Mathematisches Forschungsinstitut Oberwolfach

Report No. 17/2019

DOI: 10.4171/OWR/2019/17

Mini-Workshop: Reflection Groups in Negative Curvature

Organized by
Mikhail Belolipetsky, Rio de Janeiro
Vincent Emery, Bern
Ruth Kellerhals, Fribourg

7 April – 13 April 2019

Abstract. Discrete groups generated by reflections constitute an important source of examples of lattices in simple Lie groups of real rank 1 (whose associated symmetric spaces are negatively curved). Yet a classification for them is far from being achieved, even in the case of hyperbolic geometry. The goal of this mini-workshop was to stimulate the research by bringing together specialists of different aspects of the theory.

Mathematics Subject Classification (2010): 20F55, 20F65, 22E40, 51F15, 53C35, 57R18.

Introduction by the Organizers

This mini-workshop *Reflection groups in negative curvature* was attended by 18 participants, including the three organizers: Misha Belolipetsky (IMPA), Vincent Emery (Bern), and Ruth Kellerhals (Fribourg). The participants had responded to the invitation with interest and enthusiasm.

Hyperbolic reflection groups are discrete Coxeter groups – generated by reflections – acting on the real hyperbolic n-space $\mathbf{H}^n_{\mathbb{R}}$. They are direct analogues of spherical and Euclidean reflection groups, and provide an important source of lattices in the isometry group $\mathrm{PO}(n,1)$ of $\mathbf{H}^n_{\mathbb{R}}$. Their fundamental importance can be explained by the following:

(1) they provide very concrete examples of lattices, from which explicit volume computations, topological constructions, etc., can be realized;
(2) some of them provide examples of lattices that are non-arithmetic.

A systemic study of hyperbolic reflection groups was initiated by Vinberg in the 1960's; in particular the first results about non-arithmeticity (Makarov, Vinberg) were obtained at that time. On the other hand, the study of arithmetic subgroups has been an important tool for constructing examples of discrete reflection lattices in $\mathrm{PO}(n, 1)$ (work of Vinberg, Kaplinskaya, Bugaenko).

Due to their importance it is clear that a classification of hyperbolic reflection groups is desirable. However, the latter is far from being completed. To illustrate, a famous theorem by Prokhorov asserts that no discrete reflection lattices exist in $\mathrm{PO}(n, 1)$ for $n > 996$; yet the highest dimensional known example is $n = 21$ (Borcherds, 1987).

The definition of hyperbolic reflection group can be adapted to the case of the *complex* hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$. There also constructions are possible in low dimensions, and the fundamental facts (1) and (2) expressed above in the real case remain correct. Moreover, a classification in this case appears even more elusive.

The aim of the mini-workshop was to bring together (senior and younger) researcher that are specialist on the reflection groups in order to exchange knowledge, stimulate the research, and facilitate further and new collaborations. Four of the participants were asked in advanced to prepare a mini-course (2×50 minutes) on a specific subject related to their research, and of general interest for this workshop:

- Martin Deraux: Non-arithmetic lattices in $\mathrm{PU}(2, 1)$ and $\mathrm{PU}(3, 1)$.
- Anna Felikson: Cluster algebras and reflection groups.
- Matthew Stover: Recent rigidity results in $\mathrm{PO}(n, 1)$.
- Anne Thomas: Geometry of right-angled Coxeter groups.

In addition almost everyone in the rest of the participants gave an individual talk on his research (10 talks of 50 minutes).

Free time for discussion was scheduled in the early afternoon. The first three days of the workshop were concluded by problem sessions, at which many specific research problems have been exposed and discussed. On Friday afternoon, during a session dedicated to computer experiments, the idea formulated by Anna Haensch and Jeffrey Meyer to organize a workshop Sage Days related to hyperbolic geometry (in particular Coxeter groups) was discussed with great interest. As a result a wishlist for useful functions to implement in Sage has been elaborated, as well as practical aspects for the organization of such a workshop.

The organizers wish to thank all the participants for their enthusiasm and commitment during this week at Oberworlfach, as well as the MFO staff and the MFO administration for the excellent working conditions.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, “US Junior Oberwolfach Fellows”.

Mini-Workshop: Reflection Groups in Negative Curvature

Table of Contents

Abstracts

Non-arithmetic complex hyperbolic lattices

Martin Deraux
(joint work with John R. Parker, Julien Paupert)

We consider lattices (discrete subgroups of finite covolume) in the isometry groups of symmetric spaces of non-compact type. It follows from important work of Margulis, Corlette, Gromov-Schoen that for most symmetric spaces, all lattices are arithmetic, i.e. they can be obtained (up to commensurability and taking the image by a surjective homomorphism with compact kernel) as the set of matrices with integer entries in a linear algebraic group defined over the rationals. For these symmetric spaces, this gives a satisfactory classification of lattices, since forms over totally real number fields of real algebraic groups have been classified up to commensurability by work of Weil and Tits.

The exceptions to the above general arithmeticity theorems are real and complex hyperbolic spaces, where some non-arithmetic lattices are known to exist. In the real hyperbolic case, there is a general construction due to Gromov and Piatetski-Shapiro, based on gluing pieces of arithmetic manifold along totally geodesic submanifolds, which produces infinitely many commensurability classes of non-arithmetic lattices in every dimension. Although these examples are well understood, and the construction has been generalized in several directions, there is currently no general structure theory of lattices in $PO(n,1)$.

In the complex hyperbolic case, the situation is even more mysterious. A few non-arithmetic lattices have been constructed, only in very low dimension. The first examples were constructed by Mostow, and his work was generalized by Deligne and Mostow. For a long time, these were the only known examples, even though some alternative constructions were given (by Thurston, Barthel-Hirzebruch-Höfer, for example). In joint work with J.Parker and J. Paupert [1], [2], the author used a variation on the original Mostow construction to produce more examples of non-arithmetic lattices in $PU(2,1)$. The main technique is the construction of explicit polytopes in the complex hyperbolic plane that serve as a fundamental domain for the action of the group, the latter being verified by applying the Poincaré polyhedron theorem. Our verification of the hypotheses of the Poincaré polyhedron theorem relies on heavy computer usage, but most examples have been given alternative construction, using orbifold uniformization (hence avoiding the use of computers), see [3], [4].

It turns out that our examples contain 22 commensurability classes. In order to determine the precise number of commensurability classes obtained by our construction, we used basic commensurability invariants (cocompactness, adjoint trace field) and also volume estimates in conjunction with the Margulis commensurator theorem. It would be interesting to find general, effective techniques to determine whether lattices are commensurable. More recently, the author studied the arithmeticity of the lattices constructed by Couwenberg, Heckman and

Looijenga, and showed that the list contains precisely one non-arithmetic lattice in $PU(n,1)$ with $n \geq 3$ which is not commensurable with any Deligne-Mostow lattices by Deligne-Mostow, see [5].

The main open question is whether there are infinitely many commensurabily classes of non-arithmetic lattices in $PU(n,1)$ for $n \geq 2$. There is no straightforward analog of the Gromov-Piatetski Shapiro construction in complex hyperbolic space, since there are no totally geodesic real hypersurfaces (in any dimension $n \geq 2$).

To this day, three classes of techniques have been used to produce lattices, namely arithmetic constructions, fundamental domains, and uniformization. The last class has two slightly different flavors, the first one being orbifold uniformization, where the complex hyperbolic metric comes from the existence of a Kähler-Einstein metric (this is the approach taken in BHH). The second one is the uniformization via period maps arising from certain moduli spaces (this is the approach taken in Deligne-Mostow). It would be interesting to develop systematic techniques to go from one of the three descriptions of a lattice to another.

References

[1] M. Deraux, J .R. Parker, J. Paupert, *New non-arithmetic complex hyperbolic lattices*, Invent. Math. **203** (2016), 681–771.

[2] M. Deraux, J .R. Parker, J. Paupert, *New non-arithmetic complex hyperbolic lattices, II*, arXiv:1611.00330.

[3] M. Deraux, *Non-arithmetic lattices and the Klein quartic*, to appear in J. Reine Angew. Math., doi.org/10.1515/crelle-2017-0005.

[4] M. Deraux, *Non-arithmetic ball quotients from a configuration of elliptic curves in an Abelian surface*, Comment. Math. Helv. **93** (2018), 533-554.

[5] M. Deraux, *A new non-arithmetic lattice in* $PU(3,1)$, arXiv:1710.04463.

A hyperbolic counterpart to Rokhlin's cobordism theorem

Alexander Kolpakov

(joint work with Michelle Chu)

A classical result by V. Rokhlin states that every compact orientable 3-manifold bounds a compact orientable 4-manifold, and thus the three-dimensional cobordism group is trivial. One can recast the question of bounding in the setting of hyperbolic geometry, which generated plenty of research directions over the past decades.

A hyperbolic manifold is a manifold endowed with a Riemannian metric of constant sectional curvature -1. Here and below all manifolds are assumed to be connected, orientable, complete, and of finite volume, unless otherwise stated. We refer to [14] for the definition of an arithmetic hyperbolic manifold.

A hyperbolic n-manifold $\mathcal{M}$ bounds geometrically if it is isometric to $\partial\mathcal{W}$, for a hyperbolic $(n+1)$-manifold $\mathcal{W}$ with totally geodesic boundary.

Indeed, some interest in hyperbolic manifolds that bound geometrically was kindled by the works of D. Long, A. Reid [10, 11, 12] and B. Niemershiem [15], motivated by a preceding work of M. Gromov [5, 6] and a question by F. Farrell and S. Zdravkovska [4]. This question is also related to hyperbolic instantons, as described by J. Ratcliffe and S. Tschantz [18, 19].

As [10] shows many closed hyperbolic 3-manifolds do not bound geometrically: a necessary condition is that the eta invariant of the 3-manifold must be an integer. The first known closed hyperbolic 3-manifold that bounds geometrically was constructed in [18] and has volume of order 200.

The first examples of knot and link complements that bound geometrically were produced by L. Slavich in [16, 17]. However, [8] implies that there are plenty of cusped hyperbolic 3-manifolds that cannot bound geometrically, with the obstruction being the geometry of their cusps.

In [1], M. Belolipetsky, T. Gelander, A. Lubotzky, and A. Shalev showed that the asymptotic growth rate of the number $\alpha_n(v)$ of all orientable arithmetic hyperbolic manifolds, up to isometry, with respect to volume v is super-exponential, in all dimensions $n \geq 3$. That is, there exist constants $A, B, C, D > 0$ such that $Av^{Bv} \leq \alpha_n(v) \leq Cv^{Dv}$. In our present work, we use the ideas of [1, 12, 13] together with the more combinatorial colouring techniques from [9] in order to prove the following facts:

Theorem 1. *Let $\beta_n(v) =$ the number of non-isometric compact arithmetic hyperbolic n-manifolds of volume $\leq v$ that bound geometrically. Then, if $2 \leq n \leq 8$, we have that $Av^{Bv} \leq \beta_n(v) \leq Cv^{Dv}$, for some constants $A, B, C, D > 0$.*

Theorem 2. *Let $\gamma_n(v) =$ the number of non-isometric cusped arithmetic hyperbolic n-manifolds of volume $\leq v$ that bound geometrically. Then, if $2 \leq n \leq 19$, we have that $Av^{Bv} \leq \gamma_n(v) \leq Cv^{Dv}$, for some constants $A, B, C, D > 0$.*

The proofs of both theorems above rely heavily on reflectivity of certain quadratic forms studied by È. Vinberg, I. Kaplinskaya [7, 20] and V. Bugaenko [2, 3].

References

[1] M. Belolipetsky – T. Gelander – A. Lubotzky – A. Shalev, *Counting arithmetic lattices and surfaces*, Ann. of Math. (2) **172** (2010), 2197–2221.

[2] V. O. Bugaenko, *Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring* $\mathbb{Z}\left[\frac{1+\sqrt{5}}{2}\right]$, Vestnik Moskov. Univ. Ser. I Mat. Mekh., **5** (1984), 6–12.

[3] ——, *Arithmetic crystallographic groups generated by reflections, and reflective hyperbolic lattices*, Lie groups, their discrete subgroups, and invariant theory, 33–55, Adv. Soviet Math., 8, Amer. Math. Soc., Providence, RI, 1992.

[4] F. T. Farrell – S. Zdravkovska, *Do almost flat manifolds bound?*, Michigan Math. J. **30** (1983), 199–208.

[5] M. Gromov, *Manifolds of negative curvature*, J. Differential Geom. **13** (1987), 223–230.

[6] ——, *Almost flat manifolds*, J. Differential Geom. **13** (1978), 231–241.

[7] I. M. Kaplinskaya, È. B. Vinberg, *On the groups $O_{18,1}(\mathbb{Z})$ and $O_{19,1}(\mathbb{Z})$*, Dokl. Akad. Nauk SSSR **238** (1978), 1273–1275.

[8] A. Kolpakov – A. Reid – S. Riolo, *Many cusped hyperbolic 3-manifolds do not bound geometrically*, arXiv:1811.05509.

[9] A. Kolpakov – L. Slavich, *Hyperbolic 4-manifolds, colourings and mutations*, Proc. LMS **113** (2016), 163–184.

[10] D. D. Long – A. W. Reid, *On the geometric boundaries of hyperbolic 4-manifolds*, Geom. Topol. **4** (2000) 171–178.

[11] ———, *All flat manifolds are cusps of hyperbolic orbifolds*, Alg. Geom. Topol. **2** (2002), 285–296.
[12] ———, *Constructing hyperbolic manifolds which bound geometrically*, Math. Research Lett. **8** (2001), 443–456.
[13] A. Lubotzky, *Subgroup growth and congruence subgroups*, Invent. Math. **119** (1995), 267–295.
[14] C. Maclachlan – A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Graduate Texts in Mathematics **219**, Springer-Verlag, New York (2003).
[15] B. E. Nimershiem, *All flat three-manifolds appear as cusps of hyperbolic four-manifolds*, Topology and its Appl. **90** (1998), 109–133.
[16] L. Slavich, *A geometrically bounding hyperbolic link complement*, Alg. Geom. Topol. **15** (2015), 1175–1197.
[17] ———, *The complement of the figure-eight knot geometrically bounds*, Proc. AMS **145** (2017), 1275–1285.
[18] J. G. Ratcliffe – S. T. Tschantz, *Gravitational instantons of constant curvature*, Class. Quantum Grav. **15** (1998), 2613–2627.
[19] ———, *On the growth of the number of hyperbolic gravitational instantons with respect to volume*, Class. Quantum Grav. **17** (2000), 2999–3007.
[20] È. B. Vinberg, *On groups of unit elements of certain quadratic forms*, Math. USSR – Sbornik **16** (1972), 17–35.

Cluster algebras and reflection groups

Anna Felikson
(joint work with Pavel Tumarkin)

Cluster algebras were introduced by Fomin and Zelevinsky [3] in 2002, since then it turned out that the notion is connected to numerous fields in mathematics (such as hyperbolic geometry, Teichmüller theory, dilogarithm identities, Poisson geometry, representation theory, integrable systems, combinatorics of polytopes, probability theory and many others). In this mini-course we introduce cluster algebras and show a number of their connections to reflection groups.

From the very birth of the theory of cluster algebras it was known due to Fomin and Zelevinsky [4] that cluster algebras of finite type are tightly related to root systems. More precisely, cluster algebras of finite type are described by Dynkin diagrams. Moreover, the cluster variables (key elements in the cluster algebra construction) in cluster algebras of finite type are in natural correspondence with almost positive roots (where by the set of almost positive roots one means the union of positive roots and negative simple roots).

Further classificational result, the description of cluster algebras of finite mutation type [5], showed some more implicit connections to reflection groups: the methods used for this classification were inspired by the strategies used for investigation of hyperbolic Coxeter polytopes (see [10] for the overview of the known developments on hyperbolic Coxeter polytopes).

In [2], Barot and Marsh proposed a method to construct a series of presentations of a finite reflection group arising from a cluster algebra of finite type. More precisely, a cluster algebra is defined by a quiver (i.e. an oriented graph), and a presentation of the corresponding reflection group can be read off the quiver. The quiver defining a cluster algebra is determined up to an operation of mutation. Barot and Marsh show that the reflection group defined from the quiver does not depend on the choice of the quiver in the mutation class.

In [7] we use a geometric interpretation of the Barot-Marsh presentations to construct hyperbolic manifolds of finite volume with actions of large finite groups. We also generalise results of [2] to affine quivers, quivers arising from unpunctured surfaces and exceptional mutation-finite quivers, see [6]. When the quiver is not finite or affine, however, we obtain a quotient of a Coxeter group instead of a Coxeter group.

In the case when a cluster algebra is acyclic (i.e. when one of the quivers defining this cluster algebra contains no oriented cycles), one can construct a linear reflection group defined by the corresponding quiver. Mutation of the quiver will then correspond to the change of generators of the group, see [1]. Moreover, similar geometric construction works for every quiver in case of rank 3, see [8]. Furthermore, the geometric constructions under consideration work also for quivers with non-integer weights of arrows. This allows us to classify quivers of finite mutation type in rank 3 [8] as well as in all other ranks [9]. The classification turns out to be also tightly connected to reflection groups.

References

[1] M. Barot, C. Geiss, A. Zelevinsky, *Cluster algebras of finite type and positive symmetrizable matrices*, J. London Math. Soc. (2) **73** (2006), 545–564.

[2] M. Barot, R. J. Marsh, *Reflection group presentations arising from cluster algebras*, Trans. Amer. Math. Soc. **367** (2015), 1945–1967.

[3] S. Fomin, A. Zelevinsky, *Cluster algebras. I. Foundations.* J. Amer. Math. Soc., **15**, 2 (2002): 497–529.

[4] S. Fomin, A. Zelevinsky, *Cluster algebras. II. Finite type classification.* Invent. Math., **154**, 1 (2003): 63–121.

[5] A. Felikson, M. Shapiro, P. Tumarkin, *Skew-symmetric cluster algebras of finite mutation type*, J. Eur. Math. Soc. **14** (2012), 1135–1180.

[6] A. Felikson, P. Tumarkin, *Coxeter groups and their quotients arising from cluster algebras*, Int. Math. Res. Notices (2016), 5135–5186.

[7] A. Felikson, P. Tumarkin, *Coxeter groups, quiver mutations and geometric manifolds*, J. London Math. Soc., **94** (2016), 38–60.

[8] A. Felikson, P. Tumarkin, *Geometry of mutation classes of rank 3 quivers*, Arnold Math. J., `https://doi.org/10.1007/s40598-019-00101-2`.

[9] A. Felikson, P. Tumarkin, *Mutation-finite quivers with real weights*, arXiv: 1902.01997.

[10] `http://www.maths.dur.ac.uk/users/anna.felikson/Polytopes/polytopes.html`, webpage on Hyperbolic Coxeter polytopes

(Volumes of) hyperbolic manifolds

Brent Everitt

(joint work with Bob Howlett, John Ratcliffe, Steve Tschantz)

This will be an elementary survey of some ideas for finding small volume hyperbolic manifolds in various dimensions.

The presence of an Euler characteristic creates a big difference between even and odd dimensions: in even dimension the volume of a hyperbolic manifold is a constant multiple of the Euler characteristic by the Gauss-Bonnet theorem. As the Euler characteristic takes integer values, the most obvious place to look for minimal volume manifolds is when the characteristic is 1 (in absolute value). A compact orientable manifold has even characteristic, so the minimum volume is most likely achieved by a non-compact manifold. In odd dimensions the Euler characteristic vanishes, and so a different approach must be found. For these reasons, progress in even dimensions has been more rapid.

The underlying principle is to consider the quotient of hyperbolic space by the action of groups acting freely and properly discontinuously. Algebraically, these are discrete, torsion-free subgroups of the isometry group of hyperbolic space. The greatest control is to be had when working inside reflection groups – the torsion is well understood, and the problem of finding subgroups that avoid the torsion becomes combinatorial.

The talk will report a construction of small volume manifolds in $4, 6$ and 8-dimensions. The techniques use all right-angles polytopes and the action of finite Weyl groups (of types A_4, E_6 and E_8) on their root and weight lattices.

References

[1] Brent Everitt, John Ratcliffe and Steve Tschantz, *Right-angled Coxeter polytopes, hyperbolic six-manifolds, and a problem of Siegel*, Math. Ann. **354** (2012), 871–905.

[2] Brent Everitt, Robert Howlett, *Weyl groups, lattices and geometric manifolds*, Geom. Ded. **142** (2009), 1–21.

Commensurability of hyperbolic Coxeter groups and quadratic forms

Edoardo Dotti

Let $\mathbb{H}^n$ be the real hyperbolic space of dimension n. Consider two discrete subgroups $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$. They are said to be *commensurable* (in the wide sense) if there exists $h \in \mathrm{Isom}(\mathbb{H}^n)$ such that $\Gamma_1 \cap h\Gamma_2 h^{-1}$ has finite index in both Γ_1 and $h\Gamma_2 h^{-1}$. This is an equivalence relation. We are interested in the classification of hyperbolic Coxeter groups into commensurability classes where a Coxeter hyperbolic group is a cofinite discrete group generated by reflections on the bounding hyperplanes of a suitable polyhedron. The complete commensurability classification of hyperbolic Coxeter groups has been achieved for simplices [5] and pyramids [3].

We start by looking at arithmetic groups in $\mathrm{Isom}(\mathbb{H}^n)$. A group is arithmetic if it is commensurable to the automorphism group of a $\mathcal{O}_k$-lattice in the ambient group, where $\mathcal{O}_k$ is the ring of integers of a totally real number field k. Such a group has always an associated number field and quadratic form. By a result of Gromov and Piatetski-Shapiro [2], it is known that two hyperbolic arithmetic groups are commensurable if and only if the two associated fields are equal and the quadratic forms are similar. However, for non-arithmetic groups this is false, and a necessary and sufficient commensurability criterion does not exist to date.

Following work of Vinberg [6, §6], we are able to associate a number field and a quadratic form even to non-arithmetic hyperbolic Coxeter groups, the *Vinberg field* and the *Vinberg form*. We then obtain a new necessary condition for commensurability of non-arithmetic hyperbolic Coxeter groups.

Theorem 1 (Dotti, [1]). *Let $\Gamma_1, \Gamma_2 < \mathrm{Isom}(\mathbb{H}^n)$ be two commensurable Coxeter groups in $\mathrm{Isom}(\mathbb{H}^n)$, $n \geq 2$. Then their Vinberg fields coincide and the two associated Vinberg forms are similar over it.*

In the even dimensional case, we present a concrete criterion for similarity of the Vinberg form, in terms of the Hasse invariant for example, and we apply it to the following example. Consider two Napier cycles giving rise to the Coxeter groups Γ_1, Γ_2 (see [4]) acting cocompactly on $\mathbb{H}^4$ represented by their Coxeter graphs as shown in Figure 1.

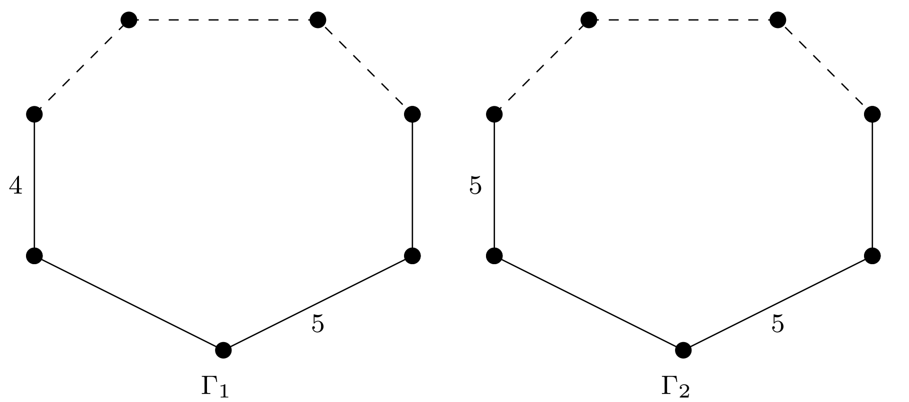

FIGURE 1. The groups Γ_1 and Γ_2 acting cocompactly on $\mathbb{H}^4$.
1

These groups have both $k = \mathbb{Q}(\sqrt{5})$ as Vinberg field. The diagonalized associated Vinberg forms over k are $q(\Gamma_1) = (4, 4, 4, -2 - 2\sqrt{5}, 20 + 8\sqrt{5})$ and $q(\Gamma_2) = (4, \frac{5}{2} + \frac{1}{2}\sqrt{5}, 2 + \frac{2}{5}\sqrt{5}, \frac{-37}{2} - \frac{17}{2}\sqrt{5}, \frac{312}{19} + \frac{136}{19}\sqrt{5})$, respectively. These forms are not similar, and therefore the two groups are not commensurable.

References

[1] E. Dotti, *PhD. Thesis*, University of Fribourg (Switzerland). In progress.
[2] M. Gromov and I. Piatetski-Shapiro, *Non-arithmetic groups in Lobačevskiĭ spaces*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques **66**(1) (1987), 93–103.
[3] R. Guglielmetti, M. Jacquemet and R.Kellerhals, *On commensurable hyperbolic Coxeter groups*, Geometriae Dedicata **183**(1) (2016), 143–167.
[4] H. Im Hof, *Napier cycles and hyperbolic Coxeter groups*, Bull. Soc. Math. Belg. Sér. A **42**(3) (1990), 523–545.
[5] N.W. Johnson, R. Kellerhals, J.G. Ratcliffe and S.T. Tschantz, *Commensurability classes of hyperbolic Coxeter groups*, Linear Algebra and its Applications **347**(1-3) (2002), 119–147.
[6] E. Vinberg, *Discrete groups generated by reflections in Lobačevskiĭ spaces*, Sbornik: Mathematics **1**(3) (1967), 429–444.

On finite upper half plane graphs

Yohei Komori

Let F_q be a finite field with q elements where q is an odd number. After fixing a non-square element $\delta \in F_q$, we define the finite upper half plane as

$$H_q = F_q(\sqrt{\delta}) - F_q.$$

For $z = x + y\sqrt{\delta}$, we use the following notations

$$\overline{z} = z^q = x - y\sqrt{\delta}, \ \ \mathrm{Im}\, z = y, \ \ N(z) = z\overline{z} = z^{q+1}.$$

To make H_q as a graph, we define the incidence relation among elements of H_q as follows [1, 6]; for $z, w \in H_q$, the "distance" between z and w is

$$d(z,w) = \frac{N(z-w)}{\mathrm{Im\ z\ Im\ w}}.$$

For $a \in H_q$, the graph $X_q(\delta, a)$ consists of H_q as its vertex set, and $z, w \in H_q$ are connected by an edge if the "distance" between z and w is equal to a i.e. $d(z,w) = a$. Then $X_q(\delta, a)$ is isomorphic to the Cayley graph of the affine group

$$\mathrm{Aff}(q) = \{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \mid x, y \in F_q,\ y \neq 0\}$$

with its symmetric system of generators

$$S_q(\sqrt{\delta}, a) = \{ \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} \in \mathrm{Aff}(q) \mid x^2 = ay + \delta(y-1)^2\},$$

which implies that $X_q(\delta, a)$ is a $(q+1)$-regular connected graph provided that $a \neq 0,\ 4\delta$. As a conclusion the spectrum $Sp(X_q(\delta, a))$ of $X_q(\delta, a)$, the set of eigenvalues of the adjacency matrix of $X_q(\delta, a)$ contains $q+1$ with multiplicity one which is called the trivial eigenvalue. Then every non-trivial eigenvalue $\lambda \in Sp(X_q(\delta, a))$ satisfies the following remarkable inequality

$$\frac{\lambda}{\sqrt{q}} \in [-2, 2]$$

which implies that $X_q(\delta, a)$ is a Ramanujan graph [2, 4, 5]. In my talk I reported some results [3, 7] on the asymptotic behavior of k-th moments of the distribution of non-trivial eigenvalues

$$\lim_{q\to\infty}\sum_{i=1}^{q-1}(\frac{\lambda_i}{\sqrt{q}})^k\frac{m_i}{q(q-1)}=\int_{-2}^{2}x^k\frac{\sqrt{4-x^2}}{2\pi}dx$$

where m_i is the multiplicity of the non-trivial eigenvalue λ_i.

References

[1] J. Angel, *Finite upper half planes over finite fields*, Finite Fields Appl. **2** (1996) 62–86.

[2] R. Evans, *Character sums as orthogonal eigenfunctions of adjacency operators for Cayley graphs*, in: Con- temp. Math., vol. **168**, 1994, pp. 33–50.

[3] A. Katamoto, *On 3rd and 4th moments of finite upper half plane graphs*, Finite Fields and Their Applications **13** (2007) 249–258.

[4] N. Katz, *Estimates for Soto-Andrade sums*, J. Reine Angew. Math. **438** (1993) 143–161.

[5] A. Lubotzky, R. Phillips and P. Sarnak, *Ramanujan graphs*, Combinatorica **8** (1988) 261–277.

[6] A. Terras, *Fourier Analysis on Finite Groups and Applications*, Cambridge Univ. Press, 1999.

[7] Y. Komori and T. Yasui, *On finite upper half plane graphs of characteristic two*, in preparation.

Large-scale geometry of right-angled Coxeter groups

Anne Thomas

(joint work with Pallavi Dani, Emily Stark)

This mini-course consisted of two talks on various aspects of geometric group theory relating to right-angled Coxeter groups. General references are [5] and [7], while many recent results and open questions are discussed in the survey [2].

Given a finite simplicial graph Γ with vertex set S, the associated right-angled Coxeter group W_Γ has generating set S, and relations $s^2 = 1$ for all $s \in S$ together with $st = ts$ whenever s and t are adjacent vertices of Γ. Right-angled Coxeter groups include some hyperbolic reflection groups, for example the group generated by reflections in the sides of a right-angled hyperbolic pentagon.

Right-angled Coxeter groups are often studied via their action on the associated Davis complex Σ_Γ. This is a cube complex with 1-skeleton the Cayley graph of W_Γ with respect to S, and then for every n-element subset of pairwise commuting elements of S, the corresponding n-cube "filled in". The group W_Γ acts properly discontinuously and cocompactly on the associated Davis complex Σ_Γ.

We recalled the notion of a CAT(0) space, due to Gromov. Gromov proved that the Davis complex for a right-angled Coxeter group is a CAT(0) space. Consequently, it has many nice properties, for example it is contractible, uniquely geodesic, and has a visual boundary. We then recalled the definition of a quasi-isometry, also due to Gromov, and explained that a major program in geometric group theory is to classify finitely generated groups up to quasi-isometry. By the Milnor–Schwarz Lemma, the group W_Γ is quasi-isometric to its Davis complex Σ_Γ.

We then recalled various notions of "hyperbolicity" due to Gromov. We stated Moussong's Theorem, which characterizes word-hyperbolicity, in the case of right-angled Coxeter groups. We then discussed joint work with Pallavi Dani [3] and with Pallavi Dani and Emily Stark [4] on the quasi-isometry and abstract commensurability classification of certain word-hyperbolic right-angled Coxeter groups. (If two groups are abstractly commensurable, meaning that they have isomorphic finite index subgroups, then they are quasi-isometric.)

The quasi-isometry invariant used in [3] is Bowditch's JSJ tree [1]. We recalled the definition of the visual boundary of a word-hyperbolic group, and the result due to Gromov that the homeomorphism type of the visual boundary is a quasi-isometry invariant for such groups. Bowditch's JSJ tree is constructed using only topological features of the visual boundary, so this tree is also a quasi-isometry invariant. The main results of [3] give an explicit construction of Bowditch's JSJ tree for a class of right-angled Coxeter groups, and show that if the graph Γ has no K_4 minor, then this tree is a complete quasi-isometry invariant.

For the work on abstract commensurability in [4], we first prove that the right-angled Coxeter groups we consider have finite-index, torsion-free subgroups which are geometric amalgams of free groups. This allows us to apply a topological rigidity theorem of Lafont [6] and so formulate our necessary conditions. These conditions involve vectors of Euler characteristics of certain parabolic subgroups. To show that our conditions are sufficient involves delicate constructions of finite coverings.

References

[1] B. H. Bowditch, *Cut points and canonical splittings of hyperbolic groups*, Acta Math. 180 (1998) 145–186.

[2] Pallavi Dani, *The large-scale geometry of right-angled Coxeter groups, (a survey of recent results on right-angled Coxeter groups)*, to appear in Handbook of Group Actions, (ed. L. Ji, A. Papadopoulos and S.T. Yau), Volume V, International Press and Higher Education Press, 2018.

[3] Pallavi Dani and Anne Thomas, *Bowditch's JSJ tree and the quasi-isometry classification of certain Coxeter groups*, J. Topol. 10 (2017), no. 4, 1066–1106.

[4] Pallavi Dani, Emily Stark and Anne Thomas, *Commensurability for certain right-angled Coxeter groups and geometric amalgams of free groups* Groups Geom. Dyn. 12 (2018), no. 4, 1273–1341.

[5] Michael W. Davis, *The geometry and topology of Coxeter groups*, London Mathematical Society Monographs Series, 32. Princeton University Press, Princeton, NJ, 2008.

[6] J.-F. Lafont, *Diagram rigidity for geometric amalgamations of free groups* J. Pure Appl. Algebra 209 (2007), no. 3, 771–780.

[7] Anne Thomas, *Geometric and topological aspects of Coxeter groups and buildings*, Zurich Lectures in Advanced Mathematics. European Mathematical Society (EMS), Zürich, 2018.

Geodesic submanifolds of hyperbolic manifolds

Matthew Stover
(joint work with Uri Bader, David Fisher, Jean-François Lafont, Nicholas Miller)

It is a basic fact that closed geodesics on finite-volume hyperbolic manifolds are uniformly dense. The higher-dimensional analogue of a geodesic is a *totally geodesic subspace*, i.e., an immersed, relatively closed, finite-volume, codimension k submanifold $N \looparrowright M$ for which the map on universal covers $\widetilde{N} \hookrightarrow \widetilde{M}$ is realized as an isometric embedding $\mathbb{H}^{n-k} \hookrightarrow \mathbb{H}^n$. In general, it is not expected that a negatively curved manifold will have totally geodesic subspaces other than geodesics. Remarkably, each known construction of hyperbolic manifolds runs counter to this expectation. Specifically, for $n \geq 4$ there are basically three known constructions – arithmetic groups, reflection groups, and cut-and-paste methods – and all contain geodesic submanifolds of some codimension $1 \leq k \leq n-2$.

When an arithmetic manifold contains one geodesic submanifold of codimension k, then it contains infinitely many and they are everywhere dense. One can easily prove this using the well-known fact due to Borel that the *commensurator* of $\pi_1(M)$ in $\mathrm{PO}_0(n,1)$ is analytically dense. All known nonarithmetic hyperbolic n-manifolds for $n \geq 4$ contain a totally geodesic hypersurface. However the above commensurator trick for turning one into infinitely many fails. Indeed, it is a famous result of Margulis that $\pi_1(M)$ is arithmetic *if and only if* its commensurator in $\mathrm{PO}_0(n,1)$ is dense [4, p. 2]. Motivated by this failure, Alan Reid and Curtis McMullen (for $n = 3$) asked: *If M is a hyperbolic n-manifold containing infinitely many totally geodesic hypersurfaces, is $\pi_1(M)$ arithmetic?*

This was asked without a single example for which the set of totally geodesic hypersurfaces was known to be finite but nonempty. (There were known examples of nonarithmetic hyperbolic 3-manifolds containing no totally geodesic surfaces; see [3, §6.1] for examples.) The first examples were provided in joint work with Fisher, Lafont, and Miller:

Theorem 1 ([3])**.** *There exist infinitely many commensurability classes of finite-volume nonarithmetic hyperbolic n-manifolds, $n \geq 3$, for which the collection of totally geodesic hypersurfaces is finite but nonempty.*

This theorem includes all the famous hybrid examples of Gromov and Piatetski-Shapiro. See [3] for a careful statement that includes higher codimensions. The proof applies to certain cut-and-paste manifolds that contain "arithmetic pieces". In particular, the proof can be made effective in that one could possibly enumerate the finite collection of geodesic submanifolds in certain cases. For example, we also show that there are nonarithmetic reflection lattices (e.g., in dimensions 3 and 6) that are not commensurable with lattices constructed by the classical cut-and-paste constructions [3, §6.2].

More recent work with Bader, Fisher, and Miller answers Reid and McMullen's question completely.

Theorem 2 ([1])**.** *Let M be a hyperbolic n-manifold containing infinitely many codimension k totally geodesic submanifolds for some $1 \leq k \leq n-2$ that are not properly contained in a totally geodesic submanifold of smaller codimension. Then $\pi_1(M)$ is arithmetic.*

For example, it follows that if $n \geq 4$ is even and M is a hyperbolic n-manifold, then M contains infinitely many totally geodesic hypersurfaces if and only if $\pi_1(M)$ is arithmetic. Theorem 2 also implies that if K is a knot in S^3 such that $S^3 \smallsetminus K$ admits a complete hyperbolic structure of finite volume for which $S^3 \smallsetminus K$ contains infinitely many immersed totally geodesic surfaces, then K is the figure-eight knot. The proof is inspired by Margulis's proof of arithmeticity of lattices in higher-rank semisimple Lie groups [4]. In particular, we prove a *superrigidity theorem* for certain representations of lattices in $\mathrm{PO}_0(n,1)$.

References

[1] U. Bader, D. Fisher, N. Miller, M. Stover, *Arithmeticity, superrigidity, and totally geodesic submanifolds*, arXiv:1903.08467

[2] A. Borel, *Density and maximality of arithmetic subgroups*, J. Reine Angew. Math. **224** (1966), 78–89.

[3] D. Fisher, J.-F. Lafont, N. Miller, M. Stover, *Finiteness of maximal geodesic submanifolds in hyperbolic hybrids*, arXiv:1802.04619

[4] G. A. Margulis, Discrete subgroups of semisimple Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) **17**, Springer-Verlag, 1991.

Pseudo-arithmetic reflection groups

Vincent Emery

(joint work with Olivier Mila)

For $n \geq 4$ there are two known sources of nonarithmetic lattices in $\mathrm{PO}(n,1)$:

(1) Reflection groups (Makarov, Vinberg; see [4])
(2) Manifolds obtained by gluing or "hybridisation" of arithmetic pieces (Gromov and Piatetski-Shapiro [2], and more recent generalisations)

This second source is systematic in the sense that it provides (infinitely many) examples in every dimension $n > 2$. On the other hand, by nature the reflection groups can only be a "sporadic" source.

For any lattice $\Gamma \subset \mathrm{PO}(n,1)$ there exists a minimal number field $K \subset \mathbb{R}$ (the *trace field*) and an algebraic K-group $\mathbf{G}$ such that $\Gamma \subset \mathbf{G}(K)$ and $\mathbf{G}(\mathbb{R})$ identifies with $\mathrm{PO}(n,1)$. We call the group $\mathbf{G}$ (which is uniquely determined by Γ up to K-isomorphism) the *ambient group* of Γ. In the recent paper [1] the authors introduced the following notion, which generalizes the notion of arithmetic lattices:

Definition 1 (Pseudo-arithmeticity)**.** A lattice $\Gamma \subset \mathrm{PO}(n,1)$ $(n \geq 4)$ is called pseudo-arithmetic if its trace field K is a multi-quadratic extension K/F and its ambient group is obtained the scalar extension $\mathbf{G}_K$ of an F-group $\mathbf{G}$ such that $\mathbf{G}(F \otimes_{\mathbb{Q}} \mathbb{R}) = \mathrm{PO}(n,1) \times \text{compact}$.

The introduction of this notion is motivated by the following theorem, proved in the same paper.

Theorem 1. *Let* $M = \mathbf{H}^n_{\mathbb{R}}/\Gamma$ *be a hyperbolic manifold obtained by gluing of arithmetic pieces. Then* Γ *is pseudo-arithmetic.*

Extending the methods of Vinberg's arithmeticity criterion [4] we have been able to check that all *currently known* hyperbolic reflection lattices in dimension $n \geq 4$ are also pseudo-arithmetic. This is a rather surprising observation, in particular since many of the known (non-arithmetic) examples are constructed essentially by combinatorial means (see for example Roberts [3]). This brings to ask:

Problem 1. Let $\Gamma \subset \mathrm{PO}(n,1)$ be lattice, with $n \geq 4$. Is Γ necessarily pseudo-arithmetic?

References

[1] V. Emery, O. Mila, *Hyperbolic manifolds and pseudo-arithmeticity*, preprint `arXiv:1810.12837`.

[2] M. Gromov, I. Piatetski-Shapiro, *Non-arithmetic groups in Lobachevsky spaces*, Publ. Math. IHES **66** (1987), 93–103.

[3] M. Roberts, *A classification of non-compact Coxeter polytopes with* $n+3$ *facets and one non-simple vertex*, preprint `arXiv:1511.08451`.

[4] E.B. Vinberg, *Discrete groups generated by reflections in Lobacevskii spaces*, Math. USSR-Sbornik **1** (1967), 429–444.

A Vinberg-like algorithm that's not Vinberg's algorithm

Alice Mark

Vinberg's algorithm has been one of the main tools in the classification of reflective hyperbolic lattices since he introduced it in the 1970s [14]. These are lattices in $O^+(n,1)$ generated by finitely many real hyperbolic reflections. Unlike finite and affine Coxeter groups, which exist in all dimensions, these do not exist in high dimensions. On the other hand, in dimensions up through 19, there are infinitely many [3]. A lattice in $O^+(n,1)$ is called reflective if it is generated up to finite index by reflections. A class of lattice with nice properties is the arithmetic lattices, and we are particularly interested in them because there are finitely many maximal arithmetic hyperbolic reflective lattices in all dimensions [2, 8, 1, 12].

The reflecting hyperplanes of a hyperbolic lattice Γ carve up hyperbolic space H^n into a tiling by copies of a fundamental polygon. Fix a copy P of that polygon. The lattice Γ is reflective if and only if P has finitely many sides and finite volume. The roots pointing outward from the walls of P form a system of simple roots for Γ. Vinberg's algorithm is a method for finding all those roots. Currently existing implementations are depend on the context of the specific problems they were written to solve. The dimension, ground field, and 'shape' of the quadratic form all come into play in an implementation. Our implementation for 2-dimensional lattices defined over $\mathbb{Z}[\sqrt{2}]$ was taking too long to run on some lattices, so we needed other techniques for finding simple roots [9].

Our approach is based around the idea that under a reasonable set of assumptions, codimension $n-1$ subspace of H^n containing a 1-dimensional facet of P admits a translation that is an element of Γ. A fast way to find this translation is to use the classical correspondence, due to Gauss and Dedekind, between integral quadratic forms defined over a field F and ideals in quadratic extensions of F. PARI [13] can identify a unit in such and extension field, and therefore a translation of φ minimal length, almost instantaneously.

We begin by assuming that we know simple roots for the stabilizer of a corner of P. This stabilizer is a finite Coxeter group of rank n. Any edge emanating from the corner is stabilized by a corresponding Coxeter subgroup of rank $n-1$. Our search for simple roots proceeds by picking one of those edges E and looking for the next wall of P one would bump into by walking along E. The shortest translation φ along E decomposes as a product of a pair of reflections orthogonal to E.

The next wall extends the rank $n-1$ group stabilizing E to a rank n group stabilizing the corner at the other end of E. We know from the classification of finite Coxeter groups that there are finitely many possibilities for what that rank n extension could be. It is always possible that the next wall will intersect E orthogonally. Any other possibilities will depend on the ground field, n, and the stabilizer of E. In the nicest cases, the next wall meets ℓ at the same vertex as one of a pair of reflections whose composition is the translation φ, and by finding φ we find that wall immediately. In other cases, finding the nearest translate gives us a bound on how far along ℓ we will need to search to find the next wall.

This algorithm is most useful for cocompact lattices, where all of P's vertices are inside H^n. Luckily, arithmetic lattices defined over extensions of $\mathbb{Q}$ are always cocompact. (For cusped arithmetic lattices, Vinberg's algorithm is effective, and you don't need these sorts of time saving strategies.) It has so far only been implemented in dimension 2, and over the field $\mathbb{Q}[\sqrt{2}]$, but we would like to implement it more generally, for other number fields and in higher dimensions.

References

[1] I. Agol, *Finiteness of arithmetic Kleinian reflection groups*, Proceedings of the International Congress of Mathematicians: Madrid, August 22-30, 2006: invited lectures (2006), 951–960

[2] I. Agol, M. Belolipetsky, P. Storm, and K. Whyte, *Finiteness of arithmetic hyperbolic reflection groups*, Groups Geom. Dyn. **2** (2018), 481–498

[3] D. Allcock, *Infinitely many hyperbolic Coxeter groups through dimension 19*, Geometry & Topology vol. 10 **12** (20o6), 737–758

[4] D. Allcock, *The Reflective Lorentzian Lattices of Rank 3*, American Mathematical Soc. (2012)

[5] V.O. Bugaenko, *Groups of automorphisms of unimodular hyperbolic quadratic forms over the ring* $\mathbb{Z}\left[\frac{\sqrt{5}+1}{2}\right]$, Moscow Univ. Math. Bull. vol. 39 (1984), 6-14

[6] V.O. Bugaenko, *On reflective unimodular hyperbolic quadratic forms*, Selecta Math. Soviet vol. 9 **3** (1990), 263-271

[7] H.S.M. Coxeter, *Discrete groups generated by reflections*, Annals of Mathematics (1934), 588–621

[8] D.D. Long, C. Maclachlan, and A. Reid, *Arithmetic Fuchsian groups of genus zero*, Pure and Applied Mathematics Quarterly vol. 2 **2** (2006), 569-599
[9] A. Mark, *The classification of rank 3 reflective hyperbolic lattices over* $\mathbb{Z}[\sqrt{2}]$, Math. Proc. Camb. Phil. Soc. vol. 164 **2** (2018), 221–257
[10] V.V. Nikulin, *On Arithmetic Groups Generated by Reflections in Lobachevsky Spaces*, Mathematics of the USSR-Izvestiya Vol. 16 **3** (1981), 573
[11] V.V. Nikulin, *On the classification of hyperbolic root systems of rank three*, Proceedings of the Steklov Institute of Mathematics. Maik Nauka/Interperiodica Pub. (2000)
[12] V.V. Nikulin, *Finiteness of the number of arithmetic groups generated by reflections in Lobachevsky spaces*, Izvestiya: Mathematics Vol. 71 **1** (1981), 53
[13] The PARI Group, *PARI/GP version* 2.5.4, Bordeaux (2014), available from http://pari.math.u-bordeaux.fr/
[14] È. B. Vinberg, *On groups of unit elements of certain quadratic forms*, Sbornik: Mathematics **16** (1972), 17–35

Problems of computation and enumeration with quadratic forms

Anna Haensch
(joint work with Mikhail Belolipetsky, Wai Kiu Chan, Benjamin Linowitz, Jeffrey S. Meyer)

The reflections of a polyhedron in n-dimensional hyperbolic space, $\mathbb{H}^n$, form a hyperbolic reflection group. Such groups are intimately connected to the theory of quadratic forms, in that certain classes of hyperbolic reflection groups are defined by the automorphism groups of quadratic forms satisfying explicit arithmetic conditions. Consequently, it is possible to realize enumerations of certain hyperbolic reflection groups by leveraging computational results in the arithmetic theory of quadratic forms. Specifically, we consider the case of congruence arithmetic hyperbolic reflection groups and their correspondence to the integral automorphisms of quadratic forms satisfying certain arithmetic conditions.

Let k be a totally real number field with ring of integers $\mathfrak{o}_k$, and let f be a quadratic form on k which has signature $(n, 1)$ and for which f^σ is positive definite for every non-identity embedding $\sigma : k \hookrightarrow \mathbb{R}$. Then the group $\Gamma = O(f, \mathfrak{o}_k)$, the group of integral automorphism of f, form a discrete subgroup of $\mathrm{Isom}(\mathbb{H}^n) \cong PO(n, 1)$. A subgroup of $\mathrm{Isom}(\mathbb{H}^n)$ which arises in such a way (or one which is commensurable to a group arising in such a way) is called arithmetic. It is a well-known result of Vinberg [8] that there are no arithmetic groups in dimension $n \geq 30$. Moreover, results of Agol, Belolipstsky, Storm and White [1] bound the volume of the orbifold $\mathbb{H}^n/\Gamma$ associated to any arithmetic reflection group, Γ, thereby determining that there are only finitely many arithmetic reflection groups which are maximal. If we further impose a condition of congruency, that is, we require that the group Γ contain a principal congruence subgroup, then we obtain improved bounds on the possible field of definition via spectral theory. In particular, it is know explicitly from work of Vigneras [7] and Burger – Sarnak [4] (as well as improved conjectural bounds by Ramanujan and Selberg) that the minimal non-zero eigenvalue of the Laplacian on $\mathbb{H}^n/\Gamma$ is bounded from below by a term that grown linearly with n. Subsequent work of Belolipetsky [2], Linowitz [6], and Belolipetsky – Linowitz [3] effectively bound the degree and possible discriminants of the field of definition, k, of any arithmetic congruence hyperbolic reflection groups, Γ. The techniques used to limit the fields of definition involves a strict bound on the norm of primes that can divide the discriminant of the defining quadratic forms, as well as explicit Hasse invariant conditions. With these local considerations in place, the candidates for defining quadratic forms can then be produced using an algorithm of Kirschmer [5] which generates and diagonal quadratic form with coefficients from the ring of integers, satisfying the desired determinant and Hasse invariant conditions. In this way we have produced a candidate list for all possible congruence arithmetic hyperbolic reflection groups.

References

[1] I. Agol, M. Belolipetsky, P. Storm, and K. Whyte. Finiteness of arithmetic hyperbolic reflection groups. *Groups Geom. Dyn.*, 2(4):481–498, 2008.
[2] M. Belolipetsky. Finiteness theorems for congruence reflection groups. *Transform. Groups*, 16(4):939–954, 2011.
[3] M. Belolipetsky and B. Linowitz. On fields of definition of arithmetic Kleinian reflection groups II. *Int. Math. Res. Not. IMRN*, (9):2559–2571, 2014.
[4] M. Burger and P. Sarnak. Ramanujan duals. II. Invent. Math., 106(1):1–11, 1991.
[5] M. Kirschmer. Definite quadratic and hermitian forms with small class number. Habilitationsschrift, RWTH Aachen University, 2016.
[6] B. Linowitz. Bounds for arithmetic hyperbolic reflection groups in dimension 2. *Transform. Groups*, 23(3):743–753, 2018.
[7] M.-F. Vignéras. Quelques remarques sur la conjecture $\lambda_1 \geq \frac{1}{4}$. In Seminar on number theory, Paris 1981–82 (Paris, 1981/1982). volume 38 of Progr. Math., pages 321–343. Birkhäuser Boston, Boston, MA, 1983.
[8] È. B. Vinberg. Absence of crystallographic groups of reflections in Lobachevskiĭ spaces of large dimension. *Trudy Moskov. Mat. Obshch.*, 47:68–102, 246, 1984.

Fully Augmented Links, Reflection Groups, and Arithmeticity

Jeffrey S. Meyer
(joint work with Christian Millichap, Rolland Trapp)

Every link $L \subset \mathbb{S}^3$ determines a link complement $M_L := \mathbb{S}^3 \setminus L$. If M admits a metric of constant curvature -1, we say that both M_L and L are hyperbolic, and if such a hyperbolic structure exists, then it is unique (up to isometry) by Mostow–Prasad rigidity. In such cases, the link group, $\Gamma_L := \pi_1(M_L)$, can be realized as a lattice in $\mathrm{PSL}_2(\mathbb{C})$, thereby enabling us to analyze arithmetic properties of the link complement. Given a lattice $\Gamma \subset \mathrm{PSL}_2(\mathbb{C})$, its *invariant trace field* $k\Gamma$ is the field generated by the traces of squares of elements of Γ.

Question 1. What number fields arise as the invariant trace fields of link complements?

Neumann has conjectured [9] that every non-real number field arises as the invariant trace field of some hyperbolic 3-manifold, yet to date, a relatively small collection of such fields have been verified to arise in this way. A link L is said to be *arithmetic* if M_L is arithmetic, or in other words, if the link group Γ_L is commensurable with a Bianchi group $\mathrm{PSL}(\mathcal{O}_d)$, where d is a negative square-free integer and $\mathcal{O}_d$ is the ring of integers in $\mathbb{Q}(\sqrt{d})$.

Question 2. Which link complements are arithmetic?

Alan Reid [11] showed that the only arithmetic knot is the figure-eight. Recently the culmination of the work of Baker, Goerner, and Reid [1] [2] [3] answered a question of Thurston by providing a complete list of all 48 principal congruence arithmetic link complements.

In joint work with Christian Millichap and Rolland Trapp [8], we analyzed these questions for a class of links known as *fully augmented links* (FALs) (see [10]). As first described by Agol and Thurston in the appendix of [6], any fully augmented link complement M_L may be decomposed into two copies of a single right-angled hyperbolic polyhedron P_L, together with gluing instructions. In [4], Chesebro, Deblois, and Wilton give a sufficient condition for the link group Γ_L to be commensurable with the reflection group $\Gamma(P_L)$, generated by reflections through the walls of the polyhedron P_L. Combining these results, we analyze the commensurability invariants of link groups via reflection group techniques.

As an example of our techniques, we compute the arithmetic invariants of fully augmented pretzel links. A *fully augmented pretzel link* with $n \geq 3$ planar circles and n crossing circles is a fully augmented link FAPL_n with $2n$ components arranged as in Figure 1. In Thurston's notes [12], he refers to these links as D_{2n}.

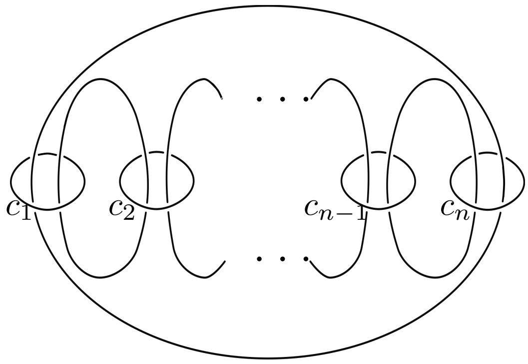

FIGURE 1. FAPL_n, with crossing circles c_i, $i = 1, \dots, n$.

We show that for each $n \geq 3$, FAPL_n satisfies the symmetry conditions of CDW [4] and hence its link group Γ_n is commensurable with the reflection group $\Gamma(P_n)$. We then use the geometry of the circle packing associated to the polyhedra P_n, using, for example, the Pedoe product, to explicitly compute its cusp shape and its Gram matrix. Using results of Flint [5] we compute the invariant trace field of Γ_n and then use Vinberg's arithmeticity criterion [7, 10.4.5] to check arithmeticity.

Theorem 1. *For $n \geq 3$, let Γ_n denote the n^{th} fully augmented pretzel link group. Then $k\Gamma_n = \mathbb{Q}(\cos(\pi/n)i)$. Furthermore, Γ_n is arithmetic if and only if $n = 3, 4$.*

References

[1] M. D. Baker, A. W. Reid *Principal Congruence Link Complements*, Ann. Fac. Sci. Toulouse Math. (6) 23 (2014), no. 5, 1063–1092.

[2] M. D. Baker, A. W. Reid *Principal congruence links: class number greater than 1*, Exp. Math. 27 (2018), no. 4, 395–413.

[3] M. D. Baker, M. Goerner, A. W. Reid *All Principal Congruence Link Groups*, arXiv:1811.00679.

[4] E. Chesebro, J. DeBlois, H. Wilton, *Some virtually special hyperbolic 3-manifold groups*, Comment. Math. Helv. 87 (2012), no. 3, 727–787.

[5] R. Flint, *Intercusp Geodesics and Cusp Shapes of Fully Augmented Links*, ProQuest LLC, Ann Arbor, MI, 2017, Thesis (Ph.D.) - City University of New York.

[6] M. Lackenby, *The volume of hyperbolic alternating link complements*, Proc. London Math. Soc. (3) 88 (2004), no. 1, 204–224, With an appendix by Ian Agol and Dylan Thurston.

[7] C. Maclachlan and A. W. Reid, *The Arithmetic of Hyperbolic 3–Manifolds*, Graduate Texts in Mathematics, vol. 219, Springer-Verlag, New York, (2003).

[8] J. S. Meyer, C. Millichap, R. Trapp *Arithmeticity and Hidden Symmetries of Fully Augmented Pretzel Link Complements*, arXiv:1811.00679.

[9] W. Neumann, *Realizing arithmetic invariants of hyperbolic 3-manifolds*, Interactions between hyperbolic geometry, quantum topology and number theory, 233–246, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.

[10] J. Purcell, *An introduction to fully augmented links*, Interactions between hyperbolic geometry, quantum topology and number theory, 205–220, Contemp. Math., 541, Amer. Math. Soc., Providence, RI, 2011.

[11] A. W. Reid *Arithmeticity of knot complements*, J. London Math. Soc. (2) 43 (1991), no. 1, 171–184.

[12] W. Thurston, *The Geometry and Topology of 3-Manifolds*, Lecture Notes, Princeton University Math. Dept. (1978).

Fields of definition of arithmetic hyperbolic reflection groups

Benjamin Linowitz

A *hyperbolic reflection group* is a discrete subgroup of the isometry group of hyperbolic n-space which is generated by reflections in the faces of a hyperbolic polyhedron. In this talk we will be concerned with hyperbolic reflection groups which are *arithmetic*. It was proven by Agol, Belolipetsky, Storm and Whyte [1], and independently by Nikulin [7], that there are only finitely many conjugacy classes of arithmetic maximal hyperbolic reflection groups, where a hyperbolic reflection group is said to be *maximal* if it is not properly contained in another hyperbolic reflection group. This finiteness result makes the enumeration of the (conjugacy classes of) maximal hyperbolic reflection groups feasible, at least in theory. A first step towards such a classification is to determine the potential *fields of definition* of such groups.

In [7] Nikulin proved that the maximum degree of the field of definition of an arithmetic hyperbolic reflection does not exceed the maximum degree of such fields in dimensions $n = 2, 3$, and a certain transition constant which was proven to be bounded above by 25 in [8]. This result makes the problem of obtaining bounds for the degree of the fields of definition in dimensions $n = 2, 3$ especially important.

To that end, we present the following results (proven in [4] and [3], the latter representing joint work with M. Belolipetsky):

Theorem 1. *The field of definition of the quaternion algebra associated to an arithmetic Fuchsian group of genus* 0 *is at most* 7*. In particular the degree of the field of definition of an arithmetic hyperbolic reflection group in* $n = 2$ *is at most* 7*.*

Theorem 2. *The degree of the (totally real) field of definition of an arithmetic hyperbolic reflection group in dimension* 3 *is at most* 9*.*

We note that the first theorem proves a conjecture of Long, Maclachlan and Reid [5] and improves upon a previously known bound of Maclachlan [6], while the second theorem improves upon previous work of Belolipetsky [2].

Note that in [5], Long, Maclachlan and Reid actually construct arithmetic Fuchsian groups of genus 0 whose field of definitions have degree 7. This construction motivates the following problem:

Problem 1. Construct an arithmetic hyperbolic reflection group in dimension $n = 2$ whose field of definition has degree 7.

References

[1] I. Agol, M. Belolipetsky, P. Storm, and K. Whyte. *Finiteness of arithmetic hyperbolic reflection groups*, Groups Geom. Dyn. **2** (2008), 481–498.

[2] M. Belolipetsky. *On fields of definition of arithmetic Kleinian reflection groups,* Proc. Amer. Math. Soc. **137** (2009), 1035–1038.

[3] M. Belolipetsky and B. Linowitz. *On fields of definition of arithmetic Kleinian reflection groups II,* Int. Math. Res. Not. IMRN 2014, **9**, 2559–2571.

[4] B. Linowitz. Bounds for arithmetic hyperbolic reflection groups in dimension 2. *Transform. Groups*, 23(3):743–753, 2018.

[5] D. Long, C. Maclachlan, A. Reid, *Arithmetic Fuchsian groups of genus zero*, Pure Appl. Math. Q., **2** (2006), no. 2, 569–599.

[6] C. Maclachlan, *Bounds for discrete hyperbolic arithmetic reflection groups in dimension 2*, Bull. Lond. Math. Soc., **43** (2011), 111–123.

[7] V. V. Nikulin. *Finiteness of the number of arithmetic groups generated by reflections in Lobachevskiĭ spaces,* Izv. Ross. Akad. Nauk Ser. Mat. **71** (2007), no. 1, 55–60

[8] V. V. Nikulin. *The transition constant for arithmetic hyperbolic reflection groups,* Izv. Ross. Akad. Nauk Ser. Mat. **75** (2011), no. 5, 103–138.

Hyperbolic manifolds without a spin structure

Leone Slavich

(joint work with Bruno Martelli, Stefano Riolo)

A smooth, orientable, compact n-manifold M is said to be spinnable if the second Stiefel-Whitney class of its tangent bundle vanishes, i. e.

$$\omega_2(M) = 0 \in H^2(M, \mathbb{Z}/2\mathbb{Z}).$$

A more topological definition of spinnable manifold is as follows: suppose that M is endowed with a cell complex structure. Then M is spinnable if there is a trivialization of its tangent bundle on the 2-skeleton of M (this does not depend on the choice of the cell decomposition). This property is not very interesting in the low-dimensional cases, since all orientable surfaces and 3-manifolds are spinnable. Things become more intersting from dimension 4 upwards, where there are both spin and non-spin examples.

One might wonder what happens if we restrict our attention to finite volume hyperbolic manifolds. By work of Deligne and Sullivan [1] the following holds:

Theorem 1. *Let M be a finite volume hyperbolic manifold. Then M is virtually spinnable.*

In other words, every hyperbolic manifold M admits a finite-index cover which is spinnable. This theorem provides an abundance of spinnable hyperbolic manifolds, and it reasonable to ask if the virtual hypothesis is really needed. Indeed, all previously known examples of closed, orientable hyperbolic 4-manifolds for which the property of being or not spinnable could be checked turned out to be spinnable. We prove the following:

Theorem 2. *There are compact orientable hyperbolic manifolds that do not admit any spin structure, in all dimensions $n \geq 4$.*

Notice that a necessary condition for a 4-manifold to admit a spin structure is that its intersection form is even, which means that for all $S \in H_2(M, \mathbb{Z})$, the algebraic self-intersection $S \cdot S$ is even. We first prove the following:

Theorem 3. *There is a compact oriented arithmetic hyperbolic 4-manifold M that contains a π_1-injective embedded surface S with genus 3 and $S \cdot S = 1$.*

The 4-manifold M above is therefore non-spinnable. We then build a sequence of non-spinnable hyperbolic n-manifolds M^n, by repeatedly embedding each M^n in M^{n+1} in a totally geodesic way as in [2]. The normal bundle of M^n in M^{n+1} will then be trivial, and by standard properties of the Stiefel-Whitney classes, if M^n is not spinnable also M^{n+1} will not be spinnable.

Notice that if S is a totally geodesic immersed surface, its self-intersection $S \cdot S$ is necessarily even. Therefore Theorem 3 implies the following:

Corollary 4. *There exists a hyperbolic 4-manifold M such that $H_2(M, \mathbb{Z})$ is not generated by totally geodesic immersed hypersurfaces.*

For the construction of the 4-manifold M of Theorem 3, we refer the reader to [3]. Here we simply remark that the 4-manifold M is built by glueing together a finite number of copies of the *hyperbolic right angled* 120-*cell*, which is a regular 4-dimensional hyperbolic polytope. This construction guarantees the arithmeticity of M. The surface S with self-intersection equal to 1 lies in the 2-skeleton of M. It is a pleated surface, tessellated by right-angled hyperbolic pentagons.

Finally, we remark that the surface S is π_1-injective in M. The quotient $\mathbb{H}^4/\pi_1(S)$ is geometrically finite, and its topology is that of a rank 2 real vector bundle over S with Euler number one. The existence of complete hyperbolic structures on non-trivial bundles over surfaces was first discovered by Gromov – Lawson – Thurston [4] in 1988. The following consequence seems also new:

Corollary 5. *There are non-trivial bundles over surfaces that cover some compact hyperbolic 4-manifolds.*

References

[1] P. Deligne, D. Sullivan, *Fibrés vectoriels complexes à groupe structural discret*, C. R. Acad. Sci. Ser., **281** (1975), 1081–1083.

[2] A. Kolpakov, A. Reid, L. Slavich, *Embedding arithmetic hyperbolic manifolds*, Math. Res. Lett. **25** (2018), 1305–1328.

[3] B. Martelli, S. Riolo, L. Slavich, *Compact hyperbolic manifolds without spin structures*, preprint `arXiv:1904.12720`.

[4] M. Gromov, B. Lawson, W. Thurston, *Hyperbolic 4-manifolds and conformally flat 3-manifolds*, Publ. Math., Inst. Hautes Étud. Sci. **68** (1988), 27–45.

Selected Open Problems

Vincent Emery, Martin Deraux, Matthew Stover, Edoardo Dotti, Mikhail Belolipetsky, Pavel Tumarkin, Anna Felikson, Anne Thomas

Several problems were discussed during the three problem sessions; some of them could find answers (at least partially) and stimulated further discussion. We list here a selection of some problems that were proposed and remained open.

Problem 1 (Deraux)**.** Are there infinitely many commensurability classes of non-arithmetic lattices in $\mathrm{PU}(2,1)$?

$\rightarrow$ It appears that a positive answer would require a completely new approach for producing nonarithmetic lattices. This same comment holds for the next question:

Problem 2 (Stover)**.** Does there exist a hyperbolic n-manifold ($n \geq 4$) containing no arithmetic totally geodesic m-submanifold for each $1 < m < n$?

Problem 3 (Dotti)**.** Let Γ be a (non-arithmetic) reflection lattice in $\mathrm{PO}(n,1)$ with $n \geq 4$. Is the trace field of Γ necessarily totally real?

$\rightarrow$ Note that the problem is more generally open for Γ *any* lattice in $\mathrm{PO}(n,1)$ ($n \geq 4$).

Problem 4 (Belolipetsky)**.** Find examples of non-isomorphic hyperbolic Coxeter groups Γ_1 and Γ_2 such that their quotients $\mathbf{H}^n_{\mathbb{R}}/\Gamma_i$ ($i = 1, 2$) are isospectral.

$\rightarrow$ The basic idea would be to apply Sunada's method (so that in particular the Coxeter groups would be commensurable).

Problem 5 (Tumarkin)**.** Let (W, S) a Coxeter system, and $\Gamma \subset W$ a reflection subgroup of finite index; then $\mathrm{rank}\Gamma \geq \mathrm{rank}W$. When does equality holds?

$\rightarrow$ Tumarkin and Thomas point out that examples with equality (other than "doubling") notably exist for affine and hyperbolic groups.

Problem 6 (Belolipetsky)**.** Visualize the Coxeter diagram of Borcherds' Coxeter polytope of dimension $n = 21$.

Problem 7 (Felikson)**.** Give a geometric explanation for the classification of quivers of finite mutation type.

Problem 8 (Thomas)**.** Develop new (abstract) commensurability invariants for Coxeter groups.

Problem 9 (Thomas)**.** Compute the abstract commensurator for (certain) right-angled Coxeter groups.

→ The abstract commensurator is an abstract commensurability invariant, which has been studied for some important classes of groups including right-angled Artin groups, but very little seems to be known for right-angled Coxeter groups.

Wishlist for Sage functions

Vincent Emery

During the week emerged the idea (by Anna Haensch and Jeff Meyer) to organize a workshop Sage Days with the goal of implementing functions related to hyperbolic geometry. This idea was welcomed by the participants, and four of them have accepted to be responsible for pushing forward its realization. A list of possible experts in Sage with competence in hyperbolic geometry was prepared, and possible places for the workshop were discussed. The discussion also provided a wishlist for Sage functions. Here is a sample, divided in three categories:

Functions on Coxeter groups:

- Vinberg's algorithm
- Geometric realization
- Volume computation (ad-hoc for Coxeter groups)
- Coxeter subgroups of Coxeter groups

Functions on lattices:

- Finding torsion-free subgroups
- Construction of fundamental domains (e.g. Dirichlet domains)
- Volume computation (integration)

Functions of arithmetic nature:

- Manipulation of Hermitian forms
- Checking if an arithmetic subgroup is congruence
- Volume computation (Prasad's formula)
- Circle packing (e.g., draw packings)
- Continued fractions
- Checking pseudo-arithmeticity

Reporter: Ruben Bär

Participants

Ruben B. Bär
Mathematisches Institut
Universität Bern
Sidlerstrasse 5
3012 Bern
SWITZERLAND

Prof. Dr. Mikhail Belolipetsky
IMPA
Estrada Dona Castorina 110
Rio de Janeiro 22460-320
BRAZIL

Prof. Dr. Martin Deraux
Institut Fourier
Université Grenoble Alpes
100, Rue des Maths
38610 Gières
FRANCE

Dr. Edoardo Dotti
Département de Mathématiques
Université de Fribourg
Perolles
Chemin du Musee 23
1700 Fribourg
SWITZERLAND

Prof. Vincent Emery
Mathematisches Institut
Universität Bern
Sidlerstrasse 5
3012 Bern
SWITZERLAND

Dr. Brent Everitt
Department of Mathematics
University of York
Heslington, York YO10 5DD
UNITED KINGDOM

Prof. Dr. Anna Felikson
Department of Mathematical Sciences
Durham University
Science Laboratories
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Anna Haensch
Department of Mathematics and
Computer Science
Mc Anulty College
600 Forbes Avenue
Pittsburgh, PA 15282
UNITED STATES

Prof. Dr. Ruth Kellerhals
Département de Mathématiques
Université de Fribourg
Perolles
Chemin du Musee 23
1700 Fribourg
SWITZERLAND

Prof. Dr. Alexander Kolpakov
Institut de Mathématiques
Université de Neuchatel
Unimail
Rue Emile Argand 11
2000 Neuchâtel
SWITZERLAND

Prof. Dr. Yohei Komori
Department of Mathematics
School of Education
Waseda University
Shinjuku-ku
Tokyo 169-8050
JAPAN

Prof. Dr. Benjamin Linowitz
Mathematics Department
Oberlin College
Oberlin OH 44074
UNITED STATES

Dr. Alice Mark
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. Jeffrey S. Meyer
Department of Mathematics
California State University, San Bernardino
5500 University Parkway
San Bernardino CA 92407
UNITED STATES

Dr. Leone Slavich
Dipartimento di Matematica "L.Tonelli"
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Prof. Matthew Stover
Department of Mathematics
Temple University
TU 038-16
1801 N. Broad Street
Philadelphia, PA 19122
UNITED STATES

Dr. Anne Thomas
School of Mathematics and Statistics
The University of Sydney
Sydney NSW 2006
AUSTRALIA

Prof. Dr. Pavel Tumarkin
Department of Mathematical Sciences
Durham University
Science Laboratories
South Road
Durham DH1 3LE
UNITED KINGDOM