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Mini-Workshop: Recent Progress in Path Integration on Graphs and Manifolds

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ABSTRACT. Ever since Richard Feynman's PhD thesis, path integrals have played a decisive role in mathematical physics. While it is well-known that such formulae can hold only formally, it was Mark Kac who realized that by replacing the unitary group by the heat semigroup, one obtains well-defined and rigorous formulae. Following this pioneering work, Feynman-Kac path integral formulae have been adapted to several situations and generalized into several directions providing the central focus of this workshop.

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Introduction by the Organizers

The workshop *Recent Progress in Path Integration on Graphs and Manifolds* has been organised by Batu Güneysu (Bonn), Matthias Keller (Potsdam), Kazumasa Kuwada (Sendai) and Anton Thalmaier (Luxembourg). It was attended by 14 participants across Europe and Australia and it allowed young researchers to meet leading experts in the field. Especially, the diverse backgrounds – analysis, geometry, mathematical physics and probability theory – created a productive atmosphere leading to discussions on new developments, advance projects and establishing new collaborations.

The workshop activities consisted of 11 extended talks which were enhanced by intensive collaboration of the participants in the late afternoon and the evenings. Each of the talks and especially the ones contributed by the young researchers were followed by a lively discussion of the new developments.

Path integral formulae are a universal phenomenon playing a central role in various fields of mathematics: Let us consider the Schrödinger operator $-\hbar^2\Delta + V$ in the Hilbert space $L^2(\mathbb{R}^m)$ of square integrable functions $\Psi : \mathbb{R}^m \rightarrow \mathbb{C}$, where $V : \mathbb{R}^m \rightarrow \mathbb{R}$ is an (electric) potential. Then, following R. Feynman (1948), for all $\Psi \in L^2(\mathbb{R}^m)$, $t \geq 0$, $x \in \mathbb{R}^m$, one expects a path integral formula of the form

$$(1) \quad e^{it(\hbar^2\Delta - V)}\Psi(x) = \frac{1}{N(t)} \int \Psi(\gamma(t)) e^{\frac{i}{\hbar}S_t(\gamma)} \mathcal{D}^x\gamma,$$

where

- $i = \sqrt{-1}$ is the imaginary unit,
- $\Delta = \sum_{j=1}^m \partial_j^2$ is the Laplace operator,
- $S_t(\gamma)$ is the classical action functional

$$S_t(\gamma) = \int_0^t |\dot{\gamma}(s)|^2 ds - \int_0^t V(\gamma(s)) ds,$$

- \mathcal{D}^x is the “Lebesgue measure” on the space $C([0, \infty), \mathbb{R}^m)$ of *continuous* paths $\gamma : [0, \infty) \rightarrow \mathbb{R}^m$, where \mathcal{D}^x is assumed to be concentrated on paths γ with $\gamma(0) = x$,
- $N(t)$ is a normalization constant such that

$$\frac{1}{N(t)} \int e^{\frac{i}{\hbar}S_t(\gamma)} \mathcal{D}^x\gamma = 1.$$

Note that

$$\mathbb{R}^m \times (0, \infty) \ni (x, t) \mapsto \Psi_t(x) := e^{it(\hbar^2\Delta - V)}\Psi(x) \in \mathbb{C}$$

is the unique solution of the Schrödinger equation

$$\partial_t \Psi_t(x) = i(\hbar^2\Delta - V(x))\Psi_t(x), \quad \Psi_0(x) = \Psi(x),$$

so that according to the basic axiomatic framework of quantum mechanics, Ψ_t is interpreted as the state at time t , given the initial state has been Ψ . Feynman’s path integral gives a brilliant intuition behind the rather abstract (but mathematically rigorous) operator theoretic framework behind quantum mechanics: in order to calculate the quantum mechanical state at the time t , one has to integrate over all possible trajectories, where each trajectory γ is weighted according to the complex number $e^{\frac{i}{\hbar}S_t(\gamma)}$. Moreover, as $\hbar \rightarrow 0$ the expression $e^{\frac{i}{\hbar}S_t(\gamma)}$ oscillates rapidly and the γ ’s (which are precisely the classical trajectories of the system!) make the action stationary and give the main contribution in the semiclassical limit.

While Feynman’s path integral formula is very illustrative, it lacks of mathematical rigour for several reasons:

- the action $S_t(\gamma)$ of an arbitrary *continuous* path γ is infinite,
- the normalization constant $N(t)$ is infinite, when interpreted literally in Feynman’s derivation of the formula,
- it can be proven that there exists no translation invariant measure on $C([0, \infty), \mathbb{R}^m)$, so the formal Lebesgue measure \mathcal{D}^x does not exist.

One way to prevent these mathematical difficulties has been noted by M. Kac in 1949, based on N. Wiener's prior mathematical work on measure and probability theory. Let us set $\hbar = 1$ for simplicity. In Kac's approach, one switches from

$$(e^{it(\Delta/2-V)})_{t \geq 0} \subset \mathcal{L}(L^2(\mathbb{R}^m)) := \text{bounded operators in } L^2(\mathbb{R}^m),$$

to the *heat semigroup*

$$(e^{t(\Delta/2-V)})_{t \geq 0} \subset \mathcal{L}(L^2(\mathbb{R}^m)),$$

which is also called a *Schrödinger semigroup* in the mathematical physics literature, thus making the substitution $t \rightarrow it$ ('Wick rotation'). Formally, the RHS of (1) becomes

$$\frac{1}{N(t)} \int_{C([0,\infty), \mathbb{R}^m)} \Psi(\gamma(t)) \exp\left(-\int_0^t |\dot{\gamma}(s)|^2 ds - \int_0^t V(\gamma(s)) ds\right) \mathcal{D}^x \gamma.$$

In a sense that can be made precise, the product of the three 'bad' expressions

$$\frac{1}{N(t)}, \quad \exp\left(-\int_0^t |\dot{\gamma}(s)|^2 ds\right), \quad \mathcal{D}^x \gamma,$$

becomes a well-defined probability measure \mathbb{P}^x on $C([0, \infty), \mathbb{R}^m)$, the *Wiener measure*. It is the law of a Brownian motion in \mathbb{R}^m starting from x and thus directly linked with probability theory. The outcome of these observations is the mathematically completely rigorous *Feynman-Kac path integral formula*

$$e^{t(\Delta/2-V)} \Psi(x) = \int \Psi(\gamma(t)) \exp\left(-\int_0^t V(\gamma(s)) ds\right) \mathbb{P}^x(d\gamma).$$

Using the (unnormalized) pinned Wiener measures $\mathbb{P}_t^{x,y}$, so that $\mathbb{P}_t^{x,y}$ is essentially the law of a Brownian bridge from starting in x and ending up in y at the time t , a formula which is equivalent to the Feynman-Kac formula can be given at the level of integral kernels:

$$e^{t(\Delta/2-V)}(x, y) = \int \exp\left(-\int_0^t V(\gamma(s)) ds\right) \mathbb{P}_t^{x,y}(d\gamma).$$

In the past decades this formula has been generalized to several directions. The central observation behind these extensions is as follows: in order to define the analog of Brownian motion or, equivalently, the family of Wiener measures \mathbb{P}^x , one starts from a (densely defined) closed nonnegative quadratic form Q in $L^2(X, \mu)$, where X is a locally compact space and μ is a Radon measure on X . By Kato's abstract theory, such a Q canonically induces a self-adjoint operator $H \geq 0$ in $L^2(X, \mu)$. Then under mild additional analytic assumptions on Q that turns the latter into a so called *regular Dirichlet form*, Fukushima's theory shows the existence of an (essentially uniquely determined) strong Markov family of probability measures \mathbb{P}^x , $x \in X$, on the space of right-continuous paths $\gamma : [0, \infty) \rightarrow \hat{X}$ having left-limits, such that

$$(2) \quad e^{-tH} \Psi(x) = \int \Psi(\gamma(t)) \mathbb{P}^x(d\gamma).$$

Above, $\hat{X} = X \cup \{\infty\}$ denotes a one-point compactification of X . Formula (2) suggests to consider H as some abstract Laplace operator, and in fact the Feynman-Kac formula remains to hold in the form

$$e^{-t(H+V)}\Psi(x) = \int \Psi(\gamma(t)) \exp\left(-\int_0^t V(\gamma(s))ds\right) \mathbb{P}^x(d\gamma),$$

for a very large class of potentials $V : X \rightarrow \mathbb{R}$. For example, if X is a (weighted) Riemannian manifold with μ its (weighted) volume measure, then one can take

$$Q(f) := \int |\nabla f|^2 d\mu, \quad \text{dom}(Q) = W_0^{1,2}(X),$$

and then H corresponds to the Friedrich's realization of the Laplace-Beltrami operator $-\Delta$ and the \mathbb{P}^x 's correspond to the Riemannian Brownian motion (having a drift in the weighted case). The main strength of Fukushima's theory, however, is that one can deal simultaneously with local and nonlocal situations such as for example the Laplacian on a weighted infinite graph, or a sufficiently well-behaved pseudodifferential operator (leading to processes with jumps in both cases). An important question in the above abstract context is that of *stochastic completeness*, namely whether or not one has

$$\mathbb{P}^x\{\gamma : \gamma(t) \in X\} = 1 \quad \text{for all } x \in X, t \geq 0.$$

For example, there exist geodesically complete Riemannian manifolds which are not stochastically complete. On the other hand, a very general result by A. Grigor'yan from 1987 states that on geodesically complete Riemannian manifolds, stochastic completeness boils down to a volume growth assumption, which is satisfied if the Ricci curvature is sufficiently bounded from below, say by a constant. The validity of a natural discrete analog of Grigor'yan's result on an infinite weighted graph has turned out to be a very subtle business and could only be established recently.

In many geometric situations one can obtain Feynman-Kac formulae for more general operators than those of the form $-\Delta + V$: most importantly, there is a Feynman-Kac formula for the semigroups $e^{-t(\nabla^* \nabla + V)}$ that are induced by covariant Schrödinger operators of the form $\nabla^* \nabla + V$ acting on metric vector bundles over Riemannian manifolds (and even for bundles over weighted graphs). This class includes the physically very relevant class of Schrödinger operators with magnetic fields, in which case the corresponding Feynman-Kac formula is called *Feynman-Kac-Ito formula*.

Moreover, there even exist probabilistic formulae for the derivatives $\nabla e^{-t(\nabla^* \nabla + V)}$, usually referred to as *Bismut derivative formula*. Some of the state-of-the-art results for geometric inequalities on noncompact Riemannian manifolds, such as the Calderon-Zygmund inequality or the parabolic Harnack inequality, are based on such derivative formulae.

Finally, we would like to mention that (covariant) Feynman-Kac formulae have played an important role in topology, too: for example, in 1984, J.-M. Bismut has given a probabilistic proof of the Atiyah-Singer index theorem, which is based on

such a covariant Feynman-Kac formula. Moreover, around 1985, Atiyah, Bismut and Witten have found a very surprising heuristic connection between a hypothetical Duistermaat-Heckman type localization formula on the loop space of a spin manifold and the Atiyah-Singer index theorem. A variant of this heuristic localization formula has recently been implemented using methods from cyclic homology and Chen's iterated integrals. Moreover, in the past decade the heuristic Duistermaat-Heckman localization formula has led to the development of the hypoelliptic Laplacian and variants thereof, providing a new powerful machinery in geometric analysis.

Filled with deep sadness we dedicate this mini-workshop to our friend, colleague and coorganizer Kazumasa Kuwada who unexpectedly passed away at the end of last year.

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Abstracts

Loop spaces and index theory

JEAN-MICHEL BISMUT

The purpose of the talk was to give a historical account of the connections between loop spaces and index theory as seen through the speaker's eyes, and to explain how these two subjects have cross-fertilized each other.

The starting point was a talk given by M.F. Atiyah at a conference in honor of L. Schwartz in 1983, which was published in [A85], in which Atiyah described ideas communicated to him by Witten. If X is a compact oriented spin manifold of even dimension, let $S^{TX} = S_+^{TX} \oplus S_-^{TX}$ be the corresponding \mathbf{Z}_2 -graded vector bundle of spinors. Let D^X be the Dirac operator acting on $C^\infty(X, S^{TX})$, that exchanges positive and negative spinors. The McKean-Singer formula [MS67] asserts that if $\text{Ind } D_+^X$ is the index of D_+^X , for any $t > 0$,

$$\text{Ind } D_+^X = \text{Tr}_s [\exp(-tD^{X,2}/2)].$$

Let LX be the smooth loop space of X , i.e., LX is the set of smooth maps $x : S^1 \rightarrow X$. Then LX can be made into a Riemannian manifold, equipped with the obvious L_2 metric. Also S^1 acts isometrically on LX , and $K(x) = \dot{x}$ is the corresponding Killing vector field, whose zero set is exactly the manifold X . Let K' be the associated 1 form. Let $d_K = d + i_K$ be the equivariant de Rham operator, which is such that $d_K^2 = L_K$. Since $L_K K' = 0$, then $d_K(d_K K')$. Also

$$d_K K' = |K|^2 + dK'.$$

If $E(x)$ is the energy of $x \in LX$, then $E(x) = |K|^2/2$.

For $t > 0$, set

$$\alpha_t = \exp(-d_K K'/2t).$$

Then $d_K \alpha_t = 0$.

The starting point of Atiyah's argument is that

$$\text{Tr}_s [\exp(-tD^{X,2}/2)] = \int_{LX} \alpha_t.$$

This equality has to be taken with a touch of salt. Indeed the left-hand side is well defined, the right-hand side not at all, or not so much. Passing from the left-hand side to the right-hand side is done by expressing the supertrace in the left-hand side as the integral on the continuous loop space of a well-defined S^1 -invariant measure μ_t using standard Brownian motion on X , and by evaluating the supertrace of the parallel transport on S^{TX} along a loop x as an infinite dimensional Pfaffian. While μ_t is just S^1 -invariant, α_t vanishes under d_K , which is much stronger.

In [B85], I showed that if (E, g^E, ∇^E) is a Hermitian vector bundle with connection, and if D^X still denotes the Dirac operator acting on $C^\infty(X, S^{TX} \otimes E)$,

we still have a more general formal equality,

$$\mathrm{Tr}_s [\exp (-tD^{X,2}/2)] = \int_{LX} \alpha_t \wedge \beta,$$

where β is a canonical d_K -closed form on LX associated with the loop vector bundle LE on LX . The restriction of β to X turns out to be the Chern character form $\mathrm{ch}(E, \nabla^E)$. Also, we still have $d_K(\alpha_t \wedge \beta) = 0$.

For the above formulas to make sense, the loop space LX should be thought as even dimensional and orientable. In the talk, I showed that the orientability of X makes that LX is even-dimensional, and I referred to [A85] for a proof that X is spin implies the orientability of LX .

The point about these formal equalities is that they have a surprising predictive power. Indeed, assume temporarily that LX is a compact even dimensional oriented Riemannian manifold, equipped as above with an isometric action of S^1 generated by a Killing vector field K . Let α_t be as above, and let β be a d_K -closed form on LX . The localization formulas of Duistermaat-Heckman [DH82], Berline-Vergne [BeV83] assert that

$$\int_{LX} \alpha_t \wedge \beta = \int_X \frac{\beta}{e_K(N_{X/LX})},$$

where $e_K(N_{X/LX})$ is the equivariant Euler class of the normal bundle $N_{X/LX}$.

As shown in [A85, B85], if LX is indeed the loop space of X , we have the identity

$$\frac{1}{e_K(N_{X/LX})} = \widehat{A}(TX),$$

so that a formal application of the above formula leads to

$$\mathrm{Ind}(D_+^X) = \mathrm{Tr}_s [\exp (-tD^{X,2})] = \int_X \widehat{A}(TX) \mathrm{ch}(E),$$

a true formula.

This was already puzzling enough. One seemed to have at our disposal two kinds of proofs for the Atiyah-Singer index theorem for Dirac operators: the heat equation proof in all its variants [Gi74, ABP73], that include Getzler's proof [G86], based on the asymptotics as $t \rightarrow 0$ of the heat kernel on the diagonal, and the 'fantastic cancellations' as $t \rightarrow 0$ of the supertrace of the heat kernel anticipated by McKean-Singer [MS67], and a direct formal proof based on the above localization formulas.

In [B86b], we reconciled the two points of view. While originally, we had tried to import in an infinite dimensional setting the existing proofs of localization in equivariant cohomology, the opposite turns out to be true: when interpreted geometrically, the heat equation method is the universal model of a proof of localization formulas in equivariant cohomology that also works in finite dimensions. In a finite dimensional context, we showed that as $t \rightarrow 0$, we have the convergence

of currents on LX ,

$$\alpha_t \rightarrow \frac{\delta_X}{e_K(N_{X/LX}, \nabla^{X/LX})},$$

where $e_K(N_{X/LX}, \nabla^{X/LX})$ is the Chern-Weil representative of the Euler class $e_K(N_{CX/LX})$ associated with the connection $\nabla^{N_{X/LX}}$ that comes from the Levi-Civita connection on TLX . When properly interpreted in infinite dimensions, the above ‘implies’ the ‘fantastic cancellations’ anticipated by McKean-Singer [MS67]. In parallel, we gave a probabilistic proof of the local index theorem [B84a, B84b].

Even though 35 years have passed, there is still a basic misunderstanding on the significance of the above. The point is not to give a nonrigorous easy proof of a known difficult result, but to reinterpret a mysterious known analytic result in geometric terms, and to build up on this geometric understanding to prove new results.

For an illustration of some of the applications of the above line of thought, we refer to our review paper [B11a], in which applications to the local families index theorem and to holomorphic torsion are outlined.

We briefly explained some of these applications. One is the local index theorem for families of Dirac operators [B86a] based on Quillen’s superconnections [Q85b].

If LX is instead taken to be compact and finite dimensional, in [B11a], the canonical construction of an odd current ϵ on LX is given that solves the equation of currents

$$d_K \epsilon = \frac{\delta_X}{e_K(N_{X/LX}, \nabla^{N_{X/LX}})} - 1,$$

that refines on the localization formulas. If we consider a family of such X , and integrate the current ϵ long the fiber, we get odd forms on the base.

By going back to the original infinite dimensional setting, these forms, which should be thought as integrals of currents on the loop space LX , have important analytic and geometric significance. Considerable knowledge can be gained through this loop space interpretation. The degree 1 component of such forms can be viewed as a connection form on the determinant bundle of the family [Q85a, BF86a, BF86b].

If in the above X is instead odd dimensional, LX is now ‘odd dimensional’. As ‘shown’ in [B11a], the integral on LX of ϵ is a fundamental spectral invariant of X , the eta invariant introduced in Atiyah-Patodi-Singer [APS75].

If X is a complex Kähler manifold, LX is also a complex Kähler manifold, and K is a holomorphic vector field. We have a natural splitting $d_K = \bar{\partial}_K + \partial_K$. As shown in [B11a], the above equation of currents is replaced by a kind of Poincaré-Lelong equation, where d_K is replaced by $\bar{\partial}_K \partial_K$, and the integral on LX of the corresponding current is the holomorphic analytic torsion of LX of Ray-Singer [RS73]. Again this knowledge, which seems a first sight to be formal, has considerable predictive power.

Along such lines, we were led to the construction of the hypoelliptic Laplacian, and its applications to the evaluation of semisimple orbital integrals [B05, BL08, B11b], subjects on which I could not say a word, in which ideas coming from path

integrals, probability theory, and the Malliavin calculus play again a fundamental role.

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Ruelle zeta function, resonance, and Anosov flow

SHU SHEN

1. THE FRIED CONJECTURE

Let X be a closed manifold. Recall that a fixed point $x \in X$ of a diffeomorphism $\phi : X \rightarrow X$ is called non degenerate if $\det(1 - D\phi(x))|_{T_x X} \neq 0$. The Lefschetz formula tells us that if ϕ is a diffeomorphism of X with only non degenerate fixed points, then

$$(1) \quad \sum_{i=0}^{\dim X} (-1)^i \text{Tr} [\phi^*|_{H^i(X)}] = \sum_{x \in \text{Fix}(\phi)} \epsilon_x,$$

where ϵ_x is the Lefschetz index defined by

$$(2) \quad \epsilon_x = \text{sgn} \det(1 - D\phi(x))|_{T_x X}.$$

Fried asked if there is a Lefschetz-like formula for flows $(\phi_t)_{t \in \mathbf{R}}$, i.e.,

$$(3) \quad \text{certain topological invariant} = \sum_{\gamma: \text{closed orbits}} \text{certain index of } \gamma.$$

Note that there are two kinds of closed orbits: the trivial closed orbits, i.e., $\phi_t x = x$ for all $t \in \mathbf{R}$, and the non trivial closed orbits. For a non trivial closed orbit γ , denote by $\ell_\gamma \in (0, \infty)$ and $m_\gamma \in \mathbf{N}$ its period and its multiplicity.

Assume:

- A.1) The flow $(\phi_t)_{t \in \mathbf{R}}$ has no trivial closed orbits;
- A.2) The flow $(\phi_t)_{t \in \mathbf{R}}$ has only non degenerate non trivial closed orbits. It means that if γ is a non trivial closed orbit with period ℓ_γ , then for any $x \in \gamma$, $\det(1 - D\phi_{\ell_\gamma}(x))|_{T_x X/\mathbf{R}V(x)} \neq 0$, where V is the generating vector field of $(\phi_t)_{t \in \mathbf{R}}$;
- A.3) There is $C > 0$, for any $T \geq 0$, we have

$$(4) \quad |\{\gamma : \ell_\gamma \leq T\}| \leq C e^{CT}.$$

Note that Assumption A.1) implies that the Euler characteristic number of X vanishes. For a flow $(\phi_t)_{t \in \mathbf{R}}$ under Assumptions A.1)-A.3), Fried conjectured a Lefschetz-like formula (or more precisely a twisted Lefschetz-like formula): if $\rho : \pi_1(X) \rightarrow \text{U}(r)$ is a unitary representation of the fundamental group of X , then

$$(5) \quad \log T_X(F) = \sum_{\gamma} \frac{\epsilon_\gamma}{m_\gamma} \text{Tr}[\rho(\gamma)],$$

where $T_X(F)$ is the analytic torsion of the unitarily flat vector bundle F associated to ρ . However, the sum in the above formula does not necessarily converge. To regularize the sum, we introduce the Ruelle zeta function for the dynamical system $(\phi_t)_{t \in \mathbf{R}}$ with twist $\rho : \pi_1(X) \rightarrow \text{GL}_r(\mathbf{C})$ defined for $\text{Re}(s) \gg 1$ by

$$(6) \quad R_\rho(s) = \exp \left(\sum_{\gamma} \frac{\epsilon_\gamma}{m_\gamma} \text{Tr}[\rho(\gamma)] e^{-s\ell_\gamma} \right).$$

Now, we can state the Fried conjecture [F87, F95] in a precise way.

- B.1) for any representation $\rho : \pi_1(X) \rightarrow \mathrm{GL}_r(\mathbf{C})$, the Ruelle zeta function $R_\rho(s)$ has a meromorphic extension to \mathbf{C} ;
- B.2) if the representation $\rho : \pi_1(X) \rightarrow \mathrm{U}(r)$ is unitary such that $H^*(X, F) = 0$, then the Ruelle zeta function $R_\rho(s)$ is regular at $s = 0$, so that

$$(7) \quad |R_\rho(0)| = T_X(F).$$

The above conjecture was solved by Fried [F86a, F86b] for the geodesic flow on the unit tangent bundle of a hyperbolic manifold. In [S18], following previous contributions by Moscovici-Stanton [MSt91], using Bismut's orbital integral formula [B11], the author affirmed the Fried conjecture for the geodesic flow on the unit tangent bundle of a closed locally symmetric manifold. In [SY17], the authors made a further generalisation to closed locally symmetric orbifolds.

2. ANOSOV FLOW

A flow $(\phi_t)_{t \in \mathbf{R}}$ with generating vector field V is called Anosov, if there is a ϕ_t -invariant continuous splitting

$$(8) \quad TX = \mathbf{R}V \oplus E^u \oplus E^s$$

of C^0 -vector bundles on X and there exist $C > 0, \theta > 0$, and a Riemannian metric on X such that for $v \in E_x^u, v' \in E_x^s$, and $t \geq 0$, we have

$$(9) \quad |D\phi_{-t}(x)v| \leq Ce^{-\theta t} |v|, \quad |D\phi_t(x)v'| \leq Ce^{-\theta t} |v'|.$$

By (8) and (9), Assumptions A.1) and A.2) are satisfied. By a simple estimate on recurrence, Assumption A.3) is also satisfied.

Take a flat vector bundle F with holonomy $\rho : \pi_1(X) \rightarrow \mathrm{GL}_r(\mathbf{C})$. The Lie derivation L_V is a first order differential operator acting on $\Omega^*(X, F)$. Fix a Riemannian metric on X and a Hermitian metric on F . Let $\langle \cdot, \cdot \rangle_{L^2}$ be the L^2 -metric on $\Omega^*(X, F)$. It is elementary to see that there is $C \geq 0$ such that for any $u \in \Omega^*(X, F)$,

$$(10) \quad \mathrm{Re} \langle L_V u, u \rangle_{L^2} \geq -C |u|_{L^2}^2,$$

and also that L_V has a unique natural closed extension in the sense that the minimal and the maximal closed extensions coincide. In particular, for $\mathrm{Re}(s) \gg 1$,

$$(11) \quad (s + L_V)^{-1} : L^2(X, \Lambda^*(T^*X) \otimes F) \rightarrow L^2(X, \Lambda^*(T^*X) \otimes F)$$

is a holomorphic family of bounded operators.

We can state the main results :

- C.1) The operator $(s + L_V)^{-1} : \Omega^*(X, F) \rightarrow \mathcal{D}'(X, \Lambda^*(T^*X) \otimes F)$, which is well defined when $\mathrm{Re}(s) \gg 1$, has a meromorphic extension to \mathbf{C} . The poles are called resonances.
- C.2) The Ruelle zeta function $R_\rho(s)$ has a meromorphic extension to \mathbf{C} , whose zeros and poles are contained in the resonance set. In particular, if 0 is not a resonance, then $R_\rho(0)$ is a well defined non zero complex number.

C.3) Assume that 0 is not a resonance of V . If V' is a vector field close enough to V in the C^∞ -sense, then¹

$$(12) \quad R_\rho^{V'}(0) = R_\rho^V(0),$$

where R_ρ^V and $R_\rho^{V'}$ are the Ruelle zeta functions for the flows induced respectively by V and V' .

The result C.1) was shown by Butterley-Liverani [BuL07], Faure-Sjöstrand [FaSj11], and Adam [Ad18]. All the cited proofs are based on the introduction of Anisotropic spaces. The proofs of C.2) were given by Giulietti-Liverani-Pollicott [GiLP13] and also by Dyatlov-Zworski [DyZ16]. Both of the proofs use the Atiyah-Bott-Guillemin trace formula in an essential way. The stability property C.3) is due to Dang-Guillarmou-Rivière-Shen [DGRS18]. It is based on a variation methods, which can also be obtained by a super symmetric argument [S19].

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¹The flow V' is still Anosov [A67] such that 0 is not a resonance of V' [BuL07].

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Dimension-free Harnack inequalities for Feynman-Kac semigroups

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Using tools from stochastic analysis, we prove various derivative formulas, gradient estimates and Harnack inequalities for Feynman-Kac semigroups with possibly unbounded potentials. The setting is that of a complete and connected Riemannian manifold M with Laplace-Beltrami operator Δ . We suppose V is a smooth function which is, for simplicity, bounded below and denote by $P_t^V f$ the minimal semigroup generated by the operator $\frac{1}{2}\Delta - V$ acting on bounded measurable functions f via the Feynman-Kac formula

$$P_t^V f(x) = \mathbb{E} \left[e^{-\int_0^t V(X_s(x)) ds} f(X_t(x)) \mathbf{1}_{\{t < \zeta(x)\}} \right]$$

for all $t \geq 0$, where $X(x)$ denotes a Brownian motion on M starting at $x \in M$ with explosion time $\zeta(x)$. Reasoning at the level of local martingales, we start by proving Bismut-type differentiation formulas for $P_t^V f$. In particular, denote by $//$ the stochastic parallel transport along the paths of $X(x)$ and by B the anti-development of $//$ to $T_x M$. Then B is a Brownian motion on $T_x M$ starting at the origin. Denote by W the solution to the covariant ordinary differential equation $DW_s = -\frac{1}{2}\text{Ric}^\sharp W_s$ along the paths of $X(x)$ with initial condition $W_0 = \text{id}_{T_x M}$. Suppose D is a regular domain in M and denote by $\tau_D(x)$ the first exit time of $X(x)$ from D . Set $\mathbb{V}_s^x := e^{-\int_0^s V(X_r(x)) dr}$ for $s \geq 0$. Then, using Itô's formula, we prove that if h a bounded adapted process with paths belonging to the Cameron-Martin space $L^{1,2}([0, t]; \text{Aut}(T_x M))$, such that $h_0 = 1$, $h_s = 0$ for $s \geq \tau_D(x) \wedge t$ and $\mathbb{E}[\int_0^{\tau_D(x) \wedge t} |\dot{k}_s|^2 ds] < \infty$, then

$$(dP_t^V f)_x = -\mathbb{E} \left[\mathbb{V}_t^x f(X_t(x)) \mathbf{1}_{\{t < \zeta(x)\}} \int_0^t \langle W_s \dot{h}_s, //_s dB_s \rangle + dV(W_s h_s) ds \right]$$

for all $t \geq 0$. For suitable f , this formula can be used to prove the uniform boundedness of the derivative $dP_t^V f$ which in turn yields a gradient estimate of the form

$$|\nabla P_t^V f| \leq e^{-Kt} P_t^V |\nabla f| + \|\nabla V\|_\infty \left(\frac{1 - e^{-Kt}}{K} \right) P_t^V |f|$$

which is equivalent to $\text{Ric} \geq 2K$ for some $K \in \mathbb{R}$. We prove that for non-negative V , this gradient estimate implies that for all bounded non-negative measurable functions f and $p > 1$ we have

$$(P_t^V f)^p(x) \leq (P_t^V f^p)(y) \exp \left[\frac{pd^2(x, y)}{2(p-1)C_1(t, K)t} + \frac{td(x, y)\|\nabla V\|_\infty}{2C_2(t, K)} \right]$$

for all $x, y \in M$ and $t > 0$, where

$$C_1(t, K) := \frac{e^{2Kt} - 1}{2Kt}, \quad C_2(t, K) := \frac{Kt}{2} \coth\left(\frac{Kt}{2}\right)$$

and where d denotes the Riemannian distance. This is the first version of F.-Y. Wang's dimension-free Harnack inequality [4] to be proven for Feynman-Kac semigroups. We can similarly derive dimension-free shift-Harnack inequalities [5], using an integration by parts formula for the Feynman-Kac semigroup acting on the codifferential of a smooth 1-form.

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Counter-intuitive approximations

CHRISTIAN BÄR

(joint work with Bernhard Hanke)

This talk describes the phenomenon that many maps can be approximated by maps with slightly worse regularity but which have unexpected additional properties. The prototype for such a result is the classical Nash-Kuiper theorem which ensures that short embeddings of Riemannian manifolds into Euclidean space can be approximated by *isometric* C^1 -embeddings [4, 3].

The talk is based on [1].

To formulate the result precisely, we denote by

- V a smooth manifold;
- $\pi : X \rightarrow V$ a smooth vector bundle;
- $k \in \mathbb{N}$ a positive integer;
- $\pi_k : J^k X \rightarrow V$ the k -jet bundle;
- Γ a subsheaf of the sheaf of C^k -sections of X ;
- f a C^k -section on V ;
- N a neighborhood of f in the strong C^{k-1} -topology.

Recall the natural commutative diagram of jet bundles:

$$\begin{array}{ccc} J^k X & \xrightarrow{\pi_{k,k-1}} & J^{k-1} X \\ & \searrow \pi_k & \swarrow \pi_{k-1} \\ & & V \end{array}$$

Here π_k and π_{k-1} are vector bundle projections while $\pi_{k,k-1} : J^k X \rightarrow J^{k-1} X$ is an affine bundle.

We set $J^k \Gamma := \{j^k \gamma(p) \mid \gamma \text{ is a local section of } \Gamma, \text{ defined near } p, p \in V\} \subset J^k X$.

Theorem. *Assume that for each $p \in V$ there is an open neighborhood W of $j^{k-1} f(p)$ in $J^{k-1} X$ and a continuous map $\sigma_W : W \rightarrow J^k X$ such that*

- $\pi_{k,k-1} \circ \sigma_W = \text{id}_W$;
- $\sigma_W(\omega) \in J^k \Gamma$ for each $\omega \in W$.

Then there exists a section \hat{f} of $X \rightarrow V$ with the following properties:

- $\hat{f} \in C_{\text{loc}}^{k-1,1}(V, X)$;
- $\hat{f} \in N$;
- $\hat{f}|_{\mathcal{U}} \in \Gamma(\mathcal{U})$ where $\mathcal{U} \subset V$ is open and dense.

Example 1. We apply the theorem with the following choices: $V = \mathbb{R}$, X is the trivial line bundle so that sections are nothing but real-valued functions, $k = 1$, and Γ is the sheaf of smooth functions with constant derivative K where K is a given constant. The strong C^0 -neighborhood of f is given by $N = \{h \in C^0(\mathbb{R}) \mid |f - h| < \varepsilon\}$. The theorem yields:

For any C^1 -function $f : [0, 1] \rightarrow \mathbb{R}$, any $\varepsilon > 0$ and any $K \in \mathbb{R}$ there exists a Lipschitz function $\hat{f} : [0, 1] \rightarrow \mathbb{R}$ such that:

- $|f - \hat{f}| < \varepsilon$;
- \hat{f} is smooth and satisfies $\hat{f}' = K$ on an open dense subset of $[0, 1]$.

If we apply this to $f(t) = t$, $K = 0$ and $\varepsilon = 0.0001$ then we get a Lipschitz function $\hat{f} : [0, 1] \rightarrow \mathbb{R}$ with $\hat{f}(0) < 0.0001$, $\hat{f}(1) > 0.9999$ and $\hat{f}' = 0$ on an open dense subset. Note that Lipschitz functions are differentiable almost everywhere by Rademacher's theorem and the fundamental theorem of calculus holds. Thus we have

$$\int_0^1 \hat{f}'(x) dx = \hat{f}(1) - \hat{f}(0) > 0.9998$$

which, at first glance, seems to violate $\hat{f}' = 0$ on the open dense subset. The point is that open dense subsets need not have full measure, so there is no contradiction. Clearly, \hat{f} cannot be C^1 in this case.

This function is not to be confused with the Cantor function. The Cantor function is a Hölder continuous function $[0, 1] \rightarrow [0, 1]$ with Hölder exponent $\alpha = \ln 2 / \ln 3$. It has vanishing derivative on an open subset of full measure but it is not absolutely continuous. Hence the fundamental theorem of calculus cannot be applied and the Cantor function is not Lipschitz.

Example 2. We apply the theorem with the following choices: Let V be an analytic surface, let X be the trivial \mathbb{R}^3 -bundle so that sections are maps $V \rightarrow \mathbb{R}^3$. Let $k = 2$ and Γ be the sheaf of analytic embeddings with induced Gauss curvature K where, again, K is a given constant. The theorem yields:

Let $f : V \hookrightarrow \mathbb{R}^3$ be a C^2 -embedding and let N be a neighborhood of f in the strong C^1 -topology. Then there exists a $C^{1,1}$ -embedding $\hat{f} : V \hookrightarrow \mathbb{R}^3$ in N which is analytic and has constant Gauss curvature K on an open dense subset of V .

The Gauss-Bonnet theorem holds for \hat{f} but does not lead to a contradiction because the open dense subset need not have full measure. The regularity of \hat{f} cannot be improved from $C^{1,1}$ to C^2 because then the Gauss curvature would be continuous and hence equal to K everywhere which then would contradict Gauss-Bonnet.

Example 3. As a third application we show:

Let $K \in \mathbb{R}$. Each differentiable manifold of dimension ≥ 2 has a complete $C_{\text{loc}}^{1,1}$ -Riemannian metric which is smooth and has constant sectional curvature $\equiv K$ on an open dense subset.

Again, such a metric cannot be C^2 in most cases.

The proof of the theorem is based on “weak flexibility”, a concept introduced by Gromov in his book [2], see the exercise on p. 111. A full solution to this exercise can also be found in [1].

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Recent Progress in Supersymmetric Path Integrals

MATTHIAS LUDEWIG

(joint work with Batu Güneysu, Florian Hanisch)

Introduction. More than 30 years ago, it was observed by Alvarez-Gaumé [AG84], Atiyah [Ati85] and Witten that there is a very short and conceptual, but formal, i.e. non-rigorous, proof of the Atiyah-Singer index theorem using a supersymmetric version of the Feynman path integral. Reformulating the supergeometry appearing in the work of Alvarez-Gaumé in the language of differential forms, Atiyah is led to consider the differential form integral

$$(1) \quad I(\theta) \stackrel{\text{formally}}{:=} \int_{\text{LX}} e^{-S+\omega} \wedge \theta$$

over the loop space of a Riemannian (spin) manifold X , for suitable differential forms $\theta \in \Omega(\text{LX})$, where

$$(2) \quad S(\gamma) = \frac{1}{2} \int_{S^1} |\dot{\gamma}(t)|^2 dt \quad \text{and} \quad \omega_\gamma[v, w] := \int_{S^1} \langle v(t), \nabla_{\dot{\gamma}} w(t) \rangle dt$$

are the usual energy functional, respectively the canonical two-form on the loop space. Atiyah proceeds with a series of formal manipulations allowing him to write (1) as a Wiener integral; then, using the Feynman-Kac formula, he identifies this integral with the supertrace of the heat semigroup associated to the Dirac operator and thus (via the McKean-Singer formula) with the index of the Dirac operator.

On the other hand, the loop space has a natural S^1 -action by rotation of loops, so if the differential form $e^{-S} \wedge \theta$ is closed with respect to the equivariant differential $d_K = d + i_K$ (where $K = \dot{\gamma}$ is the generating vector field of the action), one formally applies a Duistermaat-Heckmann type formula to this infinite-dimensional situation, in order to localize the integral to the fixed point set (i.e. the set of constant loops) with respect to the action. Now there is an obvious inclusion map $\iota : X \rightarrow LX$ identifying X with these fixed points, which yields the *localization formula*

$$(3) \quad I(\theta) = \int_X \widehat{A}(X) \wedge \iota^* \theta.$$

It was observed by Bismut [Bis85] that this can be used to (formally) prove the twisted Atiyah-Singer theorem, by considering special differential forms β on LX defined from the data of a vector bundle with connection on X , which today are called *Bismut-Chern characters*.

Our work. At the Mini-Workshop in Oberwolfach, we reported on a recent project that carries out a rigorous construction of such a supersymmetric path integral map. The map should have the following properties.

- (i) It should be defined on some large subset $\Omega_{\text{int}}(LX)$ of *integrable forms*, which at least includes the Bismut-Chern characters β defined by Bismut.
- (ii) For any form $\theta \in \Omega_{\text{int}}(LX)$ with $d_K \theta = 0$, I should satisfy the localization formula (3).

We remark that in particular, (ii) implies that I is coclosed with respect to d_K . Of course, these properties do not fix I uniquely, since e.g. the functional $I_0(\theta)$, defined as the right hand side of (3) satisfies both requirements tautologically (even with $\Omega_{\text{int}}(LX) = \Omega(LX)$). To make this a reasonable problem, we therefore require the following rather heuristic property.

- (iii) I is given by formula (1) in a suitable sense.

Our construction of such a map I consists of two parts that will be sketched now.

The first construction. In a first series of papers with Florian Hanisch [HL17a, HL17b], we construct an integral map I satisfying (iii): Following along the lines of Atiyah's original exposition, we replace the differential form integral (1) by an integral over the measure $e^{-S} d\gamma$ (which is identified with the Wiener measure). It then remains to determine the "top degree part" of the mixed differential form $e^\omega \wedge \theta$, for suitable differential forms θ . Under the condition that X is spin, we

arrive at the expression

$$[e^\omega \wedge \theta_1 \wedge \cdots \wedge \theta_N]_{\text{top}} = \sum_{\sigma \in S_N} \text{sgn}(\sigma) \int_{\Delta_N} \text{str} \left([\gamma]_{\tau_N}^1 \prod_{j=1}^N \mathbf{c}(\theta_{\sigma_j}(\tau_j)) [\gamma]_{\tau_{j-1}}^{\tau_j} \right) d\tau,$$

where S_N is the N -th symmetric group, $[\gamma]_s^t$ denotes the parallel transport in the spinor bundle along γ , and \mathbf{c} denotes Clifford multiplication. Interpreting the parallel transport in the stochastic sense, we can integrate with respect to the Wiener measure \mathbb{W} on the continuous loop space $L_c X$ to obtain the definition

$$(4) \quad I(\theta) := \int_{L_c X} [e^\omega \wedge \theta]_{\text{top}} \exp \left(-\frac{1}{8} \int_{S^1} \text{scal } \text{bigl}(\gamma(s)) ds \right) d\mathbb{W}(\gamma)$$

for suitable differential forms θ . This generalizes a definition earlier given by John Lott [Lot87].

The second construction. It is then a challenge to show that the map I defined in (4), which by construction satisfies (iii), in fact satisfies the properties (i)-(ii) as well. To this end, in a joint paper with Batu Güneysu, we develop a second, more algebraic description of the integral map. A surprising result of our endeavors is that the path integral map turns out to be precisely the *Chern character* of the Dirac operator, in a certain setting of non-commutative geometry. This version of the path integral map will be a cochain on the *bar complex*¹

$$(5) \quad \mathbf{B}(\Omega(X)) = \bigoplus_{N=0}^{\infty} \Omega(X)[1]^{\otimes N}.$$

This complex has two differentials: One coming from the usual de-Rham differential on $\Omega(X)$ and one, the *bar differential*, taking into account the multiplication on $\Omega(X)$. It is related to the space of differential forms on the loop space by Chen's *iterated integral map*

$$\rho : \mathbf{B}(\Omega(X)) \longrightarrow \Omega(LX),$$

which is a chain map when restricted to the subspace $\mathbf{B}^\sharp(\Omega(X)) \subset \mathbf{B}(\Omega(X))$ of cyclic chains (c.f. [Che73]). Using this map, a closed cochain on $\mathbf{B}(\Omega(X))$ is then essentially the same as an equivariantly closed cochain on $\text{im}(\rho) \subset \Omega(LX)$.

Now the bar complex $\mathbf{B}(\Omega)$ can be defined for any abstract dg algebra Ω , by the same formula (5). In the paper [GL19], we develop a general theory of Chern characters for so-called *Fredholm modules* over Ω , a generalization of the well-known concept of Fredholm modules introduced by Atiyah, which abstracts the notion of an elliptic operator. Now if X is a spin manifold, there is a canonical Fredholm module over $\Omega(X)$ determined by the Dirac operator D over X , and we can form its Chern character $\text{Ch}(D)$, which is a closed cochain on $\mathbf{B}^\sharp(\Omega(X))$.

We prove that $\text{Ch}(D)$ vanishes identically on $\ker(\rho)$, hence we can define its push-forward $\rho_* \text{Ch}(D)$. This cochain satisfies property (ii) of the iterated integral map by construction (since $\text{Ch}(D)$ is closed and ρ is a chain map), but not yet property one. For the latter, we have to pass to a certain *extended* bar complex

¹Here $\Omega(X)[1]$ is the same as $\Omega(X)$ as a vector space, but with degrees shifted by one.

$B(\Omega_{\mathbb{T}}(X))$ and consider the *extended iterated integral map* $\rho_{\mathbb{T}}$ defined by Getzler, Jones and Petrack [GJP91]. It follows from their work (together with some analytic considerations from our paper) that the image of $\rho_{\mathbb{T}}$ contains the Bismut-Chern characters β ; moreover, the Chern character $\text{Ch}(D)$ extends to this larger extended complex. The result is then the following.

Theorem. *The domain of the integral map I defined in (4) contains the image of $\rho_{\mathbb{T}}$. Moreover, the (extended) Chern character $\text{Ch}(D)$ coincides with the pullback $\rho_{\mathbb{T}}^*I$ of I via the extended iterated integral map.*

In other words, the two constructions of the supersymmetric path integral yield the same result. By the properties of $\text{Ch}(D)$, this integral verifies the properties (i)-(iii); at the same time, our work uncovers intricate mathematics working in the background of the supersymmetric path integral and displays the integral as a Chern character, a quite fundamental object associated to the Dirac operator.

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Scattering theory for the Hodge-Laplacian without assumptions on the injectivity radius

ROBERT BAUMGARTH

We report on an ongoing research project. We prove only using an integral criterion the existence and completeness of the Wave operators

$$W_{\pm}(\Delta_h^{(j)}, \Delta_g^{(j)}, I_{g,h}) = \lim_{t \rightarrow \pm\infty} e^{it\Delta_h^{(j)}} I_{g,h} e^{-it\Delta_g^{(j)}} P_{\text{ac}}(-\Delta_g^{(j)})$$

corresponding to the Friedrich's extension of the Hodge-Laplacian $\Delta_{\nu}^{(j)}$ acting on differential j -forms, for $\nu \in \{g, h\}$, induced by two quasi-isometric Riemannian

metrics g and h on an open smooth manifold M . In particular, this result provides a criterion for the absolutely continuous spectra

$$\sigma_{\text{ac}}(\Delta_g^{(j)}) = \sigma_{\text{ac}}(\Delta_h^{(j)})$$

of $\Delta_g^{(j)}$ to coincide. The proof is based on a gradient estimate obtained by a probabilistic Bismut-type formula for the exterior derivative and the codifferential of the heat semigroup defined by spectral calculus. By this localised formulae, the integral criterion only requires on a function defined in terms of local bounds on the Weitzenböck curvature term and some upper local control on the heat kernel acting on functions, but no control on the injectivity radii.

Let us omit the metric in what follows. We work on the special bundle $E = \wedge \mathbb{T}^*M \rightarrow M$ with its natural metric

$$\nabla := \nabla \wedge \mathbb{T}^*M := \bigoplus_{j=0}^{\dim M} \nabla \wedge^j \mathbb{T}^*M$$

and Clifford action $c : \mathbb{T}M \rightarrow \text{End}(\wedge \mathbb{T}^*M)$, $c(\alpha)\beta := \alpha \wedge \beta - \alpha^\sharp \lrcorner \beta$. The Hodge-Laplacian Δ is related to the Connection Laplacian $\text{tr} \nabla^2$ by the Weitzenböck formula

$$(1) \quad \Delta = \text{tr} \nabla^2 + \mathcal{R},$$

where $\mathcal{R} \in \Gamma(\text{End} \Omega_{C^\infty}(M))$ is a symmetric field of endomorphisms. Acting on differential j -forms, the field of endomorphisms is specified by an index $\mathcal{R}^{(j)} := \mathcal{R}|_{\Omega_{C^\infty}^j(M,g)}$. Note that $\mathcal{R}^{(1)} = \text{Ric}$ and $\mathcal{R}^{(0)} = 0$. Let $\mathcal{Q} \in \text{End}(\wedge \mathbb{T}_x^*M)$ be the solution to the ordinary differential equation

$$\frac{d}{dt} \mathcal{Q}_t = -\frac{1}{2} \mathcal{R}_{\parallel_t} \mathcal{Q}_t, \quad \mathcal{Q}_0 = \text{id}_{\wedge \mathbb{T}_x^*M},$$

along the paths of $X(x)$, where $\mathcal{R}_{\parallel_t} := \parallel_t^{-1} \mathcal{R} \circ \parallel_t$. The composition $\mathcal{Q} \circ \parallel^{-1}$ is called the **damped parallel transport** along the paths of $X(x)$.

Let $X(x)$ be a Brownian motion on M starting at $x \in M$ and $\zeta(x)$ its maximal lifetime. Then [2, Theorem 6.1], for any $\alpha \in \Omega_{L^2}(M, g)$ and $v \in \wedge \mathbb{T}_x M$, we have the following Bismut-type formulae

$$(2) \quad \langle (\mathbf{d}P_s \alpha)_x, v \rangle = -\mathbb{E} \left\langle \parallel_s^{-1} \alpha(X_s(x)) \mathbb{1}_{\{s < \zeta(x)\}}, \mathcal{Q}_s \int_0^s \mathcal{Q}_r^{-1} (dB_r \lrcorner \mathcal{Q}_r \dot{\ell}_r) \right\rangle,$$

$$(3) \quad \langle (\delta P_s \alpha)_x, v \rangle = -\mathbb{E} \left\langle \parallel_s^{-1} \alpha(X_s(x)) \mathbb{1}_{\{s < \zeta(x)\}}, \mathcal{Q}_s \int_0^s \mathcal{Q}_r^{-1} (dB_r \wedge \mathcal{Q}_r \dot{\ell}_r) \right\rangle,$$

where

- $\tau(x) < \zeta(x)$ is the first exit time of $X(x)$ from a relatively compact neighbourhood D in M ,
- $dB := \parallel^{-1} \circ dX(x)$ is a Brownian motion in $\mathbb{T}_x M$, i.e. the associated antidevelopment of the Brownian motion $X(x)$,

- $(\ell_r)_{r \in [0, s]}$ is any adapted process in $\bigwedge \mathbb{T}_x M$ with absolutely continuous paths such that for some arbitrary small $\varepsilon > 0$

$$\mathbb{E} \left(\int_0^{(s-\varepsilon) \wedge \tau(x)} |\dot{\ell}_r|^2 dr \right)^{1/2} < \infty \quad \text{and} \quad \ell_0 = v, \quad \ell_r = 0 \quad \text{for all } r \geq (s - \varepsilon) \wedge \tau(x).$$

Let us consider the special case of two quasi-isometric metrics, a conformal metric change, namely, given a smooth function $\psi : M \rightarrow \mathbb{R}$, we define another metric by $g_\psi := e^{2\psi} g$.

Then our main result reads as follows: Setting $\Psi_g(x, s)$ as a function defined only in terms of local bounds on \mathcal{R} and $\Phi_g(x, s) = \sup_{y \in M} p_s(x, y) < \infty$ the heat kernel on functions. Let $\psi, |\mathbf{d}\psi|_g$ be bounded and for some $s \in (0, \infty)$ and both $\nu \in \{g, h\}$

$$(4) \quad \int \sinh |2\psi(x)| \Psi_\nu(x, s) \Phi_\nu(x, s) \text{vol}_\nu(dx) < \infty.$$

Then the wave operators $W_\pm(\Delta_{g_\psi}, \Delta_g, I_{g, g_\psi})$ exist and are complete, and one has $\sigma_{\text{ac}}(\Delta_g) = \sigma_{\text{ac}}(\Delta_{g_\psi})$.

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Couplings, gradient estimates and Logarithmic Sobolev inequality for Langevin bridges

MAX VON RENESSE

(joint work with Giovanni Conforti)

We present quantitative results about the bridges of the Langevin dynamics and the associated reciprocal processes. These results can be seen as counterparts of well known characterizations of convexity properties of drift potentials for stochastic differential equations but now for the associated bridge processes. They include an equivalence between gradient estimates for bridge semigroups and couplings, comparison principles, bounds of the distance between bridges of different Langevin dynamics, and a Logarithmic Sobolev inequality for bridge measures. In contrast to the SDE case these estimates are shown to characterize a convexity of an effective potential appearing in a novel Feynman-Kac representation for the

bridge measure on path space. – More specifically, we consider the family of bridge measures $P_{-T,T}^{x,y}$ induced from solutions to the SDE

$$dX_t = -\nabla U(X_t)dt + dB_t, \quad X_{-T} = x$$

conditioned to the event $\{X_T = y\}$. Introducing the derived potential

$$\mathcal{U} := \frac{1}{2}|\nabla U|^2 - \frac{1}{2}\Delta U$$

one of our main results reads as follows.

Theorem 1. The following are equivalent for $\alpha > 0$:

(i) \mathcal{U} is $\frac{\alpha^2}{2}$ convex. That is,

$$\inf_{z,v \in \mathbb{R}^d: |v|=1} \nabla^2 \mathcal{U}(z)[v,v] \geq \alpha^2.$$

(ii) For any $T > 0, t \in [-T, T]$ and any smooth function f the following gradient estimates hold:

$$\forall x \in \mathbb{R}^d, \quad |\nabla_x E_{P_{-T,T}^{x,y}}(f(\omega_t))| \leq \frac{\sinh(\alpha(T-t))}{\sinh(2\alpha T)} E_{P_{-T,T}^{x,y}} |\nabla f(\omega_t)|$$

$$\forall y \in \mathbb{R}^d, \quad |\nabla_y E_{P_{-T,T}^{x,y}}(f(\omega_t))| \leq \frac{\sinh(\alpha(T+t))}{\sinh(2\alpha T)} E_{P_{-T,T}^{x,y}} |\nabla f(\omega_t)|$$

(iii) For any $x_1, y_1, x_2, y_2 \in \mathbb{R}^d$ there exists a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ and maps $X^i, i = 1, 2$ defined on it with the property that $X^i \# \tilde{P} = P_{-T,T}^{x_i, y_i}$ and

$$\tilde{P} \left(|X_t^1 - X_t^2| \leq \frac{\sinh(\alpha(T-t))}{\sinh(2\alpha T)} |x_2 - x_1| + \frac{\sinh(\alpha(T+t))}{\sinh(2\alpha T)} |y_2 - y_1| \quad \forall t \in [-T, T] \right) = 1.$$

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Heat kernel estimates, comparison and second variation

XUE-MEI LI

(joint work with Xin Chen and Bo Wu)

We discuss Hessian formulas for the second order derivatives of the solution to a parabolic equation (with or without a potential). In a joint work with X. Chen and B. Wu, we introduced a new variation for stochastic equations on the orthonormal frame bundle. This is different from those used previously for such analysis (e.g. by Bismut, Driver, et al.). We also introduced a second order variation ensuring vanishing of the second order derivatives of the projection. We also constructed a family of local vector fields on the path spaces. Together with a localising technique, and comparison theorems, this allows us to obtain a clean and neat formula for the gradient and for the Hessian of the logarithmic heat kernel, leading to nice estimates on the gradient and on the Hessian. This was shown to hold for any complete Riemannian manifolds without curvature restrictions. This has further applications to analysis on path spaces.

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Heat kernel estimates for non-local Schrödinger operators

RENÉ L. SCHILLING

(joint work with Kamil Kaleta)

We report on recent progress achieved in the joint paper with K. Kaleta (Wrocław University of Technology) *Progressive intrinsic ultracontractivity and heat kernel estimates for non-local Schrödinger operators* (<https://arxiv.org/abs/1903.12004>).

We are interested in the long-time asymptotic behaviour of semigroups generated by non-local Schrödinger operators of the form $H = -L + V$; the free operator L is the generator of a symmetric Lévy process in \mathbb{R}^d , $d > 1$ (with non-degenerate jump measure) and V is a locally bounded, confining potential. The assumptions below will ensure that the L^2 -semigroup $\{e^{-tH}, t \geq 0\}$ generated by H consists of compact operators which are also integral operators of the form $e^{tH}w(x) = \int u_t(x, y)w(y) dy$ for $w \in L^2$, say. Our aim is to study the behaviour of $u_t(x, y)$ as $t \rightarrow \infty$. Since the spectrum $\sigma(H)$ of H is discrete, we define $\lambda_0 := \inf \sigma(H)$ and we denote by $\varphi_0 \in L^2$ the corresponding eigenfunction such that $\|\varphi_0\|_{L^2} = 1$.

In the literature the (asymptotic) intrinsic ultracontractivity condition (a)IUC is used to describe the large time behaviour of $u_t(x, y)$. These are conditions on U_t which can be conveniently stated in the following form

$$(IUC) \quad \forall t_0 > 0 \exists C = C(t_0) \geq 1 \forall t \geq t_0 \forall x, y \in \mathbb{R}^d :$$

$$u_t(x, y) \stackrel{C}{\asymp} e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y),$$

$$(aIUC) \quad \exists t_0 > 0 \exists C = C(t_0) \geq 1 \forall t \geq t_0 \forall x, y \in \mathbb{R}^d :$$

$$u_t(x, y) \stackrel{C}{\asymp} e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y).$$

We add a further condition to this list which we call *progressive intrinsic ultracontractivity* — pIUC for short —

$$(pIUC) \quad \exists t_0 > 0 \exists r : [t_0, \infty) \rightarrow (0, \infty] \text{ increasing, } \lim_{t \rightarrow \infty} r(t) = \infty, \exists C > 0 :$$

$$u_t(x, y) \stackrel{C}{\asymp} e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad |x| \wedge |y| < r(t), \quad t \geq t_0.$$

(“ $\stackrel{C}{\asymp}$ ” denotes a two-sided comparison with the constants $1 \leq C < \infty$ and C^{-1} .) Note that IUC always implies aIUC, and aIUC always implies pIUC (with threshold function $r \equiv \infty$).

The IUC and aIUC regimes are well-understood and we want to concentrate on the pIUC case; *en passant* our results recover some of the known results with different proofs and — sometimes — with a greater generality. Our main result are sharp two-sided large-time estimates for the kernel $u_t(x, y)$. Let us first state the result and then discuss the assumptions **(A1)**–**(A3)** appearing in the statement.

Theorem. *Let L be the generator of a symmetric Lévy process with Lévy measure $\nu(dx) = \nu(x) dx$ and diffusion matrix $A = (a_{ij})_{i,j=1,\dots,n}$, and let V be a confining potential. Denote by $H = -L + V$ the Schrödinger operator and assume **(A1)**–**(A3)** with $t_b > 0$, $R_0 > 0$ and the profile functions $f(|x|)$ and $g(|x|)$ which control $\nu(x)$ and $V(x)$, respectively. Write λ_0 and φ_0 for the ground-state eigenvalue and eigenfunction, and $u_t(x, y)$ for the density of the operator $U_t = e^{-tH}$. There exist constants $C \geq 1$ and $R > R_0$ such that for every $t > 30t_b$ the following assertions hold.*

(1) *If $|x|, |y| \leq R$, then*

$$\frac{1}{C} e^{-\lambda_0 t} \leq u_t(x, y) \leq C e^{-\lambda_0 t}.$$

(2) *If $|x| > R$ and $|y| \leq R$, then*

$$\frac{1}{C} e^{-\lambda_0 t} \frac{\nu(x)}{V(x)} \leq u_t(x, y) \leq C e^{-\lambda_0 t} \frac{\nu(x)}{V(x)};$$

by symmetry, if $|x| \leq R$ and $|y| > R$, then

$$\frac{1}{C} e^{-\lambda_0 t} \frac{\nu(y)}{V(y)} \leq u_t(x, y) \leq C e^{-\lambda_0 t} \frac{\nu(y)}{V(y)}.$$

(3) *If $|x|, |y| > R$, then*

$$\frac{1}{C} \frac{F(\mathbf{K}t, x, y) \vee e^{-\lambda_0 t} \nu(x) \nu(y)}{V(x) V(y)} \leq u_t(x, y) \leq C \frac{F\left(\frac{t}{\mathbf{K}}, x, y\right) \vee e^{-\lambda_0 t} \nu(x) \nu(y)}{V(x) V(y)},$$

*where $\mathbf{K} = 4C_6 C_7^2$ — the constants C_6, C_7 are from **(A3)** — and*

$$F(\tau, x, y) := \int_{R-1 < |z| < |x| \vee |y|} (f(|x-z|) \wedge 1) (f(|z-y|) \wedge 1) e^{-\tau g(|z|)} dz.$$

Let us now elaborate on the statement of the theorem. A Lévy process on \mathbb{R}^d is a stochastic process $(X_t)_{t \geq 0}$ with values in \mathbb{R}^d , independent and stationary increments, and càdlàg (right-continuous, finite left limits) paths. Any Lévy process is a Markov process whose transition semigroup is a semigroup of convolution operators

$$P_t u(x) = \mathbb{E} u(X_t + x) = u * \tilde{\mu}_t(x), \quad \tilde{\mu}_t(dy) = \mathbb{P}(-X_t \in dy)$$

which is a strongly continuous contraction semigroup on $L^2 = L^2(\mathbb{R}^d, dx)$. Using the Fourier transform we can describe P_t as a Fourier multiplication operator

$$P_t u(x) = \mathcal{F}^{-1} (e^{-t\psi} \mathcal{F}u) (x)$$

with symbol (multiplier) $e^{-t\psi(\xi)}$. The semigroup $\{P_t : t \geq 0\}$ is symmetric in L^2 if, and only if, X_t is a symmetric Lévy process (i.e. $\mathbb{P}(X_t \in dy) = \mathbb{P}(-X_t \in dy)$, $t \geq 0$) which is equivalent to $e^{-t\psi}$ or ψ being real. All real characteristic exponents are given by the Lévy–Khintchine formula

$$\psi(\xi) = \xi \cdot A\xi + \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(\xi \cdot z)) \nu(dz), \quad \xi \in \mathbb{R}^d.$$

where A is a symmetric non-negative definite matrix (which may be degenerate or even vanish completely), and ν is a symmetric Lévy measure, i.e. a Radon measure on $\mathbb{R}^d \setminus \{0\}$ satisfying $\nu(E) = \nu(-E)$ and $\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |z|^2) \nu(dz) < \infty$. We always assume that $\nu \neq 0$. The matrix A describes the diffusion part of $(X_t)_{t \geq 0}$ while ν is the jump measure. We assume that the jump activity is infinite and ν is absolutely continuous with respect to Lebesgue measure, i.e.

$$\nu(\mathbb{R}^d \setminus \{0\}) = \infty \quad \text{and} \quad \nu(dx) = \nu(x) dx.$$

The generator L is a non-local self-adjoint pseudo-differential operator given by $\mathcal{F}[Lu](\xi) = -\psi(\xi)\mathcal{F}u(\xi)$, $\xi \in \mathbb{R}^d$, $u \in \mathcal{D}(L) := \{v \in L^2(\mathbb{R}^d) : \psi\mathcal{F}v \in L^2(\mathbb{R}^d)\}$,

Examples of non-local operators (and related jump processes) are *fractional Laplacians* $L = -(-\Delta)^{\alpha/2}$, $\alpha \in (0, 2)$ (*isotropic α -stable processes*) and *quasi-relativistic operators* $L = -(-\Delta + m^{2/\alpha})^{\alpha/2} + m$, $\alpha \in (0, 2)$, $m > 0$ (*isotropic relativistic α -stable processes*) which play an important role in mathematical physics.

Our assumptions on ν guarantee that the process $(X_t)_{t \geq 0}$ is a strong Feller process, i.e. P_t maps bounded measurable functions into continuous functions; this means that its one-dimensional distributions are absolutely continuous with respect to Lebesgue measure, i.e. there exists a transition density $p_t(x, y) = p_t(y - x)$ such that $\mathbb{P}^0(X_t \in E) = \int_E p_t(x) dx$ for every Borel set $E \subset \mathbb{R}^d$.

Here are the assumptions appearing in the statement of the theorem.

(A1) Lévy density. There exists a profile function $f : (0, \infty) \rightarrow (0, \infty)$ such that

- (1) there is a constant $C_1 \geq 1$ such that $C_1^{-1}f(|x|) \leq \nu(x) \leq C_1 f(|x|)$ for all $x \in \mathbb{R}^d \setminus \{0\}$;
- (2) f is decreasing and $\lim_{r \rightarrow \infty} f(r) = 0$;
- (3) there is a constant $C_2 \geq 1$ such that $f(r) \leq C_2 f(r + 1)$ for all $r \geq 1$;
- (4) f has the *direct jump property*: there exists a constant $C_3 > 0$ such that

$$\int_{\substack{|x-y|>1 \\ |y|>1}} f(|x-y|)f(|y|) dy \leq C_3 f(|x|), \quad |x| \geq 1.$$

The convolution property **(A1.d)** is fundamental for us. It has a very suggestive probabilistic interpretation: *the probability to move from 0 to x in “two large jumps in a row” is smaller than with a “single direct jump”*.

(A2) Transition density of the free process. The function $(t, x) \mapsto p_t(x)$ is continuous on $(0, \infty) \times \mathbb{R}^d$ and there exists some $t_b > 0$ such that the following conditions hold.

(1) There are constants $C_4, C_5 > 0$ such that

$$p_t(x) \leq C_4 \left([e^{C_5 t} f(|x|)] \wedge 1 \right), \quad x \in \mathbb{R}^d \setminus \{0\}, \quad t \geq t_b;$$

(2) For every $r \in (0, 1]$ we have

$$\sup_{t \in (0, t_b]} \sup_{r \leq |x| \leq 2} p_t(x) < \infty.$$

A sufficient condition for the time-space continuity of the density $p_t(x)$ is

$$e^{-t\psi(\xi)} \in L^1(\mathbb{R}^d, d\xi) \quad \text{for all } t > 0.$$

This condition trivially if ψ has a nondegenerate Gaussian part, i.e. $\det A \neq 0$. The other assumptions in **(A2)** govern the asymptotic behaviour of the transition density $p_t(x)$ for the free operator L and they should be seen as the minimal regularity requirement for the density of the free semigroup. The upper bound on $p_t(x)$ in **(A2.a)** is known for a wide range of operators L whose Lévy measures satisfy **(A1)**. The condition **(A2.b)** is a small time off-diagonal boundedness property which holds for a large class of semigroups. Under **(A2.a)** we know that $\sup_{x \in \mathbb{R}^d} p_{t_b}(x) = p_{t_b}(0) < \infty$ — this extends to all $t \geq t_b$ — and the function $p_t(\cdot)$ is smooth for all $t > t_b$; this is a consequence of the fact that p_t is the convolution of $p_{t-t_b} \in L^1(\mathbb{R}^d)$ and $p_{t_b} \in L^\infty(\mathbb{R}^d)$.

(A3) Confining potential. Let $V \in L^\infty_{\text{loc}}(\mathbb{R}^d)$ be such that $\lim_{|x| \rightarrow \infty} V(x) = \infty$ and assume that there exist constants $C_6 \geq 1$ and $R_0 > 0$, and a profile function $g : [0, \infty) \rightarrow (0, \infty)$ such that

(1) $g|_{[0, R_0)} \equiv 1$ and $C_6^{-1}g(|x|) \leq V(x) \leq C_6g(|x|)$, $|x| \geq R_0$;

(2) g is increasing on $[R_0, \infty)$;

(3) there exists a constant $C_7 \geq 1$ such that $g(r+1) \leq C_7g(r)$, $r \geq R_0$.

The uniform growth condition **(A3.c)** excludes profiles growing like $\exp(r^2)$ or $\exp(e^r)$, but exponentially and slower growing potentials — for example growth orders $\log \log r$, $\log(r)^\beta$, r^β and $e^{\beta r}$, with $\beta > 0$ — are admissible.

These assumptions guarantee the setup on H and e^{tH} (self-adjointness, mapping properties, spectral properties etc.) as explained above.

Our main tool is the Feynman-Kac formula

$$e^{-tH} f(x) := \mathbb{E}^x \left[e^{-\int_0^t V(X_s) ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d), \quad t > 0.$$

In order to get estimates, we “keep track of the path” as it goes from $x \rightarrow y$ where x is large. This is done by an elaborate pathwise decomposition which allows us to use **(A2)**, **(A3)** to estimate the expression in the Feynman–Kac formula. Since

we deal with a jump process, the process typically exits sets with jumps and this can be controlled by the following Ikeda–Watanabe formula:

$$\mathbb{P}^x(\tau_D \in dt, X_{\tau_D-} \in dy, X_{\tau_D} \in dz) = p_D(t, x, y) dt \mathbf{1}_{\{|z-y|>0\}}(y, z) \nu(z-y) dy dz;$$

here D is some domain, τ_D the first exit time from this domain and $p_D(t, x, y)$ is the density of the free process $(X_t)_{t \geq 0}$ killed upon exiting D .

If we make a further structural assumption on the profile function g , we can improve our results, splitting the estimates in two distinct scenarios: the *aIUC regime* (including the *IUC regime*) and the *non-aIUC regime*.

(A4) V is a potential satisfying **(A3)** with the profile g and $R_0 > 0$ such that $f(R_0) < 1$ and

$$g(r) = h(|\log f(r)|), \quad r \geq R_0,$$

for some increasing function $h : [|\log f(R_0)|, \infty) \rightarrow (0, \infty)$ such that $h(s)/s$ is monotone.

The large time estimates of the heat kernel $u_t(x, y)$ in the the aIUC and the non-aIUC (=pIUC) regime are substantially different. This is due to the intricate asymptotic behaviour of the function $F(\tau, x, y)$. In the non-aIUC regime the following result holds true

Corollary. *For every confining potential – no matter how slowly V grows at infinity – there is an increasing function $r : (0, \infty) \rightarrow (0, \infty]$ such that $\lim_{t \rightarrow \infty} r(t) = \infty$ and such that the following estimate holds: There is a constant $C \geq 1$ such that for sufficiently large values of t we have*

$$u_t(x, y) \underset{C}{\asymp} e^{-\lambda_0 t} \frac{f(|x|)}{g(|x|)} \frac{f(|y|)}{g(|y|)}, \quad |x|, |y| > R, \quad |x| \wedge |y| < r(t).$$

These estimates are equivalent to saying that there is a constant such that for sufficiently large values of t we have

$$\frac{1}{C} e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y) \leq u_t(x, y) \leq \tilde{C} e^{-\lambda_0 t} \varphi_0(x) \varphi_0(y), \quad |x| \wedge |y| < r(t).$$

The estimates for $u_t(x, y)$ are essentially different if $|x|, |y| > r(t)$.

On the uniqueness class, stochastic completeness and volume growth for graphs

MARCEL SCHMIDT

(joint work with Xueping Huang and Matthias Keller)

In 1980 Azencott [1] gave an example of a complete Riemannian manifold on which Brownian motion has finite lifetime. Such manifolds are referred to as stochastically incomplete and typically are of very large volume growth. On the other hand it was shown that stochastic completeness is guaranteed under certain volume bounds which were improved over the years, see Gaffney [4], Karp/Li [12],

Davies [2] and Takeda [14]. An optimal result was obtained by Grigor'yan [5] (see also [6]) who proved stochastic completeness of a geodesically complete manifold under the condition

$$\int^{\infty} \frac{r}{\log^{\#} \text{vol}(B_r)} dr = \infty,$$

where $\log^{\#} = \max\{\log, 1\}$. He also showed by examples that his criterion is sharp. Later, Grigor'yan's result was extended by Sturm [13] to strongly local Dirichlet forms where the phenomenon is referred to as conservativeness and distance balls are considered with respect to a so called intrinsic metric. Indeed, in spirit the proof in this more general situation follows Grigor'yan's. A remarkable feature of Grigor'yan's proof is that it not only yields stochastic completeness but directly implies a uniqueness class statement for the heat equation. Precisely, while stochastic completeness is equivalent to uniqueness of bounded solutions to the heat equation, the uniqueness class statement extends this uniqueness to a class of unbounded solutions which satisfy a certain growth bound.

In recent years the phenomenon of stochastic completeness was intensively studied for graphs. The interest in this topic was sparked by the PhD thesis [15] and follow up work [16] of Wojciechowski who presented examples of graphs of polynomial volume growth, which are stochastically incomplete. This showed that there is no analogous result to Grigor'yan's for graphs when one considers volume growth of balls with respect to the combinatorial graph distance. However, in view of the work of Sturm [13] for local Dirichlet forms, which uses intrinsic metrics, it seemed promising to consider distance balls with respect to a metric that is adapted to the heat flow on the graph. While such a theory of intrinsic metrics was developed at this time also for non-local (and thus for all regular) Dirichlet forms, this idea was used by Grigor'yan/Huang/Masamune [7] to prove a first result in this direction that guaranteed stochastic completeness of the graph provided

$$\text{vol}(B_r) \leq \exp(Cr \log r)$$

for r large enough and some constant $0 < C < 1/2$. Shortly afterwards Grigor'yan's result for manifolds was recovered for graphs using so called intrinsic (or adapted) metrics by Folz [3] and shortly after that an alternative proof was given by Huang [9]. See also [11] for results on the closely related problem of escape rates.

In spirit, the proofs of these results used techniques that relate the non-local graph to a more local object. Specifically, Folz [3] compared the heat flow on the combinatorial graph with a corresponding metric (or quantum) graph and Huang and Shiozawa [11] decreased non-locality of the graph by inserting additional vertices in the edges (which probabilistically decreased the jump size of the process). Although this was a breakthrough, there are two aspects in which the results are not completely satisfying – one of technical the other of structural nature. The technical aspect is that the results were proven under rather restrictive conditions such as local finiteness of the graphs, finite jump size of the metric and uniform lower bounds on the measure. Moreover, the only metrics considered were special

path metrics. These restrictions did not inspire much hope that the proof strategies can be carried over to more general jump processes. The second aspect, which may be seen as a shortcoming of more fundamental nature, is that the proofs do not allow to recover Grigor'yan's uniqueness class for the heat equation. Indeed, this is not a shortcoming of the proofs but the optimal uniqueness class that is known for manifolds does not hold for general graphs. In his PhD thesis [8] Huang gave an example of a nontrivial solution to the heat equation with initial value 0 on the integer line \mathbb{Z} , which showed that the corresponding uniqueness class statement of Grigor'yan is already wrong for this simple graph.

In this talk we present recent results from [10], where we amend these shortcomings. In order to obtain Grigor'yan's uniqueness class for the heat equation we introduce the class of globally local graphs (GL graphs for short). These are graphs whose jump size decays fast enough outside large balls. On GL graphs we establish Grigor'yan's uniqueness class and directly use it to obtain Grigor'yan's optimal volume growth criterion for stochastic completeness (with respect to an intrinsic metric). This part of the results is general and also applies to jump processes associated with regular Dirichlet forms. In a second step we establish the optimal volume growth criterion for stochastic completeness for general graphs under the only assumption that they admit an intrinsic metric with finite distance balls. For this we use the ideas from [11] to refine the given graph to a GL graph of the same volume growth, establish stochastic completeness of the refined graph and then use stability of stochastic completeness under refinements.

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Hypoelliptic Laplacian on symmetric spaces and twisted trace formula

BINGXIAO LIU

Let G be a connected real reductive Lie group, and let K be a maximal compact subgroup of G . Let $\theta \in \text{Aut}(G)$ denote the Cartan involution such that K is the fixed point set. The Cartan decomposition of \mathfrak{g} is

$$(1) \quad \mathfrak{g} = \mathfrak{p} \oplus \mathfrak{k}.$$

Let B be an invariant nondegenerate symmetric bilinear form on \mathfrak{g} , which is positive on \mathfrak{p} and negative on \mathfrak{k} .

Let $X = G/K$ be the associated symmetric space. Then $TX = G \times_K \mathfrak{p}$. Moreover, B induces a Riemannian metric g^{TX} on X , so that X is of nonpositive sectional curvature. Let $d(\cdot, \cdot)$ be the Riemannian distance in X .

In [3], Bismut constructed a family of hypoelliptic Laplacians $\mathcal{L}_b^X|_{b>0}$, which converges in the proper sense to a Bochner-like Laplacian \mathcal{L}^X on X as $b \rightarrow 0$. Using a geometric description of the orbital integrals associated with semisimple elements in G , Bismut obtained an explicit formula for the semisimple orbital integrals for the heat kernel and the wave kernels of \mathcal{L}^X [3, Theorems 6.1.1, 6.3.2].

Here, we introduce a twist $\sigma \in \text{Aut}(G)$. We extend Bismut's formula to the σ -twisted orbital integrals of heat kernels [7, Theorem 3.3].

An explicit formula for twisted orbital integrals. Let $\sigma \in \text{Aut}(G)$ be such that σ commutes with θ and preserves B . Then σ acts on X isometrically. If $\gamma \in G$, we say $\gamma\sigma$ to be semisimple if the displacement function $d_{\gamma\sigma}(x) := d(x, \gamma\sigma(x))$ on X can reach its infimum $m_{\gamma\sigma}$ in X [4, Section 2.19]. If $\gamma\sigma$ is semisimple, after conjugation, we have

$$(2) \quad \gamma = e^a k^{-1}, \quad a \in \mathfrak{p}, \quad k \in K, \quad \text{Ad}(k^{-1})\sigma a = a.$$

Then $m_{\gamma\sigma} = |a|$.

Let $Z_\sigma(\gamma) \subset G$ be the σ -twisted centralizer of γ , consisting of $h \in G$ such that $h\gamma\sigma(h^{-1}) = \gamma$. Then θ acts on $Z_\sigma(\gamma)$, and its Lie algebra $\mathfrak{z}_\sigma(\gamma)$ splits as

$$(3) \quad \mathfrak{z}_\sigma(\gamma) = \mathfrak{p}_\sigma(\gamma) \oplus \mathfrak{k}_\sigma(\gamma).$$

Let $\mathfrak{z}_\sigma^\perp(\gamma) = \mathfrak{p}_\sigma^\perp(\gamma) \oplus \mathfrak{k}_\sigma^\perp(\gamma)$ be the orthogonal of $\mathfrak{z}_\sigma(\gamma)$ in \mathfrak{g} . Let $X(\gamma\sigma)$ be the minimizing set of $d_{\gamma\sigma}$. Then it is a symmetric space so that

$$(4) \quad X(\gamma\sigma) = Z_\sigma(\gamma)/K_\sigma(\gamma).$$

The fibres of the orthogonal normal bundle $N_{X(\gamma\sigma)/X}$ can be identified with $\mathfrak{p}_\sigma^\perp(\gamma)$.

Let K^σ be the compact group generated by K and σ . If (E, ρ^E) is a unitary representation of K^σ , then $F = G \times_K E$ is a Hermitian vector bundle on X equipped with a Hermitian connection ∇^F and the equivariant action of σ .

Let $C^\mathfrak{g}$ be the Casimir element of (\mathfrak{g}, B) . Then it descends to an operator $C^{\mathfrak{g}, X}$ acting on $C^\infty(X, F)$. Let $\Delta^{X, F}$ be the Bochner Laplacian on F , and let $C^\mathfrak{k}$ be the Casimir element of \mathfrak{k} , which acts on E by ρ^E . Then

$$(5) \quad C^{\mathfrak{g}, F} = -\Delta^{X, F} + C^{\mathfrak{k}, E}.$$

Put

$$(6) \quad \mathcal{L}^X = \frac{1}{2}C^{\mathfrak{g}, X} + \frac{1}{2}c,$$

where c is an explicit constant from the Kostant identity [6, Theorem 2.16] for $C^\mathfrak{g}$. Then \mathcal{L}^X is a self-adjoint Bochner-like Laplacian acting on $C^\infty(X, F)$, which commutes with σ .

For $t > 0$, let $p_t^X(x, x')$ be the smooth kernel for the heat operator $\exp(-t\mathcal{L}^X)$. The twisted orbital integral $\text{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)]$ is an integration of p_t^X on $Z_\sigma(\gamma)\backslash G$. Using the above constructions associated with $\gamma\sigma$, we have a geometric interpretation for it [7, Definition 2.1], so that

$$(7) \quad \text{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \text{Tr}^F[\gamma\sigma p_t^X(e^f p1, \gamma\sigma e^f p1)]r(f)df.$$

Here $r(f)$ is a Jacobian term relating the measures on $Z_\sigma(\gamma)\backslash G$ and $\mathfrak{p}_\sigma^\perp(\gamma)$.

The main result is as follows [7, Theorem 3.3]. Note that if we take $\sigma = \text{Id}_G$, we will recover [3, Theorem 6.1.1].

Theorem 2. Set $p = \dim \mathfrak{p}_\sigma(\gamma)$, $q = \dim \mathfrak{k}_\sigma(\gamma)$. For any $t > 0$,

$$(8) \quad \text{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \frac{\exp(-|a|^2/2t)}{(2\pi t)^{p/2}} \times \int_{\mathfrak{k}_\sigma(\gamma)} J_{\gamma\sigma}(Y_0^\mathfrak{k}) \text{Tr}^E[\rho^E(k^{-1}\sigma) \exp(-i\rho^E(Y_0^\mathfrak{k}))] e^{-|Y_0^\mathfrak{k}|^2/2t} \frac{dY_0^\mathfrak{k}}{(2\pi t)^{q/2}}.$$

Here $J_{\gamma\sigma}$ is an analytic function on $\mathfrak{k}_\sigma(\gamma)$ given by

$$(9) \quad J_{\gamma\sigma}(Y_0^\mathfrak{k}) = \frac{1}{|\det(1 - \text{Ad}(\gamma\sigma))|_{\mathfrak{z}_0^\perp}^{1/2}} \frac{\widehat{A}(i\text{ad}(Y_0^\mathfrak{k})|_{\mathfrak{p}_\sigma(\gamma)})}{\widehat{A}(i\text{ad}(Y_0^\mathfrak{k})|_{\mathfrak{k}_\sigma(\gamma)})} \left[\frac{1}{|\det(1 - \text{Ad}(k^{-1}\sigma))|_{\mathfrak{z}_{\sigma,0}^\perp}^{1/2}} \frac{\det(1 - \exp(-i\text{ad}(Y_0^\mathfrak{k}))\text{Ad}(k^{-1}\sigma))|_{\mathfrak{k}_{\sigma,0}^\perp(\gamma)}}{\det(1 - \exp(-i\text{ad}(Y_0^\mathfrak{k}))\text{Ad}(k^{-1}\sigma))|_{\mathfrak{p}_{\sigma,0}^\perp(\gamma)}} \right]^{1/2}.$$

Let $\Gamma \subset G$ be a cocompact torsion-free lattice preserved by σ . Then σ acts on the compact locally symmetric space $Z = \Gamma\backslash X$. We have a twisted Selberg’s trace formula,

$$(10) \quad \text{Tr}[\sigma^Z \exp(-t\mathcal{L}^Z)] = \sum_{[\gamma]_\sigma \in [\Gamma]_\sigma} \text{Vol}(\Gamma_\sigma(\gamma) \backslash X(\gamma\sigma)) \text{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^Z)].$$

Then by our theorem, the summands in right-hand side of (10) are explicit.

To prove the theorem: hypoelliptic deformations. Put $N = G \times_K \mathfrak{k}$. Let $\widehat{\mathcal{X}}$ be the total space of $\widehat{\pi} : TX \oplus N \rightarrow X$, so that $\widehat{\mathcal{X}} \simeq X \times \mathfrak{g}$. The hypoelliptic Laplacians of Bismut are a family of hypoelliptic differential operators \mathcal{L}_b^X , $b > 0$ acting on $C^\infty(\widehat{\mathcal{X}}, \widehat{\pi}^*(\Lambda(T^*X \oplus N^*) \otimes F))$. We refer to [1, 2, 8] and [3, Introduction] for the motivations behind the theory of hypoelliptic Laplacians. Generally speaking, the hypoelliptic Laplacians interpolate between the elliptic Laplacians \mathcal{L}^X on X (as $b \rightarrow 0$) and the generator of geodesic flow on TX (as $b \rightarrow +\infty$).

By [3, Section 11.8], the heat operator $\exp(-t\mathcal{L}_b^X)$ has a smooth Schwartz kernel $q_{b,t}^X((x, Y), (x', Y'))$. Let \mathbf{P} be the projection from $\Lambda(T^*X \oplus N^*) \otimes F$ onto $\Lambda^0(T^*X \oplus N^*) \otimes F$. Then as $b \rightarrow 0$, we have

$$(11) \quad q_{b,t}^X((x, Y), (x', Y')) \rightarrow \mathbf{P} p_t^X(x, x') \pi^{-(m+n)/2} \exp(-\frac{1}{2}(|Y|^2 + |Y'|^2)) \mathbf{P}.$$

Since \mathcal{L}_b^X commutes with σ , we can extend (7) to the hypoelliptic twisted orbital integrals as follows.

$$(12) \quad \text{Tr}_s^{[\gamma\sigma]}[\exp(-t\mathcal{L}_b^X)] = \int_{\mathfrak{p}_\sigma^\perp(\gamma)} \left[\int_{TX \oplus N} \text{Tr}_s^{\Lambda(T^*X \oplus N^*) \otimes F} [\gamma\sigma q_{b,t}^X((e^f p1, Y), \gamma\sigma(e^f p1, Y))] dY \right] r(f) df.$$

By (11), (12) and a Bianchi identity for \mathcal{L}_b^X [3, (2.15.2)], we can establish a fundamental identity [7, Theorem 3.1] such that for $t > 0$, $b > 0$,

$$(13) \quad \text{Tr}^{[\gamma\sigma]}[\exp(-t\mathcal{L}^X)] = \text{Tr}_s^{[\gamma\sigma]}[\exp(-t\mathcal{L}_b^X)]$$

To get the explicit formula in (8), we make $b \rightarrow +\infty$ in the right-hand side of (13).

Indeed, as $b \rightarrow +\infty$, $\exp(-t\mathcal{L}_b^X)$ will concentrate to the geodesic flow $\{\varphi_t\}_{t \in \mathbb{R}}$ on TX in proper sense. Then the integrand in the right-hand side of (12) concentrates near the point $(x, Y^{TX}) \in TX$ such that $\varphi_t(x, Y^{TX}) = \gamma\sigma(x, Y^{TX})$, which is equivalent to $x \in X(\gamma\sigma)$. Therefore, we only need to consider a small neighbourhood $U_\beta = \{|f| < \beta\} \subset \mathfrak{p}_\sigma^\perp(\gamma)$ for some $\beta > 0$. We can apply a technique as the Getzler rescaling in local index theory. As $b \rightarrow +\infty$, we amplify U_β to $\mathfrak{p}_\sigma^\perp(\gamma)$ by b^2 . This way, we flatten the geometry along the normal fibre $N_{X(\gamma\sigma)/X}$. Correspondingly, the hypoelliptic heat kernel in the right-hand side of (12) will converges to the heat kernel of a model operator on $\mathfrak{p}_\sigma^\perp(\gamma) \times \mathfrak{g}$. Finally, we evaluate a rescaled twisted orbital integral for the heat kernel of this model operator, which gives the function $J_{\gamma\sigma}(Y_0^\mathfrak{k})$, $Y_0^\mathfrak{k} \in \mathfrak{k}_\sigma(\gamma)$.

An example. We take $G = \text{SL}_2(\mathbb{R})$, $K = \text{SO}(2)$, $\theta(A) = (A^t)^{-1}$. For $u, v \in \mathfrak{sl}_2(\mathbb{R})$, set $B(u, v) = 2\text{Tr}^{\mathbb{R}^2}[uv]$. Then $X = G/K$ is the upper half-plane $\mathbb{H} = \{z = x + iy : y > 0\}$ with the metric $ds^2 = \frac{dx^2 + dy^2}{y^2}$. We take σ to be the conjugation

by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, which verifies our assumptions. Then the action of θ on \mathbb{H} is given by $z \mapsto -\frac{1}{\bar{z}}$, and the action of σ is given by $z \mapsto -\bar{z}$. Then $X(\sigma)$ is just the upper y -axis.

In this case, $\mathcal{L}^X = -\frac{1}{2}\Delta^{\mathbb{H}} - \frac{1}{8}$. We take $\gamma = 1$, then $\mathfrak{k}_\sigma(1) = 0$, $\mathfrak{p}_\sigma(1)$ is 1-dimensional space. We apply our theorem to $\text{Tr}^{[\sigma]}[\exp(-t\mathcal{L}^X)]$, then we get

$$(14) \quad \text{Tr}^{[\sigma]}[\exp(-t\mathcal{L}^X)] = \frac{1}{\sqrt{2\pi t}} \frac{1}{2}.$$

Note that there is a classical formula (cf. [5]) for the heat kernel of $-\frac{1}{2}\Delta^{\mathbb{H}}$, by (7), (14), we deduce that for $t > 0$,

$$(15) \quad \frac{\sqrt{2}}{\pi t} \int_{f \in \mathbb{R}} \cosh(f) df \int_{2|f|}^{+\infty} \frac{se^{-s^2/2t}}{(\cosh(s) - \cosh(2f))^{1/2}} ds = 1.$$

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