Snapshots of modern mathematics
from Oberwolfach

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Quantum symmetry

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In mathematics, symmetry is usually captured using the formalism of groups. However, the developments of the past few decades revealed the need to go beyond groups: to "quantum groups". We explain the passage from spaces to quantum spaces, from groups to quantum groups, and from symmetry to quantum symmetry, following an analytical approach.

1 Symmetry and groups

Symmetry can be observed in everyday life as well as in mathematics. One prominent example from nature is the symmetry of a butterfly: Reflecting a picture of a (perfect) butterfly at the axis along its body yields the same picture again – we say that the butterfly is symmetric (Figure 1).

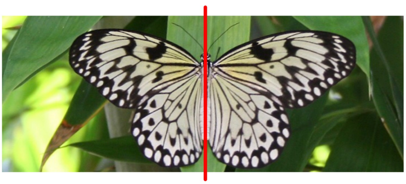

Figure 1: The symmetry of a butterfly.

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1.1 Symmetry in geometry

We now turn to a more mathematical understanding of symmetry. In geometry, the equilateral triangle, the square, and the circle are basic objects. How can we describe their symmetries? Again, we could think of their symmetry axes. In case of the triangle, there are three, the square has four while the circle has infinitely many (Figure 2).

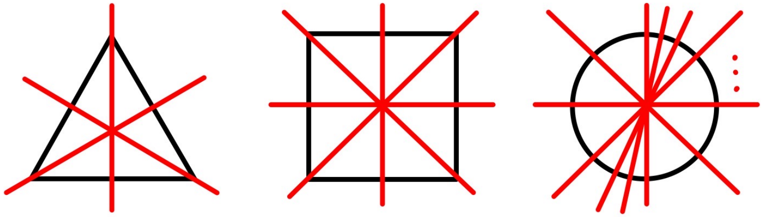

Figure 2: The reflection axes of an equilateral triangle, square, and circle.

We can also consider more abstract objects such as a set of n points or a *graph*[2] on n vertices; that is, a set of n points or *vertices* some of which are connected by *edges*. As for the points, we observe that permuting all points arbitrarily is an operation which gives back the same n points. For a graph (as depicted in Figure 3), the situation is a bit more delicate – only those permutations are allowed which satisfy the rule: "If two points are connected by an edge before applying the permutation, then so should they be after permuting the points" (Figure 3). These examples show how we can roughly capture the

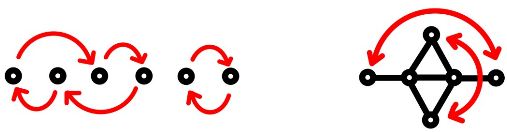

Figure 3: Permuting 6 points in a set (left) or 6 vertices in a graph (right).

idea that an object has some kind of symmetry by saying that it looks the same after a map is applied to it.

But the latter examples also begin to reveal our limits in formulating the

[2] Let's take a finite, undirected graph with no multiple edges, to be precise. As you will see, the more technical math of this snapshot is banned to the footnotes – you may simply skip all the footnotes, if you don't care about too many technical details.

notion of symmetry. If we think of way more complicated objects which might even have no pictorial representation – what should their "symmetries" be then? We introduce a precise mathematical formalism in order to capture symmetry: groups.

1.2 Groups

The maps that describe symmetry in the examples can be composed: for example, you may first reflect the triangle on one of its symmetry axes, then on another one, and the composition of these two maps also describes a symmetry of the triangle, namely a rotation. More precisely, the set of symmetries carries a structure (the composition) which makes it a "group".

In mathematics, a *group* $G = (G, \circ)$ is a set G together with a map $\circ : G \times G \to G$ such that:

1. We have *associativity*: $(a \circ b) \circ c = a \circ (b \circ c)$ for all $a, b, c \in G$.
2. There is a *neutral element*; that is, there is an $e \in G$ with $a \circ e = e \circ a = a$ for all $a \in G$. Think of it as a symmetry operation which does not change anything.
3. For every $a \in G$, there is an *inverse element* $a^{-1} \in G : \; a \circ a^{-1} = a^{-1} \circ a = e$. Hence, we are able to reverse our symmetry operation – think of reversing some reflection.

Let us consider a couple of examples of groups:

Example 1: The set of integers $\mathbb{Z}$ together with the operation $a \circ b := a + b$ is a group with neutral element $e := 0$ and inverse element $a^{-1} := -a$ for any $a \in \mathbb{Z}$.

Example 2: For any $k \in \mathbb{N}$, the set $\mathbb{Z}_k := \{0, 1, 2, \ldots, k-1\}$ becomes a group via $a \circ b := a + b \mod k$, where by $\mod k$ we denote the remainder after division by k.[3] It is called the *cyclic group (of order k).*

Example 3: The set consisting only of one element $\{e\}$ is a group, the *trivial group.* The map $\circ$ is defined as $e \circ e := e$.

Example 4: If S_n is the set of all one-to-one maps from $\{1, \ldots, n\}$ to itself, defining $\circ$ as the composition of maps, we obtain the *symmetric group.*[4]

Example 5: Given $n \in \mathbb{N}$, consider the set $M_n(\mathbb{R})$ of all $n \times n$ matrices with real numbers as entries; so $A \in M_n(\mathbb{R})$ is given by n^2 numbers $a_{ij} \in \mathbb{R}$ with

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[3] More explicitly, we obtain $a \circ b = a + b \mod k$ as follows: First compute $a + b \in \mathbb{Z}$ as in Example 1; then find a number $c \in \mathbb{Z}_k$ such that $(a + b) - c = km$ for some $m \in \{0, 1\}$; finally, put $a \circ b = c$. In other words: We identify k and 0. The neutral element is $e := 0$ and the inverse element is $a^{-1} = k - a$.

[4] The neutral element is the identity map $\text{id} : \{1, \ldots, n\} \to \{1, \ldots, n\}, \text{id}(x) := x$; the inverse element for $\sigma \in S_n$ is given by the inverse of the map σ.

$i, j \in \{1, \ldots, n\}$. We define the product $A \cdot B$ of two matrices $A = (a_{ij})$ and $B = (b_{ij})$ as the matrix $C = (c_{ij})$ whose i-j-th entry is given by

$$c_{ij} := \sum_{k=1}^{n} a_{ik} b_{kj} := a_{i1} b_{1j} + a_{i2} b_{2j} + \ldots + a_{in} b_{nj}.$$

We define the transpose of $A = (a_{ij})$ as $A^t := (a_{ji})$. We put $E_n := (\delta_{ij}) \in M_n(\mathbb{R})$, where δ_{ij} is the Kronecker delta: that is, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Now, $O_n \subset M_n(\mathbb{R})$ is the set of all matrices $A \in M_n(\mathbb{R})$ such that

$$A \cdot A^t = A^t \cdot A = E_n.$$

With the above matrix multiplication, O_n becomes a group, called the *orthogonal group*.[5]

1.3 Actions of groups

Coming back to the task of putting the notion of symmetry on more solid mathematical ground, we need to speak about *actions of groups*. So if X is a set and G is a group, we say that G *acts* on X if there is a map $\alpha : G \times X \to X$ such that:

1. We have $\alpha(e, x) = x$ for the neutral element $e \in G$ and any $x \in X$. Again, the neutral element "does not change anything".
2. We have $\alpha(ab, x) = \alpha(a, \alpha(b, x))$ for all $a, b \in G$ and all $x \in X$. Thus, first acting with b and then with a is the same as first combining a and b to a new element ab in G – and then acting with this new element.[6]

Usually, the set X has some additional structure; for example, a triangle has vertices and edges. Therefore, one requires that the group action respects that structure; in the example, a vertex should be mapped by any $\alpha(a, \cdot)$ to a vertex, not to some other point on the edge of the triangle. When we talk about groups acting on objects in what follows, it should be understood that we imply that the group action respects the structure of the object.

It is often useful to require a further property of group actions: that for different group elements a and b the corresponding mappings $\alpha(a, \cdot)$ and $\alpha(b, \cdot)$ are different from each other; thus, we still have full information about the group just by knowing its action on a given object. In this case, we say that the action is *faithful*.

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[5] The neutral element is E_n and the inverse element of a matrix A is A^t. Note that $A \cdot A^t = E_n$ amounts to $\sum_{k=1}^{n} a_{ik} a_{jk} = \delta_{ij}$, whereas $A^t \cdot A = E_n$ is equivalent to $\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij}$.

[6] To be precise, what we describe is called an action from the *left*. For an action from the *right*, replace $\alpha(ab, x) = \alpha(a, \alpha(b, x))$ by $\alpha(ab, x) = \alpha(b, \alpha(a, x))$.

Note that the cyclic group $\mathbb{Z}_3 = \{0, 1, 2\}$ of order three acts faithfully on the equilateral triangle by rotating the triangle by 120°.[7] What's more, the symmetric group S_3 acts on the equilateral triangle by permuting the three vertices. One may observe that $\mathbb{Z}_3 \subsetneq S_3$ holds, that is, $\mathbb{Z}_3$ is a subgroup of S_3.[8] Secondly, we see that $\mathbb{Z}_4$ acts faithfully on the square by rotations by multiples of 90°, but S_4 does not, since for instance permuting only the upper two vertices but not the lower two does not give back a square. As a third example, the symmetric group S_n acts on a set of n points by permutation of the points.[9]

We note that actions of groups model exactly the symmetry operations that we had in mind!

1.4 Groups as a formalism for symmetry

We may now ask: What is the "maximal" group G acting faithfully on a given object X? In other words: is there a group G acting faithfully on X such that any other group H acting faithfully on X is a subgroup of G? If we may find such a group G and if it is unique, we call it the *symmetry group* $\mathrm{Sym}(X)$ *of* X.

One can check that the symmetry group of the equilateral triangle is S_3 – the cyclic group $\mathbb{Z}_3$ also acts on the triangle, but it is not maximal. As for the square, again, the cyclic group $\mathbb{Z}_4$ is not maximal, but in this case S_4 is too big – it does not act on the square. The answer is that the symmetry group of the square is the so-called "dihedral group" D_4.[10]

Instead of considering the circle, let us consider its higher-dimensional analog, the *sphere*:

$$\mathbb{S}^{n-1} := \left\{ (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i^2 = x_1^2 + x_2^2 + \ldots + x_n^2 = 1 \right\} \subset \mathbb{R}^n.$$

For $n = 2$, the sphere $\mathbb{S}^1$ is the circle of radius 1, and for $n = 3$, it is the surface of a ball of radius 1. Now, the symmetry group of the sphere $\mathbb{S}^{n-1}$ turns out to be the orthogonal group O_n.[11]

The symmetry group of a set of n points is clearly S_n, but what about a graph Γ? Let's first be a bit more precise about what a (finite) graph is. It is a

[7] More precisely, for $a \in \mathbb{Z}_3$, the map $\alpha(a, \cdot)$ rotates the triangle by $a \cdot 120°$.

[8] A group H is a subgroup of a group G if H is a subset of G and the operation $\circ$ on G restricted to H is to the group operation of H. As for $\mathbb{Z}_3 \subset S_3$, let $\sigma_3 : \{1, 2, 3\} \to \{1, 2, 3\}$ be the map in S_3 given by $\sigma_3(1) := 2$, $\sigma_3(2) := 3$ and $\sigma_3(3) := 1$. Then $\{e, \sigma_3, \sigma_3^{-1}\} \subset S_3$ is a group with $\circ$ coming from S_3 – it is a subgroup of S_3 which is isomorphic to $\mathbb{Z}_3$ since we may label its elements as $\{0, 1, 2\}$ and recover the group operation $\circ$ of $\mathbb{Z}_3$.

[9] Let $X = \{1, \ldots, n\}$ and $G = S_n$. Define $\alpha(\sigma, x) := \sigma(x)$ for $\sigma : \{1, \ldots, n\} \to \{1, \ldots, n\}$.

[10] The dihedral group has eight elements, whereas $\mathbb{Z}_4$ has only four.

[11] A matrix $A = (a_{ij}) \in M_n(\mathbb{R})$ acts on a vector $(x_1, \ldots, x_n) \in \mathbb{R}^n$ via matrix multiplication $y := Ax \in \mathbb{R}^n$, where $y_i := \sum_{j=1}^{n} a_{ij} x_j$. Using $\sum_{k=1}^{n} a_{ki} a_{kj} = \delta_{ij}$, it is easy to check that $Ax \in \mathbb{S}^{n-1}$ whenever $x \in \mathbb{S}^{n-1}$. Thus, $\alpha(A, x) := Ax$ defines an action of O_n on $\mathbb{S}^{n-1}$.

(finite) set V of *vertices*, some of which are connected by *edges*. The set E of edges is a subset of $V \times V$;[12] we say that two vertices i and j are connected by an edge if the pair (i, j) is in E.

Note that we may view the equilateral triangle, the square, and also the set of n points as graphs,[13] so the situation of graphs covers all the previous examples except for the sphere. Given a finite graph $\Gamma = (V, E)$ with vertices $V = \{1, \ldots, n\}$ and edges $E \subset V \times V$, we say that a map $\sigma \in S_n$ is a symmetry, or, in other words, an *automorphism* of the graph, if we have: (i, j) is an edge if and only if $(\sigma(i), \sigma(j))$ is an edge. Hence, we reorder or, in other words, *permute* the vertices, but if two vertices i and j are connected by an edge, then they ought to be connected also after applying the permutation σ, and vice versa. The set of all such automorphisms forms a subgroup of S_n, and we define the *automorphism group of* Γ[14] as:

$$\mathrm{Aut}(\Gamma) := \{\sigma \in S_n \,|\, \text{for } i, j \in V : (i, j) \in E \iff (\sigma(i), \sigma(j)) \in E\} \subset S_n.$$

1.5 Take away message no. 1

We conclude that the take away message of this chapter is:

$$\textbf{symmetry} = \textbf{groups}$$

Note that symmetry can serve as a means to distinguish objects: if X and Y are two objects with different symmetry groups $\mathrm{Sym}(X) \neq \mathrm{Sym}(Y)$, then $X \neq Y$. So, distinguishing the geometrical objects X and Y becomes a problem in group theory.[15]

2 Quantum spaces

Let's go quantum now! The main feature of "quantum theories", on the mathematical side, is "noncommutativity", so let's take a look at it first.

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[12] Hence, we assume that our graph has no multiple edges.

[13] More precisely, as undirected graphs: For the triangle, take $V = \{1, 2, 3\}$ and $E = V \times V$; for the square, take $V = \{1, 2, 3, 4\}$ and $E = \{(1, 2), (2, 3), (3, 4), (1, 4)\}$; the n points are the graph $V = \{1, \ldots, n\}$ and $E = \emptyset$.

[14] Fun question: What is the smallest $n > 1$ such that there is a graph $\Gamma = (\{1, \ldots, n\}, E)$ with $\mathrm{Aut}(\Gamma)$ being the trivial group? In other words, what is the number of vertices of the smallest "non-symmetric" graph?

[15] See also the "fundamental group" in topology, the "crystallographic groups" in chemistry, or many other examples of such a strategy.

2.1 Noncommutativity

Recall the commutativity law for real numbers: Given two numbers $a, b \in \mathbb{R}$, we have:

$$a \cdot b = b \cdot a.$$

In other words, the multiplication in $\mathbb{R}$ is commutative. Is every multiplication commutative? Well, we have encountered matrices $A = (a_{ij}) \in M_n(\mathbb{R})$ before and we have defined a multiplication on it. We may easily find examples[16] of matrices $A, B \in M_n(\mathbb{R})$ such that

$$A \cdot B \neq B \cdot A.$$

We observe that the multiplication in $M_n(\mathbb{R})$ is *noncommutative*! This fact is of crucial importance, for instance in quantum physics: on the atomic level, it makes a difference whether the position is measured first and then the momentum – or vice versa. In fact, the first measurement changes the state of the system one is trying to measure; this is known as the *uncertainty principle*. One may think of measuring the position or the momentum as multiplying a matrix with a vector. Expressed in terms of matrices, the uncertainty principle means that the two matrices to not commute.

2.2 A bit of topology

Another ingredient for our journey to the quantum world comes from topology and is a bit less intuitive; nevertheless, let us take a quick glance. Recall that we call a function $f : [0, 1] \to \mathbb{R}$ on the interval $[0, 1]$ *continuous* if for any point $x \in [0, 1]$ "moving a bit to the left or right does not change $f(x)$ too much". With tools from topology, this may be put into a more robust mathematical framework [17] and it may be generalized to functions $f : X \to \mathbb{R}$ on arbitrary "topological spaces" X.

Some of these spaces X behave more nicely than others and they are called *compact spaces*. Examples of compact spaces are all finite subsets of $\mathbb{R}$, all closed intervals $[a, b]$ in $\mathbb{R}$ and all finite unions of them. The set of all positive real numbers, however, or $\mathbb{R}$ itself, are not compact.

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[16] For $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ 5 & 0 \end{pmatrix}$ we have $A \cdot B = \begin{pmatrix} 11 & 1 \\ 23 & 3 \end{pmatrix} \neq \begin{pmatrix} 4 & 6 \\ 5 & 10 \end{pmatrix} = B \cdot A$.

[17] A *topology* on Y is a set of subsets $U \subset Y$ which does not get changed by performing certain operations; in the case of $\mathbb{R}$, you may simply think of unions of (possibly infinitely many) open intervals (a, b). Now, a map $f : X \to Y$ is continuous if the preimage $f^{-1}(U)$ belongs to the topology of X whenever U belongs to the topology of Y.

2.3 Algebras of functions

We have met the multiplication of numbers as well as of matrices. We can define a multiplication on yet another set. Consider the set of continuous functions on the interval $[0, 1]$:

$$C([0,1]) := \{f : [0,1] \to \mathbb{R} \,|\, f \text{ is continuous}\}.$$

This set admits a multiplication defined as a pointwise operation:

$$(f \cdot g)(x) := f(x) \cdot g(x).$$

We also have a pointwise addition of functions, and we may multiply any function f with any number $\lambda \in \mathbb{R}$. Checking some compatibility rules for these operations, we infer that $C([0, 1])$ is an *algebra*, the algebra of continuous functions. But this set has even more structure! Namely, we have a *norm*, in this case the norm that assigns to each function its maximal value:

$$||f||_\infty := \max\{|f(x)| \,|\, x \in [0,1]\}.$$

A norm in general is a kind of a generalization of the absolute value $|x|$ of a number $x \in \mathbb{R}$.

More generally, we may define for any compact space X the set of continuous functions:

$$C(X) := \{f : X \to \mathbb{R} \,|\, f \text{ is continuous}\}$$

In fact, compactness of X guarantees that we may define a norm $|| \cdot ||_\infty$ as above.[18]

Summarizing, for a compact space X, the set $C(X)$ has the nice structure of a C^**-algebra*, that is, it is an algebra[19] equipped with a norm satisfying a few compatibility and topological properties. Furthermore, the multiplication is commutative, as clearly $f \cdot g = g \cdot f$. Now, let's be courageous and consider algebras satisfying all axioms of a C^*-algebra – without the requirement that the multiplication is commutative. For instance, the set of all $n \times n$ matrices $M_n(\mathbb{R})$ is such a *noncommutative* C^*-algebra.

Then, a "Fundamental Theorem in C^*-algebras"[1, II.2.2.4] characterizes exactly the commutative C^*-algebras:

[18] Recall that $\mathbb{R}$ is not compact, and indeed, the set $\{|f(x)| \,|\, x \in \mathbb{R}\}$ might be unbounded for certain continuous functions $f : \mathbb{R} \to \mathbb{R}$, so the maximum of a function doesn't need to exist.

[19] To be precise, we shall consider complex-valued functions and complex unital algebras; however, in order to keep it simple, let's pretend to work over the real numbers only.

> A (unital) C^*-algebra is commutative if and only if it is "isomorphic" to $C(X)$ for some compact space X.

(If two mathematical objects are *isomorphic* to each other, it means that they are essentially the same in some precisely defined sense.) This is a nice but quite abstract theorem. Let's take a breath and a step back.

2.4 "Quantum mathematics"

So, this Fundamental Theorem in C^*-algebras – what is it about? Isn't it just abstract nonsense? Well, this is one of these points where mathematics splits from its nature as a toolbox for natural sciences – and becomes a science of its own. We can take the above mentioned theorem as an abstract theorem – or as the starting point for a whole new philosophy!

Just recall what we have so far: Given a compact space X, its algebra of functions $C(X)$ is a commutative C^*-algebra. Conversely, any commutative C^*-algebra is of the form $C(X)$ for some compact space X. So, apparently, we may identify X with $C(X)$ in some sense, right? Now, let's be abstract mathematicians: once we have done this identification – what are these mysterious *noncommutative* C^*-algebras then? Somehow, in our imagination, we can identify them with *compact quantum spaces*! What does this mean? On a *technical level*, we only work with commutative or noncommutative C^*-algebras; but on an *intuitive level* we work with compact spaces or compact quantum spaces extending the above identification.

This thought looks completely crazy at first sight, but it turns out to be an extremely powerful machinery for our imagination, and it is just the starting point of a whole "noncommutative dictionary":[20]

[20] C^*-algebras [1, 3] were introduced by Israel Gelfand and Mark Naimark in the 1940s; von Neumann algebras [1, 5] were introduced by Francis Murray and John von Neumann in the 1930s; free probability theory [10, 17] was initiated by Dan-Virgil Voiculescu in the 1980s; noncommutative geometry [2, 4] is a project by Alain Connes starting in the 1980s. In principle, one could also add "free analysis" [6, 16] (originating in the 1970s) by Joseph Taylor, as a noncommutative analog of complex analysis; and "quantum information theory" [12, 19] (1980s) as an analog of information theory – but the philosophy behind these two theories is less based on "noncommutative algebras of functions". However, they should be seen as the "quantum versions" of the corresponding classical theories, having links to the other listed theories.

classical theory		*noncommutative version*
topology (compact spaces)	$\longleftrightarrow$	C^*-algebras
measure theory	$\longleftrightarrow$	von Neumann algebras
probability theory	$\longleftrightarrow$	free probability theory
differential geometry	$\longleftrightarrow$	noncommutative geometry

We summarize this chapter in form of the following take away message:

compact spaces	=	**commutative algebras of functions**
compact quantum spaces	=	**noncommutative algebras of functions**

3 Quantum symmetry and quantum groups

Given a compact space X, its symmetries can be encoded in the form of a group. Now, passing to a compact quantum space – how about its symmetries? It turns out that groups are not enough to describe them; again, we need to go quantum: to quantum groups!

3.1 Compact quantum groups

Let us recall two main ingredients of this snapshot: Firstly, a group is a set G together with a map

$$\circ : G \times G \to G$$

satisfying certain axioms. Secondly, the passage from the classical world to the quantum world was performed according to the following recipe: take a compact space X; consider the algebra $C(X)$ of continuous functions on X; extract interesting properties of this algebra; consider algebras which share all of these properties with $C(X)$ (that is, C^*-algebras), with the only difference that we allow the multiplication to be noncommutative – we obtain a theory of compact quantum spaces!

Now, let's apply the same recipe to compact groups G – groups which are also a topological space, and a compact one![21] So, we first pass to $C(G)$. What are interesting properties of this algebra (besides those for a general $C(X)$, where X is a compact space)? A compact group is not only a compact set G, but we also have the map $\circ$. How does it behave on the level of $C(G)$? By

[21] As an example, consider O_n.

composition with $\circ$, we obtain the following map:

$$\Delta : C(G) \to C(G \times G)$$
$$\Delta(f)(a, b) := f(a \circ b)$$

But how to pass from $C(G)$ to more general algebras A? Luckily, we have that $C(G \times G)$ is isomorphic to another algebra, called the "tensor product" $C(G) \otimes C(G)$ of $C(G)$ with itself. Thus, if A is any (possibly noncommutative) C^*-algebra possessing a map

$$\Delta : A \to A \otimes A$$

which shares some characteristic features with $\Delta : C(G) \to C(G) \otimes C(G)$, we may speak of it as a *compact quantum group*! A quick check then yields: every compact group is a compact quantum group – but not every compact quantum group is a compact group. Thus, compact quantum groups are an honest generalization of compact groups.[22]

3.2 Quantum symmetries in noncommutative geometry

In perfect analogy to the classical situation of Chapter 1, we may define actions of compact quantum groups on quantum spaces and we may define the *quantum symmetry group* QSym of such a quantum space.[23] Let us consider a concrete example. We view S_n as permutation matrices. [24] There is a quantum version of the symmetric group S_n called the *free symmetric quantum group* S_n^+ [18]. We may imagine S_n^+ as matrices $A = (a_{ij})$ with matrices as entries $a_{ij} \in M_m(\mathbb{R})$ satisfying the same conditions as permutation matrices. [25]

One can check that the free symmetric quantum group S_n^+ is the symmetry group of n points [20]! Wait, wasn't this S_n? True, but the question is: In which class are we trying to find our object modeling the symmetries of n points? Within the category of groups, S_n is the right object; however, within

[22] In fact, we also have a "Fundamental Theorem in compact quantum groups"[15, 5.1.3]: Let (A, Δ) be a compact quantum group. Then A is commutative if and only if A is isomorphic to $C(G)$ for some compact group G.

[23] However, depending on the choice of certain regularity assumptions for the actions, we might obtain different definitions of QSym.

[24] Given a permutation $\sigma \in S_n$, define $A_\sigma := (\delta_{i\sigma(j)}) \in M_n(\mathbb{R})$. Identifying a point $k \in \{1, \dots, n\}$ with the vector e_k consisting of 1 at the k-th entry and 0 otherwise, we observe that the matrix $A_\sigma := (\delta_{i\sigma(j)})$ acts by $A_\sigma e_k = e_{\sigma(k)}$. Thus, $\sigma \mapsto A_\sigma$ is a *representation* of S_n as matrices. Observe that every matrix $A_\sigma = (a_{ij})$ satisfies $\sum_{k=1}^n a_{ik} = \sum_{k=1}^n a_{kj} = 1$.

[25] For the entries $a_{ij} \in M_m(\mathbb{R})$, we require $\sum_{k=1}^n a_{ik} = \sum_{k=1}^n a_{kj} = E_m$. Moreover, we make the additional technical assumption $a_{ij} = a_{ij}^2 = a_{ij}^t$.

the category of quantum groups, S_n^+ is the correct one.[26] The funny thing is that S_n can be seen as a subgroup of S_n^+. [27] This means that we have more possibilities for "quantum permuting" n points than in the classical world! In fact, let us compare two examples of permutation matrices in S_4 and in S_4^+ respectively:

in S_4

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

in S_4^+

$$\begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix}$$

While the example in S_4 maps the second point to the fourth position, the example in S_4^+ maps it "partially to the third position and partially to the fourth one". This intuition is a bit shorthand, but not so far from the truth.

3.3 Take away message no. 3

Groups are an appropriate formalism to capture symmetries of classical spaces. However, passing to quantum spaces, we should use quantum groups rather than groups. Our noncommutative dictionary may be extended to:

classical theory		*noncommutative version*
compact groups	$\longleftrightarrow$	compact quantum groups

And our last take away message is:

quantum symmetry = quantum groups

Let us remark that there are cases where $\mathrm{QSym}(X) \neq \mathrm{QSym}(Y)$ holds but $\mathrm{Sym}(X) = \mathrm{Sym}(Y)$. So, quantum groups may help to distinguish X and Y in cases when groups fail.[28]

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[26] There is a canonical way to view n points as a quantum space following $X \mapsto C(X)$ [20].

[27] You basically have to check that if $m = 1$, then the conditions for the elements $a_{ij} \in M_m(\mathbb{R})$ characterize permutation matrices. So, S_n^+ is like S_n with higher-dimensional m.

[28] You may find examples for instance amongst quantum automorphism groups of graphs [14, 13], see Section 1.4 for the classical counterpart. By the way, did you find the smallest $n > 1$ such that there is a graph on n points having trivial automorphism group? It is $n = 6$, an example having edge set $\{(1,2),(2,3),(3,4),(4,5),(3,5),(4,6)\}$.

3.4 Disclaimer: algebraic vs. analytical approach

In this snapshot, we chose an analytical/topological approach [15, 11] to quantum groups, but there is also an algebraic one [7, 8, 9]. The point is that you have a choice which kind of algebras you want to "quantize". So, while we associated the algebra of continuous functions $C(G)$ to a group G and extracted its properties in order to define what a quantum group is, one could also associate the algebra of polynomials over G or some "universal enveloping algebra". Depending on this a priori choice, one obtains different approaches to quantum groups.[29]

This aspect – that the deformation of the algebras associated to groups is a major ingredient of the theory of quantum groups – makes it impossible to give a general and overall definition of what a quantum group is. This is different from the classical situation, where you first give a definition of a group and then define a compact group as a group with some additional structure – in the case of compact quantum groups, the property "compact" is already part of the definition! Hence, we may define quantum groups based on deformations of their algebra of polynomials or quantum groups based on deformations of their algebra of continuous functions – and we obtain two different definitions of quantum groups. We are only in the beginning of investigating the links between these different approaches.

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Further reading

For a snapshot on (non-quantum) groups and how to compute with them, we would like to references the reader to Snapshot 3/2018 *Computing with symmetries* by Colva M. Roney-Dougal.

[29] The analytical/topological one has been initiated by Stanisław Woronowicz [21, 22]; the algebraic one has Vladimir Drinfeld, Michio Jimbo, and Yuri Manin as three of its most famous forefathers.

Image credits

Figure 1. "Butterfly". Adapted from https://cdn.pixabay.com/photo/2016/07/18/00/18/butterfly-1525067_960_720.jpg, originally licensed under the Pixabay License, visited on November 1, 2019.

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