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Geometry and Physics of Higgs Bundles

Organized by
Lara B. Anderson, Blacksburg
Tamás Hausel, Klosterneuburg
Rafe Mazzeo, Stanford
Laura P. Schaposnik, Chicago

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Abstract. This workshop focused on interactions between the various perspectives on the moduli space of Higgs bundles over a Riemann surface. This subject draws on algebraic geometry, geometric topology, geometric analysis and mathematical physics, and the goal was to promote interactions between these various branches of the subject. The main current directions of research were well represented by the participants, and the talks included many from both senior and junior participants.

Mathematics Subject Classification (2010): 14xx, 22xx, 51xx, 53xx, 81xx.

Introduction by the Organizers

The workshop *Geometry and Physics of Higgs Bundles*, organized by Lara Anderson (Virginia Tech), Tamas Hausel (IST, Austria), Rafe Mazzeo (Stanford) and Laura Schaposnik (University of Illinois at Chicago) was attended by 46 participants, with broad representation from Europe, Asia, North and South America and India. Quite a few of these participants were either young postdocs or graduate students, and 16 of the 46 were women. Notably, the two founders of the subject, Nigel Hitchin and Carlos Simpson, both attended. The topic of Higgs bundles is distinctly interdisciplinary, drawing on algebraic geometry, geometric topology, geometric analysis, mathematical physics and beyond. The goal of this meeting was to draw together researchers working on various aspects of this general area. There were 18 talks in all, with reports from senior mathematician, leaders in the field, to promising young researchers.

One of the beguiling features of the subject is that the main object of study, the moduli space of Higgs bundles, can be interpreted both in this initial way, in the language of algebraic geometry, but can also be regarded as the space of solutions of the Hitchin equations, using the language of gauge theory, and in addition as a character variety, i.e. a class of representations of the fundamental group of a surface into a complex Lie group, modulo conjugation. These aspects appeared singly or together in all of the lectures. Several talks addressed the connection between the mathematical aspects and physical interpretations of Higgs bundles directly. This connection of course has stimulated many new and appealing problems in the subject. Other talks addressed more classical themes in the subject, including the algebro-geometric structures on various related moduli spaces, investigations of the topology of these moduli spaces, as well as some related moduli problems. Another recent theme in the subject is the study of large-scale structure of the Higgs bundle moduli space, centering around a string-theoretic conjecture by Gaiotto-Moore-Neitzke; this was well-represented by several talks. The investigation of interesting submanifolds (branes) in these moduli spaces appeared in several lectures.

There were several dramatic results announced during the meeting, including an unexpected new stratification of the moduli space by branes and a near-resolution of the Gaiotto-Moore-Neitzke conjecture. The workshop was very successful in promoting new contact between researchers as well as providing an outstanding forum for ongoing collaborations. The talks provided high-level expositions of their topics, but by and large also excellent introductions to each set of problems. Long afternoon breaks and evenings were spent in small group working sessions. The organizers were quite satisfied that the goals of this conference were achieved. Although there have been other meetings involving similar themes in the past few years, this workshop had a particularly successful distribution of people from across various areas of the subject, which led to new and useful interactions. All participants agreed that the MFO is a singularly successful venue for meetings.

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Workshop: Geometry and Physics of Higgs Bundles

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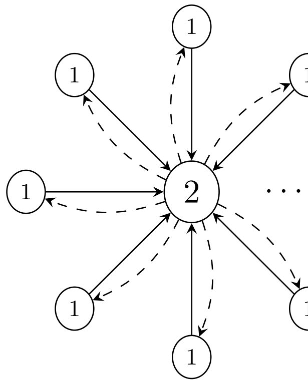
Abstracts

Hyperpolygons and Higgs bundles

Steven Rayan
(joint work with Laura Schaposnik, Hartmut Weiß)

The moduli space of semistable Higgs bundles on a fixed algebraic curve is an infinite-dimensional hyperkähler quotient that has been investigated from various points of view. These include: (1) its topology, (2) its integrable system, (3) its geometry (captured by the natural hyperkähler metric), and (4) its mirror symmetry. Nakajima quiver varieties are finite-dimensional hyperkähler quotients that share much in common with Higgs bundle moduli spaces. Hyperpolygon spaces, in particular, come closest to bridging the gap between Nakajima quiver varieties and Higgs bundles. Themes 1 and 2 above are studied for hyperpolygons in [2]. We report on 3 and 4 in joint work with H. Weiß and L. Schaposnik, respectively.

For us, a *hyperpolygon* is a representation of the *star-shaped* quiver:



The quiver has $n+1 \geq 4$ vertices in total. A representation of a solid (ingoing) arrow is a linear map $x_i \in \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)$. Once x_i is chosen, a representation of a dashed (outgoing) arrow involving the same nodes is a linear map

$$y_i \in T^*_{x_i}\mathrm{Hom}(\mathbb{C}, \mathbb{C}^2) \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}^2)^* \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}).$$

We denote a representation by $[x|y]$, where x is an n-tuple of column vectors x_i in $\mathbb{C}^2$ and y is an n-tuple of row vectors y_i in $(\mathbb{C}^2)^*$. We now choose a sufficiently generic vector $\alpha \in \mathbb{R}^n$ with positive entries and define the *hyperpolygon equations*:

$$\begin{aligned} \sum_{i=1}^{n}(x_i x_i^* - y_i^* y_i)_0 &= 0, \quad |x_i|^2 - |y_i|^2 = \alpha_i, \text{ for each } i \in \{1, \ldots, n\}, \\ \sum_{i=1}^{n}(x_i y_i)_0 &= 0, \quad y_i x_i = 0, \text{ for each } i \in \{1, \ldots, n\}, \end{aligned}$$

where the subscript 0 is an instruction to remove the trace, and norms $|x_i|$ and $|y_i|$ are the standard Euclidean ones. The left-hand sides of these equations can be interpreted as moment maps. The first $n+1$ equations are (rescaled) moment maps for the action of $G = (SU(2) \times U(1)^n)/\pm 1$ on the representation data (with the action encoded by the quiver) and the latter $n+1$ equations are moment maps for the corresponding $G^{\mathbb{C}}$-action. We define *hyperpolygon space* $\mathcal{X}_n(\alpha)$ to be the solution set of the hyperpolygon equations modulo G. The name "hyperpolygon" is motivated by the fact that, when we restrict to the level set $y = 0$, we obtain a space parametrizing equivalence classes of polygons in $\mathbb{R}^3$.

The quotient $\mathcal{X}_n(\alpha)$ is a smooth quasiprojective variety of dimension $2(n-3)$ and its hyperkähler metric is complete whenever α is sufficiently generic [12, 8, 4, 2]. As with the moduli space of Higgs bundles, the space $\mathcal{X}_n(\alpha)$ comes equipped with a Hamiltonian $U(1)$-action that acts through the rotation $[x|y] \mapsto [x|\exp(i\theta)y]$ [8, 2]. Regarding cohomology, in [2] we show that a class of Nakajima quiver varieties that includes $\mathcal{X}_n(\alpha)$ has the hyperkähler Kirwan surjectivity property.[1]

Now, choose an affine coordinate z on the complex projective line $\mathbb{P}^1$ and a divisor $D = \sum_{i=1}^n z_i$ of pairwise distinct points $z_i \neq \infty$. The map $\Phi(z) = \sum_{i=1}^n \frac{(x_i y_i)_0}{z - z_i} dz$ defines from $[x|y]$ a parabolic Higgs field for the trivial bundle $E = \mathbb{P}^1 \times \mathbb{C}^2$. The map respects stability (for sufficiently generic α) and notions of equivalence, and so we obtain an embedding of moduli spaces [2]. The target moduli space is that of β-semistable strongly parabolic Higgs bundles of rank 2 and degree 0 on $\mathbb{P}^1$ punctured along D, for some choice of parabolic weights β at the punctures (cf. [4]). The embedding map is not surjective, as only parabolic Higgs bundles with the trivial underlying bundle are obtained. The map is also not hyperkähler, as the Nakajima hyperkähler metric on $\mathcal{X}_n(\alpha)$ is complete while the Higgs bundle one pulled back to $\mathcal{X}_n(\alpha)$ is not.

Geometry. A sequence of hyperpolygons $[x^k|y^k]$ that escapes to infinity under the L^2-norm $\mu([x|y]) = \sum_{i=1}^n |y_i|^2$ will satisfy a rescaled version of the hyperpolygon equations with each α_i replaced by $\alpha_i/\sqrt{\mu([x^k|y^k])}$. The limit will thus satisfy the equations with $\alpha_i = 0$. We call these objects *limiting hyperpolygons*, which are analogous to the limiting Higgs bundles of [10]. The limiting hyperpolygons are parametrized, up to G-isomorphism, by the singular hyperkähler variety $\mathcal{X}_n(0)$. This can be regarded as the "tangent cone at infinity" to $\mathcal{X}_n(\alpha)$ with α generic. For $n = 4$, i.e. the affine D_4 quiver, the tangent cone $\mathcal{X}_4(0)$ is classically known to be $\mathbb{C}^2/\Gamma$, where $\Gamma = Q_8$ is a quaternion subgroup of order 8 in $SU(2)$. This fits neatly into the classification of ALE gravitational instantons, which can be regarded as a geometrization of the McKay correspondence. Here, a moduli space of gravitational instantons is determined by their geometry at infinity, given by the tangent cone. This is essentially the result of [9]. The geometry at infinity is a Du Val / Kleinian singularity produced by the action on $\mathbb{C}^2$ of a finite group $\Gamma < SU(2)$. This group in turn determines an (affine) ADE Dynkin type, via McKay. Taking us back from the Dynkin quiver to a gravitational instanton in the original moduli space is the Nakajima quiver variety construction [12].

[1] This has been extended recently to all Nakajima quiver varieties in [11].

For $n = 5$, we are no longer in a Dynkin type and the quotient is now an 8-manifold. However, we do know there is a stratification of $\mathcal{X}_5(0)$ by "edge collapse", as pairs (x_i, y_i) are allowed to tend to 0 now. Hence, there are 5 lower-dimensional strata corresponding to embeddings of $\mathcal{X}_4(0)$. Using this information, can we realize $\mathcal{X}_5(0)$ as $\mathbb{C}^4/\Gamma$ for some finite subgroup $\Gamma < SL(4, \mathbb{C})$? How about for general n? A positive answer to these questions will establish the decay rate of Nakajima's hyperkähler metric to the Euclidean metric as being quasi-ALE, in the sense of [6]. This is joint work in progress with H. Weiß.

Mirror Symmetry. Because $\mathcal{X}_n(\alpha)$ is a smooth, noncompact Calabi-Yau manifold for generic α, and because the Calabi-Yau structure arises from a hyperkähler structure, we can ask about the existence of different types of triple branes, as motivated by [7]. For example, a (B, A, A) brane is one that is a complex submanifold with regards to the I complex structure and Lagrangian with regards to the ω_J and ω_K symplectic forms. We note that constructions of triple branes in Nakajima quiver varieties appear in [5, 3]. As expected, they arise generally from holomorphic and anti-holomorphic involutions on the $[x|y]$ data that descend to the quotient, consistent with the picture for Higgs bundles in, for instance, [1]. For $\mathcal{X}_n(\alpha)$, we aim to characterize these branes explicitly as subvarieties containing hyperpolygons of special type (e.g. polygons with no y data). At the same time, we want to understand how mirror symmetry interacts with various kinds of hyperpolygon branes. This is joint work in progress with L. Schaposnik.

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The asymptotic geometry of the Hitchin moduli space

Laura Fredrickson

Hitchin's equations [6] are a system of gauge-theoretic equations on a Riemann surface. The Hitchin moduli space $\mathcal{M}$ – the spaces of solutions of Hitchin's equations – is a noncompact hyperkähler space that is also an algebraic completely integrable system. Near the ends of the Hitchin moduli space, we prove that the hyperkähler metric g_{L^2} on the Hitchin moduli space can be approximated up to exponentially-decaying errors by a simpler hyperkähler metric known as the semiflat metric $g_{\rm sf}$ which come from the integrable system structure. This addresses Gaiotto-Moore-Neitzke's [5, 4, 9] conjectural description of the asymptotic geometry of the Hitchin moduli space coming from $\mathcal{N} = 2$ $4d$ supersymmetric field theory.

The $SU(n)$-Hitchin moduli space $\mathcal{M}$ is parameterized by the underlying moduli space of $SL(n,\mathbb{C})$-Higgs bundles. Higgs bundles are holomorphic objects consisting of a holomorphic vector bundle $\mathcal{E} \to C$ of rank n and a holomorphic Higgs field $\varphi \in \Omega^{1,0}(C, \mathrm{End}\mathcal{E})$. Given a Higgs bundle $(\mathcal{E},\varphi)$, Hitchin's equations are a nonlinear coupled system of partial differential equations for a hermitian metric h on $\mathcal{E}$. The hyperkähler metric g_{L^2} at $(\mathcal{E},\varphi,h) \in \mathcal{M}$ is defined in terms of the variation of the Higgs bundle $(\mathcal{E},\varphi)$ but also the variation of the hermitian metric h. The main difficulty in proving results about g_{L^2} is describing the hermitian metric h for a Higgs bundle $(\mathcal{E},\varphi)$ near the ends of the Hitchin moduli space and controlling the variation of h associated to a given variation of the Higgs field $(\mathcal{E},\varphi)$. We prove the following theorem in [2].

Theorem. Fix a generic $SL(n,\mathbb{C})$-Higgs bundle $(\overline{\partial}_E,\varphi)$ in $\mathcal{M}$, and a Higgs bundle variation $\dot{\psi} = (\dot{\eta},\dot{\varphi})$. Consider the deformation $\dot{\psi}_t = (\dot{\eta}, t\dot{\varphi}) \in T_{(\overline{\partial}_E,t\varphi)}\mathcal{M}$ over the ray $(\overline{\partial}_E, t\varphi, h_t)$. As $t \to \infty$, the difference between the hyperkähler L^2-metric g_{L^2} on $\mathcal{M}$ and the semiflat metric $g_{\rm sf}$ is exponentially-decaying. In particular, there is some constant $\gamma > 0$, such that

$$g_{L^2}(\dot{\psi}_t,\dot{\psi}_t) = g_{\rm sf}(\dot{\psi}_t,\dot{\psi}_t) + O(\mathrm{e}^{-\gamma t}). \tag{1}$$

A number of previous papers made progress towards proving Gaiotto-Moore-Neitzke's conjecture when $n = 2$. Mazzeo-Swoboda-Weiss-Witt [8] showed that along a generic ray $g_{L^2} - g_{\rm sf}$ decays *polynomially* in t. Dumas-Neitzke [1] showed that—restricted to the image of Hitchin section— $g_{L^2} - g_{\rm sf}$ decays *exponentially* in t. We discuss the key ideas behind the proofs of these theorems.

Idea 1. One crucial observation is the analogy

$$\begin{array}{rcl} \text{Hitchin's hyperkähler } L^2\text{-metric } g_{L^2} & : & \text{the harmonic metric } h_t \\ & :: & \\ \text{the semiflat metric } g_{\rm sf} & : & \text{the limiting metric } h_\infty. \end{array}$$

The Hitchin moduli space consists of triples $[(\overline{\partial}_E,\varphi,h)]$ where h is the harmonic metric. Define the moduli space of limiting configurations to be the space of triples $[(\overline{\partial}_E, \varphi, h_\infty)]$ where we replace the harmonic metric with the limiting metric $h_\infty = \lim_{t\to\infty} h_t$. Just as Hitchin's hyperkähler L^2-metric $g_{\mathcal{M}}$ is the L^2-metric on $\mathcal{M}$, the semiflat metric g_{sf} is the natural hyperkähler L^2-metric on the moduli space of limiting configurations $\mathcal{M}'_\infty$, for deformations in Coulomb gauge.

Idea 2. Mazzeo-Swoboda-Weiss-Witt's result for the $SU(2)$-Hitchin moduli space in [8] relies on their description of the harmonic metrics near the ends of the Hitchin moduli space in [7]. Mazzeo-Swoboda-Weiss-Witt build a family of approximate solutions of Hitchin's equations $(\overline{\partial}_E, t\varphi, h_t^{\mathrm{app}})$ that are exponentially close to the actual solutions of Hitchin's equations $(\overline{\partial}_E, t\varphi, h_t)$.The approximate metric h_t^{app} is constructed by desingularizing the singular metric h_∞ by gluing in model solutions on disks around the zeros of $q_2 = -\det\varphi$. Analogous results for the the $SU(n)$-Hitchin moduli space appear in [3]. Fix a polystable $SL(n,\mathbb{C})$-Higgs bundle $(\mathcal{E},\varphi)$ in the regular locus $\mathcal{M}'$. In both cases, the authors prove that along the ray $(\mathcal{E}, t\varphi, h_t)$ the associated family of harmonic metrics h_t satisfies

$$h_t(w_1, w_2) = h_t^{\mathrm{app}}(\mathrm{e}^{-\kappa_t} w_1, \mathrm{e}^{-\kappa_t} w_2)$$

for κ_t satisfying $\|\kappa_t\|_{H^2(i\mathfrak{su}(E))} \le C\mathrm{e}^{-\delta t}$. I.e. the actual harmonic metric h_t is very close to the approximate metric h_t^{app}.

Thus, define the "approximate Hitchin moduli space" $\mathcal{M}'_{\mathrm{app}}$ to be the moduli space of triples $[(\overline{\partial}_E, t\varphi, h_t^{\mathrm{app}})]$. It too has a natural (non-hyperkähler) L^2-metric g_{app}. Decompose the difference $g_{\mathcal{M}} - g_{\mathrm{sf}}$ into two pieces

$$g_{\mathcal{M}} - g_{\mathrm{sf}} = (g_{\mathcal{M}} - g_{\mathrm{app}}) + (g_{\mathrm{app}} - g_{\mathrm{sf}})\,. \tag{2}$$

Because h_t and h_t^{app} are exponentially close, $g_{\mathcal{M}} - g_{\mathrm{app}}$ is exponentially-decaying.

Idea 3. The more problematic term is $g_{\mathrm{app}} - g_{\mathrm{sf}}$. Dumas-Neitzke have a clever way of proving that $g_{\mathrm{app}} - g_{\mathrm{sf}}$ is exponentially-decaying. Since $h_t^{\mathrm{app}} = h_\infty$ on the complement of disks around the zeros of q_2, the difference of the two metrics $g_{\mathrm{app}} - g_{\mathrm{sf}}$ reduces to an integral on these disks. Mazzeo-Swoboda-Weiss-Witt's possible polynomial terms are from variations in which the zeros of the quadratic differential $q_2 + \varepsilon\dot{q}_2 = -\det(\varphi + \varepsilon\dot{\varphi})$ move. Dumas-Neitzke use a local biholomorphic flow on the disks around each zero of q_2 that perfectly matches the changing location of the zero of $q_2 + \varepsilon\dot{q}_2$. Moreover, surprisingly, the most seemingly problematic piece of the integrand for the difference $g_{\mathrm{app}} - g_{\mathrm{sf}}$ turns out to be an exact form that they can control. Though Dumas-Neitzke only proved that $g_{\mathrm{app}} - g_{\mathrm{sf}}$ was exponentially-decaying for Higgs bundles in the image of the $SU(2)$-Hitchin section, this strategy can be extended to generic Higgs bundles in the $SU(n)$-Hitchin moduli space.

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Parabolic Higgs bundles and gravitational instantons

Hartmut Weiss
(joint work with Laura Fredrickson, Rafe Mazzeo, Jan Swoboda)

Parabolic Higgs bundle moduli spaces provide examples of hyper-Kähler manifolds in low dimensions. More specifically, for $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles the moduli space of ordinary Higgs bundles on a closed Riemann surface of genus $g \geq 2$ has real dimension $12g - 12 \geq 12$, whereas the moduli space of parabolic Higgs bundles on the 4-punctured sphere (or the once-punctured torus) is 4-dimensional over the reals. The construction of the hyper-Kähler metric on the ordinary Higgs bundle moduli space goes back to the seminal article of Hitchin [8], in the case of parabolic Higgs bundles it was achieved by Konno [9] in the strongly parabolic and by Nakajima [14] in the general case. The study of the asymptotic geometry of the hyper-Kähler metric on the Higgs bundle moduli space (in the following called Hitchin metric) has recently attained some attention, see [13, 4, 5] and the contribution of Laura Fredrickson to this volume. The following picture, based on predictions by Gaiotto, Moore and Neitzke [6], has been established in the above mentioned works, the final statement being contained in [5]: On the regular part of the Hitchin system the Hitchin metric is exponentially close to a semiflat model metric adapted to the torus fibration. However, it has not been possible yet to establish the predicted value of the rate of exponential decay.

4-dimensional complete hyper-Kähler manifolds are also known as gravitational instantons. Under the additional assumption of faster than quadratic curvature decay, i.e.

$$|\mathrm{Rm}(p)| \leq r(p)^{-(2+\varepsilon)}$$

for some $\varepsilon > 0$, where $r(p) = d(p, o)$ is the geodesic distance to some fixed base point o, gravitational instantons have recently been shown by Chen and Chen to admit a classification into spaces of type ALE, ALF, ALG or ALH [1, 2, 3]. ALE spaces have Euclidean volume growth and are completely described in work of Kronheimer [10, 11]. ALF, ALG and ALH spaces have cubic, quadratic and linear volume growth respectively. If one drops the assumption of faster than quadratic curvature decay, then fractional volume growth rates may occur. Examples of that nature were constructed by Hein [7]. ALG spaces also have a semiflat model geometry at infinity, which in this case is actually flat. Possible models are bundles of flat tori of modulus $\tau \in \mathbb{H}$ over the flat cone of angle $2\pi\beta$ and monodromy $e^{2i\beta}$, which fit in to the Kodaira classification of singular fibers of elliptic surface fibrations according to the following table:

D	Regular	I_0^*	II	II^*	III	III^*	IV	IV^*
β	1	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
τ	$\in \mathbb{H}$	$\in \mathbb{H}$	$e^{2\pi i/3}$	$e^{2\pi i/3}$	i	i	$e^{2\pi i/3}$	$e^{2\pi i/3}$

However, the rate of decay of the ALG-metric to the model metric is generically polynomial.

Motivating for this work was the following question, raised by Nigel Hitchin 2015 in a conference talk at the Newton Institute:

Is the moduli space of strongly parabolic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on the 4-punctured sphere a gravitational instanton of type ALG?

In this work we give a positive answer to this question. The main new technical tool is a gluing theorem for solutions of Hitchin's equation for parabolic Higgs bundles with large Higgs fields in the regular part of the Hitchin system, generalizing the one in [12] for ordinary Higgs bundles. More specifically, for t sufficiently large we wish to solve the equation

$$F^{h_t} + t^2[\Phi \wedge \Phi^{*h_t}] = 0$$

for a hermitian metric h_t which is asymptotic to

$$h_t(z) = \begin{pmatrix} |z|^{2\alpha_1} & \\ & |z|^{2\alpha_2} \end{pmatrix}$$

near the punctures. Here $0 \leq \alpha_1 < \alpha_2 < 1$ are the parabolic weights and we assumed that the parabolic degree of the Higgs bundle vanishes. This involves the construction of limiting configurations and corresponding desingularizations adapted to the parabolic structure. The actual analysis of the gluing problem then has to be carried out in weighted spaces, which requires some more advanced techniques. As in the case of ordinary Higgs bundles, we obtain that the Hitchin metric on the parabolic Higgs bundle moduli space is exponentially close to a semiflat metric on the regular part.

A distinctive feature of the moduli space of strongly parabolic $\mathrm{SL}(2,\mathbb{C})$-Higgs bundles on the 4-punctured sphere is that the discriminant locus consists of $0 \in \mathbb{C}$ alone, i.e. the regular part of the Hitchin system is the whole complement of the nilpotent cone, and the regular fibers are identified by the $\mathbb{C}^\times$-action. The nilpotent cone itself is a union of five 2-spheres, intersecting according to the affine D_4 diagram. This means that the asymptotic description of solutions and in particular the approximation of the Hitchin metric by the semiflat metric applies to a whole neigbourhood of infinity. Furthermore, it follows that the semiflat metric is the model metric corresponding to Kodaira type I_0^*, where $\beta = \frac{1}{2}$ and the modulus $\tau \in \mathbb{H}$ is determined by the conformal location of the 4 points which make up the parabolic divisor. The Hitchin metric attains a special place in the moduli space of ALG metrics since it exhibits exponential asymptotics. We're confident to be able to establish the expected rate of exponential decay in this special case very soon. The following question is the subject of ongoing joint work by the authors: Which portion of the moduli space of ALG metrics is filled out by Hitchin metrics, e.g. by varying the parabolic weights or by allowing semisimple residues?

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Limiting Configurations and Pleated Surfaces

Michael Wolf

(joint work with Andreas Ott, Jan Swoboda, Richard Wentworth)

1. Introduction

In this extended abstract, we report on ongoing joint work with **Andreas Ott, Jan Swoboda**, and **Richard Wentworth**. Let $S = S_g$ denote a closed differentiable surface of genus g on which we will put various geometric structures.

We seek to interpret a stratum in the frontier of the character variety $\chi_g = \mathrm{Hom}(\pi_1(S), \mathsf{SL}(2,\mathbb{C}))//\mathsf{SL}(2,\mathbb{C})$ of (irreducible) genus g surface group representations into $\mathsf{SL}(2,\mathbb{C})$. In particular, we refer to a recent work [MSWW16] of Mazzeo-Swoboda-Weiss-Witt. These authors fix a Riemann surface structure, say X, on S, and consider the moduli space $\mathcal{M}$ of stable $\mathsf{SL}(2,\mathbb{C}$-Higgs bundles (up to gauge equivalence) over X. They then consider those Higgs bundles for which the Higgs field, say Φ, has determinant $q = \det(\Phi) \in H^0(X, K_X^2)$, a holomorphic quadratic differential on X, to have but simple zeroes. Roughly, they continue from this restricted space to define a frontier for this portion of the moduli space by adjoining to the associated portion of χ_g a moduli space $\mathcal{M}_\infty$ consisting of (equivalence classes of) *limiting configurations*. These limiting configurations are pairs (Φ_∞, A_∞) of a singular Higgs field Φ_∞ and singular connection A_∞: together the pair satisfy a degenerate decoupled system of equations that is a limiting version of the Hitchin system. See [MSWW16] for complete details.

In this talk, we seek to address two questions:

(1) What is the dependence of this stratum of limiting conditions on the initial choice of Riemann surface X? For example, if (Φ_∞, A_∞) is a limit, under the correspondences above, of a sequence $\rho_n \in \chi_g$ of (of equivalence classes of) representations where we have chosen X as the background Riemann surface, and if $(\Phi'_\infty, A'_\infty)$ is an accumulation point of those classes of representations when we have chosen X' as a background Riemann surface, then how does $(\Phi'_\infty, A'_\infty)$ relate to (Φ_∞, A_∞)?

(2) The Hitchin theory (see [Hit87]) proceeds via consideration of ρ_n-equivariant harmonic maps $u_n : \tilde{X} \to \mathsf{SL}(2,\mathbb{C})/\mathsf{SU}(2)$. The latter symmetric space is isometric to the hyperbolic three-space $\mathbb{H}^3$, so we seek an interpretation of the limiting configuration pair (Φ_∞, A_∞) in terms of hyperbolic-geometric objects.

The theme of our work is then the reconciliation of two traditions of perspectives on the character variety χ_g: gauge theoretic, wherein the monodromy of a connection on a rank two complex bundle over the Riemann surface X lies in $\mathsf{SL}(2,\mathbb{C})$, and synthetic-hyperbolic, in which $\mathsf{SL}(2,\mathbb{C})$ is the isometry group of $\mathbb{H}^3$, and the group $\pi_1(S)$ acts equivariantly by isometries on $\mathbb{H}^3$. There are then also two notions of *bending* which we will unite: one relating to parallel transport of a connection and the other relating to the dihedral angle of two half-planes in $\mathbb{H}^3$ meeting along a geodesic.

We address the questions above by, roughly, relating limiting configurations (Φ_∞, A_∞) to classes of shearings of a pleated surfaces $\Sigma = (\tilde{f}, (S,\sigma), \rho, \lambda)$. Here the pleated surface Σ is defined by the following data: the surface S is equipped by a hyperbolic metric σ for which λ is a geodesic laminations, and the map $f : \tilde{S} \to \mathbb{H}^3$ is an isometry on complement $S \setminus \lambda$ of λ in S, as well as an isometry of λ onto its image (geodesic). For full details, see [Bon96] and the papers referenced within.

Given a pleated surface $\Sigma = (\tilde{f}, (S,\sigma), \rho, \lambda)$ and a number s, we create a pleated surface $\Sigma_s = (\tilde{f}_s, S_s, \rho_s, \lambda)$ as follows. Set Ξ_s^λ to be the transverse cocycle associated to a left earthquake of S along λ. Then set $\Sigma_s = \Sigma_{s,\mu_{q,\mathrm{ver}}} = \Xi_s^\lambda \Sigma$, the result of shearing Σ along the lamination λ for a measure of $s\mu_{q,\mathrm{ver}}$, where here $\mu_{q,\mathrm{ver}}$ denotes the measure for the vertical foliation of q. Note that this operation results in a pleated surface Σ_s with the same bending cocycle as the original surface Σ. (Naturally, a similar construction of $\Sigma_{s,n} = \Xi_s^{\lambda_n}\Sigma_n$ can be made for laminations λ_n and measures $\mu_{q_n^1,\mathrm{ver}}$.)

2. The pleated surface for a limiting configuration

Let ρ_n denote a sequence of irreducible $SL(2,\mathbb{C})$ surface group representations which leave all compact sets in the character variety χ_g, converging to a limiting configuration (Φ_∞, A_∞) relative to a choice X of Riemann surface. Let $h_n : \tilde{X} \to \mathbb{H}^3$ denote the associated family of equivariant harmonic maps from the universal cover $\tilde{X}$ to hyperbolic 3-space $\mathbb{H}^3$, normalized by some fixed choice of frames.

Let $q_n = \det(\Phi_n)$ be the Hopf differential of the harmonic map h_n; here Φ_n refers to the Higgs field. Our assumption that the limiting configuration (Φ_∞, A_∞) has $\det\Phi_\infty$ a quadratic differential with simple zeroes implies that we may assume, for n sufficiently large, that the differential q_n also has only simple zeroes. We adopt the notation that $\tilde{X}$ denotes the universal cover of X, and $\tilde{q_n}$ (respectively $\tilde{q}$) denote the lifts to the universal cover $\tilde{X}$ of q_n (resp. q), and so on. Let $q_n^1 = \frac{q_n}{\|q_n\|}$ denote the unit norm quadratic differential which is a multiple of the quadratic differential q_n. We might as well assume that $\|q\| = 1$ so that $q_n^1 \to q$.

For simplicity, we assume that q has no horizontal saddle connections: the statements in that case are both essential but also require substantial more detail, and we omit them in the talk.

Let $X^\times$ denote the complement in $\tilde{X}$ of the zeroes of $\tilde{q}$ and the horizontal leaves that emanate from those zeroes.

On $X^\times$, let π denote the natural map which takes horizontal leaves ℓ of q to their geodesic representatives $\pi(\ell)$ in the lamination λ, with an analogous definition for $\pi_n : X^\times \to \lambda_n$. We continue to denote this map by π (resp. π_n).

Proposition 1. *There is a pleated surface $\Sigma = (\tilde{f}, S, \rho, \lambda)$ with the following properties. The measured lamination λ is projectively equivalent to the measured lamination naturally associated to the horizontal measured foliation of $q = \det(\Phi)$. Let $s = s(n) = 2E(h_n)^{\frac{1}{2}}$, and let $\Sigma_s = \Xi_s\Sigma$ be as in the previous paragraph. We then have the following estimates depending upon whether the Hopf differentials q_n are proportional or not.*

(i) Suppose that q_n^1 is independent of n. Then, for every ϵ, and on any compact set $K \subset \tilde{X}$, we may choose n sufficiently large so that the images $h_n(\tilde{X})$ are within distance ϵ of $\tilde{f}_s(\tilde{X})$ on K; moreover, on the complement of any neighborhood of $q^{-1}(0)$ in K, the map h_n nearly agrees with the projection $f_s \circ \pi$ from the punctured surface $\tilde{X}$ to the lamination λ, i.e. when $d_{|q_n^1|}(p, q_n^{-1}(0)) > \epsilon$, we have for n sufficiently large that $d_{\mathbb{H}^3}(h_n(p), \pi_n(p)) < \epsilon$.

(ii) In general, with no restriction on q_n^1 other than $q_n^1 \to q$, we conclude that for every (large) constant C and every ϵ, there is an n so that we have for n sufficiently large for points p so that $d_{|q_n^1|}(p, q_n^{-1}(0)) > \epsilon$, then $d_{\mathbb{H}^3}(h_n(p), \pi_n(p)) < 2s - C$.

The proof of Proposition 1 uses estimates on high energy harmonic maps to $\mathbb{H}^2$ and $\mathbb{H}^3$ from [Wol91] and [Min92]: the constructions borrow heavily from the easier parts of Minsky's thesis [Min92].

Remarks 1.
(1) In effect, the construction in this proposition results in a family ρ_s of representations defining the pleated surfaces Σ_s that track a subsequence of the representations induced by h_n.

(2) One can understand the second statement in the proposition in the following way. A consequence of the first estimate is that if one takes a 'ray' of representations ρ_n whose Hopf differentials q_n are all multiples of a single unit quadratic differential q_n^1, then the harmonic map images $h_n(\tilde{X})$ are tracked very closely by shearings $\Sigma_s = \Xi_s\Sigma$ of a single pleated surface Σ. Thus, if one were to take a second family of representations $\rho_n^{'}$ whose Hopf differentials $q_n^{'}$ are all multiples of a single unit quadratic differential $q_n^{1'}$, then the harmonic map images $h_n^{'}(\tilde{X})$ are tracked very closely by shearings $\Sigma_s^{'} = \Xi_s\Sigma$ of a single pleated surface $\Sigma^{'}$. But those shearings Σ_s and $\Sigma_s^{'}$ are bent along measured laminations which typically make some non-zero angle with other, so even for quadratic differentials q_n and $q_n^{'}$ whose zeroes are close the distances between the images $h_n(p)$ and $h_n^{'}(p)$ of a point p far from the zeroes will distance $d_{\mathbb{H}^3}(h_n(p), h_n^{'}(p)) = 2s - O(1)$. This last estimate is because $h_n(p)$ will lie close to one geodesic and be moved by the shearing along that geodesic by a distance $s + O(e^{-cs})$ and $h_n^{'}(p)$ will lie close to another distinct geodesic and be moved by the shearing along that geodesic by a distance $s + O(e^{-cs})$. By elementary hyperbolic geometry, even if those geodesics intersect, the distance between the points $h_n(p)$ and $h_n^{'}(p)$ will be at least $2s - C_0$ for some absolute constant C_0.

(3) The reference to a compact set $K \subset \tilde{X}$ is just for convenience in the statement; the issue is that the estimates on the geometry of the harmonic map have an error estimate in them, so we can account for the imprecision in the estimates by referencing an estimate that is uniform in compacta or by slightly adjusting the approximating pleated surface.

3. Bending Cocycles

With this basic correspondence in hand, we rapidly sketch the remainder of the discussion.

The space of limiting configurations $\{(\Phi_\infty, A_\infty)\}$ fibers into Prym varieties which share a common singular Higgs field Φ_∞: in this construction a singular connection A_∞ differs from another singular connection A_∞^0 by a form $\alpha \in H^1(X^\times, L_{\Phi_\infty})$, where $L_{\Phi_\infty} = \{\gamma \in \mathfrak{su}(2) : [\gamma, \Phi_\infty] = 0\}$ is a line bundle over $X^\times$ (the Riemann surface X punctured at $\Phi_\infty^{-1}(0)$). There is an equivalence relation among the elements $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ given by an integral relation among the periods of the forms. Again, see [MSWW16] for full details on the structure of the space of limiting configurations. We note that these forms may be construed as Prym differentials on a spectral curve constructed from $(X, \det \Phi_\infty)$.

We show two results, which we summarize a bit informally, using only the terminology developed so far.

Proposition 2. *A form $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ formally defines a bending cocycle $b_{[\alpha]}$ for a geodesic lamination $\lambda \subset S$ corresponding to the horizontal foliation for Φ_∞.*

Let $(\Phi_\infty, A^0_\infty)$ denote the limiting configuration corresponding to the Hitchin section of χ_g: these are also often referred to as the Fuchsian representations. Here the associated pleated surface from Proposition 1 has a vanishing bending cocycle.

Thus, associated to a form $\alpha \in H^1(X^\times, L_{\Phi_\infty})$, we now have two pleated surfaces (or more precisely, classes of shearings of pleated surfaces. The first, Σ^α is defined via Proposition 2 by bending the Fuchsian pleated surface along $\lambda = \lambda(\Phi_\infty)$ so that the resulting bending cocycle is α.

The second pleated surface Σ_α is obtained from $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ by applying the construction of Proposition 1 to the limiting configuration $(\Phi_\infty, A^0_\infty + \alpha)$.

A main result in the talk is that

Theorem 1. *The pleated surfaces Σ_α and Σ^α agree up to shearing along λ.*

The proof involves giving a hyperbolic geometry interpretation of the bundle L_{Φ_∞} and the elements $\alpha \in H^1(X^\times, L_{\Phi_\infty})$ (cf. Donaldson [Don87]), and then combining these with some of the estimates on high energy harmonic maps as well as some elementary observations as to the geometry of highly sheared pleated surfaces.

Remark 1. The results of the proposition in the second section and the theorem in the present section is that the space of limiting configurations may be defined topologically in terms of the bending of a pleated surface along a lamination dual to the vertical measured foliation of the quadratic differential obtained as the determinant of the Higgs field defining the fiber Prym variety.

4. Limits of opers.

We recall the definition of opers (see [Dum09]), which we approach from the perspective of Schwarzian derivatives and complex projective ($\mathbb{CP}^1$) structures on a Riemann surface X. Let $\psi \in H^0(X, K_X^2)$ be a holomorphic quadratic differential on X. We consider the ordinary differential equation on X given by

$$u''(z) + \frac{1}{2}\psi(z)u(z) = 0 \tag{1}$$

and let $u_1(z)$ and $u_2(z)$ denote a basis for the solution space. The lifted map $\tilde{f} = [\tilde{u_1}(z) : \tilde{u_2}(z)] : \tilde{X} \to \mathbb{CP}^1$ develops the universal cover $\tilde{X}$ to $\mathbb{CP}^1$, and one checks that the Schwarzian derivative

$$\mathcal{S}(f) = [(\frac{f''}{f'})' - \frac{1}{2}(\frac{f''}{f'})^2]dz^2$$

(say, with respect to an initial identification of $\tilde{X}$ with a Fuchsian development) inverts (1) in the sense that $\mathcal{S}(\tilde{f}) = \psi$. A different choice of basis for the solution space to (1) would precompose f by a Mobius transformation and hence not affect

$\mathcal{S}(f)$, so the holonomy representation defined by the developing map $\tilde{f}$ is defined only by (1), hence only by ψ.

In this section, we study the asymptotics of the family of representations defined by the holonomies ρ_t associated to the oper family

$$u''(z) + \frac{1}{2}t\psi(z)u(z) = 0. \tag{2}$$

Of course, as holonomies, this family determines a family Oper($t\psi$) of (equivalence) classes of Higgs bundles, and we seek to understand, for example, its accumulation set within the space of limiting configurations.

Now, Dumas [Dum07] explains the asymptotics of these oper representations via the asymptotics of graftings and hence pleated surfaces, while our results in the prior sections relate limiting configurations to the asymptotics of pleated surfaces. The upshot is that we may interpret Dumas' work in terms of the accumulation sets of the oper families on limiting configurations.

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Unitary Representations of 3-manifold Groups and the Atiyah-Floer Conjecture

Aliakbar Daemi

(joint work with Kenji Fukaya and Maksim Lipyanskyi)

Inspired by ideas in quantum Field theory and Morse theory, Floer developed a series of important constructions in low dimensional topology and symplectic geometry. In one direction, he built on the groundbreaking work of Donaldson to construct an invariant for certain 3-manifolds which is called *instanton Floer homology* [7]. In a different direction, Floer defined an invariant, known as *Lagrangian Floer homology*, for pairs of Lagrangians in a symplectic manifold [8].

Suppose Y is a 3-manifold with the same homology groups as S^3. Suppose also E is the trivial $SU(2)$ bundle on Y. An appropriate count of the number of flat connections[1] on E produces a numerical invariant of Y, which is called *Casson invariant* [2, 14]. The simplest version of instanton Floer theory, denoted by $\mathrm{I}_*(Y)$, is a homology theory associated to the pair (Y, E), and its Euler characteristic is equal to twice the Casson invariant of Y. It is reasonable to expect that one can perform similar constructions by replacing (Y, E) with more general pairs. We say a pair (Y, E) is admissible if E is a Hermitian bundle of rank 2 on Y such that $c_1(E)$ has a no-trivial pairing with an element of $H_1(Y, \mathbb{Z})$. Then a version of instanton Floer homology, denoted by $\mathrm{I}_*(\mathrm{Y}, \mathrm{E})$, can be also defined for admissible pairs (Y, E) [9].

Suppose M is a closed manifold with a symplectic form ω. Suppose also L_0 and L_1 are Lagrangian submanifolds of M, i.e., ω vanishes on half-dimensional submanifolds L_0 and L_1 of M. In addition to his instanton homology, Floer defined a homology group $\mathrm{HF}_*(L_0, L_1)$ under some restrictive assumptions on L_0 and L_1. This homology group is called *Lagrangian Floer homology* [8], and categorifies the intersection number of L_0 and L_1. Floer's original construction has been generalized in various directions and part of the assumptions that he made on M, L_0 and L_1 subsequently have been relaxed [13, 10, 11, 1].

The Atiyah-Floer conjecture states that instanton Floer homology and Lagrangian Floer homology are related to each other. Suppose H_0 and H_1 are two handlebodies of genus g. Then the boundaries of H_0 and H_1 are diffeomorphic to Σ_g, the Riemann surface of genus g. Therefore, gluing these two 3-manifolds along their boundaries determines a closed 3-manifold Y. This decomposition of Y into two handlebodies is called a *Heegaard splitting* of Y. Any 3-manifold Y admits a Heegaard splitting as $H_0 \cup_{\Sigma_g} H_1$. Let $\chi(\Sigma_g)$ be the space of $SU(2)$-representations of $\pi_1(\Sigma_g)$ modulo the conjugation action. Let also L_i be the subspace of $\chi(\Sigma_g)$ consisting the representations that can be extended to H_i. The space $\chi(\Sigma_g)$ admits a symplectic structure and the subspaces L_0 and L_1 are Lagrangian with respect to this symplectic structure. The space $\chi(\Sigma_g)$ is called the *character variety* and L_0, L_1 are called the *handlebody Lagrangians*. According to the Atiyah-Floer conjecture, for an integral homology sphere Y, the Lagrangian Floer homology $\mathrm{HF}_*(L_0, L_1)$ is an invariant of Y which is called *symplectic instanton homology*. Furthermore, the Atiyah-Floer conjecture states that the symplectic instanton homology of Y is isomorphic to the instanton homology group $\mathrm{I}_*(Y)$ [3].

Part of the difficulty with the Atiyah-Floer conjecture is to define symplectic instanton homology. The spaces $\chi(\Sigma_g)$, L_0 and L_1 are singular and the standard definitions of Lagrangian Floer homology cannot be applied to these spaces. One way to avoid this issue is to replace the Lie group $SU(2)$ with $SO(3)$. The space of $SO(3)$-representations of $\pi_1(\Sigma_g)$, up to an appropriate conjugation action, has two connected components. One component can be identified with $\chi(\Sigma_g)$. The other component is a smooth symplectic manifold denoted by $\chi_{\mathrm{odd}}(\Sigma_g)$ which is called the *odd character variety* of Σ_g. For a possibly disconnected Riemann

[1] In general, we might need to perturb the flat equation for a connection

surface Σ, let $\chi_{\mathrm{odd}}(\Sigma)$ be the product of the odd character varieties associated to the connected components of Σ.

Suppose N is a 3-manifold with boundary Σ, and F is a Hermitian bundle of rank 2 on N. Suppose V is the bundle of skew-adjoint homomorphisms of F. Then V is a real vector bundle of rank 3 with inner product, i.e., an $SO(3)$-bundle. Assume that the restriction of V to each connected component of Σ is non-trivial. Then the space of flat connections on the bundle V gives rise to a subspace of $\chi_{\mathrm{odd}}(\Sigma)$. After a small perturbation, this subspace turns into an immersed Lagrangian submanifold denoted by L [12]. Suppose N' is another 3-manifold with boundary Σ and F' is a Hermitian bundle of rank 2 on N', satisfying the same properties as F. Suppose also L' is the immersed Lagrangian of $\chi_{\mathrm{odd}}(\Sigma)$ associated to (N', F'). The manifolds N and N' can be glued to each other along Σ to form a closed 3-manifold Y. Also, gluing F and F' defines a $U(2)$-bundle E on Y. Then (Y, E) is an admissible pair and any admissible pair can be constructed in this way.

In my talk, I mainly discussed the following theorem [4, 5], which is an instance of the Atiyah-Floer conjecture for admissible pairs, and it generalizes Dostoglou and Salamon's result in [6]:

Theorem 1. *If the immersed submanifolds L and L' of $\chi_{\mathrm{odd}}(\Sigma)$ are embedded, then they are monotone Lagrangian submanifolds of $\chi_{\mathrm{odd}}(\Sigma)$ with minimal Maslov numbers* 4*. Moreover, the Lagrangian Floer homology group* $\mathrm{HF}_*(L, L')$ *is an invariant of* (Y, E) *and is isomorphic to the instanton Floer homology group* $\mathrm{I}_*(Y, E)$*.*

A Lagrangian submanifold L of a symplectic manifold (M, ω) is *monotone*, if there exists a positive constant c such that for any $\alpha \in \pi_2(M, L)$:

$$\mu_L(\alpha) = c\omega(\alpha).$$

Here μ_L is the Maslov index associated to the Lagrangian manifold L. The minimal Maslov number of a Lagrangian is the positive integer that generates the image of μ_L. In [13], Oh defines Lagrangian Floer homology for compact monotone Lagrangians of a closed symplectic manifold whose minimal Maslov numbers are greater than 2. In particular, this version of Lagrangian Floer homology can be used to define $\mathrm{HF}_*(L, L')$ in Theorem 1.

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Higgs bundles for the geometric Langlands correspondence

Carlos Simpson
(joint work with R. Donagi, T. Pantev)

The program of Ron Donagi and Tony Pantev [1] proposes a general method for constructing the Higgs bundles over the moduli of stable bundles on a Riemann surface, that should correspond to the local systems predicted by the geometric Langlands correspondence.

We note that Donagi and Pantev have treated the case of $\mathbb{P}^1$ minus 5 points, this article is currently in preparation [2].

Here, we'll treat the case of rank 2 bundles on genus 2 curves.

Consider a smooth projective curve C of genus 2 over the complex numbers with $p \in C$ a Weierstrass point.

Let X be the moduli space of rank 2 stable bundles on C, of degree either 0 or 1 (cases denoted $X(0)$ and $X(1)$ respectively). By Narasimhan-Ramanan [6], we have

$$X(0) = \mathbb{P}^3, \quad X(1) \subset \mathbb{P}^5 \text{ intersection of two quadrics.}$$

The Hecke correspondences between these moduli spaces come from the quadric line complex.

The Hitchin moduli space has been studied by Previato and Van Geemen [7]. There are two fibrations

$$\begin{array}{ccc} M_H & \xrightarrow{f} & \mathbb{A}^3 \\ \downarrow & & \\ X & & \end{array}$$

where the vertical arrow is only a rationally defined map.

Consider a general point b in the Hitchin base $\mathbb{A}^3$, corresponding to a spectral curve $\widetilde{C} \subset T^*C$. The Hitchin fiber

$$Y_0 = f^{-1}(b) = \{\text{line bundles } L/\widetilde{C} \text{ such that } \det\pi_*(L) = \mathcal{O}_C \text{ or } \mathcal{O}_C(p)\}$$

is an abelian threefold, the Prym of $\widetilde{C}/C$. In order to obtain a map to X we need to resolve the singularities of the rational projection, by blowing up Y_0 to obtain a variety Y with exceptional divisor $E \subset Y$. The locus to be blown up is $Q \cap Y_0$ where $Q \subset M_H$ is the incoming or stable variety to the higher level fixed point locus of the $\mathbb{C}^*$ action. In $M_H(0)$ the fixed point locus consists of 16 points and in $M_H(1)$ it is a curve with a $16:1$ etale cover $\overline{C} \to C$. Then $Q \cap Y_0$ is respectively again 16 points, or the curve $\widehat{C} = \overline{C} \times_C \widetilde{C}$.

Let $\pi : Y \to X$ denote the resulting map.

The *wobbly locus* $W \subset X$ is the locus where the geometric Langlands local systems will have singularities. It is the locus where the other components of the nilpotent cone meet X, and also includes the strictly semistable locus in the case of degree 0.

Birationally, M_H identifies with the cotangent bundle of X. More concretely we get an inclusion

$$Y \hookrightarrow T^*X(\log W).$$

Therefore, choosing a line bundle $\mathcal{L}$ over Y will result in BNR spectral data needed to define a Higgs sheaf $\mathcal{V} = \pi_*\mathcal{L}$ over X with Higgs field Φ having logarithmic singularities along W.

The Donagi-Pantev program consists of the idea of adding an additional parabolic structure to $\mathcal{V}$ along the wobbly locus W, in order to obtain a parabolic logarithmic Higgs sheaf on X that will satisfy the Chern class criteria, so we can then apply Mochizuki's existence theorem [5] to get a local system on $X - W$. The second part of the program is to verify the Hecke eigensheaf property of this local system with respect to the initial point in M_H for the Langlands dual group.

In the case of a genus 2 curve, the moduli space X and wobbly locus are very classical objects. For example, in the case of degree 0 the wobbly locus inside $X(0) \cong \mathbb{P}^3$ is the union of the Kummer surface with the 16 trope planes of the Kummer 16_6 configuration. In the case of degree 1, W is the image of its normalization given by a map $\overline{C} \times \mathbb{P}^1 \to X(1)$. In this case W has cusps and nodes as its codimension 1 singularities.

We have been able to understand how to choose the parabolic structure in these cases: notably, the parabolic weight is just $\alpha = 1/2$. The quotient sheaf involved in the parabolic structure comes from the ramification of the map $Y \to X$ over the wobbly locus W. Note that there are other components of ramification that move as we move b, these don't get parabolic structures.

Furthermore, with a few special techniques we can calculate the Chern classes, notably the parabolic Δ^{par}-invariant. For this choice of parabolic structure and parabolic weight, $\Delta^{\mathrm{par}} = 0$. The Higgs sheaf is automatically stable since its spectral variety Y is irreducible. Hence we obtain a local system on $X - W$ by applying [5].

Here are a few of the current and future issues involved. A first question is how to extend across the singularities of W that are codimension 2 in X. This is all that is needed in order to get a local system. However, W is not normal crossings in codimension 2; in the odd case it has cusps as well as nodes, and in the even case the trope planes form tacnodes with the Kummer surface. For the odd case, we pass to a Kawamata-Viehweg covering space that has order two ramification along W, thereby removing the parabolic weight, and we can calculate the locally free extension of the bundle upstairs to get the Chern class calculation by GRR. In the even case, we make use of the observation by Heu and Loray [4] (following Goldman) that $X(0)$ has a degree 2 covering ramified along the Kummer surface that is the moduli of parabolic bundles on $\mathbb{P}^1$ with respect to 6 points (and the same for M_H). Upstairs, this resolves the parabolic structure along the Kummer surface, and the trope planes become normal crossings, again allowing the calculation by GRR.

A next problem is to consider the Hecke correspondences. We would like to show that the local system constructed over $X - W$ satisfies the Hecke eigensheaf property. This is work that is still in progress; it was the motivation for our preprint about higher direct images of parabolic logarithmic Higgs bundles [3].

Beyond this case of genus 2 and rank 2, several more general questions may be considered. There are several phenomena that make the higher rank and higher genus cases more difficult, including the problem of multiple weights in the $\mathbb{C}^*$ action at the tangent spaces to the fixed point loci; here weights $-1, 0, 1, 2$ contributed to the order two ramification of the map $Y \to X$ over W. Another question is to understand the singularities of W at codimension 2 in X, and whether the phenomenon of strings of fixed point loci (not present here) contributes singularities.

In the spirit of the workshop, we can isolate questions such as compatibility with real structures, monodromy groups, asymptotics of monodromy, behavior with respect to the Hitchin components, as well as the relationship with other existing constructions of the geometric Langlands local systems.

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Geometric models for moduli spaces of parabolic Higgs bundles in genus 0

Claudio Meneses

Moduli spaces of parabolic Higgs bundles on compact Riemann surfaces are very interesting examples of hyperkähler manifolds with a rich geometry that has been thoroughly studied over the last decades from multiple points of view. The introduction of parabolic structures allows the possibility of small dimensional examples [12, 5], while giving rise to new phenomena such as their dependence on parabolic weights [14].

Several exceptional features arise when the base is the Riemann sphere; in such case the rigidity of the underlying holomorphic bundles leads to rather simple and explicit geometric models for the moduli spaces in question, which we will describe briefly in the following paragraphs. The basic idea is to reduce the problem to the study of actions of groups of bundle automorphisms on suitable model spaces [6]. By construction, these models depend on a choice of parabolic weights, and such dependence is in fact the key ingredient for their explicit description: the combinatorial structure of the weight polytopes, seen through their semi-stability walls and interior open chambers, plays the role of a "set of instructions" for the model's construction, elucidating geometrically the dependence on parabolic weights and wall-crossing behavior under variations of parabolic weights.

Consider a set of $n \geq 3$ points $z_1, \ldots, z_n \in \mathbb{CP}^1$ and denote $D = z_1 + \cdots + z_n$. A parabolic Higgs bundle on $\mathbb{CP}^1$ is a pair (E_*, Φ) where E_* denotes a rank r holomorphic vector bundle $\pi : E \to \mathbb{CP}^1$ together with a collection of descending flags and associated weights

$$E|_{z_i} = E_{i1} \supseteq E_{i2} \supseteq \cdots \supseteq E_{ir} \supset \{0\}, \qquad 0 \leq \alpha_{i1} \leq \alpha_{i2} \cdots \leq \alpha_{ir} < 1,$$

where $E_{ij} = E_{ik}$ if and only if $\alpha_{ij} = \alpha_{ik}$, and Φ is a (strongly) *parabolic Higgs field*, i.e., an element $\Phi \in H^0\left(\mathbb{CP}^1, \operatorname{End}(E) \otimes K_{\mathbb{CP}^1}(D)\right)$, with nilpotent residues $\{\operatorname{Res}_{z_i} \Phi \in \operatorname{End} E|_{z_i}\}_{i=1}^n$ satisfying the compatibility condition

$$\left(\operatorname{Res}_{z_i} \Phi\right)(E_{ij}) \subseteq E_{ij+1}, \qquad \forall\, 1 \leq i \leq n,\ 1 \leq j \leq r.$$

The notion of parabolic stability, depending on a choice of parabolic weights, leads to moduli spaces of parabolic bundles $\mathscr{N}$ and parabolic Higgs bundles $\mathscr{M}$, together with a non-abelian Hodge correspondence to suitable character varieties when the parabolic degree is 0 [11, 10]. However, not every admissible set of parabolic weights determines a non-empty moduli space, leading to the notion of *weight polytopes* [1, 4]. Every weight polytope possesses a finite collection of semi-stability walls, whose complement is a finite collection of open chambers. While the complex structure of $\mathscr{M}$ and $\mathscr{N}$ is an invariant of each open chamber, further analytic invariants may be sensitive to variations of parabolic weights within any given open chamber. We will focus our attention on the natural Kähler structure whose underlying complex structure makes the embedding $\mathscr{N} \hookrightarrow \mathscr{M}$ holomorphic, and denote its Kähler form (in both cases) by Ω.

The underlying bundle of any parabolic Higgs bundle possesses an intrinsic *Harder–Narasimhan filtration*, leading to stratifications of moduli spaces. Over the Riemann sphere, Harder–Narasimhan filtrations can be understood in terms of Birkhoff–Grothendieck splittings $E \cong E_N := \mathcal{O}(m_1) \oplus \cdots \oplus \mathcal{O}(m_r)$. Hence, we can label different strata by their Birkhoff–Grothendieck splitting coefficients

$$\mathscr{M} = \bigsqcup \mathscr{M}_N, \qquad \mathscr{N} = \bigsqcup \mathscr{N}_N.$$

Our motivation is the study of the relation between the pairs $(\mathscr{N}, \Omega)$ and certain non-compact WZNW models [9]. After a choice of isomorphism $E \cong E_N$, the Mehta–Seshadri theorem can be interpreted as the equivalence between parabolic stability and the existence of a map $h : \mathbb{CP}^1 \setminus \{z_1, \ldots, z_n\} \to \mathcal{H}_r$ (valued on hermitian and positive-definite matrices), with suitable asymptotic behavior at each puncture, that satisfies the zero-curvature equation

$$\bar{\partial}\left(h^{-1}\partial h\right) = 0,$$

which is the Euler–Lagrange equation for a non-compact WZNW action. There is a Zariski open set $\mathscr{N}_0 \subseteq \mathscr{N}$ (depending on a given weight chamber, and in turn a subset of the largest stratum $\mathscr{N}_{N_0} \subseteq \mathscr{N}$) for which the map $\mathscr{S} : \mathscr{N}_0 \to \mathbb{R}$ defined as the critical values of the non-compact WZNW action satisfies

$$\frac{\sqrt{-1}}{2}\bar{\partial}\partial\mathscr{S} = \left(\Omega - \sum_{i,j}\beta_{ij}\Omega_{ij}\right)\Bigg|_{\mathscr{N}_0}$$

where Ω_{ij} are Chern forms of tautological line bundles $\mathscr{L}_{ij} \to \mathscr{N}$ [13]. The real coefficients β_{ij} depend linearly on parabolic weights. Hence, the cohomology class $[\Omega|_{\mathscr{N}_0}]$ is an analytic invariant depending explicitly on parabolic weights within the weight polytope. In fact, there exist a finite number of open chambers for which $\mathscr{N}_0 = \mathscr{N}$. In those cases, the result holds globally, and leads to the potential computation of recursive formulae for the associated symplectic volumes [16] through the intersection theory of tautological classes [15].

In order to describe the idea of a geometric model, we will consider the rank 2 case only, although the discussion generalizes to arbitrary rank with a little bit of work. From each choice of open chamber, we can list the finite collection of splittings $\{E_N\}$ that admit stable parabolic Higgs bundles for the corresponding parabolic weights. Over any such splitting E_N, we can parametrize parabolic Higgs fields explicitly in terms of their residues and an additional matrix-valued polynomial. While these structures would depend on the choice of isomorphism $E \cong E_N$, their orbits under the group of bundle automorphisms $\operatorname{Aut}(E_N)$ are an invariant of a quasi-parabolic Higgs bundle. Hence, at each marked point, let $\mathfrak{n}(E_i)$ be the subspace of nilpotent endomorphisms of the fiber $E_i := E_N|_{z_i}$. Its blow-up at 0 is naturally a line bundle $\widetilde{\mathfrak{n}(E_i)} \to \mathbb{P}(E_i)$ isomorphic to $\mathcal{O}(-2)$. Consider the model space

$$\mathscr{P}_N = \left(\widetilde{\mathfrak{n}(E_1)} \times \cdots \times \widetilde{\mathfrak{n}(E_n)} \times \mathrm{Pol}\right)_0^s$$

consisting of residue data at $z_1, \ldots, z_n$ together with a holomorphic germ on $\mathbb{CP}^1 \setminus \{\infty\}$, defining a parabolic Higgs field Φ for which (E_N, Φ) stable. Let $P\left(\mathrm{Aut}\left(E_N\right)\right) := \mathrm{Aut}\left(E_N\right)/Z\left(\mathrm{Aut}\left(E_N\right)\right)$ be the projectivization of $\mathrm{Aut}\left(E_N\right)$.

Theorem 1. *For any choice of parabolic weights within an open chamber in a given weight polytope, the action of the group $P\left(\mathrm{Aut}\left(E_N\right)\right)$ on $\mathscr{P}_N$ is free and proper. Moreover, there is an isomorphism*

$$\mathscr{M}_N \cong \mathscr{P}_N / P\left(\mathrm{Aut}\left(E_N\right)\right).$$

Further details will appear elsewhere [7, 8]. Theorem 1 should be understood within the framework of the recently developed theory of non-reductive GIT [2] and variations of non-reductive GIT [3]. However, a complete construction mechanism requires the glueing of different strata into $\mathscr{M}$, which is always a smooth complex manifold for parabolic weights inside any open weight chamber. We expect this construction to elucidate the relation between Ω and tautological forms in $\mathscr{M}$.

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Fixed point branes, singular loci and mirror symmetry

Ana Peón-Nieto
(joint work with Emilio Franco, Peter B. Gothen, André G. Oliveira)

Higgs bundles [Hi1] are a vast subject with many ramifications, of which we hereby focus on mirror symmetry. More specifically, we focus on two pieces of work by the author and collaborators [FP, FGOP1], where we consider related families of branes meaningful in topologial mirror symmetry and towards the understanding of mirror symmetry beyond the generic locus.

Let M_X be the moduli space of Higgs bundles on a Riemann surface X, that is, the scheme parametrizing pairs (E, ϕ) where $E \longrightarrow X$ is a holomorphic vector bundle of rank n and $\phi \in H^0(X, \mathrm{End}(E) \otimes K_X)$, with K_X the canonical bundle of X [Hi1, Ni]. The non abelian Hodge correspondence [S] endows it with 3 complex structures I (naturally induced by the complex structure of X), J (coming from the character variety) and $K := I \circ J$, which underlie a hyperkähler structure.

A key tool in the study of M_X is the Hitchin map, associating to a pair (E, ϕ) the characteristic polynomial of ϕ. The fibers of this map can be identified with the Jacobian of a suitable spectral cover of X [Hi2, DG], and are Lagrangian for the I-homolorphic symplectic form Ω_I. So after a hyperkähler rotation, they become special Lagrangian. Thus we have a hyperkähler (hence Calabi–Yau) manifold admitting a special Lagrangian torus fibration, which together with autoduality of Jacobian varieties gives an example of SYZ-mirror symmetry (between M_X and M_X). More generally, for any complex reductive Lie group G, the moduli space of G-Higgs bundles M_G is mirror to $\mathrm{M}_{\check{G}}$, where $\check{G}$ denotes the Langlands dual group of G [DP, HT]. Indeed, their Hitchin systems are dual Lagrangian torus fibrations, duality realised by a Fourier–Mukai equivalence $D(\mathrm{M}_G) \cong D(\mathrm{M}_{\check{G}})$ [DP].

In terms of the homological mirror symmetry conjecture [Ko], predictions from physics allow us to interpret mirror symmetry for Hitchin systems as a correspondence between branes [KW]. These can be of type A (that is Lagrangian submanifolds with a flat bundle) or B (complex submanifolds with a holomorphic bundle) in each of the distinguished Kähler structures. Particularly important are BBB-branes, and their dual BAA-branes.

All of the above applies over smooth spectral curves. But the global statement of mirror symmetry for Hitchin systems is not well understood. In [FP], we explore this phenomenon over the singular locus of M_X. From another point of view, a way to evidence for global dualities is by producing global invariants [HMP]. A remarkable example of the above is the equality of the (stringy) E-polynomials of $M_{\mathrm{PGL}(n,\mathbb{C})}$ and $M_{\mathrm{SL}(n,\mathbb{C})}$ [HT, GWZ]. The former can be expressed in terms E-polynomials of some branes in $M_{\mathrm{SL}(n,\mathbb{C})}$, investigated in [FGOP1], whose geometry brings us back to the branes of singular loci from [FP].

Singular loci and their conjectural duals

The singular locus of M_X may be covered with BBB-branes [FP]. These are given by Higgs bundles whose structure group reduces to the Levi subgroup $L_{\overline{n}}$ of the standard parabolic $P_{\overline{n}}$ associated to the partition $\overline{n}$. Hypercomplexity of these subspaces $\mathrm{M}_X(L_{\overline{r}}) \subset \mathrm{M}_X$ follows from the non abelian Hodge correspondence. For the partition $(1, \ldots, 1)$ we produce a family of hyperholomophic bundles $\mathscr{L}$ on $\mathrm{M}_X(L_{\overline{1}})$ paramerized by $\mathrm{Jac}(X)$.

We next investigate what the duals of the branes $(M(L_{\overline{1}}), \mathscr{L})$ should be, the idea being the potential existence of a stacky Fourier–Mukai transform of $\mathscr{L}$.

With this in mind, given $F_1, \ldots, F_s$ stable vector bundles of rank $\mathrm{rk}(F_i) = n_i$ (where $\sum_i n_i = n$), we consider $\mathrm{M}_X(P_{\overline{n}}) \subset \mathrm{M}_X$ given by Higgs bundles whose structure group reduces to the parabolic subgroup $P_{\overline{n}}$. Letting $U_{\overline{n}} < P_{\overline{n}}$ be the unipotent radical, let

$$D^X_{\overline{n}}(\overline{F}) = \left\{ (E, \phi) \in \mathrm{M}_X(P_{\overline{n}}) \,:\, E/U \cong \bigoplus_i F_i \right\}. \tag{1}$$

Under suitable conditions on the F_i's, $D^X_{\overline{n}}(\overline{F})$ is Lagrangian. In relation with singular loci, these Lagrangians look very much like Fourier–Mukai transforms. This, together with the existence of such a transform on the level of generic points of both branes, led us to conjecture the duality between them.

Branes in topological mirror symmetry

Let $\Gamma \subset \mathrm{Jac}(X)$ be the n-torsion subgroup. Given $\gamma \in \Gamma$, we consider its action on M_X by tensorization $\gamma \cdot (E, \phi) = (E \otimes \gamma, \phi)$. The fixed point set M^γ is easily seen to be hypercomplex [NR], thus defining the support of a brane of type BBB. These are the branes appearing in the expression of the (stringy) E-polynomial of $\mathrm{M}_{\mathrm{PGL}(n,\mathbb{C})}$, which motivated us to further study them.

When $\gamma \in \Gamma$ is of maximal order, we produce hyperholomorphic bundles $\mathscr{F}$ on M^γ parametrised by $F \in \mathrm{Jac}(X_\gamma)$, where $p_\gamma : X_\gamma \to X$ is the étale cover associated with γ. Since generic spectral curves are generically integral (although always singular), we may generically perform a Fourier–Mukai transform to compute the support of the dual branes $D^X_\gamma(F)$ using the constructions in [A]. Unfortunately, we do not have a good global understanding of $D^X_\gamma(F)$. In fact, in order to prove they are Lagrangian, we check that pullback by p_γ yields a local isomorphism from $D^X_\gamma(F)$ into $D^{X_\gamma}_{(1,\ldots,1)}(\overline{F})$ (see (1)).

More generally, given $\gamma \in \Gamma$ of order $m|n$, and a F a rank $n' := n/m$ vector bundle on m, we define an isotropic subscheme $D^X_\gamma(F)$ which is Lagrangian under suitable conditions on F. Again, isotropicity follows from the fact that

$$p_\gamma^* \,:\, D_\gamma(F)^X \longrightarrow D^{X_\gamma}_{(n',\ldots,n')}(\overline{F})$$

is a local isomorphism. Thus, in a certain sense, the geometry of singular loci and their conjectural duals determines that of fixed point branes and their duals.

Future directions

An important gap in our constructions, both in [FP] and [FGOP1], is the lack of a construction of hyperholomorphic bundles beyond the simplest case (Higgs bundles for Cartan subgroups and points fixed by maximal order line bundles). An immediate improvement (in progress) would consist in extending these results.

Another natural step is to compute stacky Fourier–Mukai duals of singular loci and check whether stability is preserved. We will undertake this in the rank two case [FP2], using [L].

Finally, it should be pointed out that the study of M^{γ} is but a first step in a long term project aiming at understanding topological mirror symmetry in terms of branes. To this aim, we carefully study in [FGOP2] the rank two case, and verify that a variation of the hyperholomorphic bundle allows to recover all the meaningful points in the nilpotent cone.

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Stratifications of the Hitchin and de Rham moduli spaces

Richard Wentworth

(joint work with Brian Collier)

In this talk we report on some of the results in the paper [1]. Let X be a closed Riemann surface of genus $g \geq 2$, and let M_H (resp. M_{dR}) denote the moduli space of $SL(n,\mathbb{C})$ Higgs bundles (resp. flat connections) on X. Then M_H and M_{dR} are holomorphic symplectic varieties, and the nonabelian Hodge correspondence gives a homeomorphism between the two. Moreover, M_H admits a $\mathbb{C}^*$-action, giving a Bialynicki-Birula stratification. The fixed points $[\mathcal{E}_0, \Phi_0]$ of the action are *complex variations of Hodge structure* (VHS). We will be interested in the *stable manifolds*

$$W^0(\mathcal{E}_0, \Phi_0) = \left\{ [\mathcal{E}, \Phi] \in M_H \mid \lim_{\xi \to 0} \xi \cdot [\mathcal{E}, \Phi] = [\mathcal{E}_0, \Phi_0] \right\}$$

While M_{dR} does not admit a $\mathbb{C}^*$-action, it has a "stratification" given by the *partial oper* structures defined by Simpson [4]. Namely, a holomorphic bundle $\mathcal{V}$ with an algebraic connection ∇ always admits a filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_\ell = \mathcal{V}$$

satisfying a Griffiths transversality condition and such that the associated graded bundle

$$\mathrm{Gr}\mathcal{V} := \oplus_{i=1}^{\ell} \mathcal{V}_i / \mathcal{V}_{i-1}$$

with the Higgs field $\Phi_{\mathcal{V}}$ induced from ∇ is semistable. We can thus define

$$W^1(\mathcal{E}_0, \Phi_0) = \{ [\mathcal{V}, \nabla] \in M_{dR} \mid [\mathrm{Gr}\mathcal{V}, \Phi_{\mathcal{V}}] = [\mathcal{E}_0, \Phi_0] \}$$

A particular case is when $[\mathcal{E}_0, \Phi_0]$ is the *Fuchsian* or *uniformizing* Higgs bundle. That is, one coming from composing the constant curvature metric on X, regarded as a solution to Hitchin's equations, with the principal embedding of $SL(2,\mathbb{C})$ in $SL(n,\mathbb{C})$. Then $W^0(\mathcal{E}_0, \Phi_0)$ is by definition the Hitchin component, and $W^1(\mathcal{E}_0, \Phi_0)$ is the space of opers. Both are parametrized by the Hitchin base of holomorphic differentials:

$$\bigoplus_{i=2}^{n} H^0(X, K_X^i)$$

In [3], Gaiotto conjectured that a certain rescaling limit of flat connections called the *conformal limit* gives a biholomorphic correspondence between the Hitchin component and the space of classical opers. This conjecture was recently proven in [2].

The goal of this talk is to explain that the conformal limit exists in much more generality and gives a correspondence between strata for every *stable* VHS.

The correspondence itself may be viewed as arising from a generalization of the Hitchin section. Namely, we identify a subcomplex of the deformation complex at a VHS, and a corresponding slice in the space of Higgs bundles, which parametrizes $W^0(\mathcal{E}_0, \Phi_0)$. The first cohomology of this subcomplex plays the role of the vector space of holomorphic differentials above. Like the Hitchin component, this parametrization is *global*. Furthermore, as in the case of opers, the same slice parametrizes $W^1(\mathcal{E}_0, \Phi_0)$. We go on to prove that this identification is precisely the conformal limit of Gaiotto. Along the way, we show that $W^0(\mathcal{E}_0, \Phi_0)$ and $W^1(\mathcal{E}_0, \Phi_0)$ are holomorphic Lagrangian submanifolds of M_H and M_{dR}, respectively.

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On type-preserving representations of the thrice punctured projective plane group

Sara Maloni

(joint work with Frederic Palesi, Tian Yang)

This talk is about dynamics on character varieties $\mathfrak{X}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)//G$. We will study the action of the outer automorphism group $\mathrm{Out}(\Gamma)$ on $\mathfrak{X}(\Gamma, G)$ given by $\theta([\rho]) = [\rho \circ \theta^{-1}]$. This question is motivated by the classical example of the action of the mapping class group $\mathcal{MCG}(\Sigma)$ on the Teichmüller space $\mathcal{T}(\Sigma)$ of a closed orientable surface Σ. Fricke proved that $\mathcal{MCG}(\Sigma)$ acts properly discontinuously on $\mathcal{T}(\Sigma)$ (that is, for every compact set K in $\mathcal{T}(\Sigma)$ there is only a finite number of elements $g \in \mathcal{MCG}(\Sigma)$ such that $K \cap g(K) \neq \emptyset$), and Goldman [2] conjectured that $\mathcal{MCG}(\Sigma)$ acts ergodically on all the other components of $\mathfrak{X}(\pi_1(\Sigma), \mathrm{PSL}_2(\mathbb{R}))$. So the geometrical (discrete vs non-discrete) and the dynamical (proper discontinuous vs ergodic action) decomposition of the character variety coincide, at least conjecturally. The conjectural picture is the same for representations $\pi_1(\Sigma) \to \mathrm{PSL}_2(\mathbb{C})$.

On the other hand, if one considers surfaces $\Sigma_{g,b}$ with non empty boundary, then the fundamental group $\pi_1(\Sigma_{g,b})$ is a free group F_n and the mapping class group $\mathcal{MCG}(\Sigma_{g,b})$ is a subgroup of $\mathrm{Out}(F_n)$. While the action of $\mathrm{Out}(F_n)$ on $\mathfrak{X}$ is well-known to be properly discontinuous on the set of discrete, faithful, convex-cocompact (i.e. Schottky) characters, the action on the complement of these characters is more mysterious. Minsky [8] proved that the set of *primitive-stable representations* $\mathfrak{X}_{\mathrm{ps}}$ is an open domain of discontinuity for the action of $\mathrm{Out}(F_n)$ which is strictly larger than the set of discrete, faithful, convex-cocompact (i.e.

Schottky) characters. Hence the geometrical and dynamical decomposition of $\mathfrak{X}(F_n, \mathrm{PSL}_2(\mathbb{C}))$ are different.

Another approach in the study of the character varieties $\mathfrak{X}(F_n, \mathrm{SL}(2,\mathbb{C}))$ was introduced by Bowditch in [1] and later generalized by Tan, Wong and Zhang [9], Maloni, Palesi and Tan [5] and Maloni and Palesi [4]. Bowditch's idea was to use a combinatorial viewpoint using trace functions on simple closed curves. They defined a domain of discontinuity $\mathfrak{X}_Q$, the Bowditch set of representations, which contains the set $\mathfrak{X}_{\mathrm{ps}}$, and hence is also strictly larger than the set of discrete, faithful, convex-cocompact (i.e. Schottky) characters.

In this talk, after discussing some general results about mapping class group actions on character varieties, we will focus on type-preserving representations of a (possibly non-orientable) punctured surface S into $\mathrm{Isom}(\mathbb{H}) = \mathrm{PGL}(2,\mathbb{R})$. A representation $\rho\colon \pi_1(S) \longrightarrow \mathrm{PGL}(2,\mathbb{R})$ is said *type-preserving* if peripheral elements are mapped to parabolic isometries and 1–sided [resp. 2–sided] elements are mapped to orientation reversing [resp. preserving] isometries. In this paper we will work on the space $\mathfrak{X}(S) = \{\text{type-preserving } \rho\colon \pi_1(S) \longrightarrow \mathrm{PGL}(2,\mathbb{R})\}/\mathrm{PSL}(2,\mathbb{R})$ of representation up to conjugation by $\mathrm{PSL}(2,\mathbb{R})$. One can obtain the $\mathrm{PGL}(2,\mathbb{R})$–character variety of S from $\mathfrak{X}(S)$ as a further quotient, which identifies certain connected components.

1.1. **Connected components of the character variety.** For each type-preserving representation ρ, one can define its Euler class $e(\rho)$ as its representation area divided by 2π. It is know that the Euler class satisfies the Milnor-Wood inequality $\chi(\Sigma_{g,n}) \leqslant e(\rho) \leqslant -\chi(S)$, see [7, 10]. For closed surfaces, they proved that the Euler class defines a one-to-one correspondence between the connected components of $\mathfrak{X}(S)$ and the integers e with $|e| \leqslant -\chi(S)$. For a punctured orientable surface $\Sigma_{g,n}$, the number of connected components of $\mathfrak{X}(\Sigma_{g,n})$ is more subtle to describe since for an integer e with $|e| \leqslant -\chi(S)$, the spaces $\mathfrak{X}_e(S)$ of (conjugacy classes of) type-preserving representations of Euler class e can either be empty or non-connected, see [3]. In the orientable case Kashaev [3] conjectured that it should be determined by the Euler class and an extra invariant which corresponds to the $\mathrm{PSL}(2,\mathbb{R})$–conjugacy classes of the holonomy representations of the boundary elements. More precisely, a parabolic element in $\mathrm{PSL}(2,\mathbb{R})$ is, up to $\pm I$, conjugate to an upper-triangular matrix, and its conjugacy class is distinguished by whether the sign of the nonzero off diagonal element is positive or negative. We respectively call the two conjugacy classes of parabolic elements the positive and the negative conjugacy classes. For a type-preserving $\rho\colon \pi_1(S) \to \mathrm{PGL}(2,\mathbb{R})$, we say that the *sign* of a puncture v is *positive* (resp. *negative*), denoted by $s(v) = +1$ (resp. $s(v) = -1$), if ρ sends a peripheral element around this puncture into a positive (resp. negative) conjugacy class of parabolic elements. For $s \in \{\pm 1\}^n$, we denote by $\mathfrak{X}_e^s(S)$ the space of conjugacy classes of type-preserving representations with Euler class e and signs of the punctures s.

Theorem 1. Let $s \in \{\pm 1\}^3$.

(1) $\mathfrak{X}_0^s(N_{1,3})$ is nonempty if and only if s contains exactly one or two $+1$'s.

(2) $\mathfrak{X}^s_{+1}(N_{1,3})$ is nonempty if and only if s contains exactly two or three $+1$'s.
(3) $\mathfrak{X}^s_{-1}(N_{1,3})$ is nonempty if and only if s contains exactly two or three -1's.
(4) All the nonempty spaces above are connected.

As a consequence, $\mathfrak{X}_0(N_{1,3})$ has six connected components, while $\mathfrak{X}_{+1}(N_{1,3})$ and $\mathfrak{X}_{-1}(N_{1,3})$ each have four connected components. Surprisingly, different connected components will have different geometric properties.

1.2. **Hyperbolicity of simple closed curves.** Bowditch [1] asked the following question: *Given a non-elementary type-preserving representation* $\rho\colon \pi_1(S) \longrightarrow \mathrm{PGL}(2,\mathbb{R})$*, is it true that if* ρ *sends every non-peripheral* 2*–sided simple closed curve to an hyperbolic element of* $\mathrm{PSL}(2,\mathbb{R})$*, then* ρ *is Fuchsian?*

Theorem 2.
(1) If $s \in \{\pm 1\}^3$ contains exactly two $+1$'s, then every type-preserving ρ in a full measure subset of $\mathfrak{X}^s_1(N_{1,3})$ and $\mathfrak{X}^{-s}_{-1}(N_{1,3})$ sends every non-peripheral simple closed curve to a hyperbolic element.
(2) Let $s = (+1,+1,+1)$. Then every representation in $\mathfrak{X}^{s+}_1(N_{1,3})$ and in $\mathfrak{X}^s_1(N_{1,3})$ and $\mathfrak{X}^{-s}_{-1}(N_{1,3})$ sends some non-peripheral 2-sided simple closed curve to a non-hyperbolic element.
(3) Every non-elementary type-preserving representation $\rho\colon \Gamma_{1,3} \longrightarrow \mathrm{PGL}(2,\mathbb{R})$ with relative Euler class $e(\rho) = 0$ sends some non-peripheral simple closed curve to a non-hyperbolic element.

In particular, the representations in $\mathfrak{X}^{s+}_1(N_{1,3})$, $\mathfrak{X}^{s-}_{-1}(N_{1,3})$ and $\mathfrak{X}_0(N_{1,3})$ are not Fuchsian, so Theorem 2 gives a negative answer to Bowditch's question for these six components. On all the other components, the answer is affirmative.

1.3. **Ergodicity of the mapping class group action.** The pure (extended) mapping class group $\mathrm{Mod}(N_{k,n})$ naturally acts on $\mathfrak{X}(N_{k,n})$ preserving the Euler class e and the sign of the boundary holonomy s. In the case of closed oriented surfaces Goldman [2] conjectured that this action is ergodic on each non-extremal and non-zero component. Marché and Wolff [6] proved that a positive answer to Bowditch's question implies Goldman conjecture and used this to prove Goldman conjecture for Σ_2. In the case of punctured surfaces, since Bowditch's Conjecture is no longer true for all the connected components, the proof of Goldman's result is more difficult.

Theorem 3. The mapping class group $\mathrm{Mod}(N_{1,3})$ acts ergodically on the connected components $\mathfrak{X}^{s+}_1(N_{1,3})$, and $\mathfrak{X}^{s-}_{-1}(N_{1,3})$ on every connected component of $\mathfrak{X}_0(N_{1,3})$.

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Equivariant minimal surfaces and their Higgs bundles.

Ian McIntosh
(joint work with John Loftin)

To fix notation, N will denote an irreducible noncompact symmetric space and G will denote the connected component of the identity of its group of isometries, with a choice of maximal compact subgroup H so that $N = G/H$. We use Σ to denote a smooth orientable surface of genus $g \geq 2$. Its universal cover (the unit disc model), equipped with the Poincaré metric, will be denoted by $\mathcal{D}$. Given a Fuchsian representation $c : \pi_1\Sigma \to PU(1,1)$, which we identify with a point in the Teichmüller space $\mathcal{T}_\Sigma$ of Σ, we denote by Σ_c the compact Riemann surface $\mathcal{D}/c$. The following definition was introduced in [7].

Definition. An *equivariant minimal surface in N* is the equivalence class $[f, c, \rho]$ of a triple (f, c, ρ) consisting of: (i) a Fuchsian representation c, (ii) an irreducible representation $\rho : \pi_1\Sigma \to G$, (iii) a minimal (i.e., conformal and harmonic) immersion $f : \mathcal{D} \to N$ which intertwines these:

$$f \circ c(\delta) = \rho(\delta) \circ f, \quad \forall \delta \in \pi_1\Sigma.$$

The natural action by conjugacy of $PU(1,1) \times G$ on (c, ρ) extends to an action on triples (f, c, ρ) and the orbit is the equivalence class $[f, c, \rho]$.

The existence theorems of Donaldson [4] and Corlette [3] for equivariant harmonic maps ensure that, since ρ is irreducible, $[f, c, \rho]$ is uniquely determined by the conjugacy classes of c and ρ. Therefore we can embed the set $\mathcal{M}(\Sigma, N)$ of equivariant minimal surfaces into the product of Teichmüller space $\mathcal{T}_\Sigma$ and the character variety $\mathcal{R}(\pi_1\Sigma, G)$,

$$F : \mathcal{M}(\Sigma, N) \to \mathcal{T}_\Sigma \times \mathcal{R}(\pi_1\Sigma, G), \quad [f, c, \rho] \mapsto ([c], [\rho]).$$

We use this embedding to equip $\mathcal{M}(\Sigma, N)$ with a topology, and a smooth structure away from singularites.

By exploiting the non-abelian Hodge correspondence, we have been able to describe some of the structure of this moduli space for the low dimensional cases where N is $\mathbb{RH}^3$, $\mathbb{RH}^4$ or $\mathbb{CH}^2$ [6, 7]. For either $\mathbb{RH}^n$ ($G = SO_0(n,1)$) or $\mathbb{CH}^n$ ($G = PU(n,1)$) the Higgs bundles are well understood (see, e.g., [1, 2]) and can be thought of as a holomorphic rank $n+1$ vector bundle $E = V \oplus 1$, the direct sum of a holomorphic rank n bundle V and the trivial line bundle 1, with Higgs field

$$\Phi \in H^0(K \otimes (\mathrm{Hom}(1,V) \oplus \mathrm{Hom}(V,1))).$$

We write $\Phi = (\Phi_1, \Phi_2)$ to denote the two components. The case $\mathbb{RH}^n$ requires additionally that V have an orthogonal structure with respect to which Φ_2 is the adjoint of Φ_1. Each stable G-Higgs bundle produces an equivariant harmonic map $f : \mathcal{D} \to N$ whose differential $\partial f : T^{(1,0)}\mathcal{D} \to T^{\mathbb{C}}N$ is essentially Φ, and thus f is an immersion when Φ is nowhere vanishing, and conformal (hence minimal) when $\mathrm{tr}(\Phi^2) = 0$. There are now two cases to consider.

Case A. When neither Φ_1 nor Φ_2 are identically zero the Higgs field determines (and is determined by) two holomorphic subbundles $\mathrm{im}(\Phi_1), \ker(\Phi_2) \subset V$ and the maps

$$0 \to K^{-1}(D_1) \xrightarrow{\Phi_1} V, \quad V \xrightarrow{\Phi_2} K(-D_2) \to 0,$$

where D_1, D_2 are the divisors of zeroes of Φ_1, Φ_2 respectively. To obtain an immersion we require $D_1 \cap D_2 = \emptyset$, otherwise the map is branched at the intersection points. For $\mathbb{RH}^n$ the orthogonal symmetry means $D_1 = D_2$ and thus both must be empty to obtain an immersion. The key now is to understand the holomorphic structure of V as a perturbation of the smooth decomposition

$$V = K^{-1}(D_1) \oplus W \oplus K(-D_2), \qquad W = \ker(\Phi_2)/\,\mathrm{im}(\Phi_1).$$

Case B. Either $\Phi_1 = 0$ (f is anti-holomorphic) or $\Phi_2 = 0$ (f is holomorphic). This can only happen for $\mathbb{CH}^n$, and without loss of generality we need only consider when $\Phi_2 = 0$, since complex conjugation of $\mathbb{CH}^n$ swaps between the two possibilities.

When the rank of V is low enough we have succeeded in analysing these cases further. This report will concentrate on the $\mathbb{CH}^2$ case, $G = PU(2,1)$ [6].

First recall that reductive representations ρ into $PU(2,1)$ come equipped with an invariant which indexes the connected components of the character variety (equally, the Higgs bundle moduli space). This is the Toledo invariant, $\tau(\rho)$ and it equals $-\frac{2}{3}\deg(V)$. It is bounded by $|\tau(\rho)| \le 2(g-1)$. One knows from Toledo [8] that $\tau(\rho) = 2(g-1)$ (resp. $\tau = -2(g-1)$) corresponds precisely to totally geodesic holomorphic (resp. anti-holomorphic) embeddings. Each corresponding representation is reducible, the product of a Fuchsian representation into $PU(1,1)$ and a representation into S^1. The definition above therefore excludes these maximal cases from $\mathcal{M}(\Sigma, \mathbb{CH}^2)$.

For Case A the pair (E, Φ) is completely determined by the two effective divisors D_1, D_2 and an extension class

$$\xi \in H^1(\mathrm{Hom}(K(-D_2), K^{-1}(D_1))) = H^1(K^{-2}(D_1 + D_2)),$$

which we can relate to the differential geometric invariants of the minimal immersion, namely, the induced metrics on $K^{-1}(-D_1)$ and $K^{-1}(-D_2)$ and a cubic holomorphic differential coming from the second fundamental form.

We showed that the Higgs bundle over Σ_c with data (D_1, D_2, ξ) is polystable if and only if the degrees d_1, d_2 of D_1, D_2 satisfy

$$2d_1 + d_2 < 6(g-1), \qquad d_1 + 2d_2 < 6(g-1).$$

Together with the conformal structure c on Σ, the data (c, D_1, D_2, ξ) gives $8g-8$ complex parameters regardless of the degrees d_1, d_2. We denote the set of this data for fixed degrees (d_1, d_2) by $\mathcal{V}(d_1, d_2)$.

Theorem A ([6]). The set $\mathcal{V}(d_1, d_2)$ has the natural structure of a non-singular complex manifold of complex dimension $8g-8$, which is a complex analytic family over $\mathcal{T}_\Sigma$. It parametrizes the open set in $\mathcal{M}(\Sigma, \mathbb{CH}^2)$ of all equivariant minimal immersions which are neither holomorphic nor anti-holomorphic, have $\tau(\rho) = \frac{2}{3}(d_2 - d_1)$ and whose normal bundle $T\Sigma^\perp$ has Euler number $2(g-1) - d_1 - d_2$.

The family $\mathcal{V}(0,0)$ consists of all minimal Lagrangian immersions. In $\mathcal{V}(0,0)$ we identified a locus of minimal Lagrangian embeddings which we call *almost Fuchsian*: each is properly embedded and ρ is quasi-Fuchsian (equally, convex cocompact) with $\mathbb{CH}^2/\rho \simeq T\Sigma^\perp$. Moreover, the almost Fuchsian embedding is the unique equivariant minimal immersion admitted by ρ. At present virtually nothing is known about the representations corresponding to points in $\mathcal{M}(\Sigma, \mathbb{CH}^2)$ outside this case, and this is an area of study which deserves serious attention.

For Case B each equivariant holomorphic (unbranched) immersion $[f, c, \rho]$ with irreducible ρ is uniquely determined by data (c, L, η) where L is a holomorphic line of degree l satisfying $0 < l < 3(g-1)$, and $\eta \in H^1(\Sigma_c, K^{-2}L)$ determines V and $\Phi = (\Phi_1, 0)$ through the extension

$$0 \to K^{-1} \stackrel{\Phi_1}{\to} V \to KL^{-1} \to 0.$$

Theorem B ([6]). The parameter space $\mathcal{W}(l)$ of data (c, L, η) is a complex manifold of dimension $9(g-1) - l$ and a complex analytic family over $\mathcal{T}_\Sigma$. It parametrizes all holomorphic minimal immersions which are equivariant with respect to an irreducible representation with Toledo invariant $\tau(\rho) = \frac{2}{3}l$.

Remarks. The definition of $\mathcal{M}(\Sigma, N)$ given above post-dates [6]: in [6] we included branched minimal surfaces in $\mathcal{M}(\Sigma, \mathbb{CH}^2)$. I now believe that excluding these makes each $\mathcal{V}(d_1, d_2)$ a connected component, and the branched immersions appear along common boundaries. How these fit together is an interesting question, intimately related to understanding the limit points of the $\mathbb{C}^\times$-action $t \cdot (E, \Phi) = (E, t\Phi)$. I believe I can show that such limit points always possess branch points (except where in the limit $\Phi = 0$, when the map has collapsed to a constant map).

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Higgs bundles and quantum enveloping algebras

Ben Davison

Let C be a projective genus g curve over $\mathbb{C}$, and let $\mathcal{H}iggs^{\mathrm{ss}}_{r,d}(C)$ denote the stack of rank r degree d semistable Higgs bundles. To start with, assume that r, d are coprime, then although $\mathcal{H}iggs^{\mathrm{ss}}_{r,d}(C)$ really is a stack (not a scheme), it is a stack in a pretty mild way — if we denote by $Higgs^{\mathrm{ss}}_{r,d}(C)$ the fine moduli scheme of semistable Higgs bundles, $\mathcal{H}iggs^{\mathrm{ss}}_{r,d}(C)$ is the stack-theoretic quotient $Higgs^{\mathrm{ss}}_{r,d}(C)/\mathbb{C}^*$ by the trivial action. We will mostly be interested in cohomology, and this fact translates to the following isomorphism

$$\mathrm{H}(\mathcal{H}iggs^{\mathrm{ss}}_{r,d}(C),\mathbb{Q}) \cong \mathrm{H}(Higgs^{\mathrm{ss}}_{r,d}(C),\mathbb{Q}) \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)$$

where $\mathrm{H}(\mathrm{pt}/\mathbb{C}^*) \cong \mathrm{H}_{\mathbb{C}^*}(\mathrm{pt}) \cong \mathbb{Q}[z]$. Say d' also satisfies $(r,d') = 1$. Let ζ_r be a primitive rth root of unity. Let $\mathcal{R}ep_r(\pi_1(C'))^{\zeta_r^e}$ be the moduli stack of representations of the fundamental group of the punctured surface C' such that the monodromy around the puncture is given by multiplication by ζ_r^e. Then we have the following square

$$\begin{array}{ccc}
\mathrm{H}(\mathcal{H}iggs^{\mathrm{ss}}_{r,d}(C),\mathbb{Q}) & \xrightarrow[nAHT]{\cong} & \mathrm{H}(\mathcal{R}ep_r(\pi_1(C'))^{\zeta_r^d},\mathbb{Q}) \\
\downarrow & & \downarrow{\scriptstyle \bullet\cong\bullet_{\mathbb{C}\sigma}} \\
\mathrm{H}(\mathcal{H}iggs^{\mathrm{ss}}_{r,d'}(C),\mathbb{Q}) & \xrightarrow[nAHT]{\cong} & \mathrm{H}(\mathcal{R}ep_r(\pi_1(C'))^{\zeta_r^{d'}},\mathbb{Q}).
\end{array}$$

The horizontal arrows are isomorphisms by the nonAbelian Hodge correspondence due to Corlette, Donaldson and Hitchin. The rightmost vertical arrow is an isomorphism after tensoring by $\mathbb{C}$ — this isomorphism comes from the deRham theorem, and the fact that the two stacks are Galois conjugate via some $\sigma \in \mathrm{Gal}(\mathbb{C}/\mathbb{Q})$. Finally the leftmost arrow is some mysterious isomorphism: we know it exists because, from the rest of the diagram we know that these two cohomologically graded vector spaces have the same graded dimensions. But we have said nothing so far that might help to explicitly construct it.

The real challenge is to understand this same phenomenon in the case $(r, d) \neq 1$. For this we need a bit more notation. Recall that *Exp* is the unique operation on formal power series making the following diagram commute

$$\begin{array}{ccc} K_0(D^b(\mathrm{Vect}_{\mathbb{N}_{>0}\oplus\mathbb{Z}})) & \xrightarrow{K_0(Sym)} & K_0(D^b(\mathrm{Vect}_{\mathbb{N}\oplus\mathbb{Z}})) \\ \downarrow{\chi_{q,t}} & & \downarrow{\chi_{q,t}} \\ t\mathbb{Z}((q^{1/2}))[[t]] & \xrightarrow{Exp} & \mathbb{Z}((q^{1/2}))[[t]] \end{array}$$

where $\chi_{q,t}(V) := \sum_{i,j} \dim(V_{i,j}) t^i q^{j/2}$ and the superscript "b" means: whatever restriction we have to make in order to make the vertical maps well defined and isomorphisms. More concretely: $Exp(\sum a_{i,j} q^{j/2} t^i) = \prod_{i,j}(1 - q^{j/2}t^i)^{-a_{i,j}}$. The slogan is: "plethystic exponential is the decategorification of passing from a graded vector space to the free supercommutative algebra generated by it".

Set $\mathcal{H}_{C,\tau} := \bigoplus_{d/r=\tau} \mathrm{H}_c(\mathcal{H}iggs^{\mathrm{ss}}_{r,d}(C), \mathbb{Q}) \otimes \mathbb{L}^{(1-g)r^2})^*$. Then Schiffmann conjectured [6], and Mellit proved [5], that if the rank-degree r piece of $\mathcal{H}_{C,\tau}$ is nonzero, then the t^r coefficient of $Log(\chi_{q,t}(\mathcal{H}_{C,\tau}))$ doesn't depend on τ. If we unravel this, it recovers the fact that for $(r, d) = (r, d') = 1$ the Poincaré polynomial of the Higgs moduli spaces are the same.

There are two mysteries here:

(1) Why does the gigantic bigraded vector space $\mathcal{H}_{C,\tau}$ seem to be isomorphic to the symmetric algebra generated by something reasonably "small" (via the above slogan regarding *Exp*), namely something of the form $\bigoplus_{r|r/d\in\mathbb{N}} \mathcal{A}_{C,r,d} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)$ for each $\mathcal{A}_{C,r,d}$ finite-dimensional?

(2) Assuming there is such a presentation, why doesn't $\mathcal{A}_{C,r,d}$ seem to depend on d?

The reason for the word "seem" in both questions is that the preceeding discussion only justifies the "decategorified" versions of these speculations — i.e. it was a discussion about generating series and plethystic exponentials, not cohomology and symmetric algebras.

To fast forward to the punchline: the first mystery is partly explained by the fact that $\mathcal{H}_{C,\tau}$ can be made into a universal enveloping algebra (then we can apply the PBW theorem). If we could solve the second mystery, that would finish the solution of the first, as it would take care of finite-dimensionality. But the second mystery remains a bit of a mystery, with the rest of the talk providing a hint for a way out, via an analogous geometric situation.

Via the BNR correspondence, the study of Higgs bundles becomes the study of sheaves on the CY2 surface $Tot(\omega_C)$. For the rest of the talk we take instead a CY2 surface that is more amenable to *noncommutative geometry* and thus to *cohomological DT theory*. Let Q' be an ADE quiver, i.e. an orientation of an ADE graph Γ, and let Q be its affine version. Then Q' corresponds to a Kleinian subgroup $G_\Gamma \subset \mathrm{SL}_2(\mathbb{C})$ and we let $X \to \mathbb{C}^2/G_\Gamma$ be the minimal resolution of the quotient singularity. The exceptional locus of this resolution is a tree of $\mathbb{P}^1$s, with incidence graph given by Γ.

Rather than study the cohomology of stacks of coherent sheaves on X directly, we study representations of $\Pi_Q = \mathbb{C}\overline{Q}/\langle\sum_{a\in Q_1}[a,a^*]\rangle$, which is a related thing to do, via the derived equivalence

$$D^b(\mathrm{Coh}(X)) \xrightarrow{\sim} D^b(\mathrm{mod}{-}\Pi_Q).$$

Actually, for the purposes of cohomological DT theory, studying the cohomology of such stacks is the wrong thing to do for a couple of reasons. Firstly, we should study sheaves on a 3-fold, not a 2-fold, and secondly we should study vanishing cycle cohomology. We resolve both of these issues at the same time: Let $\tilde{Q}$ be the quiver obtained from $\overline{Q}$ by adding a loop ω_i at i for every vertex $i \in Q_0$, and let $\tilde{W} = \sum_{i\in Q_0}\omega_i\sum_{a\in Q_1}[a,a^*]$, and let $Jac(\tilde{Q},\tilde{W})$ be the Jacobi algebra obtained by taking the quotient of $\mathbb{C}\tilde{Q}$ by all of the noncommutative derivatives of $\tilde{W}$. Then once you know what all these terms mean (see [2]) it is easy to check that there is an isomorphism $B := Jac(\tilde{Q},\tilde{W}) \cong \Pi_Q[\omega]$, the polynomial algebra in one variable with coefficients in Π_Q. In terms of the above derived equivalence, this amounts to considering $X \times \mathbb{A}^1$, obtaining a 3-fold. Furthermore, the stack of all finite-dimensional B-modules carries the vanishing cycles (perverse) sheaf $\phi_{Tr(\tilde{W})}$, and finally by [1] there is (up to a Tate twist) an isomorphism

$$\mathrm{H}(Rep_\gamma(B), \phi_{Tr(\tilde{W})}) \cong \mathrm{H}_c(Rep_\gamma(\Pi_Q), \mathbb{Q})^*$$

so that the cohomological DT theory of this 3d setup really does capture the cohomology we are interested in studying.

One extra structure that the passage to 3d buys us is a *factorization structure*. Put pithily, it is possible to show that the direct image of $\phi_{Tr(W)}$ along the map to $Sym(\mathbb{A}^1)$ taking a representation to the union (with multiplicities) of the eigenvalues of the operator $\cdot\omega$ is a factorization sheaf, and moreover an algebra in the category of such sheaves via a relative version of the Kontsevich–Soibelman critical CoHA construction [3]. Formally, it has a cocommutative coproduct, which *by itself* is enough to partially solve the first mystery, since it implies that this algebra is a universal enveloping algebra of some $\mathbb{N}^{Q_0}$-graded Lie algebra $\mathfrak{g}_{\tilde{Q},\tilde{W}} \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)$. The "quantum" in the title comes from the fact that there is an extra $\mathbb{C}^*$-parameter that scales all of the eigenvalues, inducing a non-cocommutative deformation.

Exactly the same trick applies in the case of semistable Higgs bundles of fixed slope, where instead one studies the "3d-ification" of the Hall algebra of [7].

For the analogue of "mystery 2" we need some setup. Firstly, let δ be the imaginary simple root of Q. Under the derived equivalence, this dimension vector corresponds to the numerical class of $[\mathcal{O}_x]$ for $x \in X \times \mathbb{A}^1$. Then it can be shown that $\mathfrak{g}_{\tilde{Q},\tilde{W},\delta} \cong \mathrm{H}(X) \otimes \mathrm{H}(\mathrm{pt}/\mathbb{C}^*)$, and in particular there is an element $1 \otimes z \in \mathfrak{g}_{\tilde{Q},\tilde{W},\delta}$. Then it turns out (extending old work of Nakajima and Lehn (see [4] for an overview)) that $[1\otimes z, \bullet] : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\delta}$ is an isomorphism. The analogue of this statement in terms of Higgs bundles would be exactly what we are after: that there is a (canonical) isomorphism $\mathcal{A}_{C,r,d} \cong \mathcal{A}_{C,r,d+1}$ obtained from the CoHA.

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Existence of good moduli spaces for algebraic stacks and applications

Jochen Heinloth

(joint work with Jarod Alper, Daniel Halpern-Leistner)

The aim of the joint work [2] is to give criteria for the existence of separated and proper good moduli spaces that only depend on the geometry of the moduli problem itself. Let me try to explain why this is useful. The problem to construct moduli spaces in algebraic geometry has often been solved using methods of geometric invariant theory (GIT), a very effective method. However, to use this method one usually has to pass to an auxiliary space that on the one hand does not contain all objects of the original problem and on the other hand often adds an additional boundary parameterizing objects that did not appear in the original question and sometimes lack a nice modular description. This often makes it difficult to compare stability notions formulated in terms of a moduli problem to stability notions appearing in GIT. In some sense, it is surprising that these turn out to agree in a long list of known constructions. Moreover, when starting to work on the problem we had encountered some moduli problems for which convenient GIT candidates seemed to be lacking and for these we wanted to find a way to avoid the introduction of auxiliary spaces.

As the workshop brought together researchers from different backgrounds I would like to describe the notions used in algebraic geometry to describe the geometry of moduli problems informally, before stating our the main theorem.

In general a moduli problem $\mathcal{M}$ comes as the question of identifying a certain class of objects as the points of a space, which depending on the setup could be a topological space, a manifold, a variety or a scheme. To equip the set of objects with a geometric structure one usually specifies the notion of a family of objects parametrized by a space T. A moduli space should then be a universal parameter space for the objects, i.e., giving a map from $T \to \mathcal{M}$ should be the same as specifying a family of objects parameterized by T.

The notion of a moduli stack takes this condition as a definition of $\mathcal{M}$, using the idea that to know a space $\mathcal{M}$ you only need to know what maps into it are. The precise definition is then made in such a way, that also automorphisms of families are recorded in the data, as these are essential to glue locally defined families to globally defined families.

A toy example of such a construction are quotients by group actions. If G is a group acting on say a manifold or a scheme X, sometimes the quotient X/G may be hard to construct in the category. However, for very well behaved actions the map $X \to X/G$ even turns X into a G-bundle over X. This means that for any map $T \to X/G$ the pull-back of this G-bundle is a G-bundle $P \to T$ that comes equipped with an equivariant map $P \to X$ and the data of the bundle $P \to T$ together with the equivariant map $P \to X$ uniquely determines the map $T \to X/G$. This means that whether or not a nice quotient exists we can simply define a stack $[X/G]$ by requiring that maps into this object are given by such a datum.

The notion of algebraic stacks specifies a class of stacks that at least locally look like a space. More precisely, algebraic stacks are characterized by the additional property that for any fixed object F there exists a versal family, i.e., a model family such that for every family containing F there exists a neighborhood of F that appears in the model family.

A final condition that is often used is to assume that a stack has affine diagonal, which is a condition on the automorphism groups of objects, i.e. one requires that these should be subgroups of linear groups.

Let me now state our main existence result which contains a few additional notions that will be discussed afterwards. In case the automorphism groups of objects happen to be finite, this result was known before by the Keel-Mori theorem [4].

Theorem. [2, Theorem A] *Let $\mathcal{M}$ be an algebraic stack locally of finite type with affine diagonal over a quasi-separated and locally noetherian algebraic space. Then $\mathcal{M}$ admits a good moduli space if and only if*

(1) $\mathcal{M}$ is locally linearly reductive
(2) $\mathcal{M}$ is Θ-reductive
(3) $\mathcal{M}$ has unpunctured inertia

The good moduli space is separated if and only if $\mathcal{M}$ is S-complete, and proper if and only if $\mathcal{M}$ is S-complete and satisfies the existence part of the valuative criterion for properness.

Assume in addition that $\mathcal{M}$ is defined over an algebraic space of characteristic 0 *and $\mathcal{M}$ is quasi-compact. If $\mathcal{M}$ is S-complete, then $\mathcal{M}$ admits a separated good moduli space if and only if $\mathcal{M}$ is Θ-reductive and S-complete.*

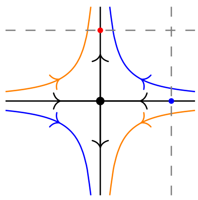
In addition to this result we also prove that if $\mathcal{M}$ arises as the semistable subspace of a larger algebraic stack equipped with a nice Θ-stratification, then it suffices to prove any of the conditions S-completeness, Θ-reductivity or the existence part of the valuative criterion for the larger stack. This gives a convenient method to prove semistable reduction theorems.

Let us briefly sketch the meaning of the additional conditions appearing above.

First, Θ refers to the quotient obtained by dividing the affine line by the action of the multiplicative group $[\mathbb{A}^1/\mathbb{G}_m]$. The symbol Θ was chosen as it resembles the standard picture of the two orbits of this action. It follows from the Rees-construction that giving a map from this stack into a moduli problem can usually be interpreted as choosing a filtration on an object and degenerating this filtration into an associated graded object.

A stack is called Θ-reductive if for any family parameterized by a disc, together with a filtration over the punctured disc, the filtration extends canonically over the puncture.

Similarly S-completeness is a condition that allows to compare opposite filtrations. For this one considers a local version of the quotient of the plane divided by the anti-diagonal action of the multiplicative group $\overline{\mathrm{ST}}_{\mathbb{A}^1} := [\mathbb{A}^2/\mathbb{G}_m, (z, z^{-1})]$:



The condition to be S-complete is then that families defined over the punctured quotient in which the origin has been removed, extend over the origin, which as in the case of Θ-reductivity is a condition on the existence of a canonical extension over a subset of codimension 2. Implicitly this condition already appears in GIT. Formulated as above, this notion allows to give a rather short proof of Cartan-Iwahori-Matsumoto decomposition in reductive groups [3].

Finally, the notion of unpunctured inertia is a condition on the specialization behavior of connected components of automorphism groups. In particular this property is automatically satisfied if all automorphism groups happen to be connected.

One of the key ingredients that allowed us to prove our result is a recent local structure theorem for algebraic stacks from [1], which as a starting point for the construction gives local descriptions of algebraic stacks as quotient stacks. For these quotient stacks good moduli spaces are available and it turns out that the conditions of the theorem allow to use these as local charts for the good moduli space of $\mathcal{M}$ that we want to construct.

In the article we also describe how these conditions can be checked for various moduli problems, such as moduli spaces of coherent sheaves, or Bridgeland semistable objects in suitable abelian categories. It also allows us to prove that analogs of Hitchin's morphism for moduli spaces of Higgs bundles with poles over curves are proper.

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Stratifications and coarse moduli spaces for the stack of Higgs bundles

Eloise Hamilton

Given a compact Riemann surface Σ of genus g and a line bundle $L \to \Sigma$ of positive degree, an L-twisted Higgs bundle on Σ is a pair $(E, \phi : E \to E \otimes L)$ where $E \to \Sigma$ is a holomorphic vector bundle. The moduli stack of L-twisted Higgs bundles of coprime rank r and degree d on Σ (called Higgs bundles from here on) is denoted by $\mathscr{H}_{r,d}(\Sigma, L)$, and the open stratum consisting of semistable Higgs bundles by $\mathscr{H}^{ss}_{r,d}(\Sigma, L)$. A moduli space for semistable Higgs bundles can be constructed using Geometric Invariant Theory (GIT) [7]. Denoted $\mathcal{M}^{ss}_{r,d}(\Sigma, L)$, it is a quasi-projective variety which is a coarse moduli space for the semistable stratum $\mathscr{H}^{ss}_{r,d}(\Sigma, L)$ and it is widely studied thanks to its rich geometric structure.

The question we aim to answer is whether there are other strata inside the stack which can similarly be identified with varieties, and if so whether these varieties admit a similarly rich structure.

1. Stratifications on the stack of Higgs bundles

Higgs Harder-Narasimhan (HHN) stratification. A Higgs bundle has a *HHN type*, which is a vector recording the slopes and ranks of the destabilising subbundles appearing in its HHN filtration. Denoting by $\mathscr{H}^{\mu}_{r,d}(\Sigma, L)$ the substack of Higgs bundles of HHN type μ and by μ^{ss} the HHN type associated to a semistable Higgs bundle, the *HHN stratification* of $\mathscr{H}_{r,d}(\Sigma, L)$ is:

$$\mathscr{H}_{r,d}(\Sigma, L) = \mathscr{H}^{ss}_{r,d}(\Sigma, L) \sqcup \bigsqcup_{\mu \neq \mu^{ss}} \mathscr{H}^{\mu}_{r,d}(\Sigma, L).$$

Harder-Narasimhan (HN) stratification. The stack of Higgs bundles can also be stratified according to the instability type of its underlying vector bundle, giving its *HN stratification*:

$$\mathscr{H}_{r,d}(\Sigma, L) = F^{-1}(\mathscr{V}^{ss}_{r,d}(\Sigma)) \sqcup \bigsqcup_{\tau \neq \tau^{ss}} F^{-1}(\mathscr{V}^{\tau}_{r,d}(\Sigma)),$$

where $\mathscr{V}_{r,d}(\Sigma)$ denotes the stack of vector bundles of rank r and degree d on Σ, $\mathscr{V}^{ss}_{r,d}(\Sigma)$ the substack of semistable vector bundles, $\mathscr{V}^{\tau}_{r,d}(\Sigma)$ that of vector bundles of Harder-Narasimhan type τ, and $F : \mathscr{H}_{r,d}(\Sigma, L) \to \mathscr{V}_{r,d}(\Sigma)$ the forgetful map. The HN stratification of the semistable stratum has been studied in [4].

Guiding questions. The first question we ask is the following: can coarse moduli spaces be constructed for the HHN and HN strata, as for the semistable stratum? Since an unstable Higgs bundle limits to its HHN graded, any coarse moduli space for a HHN stratum would have to identify a Higgs bundle with its HHN graded, to which it need not be isomorphic. Thus, in contrast to the semistable stratum, further stratification is needed to obtain coarse moduli spaces. This can be achieved using Non-Reductive GIT, and doing so raises two follow-on questions:

1) How do the refined HHN and HN stratifications compare? In particular, does the refined HN stratification take into account the HHN type, and vice versa?
2) Do the coarse moduli spaces for the refined strata admit a similarly rich structure to the moduli space of semistable Higgs bundles? In particular, which properties of the $\mathbb{C}^*$-action, the Hitchin fibration and the forgetful map carry over to the new coarse moduli spaces?

2. Refinements and coarse moduli spaces using Non-Reductive GIT

The semistable case. The GIT construction of the moduli space of semistable Higgs bundles proceeds as follows [7]. For $d \gg 1$, the semistable stratum $\mathscr{H}^{ss}_{r,d}(\Sigma, L)$ can be identified as a quotient stack for the action of a reductive group $G_{r,d}$ on a quasi-projective variety $F^{ss}_{r,d}$:

$$\mathscr{H}^{ss}_{2,d}(\Sigma, L) \cong [F^{ss}_{r,d}/G_{r,d}].$$

Moreover, and again for $d \gg 1$, the variety $F^{ss}_{r,d}$ can be embedded $G_{r,d}$-equivariantly into a projective variety $X_{r,d}$ admitting a linearised $G_{r,d}$-action, in such a way that the image is contained in the GIT-semistable locus $X^{ss}_{r,d}$. The moduli space $\mathcal{M}^{ss}_{r,d}(\Sigma, L)$ of semistable Higgs bundles is then defined as the pull-back of the GIT quotient $X_{r,d}/\!/G_{r,d}$ under the embedding $F^{ss}_{r,d} \hookrightarrow X^{ss}_{r,d}$.

The unstable case. We now fix a HHN type μ. As is the case for $\mathscr{H}^{ss}_{r,d}(\Sigma, L)$, we can show that $\mathscr{H}^{\mu}_{r,d}(\Sigma, L)$ can be identified as a quotient stack for $d \gg 1$:

$$\mathscr{H}^{\mu}_{r,d}(\Sigma, L) \cong [F^{\mu}_{r,d}/G_{r,d}],$$

where $F^{\mu}_{r,d}$ is a quasi-projective variety parametrising Higgs bundles of HHN type μ. For $d \gg 1$, the variety $F^{\mu}_{r,d}$ can also be embedded $G_{r,d}$-equivariantly inside $X_{r,d}$. The projective variety $X_{r,d}$ admits a GIT-instability stratification and we can show that there exists a correspondence between HHN types μ and GIT-instability types β so that $F^{\mu}_{r,d} \hookrightarrow X^{\beta(\mu)}_{r,d} \subseteq X_{r,d}$ where $X^{\beta(\mu)}_{r,d}$ denotes a GIT-unstable stratum for the linearised action of $G_{r,d}$ on $X_{r,d}$.

Non-Reductive GIT can be used to construct a geometric $G_{r,d}$-quotient for an open subset of $X^{\beta(\mu)}_{r,d}$. Indeed, the action of $G_{r,d}$ on $X^{\beta(\mu)}_{r,d}$ can be reduced to the action of a parabolic subgroup $P_{\beta(\mu)} \subseteq G_{r,d}$ on a quasi-projective subvariety $Y^{ss}_{\beta(\mu)} \subseteq X^{\beta(\mu)}_{r,d}$ [6]. The main theorem of Non-Reductive GIT applies to actions of such parabolic subgroups and can be used to construct an open subset of $Y^{ss}_{\beta(\mu)}$ admitting a quasi-projective geometric $P_{\beta(\mu)}$-quotient [1]. Pulling back along the inclusion $F^{\mu}_{r,d} \hookrightarrow X^{\beta(\mu)}_{r,d}$ yields an open subset $F^{\mu,\widehat{s}}_{r,d} \subseteq F^{\mu}_{r,d}$ which has a quasi-projective geometric $G_{r,d}$-quotient. It is a coarse moduli space for an open stratum $\mathscr{H}^{\mu,\widehat{s}}_{r,d}(\Sigma, L)$ of $\mathscr{H}^{\mu}_{r,d}(\Sigma, L)$.

By induction, it is possible to stratify $\mathscr{H}^{\mu}_{r,d}(\Sigma, L)$ in such a way that each stratum admits a coarse moduli space [2]. Non-Reductive GIT can similarly be applied to refine the HN stratification and obtain coarse moduli spaces for each refined stratum. These refined stratifications can be described explicitly in rank 2.

3. The rank 2 case

The refined stratifications. Given $\delta \in \mathbb{N}$, we let $\mathscr{H}^{\mu,\delta,\mathrm{indec}}_{2,d}(\Sigma, L) \subseteq \mathscr{H}^{\mu}_{2,d}(\Sigma, L)$ denote the substack of (E, ϕ) such that $(E, \phi) \ncong \mathrm{gr}(E, \phi) = (E_1, \phi_1) \oplus (E_2, \phi_2)$ and $\dim \mathrm{Hom}((E_2, \phi_2), (E_1, \phi_1)) = \delta$. We let $\mathscr{H}^{\mu,\mathrm{dec}}_{2,d}(\Sigma, L) \subseteq \mathscr{H}^{\mu}_{2,d}(\Sigma, L)$ denote the substack of $(E, \phi) \cong \mathrm{gr}(E, \phi)$. These substacks admit quasi-projective coarse moduli spaces, denoted by $\mathcal{M}^{\mu,\delta,\mathrm{indec}}_{2,d}(\Sigma, L)$ and $\mathcal{M}^{\mu,\mathrm{dec}}_{2,d}(\Sigma, L)$ respectively (this is an extension of a corresponding result for vector bundles appearing in [5]). The refined HHN stratification provided by Non-Reductive GIT is therefore given by:

$$\mathscr{H}_{2,d}(\Sigma, L) = \mathscr{H}^{ss}_{2,d}(\Sigma, L) \sqcup \bigsqcup_{\mu \neq \mu^{ss}} \bigsqcup_{\delta \in \mathbb{N}} \mathscr{H}^{\mu,\delta,\mathrm{indec}}_{2,d}(\Sigma, L) \sqcup \mathscr{H}^{\mu,\mathrm{dec}}_{2,d}(\Sigma, L).$$

Moreover, we can show that the refined HN stratification prescribed by Non-Reductive GIT is the intersection of the HN stratification with the HHN stratification. The refined stratifications and their relationship in the higher rank case are more complicated and describing them remains work in progress.

When $\Sigma = \mathbb{P}^1$. By considering L-twisted Higgs bundles on $\mathbb{P}^1$, it is possible to explicitly describe the coarse moduli spaces $\mathcal{M}^{\mu,\delta,\mathrm{indec}}_{2,d}(\mathbb{P}^1, L)$ and $\mathcal{M}^{\mu,\mathrm{dec}}_{2,d}(\mathbb{P}^1, L)$ for the HHN strata, using methods similar to those used in [8].

General Σ. We can show that for $\delta \neq 0$, the $\mathbb{C}^*$-action is circle-compact as in the semistable case, but differs in that it is not semiprojective. This is because a component of the fixed point locus can be identified with a moduli space of unstable vector bundles, constructed in [3] and more generally in [5], which is only quasi-projective. The $\mathbb{C}^*$-action can nevertheless still be used to compute suitable motivic topological invariants of $\mathcal{M}^{\mu,\delta,\mathrm{indec}}_{2,d}(\Sigma, L)$.

For $\delta = 0$, the $\mathbb{C}^*$-action is not even circle-compact, which raises the question of whether a boundary involving the moduli spaces for $\delta \neq 0$ can be constructed so that circle-compactness of the $\mathbb{C}^*$-action is preserved.

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Branes, moduli spaces, and quantization

Ingmar Saberi
(joint work with Sergei Gukov, Peter Koroteev, Satoshi Nawata, Du Pei)

Let me open with a brief expression of gratitude, both to the organizers of the Oberwolfach workshop for the opportunity to speak and for their hard work organizing such a pleasant meeting, and to the coauthors of the work in progress which I will discuss here. Any merit that these ideas may have is to be credited fully to them; any shortcoming or misunderstanding is certainly my own contribution.

Quantization has long been a source of interesting new interactions between mathematics and theoretical physics. With the advent of quantum mechanics, the study of linear unitary representations of groups found its place in physics [1, 2]; later, in the guise of the orbit method in geometric representation theory, the idea that unitary G-representations are inherently quantum-mechanical objects, and ought to correspond to G-invariant classical dynamics, was developed further in the mathematics literature. For discussion of the orbit method, as well as further references to the literature, we refer to [5].

Despite its successes and its—at least from a physical perspective—conceptual elegance, the orbit method has obvious drawbacks: it proposes an equivalence between two mathematical structures (coadjoint orbits and irreducible unitary representations) that "should" be related, but, in many cases, do not in fact precisely line up. No obvious construction or functor allows one to pass between the two types of objects; the difficulties that arise are analogous to those that appear in giving a general or functorial treatment of quantization, which no one has done, likely for good reasons. The method thus still has the status of a "damaged treasure map,"[1] rather than a complete or fully-developed theory. Furthermore, as is often the case in physics, the topic is strongly influenced by a list of important examples where the technique works relatively well—more so than is perhaps typical in mathematics.

In the physics literature, the orbit method is most closely related to *geometric quantization*, which attempts to construct the Hilbert space quantizing a symplectic manifold P as, speaking very roughly, a space of certain sections of a "prequantum" line bundle over P, equipped with a connection of curvature ω. There are other approaches to quantization in the physics literature, with other advantages and drawbacks; one notable approach is *deformation quantization*, which attempts to produce a deformation of the (commutative) Poisson algebra of classical observables—something like $A_0 = C^\infty(P)$—through a formal one-parameter

[1] The description is due to David Vogan, Jr.

family of noncommutative algebras $A_\hbar$, with noncommutativity at leading $\hbar$-order determined by the Poisson bracket. Such a formal deformation is given by the Groenewold–Moyal–Weyl star product [6], later extended to general Poisson manifolds by Kontsevich [7]; while this process is clear-cut and algorithmic in precisely the way the orbit method is not, it does not naturally produce any Hilbert space on which $A_\hbar$ acts. Moreover, the formal deformation is not guaranteed to arise from the series expansion of an actual one-parameter family.

An attempt to unite these two approaches, at least in certain examples, was given in the *brane quantization* proposal of Gukov and Witten [3]. (For related work in the mathematics literature, as well as discussion of quantization in the context of the orbit method, see [4].) The central idea of this work is to embed both constructions as part of a much larger and (at least at first sight) more baroque problem: the topological A-model on a particular target space Y [8]. Y plays the role of a "complexification" of P; in the examples of interest here, we can just take P to be the real points of an affine algebraic variety carrying an algebraic symplectic form, and Y to be the complex points of the same variety. The important structural features are that Y carries a holomorphic symplectic form Ω, whose restriction to P is the real symplectic form ω; that it is equipped with a unitary line bundle, with a connection of curvature $\mathrm{re}(\Omega)$, extending the prequantum line bundle on P; and that it is equipped with an antiholomorphic involution τ such that P is contained in the fixed-point set Y^τ. (τ plays a role only in the construction of the Hilbert-space structure on the quantization.)

The topological A-model is a topological twist of the $\mathcal{N} = (2,2)$ supersymmetric theory of maps from a Riemann surface into the target manifold Y; the salient feature of this theory is that it is expected to have a category of boundary conditions, which are referred to as A-branes, and thought of as physical objects in the target manifold. Objects in this category, which is a close relative of the Fukaya category of Y, are labeled by a coisotropic submanifold of Y together with some additional data; the general description was given in [10]. To be precise, an object consists of a coisotropic submanifold $M \subseteq Y$ up to Hamiltonian isotopy, together with a unitary line bundle $E \to M$, equipped with a connection of curvature $F \in \Omega^2(M)$. We let

$$LM = \ker\left(\omega|_M\right) \subseteq TM, \qquad FM = TM/LM; \tag{1}$$

by the coisotropic condition, FM is a bundle whose fiber dimension equals the codimension of M in Y, and ω descends to a nondegenerate section of $\wedge^2 FM$. For this data to define an A-brane, we also require that F descend to a section of $\wedge^2 FM$, and furthermore that

$$(\omega^{-1}F)^2 = -1, \tag{2}$$

when identified with an endomorphism of FM using the metric. (As such, M is a foliated manifold equipped with a transverse holomorphic structure.) As a special case, a Lagrangian submanifold equipped with a unitary line bundle with flat connection defines an (ordinary) A-brane. Morphisms between coisotropic objects were studied, among others, by [9].

Returning to our quantization problem above, one can then construct a object B_0 in the A-brane category of Y, equipped with the symplectic form $\mathrm{im}(\Omega)$, in a fairly canonical fashion: its support is all of Y, and we choose the unitary line bundle $E \to Y$ with a connection of curvature $\mathrm{re}(\Omega)$ mentioned previously. The transverse complex structure then just becomes the given complex structure on Y itself. One then expects [3] that the endomorphisms of B_0 in the A-brane category are given precisely by a "quantum" deformation of the Dolbeault cohomology of Y, which (in degree zero) precisely reproduces the deformation quantization of the coordinate ring of Y with respect to Ω:

$$\mathrm{End}(B_0) \xrightarrow[q \to 1]{} H^{0,*}(Y, \bar{\partial}). \tag{3}$$

We then expect to obtain, for every other object B of the A-brane category, a module for this algebra, simply by taking $\mathrm{Hom}(B_0, B)$ with the obvious left action by composition. P itself is Lagrangian with respect to $\mathrm{im}(\Omega)$, and so defines such an object; Gukov and Witten identify $\mathrm{Hom}(B_0, P)$ with the geometric quantization of P. However, one can also abandon the idea of P as a distinguished object, and just attempt to study representations of $\mathrm{End}(B_0)$ by studying A-branes in Y. Gukov and Witten do this to give geometric constructions of representations of $SL(2, \mathbb{R})$, generalizing the orbit method.

With all of this lengthy introduction out of the way, the logic of our investigation is hopefully clear: we intend to study the relationship between the A-brane category of a particular holomorphic symplectic manifold, on the one hand, and the category of representations of its quantized coordinate ring (or really Dolbeault cohomology) on the other. The relevant manifold is the moduli space of flat $SL(2, \mathbb{C})$ connections on the once-punctured torus; this is Hitchin's moduli space, viewed in complex structure J with holomorphic symplectic form Ω_J. As for the deformation quantization of its coordinate ring, Oblomkov [11] proved that this is nothing other than the spherical double affine Hecke algebra $\ddot{H}$, which is Morita equivalent to the full double affine Hecke algebra as studied by Cherednik [12]. $\ddot{H}$ has two parameters, one controlling the "quantum" deformation $q = \exp(2\pi i \hbar)$, and the other (t) labeling the monodromy of the connection at the puncture. (In fact, there is a five-parameter deformation of this algebra, but we do not consider that in full generality.) In general, finite-dimensional representations of $\ddot{H}$ only exist when these parameters satisfy certain special shortening conditions.

Our work identifies these finite-dimensional representations with compact branes of type (B, A, A), whose support arises as (components of) fibers of the Hitchin fibration. The shortening conditions on parameters of $\ddot{H}$ are identified with conditions for existence of the corresponding A-branes. Physically speaking, the construction can be thought of in the context of theories of class S, which arise from the six-dimensional $\mathcal{N} = (2, 0)$ theory associated to a semisimple, simply-laced Lie algebra, compactified on a Riemann surface Σ. The Coulomb branch of this theory on $S^1 \times \mathbb{R}^3$ is precisely the moduli space of G-Higgs bundles on Σ; in the presence of a so-called omega deformation, the algebra of line operators wrapping S^1 can be shown to realize the deformation quantization of the coordinate ring of the Coulomb branch with respect to Ω [13]. A similar setting was considered in [14], where the relation to the topological A-model was clarified; after identifying a plane in $\mathbb{R}^3$ with a circle bundle over $\mathbb{R}_+$, degenerating at the origin, one can think of the resulting theory as the topological A-model on $\mathbb{R} \times \mathbb{R}_+$ with target the Hitchin moduli space, and with appropriate boundary conditions or A-branes at zero and infinity.

Many open questions remain: One expects that $\mathrm{Hom}(B_0, -)$ is a functor from A-branes to $\ddot{H}$-modules, but the precise characterization of this functor remains unclear. Our hope is to be able to demonstrate an equivalence between compact A-branes and finite-dimensional $\ddot{H}$-modules, but a precise understanding of even this part of the damaged treasure map remains as work in progress. For example, the image of B_0 under this functor is clearly $\ddot{H}$ as a module over itself, but it is not even clear that B_0 is a projective object in the A-brane category! Of course, it would also be interesting to explore generalizations to other Riemann surfaces Σ; the deformation quantization of flat $G_{\mathbb{C}}$-connections should give an algebra with an action of the mapping class group of Σ by automorphisms, whose representations are connected to (B, A, A)-branes and therefore to the Hitchin fibration. We hope that these speculations will prove to be fertile stimuli for further exploration, like the original damaged treasure map itself.

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Automorphisms of moduli spaces of semistable parabolic bundles on the projective line

Inder Kaur
(joint work with Carolina Araujo, Thiago Fassarella, Alex Massarenti)

Let C be a smooth, projective curve and $p_1, \ldots, p_n \in C$ distinct points, which we call *parabolic points*. A *parabolic vector bundle* $(E, \mathbf{v})$ on C of rank 2 is a vector bundle E of rank 2 with a weighted flag on the fibre E_{p_i} over each parabolic point p_i, given by a one-dimensional subspace $V_i \subset E_{p_i}$, which we call a *parabolic direction*. Slope stability for such bundles depends on the choice of a weight vector $\mathcal{A} = (a_1, \ldots, a_n)$ of real numbers $0 \leq a_i \leq 1$. The *parabolic slope* of $(E, \mathbf{v})$ with respect to $\mathcal{A}$ is

$$\mu_{\mathcal{A}}(E) = \frac{\deg E + \sum_{i=1}^{n} a_i}{2}.$$

In [10], Mehta and Seshadri proved the existence of moduli spaces of semistable parabolic vector bundles (of any rank) over a curve of genus $g \geq 2$. The case of $g = 0$ and rank 2 was studied extensively by Bauer in [3].

Fix $n \geq 5$ distinct, general points on $\mathbf{P}^1$. We denote by $\mathcal{M}_{\mathcal{A}}$ the moduli space of rank 2 parabolic vector bundles $(E, \mathbf{v})$ on $\mathbf{P}^1$ with trivial determinant, which are parabolic semistable with respect to the weight vector $\mathcal{A} = (a_1, \ldots, a_n)$. Our goal is to determine the automorphism group of the moduli space $\mathcal{M}_{\mathcal{A}}$. We assume the points to be general because any nontrivial automorphism $\phi : \mathbf{P}^1 \to \mathbf{P}^1$ permuting the parabolic points induces a nontrivial automorphism of $\mathcal{M}_{\mathcal{A}}$ sending $(E, \mathbf{v})$ to $\phi^*(E, \mathbf{v})$. The weight vector $\mathcal{A}_F = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right)$ is especially interesting as the corresponding moduli space $\mathcal{M}_{\mathcal{A}_F}$ is a Fano variety (see [11]). It is smooth if n is odd, and has isolated singularities if n is even.

Recall that for the moduli space of rank 2 stable vector bundles with fixed determinant Λ over a smooth, projective curve C of genus $g \geq 3$, the automorphisms have been described by [9], [8] and [4]. Explicitly, an automorphism of the moduli space either sends a vector bundle E to $E \otimes L$, where L is a line bundle on C with $L^2 \simeq \mathcal{O}_C$ or to $E^\vee \otimes L$, where L is a line bundle on C with $L^2 \simeq \Lambda^2$. Since there are no 2-torsion line bundles on $\mathbf{P}^1$, a similar description for $\mathcal{M}_{\mathcal{A}}$ cannot be expected.

One way of obtaining a new parabolic vector bundle from a given bundle $(E, \mathbf{v})$ is to blow-up the ruled surface $\mathbb{P}E$ at the parabolic direction $\mathbb{P}(V_i) \in \mathbb{P}(E_{p_i})$ and then blow down the strict transform of the fibers $\mathbb{P}(E_{p_i})$. We call this operation an *elementary transformation centred at the parabolic point* p_i. For the weight vector $\mathcal{A}_F$, the elementary transformations preserve the stability condition. It is easy to show that an elementary transformation centred at even number of parabolic points (with number of points ≥ 5) induces a non-trivial automorphism of $\mathcal{M}_{\mathcal{A}_F}$. Elementary transformations are involutions and form a group under composition. Denote by **El** the group generated by elementary transformations centred at even number of points. We show the following:

Theorem 1. [2, Theorem 1.2] *Fix $n \geq 5$ general points $p_1, \dots, p_n \in \mathbf{P}^1$ and let $\mathcal{M}_{\mathcal{A}_F}$ be the moduli space of rank 2 parabolic vector bundles with trivial determinant on $\mathbf{P}^1$ which are semistable with respect to the weight vector $\mathcal{A}_F = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$. Then*

$$\left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^{n-1} \simeq \textbf{\textit{El}} = \operatorname{Aut}\big(\mathcal{M}_{\mathcal{A}_F}\big).$$

Let $\varphi \in \operatorname{Aut}(\mathcal{M}_{\mathcal{A}_F})$ be an automorphism sending a general rank 2 parabolic vector bundle $(E, \mathbf{v})$ to $(E', \mathbf{v}')$. Since $\mathbf{El} \subset \operatorname{Aut}(\mathcal{M}_{\mathcal{A}_F})$ is finite, in order to prove that the groups coincide, it suffices to show that there is an elementary transformation sending $(E, \mathbf{v})$ to $(E', \mathbf{v}')$. We do this by first showing the blow-up of $\mathbb{P}(E)$ at the finite set of points $\{\mathbb{P}(V_i)\}_{i=1,\dots,n}$ can be seen as the projectivization of the nilpotent cone associated to E. We then use the properties of the Hitchin morphism to prove that the corresponding nilpotent cones of E and E' are isomorphic over $\mathbf{P}^1$. The proof of Theorem 1 in fact holds for any genus with minor substitutions (for rank 2). For example, we use the Torelli theorem for parabolic bundles on $\mathbf{P}^1$ given in [6] but this can be substituted by a similar result for higher genus given in [5].

For $n = 5$, the moduli space $\mathcal{M}_{\mathcal{A}_F}$ is isomorphic to a del Pezzo surface of degree four and its automorphism group is classically known ([7, Section 8.6.4]). For n odd, the cardinality of the automorphism group was also proven in [1, Proposition 1.9] but using different methods. It is natural to ask whether Theorem 1 holds for other weights. Bauer described in [3] the weight polytope $\Delta \subset [0, 1]^n$ called *demi-hypercube* consisting of weight vectors $\mathcal{A}$ for which the moduli space $\mathcal{M}_{\mathcal{A}} \neq \emptyset$. The polytope Δ is generated by the even vertices of the hypercube $[0, 1]^n$, where the parity of a vertex is the parity of the set of its coordinates that equal 1. The weight vector $\mathcal{A}_F = \left(\frac{1}{2}, \dots, \frac{1}{2}\right)$ is the centre of the polytope Δ. Bauer gave a wall-and-chamber decomposition on Δ corresponding to the variation of GIT for the moduli spaces $\mathcal{M}_{\mathcal{A}}$, and described the birational maps between models corresponding to different chambers. He showed that there are weight vectors $\mathcal{A}$ for which $\mathcal{M}_{\mathcal{A}} \simeq \mathbf{P}^n$. So in general, $\mathbf{El} \neq \operatorname{Aut}\big(\mathcal{M}_{\mathcal{A}_F}\big)$.

However, we can extend Theorem 1 to certain other weights. For every vertex v of Δ, let $H_v \subset \mathbf{R}^n$ be the hyperplane spanned by those vertices of Δ that are adjacent to v. We obtain an open sub-polytope Π from Δ by chopping off each vertex v of Δ with the hyperplane H_v. The polytope Π contains in its interior weight vectors defining the same stability condition as $\mathcal{A}_F$. From the view point of birational geometry, these are the weight vectors $\mathcal{A}$ such that the corresponding moduli space $\mathcal{M}_{\mathcal{A}}$ is a small modification (i.e a birational map that restricts to an isomorphism on the complement of closed subsets of codimension at least two) of the Fano variety $\mathcal{M}_{\mathcal{A}_F}$. Moreover, for $\frac{1}{n-2} < \varepsilon < \frac{1}{n-4}$ and $\mathcal{A}_\epsilon = (1-\epsilon, \epsilon, \dots, \varepsilon)$, the moduli space $\mathcal{M}_{\mathcal{A}_\varepsilon}$ is isomorphic to the blow-up of $\mathbf{P}^{n-3}$ at n general points. We show the following:

Corollary 2. *Fix $n \geq 5$ general points $p_1, \ldots, p_n \in \mathbf{P}^1$ and let $\mathcal{A}$ be a weight vector in the interior of the polytope Π defined above. Let $\mathcal{M}_\mathcal{A}$ be the moduli space of rank 2 parabolic vector bundles with trivial determinant which are semistable with respect to the weight vector $\mathcal{A}$. Then*

$$\boldsymbol{El}_\mathcal{A} \ = \ \mathrm{Aut}\big(\mathcal{M}_\mathcal{A}\big).$$

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Intersection cohomology of the moduli space of Higgs bundles on a curve of genus 2

Camilla Felisetti

Introduction

We consider the moduli space of semistable Higgs bundles of rank 2 with trivial determinant on a curve C of genus 2 . This is a singular irreducible quasi-projective variety of complex dimension 6 which real analytic isomorphic to the character variety of representations of the fundamental group of C into $SL(2, \mathbb{C})$. In the smooth case of bundles of degree 1, corresponding to a twisted character variety, De Cataldo, Hausel and Migliorini [dCHM] have proved the so called $P = W$ conjecture, that asserts that the *Weight filtration* $W.$ coming from the natural Mixed Hodge structure on the cohomology of the character variety corresponds in the above isomorphism to another filtration, the *Perverse filtration*, on the cohomology of the Higgs moduli space. In fact the weight filtration on the Higgs side turns out to be trivial.

We are interested in finding an analogous statement in the singular case and theory suggest that we should replace ordinary cohomology groups with the *Intersection cohomology groups* introduced by Goresky and MacPherson in [GM] and [GM1]. We prove the following theorem.

Theorem 1. *Let C be a smooth projective curve of genus 2 and let $\mathcal{M}_{Dol}$ be the corresponding moduli space of Higgs bundles with trivial determinant. Then the mixed Hodge structure on the intersection cohomology groups is trivial, and the intersection Poincaré polynomial is*

$$IP_t(\mathcal{M}_{Dol}) = 1 + t^2 + 17t^4 + 17t^6.$$

Sketch of the proof

The singular locus of $\mathcal{M}_{Dol}$ consists of the strictly semistable elements. We can stratify the singular locus with respect to the stabilizer of the points and get

$$\Sigma := \left\{(L,\phi) \oplus (L^{-1}, -\phi) \mid L \in Pic^0(C), \phi \in H^0(K_C) \text{ such that } (L,\phi) \ncong (L^{-1}, -\phi)\right\}$$

$$\Omega := \left\{(L,0) \oplus (L,0) \mid L \in Pic^0(C), \text{ such that } L^2 = \mathcal{O}_C\right\}.$$

Observe that $\Sigma \cong Pic^0(C) \times H^0(K_C)/\mathbb{Z}_2$ where $\mathbb{Z}_2$ acts as the involution that exchange the two summands.

Clearly $\Omega \subset \Sigma$, moreover the points in Ω have stabilizer $SL(2,\mathbb{C})$ while the points in $\Sigma \setminus \Omega$ have stabilizer $\mathbb{C}^*$.

Following strategies by O' Grady [OG] and Kiem-Yoo [KY], we construct a semismall desingularization $\tilde{\mathcal{M}}_{Dol}$ of $\mathcal{M}_{Dol}$ and apply the decomposition theorem by Beilinson, Bernstein and Deligne [BBD] to express the cohomology of $\tilde{\mathcal{M}}_{Dol}$ as a direct sum of the intersection cohomology of $\mathcal{M}_{Dol}$ with some other summands supported respectively on Σ and Ω.

Call $\tilde{\Omega}$ and $\tilde{\Sigma}$ the preimages of Ω and Σ with respect to the desingularization map $\tilde{\pi} : \tilde{\mathcal{M}}_{Dol} \to \mathcal{M}_{Dol}$.

Proposition 3. *Keep the notation as above.*

(i) $\tilde{\Omega}$ is the union of 16 copies of a smooth irreducible hypersurface S in $\mathbb{CP}^4$;
(ii) $\tilde{\Sigma} \setminus \tilde{\Omega}$ is a $\mathbb{CP}^1$ bundle over $\Sigma \setminus \Omega$.

Applying the decomposition theorem we obtain the following decomposition of the cohomology of $\tilde{\mathcal{M}}_{Dol}$:

$$H^*(\tilde{\mathcal{M}}_{Dol}) = IH^*(\mathcal{M}_{Dol}) \oplus H^{*-2}(\Sigma, IC_\Sigma(\mathcal{L}_\Sigma)) \oplus H^{*-6}(\Omega, IC_\Omega(\mathcal{L}_\Omega))$$

with

$$IC_\Sigma(\mathcal{L}_\Sigma)_{|\Sigma^{reg}} \cong \mathbb{Q}[4](-1) \quad IC_\Omega(\mathcal{L}_\Omega) \cong \mathbb{Q}(-3)^{\oplus 16}.$$

Here Σ^{reg} is the smooth part of Σ and the shifts (-1) and (-3) correspond to the Hodge structures $\mathbb{Q}(-1)$ and $\mathbb{Q}(-3)$ of respectively $\mathbb{P}^1$ and S.

Remark 2. Observe that since Ω is nonsingular and Σ have finite quotient singularities intersection cohomology and cohomology coincide.

Using a description of the singularities due to Simpson [Sim], we can prove that any step of desingularization is obtain by blowing up a $\mathbb{C}^*$- fixed subset, as a result one can extend the natural $\mathbb{C}^*$ action on $\mathcal{M}_{Dol}$ to $\tilde{\mathcal{M}}_{Dol}$.
This tell us that the variety $\mathcal{M}_{Dol}$ is semi-projective and the contraction induced by the $\mathbb{C}^*$ action yields an isomorphism between the cohomology of $\tilde{\mathcal{M}}_{Dol}$ and the cohomology of the fixed locus, which is compact.
Now we can take advantage of some properties of the weight filtration $W.$ of a mixed Hodge structure: on the one hand, given an algebraic variety X, if X is nonsingular then $W_iH^k(X) = 0$ for all $i < k$, i.e. we have weights *higher or equal* than k; on the other hand if X is compact then $W_iH^k(X) = H^k(X)$ for any $i \geq k$, i.e. we have weights *lower or equal* than k.
In our case $\tilde{\mathcal{M}}_{Dol}$ is nonsingular $H^k(\tilde{\mathcal{M}}_{Dol})$ have weights $\geq k$. However, because of the isomorphism with the fixed locus, which is compact, the weights in $H^k(\mathcal{M}_{Dol})$ are also $\leq k$. As a result, $H^k(\mathcal{M}_{Dol})$ have precisely weight k, that is the Mixed Hodge structure on it is actually a pure Hodge structure of weight k .
Since the Hodge structure on $IH^*(\mathcal{M}_{Dol})$ is a sub-Hodge structure of that on $H^*(\tilde{\mathcal{M}}_{Dol})$, if the latter is pure so is the former and this proves the first part of the theorem.

We are now in a position to use a beautiful trick. Recall that the E-polynomial of a variety X is defined as

$$E(X)(u,v) = \sum_{h=0}^{2\dim X} (-1)^k \sum_{h,p,q} h_c^{k,p,q} u^p v^q$$

where $h_c^{k,p,q} = \dim \mathrm{Gr}_F^p Gr_{p+q}^W H_c^k(X)$ and satisfies the following properties:

(i) if $Z \subset X$ then $E(X) = E(Z) + E(X \setminus Z)$

(ii) $E(X \times Y) = E(X)E(Y)$

Similarly we can define the same invariant $IE(X)$ for intersection cohomology, however it will not satisfy the above properties.
Now we just need to compute the $E-$polynomial of $\tilde{\mathcal{M}}_{Dol}$ and subtract the contributions in the decomposition theorem coming from the summands supported on the singular locus. In this way we will end up with the intersection $E-$polynomial of $\mathcal{M}_{Dol}$. Since the mixed Hodge structure on $IH^*(\mathcal{M}_{Dol})$ is pure, in order to get the intersection Betti numbers we just need to sum up all the summands of the same weight and apply Poincaré-Verdier duality.

We have

$$E(\tilde{\mathcal{M}}_{Dol}) = E(\mathcal{M}_{Dol}^s) + E(\tilde{\Sigma}) + E(\tilde{\Sigma} \setminus \tilde{\Omega}),$$

where $\mathcal{M}_{Dol}^s$ denotes the smooth locus of $\mathcal{M}_{Dol}$ consisting of stable Higgs bundles.
The last two summands can be computed with proposition (3) while the first one is computed by stratifying it with respect to the underlying vector bundle.
Therefore we will divide stable Higgs pairs in following three strata:

- pairs (V,Φ) with V stable vector bundle;
- pairs (V,Φ) with V strictly semistable vector bundle;
- pairs (V,Φ) with V unstable vector bundle.

In the case of strictly semistable objects we have to distinguish four different cases:

(i) $V = L\oplus L^{-1}$ where $L\in Pic^0(C)$ and $L\ncong L^{-1}$;

(ii) V is a non trivial extension $0\longrightarrow L\longrightarrow V\longrightarrow L^{-1}\longrightarrow 0$ with $L\ncong L^{-1}$;

(iii) $V = L\oplus L^{-1}$ where $L\in Pic^0(C)$ and $L\cong L^{-1}$;

(iv) V is a non trivial extension $0\longrightarrow L\longrightarrow V\longrightarrow L^{-1}\longrightarrow 0$ with $L\cong L^{-1}$;

By computing the $E-$ polynomial of each stratum we have:

$$E(\tilde{\mathcal{M}}_{Dol}) = u^6v^6 + 2u^5v^5 + 21u^4v^4 + u^5v^3 + u^3v^5 + 34u^3v^3,$$
$$IE(\mathcal{M}_{Dol}) = u^6v^6 + u^5v^5 + 15u^4v^4 + u^5v^3 + u^3v^5 + 17u^3v^3.$$

Applying Poincaré duality and summing up the pieces of the same weight have

$$IP_t(\mathcal{M}_{Dol}) = 1 + t^2 + 17t^4 + 17t^6.$$

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On two generalisations of Hitchin's equations in four dimensions from Theoretical Physics

Yuuji Tanaka

This report is about two types of generalisations of Hitchin's prominent equations on compact Riemann surfaces [H] in four dimensions, both of which have the origin in $\mathcal{N}=4$ super Yang–Mills theory in Theoretical Physics.

Hitchin's equations look for a pair (A, Φ) consisting of a connection A on a hermitian vector bundle E over a Riemann surface Σ and a section Φ of $\mathrm{End}(E) \otimes \Lambda^{1,0}_\Sigma$, where $\Lambda^{1,0}_\Sigma := (T^*\Sigma \otimes \mathbb{C})^{1,0}$, satisfying

$$\bar\partial_A \Phi = 0, \quad F_A + [\Phi \wedge \Phi^*] = 0.$$

If one wishes to generalise these in higher dimensions, there could be perhaps at least two directions, as we can think of $\Lambda^{1,0}_\Sigma$ as either

(A) the canonical bundle K_Σ, which is a complex line bundle; or
(B) the cotangent bundle Ω^1_Σ, which is a vector bundle in complex dimension greater than one

They are exactly the cases incarnated by the ones coming from the $\mathcal{N} = 4$ super Yang–Mills theory, which we describe below.

To set the stage for what we discuss in this report, let X be a closed, oriented, smooth 4-manifold, and let P be a principal $SU(2)$ or $SO(3)$-bundle.

(A) Vafa–Witten equation on closed four-manifolds. Vafa and Witten [VW] introduced the following set of equations[1] for a pair (A, B) consisting of a connection on P and a section B of $\Lambda^+ \otimes \mathfrak{g}_P$:

$$d_A^* B = 0, \quad F_A^+ + \frac{1}{8}[B.B] = 0,$$

where F_A^+ is the self-dual part of the curvature of A, the bracket $[B.B]$ is again a section of $\Lambda^+ \otimes \mathfrak{g}_P$ (see [M] or [Tan3] for more detail of this). They conjectured that the generating function of invariants, which could be defined through the moduli space of solutions to the above equations, would be modular forms as a consequence of S-duality in the $\mathcal{N} = 4$ super Yang–Mills theory.

On a compact Kähler surface with Kähler form ω, the above equations become:

$$\bar\partial_A \varphi = 0, \quad F_A^{0,2} = 0, \quad F_A^{1,1} \wedge \omega + [\varphi \wedge \varphi^*] = 0,$$

where $\varphi \in \Gamma(X, \mathrm{End}(E) \otimes K_X)$, and E is the associated complex vector bundle with hermitian metric. Note that the integrability condition $[\varphi \wedge \varphi] = 0$ is automatically satisfied as K_X is a line bundle. This is the form mentioned in the above as (A), and by the Hitchin–Kobayashi correspondence [AG], [Tan2], solutions to the equations correspond to stable Higgs pairs. Furthermore, Richard Thomas and the reporter [TT1], [TT2] constructed a symmetric perfect obstruction theory on the moduli space of this and defined deformation invariants by using virtual techniques in Algebraic Geometry. We also checked the modular properties of the generating functions of the invariants conjecture by Vafa and Witten in examples.

(B) Kapustin–Witten equations on closed four-manifolds. From a different topological twist of the same $\mathcal{N} = 4$ super Yang–Mills theory, Kapustin and Witten [KW] introduced the following set of equations for a pair $(A, \mathfrak{a})$ consisting of a connection A and a section $\mathfrak{a}$ of $\Lambda^1 \otimes \mathfrak{g}_P$:

$$d_A^* \mathfrak{a} = 0, \quad (d_A \mathfrak{a})^- = 0, \quad F_A^+ - [\mathfrak{a} \wedge \mathfrak{a}]^+ = 0,$$

where "-" or "+" indicates the anti-self-dual or self-dual part respectively.

[1] To be more precise, there is one other variable $C \in \Gamma(X, \mathfrak{g}_P)$ in their equations, however we assume that this vanishes for simplicity in this report.

More precisely, there are two super charges in this twist, in fact, they introduced the equations parametrized by $\tau \in \mathbb{P}^1$, which is a linear combination of them. The above corresponds to $\tau = 0$ (or $\tau = \infty$ with orientation reversed) version of them. Note that, if τ is not real, they are overdetermined. Siqi He, Mazzeo, Walpuski and Witten made significant progress for $\tau = -1$ version of them with Nahm pole boundary condition. One other remark is that $\tau = \pm i$ version is relevant for geometric Langlands programme as originally described in [KW].

On a compact Kähler surface with Kähler form ω, the above equations become:

$$\bar{\partial}_A \phi = 0, \quad [\phi \wedge \phi] = 0, \quad F_A^{0,2} = 0, \quad \Lambda \left(F_A^{1,1} + 2[\phi \wedge \phi^*] \right) = 0,$$

where $\phi \in \Gamma(X, \mathrm{End}\,(E) \otimes \Omega_X^1)$, and $\Lambda := (\wedge \omega)^*$ (see [N], [Tan3] for its derivation). This is the form mentioned as (B), and they are the same equations considered by Simpson in [S], hence solutions to them correspond to stable Higgs bundles on X.

Non-compactness issue in the direction of Higgs fields. Both of Vafa–Witten and Kapustin–Witten equations with a gauge fixing equation form elliptic systems, thus the moduli problem in gauge theory might be well posed. However, there is a non-compactness phenomena in the direction of the Higgs fields denoted as B or $\mathfrak{a}$ in the above, in addition to the Uhlenbeck bubbling. This is because we do not have a priori bound for the L^2-norm for these Higgs fields.

If the underlying manifold is a compact Kähler surface, we could make use of $\mathbb{C}^*$-action on the moduli space and its fixed loci, as performed in [TT1], [TT2]. For general cases, perhaps a way to sort this issue out might be to take rescaling $\Phi' := \Phi / ||\Phi||_{L^2}$, where Φ is B in the Vafa–Witten case and $\mathfrak{a}$ in the Kapustin–Witten case. This idea is not quite new, for example, Simpson used this to obtain destabilizing sheaves in his renowned proof of the Hitchin–Kobayashi correspondence in [S]. A series of fascinating breakthrough in this direction has been established by Taubes, in particular, he obtained the following:

Theorem 1 ([Tau2], [Tau3])**.** *Let $\{(A_i, \mathfrak{a}_i)\}$ be a sequence of solutions to the Kapustin–Witten equations with $||\mathfrak{a}_i||_{L^2} \to \infty$. Denote by $\mathfrak{a}_i' := \mathfrak{a}_i / ||\mathfrak{a}_i||_{L^2}$. Then there exist a finite set of points $\Theta := \{p_1, \dots, p_k\}$ and a closed subset $Z \subset X$ with finite two-dimensional Hausdorff measure, such that a subsequence of $\{(A_i, \mathfrak{a}_i')\}$ converges in L_1^2 on $X \setminus \{Z \cup \Theta\}$ after gauge transformations.*

Taubes also proved a similar result for the Vafa–Witten case [Tau4]. For the original Hitichin's equations on a compact Riemann surface, Fredrickson, Mazzeo, Mochizuki, Swoboda, Weiss and Witt made wonderful progress in the same spirit. Note that this Z could be thought of as a recurrence of that appeared in $SW = Gr$ by Taubes [Tau1], in which it was a pseudo-holomorphic submanifold.

The reporter clarified the situation for the Vafa–Witten case, assuming that there are no bubbling of connections, in the following manner:

Theorem 2 ([Tan2])**.** *Assume that X is simply-connected for simplicity. Let $\{(A_i, B_i)\}$ a sequence of solutions to the Vafa–Witten equations with "no bubbling condition" with $||B_i||_{L^2} \to \infty$. Then a subsequence of $\{A_i\}$ converges to* *an anti-self-dual connection* $A_\diamond$ *weakly in the* L_1^2*-topology after gauge transformations. If the limit is not reducible, then there exists a constant* $C > 0$ *such that* $\int_X |B_i|^2 dvol_g \leq C$ *for all* i*; and a subsequence of* $\{(A_i, B_i)\}$ *converges in the* C^∞*-topology to a solution to the Vafa–Witten equations after gauge transformations.*

As for the structure of the singular set Z both in Vafa–Witten and Kapustin–Witten cases, we proved:

Theorem 3 ([Tan3])**.** *In both Vafa–Witten and Kapustin–Witten cases, the singular set* Z *has the structure of an analytic subvariety, if the underlying 4-manifold* X *is a compact Kähler surface.*

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Reporter: Mengxue Yang

Participants

Dr. Nezhla Aghaee
Institute for Theoretical Physics
Universität Bern
Sidlerstr. 5
3012 Bern
SWITZERLAND

Dr. Daniele Alessandrini
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Jørgen E. Andersen
Matematisk Institut
Aarhus Universitet
Ny Munkegade 118
8000 Aarhus C
DENMARK

Prof. Dr. Lara B. Anderson
Department of Physics
Virginia Polytechnic Institute and State University
850 West Campus Drive
Blacksburg, VA 24061
UNITED STATES

Prof. Dr. Roger Bielawski
Institut für Mathematik
Universität Hannover
Welfengarten 1
30167 Hannover
GERMANY

Prof. Dr. Indranil Biswas
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005
INDIA

Prof. Dr. Steve Bradlow
Department of Mathematics
University of Illinois
Urbana, IL 61801
UNITED STATES

Prof. Dr. Aliakbar Daemi
Simons Centre for Geometry and Physics
State University of New York at Stony Brook
Stony Brook, NY 11794-4400
UNITED STATES

Dr. Ben Davison
School of Mathematics
University of Edinburgh
James Clerk Maxwell Building
King's Buildings
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Dr. Camilla Felisetti
Département de Mathématiques
Université de Geneve
Case Postale 64
2-4 rue du Lievre
1211 Genève 4
SWITZERLAND

Prof. Dr. Laura Fredrickson
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Eloise Hamilton
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Tamas Hausel
Institute of Science and
Technology Austria (IST Austria)
Am Campus 1
3400 Klosterneuburg
AUSTRIA

Prof. Dr. Jochen Heinloth
Fachbereich Mathematik
Universität Duisburg-Essen
Universitätsstrasse 2
45117 Essen
GERMANY

Prof. Dr. Sebastian Heller
Department Mathematik
Universität Hamburg
Bundesstrasse 55
20146 Hamburg
GERMANY

Prof. Dr. Nigel J. Hitchin
Mathematical Institute
Oxford University
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Johannes Horn
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Sheldon Katz
Department of Mathematics
University of Illinois at Urbana
Champaign
301 Altgeld Hall
1409 West Green Street
Urbana, IL 61801
UNITED STATES

Prof. Dr. Ludmil Katzarkov
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Dr. Inder Kaur
Instituto de Matematica Pura e
Aplicada (IMPA)
Jardim Botanico
Estrada Dona Castorina, 110
22460 Rio de Janeiro, RJ 320
BRAZIL

Dr. Qiongling Li
Chern Institute of Mathematics
Nankai University
Weijin Road No. 94
Tianjin 300 071
CHINA

Dr. Marina Logares
School of Computing and Mathematics
University of Plymouth
Plymouth, Devon PL4 8AA
UNITED KINGDOM

Dr. Sara Maloni
Department of Mathematics
University of Virginia
Kerchof Hall
P.O.Box 400137
Charlottesville, VA 22904-4137
UNITED STATES

Prof. Dr. Rafe Mazzeo
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
UNITED STATES

Dr. Ian McIntosh
Department of Mathematics
University of York
Heslington, York YO10 5DD
UNITED KINGDOM

Dr. Claudio Meneses-Torres
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
Ludewig-Meyn-Strasse 4
24118 Kiel
GERMANY

Prof. Dr. Takuro Mochizuki
Research Institute for Mathematical Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Dr. Swarnava Mukhopadhyay
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005
INDIA

Dr. Andreas Ott
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Tony Pantev
Department of Mathematics
David Rittenhouse Laboratory
University of Pennsylvania
209 South 33rd Street
Philadelphia PA 19104-6395
UNITED STATES

Dr. Du Pei
Department of Mathematics
California Institute of Technology
1200 E. California Boulevard
Pasadena, CA 91125
UNITED STATES

Dr. Ana Peón-Nieto
Département de Mathématiques
Université de Geneve
Case Postale 64
2-4 rue du Lievre
1211 Genève 4
SWITZERLAND

Prof. Dr. Steven Rayan
Department of Mathematics and Statistics
University of Saskatchewan
Saskatoon, SK S7N 5E6
CANADA

Alejandra Rincón Hidalgo
Fachbereich Mathematik und Informatik
AG Komplexe Analysis
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Dr. Roberto Rubio
Facultat de Ciències
Universitat Autònoma de Barcelona
Cerdanyola del Vallès
C3b/156
08193 Bellaterra, Barcelona
SPAIN

Ingmar Saberi
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Laura P. Schaposnik M.
Department of Mathematics
University of Illinois
851 S. Morgan Street
Chicago, IL 60607
UNITED STATES

Prof. Dr. Carlos Simpson
Laboratoire J.-A. Dieudonne
Université de Nice
Sophia Antipolis
Parc Valrose
06108 Nice Cedex 2
FRANCE

PD Dr. Jan Swoboda
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Yuuji Tanaka
Mathematical Institute
Oxford University
24-29 St. Giles
Oxford OX1 3LB
UNITED KINGDOM

Prof. Dr. Jörg Teschner
D E S Y - Theory Group
Gebäude 2a
Notkestrasse 85
22603 Hamburg
GERMANY

Prof. Dr. Hartmut Weiss
Mathematisches Seminar
Christian-Albrechts-Universität Kiel
Ludewig-Meyn-Strasse 4
24098 Kiel
GERMANY

Prof. Dr. Richard A. Wentworth
Department of Mathematics
University of Maryland
College Park, MD 20742-4015
UNITED STATES

Prof. Dr. Graeme Wilkin
Department of Mathematics
National University of Singapore
10 Kent Ridge Cresent
Singapore 119 260
SINGAPORE

Prof. Dr. Michael Wolf
Department of Mathematics
Rice University
P.O. Box 1892
Houston, TX 77251
UNITED STATES

Mengxue Yang
Department of Mathematics, Statistics
and
Computer Science, M/C 249
University of Illinois at Chicago
851 S. Morgan Street
Chicago, IL 60607-7045
UNITED STATES