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NICOLA OSWALD AND JÖRN STEUDING

Maximal Quaternion Orders in Quadratic Extensions- in Hurwitz's Diaries

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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MAXIMAL QUATERNION ORDERS IN QUADRATIC EXTENSIONS — IN HURWITZ'S DIARIES

NICOLA OSWALD & JÖRN STEUDING

Dedicated to Prof. Dr. Klaus Volkert at the occasion of his retirement

ABSTRACT. We present and comment on some unpublished work of Adolf Hurwitz on quaternion arithmetic from his diaries.

KEYWORDS: quaternion integers, maximal order
MATHEMATICAL SUBJECT CLASSIFICATION: 11E88, 16H10, 01A60

1. A Brief Historical Introduction

Quaternions are hypercomplex numbers and were invented by William Rowan Hamilton in 1843. They form a skewfield and their lack of commutativity is not unrelated to matrix multiplication (as was observed already by Arthur Cayley in 1858). Eversince their invention quaternions have been studied intensively for several reasons. In the 1880s, Rudolf Lipschitz started his number-theoretical investigations [14] that led him to a quaternionic proof of the celebrated four squares-theorem of Joseph-Louis Lagrange that every positive integer can be represented as a sum of four squares, e.g.

$$2020 = 17^2 + 19^2 + 23^2 + 29^2.$$

Since quaternions are of the form

$$x = \alpha + \beta\mathrm{i} + \gamma\mathrm{j} + \delta\mathrm{k} \qquad \text{with } \alpha, \beta, \gamma, \delta \in \mathbb{R}$$

and three independent square roots $\mathrm{i}, \mathrm{j}, \mathrm{k}$ of -1 satisyfying $\mathrm{i}^2 = \mathrm{j}^2 = \mathrm{k}^2 = \mathrm{ijk} = -1^*$, it follows that their *norm*, defined by

$$N(x) := x \cdot x' = \alpha^2 + \beta^2 + \gamma^2 + \delta^2,$$

is a quadratic form in the coordinates of x (if we consider quaternions as a four-dimensional vector $x = (\alpha, \beta, \gamma, \delta)$ with real entries); here $x' :=$

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*which also implies non-commutativity, e.g., $\mathrm{ij} = -\mathrm{ji}$

$\alpha - \beta \mathrm{i} - \gamma \mathrm{j} - \delta \mathrm{k}$ is the conjugate of x. Hence, it is indeed natural to relate sums of four integer squares with the norm of quaternions. For this purpose, Lipschitz introduced $\mathcal{L} := \mathbb{Z}[\mathrm{i}, \mathrm{j}]$[†] as the quaternion counterpart of $\mathbb{Z}$ in $\mathbb{Q}(\mathrm{i}, \mathrm{j})$. This is in analogy to Gauss' investigations on the arithmetic of the set $\mathbb{Z}[\sqrt{-1}]$ within the field $\mathbb{Q}(\sqrt{-1})$ of complex numbers with rational real and imaginary parts.[‡] It is not only the style of writing that makes Lipschitz's approach dfficult to follow, another obstacle is the lack of a euclidean algorithm for these *Lipschitz quaternion integers*.

A decade later, Adolf Hurwitz [11] found a solution to this problem by adjoining

$$\rho := \tfrac{1}{2}(1 + \mathrm{i} + \mathrm{j} + \mathrm{k})$$

to Lipschitz's ring which then leads to a euclidean ring $\mathcal{H} := \mathcal{L}[\rho] = \mathbb{Z}[\mathrm{i}, \mathrm{j}, \rho]$. This step from the *naive* ring $\mathcal{L}$ to the larger ring $\mathcal{H}$ of *Hurwitz quaternion integers* has a historical forerunner, namely Gotthold Eisenstein's treatment of $\mathbb{Q}(\sqrt{-3})$, where $\mathbb{Z}[\sqrt{-3}]$ is not euclidean while $\mathbb{Z}[\frac{1}{2}(1 + \sqrt{-3})]$ is. It was probably Hurwitz's expertise in the development of algebraic number theory in the nineteenth century that led him to his elegant approach including the very beginnings of an ideal theory for quaternions. A few months before his early death in 1919, Hurwitz published his in some parts only sketched approach from 1896 as a detailed and elaborated textbook [12] in order to "encourage younger colleagues to promote this field by further investigations."[§] In his review of Hurwitz's booklet, the young Erich Hecke wrote that "the book gives us an idea of the treasures we might hope for when the lectures of the author, who died so young, are published."[¶]

It is the authors' intention to publish a translation of Hurwitz's *lectures on the number theory of quaternions* [12] into English with additional comments and historical remarks. Indeed, this topic is of particular interest in terms of its historical development and there is already extensive literature on the number theory of quaternions and other algebras; for example, [2, 8] (with respect to the topic we will focus below). For the

[†] since ij = k this ring equals $\mathbb{Z}[\mathrm{i}, \mathrm{j}, \mathrm{k}]$

[‡] Of course, we could also write $\mathbb{Z}[\mathrm{i}]$ or in place of i any other square root of -1 from the quaternionic setting, however, we prefer to distinguish here and in the sequel the commutative from the non-commutative case.

[§] "Es würde mich freuen, wenn diese Vorlesungen etwa jüngere Fachgenossen dazu anregen würden, jenes Gebiet durch weitere Untersuchungen zu fördern." This is from the preface of [12].

[¶] "Das Buch läßt ahnen, welche Schätze wir einmal bei der Herausgabe der Vorlesungen des so früh verstorbenen Verfassers zu erhoffen haben." [10]

sake of brevity, however, we will limit ourselves quite considerably in this note.

Hurwitz's estate is archived at the *Eidgenössische Hochschule Zürich* where he was professor since 1892; quite a few of Hurwitz's manuscripts and texts are now available in digital form, in particular his extensive diaries. During their *research in pairs* at the *Mathematisches Forschungsinstitut Oberwolfach* (MFO) in September 2019, the authors were investigating Hurwitz's diaries with respect to quaternions and some of their findings will be explained in the two subsequent sections. Moreover, the authors learned from a commemorative plaque in the library of MFO that the partner (Jack Todd) of a later researcher of quaternion arithmetic (Olga Taussky) was involved in the rescue of this extraordinary place of mathematical research after the end of Nazi-Germany.

2. Quaternions in Hurwitz's early Diaries

The first examination of quaternions in Hurwitz's diaries can be found in diary No. 14, which he kept from January 1, 1896 to February 1, 1897. An entry from August 30, entitled "*Euler'sche Identität* $\sum_{i=1}^4 x_i^2 \sum_{i=1}^4 y_i^2$", provides a new proof of Euler's four square-identity or, in other words, the norm equation for quaternions, i.e.,

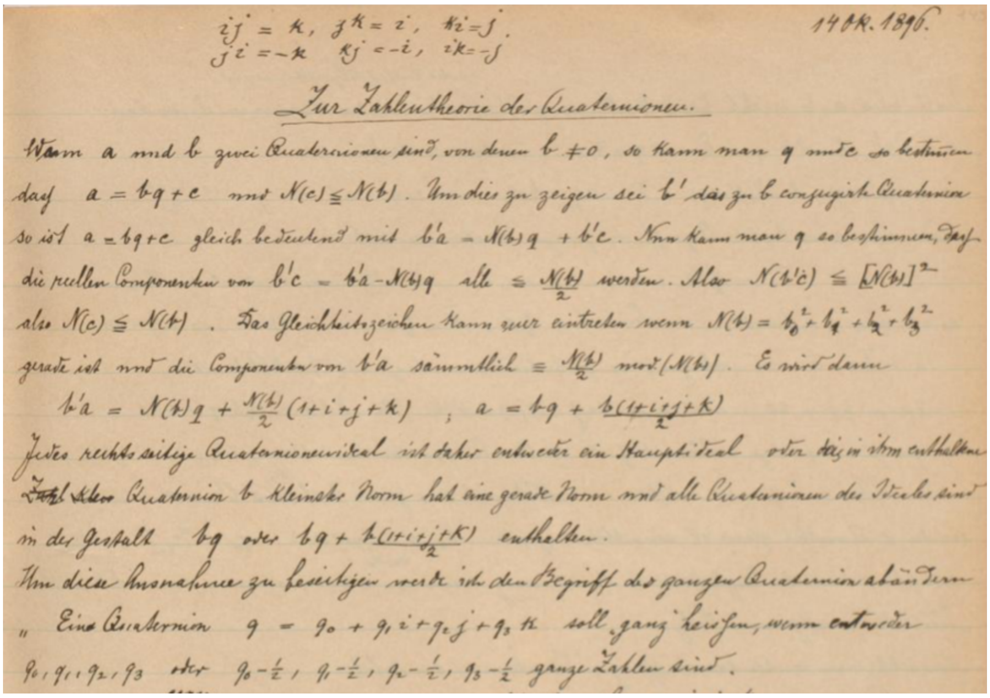
$$N(x \cdot y) = N(x) \cdot N(y) \qquad \text{for arbitrary } x, y \in \mathbb{R}[\mathrm{i}, \mathrm{j}].$$

This multiplicativity is an important ingredient in Lagrange's (or any) proof of the four square-theorem. Hurwitz's new and elegant proof relies on the vector product (for details see [15]). Probably this was the starting point for his interest in Lipschitz's work [14] on sums of four squares and quaternion arithmetic. Later entries consider, for example, "*the treatment of Lipschitz's number systems*" and "*orders of quaternions*". Obviously, Hurwitz was not satisfied with Lipschitz's approach; we can read in the diary:

> Lipschitz's proofs do not seem sufficiently clear and a direct justification would be desirable. Furthermore, the following questions should be settled: Are z_1, z_2, z_3, z_4 the bilinear [relations] of the $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, which appear in Euler's identity $\sum x^2 \sum y^2 = \sum z^2$ and is the calculation with the units e_1, e_2, e_3, e_4 determined by the demand $\sum e_i x_i \sum e_i y_i = \sum e_i z_i$; is then therewith the

system of quaternions determined? And does something similar is true for the formula $\sum_1^8 x_i^2 \sum_1^8 y_i^2 = \sum_1^8 z_i^2$?‖

On the following pages Hurwitz carried out some computations with respect to the multiplicative structure of an "extended" Lipschitz's number system (which is now called a *Clifford algebra*).



$ij = k$, $jk = i$, $ki = j$. $ji = -k$, $kj = -i$, $ik = -j$ 14 Okt. 1896.

Zur Zahlentheorie der Quaternionen.

Wenn a und b zwei Quaternionen sind, von denen $b \neq 0$, so kann man q und c so bestimmen daß $a = bq + c$ und $N(c) \leqq N(b)$. Um dies zu zeigen sei b' das zu b conjugirte Quaternion so ist $a = bq + c$ gleich bedeutend mit $b'a = N(b)q + b'c$. Nun kann man q so bestimmen, daß die reellen Componenten von $b'c = b'a - N(b)q$ alle $\leq \frac{N(b)}{2}$ werden. Also $N(b'c) \leqq [N(b)]^2$ also $N(c) \leqq N(b)$. Das Gleichheitszeichen kann nur eintreten wenn $N(b) = b_0^2 + b_1^2 + b_2^2 + b_3^2$ gerade ist und die Componenten von $b'a$ sämmtlich $\equiv \frac{N(b)}{2}$ mod. $(N(b))$. Es wird dann

$b'a = N(b)q + \frac{N(b)}{2}(1+i+j+k)$; $a = bq + \frac{b(1+i+j+k)}{2}$

Jedes rechtsseitige Quaternionenideal ist daher entweder ein Hauptideal oder das in ihm enthaltene ~~Zahl~~ Quaternion b kleinster Norm hat eine gerade Norm und alle Quaternionen des Ideales sind in der Gestalt bq oder $bq + \frac{b(1+i+j+k)}{2}$ enthalten.

Um diese Ausnahme zu beseitigen werde ich den Begriff des ganzen Quaternion abändern

„ Eine Quaternion $q = q_0 + q_1 i + q_2 j + q_3 k$ soll ganz heißen, wenn entweder q_0, q_1, q_2, q_3 oder $q_0 - \frac{1}{2}, q_1 - \frac{1}{2}, q_2 - \frac{1}{2}, q_3 - \frac{1}{2}$ ganze Zahlen sind.

FIGURE 1. Excerpt of the first page of Hurwitz's "*Zahlentheorie der Quaternionen*" in diary No. 14, p. 149. Hurwitz wrote that in order "to eliminate this exception [he] will change the notion of the integer quaternion".

The entry "*Zur Zahlentheorie der Quaternionen*", dated October 14, 1896, contains Hurwitz's approach which led to his aforementioned article [11], submitted about two months later and published almost immediately. The essential difference to Lipschitz's arithmetic becomes obvious very quickly, namely the demand of division with remainder and euclidicity, respectively. For this purpose Hurwitz used a different definition of

‖"Es scheinen die Lipschitz'schen Beweise nicht ausreichend klar und eine directe Begründung wäre wünschenswert. Auch sollten folgende Fragen erledigt werden: Sind z_1, z_2, z_3, z_4 die bilinearen [Verbindungen] der $x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4$, welche in der Euler'schen Identität $\sum x^2 \sum y^2 = \sum z^2$ vorkommen und bestimmt man die Rechnung mit den Einheiten e_1, e_2, e_3, e_4 durch die Forderung $\sum e_i x_i \sum e_i y_i = \sum e_i z_i$; ist dann dadurch das Quaternionensystem bestimmt? Und gilt etwa Ähnliches für die Fomel $\sum_1^8 x_i^2 \sum_1^8 y_i^2 = \sum_1^8 z_i^2$?"; (Diary No. 14, pp. 144) quaternion integers (as we already mentioned in the introduction; see also Figure 1).

In his treatment, Hurwitz systematically transferred the fundamental concepts of elementary and algebraic number theory to the quaternion numbers. In some places this transfer into the non-commutative structure needed considerable computational efforts. The elaborated definitions of greatest common divisor, units, group of integer quaternions coprime with a given prime number p, prime quaternions, permutations of quaternions, etc., which followed the satisfactory adaptation of the notion of quaternion integers, can later be found in Hurwitz's article [11] in a purposefully structured way, as well as the vivid treatment of Euler's *problema curiosum* [9] on magic squares of squares.

When it is about certain orthogonal substitutions, it becomes obvious that Hurwitz deliberately preferred the elementary approach wherever possible; Cayley's matrix calculus is circumvented almost everywhere. It is precisely the fact that he was aware of the different approaches and points of view that emphasizes particularly clearly his efforts to create an *elementary* number theory at this point. The extensive generalized explanations on orthogonal substitutions on the diary pages in between 166 and 182 are actually not included in Hurwitz's publication.

With regard to the subsequent treatment of "*Quaternionenordnungen*" in his diary, it is noticeable that Hurwitz later changed his terminology. These occasionally confusing changes between different notions is fortunately often clarified by precise definitions, as in this case of page 182:

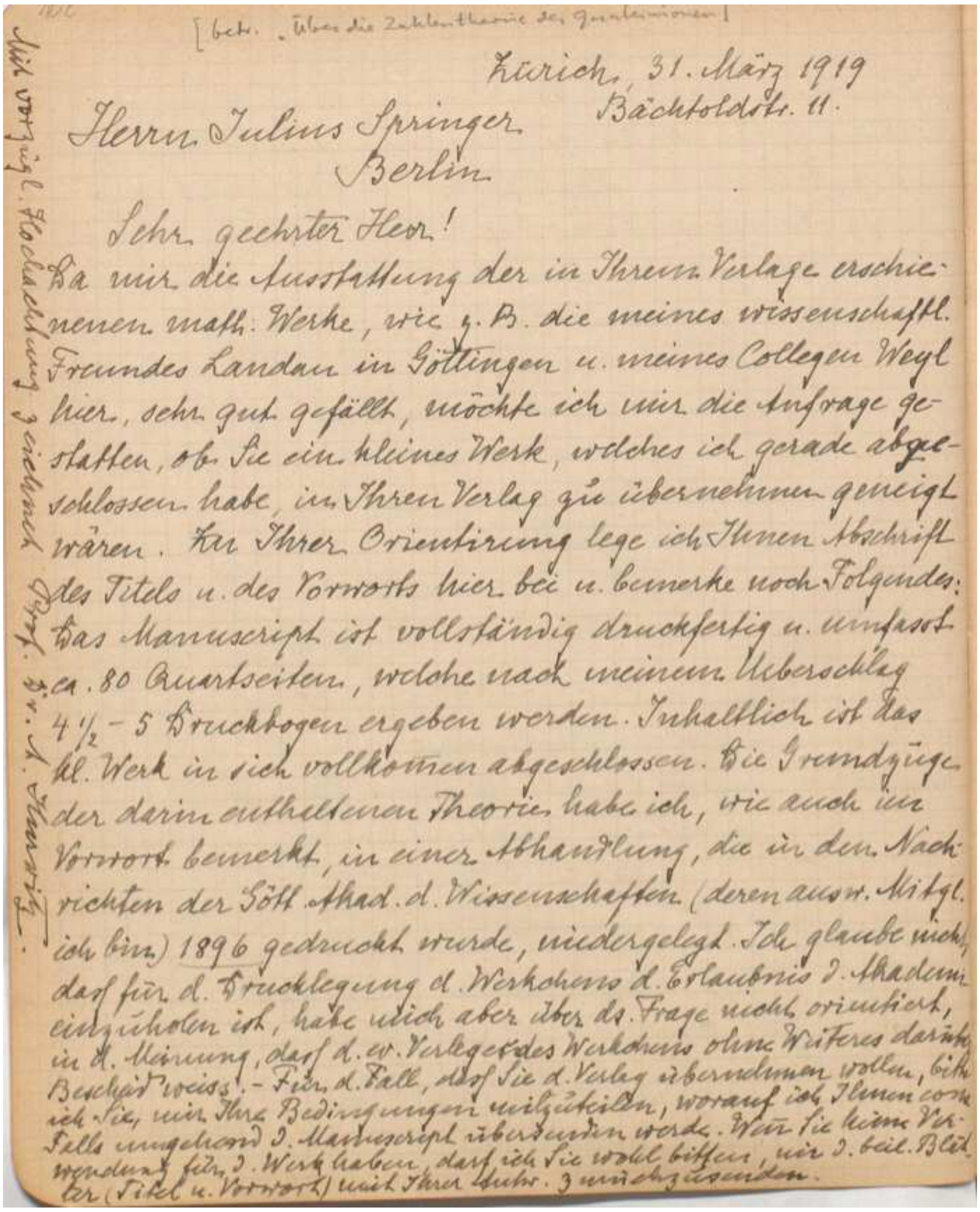
> A system of rational quaternions is called quaternion order if sum, difference and product of any two quaternions of the system are also in the system and if all quaternions of the system can be represented linearly with integer coefficients by a finite number of them.**

In modern terminology, an *order* O is a subring of some ring R such that i) R is a finite-dimensional algebra over $\mathbb{Q}$, ii) O spans R over $\mathbb{Q}$, and iii) O is a $\mathbb{Z}$-lattice (see [13]).

Further diaries indicate Hurwitz's continuous interest in quaternions between 1896 and 1919. Diary no. 15, for example, contains an entry

** "Ein System von rationalen Quaternionen heißt Quaternionenordnung, wenn sich Summe, Differenz und Produkt irgend zweier Quaternionen des Systems ebenfalls in dem System befinden und wenn alle Quaternionen des Systems linear mit ganzzahligen Coefficienten durch eine endliche Anzahl unter ihnen darstellbar sind."

(written between May and June 1897) about "*Algebraische Gebilde mit Quaternionengruppe*", and in diary No. 25 from 1911/12 one can find a short entry about "*Quaternionen + binäre Substitutionen*", where Hurwitz described the isomorphism between the two number systems of quaternions and of substitutions. And the preceding and following entries in this diary show that Hurwitz prepared notes for a seminar on (hyper-) complex number systems.



[betr. „Über die Zahlentheorie der Quaternionen"]

Zürich, 31. März 1919
Bächtoldstr. 11.

Herrn Julius Springer
Berlin

Sehr geehrter Herr!

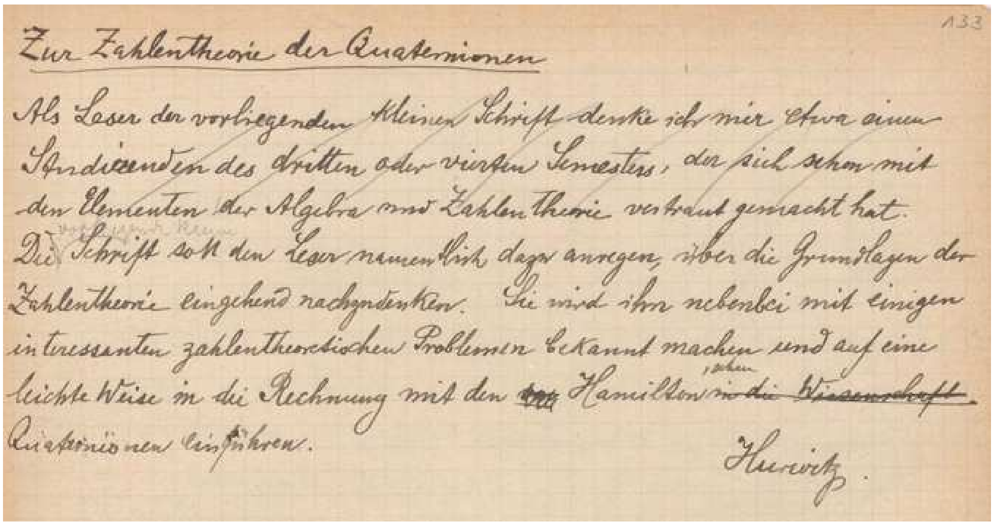
Da mir die Ausstattung der in Ihrem Verlage erschienenen math. Werke, wie z. B. die meines wissenschaftl. Freundes Landau in Göttingen u. meines Collegen Weyl hier, sehr gut gefällt, möchte ich mir die Anfrage gestatten, ob Sie ein kleines Werk, welches ich gerade abgeschlossen habe, in Ihren Verlag zu übernehmen geneigt wären. Zu Ihrer Orientirung lege ich Ihnen Abschrift des Titels u. des Vorworts hier bei u. bemerke noch Folgendes: Das Manuscript ist vollständig druckfertig u. umfasst ca. 80 Quartseiten, welche nach meinem Ueberschlag 4½–5 Druckbogen ergeben werden. Inhaltlich ist das kl. Werk in sich vollkommen abgeschlossen. Die Grundzüge der darin enthaltenen Theorie habe ich, wie auch im Vorwort bemerkt, in einer Abhandlung, die in den Nachrichten der Gött. Akad. d. Wissenschaften (deren ausw. Mitgl. ich bin) 1896 gedruckt wurde, niedergelegt. Ich glaube nicht, daß für d. Drucklegung d. Werkchens d. Erlaubnis d. Akademie einzuholen ist, habe mich aber über d. Frage nicht orientiert, in d. Meinung, daß d. ev. Verleger des Werkchens ohne Weiteres darüber Bescheid weiss. – Für d. Fall, daß Sie d. Verlag übernehmen wollen, bitte ich Sie, mir Ihre Bedingungen mitzuteilen, worauf ich Ihnen [illegible] umgehend d. Manuscript übersenden werde. Wenn Sie keine Verwendung für d. Werk haben, darf ich Sie wohl bitten, mir d. beil. Blätter (Titel u. Vorwort) mit Ihrer Antw. [illegible] zurückzusenden.

Mit vorzügl. Hochachtung [illegible] Prof. Dr. A. Hurwitz.

Figure 2. Concept of the letter to Julius Springer from March 31, 1919, on page 102 of diary No. 30.

3. Quaternions in Hurwitz's later diaries

After a few years without noticeable entries on quaternions, we can find in diary No. 28 from 1917, p. 186, the description of a topic for a diploma thesis, "*Die endlichen Körper*", for a student by the name Herter*. Hurwitz referred here to the classification of finite fields by Eliakim Hastings Moore, the teacher of Leonard Dickson who will play a certain role later on. In his last diary from 1919, No. 30, p. 100, Hurwitz came back to this task for his student. Building on results from his 1896 paper, Hurwitz formulated an open question concerning the number of solutions of certain diophantine equations modulo a prime number. This diary entry from March 21 was the starting point of about eighty subsequent pages of investigations on quaternions.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Zur Zahlentheorie der Quaternionen

Als Leser der vorliegenden kleinen Schrift denke ich mir etwa einen Studierenden des dritten oder vierten Semesters, der sich schon mit den Elementen der Algebra und Zahlentheorie vertraut gemacht hat. Die Schrift soll den Leser namentlich dazu anregen, über die Grundlagen der Zahlentheorie eingehend nachzudenken. Sie wird ihn nebenbei mit einigen interessanten zahlentheoretischen Problemen bekannt machen und auf eine leichte Weise in die Rechnung mit den ~~der~~ Hamiltonschen ~~in die Wissenschaft~~ Quaternionen einführen.

Hurwitz.

Figure 3. This is a draft of a preface from Hurwitz's last diary from summer 1919.

Certainly this last notebook is the most interesting one concerning quaternions. It seems that Hurwitz was working out ideas at a certain speed†, sometimes calculations and thoughts follow one another with no direct link or with a connection only difficult to comprehend. Besides the aforementioned outline of a topic for a thesis, and several interesting letters to Julius Springer, the editor of Hurwitz's *Vorlesungen* [12] (see Figure 2), there is an interesting first draft of a preface with quite a few differences to the later published one (see Figure 3); here Hurwitz stressed

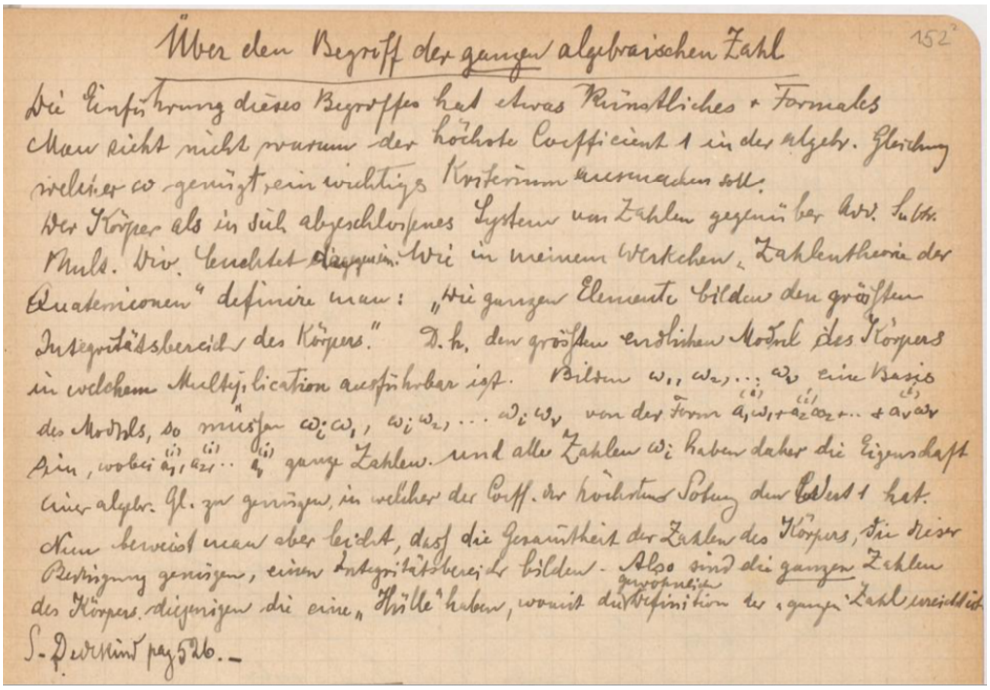
*about whom we unfortunately could not find out anything

†From a letter to Pòlya from August 31, 1919, we know that Hurwitz was already very ill. Maybe he indeed felt a certain time pressure.

that "the booklet on quaternions could stimulate the reader to think about the foundations of number theory."‡

Moreover, one can find in this diary some reflections on the notion of algebraic integers (see Figure 4) — a topic of utmost importance for Hurwitz' number theory of quaternions — and, in particular, his late and extensive investigations on maximal orders catch the eye. We shall have a closer look on these maximal orders here.

In algebraic number theory, algebraic integers are defined as roots of a monic (minimal) polynomial with integer coefficients and they form a commutative ring. Their intersection with a number field (i.e., is a finite algebraic extension of $\mathbb{Q}$) constitutes an algebraically closed ring, called the *maximal order* (or the ring of integers). For example, the maximal orders of the quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ are $\mathbb{Z}[\sqrt{-1}]$ and $\mathbb{Z}[\frac{1}{2}(1+\sqrt{-3})]$, respectively; note that $\mathbb{Z}[\sqrt{-3}]$ is also an order in $\mathbb{Q}(\sqrt{-3})$ but it is not algebraically closed (maximal). We refer to [13] for details.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Über den Begriff der ganzen algebraischen Zahl

Die Einführung dieses Begriffes hat etwas Künstliches + Formales
Man sieht nicht warum der höchste Coefficient 1 in der algebr. Gleichung
welcher ω genügt, ein wichtiges Kriterium ausmachen soll.
Der Körper als in sich abgeschlossenes System von Zahlen gegenüber Add. Subtr.
Mult. Div. leuchtet ~~dagegen~~ ein. Wie in meinem Werkchen „Zahlentheorie der
Quaternionen" definire man: „Die ganzen Elemente bilden den größten
Integritätsbereich des Körpers." D.h. den größten endlichen Modul des Körpers
in welchem Multiplication ausführbar ist. Bilden $\omega_1, \omega_2, \ldots, \omega_\nu$ eine Basis
des Moduls, so müssen $\omega_i\omega_1, \omega_i\omega_2, \ldots \omega_i\omega_\nu$ von der Form $a_1^{(i)}\omega_1 + a_2^{(i)}\omega_2 + \ldots + a_\nu^{(i)}\omega_\nu$
sein, wobei $a_1^{(i)}, a_2^{(i)}, \ldots a_\nu^{(i)}$ ganze Zahlen. und alle Zahlen ω_i haben daher die Eigenschaft
einer algebr. Gl. zu genügen, in welcher der Coeff. der höchsten Potenz den Wert 1 hat.
Nun beweist man aber leicht, daß die Gesamtheit der Zahlen des Körpers, die dieser
Bedingung genügen, einen Integritätsbereich bilden. Also sind die ganzen Zahlen
des Körpers diejenigen die eine „Hülle" haben, womit die gewöhnliche Definition der „ganzen Zahl" erreicht.
S. Dedekind pag 526. –

FIGURE 4. Reflections *Über den Begriff der* <u>*ganzen*</u> *algebraischen Zahl* on page 152 of diary No. 30.

On page 139, the correspondence between 2×2 matrices and quaternions given as linear combinations of the fundamental units is considered. We

‡"Die Schrift soll den Leser namentlich dazu anregen, über die Grundlagen der Zahlentheorie eingehend nachzudenken."

shall explain this in our notation here briefly. Quaternions may be introduced by writing complex entries $a = \alpha + \beta\sqrt{-1}, b = \gamma + \delta\sqrt{-1}$ in a 2×2 matrix, namely

$$\begin{aligned} \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix} &= \begin{pmatrix} \alpha + \beta\sqrt{-1} & \gamma + \delta\sqrt{-1} \\ -\gamma + \delta\sqrt{-1} & \alpha - \beta\sqrt{-1} \end{pmatrix} \\ &= \alpha \cdot \mathbf{1} + \beta \cdot \mathbf{I} + \gamma \cdot \mathbf{J} + \delta \cdot \mathbf{K}, \end{aligned}$$

where

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix},$$

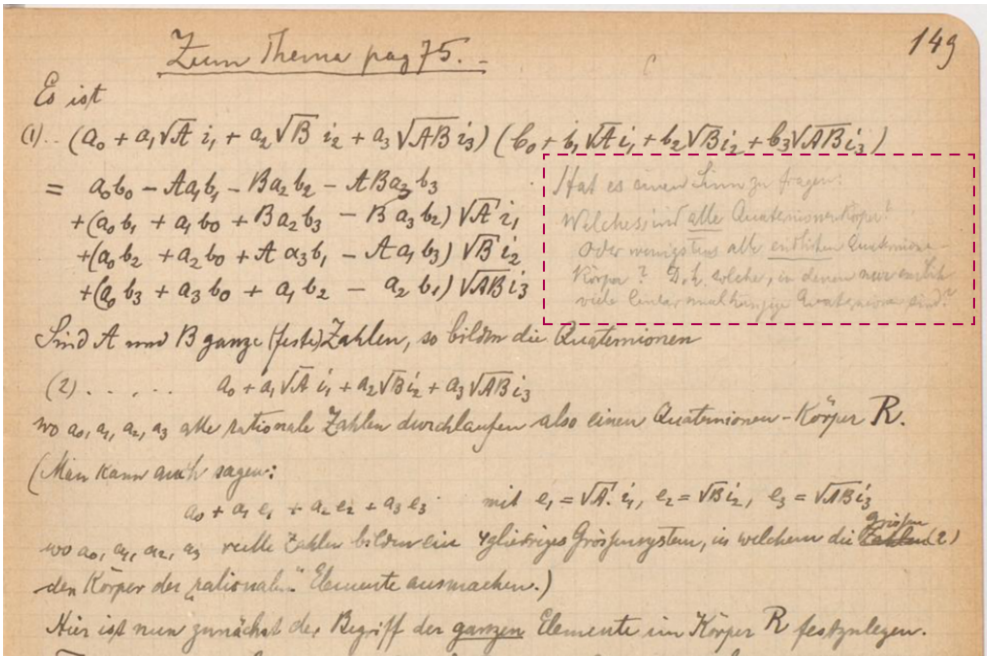
$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}$$

are skew-hermitian matrices according to the base $1, \mathrm{i}, \mathrm{j}, \mathrm{k}$ of the skewfield of quaternions. This construction is due to Cayley and the case of $\beta = \delta = 0$ leads to his matrix representation of the complex numbers. Following this approach, Hurwitz is finally led to quaternions with complex coordinates (as considered already by Hamilton as "biquaternions"). He investigated transformations of a group of such "complex" quaternions with respect to real quaternions.

This search for suitable extensions of quaternions accompanied Hurwitz throughout the final pages of his last diary. On page 149, just before his thoughts on algebraic integers, he returned to a question from page 75 about a proposed generalization of coefficients to real numbers $\sqrt{A}, \sqrt{B}, \sqrt{AB}$ (see Fig 5). Hurwitz referred here explicitly to pages 211 and 327 in the book "*Niedere Zahlentheorie*" [1] of Paul Bachmann, where sums of squares with corresponding coefficients are discussed. Such kind of quaternions had already been investigated earlier by Dickson [4]; however, the arithmetical context is new. We want to highlight some notations written with pencil next to his investigations of the multiplicative structural:

> Does it make sense zu ask: Which are all fields of quaternions? Or at least all finite fields of quaternions? i.e. those which contain only finitely many linearly independent quaternions?[§]

[§] "Hat es einen Sinn zu fragen: Welches sind alle Quaternionenkörper? Oder wenigstens alle endlichen Quaternionenkörper? D.H. welche, in denen nur endliche viele linear unabhängige Quaternionen sind?"


Zum Thema pag 75.

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Es ist

(1) $(a_0 + a_1\sqrt{A}\, i_1 + a_2\sqrt{B}\, i_2 + a_3\sqrt{AB}\, i_3)(b_0 + b_1\sqrt{A}\, i_1 + b_2\sqrt{B}\, i_2 + b_3\sqrt{AB}\, i_3)$

$= a_0b_0 - Aa_1b_1 - Ba_2b_2 - ABa_3b_3$
$+ (a_0b_1 + a_1b_0 + Ba_2b_3 - Ba_3b_2)\sqrt{A}\, i_1$
$+ (a_0b_2 + a_2b_0 + Aa_3b_1 - Aa_1b_3)\sqrt{B}\, i_2$
$+ (a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)\sqrt{AB}\, i_3$

Hat es einen Sinn zu fragen: Welches sind alle Quaternionen-Körper? Oder wenigstens alle endlichen Quaternionen-Körper? D.h. solche, in denen nur endlich viele linear unabhängige Quaternionen sind?

Sind A und B ganze (feste) Zahlen, so bilden die Quaternionen

(2) . . . . . $a_0 + a_1\sqrt{A}\, i_1 + a_2\sqrt{B}\, i_2 + a_3\sqrt{AB}\, i_3$

wo a_0, a_1, a_2, a_3 alle rationale Zahlen durchlaufen also einen Quaternionen-Körper R.

(Man kann auch sagen:

$a_0 + a_1e_1 + a_2e_2 + a_3e_3$ mit $e_1 = \sqrt{A}\, i_1$, $e_2 = \sqrt{B}\, i_2$, $e_3 = \sqrt{AB}\, i_3$

wo a_0, a_1, a_2, a_3 reelle Zahlen bilden ein 4gliedriges Größensystem, in welchem die Größen (e) den Körper der rationalen Elemente ausmachen.)

Hier ist nun zunächst der Begriff der ganzen Elemente im Körper R festzulegen.

FIGURE 5. Special coefficient for quaternions appear on page 149 of diary No. 30. The research questions, written with pencil, were framed by the authors.

The language here is difficult to understand without the context. Further "finite fields of quaternions" (*endlichen Quaternionenkörper*) are skewfields of quaternions whose coefficients are elements of finite algebraic extensions. Hurwitz's questions do indeed lead to interesting observations with regard to maximal orders. From the subsequently discussed example, the "non-commutative limits" of Dedekind's definition (see Fig. 4) become much clearer than with the quaternions of rational coefficients.

Starting from page 154, Hurwitz investigated the set of quaternions with coefficients from the Gaussian number field $\mathbb{Q}(\sqrt{-1})$ (still with the ordinary imaginary $\sqrt{-1}$ from the complex plane). As before, his aim was to find the counterpart of the ring J of integers in the skew field of quaternions with rational coefficients. His considerations were not published and since they were written in parallel to the gain in knowledge, they are – as interesting as they are – difficult to read. In the following we take a closer look at them.

Hurwitz's approach may be considered as an analogue of extending the arithmetic of rational numbers to algebraic extensions. After some lengthy computations including various case distinctions and different strategies, on page 189 (see also Figure 6), Hurwitz finally achieved a list of eight integral domains, some being contained in another, and claimed that *there are four maximal orders of integers*. This is pretty different to the situation in algebraic number theory or even Hurwitz' treatment of quaternion integers in his booklet [12] where always a *unique* maximal order exists!

Following Hurwitz's notation (closely), one of the four maximal orders is denoted by $\mathcal{I}_7$ and consists of the quaternions

$$\frac{1}{2}(\kappa_0 + \kappa_1 i_1 + \kappa_2 i_2 + \kappa_3 i_3) \tag{1}$$

with coefficients $\kappa_1, \ldots, \kappa_4$ satisfying the congruences

$$\kappa_0 \equiv \kappa_1 + \kappa_2 + \kappa_3 \mod 2 \qquad \text{and} \qquad \kappa_1 \equiv \kappa_2 \equiv \kappa_3 \mod (1+\sqrt{-1});$$

notice that the coefficients here and in the sequel are from the ring $\mathbb{Z}[\sqrt{-1}]$ of Gaussian integers and the congruences as well have to be understood with respect to this ring. The other three maximal orders come in a triple $\mathcal{I}_{8a}, \mathcal{I}_{8b}$ and $\mathcal{I}_{8c}$, where $\mathcal{I}_{8a}$ is given by the quaternions (1) for which

$$\kappa_0 \equiv \kappa_1\sqrt{-1}\ , \quad \text{and} \quad \kappa_2 \equiv \kappa_3\sqrt{-1} \mod 2$$

holds, and $\mathcal{I}_{8b}$ and $\mathcal{I}_{8c}$ result from $\mathcal{I}_{8a}$ by cyclic permutation of the units (see Figure 6). Hurwitz's calculated these maximal orders correctly by a method similar to the one in the fourth lecture of his booklet [12]. To be explicit we thus may take

$$\begin{aligned} h_1 &:= \tfrac{1}{2}(1+\sqrt{-1}i_1), \quad h_2 := h_1' = \tfrac{1}{2}(1-\sqrt{-1}i_1), \\ h_3 &:= \tfrac{1}{2}(i_2+\sqrt{-1}i_3), \quad h_4 := h_3' = \tfrac{1}{2}(i_2-\sqrt{-1}i_3) \end{aligned}$$

as base for $\mathcal{I}_{8a}$, where the conjugation here is the one in $\mathbb{Z}[\sqrt{-1}]$.

A few years later, Dickson [7] was following the same line of inquiries, however, his main result already deals with a general quadratic number field $\mathbb{Q}(\Theta)$. In the case of $\Theta^2 = d = 4n+3$ for some integer n, the bases for his maximal orders are, in his notation and writing $\mathrm{i}, \mathrm{j}, \mathrm{k}$ for Hurwitz' set of square roots i_1, i_2, i_3 of -1, given by

$$\Pi := \left\{1, \tfrac{1}{2}(1+\mathrm{i}+\mathrm{j}+\mathrm{k}), \mathrm{j}+f, n\mathrm{k}+f+g\right\},$$

where

$$f = \tfrac{1}{2}(1+\Theta)(1+\mathrm{k}) \qquad \text{and} \qquad g = \tfrac{1}{2}(1+\Theta)(1+\mathrm{j}),$$

as well as

$$\Sigma = \left\{1, \mathrm{j}, \tfrac{1}{2}(\mathrm{i}-\Theta), \tfrac{1}{2}(\mathrm{k}+\Theta\mathrm{j})\right\}$$

and two further by cyclic permutation of the units. This result actually includes Hurwitz's four maximal orders for the special case $\Theta = \sqrt{-1}$ (in which case $n = -1$). In order to see that, a short computation shows that the base $\{h_1, h_2, h_3, h_4\}$ for $\mathcal{I}_{8a}$ and $\Sigma = \{d_1, d_2, d_3, d_4\}$ (in the same

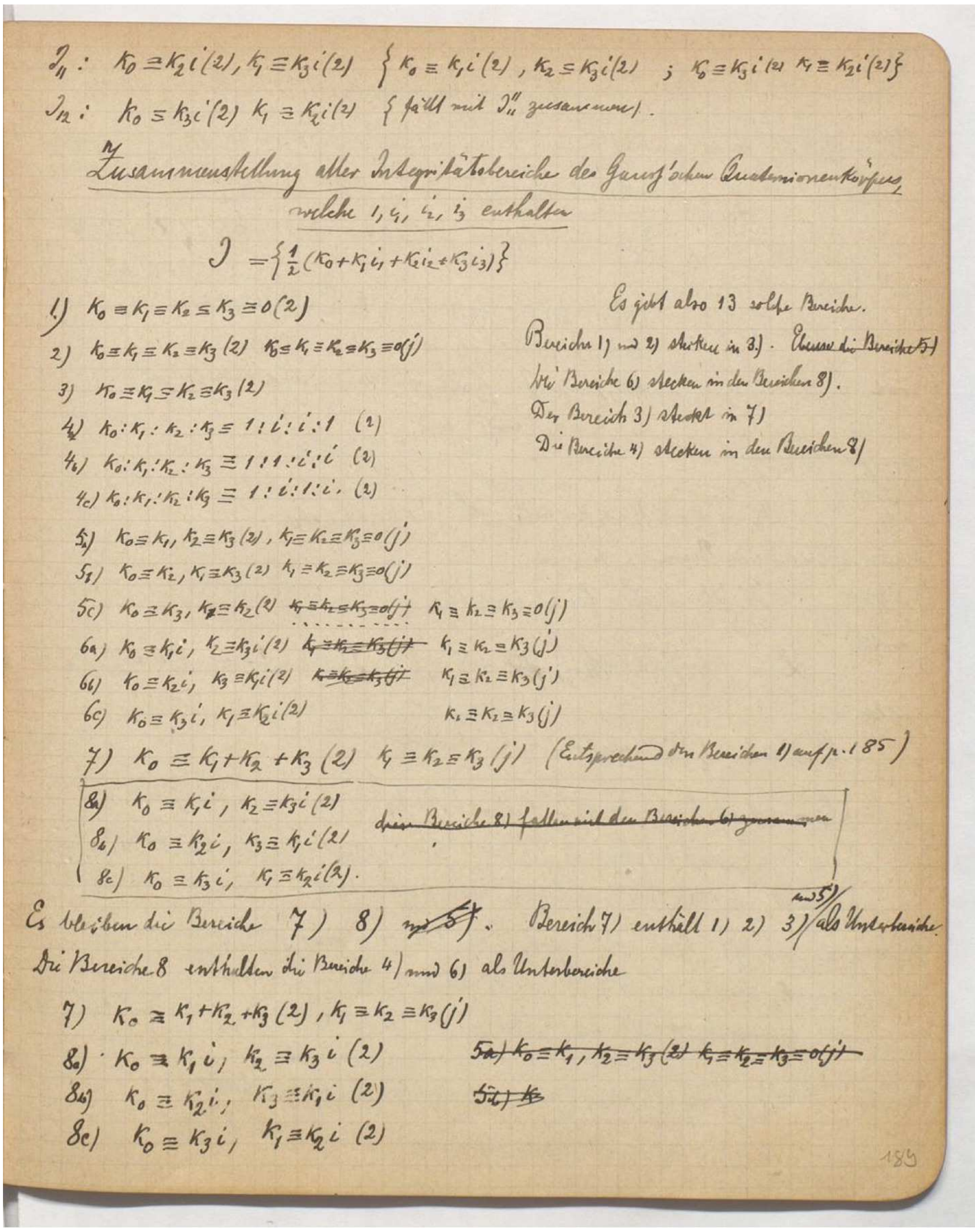

FIGURE 6. List of "Integritätsbereiche des Gaußschen Quaternionenkörpers" on page 189 of diary No. 30.

sequence as above, i.e., $d_1 = 1$ etc.) are transformed into one another according to

$$\Big(h_1, h_2, h_3, h_4\Big) = \begin{pmatrix} 0 & 0 & \sqrt{-1} & 0 \\ 1 & 0 & -\sqrt{-1} & 0 \\ 0 & 1 & 0 & \sqrt{-1} \\ 0 & 0 & 0 & -\sqrt{-1} \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{pmatrix}$$

and identifying i_1, i_2, i_3 with $\mathrm{i}, \mathrm{j}, \mathrm{k}$, resp.

$$\Big(d_1, d_2, d_3, d_4\Big) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ -\sqrt{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{-1} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{pmatrix}.$$

By cyclic permutation, this shows that the triples of Hurwitz and Dickson correspond to each other. By a similar reasoning one finds that also the remaining maximal orders coincide. Thus, *the result from the last diary and the corresponding case of Dickson's more general result coincide.* Hurwitz's attempt to consider a quaternion analogue of the for algebraic number theory and its development in the 19th century so essential topic of extending the integers was anticipating the celebrated work of Dickson, and, furthermore, Hurwitz's approach along the lines of his booklet would indeed permit the generalization Dickson had.

The very last entry of his diary deals with some techniques from the dissertation "*Zahlentheorie der Tettarionen*" [16] of his pupil Gustave Du Pasquier to the aforementioned problem (see Figure 7). In the 1920s Dickson published quite a few papers and books on the arithmetic of algebras (and quaternions in particular; see his monographs [5, 6], and we refer to [2, 8] for their mathematical impact). Later in the twentieth century, with the work of Heinrich Brandt, Martin Eichler and Marie-France Vignéras [17] to name just a few, research into the number theory of quaternions shifted more towards ideal theory. A similar phenomenon of non-unique maximal orders has been discovered in the ring of octonions by Harold Scott MacDonald Coxeter [3].

4. Coda

It is remarkable that Hurwitz, at that late time of his disease, had the ambition, energy and a cool head to continue his investigations on quaternions. Embedded in the above calculations is the copy of a very personal document, the medical certificate of his family doctor as well as his request to the President of the *Highschool Council*, K. Gnehm, "since the discomforts have increased noticeably in the last few months, to be dispensed from the lecture during the coming winter semester."¶ Hurwitz himself offered:

¶"[...] da die Beschwerden in den letzten Monaten merklich zunehmen, während des kommenden Wintersemesters von der Vorlesung dispensiert zu werden."

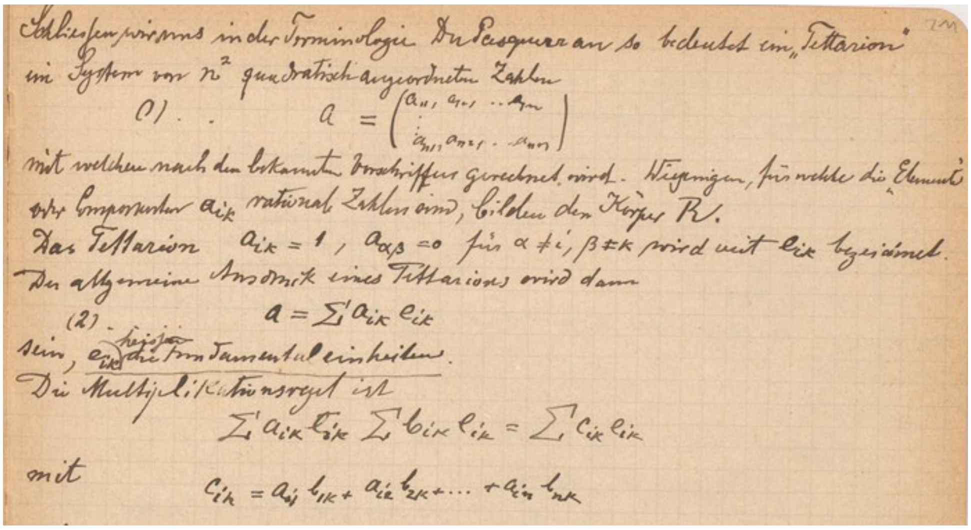

FIGURE 7. This shows the penulitmate page (page 211) in Hurwitz's diary; on page 178 appears the last written date and it is Sunday, 5th of October 1919.

> I would be very grateful if you, on the basis of the enclosed medical certificate, would release me from the obligation to give the lecture on alg. equations which I have announced, and thus agree that next winter I will only give the mathemat. Seminar.‖

This planning could not be realised. Hurwitz died from kidney malfunction on November 18, 1919. With his extensive and valuable estate he left a great treasure to the history of mathematics.

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‖ "Ich wäre Ihnen sehr dankbar, wenn Sie mich auf Grund des beiliegenden ärztlichen Zeugnisses von der Verpflichtung, die von mir angekündigte Vorlesung über alg. Gleichungen zu halten, befreien und also zustimmen würden, daß ich im kommenden Winter nur das mathemat. Seminar abhalte."; (Hurwitz, 1919, p. 168)

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Nicola M.R. Oswald
Bergische Universität Wuppertal
Fakultät für Mathematik und Naturwissenschaften
Arbeitsgruppe Didaktik und Geschichte der Mathematik
Gaußstraße 20
42 119 Wuppertal, Germany
oswald@uni-wuppertal.de

Jörn Steuding
Julius-Maximilians Universität Würzburg
Fakultät für Mathematik und Informatik
Institut für Mathematik
Emil Fischer-Str. 40
97 074 Würzburg, Germany
steuding@mathematik.uni-wuerzburg.de