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Real Algebraic Geometry with a View Toward Hyperbolic Programming and Free Probability

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Abstract. Continuing the tradition initiated in the MFO workshops held in 2014 and 2017, this workshop was dedicated to the newest developments in real algebraic geometry and polynomial optimization, with a particular emphasis on free non-commutative real algebraic geometry and hyperbolic programming. A particular effort was invested in exploring the interrelations with free probability. This established an interesting dialogue between researchers working in real algebraic geometry and those working in free probability, from which emerged new exciting and promising synergies.

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Introduction by the Organizers

In this workshop we brought together experts, as well as young researchers, working on the following themes:

(A) Real Algebraic Geometry (Positive Polynomials and Moments) and Optimisation (Polynomial, Convex),

(B) Hyperbolic Polynomials and Hyperbolic Programming,

(C) Free Non-Commutative Real Algebraic Geometry and Free Probability.

To stimulate discussions and exchanges during the workshop we scheduled 11 senior and junior speakers giving 50-minute tutorial and introductory lectures. A lecture within a given theme was technically accessible especially to attendees who are not familiar with the theme. The whole workshop and its schedule was then structured around these three themes and their respective thematic lectures. These survey-expository talks can be roughly divided according to the main themes (speakers in each area are listed in order of appearance in the schedule):

(A) Vicki Powers, Mohab Safey El Din, Monique Laurent, Frank Vallentin,
(B) Daniel Plaumann, Petter Brändén, James Saunderson,
(C) Andreas Thom, Serban Belinschi, Tobias Mai, Jurij Volčič,

though some of the talks were clearly touching more than one area in perfect accordance with the synergetic spirit of this meeting. The survey-expository talks were scheduled at the beginning of each session on the first four days of the workshop, while regular research talks of 40 minutes were scheduled in almost all of the remaining slots. Some slots were dedicated to 10 minutes short presentations, see more details below. To encourage the dialogue between the various areas we decided to keep a mixed daily thematic structure in the schedule.

The list of participants included many PhD and also Master students. Several received very short notice invitations, so we managed to have a full workshop, despite the unusual number of last minute cancellations (due to the untimely propagation of the pandemic). Citing from some testimonies given by junior participants:

The quality of talks was consistently very high. The first speakers made their talks intentionally highly accessible for listeners without specific background, which was a decision made in favour of the high number of early-stage attendants of the conference. This allowed me to follow many of the later talks in the area of Free Geometry and Free Probability despite my lack of background. However, the most valuable part of the venue was most certainly the discussions with mathematicians sharing similar interests. While in other conferences time for private discussions is often too short to be of value, both the concept of this workshop with a four hour lunch break and the self-contained nature of Oberwolfach enabled to have very productive discussions.

Let us give now a summary of the main topics discussed at the workshop.

(A) Real Algebraic Geometry [Positive Polynomials and Moments] and Optimisation [Polynomial, Convex]

In the first introductory talk of the workshop, Vicki Powers surveyed historical developments of the K–moment problem which asks for a characterization of sequences of moments of Borel measures supported on a given semialgebraic set K. The dual problem consists of characterizing all polynomials which are positive on K. The speaker then described various solutions to this latter problem, with or without compactness assumptions on K, all of them based on the Positivstellensatz, a fundamental result of real algebraic geometry relying on representations of polynomials as sums of squares (SOS). Finding SOS decompositions can be achieved with semidefinite programming (SDP), a class of optimization problems that can be solved efficiently with floating point algorithms. Later on, in the afternoon, Bachir El Khadir investigated connections between convexity and positivity which are relevant in engineering applications. He focused on the explicit construction of convex forms (homogeneous polynomials) which are not SOS. The last speaker of the day David Kimsey returned to the K–moment problem, extending known results to more general planar domains K, including for example the case of the union of an unbounded planar curve with a disk.

The second tutorial by Mohab Safey El Din was essentially computationally oriented. It described the ingredients of a toolbox for solving challenging problems of real algebraic geometry, such as for instance finding a point in each connected component of a basic semialgebraic set, or finding the critical points of the projection map onto a semialgebraic set. These algorithms are based on exact representations and computations (using integers), in contrast with alternative algorithms using floating point approximations (such as the ones used for SOS and SDP). The afternoon talks by Tobias Kuna and Patrick Michalski focused on the moment problems for measures supported on infinite-dimensional functional spaces. Interestingly, this was complemented the next morning by a talk by Konrad Schmüdgen, who also covered the historical side of the infinite-dimensional moment problem.

The third introductory talk was given by Monique Laurent who clarified the connections between the problem of moments, SOS representations of positive polynomials, and polynomial optimization (which is the problem of minimizing a polynomial on a semialgebraic set). She described the basic ingredients of the Lasserre (aka moment-SOS) hierarchy, which allows to solve globally a non-convex polynomial optimization problem at the price of solving a family of convex optimization problems (typically SDP problems). Christoph Schulze then described the geometry of the convex cone of polynomials that are locally non-negative, with connections to singularity theory.

On the fourth day, the morning survey by Frank Vallentin described the use of convex optimization relaxations (many of which based on SDP) and harmonic analysis for computing upper bounds of geometric packing problems (like the kissing number or the maximal density of translative packings of convex bodies) or energy minimization problems (like the problem of finding point configurations on the unit sphere which minimize potential energy). In the afternoon, Mareike Dressler explored a recent alternative to SOS techniques for assessing positivity of polynomials, the sums of non-negative circuit (SONC) functions. It was made clear that the cone of SONC functions coincides with another recently studied cone, namely the sums of arithmetic-geometric exponentials (SAGE). The history behind the use of the arithmetic-geometric inequality for certifying positivity was then described the next morning by Bruce Reznick. Finally, Greg Blekherman investigated the use of SOS polynomials and hence convex duality to derive algebraic inequalities satisfied by the moments of Borel measures.

(B) Hyperbolic Polynomials and Hyperbolic Programming

The three survey talks presented the different facets of hyperbolicity: the (real) algebraic geometry of hyperbolic polynomials (Daniel Plaumann), their analytic and combinatorial aspects (Petter Brändén), and the prospects and challenges of hyperbolic programming (James Saunderson). The survey talk of Daniel Plaumann on Monday introduced hyperbolic polynomials and hyperbolicity cones, including the relationship between hyperbolic polynomials and determinants and (much more elusively) between hyperbolicity cones and spectrahedral cones. This was immediately followed by the talk of Markus Schweighofer that showed how to construct spectrahedral relaxations of hyperbolicity cones, subject to some new and intriguing conjectures. One of the most important recent developments in the subject was the emergence of the class of Lorentzian polynomials; this class and its relation to hyperbolic polynomials were the main topic of the survey talk of Petter Brändén. The survey talk of James Sauderson tied hyperbolic polynomials and hyperbolic programming with the central core topic of both real algebraic geometry and polynomial optimization, that of non-negative polynomials. In a similar spirit, the concluding talk of the workshop, given by Mario Kummer, broke new grounds by using Ulrich sheaves note only for the study of hyperbolic polynomials vs. determinants (continuing the work presented at the previous workshops by Kummer himself as well as by Hanselka and Shamovich) but also for the study of non-negative polynomials vs. sums of squares.

(C) Free Non-Commutative Real Algebraic Geometry and Free Probability

Andreas Thom gave an introduction to operator algebras, in particular in the context of non-commutative real algebraic geometry. In the light of a recent preprint by Ji, Natarajan, Vidick, Wright, and Yuen, which seems to have solved the notorious Connes embedding conjecture in the negative, the talk ended with some explanations about the general idea of their proof. As this aroused quite some interest, we scheduled an extra meeting for Tuesday evening where Thom provided more details about the proof, answered many questions and engaged in a lively discussion about the implications of this result.

Serban Belinschi and Tobias Mai gave in their survey-expository talks introductions to free probability theory. Whereas Belinschi provided the basic notions of free probability and put some emphasis on "free analysis" – which is the non-commutative analytic machinery for dealing with non-commutative distributions – Mai provided some more details about the operator algebraic aspects of free probability and corresponding regularity questions for non-commutative distributions; in particular, in the context of non-commutative rational functions - which tied in very nicely with the more general context of free real algebraic geometry.

The expository talk of Jurij Volčič was on free real algebraic geometry; this subject studies non-commutative polynomials and rational functions, their evaluations on tuples of matrices, and positive definiteness thereof. An important point is that the matrix arguments are not fixed in size, but can be of any size. This connects with free probability theory, where instead of matrices of any size, one usually wants to plug in tuples of nice operators as arguments in the non-commutative functions.

More specific topics in the context of free probability were presented in the research talks of Malte Gerhold, David Jekel, and Ian Charlesworth. Gerhold addressed in his talk an open problem from dilation theory and showed how ideas from free probability theory could be used, in a joint work with Shalit, to improve estimates on best constants in this context. Jekel gave an idea of non-commutative laws and presented his results about non-commutative versions of transport for such laws. Along the way he proposed also the question about the size of finite fibers of non-commutative polynomial maps and advertised trace polynomials as an important extension of the class of non-commutative polynomials. Charlesworth returned in his talk to the general topic of regularity theorems in free probability theory, which aims to deduce properties of von Neumann algebras from probabilistic properties of a set of generators. In particular, he described two recent additions (from his joint work with Nelson) to our tool box in this direction: namely, the free Stein irregularity and the free Stein dimension.

Short Presentations

The MFO workshop format allows for circa 25 regular talks. To provide additional time, especially (but not only) for junior participants, we had two sessions of short presentations (10 minutes each). One session took place in lieu of a regular talk (on Tuesday), while the other one took place on Wednesday evening. In addition to leading to numerous discussions, the short presentations seemed to be a very positive experience for those PhD students who presented their results. Several of them told us that they will want to establish a practice of such short presentations at research seminars at their home universities.

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Workshop: Real Algebraic Geometry with a View Toward Hyperbolic Programming and Free Probability

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Abstracts

The moment problem and positive polynomials

Vicki Powers

Let $\mathbb{R}[\underline{X}]$ denote the ring of polynomials in n variables with real coefficients, and $\sum \mathbb{R}[\underline{X}]^2$ the set of sums of squares of real polynomials in n variables. If $\sigma \in \sum \mathbb{R}[\underline{X}]^2$, then f is *sos*.

Given a closed subset K of $\mathbb{R}^n$, the K–moment problem asks for a characterization of all possible moment multi-sequences of Borel measures which are supported on K, equivalently, it asks for all linear functions $L\colon \mathbb{R}[\underline{X}] \to \mathbb{R}$ for which there exist a positive Borel measure μ on K such that L is integration on K with respect to μ. The moment problem has close ties to a problem from real algebraic geometry, namely, the representation of polynomials positive on semialgebraic sets. Both problems have their roots in work from the late 19th century.

For $S = \{g_1, \dots, g_r\}$ in $\mathbb{R}[\underline{X}]$, K_S, the *basic closed semialgebraic set* generated by S, is $\{x \in \mathbb{R}^n \mid g_i(x) \geq 0 \text{ for all } i\}$.

Let K be a basic closed semialgebraic set. We say that the finite set of polynomials $\{g_1, \dots, g_r\}$ *solves the moment problem for* K if

- $K = \{x \in \mathbb{R}^n \mid g_1(x) \geq 0, \dots, g_r(x) \geq 0\}$, and
- every linear functional $L\colon \mathbb{R}[\underline{X}] \to \mathbb{R}$ comes from a Borel measure on K iff $L(f^2 g_i) \geq 0$ for all $f \in \mathbb{R}[\underline{X}]$ and all $i = 0, \dots, r$, where $g_0 = 1$.

In this case, we have a satisfactory finite characterization of the K–moment sequences. Haviland's Theorem shows the connection between the moment problem and positive polynomials:

Theorem (Haviland, 1936)**.** *The K–moment problem is solvable for L if and only if $L(f) \geq 0$ for every polynomial f which is ≥ 0 on K.*

In 1991, Schmüdgen proved the following remarkable theorem:

Theorem (Schmüdgen)**.**
Suppose $g_1, \dots, g_r \in \mathbb{R}[\underline{X}]$ such that $K := \{g_1 \geq 0, \dots, g_r \geq 0\}$ is compact. *Then the set*

$$\{g_1^{e_1} \cdot \dots \cdot g_r^{e_r} \mid (e_1, \dots, e_r) \in \{0,1\}^r\}$$

of all products of the g_i's solves the moment problem for K.

Schmüdgen's result was the first on the moment problem which covers a general class of sets K. Note that the theorem holds regardless of the generators chosen for K.

Given $S = \{g_1, \dots, g_r\} \subseteq \mathbb{R}[\underline{X}]$, let K_S be the basic closed semialgebraic set generated by S and P_S the *preorder* generated by S, the set of finite sums of elements

$$\sigma g_1^{\epsilon_1} \dots g_r^{\epsilon_r}, \epsilon_i \in \{0,1\} \text{ and } \sigma \in \sum \mathbb{R}[\underline{X}]^2.$$

Notice that $f \in P_S$ implies $f \geq 0$ on K_S and an identity

$$f = \sum_{\epsilon \in \{0,1\}^r} \sigma_\epsilon \, g_1^{\epsilon_1} \dots g_r^{\epsilon_r}$$

is a certificate of positivity for f on K_S.

The Positivstellensatz is the fundamental theorem of real algebraic geometry.

Positivstellensatz. *$S \subseteq \mathbb{R}[\underline{X}]$ is finite, $K = K_S$, $P = P_S$. Then*

(1) *$f > 0$ on $K \iff \exists\, p, q \in P$ such that $pf = 1 + q$.*
(2) *$f \geq 0$ on $K \iff \exists\, m \in \mathbb{N}$ and $p, q \in P$ such that $pf = f^{2m} + q$.*
(3) *$f = 0$ on $K \iff \exists\, m \in \mathbb{N}$ such that $-f^{2m} \in P$.*
(4) *$K = \emptyset \iff -1 \in P$.*

The proof of Schmüdgen's Theorem uses the Positivstellensatz.

A corollary is

Positivstellensatz (Schmüdgen)**.** *Suppose K_S is compact. If $f \in \mathbb{R}[\underline{X}]$ and $f > 0$ on K, then $f \in P_S$.*

What about the non-compact case?

Theorem 1. *Let $K_S \subseteq \mathbb{R}^n$, $n \geq 2$, and assume that K_S contains an open cone. Then no finite set of polynomials solves the moment problem for K_S.*

Further, there exists $f > 0$ on K_S with $f \notin P_S$.

In the case of non-compact subsets of $\mathbb{R}$, there is a positive answer. Kuhlmann and Marshall proved

Theorem 2. *Suppose $S \subseteq \mathbb{R}[X]$ and K_S is not compact. T.F.A.E.*

(1) *P_S solves the moment problem for K_S*
(2) *S contains the "natural generators" for K_S (up to scaling by positive constants)*
(3) *$f \geq 0$ on $K_S \Rightarrow f \in P_S$.*

Geometry of hyperbolic polynomials

Daniel Plaumann

This was an introductory talk on hyperbolic polyomials, intended to prepare for subsequent research talks in the workshop.

A real homogeneous polynomial $f \in \mathbb{R}[x_1, \dots, x_n]$ is called *hyperbolic* with respect to a fixed point $e \in \mathbb{R}^n$ if $f(e) \neq 0$ and the polynomial $f(a + te)$ in one variable t is real-rooted for every $a \in \mathbb{R}^n$. For $e = (1, 0, \dots, 0)$, this condition is equivalent to $f(t, a)$ being real-rooted for all $a \in \mathbb{R}^{n-1}$.

With any hyperbolic polynomial is associated the closed convex cone

$$C(f, e) = \big\{ a \in \mathbb{R}^n \mid \text{all roots of } f(a - te) \text{ are non-negative} \big\},$$

called the *hyperbolicity cone*. The notion originates in PDE theory in the work of Gårding and Petrovsky (see [4], [9]), but has also become prominent in optimization (hyperbolic programming [5]), combinatorics (real-stable polynomials and matroids [13]), and real algebraic geometry (determinantal representations [12], real fibred morphisms [8]).

Out of the wealth of examples, I presented the following:

(1) For any $(n-1)$–tuple of real-symmetric or complex-hermitian $d\times d$–matrices $A_2,\ldots,A_n$, the polynomial

$$\det(x_1 I_d + x_2 A_2 + \cdots + x_n A_n)$$

is hyperbolic with respect to $e=(1,0,\ldots,0)$, reflecting the fact that such matrices have only real eigenvalues. The hyperbolicity cone is the *spectrahedral cone*

$$\big\{a\in\mathbb{R}^n \mid a_1 I_d + a_2 A_2 + \cdots + a_n A_n \text{ is positive semidefinite}\big\}.$$

(2) The elementary symmetric polynomials $\sigma_d(x_1,\ldots,x_n)$ of degree d are hyperbolic with respect to any point in the positive orthant $\mathbb{R}^n_{>0}$, which is therefore contained in the hyperbolicity cone.

(3) Any product of real linear forms is hyperbolic with respect to any point in which it does not vanish. In this case, the hyperbolicity cones are polyhedral.

In 2005, Helton and Vinnikov proved in [6] that every hyperbolic polynomial in at most 3 variables admits a *definite determinantal representation*, which for $e=(1,0,\ldots,0)$ and $f(e)=1$ takes the form in Example (1) above. This confirmed a conjecture made by Peter Lax in 1958. While it is clear that the corresponding statement does not hold in more than three variables, a number of possible generalizations have been proposed. In 2010, Brändén showed in [1] that there exists a hyperbolic polynomial f of degree 4 in eight variables (which can reduced to four) such that no power f^r for any $r\geq 1$ admits a definite determinantal representation, exploiting the connection with the theory of matroids. Much research in real algebraic geometry has been directed at various questions of determinantal representability; see [12] for a survey.

On the other hand, the geometry of hyperbolic polynomials and their hyperbolicity cones can be studied very well without resorting to determinants. The work of Renegar in [10] on hyperbolic programming has been influential in this regard. An important role is played by the directional derivatives $D_v(f)$ of a hyperbolic polynomial f with $v\in C(f,e)$. Such a derivative is again hyperbolic and *interlaces* f, i.e., its roots are nested in between those of f. The interlacers of degree $\deg(f)-1$ form a convex cone, which has been studied in [7] and provides one way of relating hyperbolicity with non-negativity and sums-of-squares of real polynomials.

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Spectrahedral relaxations of hyperbolicity cones

Markus Schweighofer

Denote by $x = (x_1, \dots, x_\ell)$ an ℓ–tuple of distinct variables so that $\mathbb{R}[x]$ denotes the ring of real polynomials in these variables. A polynomial $p \in \mathbb{R}[x]$ is a *real zero polynomial* if for all $a \in \mathbb{R}^\ell$ and $\lambda \in \mathbb{C}$,

$$p(\lambda a) = 0 \implies \lambda \in \mathbb{R}.$$

This amounts to p being real-rooted on each line through the origin (as a univariate polynomial) and $p(0) \neq 0$ (take $a = 0$). For such a real zero polynomial $p \in \mathbb{R}[x]$, the set

$$C(p) := \{a \in \mathbb{R}^\ell \mid \forall \lambda \in [0, 1) \colon p(\lambda a) \neq 0\} \subseteq \mathbb{R}^\ell$$

is called the *rigidly convex set* defined by p. It is not obvious from the definition that rigidly convex sets are convex but this has already been known by Gårding. The prototype of real zero polynomials are products of linear polynomials with non-zero constant part. The rigidly convex sets defined by these are obviously exactly the (closed convex) *polyhedra* whose interior contain the origin.

The **generalized Lax conjecture (GLC)** [4, Section 6.1] says that the rigidly convex sets are exactly the *spectrahedra* whose interior contains the origin. A spectrahedron in $\mathbb{R}^\ell$ is a set which is defined by a *linear matrix inequality*, i.e., a set of the form

$$\{a \in \mathbb{R}^\ell \mid A_0 + a_1 A_1 + \ldots + a_\ell A_\ell \text{ is positive semidefinite}\}$$

where $A_0, A_1, \dots, A_\ell \in \mathbb{R}^{d\times d}$ are symmetric matrices of some size d. It is not hard to show that each spectrahedron whose interior contains the origin is rigidly convex. The non-trivial direction of GLC thus says that each rigidly convex set is a spectrahedron. In this talk, we presented a very partial result towards GLC that depends on two conjectures. To formulate these conjectures, we consider $m+n$ additional variables coming in two additional blocks $y = (y_1, \dots, y_m)$ and $z = (z_1, \dots, z_n)$.

The first conjecture is the **real zero amalgamation conjecture (RZAC):** Let $d \in \mathbb{N}_0$ and suppose that $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, z]$ are real zero polynomials of degree at most d with

$$p(x, 0) = q(x, 0).$$

Then there exist a real zero polynomial $r \in \mathbb{R}[x, y, z]$ of degree at most d such that

$$r(x, y, 0) = p \qquad \text{and} \qquad r(x, 0, z) = q.$$

The second is the **weak real zero amalgamation conjecture (WRZAC):** Suppose that $p \in \mathbb{R}[x, y]$ and $q \in \mathbb{R}[x, z]$ are real zero polynomials with

$$p(x, 0) = q(x, 0).$$

Then there exists a real zero polynomial $r \in \mathbb{R}[x, y, z]$ (of no matter what degree) such that the polynomials $r(x, y, 0)$ and p coincide as well as the cubic parts of $r(x, 0, z)$ and q.

Of course, RZAC implies WRZAC. Assuming WRZAC and using the Helton-Vinnikov theorem [4], we can prove the following very weak form of GLC: Given a rigidly convex set and finitely many planes (i.e., two-dimensional linear subspaces) in $\mathbb{R}^\ell$, there is a spectrahedron in $\mathbb{R}^\ell$ containing the rigidly convex set and agreeing with it on each of the planes. One could depict this by saying that one could "wrap rigidly convex sets into spectrahedra and tie them with a cord". For each rigidly convex set *defined by a cubic real zero polynomial* and for each finitely many given *three-dimensional* subspaces of $\mathbb{R}^\ell$, we can prove that there is a spectrahedron in $\mathbb{R}^\ell$ containing the rigidly convex set and agreeing with it on each of these subspaces. This uses a theorem of Buckley and Košir [2, Theorem 6.4] in place of the Helton-Vinnikov theorem.

In this talk, we presented the most important element of the proof, namely a certain spectrahedral relaxation of rigidly convex sets. For each real zero polynomial $p \in \mathbb{R}[x]$, we construct a linear matrix inequality of size $\ell + 1$ in the variables x that depends only on the cubic part of p and defines a spectrahedron $S(p)$ containing the rigidly convex set $C(p)$. The proof of the containment uses again the characterization of real zero polynomials in two variables by Helton and Vinnikov from [4].

Without going into details we reported that we have proven three special cases of RZAC where we use for each case another non-trivial ingredient. Namely we use the theory of stability preservers by Borcea and Brändén [1] to settle the case $\ell = 0$ where there are no shared variables, the Helton-Vinnikov theorem [4] to treat the case $\ell = m = n = 1$ where each block of variables consists just of a single variable and the theory of positive semidefinite matrix completion [3] to handle the case $d = 2$, i.e., to amalgamate quadratic real zero polynomials.

The use of stability preservers can also be seen as an application of a multidimensional version of the finite free additive convolution [5] which is a yet unexplored link to free probability. All details and proofs can be found in the preprint [6] which at the time of the talk is yet very preliminary. The title of this preprint refers to the more common but for our purpose more convenient homogeneous setup where real zero polynomials correspond to homogeneous polynomials and rigidly convex sets correspond to hyperbolicity cones

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On sum of squares representation of convex forms and generalized Cauchy-Schwarz inequalities

Bachir El Khadir

A convex form of degree larger than one is always nonnegative since it vanishes together with its gradient at the origin. In 2007, Parrilo asked if convex forms are always sums of squares. A few years later, Blekherman answered the question in the negative by showing through volume arguments that for high enough number of variables, there must be convex forms of degree 4 that are not sums of squares. Remarkably, no examples are known to date. In this talk, we show that all convex forms in 4 variables and of degree 4 are sums of squares. We also show that if a conjecture of Blekherman related to the so-called Cayley-Bacharach relations is true, then the same statement holds for convex forms in 3 variables and of degree 6. These are the two minimal cases where one would have any hope of seeing convex forms that are not sums of squares (due to known obstructions). A main ingredient of the proof is the derivation of certain "generalized Cauchy-Schwarz inequalities" which could be of independent interest.

The moment problem on curves with bumps

David Kimsey
(joint work with Mihai Putinar)

Given a quadratic module $Q \subseteq \mathbb{R}[x_1, \ldots, x_d]$, one says that Q has the *moment property* if every real linear functional $L\colon \mathbb{R}[x_1, \ldots, x_d] \to \mathbb{R}$ that is nonnegative on Q admits the representation

$$L(f) = \int_{\mathbb{R}^d} f \mathrm{d}\mu,$$

where μ is a positive Borel measure on $\mathbb{R}^d$. Similarly, one says that a quadratic module $Q \subseteq \mathbb{R}[x_1, \ldots, x_d]$ has the *strong moment property* if Q has the moment property and the measure μ above has the additional requirement

$$\operatorname{supp} \mu \subseteq K_Q := \{x \in \mathbb{R}^d : q(x) \geq 0 \text{ for all } q \in Q\}.$$

It is well-known that every Archimedean quadratic module $Q \subseteq \mathbb{R}[x_1, \ldots, x_d]$ (and hence the semi-algebraic set K_Q is compact) has the strong moment property. Unfortunately, this result does not hold in general when K_Q is not compact. Indeed, Scheiderer, Kuhlmann and Marshall and Powers and Scheiderer have discovered various families of quadratic modules and preorderings of $\mathbb{R}[x_1, \ldots, x_d]$ which do not have the strong moment property, or even the moment property.

On the other hand, Scheiderer and Plaumann were able to classify curves such that set of nonnegative polynomials on the curve agrees with the set of polynomials which can be written as a sums of squares in the coordinate ideal, i.e., they provided necessary and sufficient conditions on a real principal ideal (q), where $q \in \mathbb{R}[x_1, x_2]$, such that

$$\{f \in \mathbb{R}[x_1, x_2] : f|_{\mathcal{V}(q)} \geq 0\} = (q) + \Sigma^2,$$

where $\mathcal{V}(q)$ denotes the real zero set of q and Σ^2 is the set of polynomials that can be represented as a sum of squares of polynomials in $\mathbb{R}[x_1, x_2]$. A direct consequence of this classification of Scheiderer and Plaumann is that the quadratic module $(q) + \Sigma^2$ satisfies the strong moment property.

In this talk, we showed that if (q) is a non-trivial principal ideal which obeys the conditions of Scheiderer and Plaumann's classification described above and $Q \subseteq \mathbb{R}[x_1, x_2]$ is an Archimedean quadratic module, then the quadratic module

$$q\,Q + \Sigma^2 \subseteq \mathbb{R}[x_1, x_2] \text{ satisfies the strong moment property.}$$

It is worth noting that the semi-algebraic set associated with the quadratic module $\widetilde{Q} := q\,Q + \Sigma^2$ is given by

$$K_{\widetilde{Q}} = \mathcal{V}(q) \cup (K_Q \cap \{q > 0\}),$$

which need not be compact. Our result admits a higher dimensional analogue, so long as we introduce the additional assumption that $(q) + \Sigma^2$ satisfies the strong moment property. This work enlarges the realm of geometric shapes on which the power moment problem is accessible and solvable by non-negativity certificates.

Hyperbolic and Lorentzian polynomials

Petter Brändén

Hyperbolic polynomials have been studied in several different areas of mathematics such as partial differential equations, optimization, real algebraic geometry, combinatorics and computer science. A homogeneous polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ is *hyperbolic* with respect to $e \in \mathbb{R}^n$ if $f(e) > 0$, and for all $x \in \mathbb{R}^n$ the univariate polynomial

$$t \mapsto f(te - x)$$

has only real zeros. This class of polynomials have important convexity properties. The *hyperbolicity cone*

$$C(f, e) = \{x \in \mathbb{R}^n : f(te - x) \text{ has no negative zeros}\}$$

is a convex semi-algebraic cone.

It was proved in [2, 4] that there is an explicit connection between hyperbolic polynomials and so called M-convex sets, which are generalizations of matroids. A finite set $J \subset \mathbb{N}^n = \{0, 1, 2, \ldots\}^n$ is *M–convex* if for all $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ in J with $\alpha_i > \beta_i$, there exists an index j such that

$$\beta_j > \alpha_j \text{ and } \alpha - e_i + e_j \in J,$$

where $e_1, \ldots, e_n$ is the standard basis of $\mathbb{R}^n$. Hence if $J \subseteq \{0, 1\}^n$, then J is M–convex if and only if J is the set of bases of a matroid. The support of a polynomial

$$f = \sum_{\alpha \in \mathbb{N}^n} a(\alpha) x_1^{\alpha_1} \cdots x_n^{\alpha_n},$$

is

$$\text{supp}(f) = \{\alpha \in \mathbb{N}^n : a(\alpha) \neq 0\}.$$

Theorem 1 ([2, 4])**.** *Suppose f is hyperbolic and $v_1, \ldots, v_m \in C(f, e)$. Then the support of the polynomial*

$$f(x_1 v_1 + \cdots + x_m v_m)$$

is M–convex.

An M–convex set which arises as in Theorem 1 is called *hyperbolic*. Not all M–convex sets are hyperbolic [2].

In [3] the class of Lorentzian polynomials were introduced. Equivalent definitions appear in [1, 5].

Definition 2. *A homogeneous polynomial f of degree d is* strictly Lorentzian *if*

(1) all coefficients of f are positive, and
(2) the polynomial

$$g = \frac{\partial^{d-2} f}{\partial x_{i_1} \cdots \partial x_{i_{d-2}}}$$

is strictly hyperbolic with respect to $e = (1, \ldots, 1)$ for all $i_1, i_2, \ldots, i_{d-2}$, i.e., the polynomial

$$t \mapsto g(te - x)$$

has two distinct real zeros for all $x \in \mathbb{R}^n$ not parallell to e.

A homogeneous polynomial f is Lorentzian if it is the limit[1] *of strictly Lorentzian polynomials.*

Lorentzian polynomials generalize hyperbolic polynomials. Indeed if f is hyperbolic and $v_1, \ldots, v_m \in C(f, e)$, then the polynomial

$$g = f(x_1 v_1 + \cdots + x_m v_m)$$

is Lorentzian. Minkowski volume polynomials

$$\mathrm{Vol}(x_1 K_1 + \cdots + x_n K_n),$$

where $K_1, \ldots, K_n$ are convex bodies, are also Lorentzian.

Lorentzian polynomials completely characterizes M–convexity.

Theorem 3. *The support of any Lorentzian polynomial is M–convex. Conversely if $J \subset \mathbb{N}^n$, then the polynomial*

$$\sum_{\alpha \in J} \frac{x_1^{\alpha_1} \cdots x_n^{\alpha_n}}{\alpha_1! \cdots \alpha_n!}$$

is Lorentzian.

Peter Lax conjectured that if $f(x, y, z)$ has degree d and is hyperbolic with respect to $e = (a, b, c)$, then there are real symmetric $d \times d$ matrices A, B and C such that $aA + bB + cC$ is positive definite and

$$f(x, y, z) = \det(xA + yB + zC).$$

This was proved by Helton and Vinnikov in 2006. Gurvits [5] conjectured a Lorentzian analog:

Conjecture. *If $f(x, y, z)$ is a Lorentzian polynomial of degree d, then there are convex bodies $K_1, K_2, K_3 \subset \mathbf{R}^d$ such that*

$$f(x, y, z) = \mathrm{Vol}(x_1 K_1 + x_2 K_2 + x_3 K_3).$$

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Free probability

Serban T. Belinschi

The fundamental object in noncommutative probability theory is the *noncommutative probability space*: a pair $(\mathcal{A},\tau)$, where $\mathcal{A}$ is a unital algebra over the field of complex numbers $\mathbb{C}$, and $\tau\colon \mathcal{A}\to\mathbb{C}$ is a linear functional which maps the unit 1 of the algebra $\mathcal{A}$ to the complex number one: $\tau(1)=1$. In the following, we will impose supplementary conditions on $\mathcal{A}$ and/or τ, as needed. Here are a few examples that will appear below: (i) $\mathcal{A}=\mathbb{C}\langle X_1,\dots,X_n\rangle$, the algebra of polynomials in n noncommuting indeterminates, and τ is an arbitrary unit-preserving linear functional on it; (ii) $\mathcal{A}=\mathbb{C}^{n\times n}$, the algebra of $n\times n$ complex matrices, and $\tau=\mathrm{tr}_n$, the canonical normalized trace; $\mathcal{A}=L^\infty(\Omega,\Sigma,P)$, the algebra of essentially bounded measurable functions on a classical probability space (Ω,Σ,P), and $\tau(\cdot)=\int_\Omega\cdot\,\mathrm{d}P$ is the usual expectation; (iv) $\mathcal{A}=\mathcal{M}$, a W^*–algebra, and $\tau\colon\mathcal{M}\to\mathbb{C}$ a normal, faithful, tracial state (examples (ii) and (iii) are particular cases of (iv)).

Voiculescu's free probability is a noncommutative probability theory in which the role of independence from classical probability is replaced by *freeness*, or free independence (see [6]). Freeness is defined in purely algebraic terms: unital subalgebras $\mathcal{A}_i, i\in I$ of $\mathcal{A}$ are *free with respect to* τ if for any $d\in\mathbb{N}$ and $a_1,a_2,\dots,a_d$ such that $a_1\in\mathcal{A}_{i_1},a_2\in\mathcal{A}_{i_2}\dots,a_d\in\mathcal{A}_{i_d},i_1\neq i_2,i_2\neq i_3,\dots,i_{d-1}\neq i_d$, and $\tau(a_1)=\tau(a_2)=\cdots=\tau(a_d)=0$, we have $\tau(a_1a_2\cdots a_d)=0$. One can easily show that this condition provides the recipe for extending τ to the algebra generated by $\{\mathcal{A}_i\colon i\in I\}$ when the restrictions $\tau|_{\mathcal{A}_i}, i\in I$ of τ to all algebras $\mathcal{A}_i$ are known.

Comparing examples of constructions of freeness and classical independence might give a better understanding of the former. Given a discrete group G, denote by $L(G)$ the von Neumann algebra generated by the left regular representation $\lambda\colon G\to B(\ell^2(G))$ of G on the space $\ell^2(G)$ of square-summable series indexed by G. One defines the vector state $\tau\colon L(G)\to\mathbb{C},\tau(x)=\langle x\delta_e,\delta_e\rangle$, where $\delta_e\in\ell^2(G)$ is the characteristic function of the group's neutral element. Consider now the group of integers $\mathbb{Z}$. One embeds canonically two copies of $L(\mathbb{Z})$ in $L(\mathbb{Z}\times\mathbb{Z})$ by sending $\lambda(1)$ to $\lambda(1,0)$ and $\lambda(0,1)$, respectively. This way, $L(\mathbb{Z}\times\mathbb{Z})\simeq L(\mathbb{Z})\otimes L(\mathbb{Z})$ contains two copies of $L(\mathbb{Z})$ which are *classically independent* with respect to $\tau\colon L(\mathbb{Z}\times\mathbb{Z})\to\mathbb{C}$. If instead of taking $\mathbb{Z}\times\mathbb{Z}$, one considers $\mathbb{F}_2=\mathbb{Z}\star\mathbb{Z}$, the *free group* with free generators a and b, then one embeds canonically two copies of $L(\mathbb{Z})$ in $L(\mathbb{Z}\star\mathbb{Z})$ by sending $\lambda(1)$ to $\lambda(a)$ and $\lambda(b)$, respectively. It can be easily verified that these two copies satisfy the freeness relations described above with respect to $\tau\colon L(\mathbb{Z}\star\mathbb{Z})\to\mathbb{C}$. Moreover, it makes sense to write $L(\mathbb{Z}\star\mathbb{Z})\simeq L(\mathbb{Z})\star L(\mathbb{Z})$,i.e., there exists a notion of free product of von Neumann algebras endowed with states. These constructions are described in great detail in [6, Chapters 1 and 2].

We consider next the notion of distribution in noncommutative probability. Let $(\mathcal{A},\tau)$ be a noncommutative probability space and consider a tuple $(a_1,\dots,a_n)\in\mathcal{A}^n$ of possibly non-commuting random variables. The *distribution* of $(a_1,\dots,a_n)$ with respect to τ is the linear map

$$\mu_{(a_1,\dots,a_n)}\colon\mathbb{C}\langle X_1,\dots,X_n\rangle\to\mathbb{C},\quad \mu_{(a_1,\dots,a_n)}(P)=\tau(P(a_1,\dots,a_n)).$$

As in the case of classical probability measures, one is interested in distributions having properties like positivity or continuity, as it is the case for distributions with respect to states on C^*–algebras or W^*–algebras. It turns out that under such conditions, there exists a GNS-type construction. Specifically, endow $\mathbb{C}\langle X_1,\dots,X_n\rangle$ with the involution * given by $(\alpha X_{i_1}X_{i_2}\cdots X_{i_d})^*=\overline{\alpha}X_{i_d}\cdots X_{i_2}X_{i_1}$ and assume that the distribution $\mu\colon \mathbb{C}\langle X_1,\dots,X_n\rangle\to\mathbb{C}$ satisfies:

(1) $\mu(P^*P)\geq 0$ for all $P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$;
(2) $\mu(P^*)=\overline{\mu(P)}$ for all $P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$;
(3) There exists $M\in[0,+\infty)$ such that $|\mu(X_{i_1}X_{i_2}\cdots X_{i_d})|\leq M^d$ for all $d\in\mathbb{N}, i_1,i_2,\dots,i_d\in\{1,\dots,n\}$.

One defines a sesquilinear form $\langle P,Q\rangle_\mu=\mu(Q^*P)$ on $\mathbb{C}\langle X_1,\dots,X_n\rangle$ and the corresponding seminorm $\|P\|_\mu=\sqrt{\langle P,P\rangle_\mu}$. By completing with respect to $\|\cdot\|_\mu$ while factoring out the null space $\mathcal{N}_\mu$ of $\|\cdot\|_\mu$, one obtains a Hilbert space which we denote by $L^2(\mu)$. Direct computations using the Cauchy-Schwarz inequality and item (3) above show that the multiplication operator $\mathfrak{M}_Q\colon L^2(\mu)\to L^2(\mu)$ given by $\mathfrak{M}_Q(\hat{P})=\widehat{QP}$, $P\in\mathbb{C}\langle X_1,\dots,X_n\rangle$, is well-defined, linear, and continuous. One concludes by noting that the joint distribution of the *selfadjoint* variables $X_1,\dots,X_n\in B(L^2(\mu))$ with respect to $\tau\colon B(L^2(\mu))\to\mathbb{C}, \tau(T)=\langle T1,1\rangle_\mu$ is μ.

We denote the space of distributions satisfying (1)–(3) above by $\Sigma_0(n)$.As the elements of $\Sigma_0(n)$ are linear, they are determined by their evaluations on monomials, i.e., their *moments* $\mu(X_{i_1}X_{i_2}\cdots X_{i_d}), d\in\mathbb{N}, i_1,i_2,\dots,i_d\in\{1,\dots,n\}$. This allows one to identify distributions μ with the countable family of numbers $\{m_\mu(w)\}_{w\in\mathbb{F}_n^+}$, where $\mathbb{F}_n^+$ is the free semigroup with n generators (the set of all words in n letters).

We conclude with a characterization of elements of $\Sigma_0(n)$ via noncommutative analytic functions, followed by a similar characterization of freeness. Let $\mu\in\Sigma_0(n)$ be the distribution of the tuple of random variables $(a_1,\dots,a_n)$ belonging to the W^*–algebra $\mathcal{A}$ with respect to the state τ. For any $d\in\mathbb{N}$ and selfadjoint matrices $\alpha_1,\dots\alpha_n\in\mathbb{C}^{d\times d}$, one defines $\alpha_1\otimes a_1+\cdots+\alpha_n\otimes a_n\in\mathbb{C}^{d\times d}\otimes\mathcal{A}\simeq\mathcal{A}^{d\times d}$ and the conditional expectation $E_d=\mathrm{Id}_{\mathbb{C}^{d\times d}}\otimes\tau\colon\mathcal{A}^{d\times d}\to\mathbb{C}^{d\times d}, E[(a_{ij})_{1\leq i,j\leq d}]=(\tau(a_{ij}))_{1\leq i,j\leq d}$. One defines $G_\mu(\alpha_1,\dots,\alpha_n;b)=E_d\left[(b-\alpha_1\otimes a_1-\cdots-\alpha_n\otimes a_n)^{-1}\right]$, $M_\mu(\alpha_1,\dots,\alpha_n;b)=E_d\left[(1-b\alpha_1\otimes a_1-\cdots-b\alpha_n\otimes a_n)^{-1}b\right]=G_\mu(\alpha_1,\dots,\alpha_n;b^{-1})$. These functions encode the distribution $\mu=\mu_{(a_1,\dots,a_n)}$ of $(a_1,\dots,a_n)$: there are many ways to see that, but a straightforward method based on [2] provides a canonical way to identify any given moment $m_\mu(w)$. Say $w=i_1i_2\cdots i_{m-2}\in\mathbb{F}_n^+$ is given. Pick $d=n$ and $\alpha_k=(\delta_{ik}\delta_{jk})_{1\leq i,j\leq n}$ so that $\alpha_1\otimes a_1+\cdots+\alpha_n\otimes a_n=\mathrm{diag}(a_1,\dots,a_n):=\mathbf{A}$. Consider $M_\mu(b)=(\mathrm{Id}_{\mathbb{C}^{mn\times mn}}\otimes\tau)\left[b(1-(I_m\otimes\mathbf{A})b)^{-1}\right]$ (we suppress here the variables α_j from the notation, as they are now fixed) evaluated in

$$b=\begin{bmatrix}0 & b_1 & 0 & \cdots & 0\\ 0 & 0 & b_2 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & b_{m-1}\\ 0 & 0 & 0 & \cdots & 0\end{bmatrix},\qquad b_1,b_2,\dots,b_{m-1}\in\mathbb{C}^{n\times n}.$$

Then $[M_\mu(b)]_{1,m} = (\mathrm{Id}_{\mathbb{C}^{n\times n}} \otimes \tau)(b_1\mathbf{A}b_2\cdots\mathbf{A}b_{m-1})$. If we choose $b_1, b_2, \ldots, b_{m-1}$ so that $b_1\mathbf{A}b_2\mathbf{A}b_3\cdots\mathbf{A}b_{m-1} = U_1^*\mathbf{A}U_1U_2^*\mathbf{A}U_2U_3^*\cdots\mathbf{A}U_{m-2}^*$ for permutation matrices U_j that, say, interchange coordinates 1 and i_j, we find $m_\mu(w)$ in the first (upper left) entry of $[M_\mu(b)]_{1,m} \in \mathbb{C}^{n\times n}$.

It had been observed by Voiculescu that noncommutative analytic transforms characterize noncommutative distributions, and, in particular, characterize freeness. This was made explicit in [5] via the noncommutative extension of the R–transform. Here we want to illustrate a different characterization of freeness via noncommutative analytic transforms, namely via Voiculescu's analytic subordination functions [3]. We summarize this result as it follows from [1], with some details omitted. Let again $(a_1, \ldots, a_n), (c_1, \ldots, c_p)$ be two tuples of selfadjoint random variables in a noncommutative probability space $(\mathcal{A}, \tau)$, where τ is a state. Then $(a_1, \ldots, a_n)$ and $(c_1, \ldots, c_p)$ are free with respect to τ if and only if there exist noncommutative maps ω_1, ω_2 such that

$$\begin{aligned}
&\left(\omega_1\left((\alpha_j)_{j=1}^n, (\beta_k)_{k=1}^p; b\right) + \omega_2\left((\alpha_j)_{j=1}^n, (\beta_k)_{k=1}^p; b\right) - b\right)^{-1} \\
&\quad= (\mathrm{Id}_{\mathbb{C}^{d\times d}} \otimes \tau)\left[\left(\omega_1\left((\alpha_j)_{j=1}^n, (\beta_k)_{k=1}^p; b\right) - \alpha_1 \otimes a_1 - \cdots - \alpha_n \otimes a_n\right)^{-1}\right] \\
&\quad= (\mathrm{Id}_{\mathbb{C}^{d\times d}} \otimes \tau)\left[\left(\omega_2\left((\alpha_j)_{j=1}^n, (\beta_k)_{k=1}^p; b\right) - \beta_1 \otimes c_1 - \cdots - \beta_p \otimes c_p\right)^{-1}\right] \\
&\quad= (\mathrm{Id}_{\mathbb{C}^{d\times d}} \otimes \tau)\left[\left(b - \alpha_1 \otimes a_1 - \cdots - \alpha_n \otimes a_n - \beta_1 \otimes c_1 - \cdots - \beta_p \otimes c_p\right)^{-1}\right],
\end{aligned}$$

where $d \in \mathbb{N}$, the variables $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_p \in \mathbb{C}^{d\times d}$ are selfadjoint, the variable $b \in \mathbb{C}^{n\times n}$ satisfies $\Im b = \frac{b-b^*}{2i} > 0$, and $\Im\omega_j\left((\alpha_j)_{j=1}^n, (\beta_k)_{k=1}^p; b\right) > 0, j = 1, 2$. It is a remarkable result of [7] that, roughly speaking, self-maps of the noncommutative set of elements with positive imaginary part that behave like $-b^{-1}$ at infinity are in bijective correspondence with noncommutative distributions (see [7, Theorem 3.1] for the complete statement).

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Applications of free probability to dilation theory

Malte Gerhold
(joint work with Satish Pandey, Orr M. Shalit and Baruch Solel)

If a matrix (= square matrix with complex entries) $A \in M_n(\mathbb{C})$ appears as the upper left corner of a matrix $B \in M_k(\mathbb{C})$, $k \geq n$, i.e., $B = \begin{bmatrix} A & * \\ * & * \end{bmatrix}$, we call B a *dilation* of A, and A a *compression* of B, and we write $A \prec B$. We use the same terminology for d–tuples $A = (A_1, \ldots, A_d) \in M_n(\mathbb{C})^d$ and $B = (B_1, \ldots, B_d) \in M_k(\mathbb{C})^d$ if each A_i is the $n \times n$ upper left corner of the corresponding B_i. Although a bit too restrictive to serve as a good definition, this is good enough to state the *complex matrix cube problem*, which motivates most of the results that appear in this talk:

Find the smallest constant $C_d > 0$ such that every $A \in \mathbb{M}^d$ admits a normal dilation $N \succ A$ with $\|N\| \leq C_d \|A\|$;

here we say that a d–tuple of matrices $N \in M_n(\mathbb{C})^d$ is *normal* if each N_i is normal (i.e., N_i commutes with its adjoint N_i^*) and the N_i commute with each other, or to put it differently, if the N_i are simultaneously unitarily diagonalizable; $\mathbb{M}^d$ denotes the set of all d–tuples of complex square matrices of (the same) arbitrary size. The norm of an $n \times n$ matrix always means the operator norm induced by viewing $\mathbb{C}^n$ as a finite-dimensional Hilbert space with the standard inner product, i.e., $\|A_i\| = \sup_{\|x\|_2 \leq 1} \|A_i x\|_2$, and the norm of a d–tuple is $\|A\| := \max_{1 \leq i \leq d} \|A_i\|$.

Although surprising at first sight, it is quite easy to show that $C_d \leq d$, so in particular $C_d < \infty$. However it seems to be a rather difficult problem to find the precise value of C_d, even for $d = 2$. The best general known bounds are $\sqrt{d} \leq C_d \leq \sqrt{2d}$; see [7] for the lower and [6] for the upper bound. Striving to improve these bounds, we encountered some interesting phenomena related to dilations as well as some surprising applications.

Before we present the results, we will fix some notation. The notion of dilation easily generalizes to tuples of bounded operators on possibly infinite-dimensional Hilbert spaces $H \subseteq K$. We extend the dilation order even further to tuples in arbitrary (abstract) C*–algebras and write $A \prec B$ if B is a dilation of A in some arbitrarily chosen faithful representation. With an operator tuple A, we associate two sets of matrices,

- its *matrix range* $W(A) := \{\Phi(A) \mid \Phi\colon \mathrm{C}^*(1, A) \xrightarrow{\text{ucp}} M_n(\mathbb{C})\}$
- its *free spectrahedron* $\mathcal{D}(A) := \{B \in \mathbb{M}^d \mid \mathrm{Re}\left(\sum B_i \otimes A_i\right) \leq 1\}$

which are polar duals of each other in the sense of matrix convex sets if 0 is contained in both matrix ranges $W(A)$ and $W(B)$ (we will usually assume this without further mentioning). We make use of the well known equivalences

$$\begin{aligned} A \prec B &\iff W(A) \subset W(B) \iff \mathcal{D}(A) \supset \mathcal{D}(B) \\ &\iff \|X_0 \otimes 1 + \sum X_i \otimes A_i\| \leq \|X_0 \otimes 1 + \sum X_i \otimes B_i\|, \quad \forall X \in \mathbb{M}^d \\ &\iff A = \Phi(B) \text{ for some ucp-map } \Phi. \end{aligned}$$

For two operator d–tuples we define the *dilation scale*

$$c(A,B) := \inf\{c > 0 \mid A \prec cB\}.$$

Some general arguments combined with the fact that every contraction admits a unitary dilation allow us to conclude

$$C_d = \sup\{c(U,u_0) \mid U \text{ unitary } d\text{–tuple}\},$$

where u_0 denotes the d–tuple of universal commuting unitaries (these can be realized for example as the canonical generators of the group C*–algebra C$^*(\mathbb{Z}^d)$).

In [2], the lower bound for C_2 was improved to $C_2 > 1.54$ by calculating $c(u_\theta, u_0)$, where u_θ is a $e^{i\theta}$–commuting pair of unitaries. In this talk, we examine dilations involving the d–tuple u_f of *free Haar unitaries*, i.e., the canonical generators of the reduced group C*–algebra $\mathrm{C}^*_r(\mathbb{F}_d)$ of the free group on d generators. The main observation which motivates this is the following: it is quite easy to find an upper bound for $c(U,u_f)$ for any d–tuple of unitaries U (and, indeed, a quite small one) using the fact that the d–tuple $U \otimes u_f := (U_1 \otimes u_{f,1}, \dots U_d \otimes u_{f,d})$ is again (isomorphic to) a d–tuple of free Haar unitaries; this is an instance of Fell's absorption principle for unitary representations of groups. Clearly,

$$c(U,u_0) \leq c(U,u_f)c(u_f,u_0),$$

so a good bound for $c(u_f,u_0)$ could imply a better upper bound for C_d.

Free probability gives a tool to calculate the distribution of the free Haar unitaries with respect to the canonical trace. Whereas there was enough understanding of the free Haar unitaries to estimate $c(U,u_f)$ long before free probability was invented, we will need results of Franz Lehner based on operator valued free probability [5] in order to estimate $c(u_f,u_0)$. Putting everything together, today we are just able to reproduce the known upper bound $C_d \leq \sqrt{2d}$. However, there is strong evidence (based on random matrix approximation of the free unitaries [1, 3] and an algorithm to compute dilation scales between matrix-tuples [4]) that the upper bound for $c(u_f,u_0)$ we use is far from optimal, at least for small d – and any improvement of this single concrete dilation scale will now give an improvement for the upper bound on the general constant C_d.

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Some computational aspects of real algebraic geometry

Mohab Safey El Din

Real algebraic geometry deals with the sets of solutions (with coordinates in a real closed field R) of finitely many disjunctions of polynomial systems of equations and inequalities with coefficients in R and maps between such solution sets (which are called semi-algebraic sets). Tackling real fields which are not archimedean make the development of some results in this area more difficult to obtain and quantitive results are trickier to obtain as there is not an appropriate notion of degree as in classical algebraic geometry.

However, in this talk, we have shown how real algebraic geometry is important to many applications in engineering sciences such as robotics, mechanism design, biology, chemistry, computer vision (amongst many other). Computations and algorithms in real algebraic geometry are of first importance in these applications where obtaining exact and trustworthy results is sometimes crucial.

Hence, the sequel of the talk focused on computer algebra based algorithms in real algebraic geometry with first an overview of the most basic routines one should expect. The very first one is of course real root counting, and isolation when the base field is archimedean, for zero-dimensional polynomial systems (which are the ones which have finitely many solutions with coordinates in an algebraic closure of the base field). This latter problem, through effective variants of the primitive element theorem, can be reduced to the univariate case.

Next, we reviewed some more involved algorithmic problems such as computing sample points in each connected component of semi-algebraic sets, computing their dimension, the number of their connected components, or a triangulation of them. At the heart of many of these algorithms, there are important results about the properties of semi-algebraic sets and their projections. The very first one is the stability theorem by Tarski: projections of semi-algebraic sets are semi-algebraic sets. Hardt's semi-algebraic triviality theorem is a second important one as it shows that projections can be made trivial by partitioning their target space into semi-algebraic cells (see e.g., [1, 2]).

This lead us to consider a semi-algebraic version of Thom's isotopy lemma which is due to Coste and Shiota [3].

We showed how this theorem can be use to solve efficiently two important algorithmic problems. The first one is one-block quantifier elimination which consists in computing a semi-algebraic description of the projection of a semi-algebraic set. We show how to tune the critical point method to obtain practically fast algorithms. The very basic principle, which is already used in [2, 4], is to run algorithms from the existential theory over the reals with parameters.

Next, we introduce the problem of identifying the best possible degrees of the output formulas by relating them with degrees of fundamental geometric objects. This, in turn, is related to another algorithmic problem which is real root classification. It consists in solving parametric polynomial systems which have finitely many complex roots for generic values of the parameters by partitioning the parameters'space into semi-algebraic cells over which the number of real solutions remains invariant.

In a joint work with Huu Phuoc Le, we show how obtain an algorithm which computes semi-algebraic formulas describing these cells involving polynomials of optimal degree (i.e., related to the algebraic degree of some polar variety). We reported on practical expriments showing that this algorithm outperforms the state-of-art implementations/algorithms.

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Projective limits techniques for the infinite dimensional moment problem

Kuna, Tobias

(joint work with Maria Infusino, Salma Kuhlmann and Patrick Michalski)

We consider the following infinite dimensional moment problem: *when can a linear functional L on a non-finitely generated unital commutative real algebra A be represented as an integral w.r.t. a measure on the space $X(A)$ of all characters of A?* Our main idea is to construct $X(A)$ as a projective limit of all $X(S)$ with S finitely generated subalgebra of A. This gives us a mechanism to exploit results for the classical finite dimensional moment problem in the infinite dimensional case. In other words, this projective limit approach will put into focus the difference between the finite and the infinite dimensional moment problem, see [1] for more details.

$X(A)$ is the space of all (unitary) algebraic homomorphisms $\alpha\colon A \to \mathbb{R}$. For $a \in A$ denote by $\hat{a}\colon X(A) \to \mathbb{R}$ the mapping defined as $\hat{a}(\alpha) := \alpha(a)$. We endow $X(A)$ with the weak topology on $X(A)$, which is the weakest topology which makes all $\hat{a}$ continuous for $a \in A$. We show that $X(A)$ is the projective limit of the spaces $X(S)$ also equipped with the weak topology, where S varies through all finitely generated subalgebras of A. This enable us to use classical results about the construction of measures on the projective limit from their projections. We split the construction in different steps.

In fact, we show that the existence of a Radon representing measure μ_S for L restricted to each S such that $(\mu_S)_S$ fulfils the Bochner-Yamasaki condition is equivalent to the existence of a representing measure for L on $X(A)$, but defined on a smaller σ–algebra than the Borel one, namely the cylinder σ–algebra on $X(A)$, which is the weakest σ–algebra which makes all the functions $\hat{a}$ measurable. The main improvement with respect to the general theory of projective limits is that we do not require that the moment problems for the L restricted to S have a unique solution, in general the different measure μ_S may not fit well together. Indeed, we can show that then automatically a coherent system of representing measures always exists.

The Bocher-Yamasaki condition is related to the fact that a lot of characters from $X(A)$ cannot be extended to characters on A. The condition requires that the realizing measures are "supported" on characters which can be extended. If A is countable generated then Bochner-Yamasaki condition is automatically fulfilled.

The next natural question is, which conditions ensure the existence of a representing measure on the Borel σ–algebra generated by the weak topology?

Prokhorov's (ε-K) condition, which is related to the Minlos-Sazonov condition for generating functionals, characterizes when a representing measures on the cylinder σ–algebra can be extended to a Radon representing measure. An extra bonus is that under Prokhorov's (ε-K) condition the Bochner-Yamasaki condition holds automatically. If A is countably generated, Prokhorov's (ε-K) condition holds automatically. An interesting side remark is that the extension cannot be achieved by Caratheodory's extension theorem, which will only give a measure on the cylinder σ–algebra.

These results allow us to establish infinite dimensional analogues of the classical Riesz-Haviland and Nussbaum theorems as well as a representation theorem for linear functionals non-negative on a "partially" Archimedean quadratic module of A. This gives us a framework to compare and relate the abundance of exciting recent results for the infinite dimensional moment problem. In particular, applying our results to the algebra of polynomials in infinitely many variables or to the symmetric tensor algebra of an infinite dimensional vector space, we can retrieve some recent solutions to the moment problem on such algebras including the ones for constructibly Radon measures. Details can be found in [1]. In a forthcoming work, [2], we will investigate deeper into topological aspects of the problem.

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Inner-outer factorization using free analysis

Michael Hartz

(joint work with Alexandru Aleman, John E. McCarthy and Stefan Richter)

The Hardy space H^2 is a classical Hilbert space of holomorphic functions on the open unit disc $\mathbb{D} \subset \mathbb{C}$. It can be defined as

$$H^2 = \left\{ f = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{O}(\mathbb{D}) : \sum_{n=0}^{\infty} |a_n|^2 < \infty \right\}.$$

This space occupies a key place at the intersection of operator theory, complex analysis and harmonic analysis. One of the foundational results in the theory of H^2 is the inner-outer factorization. A function $g \in H^2$ is said to be *inner* if the multiplication operator $M_g \colon H^2 \to H^2$ is defined and isometric. Moreover, a function $h \in H^2$ is *outer (or cyclic)* if the space $h \cdot \mathbb{C}[z]$ is dense in H^2. (These are not the original definitions, but they are equivalent ones.) The inner-outer factorization, due to Riesz, Herglotz and Beurling, then states that every function in H^2 is a product of an inner and an outer function. Moreover, this factorization is unique up to multiplication by a unimodular constant.

The class of complete Pick spaces consists of Hilbert function spaces that mirror some of the fine structure of H^2. Examples are the Dirichlet space

$$\mathcal{D} = \{ f \in \mathcal{O}(\mathbb{D}) : f' \in L^2(\mathbb{D}) \},$$

as well as the Drury-Arveson space H^2_d, which is a multivariable generalization of H^2; see [1]. These spaces typically do not admit non-constant isometric multipliers, so the naive generalization of the inner–outer factorization fails. But there is a less naive generalization.

Definition 1. *Let $\mathcal{H}$ be a complete Pick space. A function $g \in \mathcal{H}$ is* subinner *if the multiplication operator $M_g \colon \mathcal{H} \to \mathcal{H}$ is contractive, and there exists $f \in \mathcal{H} \setminus \{0\}$ with $\|gf\| = \|g\|$.*

We also define *free outer* functions in terms of an extremal property. Free outer functions are cyclic, but the converse does not hold in general.

Theorem 2 ([2])**.** *Let $\mathcal{H}$ be a complete Pick space. Then every $f \in \mathcal{H}$ admits a factorization $f = gh$, where g is subinner and h is free outer. Moreover, the factorization is unique up to multiplication by a unimodular constant.*

If $\mathcal{H} = H^2$, this factorization agrees with the classical inner-outer factorization.

The factorization in the theorem is obtained with the help of free analysis. It is known that complete Pick spaces can be embedded into the full Fock space, which can be regarded as a Hilbert space of non-commutative holomorphic functions. The key advantage in the non-commutative world is the existence of non-constant isometric multipliers, and indeed there is a direct analogue of the inner-outer factorization in the full Fock space [3, 4]. Jury and Martin [5] observed that this leads to some factorization in complete Pick spaces. The main novelty in our theorem is an intrinsic characterization of the factors in terms of the original function space, as well as the uniqueness statement.

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When Jacobi-Prestel meets Nussbaum

Patrick Michalski
(joint work with Maria Infusino, Salma Kuhlmann and Tobias Kuna)

We deal with the following general version of the moment problem: *When can a linear functional L on a unital commutative $\mathbb{R}$–algebra A be represented as an integral with respect to a (positive) Radon measure μ on the character space $X(A)$ of A equipped with the Borel σ–algebra generated by the weak topology, i.e.,*

$$L(a) = \int_{X(A)} a(\alpha) \mathrm{d}\mu \qquad \textit{for all } a \in A?$$

Recall that $X(A)$ is the space of all algebra homomorphisms from A to $\mathbb{R}$ and it is here assumed to be non-empty. The Radon measure μ is called *K–representing for L* if it is supported in a closed subset K of $X(A)$, i.e., $\mu(X(A) \setminus K) = 0$.

For convenience we assume that $L(1) = 1$ and that L is non-negative on a quadratic module Q in A. Indeed, the first assumption ensures that all representing measures obtained are probabilities and the second is a necessary condition for the moment problem to be solvable. Constructing $X(A)$ as the projective limit of the family of all $X(S)$ with S finitely generated subalgebra of A and using the well-known Prokhorov theorem [5], we establish in [2] the following sufficient condition for the moment problem stated above to be solvable.

Theorem 1. *If for each finitely generated subalgebra S of A there exists a unique $K_{Q\cap S}$–representing measure μ_S for $L\restriction_S$ such that*

$$(1) \qquad \forall \varepsilon > 0 \; \exists \; C \subset X(A) \textit{ compact s.t. } \forall S: \; \pi_{S\#}\mu(\pi_S(C)) \geq 1 - \varepsilon,$$

then there exists a unique K_Q–representing Radon measure for L, where $K_Q = \{\alpha \in X(A) : q(\alpha) \geq 0 \text{ for all } q \in Q\}$.

Theorem 1 bridges finite dimensional (i.e., A is finitely generated) and infinite dimensional (i.e., A is not finitely generated) moment theory. Note that if the quadratic module Q is Archimedean, then K_Q is compact and (1) is always satisfied and so, Theorem 1 yields the counterpart for the moment problem of the Jacobi-Prestel Positivstellensatz [3]. Also, (1) is always satisfied when A is countably generated and, assuming $\sum_{n=1}^{\infty} \sqrt[-2n]{L(a^{2n})} = \infty$ for all $a \in A$, we obtain a K_Q−representing Radon measure for L on A (this establises an infinite dimensional analogue of the classical Nussbaum theorem [4]). In fact, both results can be combined to provide the following sufficient condition for the existence of representing measures not necessarily supported in a compact subset of $X(A)$.

Theorem 2. *If there exist subalgebras $B_{\mathrm{a}}, B_{\mathrm{c}}$ of A and a quadratic module Q in A such that $L(Q) \subseteq [0, \infty)$ and*

(1) $B_{\mathrm{a}} \cup B_{\mathrm{c}}$ generates A,
(2) $Q \cap B_{\mathrm{a}}$ is Archimedean,
(3) B_{c} is countably generated and $\sum_{n=1}^{\infty} \frac{1}{\sqrt[2n]{L(a^{2n})}} = \infty$ for all $a \in B_{\mathrm{c}}$,

then there exists a unique K_Q–representing Radon measure for L, where $K_Q = \{\alpha \in X(A) : q(\alpha) \geq 0$ for all $q \in Q\}$.

This generalizes [1, Theorem 5.4].

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On morphisms of the projective line with only real periodic points

Khazhgali Kozhasov

Let $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ be a morphism of the complex projective line $\mathbb{P}^1 = (\mathbb{C}^2 \setminus \{0\})/\mathbb{C}^*$. It is defined by two coprime homogeneous polynomials $\varphi_0, \varphi_1 \in \mathbb{C}[x_0, x_1]$ of the same degree d via $\varphi(x) = [\varphi_0(x_0, x_1) : \varphi(x_0, x_1)]$, $x = [x_0, x_1] \in \mathbb{P}^1$.

A point $x \in \mathbb{P}^1$ is called *periodic* if $\varphi^k(x) = x$ for some integer $k > 0$, where $\varphi^1(x) = \varphi(x)$ and $\varphi^{k+1}(x) = \varphi(\varphi^k(x))$. Any morphism $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree $d \geq 2$ has infinitely many periodic points [1]. Let us now assume that φ is defined over $\mathbb{Q}$, that is, one can choose $\varphi_0, \varphi_1 \in \mathbb{Q}[x_0, x_1]$ to have rational coefficients. Then the induced morphism $\varphi : \mathbb{P}^1(\mathbb{Q}) \to \mathbb{P}^1(\mathbb{Q})$ on the set $\mathbb{P}^1(\mathbb{Q})$ of rational points of $\mathbb{P}^1$ has only finitely many (rational) periodic points [3]. One of the major open problems in arithmetic dynamics is the Uniform boundedness conjecture [2] which asserts that the number of rational periodic points can be bounded by a constant depending only on the degree $d = \deg(\varphi_0) = \deg(\varphi_1)$ of $\varphi = [\varphi_0 : \varphi_1]$. The number of real periodic points of φ though is sometimes finite and sometimes infinite (depending on the choice of φ). However, even when φ has infinitely many real periodic points, it still may have some complex (in $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{R})$) ones. Thus, one can ask the following natural question.

Question 1. *Given $d \geq 2$ does there exist a morphism $\varphi\colon \mathbb{P}^1 \to \mathbb{P}^1$ of degree d that is defined over $\mathbb{R}$ (i.e., $\varphi_0, \varphi_1 \in \mathbb{R}[x_0, x_1]$) and has only real periodic points?*

In the talk we answer this question in the affirmative by showing that the morphism $[x_0 : x_1] \mapsto [x_1^d T_d(x_0/x_1) : x_1^d]$ given by the Chebyshev polynomial of the first kind $T_d(t) = \cos(d \arccos(t))$, $t \in [-1, 1]$, has only real periodic points.

Consider now the set R_d which consists of those real morphisms of degree d that have only real periodic points. It is easy to see that R_d is closed (intersection of countably many semialgebraic sets) in the space of all real morphisms of $\mathbb{P}^1$.

Question 2. *Does the set R_d has full dimension (equivalently, has nonempty interior) in the space of all real morphisms of degree d?*

This question remains open at this moment. In an ongoing joint work with Kummer we construct a d–dimensional family of morphisms in the $(2d + 1)$–dimensional R_d. We also conjecture that morphisms of $\mathbb{P}^1$ given by Hermite polynomials $H_d(t) = (-1)^d e^{t^2} \frac{\mathrm{d}^d}{\mathrm{d}t^d} e^{-t^2}$ have only real periodic points.

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Free probability: operator algebraic aspects and regularity of noncommutative distributions

Tobias Mai

The theory of von Neumann algebras has its foundation in the series of groundbreaking papers *On rings of operators* by Francis Murray and John von Neumann, the first one of which [8] appeared in 1936. A *von Neumann algebra* can be defined as a unital ∗–subalgebra M of $B(H)$, the ∗–algebra of all bounded linear operators on some complex Hilbert space H, whose *bicommutant* $M'' := (M')'$ satisfies $M'' = M$. We recall that the *commutant* S' of any subset $S \subseteq B(H)$ is defined by $S' := \{y \in B(H) \mid \forall x \in S\colon xy = yx\}$.

An interesting construction presented by Murray and von Neumann in [9] (see also [8]) associates to any discrete group G a von Neumann algebra $L(G)$ on the Hilbert space $\ell^2(G)$. The Hilbert space $\ell^2(G)$ is built in such a way that the group elements yield an orthonormal basis $(\delta_g)_{g \in G}$ of $\ell^2(G)$. Each $g \in G$

induces a unitary operator λ_g on $\ell^2(G)$ satisfying $\lambda_g \delta_h = \delta_{gh}$ for every $h \in G$; in this way, one obtains the so-called *left regular representation of G on $\ell^2(G)$*, namely $\lambda : G \to B(\ell^2(G)), g \mapsto \lambda_g$. The *group von Neumann algebra $L(G)$* is then defined as $L(G) := \lambda(G)'' \subseteq B(\ell^2(G))$. Group von Neumann algebras have the remarkable feature that they all carry a *trace* $\tau_G \colon L(G) \to \mathbb{C}$, i.e., a faithful normal state which is moreover *tracial* in the sense that $\tau_G(xy) = \tau_G(yx)$ holds for all $x, y \in L(G)$; it is simply given by $\tau_G(x) = \langle x\delta_e, \delta_e \rangle_{\ell^2(G)}$, where e stands for the identity element in G. As a consequence, we obtain that $L(G)$ is always a proper $*$–subalgebra of $B(\ell^2(G))$. Nonetheless, due to their unwieldy definition in terms of the bicommutant, it is rather intricate to distinguish (up to isomorphism) group von Neumann algebras that arise from different groups. In fact, it is not even known whether two of the group von Neumann algebras $L(\mathbb{F}_n)$ associated to the *free groups* $\mathbb{F}_n$ with $n \geq 2$ generators are isomorphic or not.

It was precisely this fundamental open question which motivated Dan-Virgil Voiculescu to launch around the year 1985 in [12] what became known as *free probability theory*. His crucial insight was that the isomorphism problem can be formulated in the language of noncommutative probability theory; this approach rests on the following two pillars.

First of all, Voiculescu noticed that $(L(G), \tau_G)$ can be seen as a space of noncommutative random variables. More generally, a *tracial W^*–probability space* (M, τ) consists of a von Neumann algebra $M \subseteq B(H)$ and a trace $\tau \colon M \to \mathbb{C}$. This rephrases classical probability spaces in an operator algebraic fashion; in fact, if $(\Omega, \mathcal{F}, \mathbb{P})$ is any probability space, then $(L^\infty(\Omega, \mathcal{F}, \mathbb{P}), \mathbb{E})$ with the usual expectation $\mathbb{E}$ that is given by $\mathbb{E}[X] := \int_\Omega X(\omega)\, \mathrm{d}\mathbb{P}(\omega)$ fits into that framework.

Voiculescu's second ingenious observation was that the "relative position" of $L(\mathbb{F}_n)$ and $L(\mathbb{F}_m)$ inside $L(\mathbb{F}_{n+m})$ resulting from the free product decomposition $\mathbb{F}_n * \mathbb{F}_m \cong \mathbb{F}_{n+m}$ can be described abstractly in terms of $\tau_{\mathbb{F}_{n+m}}$. Hence, it became possible to formulate this relation in any tracial W^*–probability space and to study it detached from the particular von Neumann algebras $L(\mathbb{F}_n)$; the condition that Voiculescu found became known as *free independence* and can be seen as a highly noncommutative analogue of the notion of independence in classical probability.

The great success of free probability theory relies in particular on its surprising connections to random matrix theory. While free independence was originally designed for operator algebraic purposes, it was observed later on by Voiculescu [13] that free independence governs the behavior of classically independent *Gaussian random matrices* when their size tends to infinity. This phenomenon of *asymptotic freeness* turned out to occur also for various other random matrix models. It not only allows to apply free probability tools to random matrix problems, but also made random matrix techniques become an important tool of free probability. Among the most striking results that were proven by making use of this connection is the following dichotomy result obtained independently by Ken Dykema [5] and Florin Rădulescu [10]: the von Neumann algebras $L(\mathbb{F}_n)$ for $n \geq 2$ are either all isomorphic or no two of them can be isomorphic.

Free probability gets much impetus from the far reaching analogy with classical probability theory. For instance, it is very natural to introduce a noncommutative counterpart of the fundamental notion of distributions: for any n-tuple $X = (x_1, \dots, x_n)$ of selfadjoint noncommutative random variables in some tracial W^*–probability space (M, τ), we define the *(joint) noncommutative distribution of* X as the linear map $\mu_X \colon \mathbb{C}\langle T\rangle \to \mathbb{C}, P \mapsto \tau(P(X))$ on the algebra $\mathbb{C}\langle T\rangle$ of all noncommutative polynomials in n formal variables $T = (t_1, \dots, t_n)$. It can be shown that μ_X uniquely determines the von Neumann algebra $W^*(X)$ that is generated by $x_1, \dots, x_n$ up to isomorphism; in fact, one can associate to μ_X a von Neumann algebra $L^\infty(\mu_X)$ such that $W^*(X) \cong L^\infty(\mu_X)$. Thus, the concept of noncommutative distributions opens a completely new perspective: all operator algebraic properties of $W^*(X)$ (such as *factoriality* [4], failure of *property* Γ [4], or absence of *Cartan subalgebras* [15]) are hidden in the purely algebraic data μ_X.

Nonetheless, it is often very delicate to read off such information from μ_X. It is therefore of fundamental interest to understand the structure of noncommutative distributions also from an analytic point of view. The major drawback is that μ_X allows in general no measure theoretic description. As an expedient, one studies instead its "push forward" under suitable "noncommutative test functions" such as (matrices of) noncommutative polynomials or rational functions; here, one makes use of the fact that the noncommutative distribution μ_y of a single noncommutative random variable $y = y^* \in M$ can be identified with the Borel probability measure on the real line $\mathbb{R}$ satisfying $\tau(y^k) = \int_{\mathbb{R}} t^k \, \mathrm{d}\mu_y(t)$ for every integer $k \geq 0$. In the recent years, much progress has been made in understanding regularity properties of noncommutative distributions; see, e.g. [11, 2, 6, 1, 7].

Regardless of whether one is interested in the operator algebraic properties of $W^*(X)$ or in the regularity properties of μ_X, it is by no means surprising that the answer most often depends on the concrete set of generators $X = (x_1, \dots, x_n)$. This leads to the question of whether one can characterize what "good" generators are in that respect. Voiculescu's work on free analogues of entropy and Fisher's information measure provides in fact a whole hierarchy of such criteria; see [17] for a survey. More precisely, we distinguish two conceptually different approaches: in the *microstates approach* [14], one quantifies "how well" X can be approximated in distribution by n–tuples of matrices of growing size, whereas the *non-microstates approach* [16] builds upon a kind of L^2–theory for "free differential operators."

An interesting feature of noncommutative random variables $X = (x_1, \dots, x_n)$ which are well-behaved in the aforementioned sense is, loosely speaking, that they behave very much like the formal variables $T = (t_1, \dots, t_n)$. More precisely, they yield operator models which allow to study algebraic properties of $\mathbb{C}\langle T\rangle$ and related objects by analytic means. It was shown in [7] that this strategy even works for the so-called *free field*. In the language of [3], the free field is the *universal skew field of fractions of* $\mathbb{C}\langle T\rangle$, and it turns out to be the correct algebraic framework for what one would intuitively call *noncommutative rational functions*. This connection has already led to some interesting insights, but the story has only just begun ....

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Polynomial optimization with sums of squares and moments

Monique Laurent

Polynomial optimization deals with the minimization of a polynomial function f over a basic closed semi-algebraic set $K = \{x \in \mathbb{R}^n : g_1(x) \geq 0, \ldots, g_m(x) \geq 0\}$, i.e., the problem of computing the global minimum

$$f_{\min} = \min\{f(x) : g_j(x) \geq 0 \text{ for } j \in [m]\}.$$

Here, $f, g_1, \ldots, g_m$ are n–variate polynomials and the set K is assumed to be compact. Clearly this can be reformulated as

$$f_{\min} = \sup\{\lambda : f(x) - \lambda \geq 0 \text{ for all } x \in K\}, \tag{1}$$

which thus amounts to checking *nonnegativity* of a polynomial over K. This is in general a hard computational problem. Here are a few special such hard instances:

(i) decide whether a given matrix M is copositive (i.e., whether the polynomial $\sum_{i,j} x_i^2 x_j^2 M_{ij}$ is nonnegative over the unit sphere), (ii) decide whether a polynomial is convex, (iii) decide whether some given integers $a_1, \ldots, a_n$ can be split into two classes with equal sums, (iv) compute the maximum cardinality of an independent set in a graph, via the minimum of the quadratic form $x^T(I + A_G)x$ over the standard simplex (with A_G the graph adjacency matrix), (v) compute the maximum cardinality of a cut in a graph, via the minimization of the quadratic form $x^T L_G x$ over the hypercube $[-1, 1]^n$ (with L_G the graph Laplacian matrix).

A strategy in order to cope with this computational hardness is to compute *bounds* for the global minimum, that are obtained from tractable relaxations based on exploiting sums of squares and the dual theory of moments. This approach was started around 2000 with foundational works by Lasserre [3] and Parrilo [9]. Since then the field has seen a rapid growth. Here we will only sketch some of the main features of this approach and some extensions to more general settings (the general problem of moments and polynomial optimization in noncommutative variables). Several monographs and overviews are available that describe this general approach, see e.g. [4, 5, 6] and further references therein.

Bounds via sums of squares of polynomials. The starting point is the fact that, while checking nonnegativity of a degree $2d$ polynomial p is a computationally hard problem, one can decide whether p can be written as a sum of squares of polynomials efficiently, using semidefinite optimization. This is the well known *Gram matrix method*, which claims that p is a sum of squares if and only if one can find a positive semidefinite matrix X (indexed by monomials up to degree d) satisfying the polynomial identity: $p = \sum_{\alpha,\beta} x^{\alpha+\beta} X_{\alpha,\beta}$.

Let Σ denote the cone of sums of squares of polynomials and let $\mathcal{M}(g) = \sum_{j=0}^{m} g_j \Sigma$ denote the quadratic module generated by $g = (g_1, \ldots, g_m)$ (setting $g_0 = 1$). Given a degree bound $2t$ the truncated quadratic module $\mathcal{M}(g)_{2t}$ consists of the polynomials $\sum_j s_j g_j$ where $s_j \in \Sigma$ with $\deg(s_j g_j) \leq 2t$. Then one can define the parameter

$$f_{\mathrm{sos},t} = \sup\{\lambda : f - \lambda \in \mathcal{M}(g)_{2t}\},$$

which can be computed with semidefinite optimization. This gives a hierarchy of (monotonically nondecreasing) lower bounds for $f_{\min}$, that converge asymptotically to $f_{\min}$ when $\mathcal{M}(g)$ is Archimedean in view of the Positivstellensatz of Putinar [10].

Bounds via moments. We may reformulate $f_{\min}$ as

$$f_{\min} = \min\left\{\int_K f(x)\mathrm{d}\mu : \mu \text{ is a Borel probability measure supported by } K\right\},$$

$$= \min\{L(f) : L \in \mathbb{R}[x]^*, L(1) = 1, L \text{ has a representing measure supported by } K\}.$$

So we arrive at the (classical, hard) moment problem, asking to characterize which linear functionals on $\mathbb{R}[x]$ arise from measures. Nonnegativity on the quadratic module $\mathcal{M}(g)$ is a necessary condition, which leads to the following (moment) bounds:

$$f_{\mathrm{mom},t} = \inf\{L(f) : L \in \mathbb{R}[x]_{2t}^*, L(1) = 1, L \geq 0 \text{ on } \mathcal{M}(g)_{2t}\}.$$

Each bound $f_{\mathrm{mom},t}$ can be computed via a semidefinite program, the dual of the semidefinite program expressing $f_{\mathrm{sos},t}$, and we have $f_{\mathrm{sos},t} \leq f_{\mathrm{mom},t} \leq f_{\min}$, which gives asymptotic convergence to $f_{\min}$ (under the Archimedean condition). An additional remarkable feature of the moment bounds is that they may permit to also find global minimizers. For this let d_K denote the largest (rounded) half-degree of the polynomials g_j and assume $t \geq d_K$ and $2t \geq \deg(f)$. For $s \leq t$ $M_s(L)$ is the *moment matrix* of L, indexed by monomials up to degree s, with (α, β)–enry $L(x^\alpha x^\beta)$.

Theorem 1. *Assume L is an optimal solution of the program defining $f_{\mathrm{mom},t}$, which satisfies the following* flatness condition*:*

$$\text{rank } M_s(L) = \text{rank } M_{s-d_K}(L) \quad \textit{for some } d_K \leq s \leq t.$$

Then, (i) the bound is exact: $f_{\mathrm{mom},t} = f_{\min}$, (ii) the common roots to the polynomials in the kernel of $M_s(L)$ are global minimizers of f in K, (iii) and these are all *the global minimizers if the rank of $M_t(L)$ is maximum among all optimal solutions.*

In addition, under the flatness condition, the global minimizers can be found efficiently through some eigenvalue computations (by applying the well known eigenvalue method for computing roots in the finite variety case). These facts belong to the remarkable properties of the moment method, that fully exploit the algebraic structure of the problem. While flatness does not always hold (e.g., it can only hold if there are finitely many global minimizers), it does hold *generically* [8] and thus the hierarchy of bounds $(f_{\mathrm{mom},t})$ has generically finite convergence.

Extension to the general moment problem. The above approach applies more generally to any linear optimization problem over measures, of the form

$$\inf\left\{\int_K f_0 \mathrm{d}\mu : \int_K f_j \mathrm{d}\mu = b_j \ (j \in [m]), \mu \text{ Borel measure on } K\right\},$$

where K is a basic closed semialgebraic set as before. This setting is very general and captures, e.g., the minimization of polynomial and rational functions, finding cubature rules, applications to control, etc. Both sums-of-squares and moment bounds extend naturally to this setting, with asymptotic convergence under the Archimedean condition (and strict feasibility) and also the finite convergence result (under flatness) still applies. We refer to [4] for a broad exposition.

Extension to noncommutative polynomial optimization. Some applications to control and quantum information deal with polynomials instantiated at matrices (of any size $d \geq 1$) instead of just scalars. This can be modelled using the set $\mathbb{R}\langle x\rangle$ of polynomials in noncommutative variables $x_1, \ldots, x_n$, equipped with the involution $*$, which reverses the order of letters in words and with $x_i^* = x_i$.

The evaluation of a symmetric polynomial f at an n–tuple of symmetric matrices returns a symmetric matrix. Then one may ask to minimize the quantity $v^T f(X)v$ (leading to *eigenvalue optimization*), or the (normalized) trace $\mathrm{Tr}(f(X))/d$ (leading to *tracial polynomial optimization*), where $d \in \mathbb{N}$, $X \in (\mathcal{S}^d)^n$, $v \in \mathbb{R}^d$ is a unit vector, and X may be constrained to satisfy constraints of the form $g_j(X) \succeq 0$ (for symmetric g_j). The moment bounds admit natural extensions (considering now linear functionals on $\mathbb{R}\langle x\rangle_{2t}$ and sums of Hermitian squares). However, as the matrix size d is variable too, the asymptotic convergence will be to the infinite dimensional analogues, where allowing X to be bounded operators on a separable Hilbert space. In the tracial case, there are tight links to the (recently disproved) Connes' embedding conjecture. For details over noncommutative polynomial optimization and applications we refer, e.g., to [1, 7, 2] and references therein.

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A new approach to the infinite-dimensional moment problem

Konrad Schmüdgen

Let A be a (not necessarily finitely generated)commutative unital real algebra. In a very general version, the infinite-dimensional moment problem asks when a linear functional on A can be written as an integral over characters of A. We present a new approach to this problem based on cylinder measures rather than measures. This approach was proposed in the author's paper Arkiv Mat. **56** (2018), 441–459.

To be more precise, let T denote the linear span of a set of algebra generators of A. We equip T with a locally convex topology and denote the corresponding locally convex space by T. Then a linear functional L on A is called a moment functional if L can be represented by an integral over a continuous cylinder measure μ on T' such that μ is concentrated on the restrictions of character of A to T. The guiding example is $A = \mathbb{R}[x_1, ..., x_d], T = \mathbb{R}^d$, equipped with the Euclidean topology.

Our main results provide such integral representations for A_+–positive linear functionals (generalized Haviland theorem) and for positive functionals fulfilling Carleman conditions. As an application we treat the moment problem for the symmetric algebra $S(V)$ of a real vector space V. If appropriate continuity assumptions concerning some nuclear topology are added, then Minlos theorem implies that we can represent L as an integral over some measure. The continuity in the Sazonov topology characterizes when a cylinder measure extends to a measure. Important and deep results of Borchers and Yngvason (1975) and by Berezansky and Sifrin (1971) are discussed.

An interesting example are Gaussian cylinder measures with respect to some continuous scalar product on a Hilbert space V. Such a scalar product can be represented by a positive self-adjoint operator, say b, on V with trivial kernel. Then the corresponding Gaussian cylinder measure yields a Borel measure if and only if b is trace class. Thus, if we choose b to be not of trace class, then we obtain a moment sequence on $S(V)$ which has a unique representing continuous cylinder measure, but no representing measure.

A Jordan decomposition theorem for noncommutative kernels

Joseph A. Ball

(joint work with Gregory Marx and Victor Vinnikov)

A *completely positive (cp) kernel* on a point set Ω with values in $\mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$ (bounded linear operators from the C^*–algebra $\mathcal{A}$ to bounded linear operators on the Hilbert space $\mathcal{Y}$) in the sense of Barreto-Bhat-Liebscher-Skeide (see [5]) is a function $k\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))$ which satisfies the cp condition:

$$\sum_{i,j=1}^{N} y_i^* k(\omega_i, \omega_j)(a_i^* a_j) y_j \geq 0$$

for all ω_i's in Ω, a_i's in $\mathcal{A}$, y_i's in $\mathcal{Y}$ for $i = 1, \ldots, N$, $N = 1, 2, \ldots$. A *free noncommutative (nc) cp kernel* is a quantized version of the BBLS cp kernel, whereby one allows the point set to include matrices over the level-1 set of points Ω and demands that the kernel function respect direct sums and similarities via complex matrices for the matrix-point arguments in a natural way (see [3]). Here the level-1 points are assumed to come from a vector space (i.e., a bimodule over $\mathbb{C}$), so that multiplication on the left or on the right by a scalar matrix acting on a matrix over the level-1 points makes sense as long as the sizes are compatible (see [9] for a comprehensive axiomatic treatment of free noncommutative functions and kernels). Such kernels play the role of the classical Pick matrix in the solution of the free nc version of the Nevanlinna-Pick interpolation problem as presented in [4]. When one drops the cp condition on such a kernel K, one is led to the notion of a *free nc kernel.* We expect the following *Jordan decomposition theorem* for free nc kernels to hold: *any such nc kernel over a finite point set* Ω *such that each value* $K(Z,W)$ *is completely bounded* (for each $Z, W \in \Omega$) *can be written as a four-fold linear combination of cp nc kernels.* Let us note that special cases of this result have already appeared in the literature: in the case where $\Omega = \{\omega_0\}$, the result is due to Wittstock [13]. In case all points are at level 1,the result (with appropriate hypotheses) is due to Bhatacharyya-Dritschel-Todd [6]; let us note that some hypotheses are necessary for such a result, due to the counterexample discussed in [1] for the case where $\mathcal{A} = \mathbb{C}$ and all points are at level 1.

The proof is more involved than what one might expect from the statement. The idea behind the proof is as follows. Suppose that we are given a kernel K on a finite noncommutative point set $\Omega = \{Z_1, \dots, Z_d\}$ and a function $K\colon \Omega \times \Omega \to \mathcal{L}(\mathcal{A}, \mathcal{L}(\mathcal{Y}))_{\rm nc}$. Set $Z^{(0)} = \bigoplus_{i=1}^N Z_i$. If each point Z_i say has size $n_i \times n_i$, we then see that $Z^{(0)}$ has size $N_0 \times N_0$ where $N_0 = \sum_{i=1}^N n_i$. Let us set ϕ equal to the linear map from $\mathbb{A} := \mathcal{A}^{N_0 \times N_0}$ to $\mathbb{L} := \mathcal{L}(\mathcal{Y})^{N_0 \times N_0}$ given by $\phi = K(Z^{(0)}, Z^{(0)})$. Then, by an idea already appearing in [10], the property of K being a noncommutative kernel can be encoded as a property of the map ϕ: ϕ must be an $(\mathcal{S}, \mathcal{S}^*)$–bimodule map from $\mathbb{A}$ to $\mathbb{L}$, where $\mathcal{S}$ is the subalgebra of $\mathbb{C}^{N_0 \times N_0}$ consisting of all complex matrices α which intertwine the point $Z^{(0)}$ with itself: $\alpha Z^{(0)} = Z^{(0)} \alpha$. Let us introduce the notation

$$\mathcal{D} = C^*(\mathcal{S}) \tag{1}$$

for the C^*–algebra generated by $\mathcal{S}$ inside $\mathbb{C}^{N_0 \times N_0}$. Then we may consider the operator space $\mathbb{S} = \left[\begin{smallmatrix} \mathcal{D} \otimes 1_{\mathbb{A}} & \mathbb{A} \\ \mathbb{A}^* & \mathcal{D} \otimes 1_{\mathbb{A}} \end{smallmatrix}\right]$ and the map $\Phi\colon \mathbb{S} \to \mathbb{L}$ given by

$$\Phi\colon \begin{bmatrix} \alpha 1_{\mathbb{A}} & P_1 \\ P_2^* & \beta 1_{\mathbb{A}} \end{bmatrix} \mapsto \begin{bmatrix} \alpha 1_{\mathbb{L}} & \phi(P_1) \\ \phi(P_2)^* & \beta 1_{\mathbb{L}} \end{bmatrix} \tag{2}$$

From the fact that ϕ is an $(\mathcal{S}, \mathcal{S}^*)$–bimodule map, it is easily seen that Φ is also a $(\mathcal{S}, \mathcal{S}^*)$–bimodule map. Once we show that Φ is also cp, by results of Arveson [2] it is known that Φ can be extended to a map $\widetilde{\Phi}\colon \mathbb{A}^{2\times 2} \to \mathbb{L}^{2 \times 2}$ which also is a $(\mathcal{S}, \mathcal{S}^*)$–bimodule map. Unwinding how the $(\mathcal{S}, \mathcal{S}^*)$–property for a map Φ encodes the fact that Φ has the form $\Phi = K(Z^{(0)}, Z^{(0)})$ for a nc kernel on Ω, we see that there is a nc kernel $\mathbb{K}$ on Ω so that $\Phi = \mathbb{K}(Z^{(0)}, Z^{(0)})$. Furthermore, as by construction Φ is also known to be cp, it follows that $\mathbb{K}$ is in fact a cp nc kernel. Furthermore one can arrange that $\mathbb{K}$ has the form

$$\mathbb{K}(Z,W)\left(\left[\begin{smallmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{smallmatrix}\right]\right) = \begin{bmatrix} \mathbb{K}_{11}(Z,W)(P_{11}) & K(Z,W)(P_{12}) \\ \phi(Z,W)(P_{21}^*)^* & \mathbb{K}_{22}(Z,W)(P_{22}) \end{bmatrix}$$

As K appears as the $(1,2)$–corner in a 2×2–block kernel which is cp, by adapting [6, Lemma 3.2] to the quantized setting here, it follows that indeed K can be decomposed as $K = K_1 - K_2 + i(K_3 - K_4)$ where each of K_1, K_2, K_3, K_4 is a cp nc kernel.

The only remaining gap is to verify that the map Φ given by (2) is cp. If it were the case that ϕ were not only $(\mathcal{S}, \mathcal{S}^*)$–bimodule map but even a $\mathcal{D}$–bimodule map (where $\mathcal{D}$ is the C^*–algebra as in (1)), then one would be able to proceed by using the $\mathcal{D}$–bimodule structure to follow the proof of Lemma 8.1 in [12] to obtain the result. Even though ϕ itself in general is not a $\mathcal{D}$–bimodule map, by the results of Blecher [7], if we choose $\mathcal{C}$ to be any C^*–algebra which is generated by a completely isometric copy of $\mathcal{S}$ inside it, then one can embed the C^*–algebras $\mathbb{A}$ and $\mathbb{L}$ completely contractively into their $\mathcal{C}$–dilations

$$\mathcal{C} \otimes_{h\mathcal{S}} \mathbb{A} \otimes_{h\mathcal{S}^*} \mathcal{C}, \quad \mathcal{C} \otimes_{h\mathcal{S}} \mathbb{L} \otimes_{h\mathcal{S}^*} \mathcal{C}$$

respectively (where $\otimes_{h\mathcal{S}}$ indicates the Haagerup tensor product balanced over $\mathcal{C}$ as discussed in [8]) so that $\mathcal{C} \otimes_{h\mathcal{S}} \mathbb{A} \otimes_{h\mathcal{S}^*} \otimes \mathcal{C}$ is a $\mathcal{C}$–bimodule with module action which extends the left/right bimodule action of $(\mathcal{S}, \mathcal{S}^*)$. The particular case where $\mathcal{C}$ is the C^*–envelope of $\mathcal{S}$ is done in the work of Muhly-Na [11]. The additional observation in [7] is that by choosing $\mathcal{C}$ to be the maximal C^*–algebra $\mathcal{C}_{\max} := C^*_{\max}(\mathcal{S})$ containing $\mathcal{S}$ (so a copy of any other C^*–algebra $\mathcal{C}$ containing $\mathcal{S}$ is contained in $\mathcal{C}_{\max}$ completely isometrically), it happens that the embedding of $\mathbb{A}$ and $\mathbb{S}$ into

$$\widehat{\mathbb{A}} := \mathcal{C}_{\max} \otimes_{h\mathcal{S}} \mathbb{A} \otimes_{h\mathcal{S}^*} \mathcal{C}_{\max}, \quad \widehat{\mathbb{L}} := \mathcal{C}_{\max} \otimes_{h\mathcal{S}} \mathbb{L} \otimes_{h\mathcal{S}^*} \mathcal{C}_{\max},$$

respectively, are actually completely isometric, and the induced mapping

$$\widehat{\phi} = 1_{\mathcal{C}_{\max}} \otimes \phi \otimes 1_{\mathcal{C}_{\max}} \colon \widehat{\mathbb{A}} \to \widehat{\mathbb{L}}$$

is a $\mathcal{C}_{\max}$–bimodule map which extends (after some completely isometric identifications) the $(\mathcal{S}, \mathcal{S}^*)$–bimodule map ϕ (with the $(\mathcal{S}, \mathcal{S}^*)$–bimodule action extended to the $\mathcal{C}_{\max}$–bimodule action). We then take $\widehat{\mathbb{S}}$ to be the operator system $\widehat{\mathbb{S}} = \begin{bmatrix} \mathcal{C}_{\max}\cdot 1_{\mathbb{A}} & \widehat{\mathbb{A}} \\ \widehat{\mathbb{A}}^* & \mathcal{C}_{\max}\cdot 1_{\mathbb{A}} \end{bmatrix}$ and consider the map $\widehat{\Phi} \colon \widehat{\mathbb{S}} \to \widehat{\mathbb{L}}^{2\times 2}$ given by

$$\widehat{\Phi} \colon \begin{bmatrix} \widehat{\alpha}\cdot 1_{\mathbb{A}} & \widehat{P}_1 \\ \widehat{P}_2 & \widehat{\beta}\cdot 1_{\mathbb{A}} \end{bmatrix} \mapsto \begin{bmatrix} \widehat{\alpha}\cdot 1_{\mathbb{L}} & \widehat{\phi}(P_1) \\ \widehat{\phi}(P_2^*)^* & \widehat{\beta}\cdot 1_{\mathbb{L}} \end{bmatrix}. \tag{3}$$

One can then follow the argument sketched above for the case where ϕ was assumed to be a $\mathcal{D}$–bimodule map (using here instead the property that $\widehat{\phi}$ is a $\mathcal{C}_{\max}$–bimodule map) to see that $\widehat{\Phi}$ is completely positive. But we recover Φ as the restriction of $\widehat{\Phi}$ to $\mathbb{S} \subset \widehat{\mathbb{S}}$; hence we finally arrive at Φ having the cp property as required.

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Cones of locally non-negative polynomials

Christoph Schulze

We consider the ring of polynomials $\mathbb{R}[\underline{x}] := \mathbb{R}[x_1, \ldots, x_n]$ and the subset $\mathcal{P}^{Loc}$ of polynomials f for which there exists some neighbourhood of $0 \in \mathbb{R}^n$ (which may depend on f) where f takes only non-negative values. This means, we have either $f(0) > 0$ or f has a local minimum at $0 \in \mathbb{R}^n$ with $f(0) = 0$. Therefore, $\mathcal{P}^{Loc}$ is strongly connected to the study of local minima (with value 0). One may easily see that $\mathcal{P}^{Loc}$ is a convex cone. The inclusion of two faces of this cone may be interpreted geometrically: the smaller face is represented by elements in its relative algebraic interior which (up to scalars) approach the value 0 at least as fast as respective elements of the larger face (in some neighbourhood of $0 \in \mathbb{R}^n$). Replacing $\mathbb{R}[\underline{x}]$ by the ring of convergent power series $\mathbb{R}\{\underline{x}\} := \mathbb{R}\{x_1, \ldots, x_n\}$, we obtain an analytic version $\mathcal{P}^{Loc}_{ana}$. One may also define a corresponding convex cone $\mathbb{R}[[\underline{x}]]_+$ in the ring of formal power series $\mathbb{R}[[\underline{x}]] := \mathbb{R}[[x_1, \ldots, x_n]]$ as the intersection of all positive cones (orderings) of $\mathbb{R}[[\underline{x}]]$. The inclusions $\mathcal{P}^{Loc} \subseteq \mathcal{P}^{Loc}_{ana} \subseteq \mathbb{R}[[\underline{x}]]_+$ induce bijections between the sets of faces of finite codimension of these three cones. This shows that the study of the most frequent minima does not depend on the considered rings.

The most frequent minima were classified up to stable equivalence (especially local analytic coordinate changes) in [1]. We study the analogous (weaker) classification for faces of the mentioned cones but we consider additionally the inclusion structure of the faces. In the following, we say that a face belongs to layer $i+1$ if it is a maximal proper face of a face of layer i - in layer 0, the only non-proper layer, there is only the whole cone itself. For $n = 2$ we give a complete classification of the first 8 proper layers (up to codimension 27). In the case $n = 3$ we classify the first 7 proper layers. Furthermore, for $n = 2$ a coarser but complete classification of faces of finite codimension is possible via “real Enriques diagrams” (this uses blowing up). These classifications were visualized in diagrams.

We apply the classification to the question arising from Scheiderer's local-global principle if a given $f \in \mathcal{P}^{Loc}$ is a sum of squares in $\mathbb{R}[[\underline{x}]]$. Using results from [2] and some calculations, we show for all faces which were classified in the case $n = 3$ that all elements in their relative algebraic interiors are sums of squares in $\mathbb{R}[[\underline{x}]]$. We mention two examples of local data assigned to polynomials $f \in \mathcal{P}^{Loc}$ with $f(0) = 0$ which are invariants under replacing the polynomial by another polynomial representing the same face of the local cone. This shows that problems on local minima (with value 0) are often problems on faces of the local cones.

The study of the mentioned local cones originated from the study of faces of cones of globally non-negative homogeneous polynomials and there are strong connections. Also, the convex structure of the local cones is very interesting on its own terms and related to singularity theory. Furthermore, they may be considered in the more general setting of polynomials which are non-negative on the intersection of some neighbourhood of $0 \in \mathbb{R}^n$ with some given semi-algebraic set (having 0 on its boundary).

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Some applications of free probability to non-commutative algebra theory

Sheng Yin

(joint work with Tobias Mai, Roland Speicher and Thomas Schick)

A basic idea to study the polynomials ring $\mathbb{C}[x_1, \dots, x_d]$ of commutating variables $x_1, \dots, x_d$ is evaluating polynomials at points in $\mathbb{C}^d$. Namely, we may regard each variable x_i as the i–th coordinate function $X_i \colon \mathbb{C}^d \to \mathbb{C}$. However, when it comes to the non-commutative polynomial ring $\mathbb{C}\langle x_1, \dots, x_d \rangle$, this idea becomes more complicated. For example, to distinguish non-commutative polynomials, we need their evaluations at tuples of $n \times n$ matrices over $\mathbb{C}$ for every integer n. This tells us that realizing $\mathbb{C}\langle x_1, \dots, x_d \rangle$ by "non-commutative coordinate functions" is not a trivial task any more. Recent results [6, 3, 4] from free probability theory suggest that we may also regard non-commuting indeterminates $x_1, \dots, x_d$ as non-commutative random random variables. Because it was showed that many non-commutative random variables (for example, freely independent semicircular random variables) generate the algebra of non-commutative polynomials. Moreover, such results was pushed further to the case of free field (aka the universal skew field consisting of non-commutative rational functions) in [5]. Namely, non-commutative random variables $X_1, \dots, X_d$ satisfying certain regularity property actually generate the free field.

Along the way of identifying non-commutative polynomials and rational functions as random variables, we also found that several other algebraic notions can be identified with their counterparts in free probability theory. Such identifications turn out to be mutually beneficial: they allow us to convert a probabilistic/analytic problem to an algebraic one and vice verse. In the following we we will present two purely algebraic theorem that can be proved in an analytic or probabilistic way with the help of these identifications. The first result reads as follows.

Theorem 1. *(*[1, Proposition 8.4.1]*) Any $n \times n$ matrix over the free field has at most n central eigenvalues.*

Here a complex number λ is called a *central eigenvalue* of a matrix A over the free field if $\lambda - A$ is not invertible. Let us consider an $n \times n$ matrix A over the free field of d variables $x_1, \dots, x_d$. If there exists any point $X \in \mathbb{C}^d$ such that $A(X)$ is well-defined, i.e., X lies in the domain of every entry of A, then we see that each central eigenvalue of A has to be an eigenvalue of $A(X) \in M_n(\mathbb{C})$. In such a case, clearly A has at most n central eigenvalues. However, since A may have an entry like $(x_1x_2 - x_2x_1)^{-1}$ that has no well-defined evaluations for any point $X \in \mathbb{C}^d$, we may need to plug in matrices of size larger than 1 in order to find the central eigenvalues of A. Therefore it is probably not very clear why in general a matrix A over the free field always has at most n central eigenvalues. But now we can convert this algebraic question on A to a question on the random variable $A(X)$, where we can take X as a tuple of freely independent semicircular random variables. (The choice of random variables here is not really important as long as they satisfy some regularity property.) Then our question amounts to ask what is the cardinality of the point spectrum of an operator $A(X)$ or how many atoms does the (probability) distribution $\mu_{A(X)}$ of $A(X)$ have when $A(X)$ is a normal operator. In the case that $A(X)$ is normal, the reason that the number of atoms of $\mu_{A(X)}$ is at most n becomes very intuitive: it is because $\mu_{A(X)}$ is a probability measure as well as each atom of $\mu_{A(X)}$ has at least mass $1/n$ (due to a rank equality between A and $A(X)$). For the general case that $A(X)$ may not be normal, it requires a bit more technical work but the reason more or less stays the same. See [5, Proposition 5.17] for the details for the above argument.

The second result concerns linear matrices over $\mathbb{C}\langle x_1, \dots, x_d\rangle$, i.e., matrices over non-commutative polynomials that have at most degree 1. Before we state this theorem, let us recall that an $n \times n$ matrix A over $\mathbb{C}\langle x_1, \dots, x_d\rangle$ is called *full* if there is no matrix factorization $A = PQ$ with column number of P is less than n.

Theorem 2 ([2, Corollary 6.3.6]). *Let A be an $n \times n$ linear matrix over the algebra $\mathbb{C}\langle x_1, \dots, x_d\rangle$. If A is not full, then there exist invertible matrices U and V over $\mathbb{C}$ such that*

$$UAV = \begin{pmatrix} B & 0 \\ C & D \end{pmatrix},$$

where the zero block in the right hand side has size of $r \times s$ with $r + s > n$.

An $n \times n$ matrix A that has an $r \times s$ zero block with $r + s > n$ is called *hollow*. It is not difficult to see that a hollow matrix cannot be full. So the above theorem actually tells us that the converse of this fact holds for linear matrices up to scalar-valued invertible matrices. Namely, linear non-full matrices can always be turned into a hollow structure. However, to construct these scalar-valued matrices U and V seems to be quite non-trivial (at least in [2] their construction relies on several other involved structure theorems on matrices over polynomials). Now, again let us consider the operator $A(X)$ where X is a tuple of freely independent semicircular random variables. Then the non-fullness of A implies that $A(X)$ has a non-trivial kernel, which is a non-trivial subspace of some Hilbert space H^n that $A(X)$ acts on. This means that there exists some non-zero vector e in H^n such that $A(X)e = 0$. Moreover, we know that there exists also a non-zero vector f in the kernel of $A(X)^*$, i.e., $A(X)^*f = 0$ (since $A(X)$ lives in a finite von Neumann algebra). Then we can construct invertible matrices U and V out of e and f such that UAV has to be hollow. Actually, we found, in a recent joint work with Thomas Schick (in progress), that U and V can be taken as the matrices that erase the linearly dependent entries of e and f. Since these vectors e and f come from the kernels of operators, we see again that our new proof is in some kind of analytic/operator-algebraic way.

In summary, we found that those interactions between the non-commutative algebra theory and free probability theory are quite interesting and worth further investigation. In particular, we are looking forward to more questions like above that can be converted to one side to the other side.

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Cones in and around the sums of squares cone

Charu Goel

In this note, we first introduce cones of sums of squares of k–term forms, which are more restrictive than the sums of squares (sos) cone. We consider the dual of these cones and show how these cones relate to each other. Finally, we describe a nice filtration of intermediate cones of forms (between the sos cone and the cone of positive semidefinite forms) characterised by the existence of quadratic forms which are nonnegative on a given semi algebraic subset of $\mathbb{R}^{N_0}$.

Define $F_{n,2d}$ as the vector space of forms of degree $2d$ in n variables called n–ary $2d$–ic forms. Let $\mathcal{P}_{n,2d}$ and $\Sigma_{n,2d}$ denote the cone of psd and sos n–ary $2d$–ic forms.

Sum of squares of k–term forms. Define $F^k_{n,2d}$ as the subset of $F_{n,2d}$ with at most k terms. A form f is a *sum of binomial squares (sobs)* if it is a sum of squares of the form $(a\underline{x}^{\underline{\alpha}} - b\underline{x}^{\underline{\beta}})^2$, where $\underline{\alpha}, \underline{\beta} \in \mathbb{N}^n; a, b \in \mathbb{R}$. Even symmetric sextics were studied in [3] and necessary and sufficient conditions were given for such a form to be psd, to be sos and to be sobs. Moreover, coefficient tests for sobs were recently given in [4] and using them similar conditions were deduced in [5] for the special case of even symmetric forms. In a joint work with Bruce Reznick [6], we generalize the restriction on summands to be a sum of at most k terms. For each $(n, 2d)$ and integer $k \geq 1$, we define

$$\sum\nolimits^k_{n,2d} := \text{the set of sums of squares of } n\text{–ary } d\text{–ic forms with at most } k \text{ terms.}$$

Clearly $\sum^k_{n,2d}$ is a convex cone. We prove in [6] that it is in fact a *closed* convex cone. Let $N(n,d) = \binom{n+d-1}{n-1}$ denote the dimension of $F_{n,d}$, so that we have $\sum^{N(n,d)}_{n,2d} = \sum_{n,2d}$. We wish to show that for $k < N(n,d)$, we have $\sum^k_{n,2d} \subsetneq \sum^{k+1}_{n,2d}$.

For $k = 1$, this is simple: $\sum^1_{n,2d}$ is the sum of squares of monomials, and so consists of even forms, all of whose coefficients are non-negative. For $k = 2$: $\sum^2_{n,2d}$ is the cone of sobs; evidently, $(x_1^d - x_2^d)^2 \in \sum^2_{n,2d} \setminus \sum^1_{n,2d}$.

More generally, if we can find an indefinite irreducible n–ary d–ic form h with exactly k terms, then for any representation $h^2 = \sum_j h_j^2$, we have $h_j = \alpha_j h$, so $h^2 \in \sum^k_{n,2d} \setminus \sum^{k-1}_{n,2d}$. For example, for $n = 2$, given d, given $k \leq N(2,d) = d+1$, we have $P_{k,d}(x,y) = x^{2d+2-2k}(x+y)^{2k-2} \in \sum^k_{2,2d} \setminus \sum^{k-1}_{2,2d}$.

A general principle called *perturbation* can be used to show that there are irreducible forms with any number of terms, i.e., if we have a single form with at least two terms and no repeated factor, we can perturbate it to get higher ones and hence we are done. For example, for $n = 3$, $k = 3$, given d, we have $h^2 = ({x_1}^d + {x_2}^d - {x_3}^d)^2 \in \sum^3_{3,2d} \setminus \sum^2_{3,2d}$ and this h can be perturbated to irreducible forms with $k \geq 4$ terms.

Using the work in [9], we describe in [6] the dual cone to $\sum^k_{n,2d}$ in a particularly simple way. This requires first some notation. For integers $r \leq N$, given a form $g(x_1, \ldots, x_N)$ in N variables, we may define $\binom{N}{r}$ forms in r variables $u_1, \ldots, u_r$, by choosing $1 \leq i_1 < i_2 \cdots < i_{r-1} < i_r \leq N$, and then setting $x_{i_j} = u_j$ and $x_\ell = 0$ otherwise. Let us say that a quadratic form $q(t_1, \ldots t_N)$ is *r–psd* provided each of these $\binom{N}{r}$ quadratic forms is psd. Now, associated to $p \in F_{n,2d}$, there is a quadratic form H_p in $N(n,d)$ variables, and $(\sum_{n,2d})^* = \{p : H_p \text{ is psd}\}$; (see [9, pp. 40–41]). We prove in [6] that $(\sum^k_{n,2d})^* = \{p : H_p \text{ is } k\text{–psd}\}$. It is easy to see that there exist quadratic forms which are r–psd and not $(r+1)$–psd, but unfortunately not every quadratic form is H_p for some p. If for every k there exists p_k such that H_{p_k}

is k–psd and not $(k+1)$–psd, then $p_k \in \left(\sum_{n,2d}^{k}\right)^* \setminus \left(\sum_{n,2d}^{k+1}\right)^*$. This will give us $\left(\sum_{n,2d}^{k+1}\right)^* \subsetneq \left(\sum_{n,2d}^{k}\right)^*$, which will imply $\sum_{n,2d}^{k} \subsetneq \sum_{n,2d}^{k+1}$.

This work could be useful in deriving new results on certificates of nonnegativity of forms. For instance, a recent work by Gouveia-Kovačec-Saee [7] that has been brought to our notice after this talk at MFO. To conclude, we would like to emphasize that noting the facts that $\sum_{n,2d}^{1}$ corresponds to *linear programming*, $\sum_{n,2d}^{2}$ corresponds to *second-order cone programming* (see [1]) and $\sum_{n,2d}^{N(n,d)} = \sum_{n,2d}$ corresponds to *semidefinite programming*, it would be interesting to investigate "what the cones $\sum_{n,2d}^{k}$ for $3 \leq k < N(n,d)$ would correspond to?".

Intermediate cones of forms between the sos and psd cones. Given a form $f \in F_{n,2d}$ and a corresponding Gram matrix $G \in \mathrm{Sym}_{N_0 \times N_0}(\mathbb{R})$, associate to it a quadratic form $q_G \in \mathbb{R}[x_1, \ldots, x_{N_0}]$ defined by $q_G(u_1, \ldots, u_{N_0}) := (u_1, \ldots, u_{N_0})\ G\ (u_1, \ldots, u_{N_0})^T$, where $N_0 = N(n,d)$. In [5], we explored the fact that sos and psd forms are characterised by the existence of quadratic forms q_G that are nonnegative on $\mathbb{R}^{N_0}$ and the Veronese variety $\nu_d(\mathbb{R}^n)$ respectively. This motivated Salma Kuhlmann and myself to describe a filtration of intermediate cones of forms (between the sos and the psd cones) characterised by the existence of quadratic forms which are nonnegative on a given semi algebraic subset of $\mathbb{R}^{N_0}$. This led us to propose *a generalization of Hilbert's 1888 theorem* [8] *along the varieties containing the Veronese variety* $\nu_d(\mathbb{R}^n)$. A similar approach has been taken up by Blekherman-Smith-Velasco, and relates to their recent work in [2], in which they produced a complete list of varieties for which nonnegative quadratic forms are sos. Further, since the Veronese variety is described by finitely many quadratic forms (see for instance [5, Lemma 2.37]), we reduced our problem to *checking nonnegativity of quadratic forms on a variety defined by finitely many quadratic forms.* That is, "if $\nu_d(\mathbb{R}^n)$ is defined by v many quadratic forms say $\{q_1, \ldots, q_v\}$, can one separate the $\exists$*-cones* $C_i := C_i(q_1, \ldots, q_i) := \{f \in \mathcal{F}_{n,2d} : \exists\ q_G$ nonnegative on the variety $\mathbb{K}_i$ defined by $\{q_1, \ldots, q_i\}\}$?". Using a result from [2], we recently deduced that strict inclusion of the first two cones in the row holds, i.e., $\Sigma_{n,2d} = C_1 \subsetneq C_2$.

The following question also arises naturally: "Let S be a subset of $\mathbb{R}^{N_0}$ and q a quadratic form nonnegative on S. When can we find another quadratic form q' such that $q = q'$ on S and q' is psd?". When $S = \nu_d(\mathbb{R}^n)$, an answer to this question would analyze the situation when a psd form $f \in \mathcal{F}_{n,2d}$ has a Gram matrix G necessarily nonnegative on $\nu_d(\mathbb{R}^n)$ and another Gram matrix G' which is psd, revealing that f is a sos. This work is in progress and we believe that these ideas are particularly exciting and promising.

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LP and SDP bounds for packing and energy minimization

Frank Vallentin

In this introductory lecture the starting point is the polynomial optimization formulation of the independence number of a finite graph $G = (V, E)$:

$$\alpha(G) = \max\left\{\sum_{i \in V} x_i : x_i \geq 0, x_i^2 - x_i = 0, x_i x_j = 0 \text{ if } ij \in E\right\}.$$

This formulation allows to apply moment relaxation techniques in order to derive a complete semidefinite programming (SDP) proof system for the independence number. Following Laurent (2003), such an SDP proof system can be given by the *t–th step of Lasserre's hierarchy*:

$$\mathrm{las}_t(G) = \max\left\{\sum_{x \in V} y_{\{x\}} : y \in \mathbb{R}^{I_{2t}}_{\geq 0},\ y_\emptyset = 1,\ M_t(y) \text{ is positive semidefinite}\right\},$$

where I_t is the set of all independent sets with at most t elements and where $M_t(y) \in \mathbb{R}^{I_t \times I_t}$ is the moment matrix defined by the vector y: Its (J, J')–entry equals

$$(M_t(y))_{J,J'} = \begin{cases} y_{J \cup J'} & \text{if } J \cup J' \in I_{2t}, \\ 0 & \text{otherwise.} \end{cases}$$

The first step in Lasserre's hierarchy coincides with the ϑ'–number, the strengthened version of Lovász ϑ–number due to Schrijver (1979). Furthermore, the SDP proof system is complete, in the following sense:

$$\vartheta'(G) = \mathrm{las}_1(G) \geq \mathrm{las}_2(G) \geq \ldots \geq \mathrm{las}_{\alpha(G)}(G) = \alpha(G).$$

This SDP proof system can be generalized to infinite topological packing graphs, where the vertex set is a Hausdorff topological space and each finite clique is contained in an open clique. An *open clique* is an open subset of the vertex set where every two vertices are adjacent. Then the *t–th step of the generalized hierarchy* of a topological packing graph is

$$\mathrm{las}_t(G) = \sup \Big\{ \lambda(I_{=1}) : \lambda \in \mathcal{M}(I_{2t})_{\geq 0},\ \lambda(\{\emptyset\}) = 1,\ A_t^* \lambda \in \mathcal{M}(I_t \times I_t)_{\succeq 0} \Big\},$$

where $\mathcal{M}(I_{2t})_{\geq 0}$ denotes the cone of positive Radon measures on I_{2t}, and where condition $A_t^* \lambda \in \mathcal{M}(I_t \times I_t)_{\succeq 0}$ says that measure λ satisfies a moment condition, see de Laat, Vallentin (2015) for the technical details.

Evaluating the first steps of the hierarchy allows for computing upper bounds of geometric packing problems like the kissing number or determining the maximal density of translative packings of convex bodies in n–dimensional Euclidean spaces. For instance, the latter can be modeled by considering independent sets in the infinite Cayley graph Cayley$(\mathbb{R}^n, \mathcal{K}^\circ)$ where the vertex set is $\mathbb{R}^n$ and where two vertices x, y are adjacent whenever $x - y$ lies in the interior of the convex body $\mathcal{K}$. Then independent sets in Cayley$(\mathbb{R}^n, \mathcal{K}^\circ)$ correspond to translative packings of the scaled body $\frac{1}{2}\mathcal{K}$.

Computing the first step $\mathrm{las}_1(\mathrm{Cayley}(\mathbb{R}^n, \mathcal{K}^\circ))$ is equivalent (after symmetrization and dualization) to the linear programming (LP) bound of Cohn, Elkies (2003):

$$\begin{aligned}\delta^t(\mathcal{K}) \leq \inf \big\{ & f(0) : f \in L^1(\mathbb{R}^n) \text{ continuous}, \widehat{f}(0) \geq \mathrm{vol}\, \mathcal{K}, \\ & \widehat{f}(u) \geq 0\ \forall u \in \mathbb{R}^n \setminus \{0\}, f(x) \leq 0\ \forall x \notin (\mathcal{K}^\circ - \mathcal{K}^\circ) \big\}\end{aligned}$$

where $\delta^t(\mathcal{K})$ is the largest possible density of a translative packing of $\mathcal{K}$. In the general setting, one has (after symmetrization of the convex optimization problem $\mathrm{las}_t(G)$) to characterize the cone of positive type functions explicitly for which one can use classical results from abstract harmonic analysis which identify the extreme rays of the cone of positive type functions with irreducible unitary representations. In the setting above Bochner's theorem is used which expresses positive type functions f via the nonnegativity of its Fourier transform $\widehat{f}$.

In some cases already the first step gives exact results. Originally, Cohn, Elkies (2003) used the LP bound for a "numerical solution" of the sphere packing problem in dimensions 8 and 24. Viazovska (2017) made a breakthrough when she constructed the optimal f (the magic function, it is related to interpolation formulas of radial Schwartz functions and to solutions of functional equations using the theory of modular forms) to solve the 8–dimensional case. One week later, Cohn, Kumar, Miller, Radchenko, Viazovska (2017) settled the 24–dimensional case.

de Laat, Oliveira, Vallentin (2014) applied the LP bound (using SDP) to packings of spheres of several radii. Dostert, Guzman, Oliveira, Vallentin (2017) found new upper bounds for translative tetrahedra packings. Thereby they characterized sum of squares which are invariant under reflection groups.

In most cases the first step does not give exact results and one can improve upon the first step by going to higher steps. This was pioneered by Schrijver (2005) in the framework of error correcting codes over binary alphabets. He was able to strengthen ϑ' by considering obstructions between triples of points, a 3–point bound. This 3–point bound lie between the first step las_1 and the second step las_2. Bachoc, Vallentin (2008) transferred Schrijver's approach to the unit sphere S^{n-1} and found many new upper bounds for kissing numbers. This 3-point bound was useful in several occasions: Bachoc, Vallentin (2009) proved the optimality of a 10 point spherical code on S^3. Cohn, Woo (2012) showed the universal optimality of a 7 point configuration in $\mathbb{R}P^2$. de Laat (2019) determined a 4–point bound for "numerically solving" the energy minimization problem of 5 particles on S^2. de Laat, Machado, Oliveira, Vallentin (2018) computed 6–point bounds for equiangular lines. Dostert, de Laat, Moustrou (2020) showed the optimality of a 183 point code on the S^7–hemisphere.

LP and SDP bounds can also be used in the context of energy minimization problems. Cohn, Kumar (2007) determined point configurations on the unit sphere S^{n-1} which minimize potential energy for all completely monotonic potential functions via LP bounds. Very recently, Cohn, Kumar, Miller, Rachenko, Viazovska (2019) were able to prove corresponding results for point configurations in Euclidean spaces of 8 and 24 dimensions.

Free real algebraic geometry, with a focus on convexity

Jurij Volčič

One of the noncommutative analogs of the classical real algebraic geometry is free real algebraic geometry (FRAG), which studies noncommutative polynomials and rational functions, their evaluations on tuples of matrices, and positive definiteness thereof. The adjective "free" signals that one is interested in variables that are relation-free and matrix arguments of arbitrary sizes. The complex free algebra $\mathbb{C}\langle x\rangle$ over noncommuting variables $x = (x_1, \dots, x_d)$ comes with a natural involution $*$ that fixes x_j. This involution also naturally extends to matrices over $\mathbb{C}\langle x\rangle$. A noncommutative polynomial $f \in M_\delta(\mathbb{C}\langle x\rangle)$ is hermitian if $f^* = f$. The central geometric object of FRAG is the *positivity domain* of such an f,

$$\mathcal{D}_f = \bigcup_{n\in\mathbb{N}} \mathcal{D}_f(n), \qquad \mathcal{D}_f(n) = \{X \in H_n(\mathbb{C})^d \colon f(X) \succ 0\}$$

where $H_n(\mathbb{C})$ denotes the real space of $n \times n$ hermitian matrices. For the sake of normalization let $0 \in \mathcal{D}_f$ be a quiet assumption from hereon; that is, $f(0) \succ 0$. Matricial sets of the form $\mathcal{D}_f$ are also called *(basic open) free semialgebraic sets*. They naturally appear in free analysis, operator systems and algebras, control and systems theory, relaxation schemes for polynomial optimization, and quantum information theory. Most of the arising questions concern convexity. Here $\mathcal{D}_f$ is *convex* if $\mathcal{D}_f(n)$ is a convex subset of $H_n(\mathbb{C})^d$ for every $n \in \mathbb{N}$.

Convex free semialgebraic sets have exceptional structural features in comparison with their classical cousins. An apparent example of a convex free semialgebraic set is a *free spectrahedron* $\mathcal{D}_L$, or a *linear matrix inequality (LMI) domain*, where $L = I + A_1x_1 + \cdots + A_dx_d$ with $A_j \in H_\delta(\mathbb{C})$ is a monic hermitian pencil (an LMI representation of $\mathcal{D}_L$). It turns out [7] that every convex free semialgebraic set is a free spectrahedron. Furthermore, free spectrahedra admit a perfect Positivstellensatz [3]: a noncommutative polynomial f is positive semidefinite on $\mathcal{D}_L$ if and only if it belongs to the quadratic module generated by L, i.e.,

$$f = s_1^* s_1 + \cdots + s_\ell^* s_\ell + s_{\ell+1}^* L s_{\ell+1} + \cdots + s_m^* L s_m$$

and $2 \deg s_i \leq \deg f$. These two results have profound consequences for optimization over convex free semialgebraic sets. Namely, such optimization problems can be formulated as semidefinite programs, which can be efficiently solved using interior point methods with various implementations in computational software.

It is thus natural to ask how to check whether a free semialgebraic set is convex, and how to find its LMI representation if that is the case. It is worth pointing out that neither the original functional-analytic proof of the existence of an LMI representation nor the subsequent real-algebraic proofs are constructive. Fortunately, invariant and representation theory entered the picture in the last few years via the *free locus* of f,

$$\mathscr{Z}_f = \bigcup_{n \in \mathbb{N}} \mathscr{Z}_f(n), \qquad \mathscr{Z}_f(n) = \{X \in M_n(\mathbb{C})^d \colon \det f(X) = 0\}.$$

One should think of $\mathscr{Z}_f$ as of the "Zariski closure" of the boundary of $\mathcal{D}_f$. In [8, 6] it was shown that persistent (as n grows) irreducible components of $\mathscr{Z}_f$ are in one-to-one correspondence with certain equivalence classes of factors of f over $\mathbb{C}\langle x \rangle$. In particular, f is irreducible if and only if $\mathscr{Z}_f(n)$ is a reduced irreducible hypersurface in $M_n(\mathbb{C})^d$ for large enough n. Together with the realization theory for noncommutative rational functions, these results were crucial for procedural study of semialgebraic convexity. Namely, in [4] an efficient algorithm (based on linear algebra, probabilistic methods and semidefinite programming) was designed for checking convexity of $\mathcal{D}_f$, and constructing its LMI representation. The derived machinery also has surprising theoretical consequences. Firstly, the intersection of a finite family of free semialgebraic sets with irreducible boundaries is convex if and only if each member of the family is convex. Secondly, if $f \in \mathbb{C}\langle x \rangle$ is hermitian and irreducible, and $\mathcal{D}_f$ is proper and convex, then f is a concave quadratic. Moreover, if f is hermitian and $\mathcal{D}_{f+\varepsilon}$ is proper and convex for all small enough $\varepsilon > 0$, then f is a composite of a univariate polynomial with a concave quadratic [10]. Similar methods were also used to analyze free stability on the matricial positive orthant and to prove the existence of determinantal representations for Hurwitz stable noncommutative polynomials and rational functions [9].

The success with convexity indicates a natural future quest: devise a (computationally efficient) procedure that determines which non-convex free semialgebraic sets can be analytically transformed into convex ones. Such a procedure would have important consequences for optimization in control theory, as it would accept a "hard" (non-convex) problem and return an equivalent "easy" (convex) problem. There are two kinds of partial results towards this goal. By [2], $\mathcal{D}_f$ admits a proper noncommutative rational map into a free spectrahedron if and only if there is a *plurisubharmonic* noncommutative rational function r such that $\mathcal{D}_f = \mathcal{D}_{-r}$.

This gives a geometric condition for being transformable into a convex set. On the other hand, any deterministic non-convex–to–convex procedure would ostensibly rely on the output being rather unique: that is, one would hope that there are not many analytic maps between free spectrahedra. This is indeed true at least for spectrahedra with certain genericity assumptions [1] or symmetries [5]. Roughly speaking, if there is a bianalytic map $f\colon \mathcal{D}_L \to \mathcal{D}_M$ between two free spectrahedra (with the required features), then f is actually a *convexotonic map* (a birational map with lots of structure), and there is a pair of unitaries that intertwines the coefficients of the monic hermitian pencils L and M. The future research in FRAG will likely focus on extending these results to arbitrary free semialgebraic sets.

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Results and questions related to entropy and transport for non-commutative laws

David Jekel

A non-commutative law is a positive unital tracial linear functional $\mathbb{C}\langle x_1, \dots, x_n\rangle \to \mathbb{C}$ satisfying the radius bounds $|\lambda(x_{i_1} \dots x_{i_\ell})| \leq R^\ell$ for some $R > 0$, where $\mathbb{C}\langle x_1, \dots, x_n\rangle$ denotes the non-commutative polynomial algebra in n variables, with the $*$–structure given by $x_j = x_j^*$. These non-commutative laws are the non-commutative analogue of probability measures supported on $[-R, R]^n$ viewed as linear functionals on the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$.

Dan-Virgil Voiculescu defined an analogue of the entropy $-\int \rho \log \rho$ for non-commutative distributions in [9, 10, 11, 12]. The microstates free entropy $\chi(\lambda)$ is defined as the (lim sup) exponential growth rate of the volume of microstate spaces associated to the law λ. A microstate space is a set of tuples of self-adjoint matrices $\mathbf{A} = (A_1, \dots, A_n)$ satisfying $\|A_j\| \leq R$ and $(1/k)\mathrm{Tr}(p(A_1, \dots, A_n))$ is close to $\lambda(p)$ (with some specified error tolerance) for some finitely many polynomials $p \in \mathbb{C}\langle x_1, \dots, x_n \rangle$. In particular, many of these microstate spaces are open semialgebraic sets (where the variables are the real and imaginary parts of the entries of A_j) defined by inequalities given by traces of non-commutative polynomials.

The free entropy $\chi(\lambda)$ satisfies a change-of-variables formula $\chi(\mathbf{f}_* \mu) = \chi(\mu) +$ "trace of log of $D\mathbf{f}$" when $\mathbf{f}$ is an n–tuple of self-adjoint non-commutative polynomials with a "good" inverse function. In hopes of removing the hypothesis of global invertibility of $\mathbf{f}$, I proposed the problem of finding bounds for the sizes of the finite fibers of a non-commutative polynomial map $M_k(\mathbb{C})_{sa}^n \to M_k(\mathbb{C})_{sa}^n$, which I heard from Dimitri Shlyakhtenko. The question is whether we can do any better than applying the classical Bézout bounds to $\mathbf{f}$ as a function of the matrix entries' real and imaginary parts. As long as the fibers have cardinality much smaller than $\exp(Ck^2)$, then we would be able to vastly generalize the change of variables for entropy. We know that this works in the case of one matrix, but it is unclear whether it holds for $n \geq 2$.

I also explained the result from [2, 4, 5] and ongoing work that for non-commutative laws arising from random matrix theory, there is a "non-commutative C^k function $\mathbf{f}$" that pushes forward that the law μ to the "free Gaussian" law, and in fact, $\mathbf{f} = (f_1, \dots, f_n)$ can be chosen to be "lower triangular" in the sense that f_k only depends on $x_1, \dots, x_k$ for each $k = 1, \dots, n$.

These non-commutative C^k functions are modeled not on non-commutative polynomials, but on *trace polynomials*, functions such as

$$f(x_1, x_2) = x_1 \tau(x_2) + x_2 x_1 \tau(x_1 x_2) + \tau(x_1)\tau(x_2^2)x_1^2.$$

The trace polynomials have been studied by various authors in various contexts [1, 2, 3, 6, 7, 8]. Particularly relevant to my own work, they have good closure properties under algebraic operators, composition, and differentiation [1, 2, 5].

Note: The slides from my talk are available on my website (davidjekel.com/research/). During the conference, Jurij Volčič brought my attention to further references that study the algebra and algebraic geometry of trace polynomials (for a fixed matrix size k). We have started to study some examples to study the problem about a non-commutative Bézout bound. I also met with Monique Laurent and Victor Magron to discuss the problem of minimizing the trace of a non-commutative polynomial over a region defined by matrix polynomial inequalities.

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Hyperbolic programming and certifying nonnegativity

James Saunderson

Let $p \in \mathbb{R}[x_1, \ldots, x_n]_d$ be a homogeneous polynomial in n variables of degree d with real coefficients. If $e \in \mathbb{R}^n$ then p is *hyperbolic with respect to* e if $p(e) > 0$ and, for all $x \in \mathbb{R}^n$, the univariate polynomial $t \mapsto p(te - x)$ has only real zeros. Associated with a hyperbolic polynomial p and direction e is the closed hyperbolicity cone

$$\Lambda_+(p, e) = \{x \in \mathbb{R}^n : \text{all zeros of } t \mapsto p(te - x) \text{ are non-negative}\},$$

which is convex [1]. An important special case is when $p(x) = \det(\sum_{i=1}^n A_i x_i)$, where the A_i are symmetric $m \times m$ matrices and e is such that $\sum_{i=1}^n A_i e_i$ is positive definite. Then $\Lambda_+(p, e)$ is a *spectrahedron*, the intersection of the cone of $m \times m$ positive semidefinite matrices with a linear space.

Hyperbolic programming. If K is a closed convex cone, the associated *conic program* is a linear optimization problem of the form

$$\text{minimize}_x \ \langle c, x \rangle \text{ subject to } x \in K \cap L$$

where L is an affine space. A *hyperbolic program* is a conic program in which the cone K is a hyperbolicity cone. Linear, second-order cone, and semidefinite programs are special cases of hyperbolic programs. Hyperbolic programs are of interest because they admit efficient algorithms via interior point methods [2] as long as the hyperbolic polynomial can be evaluated efficiently.

It is unclear how much more general hyperbolic programming is when compared to semidefinite programming. One celebrated question in this direction is the *generalized Lax conjecture*, which conjectures that every hyperbolicity cone is a spectrahedron. From the point of view of conic optimization, it is more natural to consider the weaker *projected Lax conjecture*, which conjectures that every hyperbolicity cone is the projection of a spectrahedron. If this were true then every hyperbolic program could be reformulated as a (possibly much more complicated) semidefinite program. This conjecture is true, for instance, for smooth hyperbolicity cones [4].

Certifying hyperbolicity. Given a homogeneous polynomial p, and a direction e, how can we decide if p is hyperbolic with respect to e? Classical criteria for the real-rootedness of univariate polynomials (in terms of the Hermite matrix or certain Bézoutians) allow us to decide hyperbolicity by checking whether a certain symmetric matrix with polynomial entries is positive semidefinite.

In the case of quadratics and cubics, this reduces to checking nonnegativity of the discriminant. For example, a cubic of the form $t \mapsto t^3 - 3at + 2b$ has real roots if and only if $a^3 - b^2 \geq 0$. It turns out that a cubic of the form $p(x_0, x) = x_0^3 - 3\|x\|^2 x_0 + 2q(x)$ is hyperbolic with respect to $e = (1, \mathbf{0})$ if and only if $\max_{\|x\| \leq 1} q(x) \leq 1$, i.e., the cubic q takes value at most one on the unit sphere. An existing hardness result for maximizing cubic forms on the sphere, due to Nesterov, can be used to show that it is co-NP hard to decide whether a homogeneous cubic is hyperbolic with respect to a given direction [5].

Certifying nonnegativity. On one hand, hyperbolicity can be certified by checking polynomial inequalities. On the other hand, the hyperbolicity of p implies many polynomial inequalities. One family of these arises from interlacing properties of hyperbolic polynomials and directional derivatives in directions contained in the hyperbolicity cone. Interlacing conditions can be expressed in terms of polynomial inequalities (via, e.g., Bézoutians). For instance, Kummer, Plaumann, and Vinzant [3] showed that if p is square-free and hyperbolic with respect to e then

$$D_e p(x) D_u p(x) - p(x) D_e D_u p(x) \geq 0 \text{ for all } x \in \mathbb{R}^n \iff u \in \Lambda_+(p, e).$$

We denote by $\phi_{p,e}(x)[u] := D_e p(x) D_u p(x) - p(x) D_e D_u p(x)$, which is homogeneous of degree $2d-2$ in x, linear in u, and nonnegative whenever $u \in \Lambda_+(p, e)$. Further nonnegative polynomials can be produced by composition of $x \mapsto \phi_{p,e}(x)[u]$ with a polynomial map $f \colon \mathbb{R}^m \to \mathbb{R}^n$.

For example the quadratic $p(x_1, x_2, \ldots, x_n) = \frac{1}{2}\left[x_1^2 - (x_2^2 + \cdots + x_n^2)\right]$ is hyperbolic with respect to $e = (1, 0, \ldots, 0)$ and

$$\phi_{p,e}(x)[u] = u_1(x_1^2 + x_2^2 + \cdots + x_n^2) - 2x_1(x_2 u_2 + \cdots + x_n u_n)$$

is nonnegative whenever $\sqrt{u_2^2 + \cdots + u_n^2} \leq u_1$. Putting $u = e$ and composing with $f(z) = (f_1(z), \ldots, f_n(z))$, we obtain an arbitrary sum of squares:

$$\phi_{p,e}(f(z))[e] = f_1(z)^2 + f_2(z)^2 + \cdots + f_n(z)^2.$$

In general, if we can write $q \in \mathbb{R}[z_1, \ldots, z_m]_{2d-2}$ as where p is hyperbolic with respect to e and f is a polynomial map, we say q has a *hyperbolic certificate of nonnegativity.* For a fixed choice of p, e, and f, we can search for such a representation of a polynomial by solving a hyperbolic programming feasibility problem with respect to the cone $\Lambda_+(p, e)$.

If q is a sum of squares, then we have seen that it has a hyperbolic certificate of nonnegativity using a quadratic hyperbolic polynomial. In general, there are polynomials that have hyperbolic certificates of nonnegativity but are not sums of squares [5], but few explicit examples are currently known.

- Do forms whose nonnegativity follows from the arithmetic-geometric mean inequality, such as the Motzkin form $x^2y^4 + x^4y^2 + z^6 - 3x^2y^2z^2$, have hyperbolic certificates of nonnegativity?
- More generally, in which degrees and numbers of variables are there nonnegative forms that do not have hyperbolic certificates of nonnegativity?

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Free convex hulls of quantum permutation matrices

Tim Netzer

A magic square, also known as a doubly stochastic matrix, is a square matrix with nonnegative entries, all of whose rows and columns sum to 1. The famous Birkhoff-von Neumann Theorem states that the permutation matrices constitute the vertices of the polyhedron of all magic squares. In particular, every magic square is a convex combination of permutation matrices.

When passing from classical semialgebraic geometry to free (=noncommutative) semialgebraic geometry, real numbers are replaced by hermitian matrices, and being positive semidefinite is the right notion of nonnegativity. Usually one also considers matrices of all sizes simultaneously. So fix some $d \geq 1$, and for $s \geq 1$ define the set of *quantum magic squares at level s* as

$$\mathcal{M}_d(s) = \left\{ (M_{ij})_{i,j=1}^d \mid M_{ij} \in \mathrm{Her}_s(\mathbb{C}), M_{ij} \geqslant 0, \sum_i M_{ij} = \sum_j M_{ij} = I_s \right\}$$

and the set of all quantum magic squares as

$$\mathcal{M}_d = \bigcup_{s \geq 1} \mathcal{M}_d(s).$$

The set $\mathcal{M}$ is a free spectrahedron, i.e. definable by a linear matrix inequality. The role of permutation matrices is taken by *quantum permutation matrices*, which are quantum magic squares in which each entry is an orthogonal projection, i.e. an idempotent hermitian matrix:

$$\mathcal{P}_d(s) = \left\{ (M_{ij})_{i,j} \in \mathcal{M}_d(s) \mid M_{ij}^2 = M_{ij} \right\}, \quad \mathcal{P}_d = \bigcup_{s \geq 1} \mathcal{P}_d(s).$$

Quantum permutation matrices appear as representations of the quantum permutation group, and are useful in quantum information theory, among others.

Now a possible free Birkhoff-von Neumann Theorem clearly requires the notion of convexity to be adapted to the noncommutative setup. Therefore let

$$\mathcal{S} = \bigcup_{s\geq 1} \mathcal{S}(s) \quad \text{with } \mathcal{S}(s) \subseteq \mathrm{Her}_s(\mathbb{C})^n$$

be a free set, where we usually also require that block-sums from elements from $\mathcal{S}$ are again in $\mathcal{S}$. Then $\mathcal{S}$ is called *matrix convex*, if whenever $(A_1, \ldots, A_n) \in \mathcal{S}(s)$ and $V \in \mathrm{Mat}_{s,t}(\mathbb{C})$ with $V^*V = I_t$, then

$$(V^*A_1V, \ldots, V^*A_nV) \in \mathcal{S}(t).$$

This implies classical convexity of each $\mathcal{S}(s)$, but is a stronger property in general. Since the intersection of matrix convex sets is matrix convex, the matrix convex hull of a set is well-defined. Therefor a generalization of the Birkhoff-von Neumann Theorem would state that $\mathcal{M}_d$ is the matrix convex hull of $\mathcal{P}_d$. Unfortunately this fails, already in the first nontrivial case:

Theorem 1. *For each $d \geq 3$ and $s \geq 2$ there exists some $M \in \mathcal{M}_d(s)$ which is not in the matrix convex hull of $\mathcal{P}_d$. Further, every quantum permutation matrix is an Arveson boundary point of the free spectrahedron $\mathcal{M}_d$, but not every such boundary point is a quantum permutation matrix.*

But there is also a partial positive result:

Theorem 2. *If $M = (M_{ij})_{i,j} \in \mathcal{M}_d(s)$ fulfills*

$$\sum_{i=1}^{d} M_{i\pi(i)} \geqslant \frac{d-2}{d-1} \cdot I_s$$

for all permutations $\pi \in S_d$, then M is in the matrix convex hull of $\mathcal{P}_d$. M then even admits a dilation to a quantum permutation matrix with commuting entries.

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Global optimization via the dual SONC cone and linear programming

Mareike Dressler

(joint work with Janin Heuer, Helen Naumann and Timo de Wolff)

Let $A \subseteq \mathbb{R}^n$ be a finite set and let $\mathbb{R}^A$ denote the space of all *(sparse) exponential sums* supported on A. These are of the form

$$f = \sum_{\boldsymbol{\alpha}\in A} c_{\boldsymbol{\alpha}} e^{\langle \mathbf{x},\boldsymbol{\alpha}\rangle} \in \mathbb{R}^A, \quad c_{\boldsymbol{\alpha}} \in \mathbb{R} \text{ for all } \boldsymbol{\alpha} \in A.$$

We consider the following global optimization problem

$$(1) \qquad \inf_{\mathbf{x}\in\mathbb{R}^n} f(\mathbf{x}),$$

which is the unconstrained version of a *signomial optimization problem.* Signomial programs are a rich class of nonconvex optimization problems with a broad range of applications; see e.g., [1] for an overview.

If $A \subseteq \mathbb{N}^n$, then $\mathbb{R}^A$ coincides with the space of real polynomials on the positive orthant supported on A. Thus, (1) also represents all unconstrained *polynomial optimization problems* on $\mathbb{R}^n_{>0}$; see e.g., [2, 10] for an overview about polynomial optimization problems and their applications.

Under the assumption that (1) has a finite solution, minimizing $f \in \mathbb{R}^A$ is equivalent to adding a minimal constant γ such that $f + \gamma \geq 0$. Hence, we consider the (convex, closed) *sparse nonnegativity cone* in $\mathbb{R}^A$, which is defined as

$$\mathcal{P}_A^+ = \{f \in \mathbb{R}^A : f(\mathbf{x}) \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n\}.$$

It is well-known that deciding nonnegativity is NP-hard even in the polynomial case. Thus, a common way to attack (1), is to search for *certificates of nonnegativity.* These conditions, which imply nonnegativity, are easier to test than nonnegativity itself, and are satisfied for a vast subset of $\mathcal{P}_A^+$. In the polynomial case, a well-known example of a certificate of nonnegativity are *sums of squares (SOS)*, which can be tested via *semidefinite programming.* Unfortunately, SOS decompositions do not preserve the sparsity of A.

Another certificate of nonnegativity is a decomposition of f into *sums of nonnegative circuit functions (SONC)*, which were introduced by Iliman and de Wolff for polynomials [8] generalizing work by Reznick [13]. We build on a recent, generalized notion by Forsgård and de Wolff [7]. A *circuit function* is a function, which is supported on a minimally affine dependent set, its support forms a simplex, and the coefficients associated to the vertices of the convex hull of its support are positive. For these kind of functions nonnegativity can effectively be decided by solving a system of linear equations (via the so-called *circuit number*).

SONCs form a closed full-dimensional convex cone $\mathcal{S}_A^+ \subseteq \mathcal{P}_A^+$. This cone and the functions therein respectively were investigated independently by other authors using a separate terminology. The perspective of considering $\mathcal{S}_A^+$ as a subclass of nonnegative signomials was originally introduced by Chandrasekaran and Shah [3] under the name *SAGE*, which was later generalized by Chandrasekaran, Murray, and Wiermann [12, 11]. It turns out, that the SAGE cone coincides with the SONC cone. For further details about the SONC cone see [5, 4].

A common, tractable approach to minimize a function using the SONC approach is to restrict to the *signed SONC cone* $\mathcal{S}^+_{A^+,A^-}$, which is the intersection of $\mathcal{S}_A^+$ with a particular orthant indicated by the pair (A^+, A^-), with $\emptyset \neq A^+ \subseteq \mathbb{R}^n$, corresponding to positive coefficients, and $A^- \subseteq \mathbb{R}^n$ corresponding to the remaining nonpositive coefficients of a given function. Our **key idea** is to relax the problem (1) via optimizing over the *dual SONC cone* $\check{\mathcal{S}}^+_{A^+,A^-}$. Our approach is motivated by the recent works [6], [12], and [9], and builds on two key observations:

(1) The dual SONC cone is contained in the primal one.
(2) Optimizing over the dual cone can be carried out by solving a linear program.

We emphasize that neither the primal nor the dual SONC cone is polyhedral. The approach works as follows: First, we investigate a lifted version of the dual cone involving additional linear auxiliary variables. Second, we show that the coefficients of a given exponential sum can be interpreted as variables of the dual cone. Third, we observe that fixing these coefficient variables yields an optimization problem only involving the linear auxiliary variables.

Based on our two key observations stated above, we present a linear program (LP) solving a relaxation of (1). We implemented the proposed algorithm and provide a collection of examples showing that (LP) works in practice. Using the software POEM [14], we compare our approach exemplarily to existing algorithms for finding SONC and SAGE decompositions via the primal cone $\mathcal{S}_A^+$.

Conclusion: The dual SONC cone is a proper subset of the corresponding primal cone, we observe that, as expected, our linear program (LP) yields, in general, worse results than the SONC and the SAGE approach. Since our new approach only relies on solving LPs it is, however, computationally more stable with promising runtimes, and it gives a result whenever a solution in the dual cone exists. In particular, we obtain an algorithm which yields a bound computed independently of the existing primal SONC and SAGE algorithms.

Open questions and possible directions for future research resulting from this work are:

(1) The constraints guaranteeing containment in the dual cone are very restrictive, occasionally leading to an infeasible linear program. In this case, one can solve a relaxed version of (LP) allowing its constraints to be violated by some tolerance $\mathsf{tol} \geq 0$. This relaxed problem would still be an LP. Such an approach yields a solution in a relaxed version of the dual SONC cone, and since the dual SONC cone is contained in the primal, this relaxation also leads to a certificate that the found solution is contained in a relaxed version of the primal SONC cone and therefore also in a relaxation of the nonnegativity cone.
(2) Investigate the polyhedron obtained by fixing the non-auxiliary variables. From duality theory we know that there has to exist a primal polyhedron as well. The primal SONC cone itself is, however, not polyhedral. What is the relation of this primal polyhedron to the SONC cone?
(3) What is the relation of the dual SONC cone to sparse moment cones?

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Free Stein irregularity

Ian Charlesworth
(joint work with Brent Nelson)

Regularity theorems in free probability, very broadly, aim to deduce properties of von Neumann algebras from probabilistic properties of a set of generators. The approach dates back to several works of Voiculescu from the 1990's, in which the analogues of entropy and Fisher's information measure from information theory were introduced to the setting of free probability; see, e.g., [6] for a summary. There are two different ways of formulating a free probabilistic analogue of entropy; we focus here on the so-called non-microstates approach, introduced in [5], which attempts in some sense to measure how amenable a non-commutative law is to a sort of differential calculus.

Given a tuple of self-adjoint operators $X = (x_1, \ldots, x_n)$ generating a tracial von Neumann algebra (M, τ), we define their non-commutative law as the linear functional μ_X defined on the space $\mathbb{C}\langle T_1, \ldots, T_n\rangle$ of polynomials in n non-commuting indeterminates by evaluation in X followed by an application of τ; hence, for example, $\mu_X(T_1T_2 + T_2T_1) = \tau(x_1x_2 + x_2x_1)$. We define the free difference quotients as the maps

$$\partial_i : \mathbb{C}\langle T_1, \ldots, T_n\rangle \to \mathbb{C}\langle T_1, \ldots, T_n\rangle \otimes \mathbb{C}\langle T_1, \ldots, T_n\rangle$$

defined by linearity, the Leibniz rule, and the condition $\partial_i T_j = \delta_{i=j} 1 \otimes 1$; the non-commutative Jacobian may be defined on a tuple $p = (p_i)_i$ of polynomials by $\mathscr{J} p = (\partial_j p_i)_{i,j}$. Evaluation in X gives us a corresponding densely defined relation from $L^2(M,\tau)^n$ to $M_n\left(L^2(M,\tau) \otimes L^2(M,\tau)\right)$ and so we can consider their adjoints in this setting. Last, let us denote by $\mathbb{1}$ the matrix $\mathrm{id}_n \otimes 1 \otimes 1 \in M_n\left(L^2(M,\tau) \otimes L^2(M,\tau)\right)$.

If $\mathbb{1}$ lies in the domain of $\mathscr{J}^*$, its image $\Xi = \mathscr{J}^* \mathbb{1}$ is said to be the tuple of conjugate variables to X [5]. The conjugate variables carry a lot of information about X and M. For example: the condition $X = \lambda \Xi$ characterizes the free semicircular law, the central limit law of free probability; if Ξ exists and $n > 1$, M does not have property Γ [3]; if Ξ exists, then every non-constant self-adjoint polynomial in X is diffuse [2, 4] (in fact, this is true under weaker assumptions, such as that the free entropy dimension achieves its maximum: $\delta^*(X) = n$).

To overcome some difficulties of working with the conjugate variables, we instead study properties of the domain of $\mathscr{J}^*$. In joint work with Brent Nelson [1], we define two related quantities: the free Stein irregularity $\Sigma^*(X)$, the distance in the Hilbert-Schmidt norm on $M_n\left(L^2(M,\tau) \otimes L^2(M,\tau)\right)$ from $\mathbb{1}$ to $\overline{\mathrm{dom}(\mathscr{J}^*)}$; and the free Stein dimension $\sigma(X)$, one n–th of the dimension of $\overline{\mathrm{dom}(\mathscr{J}^*)}$ as an $M \bar{\otimes} M^{\mathrm{op}}$ module. These are related by the formula $\Sigma^*(X)^2 + \sigma(X) = n$, and can be seen as providing a qualitative sense of how well behaved $\mathscr{J}^*$ is, or how strong the obstructions to $\mathbb{1}$ being in its domain are.

We are able to show that for tuples X, Y of variables, $\Sigma^*(X)^2 + \Sigma^*(Y)^2 \leq \Sigma^*(X,Y)^2$ with equality if X and Y are freely independent. Moreover, the free Stein dimension bounds the free entropy dimension from below — $\sigma(X) \leq \delta^*(X)$ — and as a consequence, $\sigma(X) = n$ implies the same result about non-constant polynomials being diffuse mentioned above. In fact, in every case where the values of $\sigma(X)$ and $\delta^*(X)$ are both known explicitly, they agree; this raises the question of whether they in fact always agree, which would be a striking result if true.

The domain of $\mathscr{J}^*$, which implicitly depends on the tuple X, can be put in correspondence with the set of derivations on the polynomial algebra generated by X taking values in $L^2(M,\tau) \otimes L^2(M,\tau)$ whose adjoints contain $1 \otimes 1$. The fact that this space of derivations depends only on the algebra generated by X and not the variables themselves allows us to argue that σ is an algebra invariant, a property which δ^* is not known to enjoy.

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Lattice points in dilated simplices and sums of squares

Bruce Reznick

This talk is, in some sense, a remake of a talk I gave here in 1987, but was given with improved notation, better jokes and, unfortunately, less hair. The main theorem was proved in [2], though one announced result was not proved. This gap was remedied by Vicki Powers and the author in the recent [1]. All proofs here, except for the claim at the very end, can be found in [2] and [1].

There are two fundamental forms which arise as examples in Hilbert's 17th problem. The first is the Motzkin form

$$M(x,y,z) := x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2.$$

which is the first known example of a psd form which is not sos. (See [3] for historical details.) On the other hand, a form of a type Hurwitz studied:

$$H(x,y,z) := x^6 + y^6 + z^6 - 3x^2y^2z^2$$
$$= \tfrac{3}{2}(x^2y - yz^2)^2 + (x^3 - xy^2)^2 + \tfrac{1}{2}(x^2y - y^3)^2 + (z^3 - y^2z)^2 + \tfrac{1}{2}(yz^2 - y^3)^2$$

is evidently a sum of squares. Both these forms arise from a monomial substitution into the arithmetic-geometric inequality (AGI), and so both are psd, but M is not sos and H is sos. Why?

The answer turns out to rely on the pattern of the lattice points within the simplex determined by the vectors of monomials in the substitution. To be specific, suppose we have the AGI

$$\lambda_1 t_1 + \cdots + \lambda_n t_n \geq t_1^{\lambda_1} \cdots t_n^{\lambda_n},$$

where $t_i \geq 0$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. We suppose that $\{u_1, \dots, u_n\}$ with $u_i \in (2\mathbb{Z}_{\geq 0})^n$ and $\sum_{j=1}^n u_{ij} = 2d$. We further assume that $\mathcal{U} = \mathrm{cvx}(\{u_1, \dots, u_n\})$ is a simplex, and that $w \in \mathcal{U} \cap \mathbb{Z}^n$ has the (unique) barycentric representation $w = \sum \lambda_i u_i$, $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$. In this way, the substitution $\{t_i = x^{u_i}\}$ into the AGI yields a psd form of degree $2d$,

$$p(\mathcal{U}, w)(x) := \lambda_1 x^{u_1} + \cdots + \lambda_n x^{u_n} - x^w.$$

In the two cases above, up to a multiple of 3, we have $n = 3$, $\lambda_j = 1/3$ and $\mathcal{U}_1 = \{(4,2,0),(2,4,0),(0,0,6)\}$ and $\mathcal{U}_2 = \{(6,0,0),(0,6,0),(0,0,6)\}$

Suppose $\mathcal{U}$ is defined as above and let $S \subset \mathcal{U} \cap \mathbb{Z}^n$ be a set of lattice points containing the u_i's. Then S is $\mathcal{U}$*–mediated* if for every $y \in S$, either $y = u_i$ for some i, or there exist $z_1 \neq z_2 \in S \cap (2\mathbb{Z})^n$ so that $y = \frac{1}{2}(z_1 + z_2)$. In other words, S is $\mathcal{U}$–mediated if every point in S is either a vertex of $\mathcal{U}$ or an average of two different even points in $\mathcal{U}$. (This definition generalizes naturally to polytopes, but there are no known applications.)

The following was proved in [2]: Theorem: As defined above, $p(\mathcal{U}, w)$ is sos if and only if there is a $\mathcal{U}$–mediated set containing w. The proof given in the talk evoked the formal inverse of a matrix perturbing I_m, as well as the Coen Brothers.

The paper [2] had its day in the sun and then in the way of most papers sank to the bottom of the ocean. Then the recent interest in circuit polynomials recalled it to life: mediated sets found their inner Godzilla and resurfaced. In particular, an assertion was made in [2] and not proved: For every integer $k \geq \max\{2, n-2\}$, $k\mathcal{U} \cap \mathbb{Z}^n$ is $(k\mathcal{U})$–mediated. As a corollary, each form $p(\mathcal{U}, w)(x_1^k, \ldots, x_n^k)$ is sos.

The reason this wasn't proved in [2] is that the speaker had optimistically generalized, and erroneously conjectured that such a property holds for every psd form f; in particular, this would imply that f is a sum of squares of forms in the variables $x_j^{1/k}$ for sufficiently large k.

In [1], Vicki and I give the proof of the assertion (corrected from the unpublished proof of thirty years earlier), as well as showing that this conjecture is false for the Horn form $F(x_1, \ldots, x_5)$. In the talk, a proof of a weaker version of the dilation theorem was presented: $k \geq n-1$.

What's new in the talk was the observation, based on evidence, that for the full psd-not-sos example M_n (note that $M_3 = M$) given by Motzkin in 1967:

$$M_n(x_1, \ldots, x_n) :=$$
$$x_1^4 x_2^2 \cdots x_{n-1}^2 + x_1^2 x_2^4 \cdots x_{n-1}^2 + \cdots + x_1^2 x_2^2 \cdots x_{n-1}^4 + x_n^{2n} - n x_1^2 x_2^2 \cdots x_{n-1}^2 x_n^2,$$

the lower bound $k \geq \max\{2, n-2\}$ is in fact best possible.

I want to thank the organizers for the chance to speak. I also want to thank this research community for its friendliness and openness. Young mathematicians should realize that this is not automatic! This talk was given during the penultimate Oberwolfach workshop before it was overtaken by the rough coronaviral beast which is now slouching everywhere.

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What moment inequalities can be learned via sums of squares?

Grigoriy Blekherman

(joint work with Felipe Rincon, Rainer Sinn, Cynthia Vinzant and Josephine Yu)

Introduction. Sums of squares provide an inner approximation to nonnegative polynomials, and on the dual side moment sequences with positive semidefinite moment matrices provide an outer approximation to true moment sequences of measures. We ask what moment inequalities can be deduced from positive semidefiniteness of the moment matrices? In case of a basic semi-algebraic set we also get to use positive semidefiniteness of localization matrices of larger and larger degrees coming from the associated quadratic module or preorder.

We consider two sets in detail, the nonnegative orthant and the unit cube. We show examples of moment inequalities that cannot be deduced via sums of squares even if we allow arbitrarily high degree. These inequalities are of AM/GM-type, providing more motivation to Motzkin's original construction of a nonnegative polynomial that is not a sum of squares [5] and subsequent work on AGI forms by Reznick [6], and SONC polynomials by de Wolff, Dressler, Iliman and Theobald [3, 4]. The technique used for the proofs is **tropicalization**, which has been used extensively in (complex and real) algebraic geometry, but not in semialgebraic real algebraic geometry.

Results in Detail. Let $\mathcal{S}$ be a subset of the nonnegative orthant $\mathbb{R}^n_{\geq 0}$. The *tropicalization* (or logarithmic limit-set) of $\mathcal{S}$ is defined to be

$$\operatorname{trop}\mathcal{S} = \lim_{t\to 0} \log_{\frac{1}{t}}(\mathcal{S}\cap \mathbb{R}^n_{>0}).$$

Tropicalization of a semialgebraic set is known to be a rational polyhedral complex, but it does not have to be a convex set [1, 2]. We say that a set $\mathcal{S}$ has the *Hadamard property* if $\mathcal{S}$ is closed under coordinatewise multiplication: if $(a_1,\dots,a_n)$, $(b_1,\dots,b_n)$ are in $\mathcal{S}$ then so is $(a_1b_1,\dots,a_nb_n)$. If $\mathcal{S}$ is a semialgberaic set with the Hadamard property then tropicalization $\operatorname{trop}\mathcal{S}$ is a rational polyhedral cone. The defining inequalities of $\operatorname{trop}\mathcal{S}$ correspond to *pure binomial inequalities* $x^\alpha \geq x^\beta$ valid on $\mathcal{S}$.

Let $P_{n,d}$ be the cone of polynomials of degree at most d nonnegative on $\mathbb{R}^n_{\geq 0}$ and $\Sigma_{n,d}$ be the cone of polynomials that lie in the preorder generated by $x_1,\dots,x_n$. The dual cones $P^*_{n,d}$ and $\Sigma^*_{n,d}$ have coordinates indexed by moments m_α with $\alpha = (\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n$ and $|\alpha| = \alpha_1+\cdots+\alpha_n \leq d$. With the moment coordinates, the dual cones $P^*_{n,d}$ and $\Sigma^*_{n,d}$ have the Hadamard property, and so their tropicalizations $\operatorname{trop}P^*_{n,d}$ and $\operatorname{trop}\Sigma^*_{n,d}$ are rational polyhedral cones. The defining inequalities of $\operatorname{trop}P^*_{n,d}$ correspond to *pure binomial inequalities* in moments of measures that are supported on the nonnegative orthant. The defining inequalities of the larger cone $\operatorname{trop}\Sigma^*_{n,d}$ correspond to pure binomial inequalities in moments that *can be deduced via sums of squares.*

We show that defining inequalities of $\operatorname{trop}\Sigma^*_{n,d}$ are given by $\tilde{m}_{\alpha_1}+\tilde{m}_{\alpha_2}\geq 2\tilde{m}_\beta$ where $\beta = \frac{1}{2}(\alpha_1+\alpha_2)$, where $\tilde{m}_i = \log m_i$ and $|\alpha_1|\leq d$, $|\alpha_2|\leq d$, $|\beta|\leq d$. These correspond to binomial moment inequalities in moments $m_{\alpha_1} m_{\alpha_2} \geq m_\beta^2$. The defining inequalities of trop $P^*_{n,d}$ are given by $a_1 \tilde{m}_{\alpha_1} + \cdots + a_k \tilde{m}_{\alpha_k} \geq \tilde{m}_\beta$, where β is a convex combination of α_i's with weights a_k: $\beta = a_1\alpha_1 + \cdots + a_k\alpha_k$.

The above results can generalized to arbitrary set of moments/supports $\mathcal{A}$. In particular, if $\mathcal{A}$ is the Motzkin support configuration $(0,0), (1,2), (2,1), (1,1)$, then there are no midpoint lattice points and the dual cone trop $\Sigma^*_{\mathcal{A}}$ is the entire space $\mathbb{R}^n$, while the cone trop $P^*_{\mathcal{A}}$ has one defining inequality $m_{00} + m_{12} + m_{21} \geq 3m_{11}$, which cannot be deduced via sums of squares.

We generalize these results to tropicalizations of polynomials nonnegative on the unit cube $[0,1]^n$, where sums of squares are replaced by the preorder of the unit cube. Interestingly in this case there also are pure binomial inequalities in moments that cannot be deduced from the preorder regardless of the degree.

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Sparse (non)commutative polynomial optimization

Victor Magron

(joint work with Jean-Bernard Lasserre, Hoang Anh Ngoc Mai, Igor Klep and Janez Povh)

This work focuses on optimization of polynomials in (non)commuting variables, while taking into account sparsity in the input data.

Given $f, g_1, \ldots, g_m \in \mathbb{R}[x]$, and the basic semialgebraic set $S(g) := \{x \in \mathbb{R}^n : g_j(x) \geq 0\,,\, j = 1, \ldots, m\}$, with $g := \{g_1, \ldots, g_m\}$, *polynomial optimization* is concerned with computing $f^\star := \inf\{f(x) : x \in S(g)\}$. A basic idea is to rather consider $f^\star = \sup\{\lambda \in \mathbb{R} : f - \lambda > 0 \text{ on } S(g)\}$ and replace the difficult constraint "$f - \lambda > 0$ on $S(g)$" with a more tractable sums of squares (SOS) based decomposition of $f - \lambda$, thanks to various certificates of positivity on $S(g)$. Let $\Sigma[x]$ be the set of SOS polynomials. If $S(g)$ is compact and satisfies the so-called Archimedean assumption[1], Putinar's Positivstellensatz [7] provides the decomposition $f - \lambda = \sigma_0 + \sum_{j=1}^m \sigma_j g_j$, with $\sigma_j \in \Sigma[x]$. Then one obtains the monotone non-decreasing sequence $(\rho_k)_{k\in\mathbb{N}}$ of lower bounds on $f^\star$ defined by:

$$\rho_k := \sup_{\lambda,\sigma_j} \left\{ \lambda : f - \lambda = \sigma_0 + \sum_{j=1}^m \sigma_j g_j,\ \sigma_j \in \Sigma[x]\,,\ \deg(\sigma_j g_j) \leq 2k \right\}. \tag{1}$$

For each fixed k, (1) is a semidefinite program and therefore can be solved efficiently. Moreover, by invoking Putinar's Positivstellensatz, one obtains the convergence $\rho_k \uparrow f^\star$ as k increasees. In Table 1 are listed several useful Positivstellensätze that guarantee convergence of similar sequences $(\rho_k)_{k\in\mathbb{N}}$ to $f^\star$ (where now in (1) one uses the appropriate positivity certificate).

TABLE 1. Several Positivstellensätze applicable in practice.

Author(s)	Statement
Schmüdgen [9]	If f is positive on $S(g)$ and $S(g)$ is a compact set, then $f = \sum_{\alpha\in\{0,1\}^m} \sigma_\alpha \prod_{j=1}^m g_j^{\alpha_j}$ for some $\sigma_\alpha \in \Sigma[x]$.
Putinar [7]	If a polynomial f is positive on $S(g)$ satisfying Archimedian assumption[1], then $f = \sigma_0 + \sum_{j=1}^m \sigma_j g_j$ for some $\sigma_j \in \Sigma[x]$.
Reznick [8]	If f is a positive definite form, then $\|x\|_2^{2k} f \in \Sigma[x]$ for some $k \in \mathbb{N}$.
Polya [6]	If f is a homogeneous form and $f > 0$ on $\mathbb{R}^n_+\backslash\{0\}$, then $(\sum_j x_j)^k f$ has nonnegative coefficients for some $k \in \mathbb{N}$.
Krivine-Stengle [3]	If a polynomial f is positive on $S(g)$, $S(g)$ is compact and $g_j \leq 1$ on $S(g)$, then $f = \sum_{\alpha,\beta\in\mathbb{N}^m} c_{\alpha,\beta} \prod_{j=1}^m (g_j^{\alpha_j}(1-g_j)^{\beta_j})$ for some $c_{\alpha,\beta} \geq 0$.

However their associated so-called *dense hierarchies* of linear/semidefinite programs are only suitable for modest size problems (e.g., $n \leq 10$ and $\deg(f) \leq 10, \deg(g_j) \leq 10$). Indeed, for instance, even though (1) is a semidefinite program, it involves $\binom{n+2k}{n}$ variables and semidefinite matrices of size up to $\binom{n+k}{n}$, a clear limitation for state-of-the-art solvers.

Therefore a scientific challenge with important computational implications is to develop alternative positivity certificates that scale well in terms of computational complexity, at least in some identified class of problems.

Fortunately as we next see, we can provide such alternative positivity certificates for the class of problems where some structured sparsity pattern is present in the problem description (as often the case in large-scale problems). Indeed this sparsity pattern can be exploited to yield a positivity certificate in which the sparsity pattern is reflected, thus with potential significant computational savings.

For $n, m \in \mathbb{N}^{>0}$, let $I := \{1, \ldots, n\}$ and $J := \{1, \ldots, m\}$. For $T \subset I$, denote by $\mathbb{R}[x(T)]$ (resp. $\Sigma[x(T)]$) the ring of polynomials (resp. the subset of SOS polynomials) in the variables $x(T) := \{x_i : i \in T\}$. Also denote by $\mathbb{R}[x(T)]_t$ (resp. $\Sigma[x(T)]_t$) the restriction of $\mathbb{R}[x(T)]$ (resp. $\Sigma[x(T)]$) to polynomials of degree at most t (resp. $2t$). For $R \subset J$, we note $g_R := \{g_j : j \in R\}$.

Designing alternative hierarchies for solving $f^\star := \inf\{f(x) : x \in S(g)\}$*, significantly (computationally) cheaper than their dense version* (1)*, while maintaining convergence to the optimal value* $f^\star$ *is a real challenge with important implications.*

[1]There are $\sigma_j \in \Sigma[x]$ such that $S(\{\sigma_0 + \sum_{j=1}^m \sigma_j g_j\})$ is compact.

One first such successful contribution is due to Waki et al. [11] when the input polynomial data f, g_j are sparse, where by sparse we mean the following:

Assumption 1. *The following conditions hold:*

(i) Running intersection property (RIP): $I = \bigcup_{l=1}^{p} I_l$ *with* $p \in \mathbb{N}^{\geq 2}$, $I_l \neq \emptyset$, $l = 1, \ldots, p$, *and for every* $l \in \{2, \ldots, p\}$, *there exists* $s_l \in \{1, \ldots, l-1\}$, *such that* $I_l \cap \left(\bigcup_{j=1}^{l-1} I_j\right) \subset I_{s_l}$.

(ii) Structured sparsity pattern for the objective function[2]*:* $f = \sum_{l=1}^{p} f_l$ *where* $f_l \in \mathbb{R}[x(I_l)]_{\deg(f)}$, $l = 1, \ldots, p$.

(iii) Structured sparsity pattern for the constraints: $J = \bigcup_{l=1}^{p} J_l$ *and for every* $j \in J_l$, $g_j \in \mathbb{R}[x(I_l)]$, $l = 1, \ldots, p$.

(iv) Additional redundant quadratic constraints: There exists $L > 0$ *such that* $\|x\|_2^2 \leq L$ *for all* $x \in S(g)$ *and* $L - \|x(I_l)\|_2^2 \in g_{J_l}$, $l = 1, \ldots, p$.

With τ ($\leq n$) being the maximum number of variables appearing in each index subset I_l of f, g_j, i.e., $\tau := \max\{|I_l| : l = 1, \ldots, p\}$, Table 2 displays the respective computational complexity of the sparse hierarchy of Waki et al. [11] and the dense hierarchy of Lasserre [4] for semidefinite programs with same order $k \in \mathbb{N}$.

TABLE 2. Comparing computational complexity of sparse and dense hierarchies.

order k	sparse hierarchy	dense hierarchy
number of variables	$O(\tau^{2k})$	$O(n^{2k})$
largest size of semidefinite matrix	$O(\tau^k)$	$O(n^k)$

Obviously the sparse hierarchy provides a potentially high computational saving when compared to the dense one. In addition, convergence of the hierarchy of Waki et al. to the optimal value of the original problem was proved in [5], resulting in the following sparse version of Putinar's Positivstellensatz:

Theorem 2. *(Lasserre, Waki et al.) Let Assumption 1 holds. If a polynomial* f *is positive on* $S(g)$, *then there exist* $\sigma_{0,l} \in \Sigma[x(I_l)]_k$, $\sigma_{j,l} \in \Sigma[x(I_l)]_{k-u_j}$ *with* $u_j := \lceil \deg(g_j)/2 \rceil$, $j \in J_l$, $l = 1, \ldots, p$ *such that* $f = \sum_{l=1}^{p} (\sigma_{0,l} + \sum_{j \in J_l} \sigma_{j,l} g_j)$.

Shortly after, Grimm et al. [1] provided another (simpler) proof where $\mathrm{int}(S(g)) \neq \emptyset$ is not needed, but where compactness of $S(g)$ is still a crucial assumption.

In the first part of this work, we focus on a converging hierarchy of semidefinite relaxations for eigenvalue and trace optimization [2]. This hierarchy is a noncommutative analogue of results due to Lasserre [5] and Waki et al. [11]. The Gelfand-Naimark-Segal (GNS) construction is applied to extract optimizers if flatness and irreducibility conditions are satisfied. Among the main techniques used are amalgamation results from operator algebra.

In the second part, we focus on decompositions of positive definite forms as SOS of sparse rational functions [10]. If f is a positive definite form, Reznick's

[2]If there are f_l in the sum f such that $\deg(f_l) > \deg(f)$, we can always remove the high degree redundant term in f_l which cancel with each other to make degree of f_l at most $\deg(f)$.

Positivstellensatz [8] states that there exists $k \in \mathbb{N}$ such that $\|x\|_2^{2k} f$ is a sum of squares of polynomials. Assuming that f can be written as a sum of forms $\sum_{l=1}^{p} f_l$, where each f_l depends on a subset of the initial variables, and assuming that these subsets satisfy the so-called *running intersection property*, we provide a sparse version of Reznick's Positivstellensatz. Namely, there exists $k \in \mathbb{N}$ such that $f = \sum_{l=1}^{p} \sigma_l / H_l^k$, where σ_l is a sum of squares of polynomials, H_l is a uniform polynomial denominator, and both polynomials σ_l, H_l involve the same variables as f_l, for each $l = 1, \dots, p$. In other words, the sparsity pattern of f is also reflected in this sparse version of Reznick's certificate of positivity.

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Positive Ulrich bundles

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(joint work with Christoph Hanselka)

A recurring question in real algebraic geometry is to what extend certain geometric objects can be represented in a specific algebraic way that makes some of its geometric properties apparent. Two prominent examples are the questions of which nonnegative polynomials can be written as a sum of squares and which hyperbolic polynomials have a symmetric, linear and definite determinantal representation. In this talk we provide a common framework that covers both of these two situations. The geometric property that we consider is the following:

Definition 1. *A morphism $f\colon X \to Y$ between real varieties is called* real-fibered *if $f^{-1}(Y(\mathbb{R})) = X(\mathbb{R})$. For simplicity, all our varieties are assumed to be irreducible.*

The following will turn out to be a natural certificate for real-fiberedness:

Definition 2. *Let $f\colon X \to Y$ be a morphism between real varieties. A coherent sheaf $\mathcal{F}$ on X is called f–*Ulrich *if $f_*\mathcal{F} = \mathcal{O}_Y^N$ for some natural number N. Furthermore, such $\mathcal{F}$ is called* positive *or* sesquipositive *if there is a nondegenerate symmetric or hermitian bilinear form $\mathcal{F} \otimes \mathcal{F} \to f^!\mathcal{O}_Y$ such that the corresponding $\mathcal{O}_Y$–valued form on $f_*\mathcal{F}$ is positive definite. See* [5, §5] *for more details.*

Remark 3. *If $f\colon X \to Y$ is a finite morphism and Y is smooth, then the existence of a positive or sesquipositive f–Ulrich sheaf implies that f is real-fibered* [5, §5].

Here are the main examples for the above introduced notions.

Example 4. *Let $h \in \mathbb{R}[x_0, \ldots, x_n]_{2d}$ be a homogeneous polynomial of degree $2d$. Let $X = \mathcal{V}(y^2 - h) \subset \mathbb{P}(d, 1, \ldots, 1)$ be the zero set of h in the weighted projective space which has one coordinate y that is homogeneous of degree d and $n+1$ coordinates $x_0, \ldots, x_n$ that are homogeneous of degree one. The projection*

$$f\colon X \to \mathbb{P}^n,\ (y : x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n)$$

onto the x_i–coordinates is real-fibered if and only if $h(p) \geq 0$ for all $p \in \mathbb{R}^{n+1}$. Furthermore, there is a positive (or sesquipositive) f–Ulrich sheaf if and only if h can be written as a sum of squares $h = g_1^2 + \cdots + g_r^2$ of polynomials g_i.

Example 5. *Let $h \in \mathbb{R}[x_0, \ldots, x_n]_d$ be a homogeneous polynomial of degree d that is positive in $e \in \mathbb{R}^{n+1}$. Let $X = \mathcal{V}(h) \subset \mathbb{P}^n$ be the zero set of h and $f : X \to \mathbb{P}^{n-1}$ the linear projection with centre $[e] \in \mathbb{P}^n$. Then h is* hyperbolic with respect to e *if and only if f is real-fibered. See for example* [2] *for the original definition of hyperbolic polynomials. Furthermore, there is a positive (or sesquipositive) f–Ulrich sheaf if and only if a power of h is the determinant $h^r = \det A(x)$ of some symmetric (or hermitian) linear matrix pencil $A(x) = x_0A_0 + \cdots + x_nA_n$ for which $A(e)$ is positive definite. Here r is the rank of the Ulrich sheaf.*

We have the following existence result for positive Ulrich sheaves.

Theorem 6. *Let $f : X \to Y$ be a finite surjective morphism of real projective varieties and $\mathcal{F}$ a coherent, torsion-free sheaf on X which is (sesqui-)positive. Then the following are equivalent:*

(1) $\dim \Gamma(X, \mathcal{F}) \geq \deg(f) \cdot \operatorname{rank}(\mathcal{F})$,
(2) $\mathcal{F}$ is f–Ulrich.

The following corollaries can now be easily deduced from the theorem after choosing the correct line bundle on X.

Corollary 7. *Let X be a real curve and $f : X \to \mathbb{P}^1$ a real-fibered morphism. Then there is a positive f–Ulrich line bundle on X. This implies in particular the Helton-Vinnikov theorem* [3] *on symmetric, definite determinantal representations of ternary hyperbolic polynomials as well as its generalization in* [6].

Corollary 8. *Let X be a smooth Del Pezzo surface and $f\colon X \to \mathbb{P}^2$ be a real-fibered morphism such that $f^*\mathcal{O}_{\mathbb{P}^2}(1)$ is the anticanonical line bundle on X. Then there is a sesquipositive f–Ulrich line bundle on X. If X has degree two, this implies Hilbert's theorem on sums of squares representations of nonnegative ternary quartics* [4]*. If X has degree three, this implies the result by Buckley and Košir on hermitian, definite determinantal representations of quarternary hyperbolic polynomials of degree three* [1]*.*

In these examples, the (sesqui-)positivity follows rather directly from the choice of the line bundle. The condition on global sections can then be checked using Riemann-Roch.

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