge f in the middle and both vertices adjacent to f are starting points of two halflines going to ∞. Assume that the coupling values are independent and uniformly distributed on the interval $(1, 3)$. Note that this is not a distribution of linear growth, so the above Theorem does not apply to this case. For this tree the number of ground states depends on the coupling value J_f: If $J_f > 2$, then the natural ground states, i.e. $\sigma_x = +1 \forall x \in V$ or $\sigma_x = -1 \forall x \in V$, are the only ground states almost surely. If $J_f \leq 2$, then there exist two further ground states: The spin configurations which are $+1$ on the right-hand side of the graph and -1 on the left-hand side, or vice versa. In these ground states all edges besides f are satisfied. So in particular $\mathbb{P}(|\mathcal{G}(J)| = 2) = \mathbb{P}(|\mathcal{G}(J)| = 4) = 0.5$ for this tree and coupling distribution.

To conclude we want to draw attention to several naturally arising open problems:

Problem 1. *Does* $|\mathcal{G}(J)| \in \{2, \infty\}$ *hold for all graphs and all distributions of linear growth?*

Problem 2. *Is there a connection between random walks and spin glass ground states for more general graphs than trees?*

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