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Geometric Structures in Group Theory (hybrid meeting)

Organized by
Martin Bridson, Oxford
Cornelia Drutu, Oxford
Linus Kramer, Münster
Bertrand Rémy, Palaiseau
Petra Schwer, Magdeburg

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Abstract. The conference focused on the use of geometric methods to study infinite groups and the interplay of group theory with other areas. One of the central techniques in geometric group theory is to study infinite discrete groups by their actions on nice, suitable spaces. These spaces often carry an interesting large-scale geometry, such as non-positive curvature or hyperbolicity in the sense of Gromov, or are equipped with rich geometric or combinatorial structure. From these actions one can investigate structural properties of the groups. This connection has become very prominent during the last years. In this context non-discrete topological groups, such as profinite groups or locally compact groups appear quite naturally. Likewise, analytic methods and operator theory play an increasing role in the area.

Mathematics Subject Classification (2020): 20Exx, 20Fxx, 20Jxx, 47Bxx, 57Mxx.

Introduction by the Organizers

A unifying theme in the modern study of infinite groups is to study actions of such groups on spaces with an interesting geometric structure. There is a rich interplay between geometric properties of the space that a group can (or cannot) act on, and algebraic properties of the group. This meeting focused on thriving areas within this setting.

The spaces in question often have specific curvature properties, such as the CAT(0) condition, or Gromov's notion of hyperbolicity. Results about group actions on

CAT(0) cube complexes have lead, for example, to the positive solution of several points in Thurston's program for 3-manifolds. Several talks during this workshop highlighted new developments in the theory of groups acting on CAT(0) cube complexes and on hyperbolic or nonpositively curved spaces.

But other geometric conditions have also emerged during the last years. The notion of acylindrically hyperbolic groups has opened the door for studying a large class of non-hyperbolic groups by geometric means. Properties of random groups or growth properties of groups are topics of continuing interest in the area, and several lectures reported on them.

Another recurring theme is to study the automorphism groups of interesting groups, such as free groups, Coxeter groups, and Artin groups. One major recent breakthrough in this area, which was also highlighted in the conference, is the construction of an outer space for right-angled Artin groups.

More on the analytic side the following topics were covered. Measure-theoretic properties of group actions were discussed, such as measure equivalence and rigidity results related to them. The notion of amenability played a central role in various talks. It was in particular used to study questions about group C^*-algebras. Moreover new results on L^2-Betti numbers were presented. Kazhdan's property (T) as well as strong analogs played a prominent role in several lectures at the conference. In addition nonstandard analytic methods were shown to lead to compactifications of character varieties with good properties.

This workshop took place under special conditions. In the implemented hybrid format, both in-person and virtual participants attended the meeting. This was the first Oberwolfach workshop after the institute had been shut down due to the COVID-19 pandemic for more than 3 months. A reduced number of 20 participants was present in Oberwolfach, while roughly the same number participated virtually. The lectures were broadcast live and provided as video recordings for all those participants who could not come in person. Besides eleven lectures presented by participants present at the workshop, we had four video lectures from participants in Japan, Canada, the United States, and Great Britain. This mixture worked remarkably well and the conference benefited immensely from the video lectures.

Even though the size of the workshop was reduced the interaction between the participants was, as always, very lively. The schedule provided enough time for discussions in smaller groups during the early afternoons. It is safe to say that we can expect to hear about several interesting projects that emerged during this week at future group theory conferences.

The whole staff in Oberwolfach succeeded in meeting the necessary hygiene measures while at the same time providing excellent working conditions and a relaxed and enjoyable atmosphere as usual. We would especially like to mention the supportive and helpful work of the IT group. The recording and video equipment set up in the lecture hall worked remarkably well and we had plenty of support during the week. We thank the staff in Oberwolfach very much for providing such excellent working conditions during these difficult times.

We are glad that this workshop could take place in this reduced form. Although the virtual attendance of many participants was (obviously) not a replacement for being in Oberwolfach, we got the impression that overall this conference was very successful and has fully met the high standards of the Oberwolfach workshops. The scientific exchange between the participants in Oberwolfach and elsewhere will certainly lead to exciting new developments and collaborations.

Workshop (hybrid meeting): Geometric Structures in Group Theory

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Abstracts

The real spectrum compactification of character varieties: characterizations and applications

Alessandra Iozzi
(joint work with Marc Burger, Anne Parreau, Maria Beatrice Pozzetti)

We consider the *real spectrum compactification* $\Xi^{\mathrm{RSp}}(\Gamma, G)$ of the character variety $\Xi^{\mathrm{RSp}}(\Gamma, G) = \mathrm{Hom}(\Gamma, G)/\sim$, where Γ is finitely generated, $G < \mathrm{SL}_n$ is a semisimple algebraic group defined over the field $\overline{\mathbb{Q}}^{\mathrm{r}}$ of the real algebraic numbers and $\sim$ is G-conjugation. We give properties of $\Xi^{\mathrm{RSp}}(\Gamma, G)$ and of its closed points $\Xi^{\mathrm{RSp}}_{\mathrm{cl}}(\Gamma, G)$ and we characterize $\Xi^{\mathrm{RSp}}(\Gamma, G) \setminus \Xi^{\mathrm{RSp}}(\Gamma, G)$ in terms of (equivalence classes of) representations $\rho\colon \Gamma \to G_{\mathbb{F}}$, where $\mathbb{F}$ is a non-Archimedena minimal real closed field on which there exists an order compatible valuation $v\colon \mathbb{F} \to \mathbb{R}_{\geq 0}$. Inspired by work by Kramer and Tent (see [4] for an announcement and [1] for details), we show that associated to any representation $(\rho, \mathbb{F})$ there is a Λ-building $\mathcal{B}_{G_{\mathbb{F}}}$, where $\Lambda := v(\mathbb{F}^{\times})$, on which Γ acts isometrically. We can hence further characterize the closed points $(\rho, \mathbb{F}) \in \Xi^{\mathrm{RSp}}_{\mathrm{cl}}(\Gamma, G) \setminus \Xi^{\mathrm{RSp}}(\Gamma, G)$ as those on which Γ acts without fixed points or, equivalently, those for which there exists an element of finite length $\leq 2^n - 1$ (where n is the cardinality of a finite generating set of Γ) that has positive translation length. Any such representation $(\rho, \mathbb{F})$ can also be realized as (equivalent to) an $(\omega, \boldsymbol{\lambda})$-limit of a sequence $\rho_k\colon \Gamma \to G$, where ω is a non-principal ultrafilter on $\mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_k)_{k \geq 1}$ is a sequence of scales defined in terms of the displacement functions of $(\rho_k)_{k \geq 1}$. Furthermore we show that there is a field embedding $\mathbb{F} \hookrightarrow \mathbb{R}^{\omega}_{\boldsymbol{\lambda}}$, where $\mathbb{R}^{\omega}_{\boldsymbol{\lambda}}$ is the Robinson filed associated to ω and $\boldsymbol{\lambda}$, and an embedding of buildings $\overline{\mathcal{B}_{G_{\mathbb{F}}}} \hookrightarrow \mathcal{B}_{G_{\mathbb{R}\omega_{\boldsymbol{\lambda}}}}$ that identifies $\overline{\mathcal{B}_{G_{\mathbb{F}}}}$ with a totally geodesic CAT(0)-complete subspace of the affine building $\mathcal{B}_{G_{\mathbb{R}\omega_{\boldsymbol{\lambda}}}}$. Since $\mathcal{B}_{G_{\mathbb{R}\omega_{\boldsymbol{\lambda}}}}$ can be identified isometrically with an asymptotic cone of symmetric spaces, $\overline{\mathcal{B}_{G_{\mathbb{F}}}}$ inherits this CAT(0) metric.

This whole machinery can be used to show also that if $(\rho, \mathbb{F}) \in \Xi^{\mathrm{RSp}}(\Gamma, G) \setminus \Xi^{\mathrm{RSp}}(\Gamma, G)$, then the action of Γ on $\overline{\mathcal{B}_{G_{\mathbb{F}}}}$ is proper in the sense of Korevaar and Schoen, [3, § 3], and this implies the existence of a $\Gamma = \pi_1(M)$-equivariant Lipschitz harmonic map $\widetilde{M} \to \overline{\mathcal{B}_{G_{\mathbb{F}}}}$, where M is a complete Riemannian manifold.

Finally we show that if $\rho : \Gamma \to G_{\mathbb{F}}$ is any representation, where $\mathbb{F}$ is a real closed filed with an order compatible evaluation, the building $\mathcal{B}_{G_{\mathbb{F}}}$ arises as a quotient of the symmetric space associated to $G_{\mathbb{F}}$.

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Group C^*-algebras of locally compact groups acting on trees

Tim de Laat
(joint work with Dennis Heinig and Timo Siebenand)

There are several ways to construct C^*-algebras from locally compact groups. Such group C^*-algebras typically encode part of the (unitary) representation theory of the underlying group. The best known are the universal group C^*-algebra $C^*(G)$ and the reduced group C^*-algebra $C^*_r(G)$ of a locally compact group G. It is well known that these two algebras coincide if and only if G is amenable.

For our purposes, a group C^*-algebra of G is a completion A of $C_c(G)$ with respect to a C^*-norm $\|.\|_\mu$ satisfying $\|f\|_u \geq \|f\|_\mu \geq \|f\|_r$ for all $f \in C_c(G)$. Here, $\|.\|_u$ and $\|.\|_r$ denote the universal and the reduced C^*-norm, respectively. For such a group C^*-algebra A, the identity map from $C_c(G)$ to itself induces canonical surjective *-homomorphisms $C^*(G) \twoheadrightarrow A$ and $A \twoheadrightarrow C^*_r(G)$. The algebra A is called an exotic group C^*-algebra if both the quotient map $C^*(G) \twoheadrightarrow A$ and the quotient map $A \twoheadrightarrow C^*_r(G)$ are non-injective.

It is an open problem whether every non-amenable locally compact group admits exotic group C^*-algebras. For (non-amenable) countable discrete groups having a non-abelian free subgroup, it follows from [5] that there is always a continuum of exotic group C^*-algebras.

Let G be a locally compact group. A natural construction of potentially exotic group C^*-algebras comes from L^p-integrability properties of matrix coefficients of unitary representations. Let $p \in [1,\infty]$. A unitary representation $\pi \colon G \to \mathcal{U}(\mathcal{H})$ is called an L^p-representation if there exists a dense subspace $\mathcal{H}_0 \subset \mathcal{H}$ such that for all $\xi, \eta \in \mathcal{H}_0$, the matrix coefficient $\pi_{\xi,\eta}$, i.e. the function from G to $\mathbb{C}$ defined by $\pi_{\xi,\eta}(s) = \langle \pi(s)\xi, \eta \rangle$, is an element of $L^p(G)$. The representation π is an L^{p+}-representation if for all $\varepsilon > 0$, it is an $L^{p+\varepsilon}$-representation.

For $p \in [2,\infty]$, let $C^*_{L^{p+}}(G)$ denote the group C^*-algebra obtained as the completion of $C_c(G)$ with respect to the norm

$$\| \cdot \|_{L^{p+}} \colon C_c(G) \to [0,\infty),\ f \mapsto \sup\{\|\pi(f)\| \mid \pi \text{ is a } L^{p+}\text{-representation}\}.$$

This essentially goes back to [1], where algebras coming from L^p-representations of countable discrete groups were studied.

In [3], we study the group C^*-algebras $C^*_{L^{p+}}(G)$ for classes of (non-discrete) totally disconnected locally compact groups G acting on trees. A tree T is called semihomogeneous of degree (d_0, d_1) if every vertex of T has degree d_0 or d_1 and for every two adjacent vertices, one of them has degree d_0 and the other one has degree d_1. Furthermore, it is assumed that $d_0, d_1 \geq 2$ and that one of them is at least 3. If $d_0 = d_1$, then the tree is called homogeneous, in which case we assume that the degree is at least 3.

Our first result shows that given an appropriate locally compact group G acting on a (semi)homogeneous tree, the group C^*-algebras $C^*_{L^{p+}}(G)$ are canonically pairwise distinct for $p \in [2, \infty]$. The following is [3, Theorem A].

Theorem 1. *Let G be a non-compact, closed subgroup of the automorphism group* $\mathrm{Aut}(T)$ *of a (semi)homogeneous tree T, and suppose that G acts transitively on the boundary ∂T. For $2 \leq q < p \leq \infty$, the canonical quotient map*

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

is not injective.

This result reproves a result of Samei and Wiersma [6]. Our approach is similar to theirs, in the sense that it relies on establishing the so-called integrable Haagerup property for the groups under consideration. This property, combined with the Kunze-Stein property, which is known for these groups, leads to the theorem. Our approach towards the integrable Haagerup property, however, relies more on harmonic analysis and representation theory rather than on geometric arguments, which was the case in [6].

In [3], we also consider the question whether the algebras $C^*_{L^{p+}}(G)$ are the only group C^*-algebras coming from a G-invariant ideal of the Fourier-Stieltjes algebra $B(G)$ of G. The latter consists of all matrix coefficients of unitary representations of G. The following result is [3, Theorem B].

Theorem 2. *Let G be a non-compact, closed subgroup of the automorphism group* $\mathrm{Aut}(T)$ *of a homogeneous tree T satisfying Tits' independence property, and suppose that G acts transitively on T and on the boundary ∂T. If $C^*_\mu(G)$ is a group C^*-algebra of G such that its dual space $C^*_\mu(G)^*$ is a G-invariant ideal in $B(G)$, then there exists a unique $p \in [2, \infty]$ such that $C^*_\mu(G)^* = B_{L^{p+}}(G)$, where $B_{L^{p+}}(G) := C^*_{L^{p+}}(G)^*$.*

Our results have nice consequences for certain classes of groups. For instance, by the work of Bruhat and Tits, it is known that simple algebraic rank one groups over non-Archimedean local fields admit a natural boundary-transitive action on their Bruhat-Tits tree [2]. It follows that Theorem 1 in particular applies to these groups.

Let us point out that group C^*-algebras constructed from L^p-integrability properties of matrix coefficients were already studied extensively before for countable discrete groups and for Lie groups. The systematic study of such algebras (in the setting of discrete groups) was initiated in [1]. As already mentioned above, for countable discrete groups containing a non-abelian free subgroup, it is known that there always exists a continuum of exotic group C^*-algebras constructed from L^p-integrability properties [5]. In the setting of Lie groups, interesting results were obtained by Wiersma [7] and by Samei and Wiersma [6]. In particular, the analogue of Theorem 1 for the groups $\mathrm{SO}_0(n, 1)$ and $\mathrm{SU}(n, 1)$ was proved by Samei and Wiersma. In [4], Siebenand and the author generalised this to all classical simple Lie groups with real rank one, including the ones with property (T), which could not be dealt with before. This culminated in the following theorem (see [4, Theorem B]), which is the analogue of Theorem 1 for Lie groups of real rank one.

Given a locally compact group G, we define $\Phi(G)$ by

$$\Phi(G) := \inf\{p \in [1,\infty] \mid \forall\, \pi \in \widehat{G} \setminus \{\tau_0\},\, \pi \text{ is an } L^{p+}\text{-representation}\},$$

where τ_0 denotes the trivial representation of G. For the classical Lie groups with real rank one, the constant $\Phi(G)$ is known and given by

$$\Phi(G) = \begin{cases} \infty & \text{if } G = \mathrm{SO}_0(n,1), \\ \infty & \text{if } G = \mathrm{SU}(n,1), \\ 2n+1 & \text{if } G = \mathrm{Sp}(n,1). \end{cases}$$

Theorem 3. *Let G be a (connected) classical simple Lie group with real rank one. Then for $2 \leq q < p \leq \Phi(G)$, the canonical quotient map*

$$C^*_{L^{p+}}(G) \twoheadrightarrow C^*_{L^{q+}}(G)$$

is not injective. Furthermore, for every $p, q \in [\Phi(G), \infty)$, we have

$$C^*_{L^{p+}}(G) = C^*_{L^{q+}}(G).$$

Note that in case the Lie group has property (T), which is the case for $\mathrm{Sp}(n,1)$, we see that from a certain p_0 onwards, the chain $C^*_{L^{p+}}(.)$ "stabilizes". This behaviour is very different from the behaviour of groups with the (integrable) Haagerup property.

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Tits Alternative in dimension 2

Piotr Przytycki
(joint work with Damian Osajda)

Tits proved that every finitely generated linear group is either virtually solvable or contains a nonabelian free group. In other words, each linear group $\mathbf{GL}_n(\mathbf{k})$ satisfies the *Tits Alternative*, saying that each of its finitely generated subgroups is virtually solvable or contains a nonabelian free group. Whether CAT(0) groups satisfy the Tits Alternative remains an open question, even in the case of groups acting properly and cocompactly on 2-dimensional CAT(0) complexes. We prove the following.

Theorem 1. *Let X be a 2-dimensional building or a 2-dimensional systolic complex. Suppose that G acts on X properly and there is a bound on the order of its finite subgroups (e.g. G acts on X properly and cocompactly). Then G satisfies the Tits Alternative.*

Same methods allow also to extend Theorem 1 to X a simply connected $B(6)$-small cancellation complex. With Jon McCammond we also extend Theorem 1 to X the Cayley complex for the standard presentation of an Artin group A_Γ of large type.

Coherence of random groups

Dawid Kielak
(joint work with Robert Kropholler and Gareth Wilkes)

In 1973 Scott proved the following remarkable theorem.

Theorem 1 ([11])**.** *Every finitely generated fundamental group of a (not necessarily compact) 3-manifold is finitely presented.*

In response to Scott's findings Stallings formulated the following definition.

Definition 2. A group is *coherent* if and only if all of its finitely generated subgroups are finitely presented.

(In today's terminology one might call such groups *locally finitely presented*.) Coherence of fundamental groups of compact 3-manifolds is an immediate corollary of Scott's theorem.

It was known since 1961 that not all finitely presented groups are coherent – one can use the Higman embedding theorem to construct a finitely presented group containing a subgroup (like $\mathbb{Z} \wr \mathbb{Z}$) which is finitely generated, recursively presented, but not finitely presented. The first explicit example of a finitely presented incoherent group was given by Stallings in 1963 [12]. A simpler example was given again by Stallings in 1977 [13] in reaction to Scott's paper: we take $G = F_2 \times F_2$, and consider an epimorphism $\phi\colon F_2 \to \mathbb{Z}$ which sends the elements of chosen basis of the F_2 factors to a generator in $\mathbb{Z}$. The kernel of ϕ is finitely generated but not finitely presented.

In view of the above, one might be tempted to ask: what is the *generic* picture for finitely presented groups? This is the motivation behind our work. To study the problem we decided to work in the *few relators model* of Arzhantseva–Ol'shanskii [2]. The model is constructed as follows. We fix a number of generators $n \geqslant 2$ and a number of relators m. Now for every $l \in \mathbb{N}$ we consider a group-valued random variable

$$G(n, m, l) = \langle a_1, \ldots, a_n \mid r_1, \ldots, r_m \rangle$$

where the relators are chosen at random from among the cyclically reduced words of length l in the free group $F(a_1, \ldots, a_n)$.

Definition 3. Given a property $\mathcal{P}$ of groups, we say that *random groups (in the (n, m)-few relators model) satisfy $\mathcal{P}$*

- *with asymptotic probability* 1 if and only if

$$\lim_{l \to \infty} \mathbb{P}\big(G(n, m, l) \text{ satisfies } \mathcal{P}\big) = 1$$

- *with positive asymptotic probability* if and only if

$$\liminf_{l \to \infty} \mathbb{P}\big(G(n, m, l) \text{ satisfies } \mathcal{P}\big) > 0$$

It follows from the work of Gromov [6] that presentation complexes of random groups (in the above sense) are aspherical with asymptotic probability 1, and hence with the same probability the Euler characteristic of a random group is $1 - n + m$. We can now state our main theorem.

Theorem 4. *A random group in the (n, m)-few relators model with Euler characteristic χ*

(1) *is coherent with asymptotic probability* 1 *when* $\chi < 0$*;*
(2) *is coherent with positive asymptotic probability when* $\chi = 0$*;*
(3) *is incoherent with asymptotic probability* 1 *when* $\chi > 0$*, provided that* $n = 2$*.*

The theorem comes coupled with a conjecture.

Conjecture 5. *A random group in the (n, m)-few relators model with Euler characteristic χ is*

(1) *coherent when* $\chi \leqslant 0$*;*
(2) *incoherent when* $\chi > 0$*;*

both with asymptotic probability 1*.*

Our key input is a computation of the first ℓ^2 Betti numbers of random groups.

Theorem 6. *A random group in the (n, m)-few relators model with Euler characteristic χ has the first ℓ^2 Betti number $\beta_1^{(2)}$*

(1) *equal to χ with asymptotic probability* 1 *when* $\chi < 0$*;*

(2) *equal to* 0 *with positive asymptotic probability when* $\chi = 0$ *and with asymptotic probability* 1 *when both* $\chi > 0$ *and* $n = 2$.

To perform the computation we use the technology of Novikov rings, which are sufficiently combinatorial to be amenable to probabilistic arguments.

When the fist ℓ^2 Betti numbers vanish, we proceed as follows: With asymptotic probability 1, random groups are $C'(\frac{1}{6})$ small cancellation [2]. Such groups are aspherical (and hence torsion-free) [5], virtually RFRS [15, 1, 7], and satisfy the Atiyah conjecture [10]. Random groups are also infinite. Infinite virtually RFRS groups with vanishing $\beta_1^{(2)}$ are virtually algebraically fibred [8].

Now the proof splits into two cases, depending on χ. When $\chi = 0$, we see that also the second ℓ^2 Betti number vanishes [9]. Hence, the infinite virtually RFRS group in question virtually maps onto $\mathbb{Z}$ with kernel of type $\mathtt{FP}_2$ [8]. The kernel must therefore be of cohomological dimension 1 [3], and so must be free [14]. This shows that the random group is virtually a free-by-cyclic groups, and such groups are coherent [4]. Coherence easily passes to finite index overgroups.

When $\chi > 0$, we see that $\beta_2^{(2)} \neq 0$. This implies that our random group cannot virtually map onto $\mathbb{Z}$ with finitely presented kernel [9]. But we know that the group virtually algebraically fibres, and hence virtually maps onto $\mathbb{Z}$ with a finitely generated kernel. This kernel now certifies that the random group was not coherent.

The case of negative Euler characteristic follows from a careful construction of an embedding of random groups with negative Euler characteristic into well-controlled random groups of Euler characteristic 0.

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Coxeter quotients of the automorphism group of a Coxeter group

Olga Varghese

One fascinating property of a group is property (T). It was defined by Kazhdan for topological groups in terms of unitary representations and was reformulated by Delorme and Guichardet in geometric group theory. A countable group G has Kazhdan's property (T) if every action of G on a real Hilbert space by isometries has a global fixed point ([1, Theorem 2.12.4]). Examples of groups satisfying this property are finite groups ([1, Proposition 1.1.5]), the general linear groups $\mathrm{GL}_n(\mathbb{Z})$ for $n \geq 3$ ([1, Theorem 4.2.5]), the automorphism groups of free groups $\mathrm{Aut}(F_n)$ for $n \geq 5$ ([2], [3]).

We focus on groups which are defined in combinatorial way. Given a finite simplicial graph $\Gamma = (V, E)$ with an edge-labelling $\varphi : E \to \mathbb{N}_{\geq 2}$, the Coxeter group W_Γ associated to Γ is the group with the presentation

$$W_\Gamma = \langle V \mid v^2, (vw)^{\varphi(\{v,w\})} \text{ for all } v \in V \text{ and } \{v, w\} \in E \rangle.$$

If Γ is disconnected with connected components $\Gamma_1, \ldots, \Gamma_n$, then W_Γ is the free product $W_{\Gamma_1} * \ldots * W_{\Gamma_n}$ and if Γ is a join $\Gamma = \Gamma_1 * \Gamma_2$ and $\varphi(\{v, w\}) = 2$ for all $v \in V_1$ and $w \in V_2$, then W_Γ is the direct product $W_{\Gamma_1} \times W_{\Gamma_2}$. Coxeter groups are fundamental, well understood objects in geometric group theory, but there are many open questions concerning their automorphism groups.

We address the following conjecture:

Conjecture. *For every infinite Coxeter group W_Γ, the group $\mathrm{Aut}(W_\Gamma)$ virtually maps onto some infinite Coxeter group.*

If this conjecture is true, then we know that

Corollary. *The automorphism group of an infinite Coxeter group does not satisfy Kazhdan's property (T).*

Our goal is to verify the above conjecture for a large class of Coxeter groups.

Theorem A. *Let W_Γ be an infinite Coxeter group. If Γ has a maximal complete subgraph Δ such that the center of W_Δ is trivial, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto W_Γ.*

The next large class of Coxeter groups on which we want to focus is the class consisting of *even* Coxeter groups. This class of groups is known as a generalization of right-angled Coxeter groups. A Coxeter group W_Γ is called *even*, if all edge labels are even. A vertex $v \in V$ is called *even*, if all edge labels of $e \in E$ with $v \in e$ are even.

Theorem B. *Let W_Γ be a Coxeter group. If there exist two non-adjacent even vertices $v, w \in V$, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto $\mathbb{Z}_2 * \mathbb{Z}_2$.*

Theorem C. *Let W_Γ be an infinite Coxeter group and let $\Gamma_1, \ldots, \Gamma_n$ be the connected components of Γ. If $n \geq 2$, then the automorphism group $\mathrm{Aut}(W_\Gamma)$ virtually surjects onto $W^{ab}_{\Gamma_i} * W^{ab}_{\Gamma_j}$ for all $i \neq j, i, j \in \{1, \ldots, n\}$.*

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Outer space for RAAGs

Karen Vogtmann
(joint work with Corey Bregman and Ruth Charney)

A right-angled Artin group (RAAG) based on a finite simplicial graph Γ is the group

$$A_\Gamma = \langle \text{vertices}(\Gamma) \mid vw = wv \text{ if } v \text{ and } w \text{ are joined by an edge of } \Gamma \rangle.$$

For any RAAG A_Γ we construct a finite-dimensional space $\mathcal{O}_\Gamma$ on which the group $Out(A_\Gamma)$ of outer automorphisms of A_Γ acts properly. We prove that $\mathcal{O}_\Gamma$ is contractible, so that the quotient is a rational classifying space for $Out(A_\Gamma)$.

Free groups and free abelian groups are special examples of RAAGs. The space $\mathcal{O}_\Gamma$ blends features of the symmetric space of lattices in $\mathbb{R}^n$, with its action of $Out(\mathbb{Z}^n) = GL(n, \mathbb{Z})$, and those of Outer space for the free group F_n, with its action by $Out(F_n)$. Both of those spaces can be described as spaces of marked metric objects, namely flat tori T^n marked by an isomorphism of $\pi_1(T^n)$ with $\mathbb{Z}^n$, and metric graphs G marked by an isomorphism of $\pi_1(G)$ with F_n.

Points in $\mathcal{O}_\Gamma$ are also marked metric spaces with fundamental group A_Γ. These metric spaces are homeomorphic (but not isometric) to certain locally CAT(0) cube complexes called *Salvetti blowups*. Recall that the usual Salvetti complex associated to A_Γ is the cube complex with one vertex and one k-dimensiohnal cube for every k-clique; for example if $A_\Gamma = F_n$, the Salvetti complex is a rose with n petals and if $A_\Gamma = \mathbb{Z}^n$ the Salvetti complex is an n-dimensional torus. In general the Salvetti complex has fundamental group A_Γ and is locally CAT(0) so its universal cover is contractible. Salvetti blowups are the RAAG analog of graphs. In fact for $A_\Gamma = F_n$ they are graphs, namely those with no univalent or bivalent vertices and no separating edges; they are obtained from a rose by "blowing up" the vertex into a maximal tree. General Salvetti blowups were introduced in a previous paper of the second two authors and N. Stambaugh [1]. In that paper marked Salvetti blowups formed the vertices of a contractible simplicial complex K_Γ with a proper action by a subgroup $U(A_\Gamma)$ of $Out(A_\Gamma)$.

The subgroup $U(A_\Gamma)$ does not contain *twists*, i.e. transvections $v \mapsto vw$ where $vw = wv$, so is called the *untwisted subgroup*. To incorporate twists, in this paper we endow Salvetti blowups with locally CAT(0) metrics in which the "cubes" are isometric to Euclidean parallelotopes. This presents difficulties since many standard results about CAT(0) cube complexes assume that all k-cubes are isometric to $[0,1]^k$. The steps in the proof are as follows

(1) Embed the contractible simplicial complex K_Γ into a space Σ_Γ of marked *cuboid* Γ-*complexes*, where "cubes" are still rectilinear but can have different edge-lengths. The action of $U(A_\Gamma)$ on K_Γ extends to Σ_Γ. Show that the image of K_Γ is a deformation retract of Σ_Γ, so Σ_Γ is contractible.
(2) Embed Σ_Γ into a space $\mathcal{T}_\Gamma$ of marked *skewed* Γ-*complexes*, where "cubes" may now be Euclidean parallelotopes but the metric on the skewed Γ-complex as a whole must still be locally CAT(0). The action of $U(A_\Gamma)$ on Σ_Γ extends to an action on $\mathcal{T}_\Gamma$. Show that the image of Σ_Γ is a deformation retract of $\mathcal{T}_\Gamma$.
(3) Points in $\mathcal{O}_\Gamma$ are certain locally CAT(0) metric spaces, marked by an arbitrary isomorphism of their fundamental group with A_Γ. Each of these metric spaces is isometric to some skewed Γ-complex, but it may support many different Γ-complex structures. The entire group $Out(A_\Gamma)$ acts by changing the marking. Map $\mathcal{T}_\Gamma$ to $\mathcal{O}_\Gamma$ by forgetting the combinatorial structure. Prove that this map is a fibration with contractible fibers.

The first step is easy, the second requires some work, and the third step is the hardest. This third step is accomplished by analyzing the relation between the combinatorial and metric structures of a skewed Γ-complex. Specifically, we must show that the map is surjective (since the markings in $\mathcal{T}_\Gamma$ are of a restricted type, admitting only an action of $U(A_\Gamma)$, while there are no restrictions on the markings in $\mathcal{O}_\Gamma$), then determine the set of all possible skewed Γ-complex structures one can put on a given marked metric space, and finally show that the map is a fibration, i.e. one can lift small neighborhoods in $\mathcal{O}_\Gamma$ to $\mathcal{T}_\Gamma$.

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The rates of growth in a hyperbolic group.

Koji Fujiwara
(joint work with Zlil Sela)

In [3], we study the set of rates of growth of a hyperbolic group with respect to all its finite generating sets. We prove that the set is well-ordered, and that every real number can be the rate of growth of at most finitely many generating sets up to automorphism of the group. We prove that the ordinal of the set of rates of growth is at least ω^ω , and in case the group is a limit group (e.g., free and surface groups), it is ω^ω.

In this talk, I give a brief background of the subject (for example see [2]), describe an application of the main results to show a hyperbolic groups is Hopf using the growth tightness by [1], which was known by a different argument in [4], then explain the strategy to prove the theorems. We use techniques from limit groups that was developed by Sela, [5]. This is a joint work with Zlil Sela.

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Strong property (T) for $\tilde{A}_2$-lattices

Stefan Witzel
(joint work with Jean Lécureux and Mikael de la Salle)

I presented a result showing that $\tilde{A}_2$-lattices satisfy Lafforgue's strong property (T), see Theorem 12 below. A talk on the same result, but with a very different focus, has been given by de la Salle during the Oberwolfach workshop 1933 in 2019.

1. $\tilde{A}_2$-lattices

We begin by introducing $\tilde{A}_2$-lattices and discussing their relevance. In the theory of reductive groups over local fields, the role that is played by symmetric spaces for Archimedean fields ($\mathbb{R}$ and $\mathbb{C}$) is played by locally finite Euclidean buildings for non-Archimedean fields (such as $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$). In type A there is the following correspondence in which the first part follows from Bruhat–Tits theory [BT72, BT84] and the second from the classification by Tits–Weiss [Tit79], [Wei09, Tables 28.5, 28.6].

Fact 1. *If K is a non-Archimedean local field (possibly non-commutative), there is a locally finite building of type $\tilde{A}_n$ on which $\mathrm{PGL}_{n+1}(K)$ acts properly and cocompactly. Conversely, if $n \geq 3$ then every locally finite building of type $\tilde{A}_n$ is of that form.*

We call buildings that correspond to $\mathrm{PGL}_{n+1}(K)$ as above *Bruhat–Tits*. There are infinitely many buildings of type $\tilde{A}_2$ that are not Bruhat–Tits [VM87], even with cocompact automorphism group [BCL19, Section 10], [Rad19, Corollary E]. If X is a building of type $\tilde{A}_n$ we say that a group that acts with finite stabilizers and cocompactly on X is an *$\tilde{A}_n$-lattice*.

An important result in the linear case is the Margulis Normal Subgroup Theorem which we formulate in a special case:

Theorem 2 ([Mar91, Theorem IV.4.9])**.** *If $\Gamma < \mathrm{PGL}_n(K)$ is a lattice then every proper quotient of Γ is finite.*

A complementary result is due to Bader, Caprace, and Lécureux:

Theorem 3 ([BCL19])**.** *If Γ is an $\tilde{A}_2$-lattice that is not virtually contained in any $\mathrm{PGL}_3(K)$ then every linear quotient of Γ is finite.*

The statement of Theorem 2 for general $\tilde{A}_2$-lattices had been announced by Shalom and Steger but was never published. A new proof is currently being written up by Bader, Caprace, Furman, and Lécureux. Although it will not be important in what follows, we remark in passing that the division line between the $\tilde{A}_2$-lattices in Theorems 2 and 3 is not quite the same as between the cases in Fact 1: a local field of positive characteristic K has an infinite automorphism group that acts on the Bruhat–Tits building and it is not clear that a lattice on the building will be virtually contained in $\mathrm{PGL}_n(K)$, see [BCL19] for details.

The results we have mentioned so far may seem to suggest that general $\tilde{A}_2$-lattices are not be very different from those virtually contained in some $\mathrm{PGL}_3(K)$. A fundamental difference, however, is that the latter are residually finite (being finitely generated, linear) while for a general $\tilde{A}_2$-lattice it is not clear how to obtain finite quotients. In fact it is conjectured

Conjecture 4 ([BCL19, Conjecture 1.5])**.** *An $\tilde{A}_2$-lattice whose associated building is not Bruhat–Tits is virtually simple.*

We take this conjecture as an important motivation for studying general $\tilde{A}_2$-lattices.

2. Strong property (T)

Strong property (T) was introduced by Lafforgue [Laf08, Laf09] as a strengthening of Kazhdan's property (T). We introduce it by analogy, giving a definition and two consequences first for (T) and then for strong (T).

Definition 5 (Property (T))**.** A group Γ has *property (T)* if there exists a sequence $(\mu_n)_{n\in\mathbb{N}}$ of finitely supported probability measures on Γ such that for every unitary representation $\pi\colon \Gamma \to \mathcal{U}(H)$ the sequence $\pi(\mu_n) = \sum_{\gamma\in\Gamma} \mu_n(\gamma)\pi(\gamma)$ converges in norm to a projection on the space of π-invariant vectors.

Definition 6 (Strong property (T))**.** Let $\mathcal{E}$ be a family of Banach spaces. A finitely generated group Γ has *strong property (T) with respect to* $\mathcal{E}$ if there exists an $\alpha > 0$ and a sequence $(\mu_n)_{n\in\mathbb{N}}$ of finitely supported probability measures on Γ such that for representation $\pi\colon \Gamma \to \mathcal{B}(E)$ with $E \in \mathcal{E}$ satisfying $\sup_{g\in\Gamma}\|\pi(\gamma)\| \exp(-\alpha\ell(g)) < \infty$ the sequence $\pi(\mu_n) = \sum_{\gamma\in\Gamma} \mu_n(\gamma)\pi(\gamma)$ converges in norm to a projection on the space of π-invariant vectors.

Here $\ell(g)$ denotes the word length of g with respect to some finite generating set of Γ. If the class of Banach spaces is not specified it is understood to be Hilbert spaces.

It is well known that property (T) implies Serre's property (FA):

Theorem 7 ([Wat82])**.** *If Γ has property (T) and acts isometrically on a tree then it has a global fixed point.*

Strong property (T) extends this phenomenon to arbitrary hyperbolic spaces:

Theorem 8 ([Laf08, Théorème 1.4])**.** *If Γ has strong property (T) and acts isometrically on a coarsely geodesic, uniformly locally finite hyperbolic space then it has a bounded orbit.*

Another fixed point property concerns affine actions:

Theorem 9 ([Del77, Théorème V.1])**.** *If Γ has property (T) then every affine isometric action on a Hilbert space has a fixed point.*

Theorem 10 ([Laf09, Proposition 5.6])**.** *If Γ has strong property (T) with respect to $\mathcal{E}$ then every affine isometric action on a space $E \in \mathcal{E}$ has a fixed point.*

3. Results

When introducing strong property (T) Lafforgue proved it for the following groups. The statement involves strong property (T) for non-discrete groups, which we did not introduce.

Theorem 11 ([Laf08])**.** *If K is a local field, Archimedean or not, and $\mathbf{G}$ is a simple algebraic group whose Lie-algebra contains $\mathfrak{sl}_3$ then $\mathbf{G}(K)$ as well as its uniform lattices have strong property (T).*

Jean Lécureux, Mikael de la Salle and I extended this result to include general $\tilde{A}_2$-lattices:

Theorem 12 (Lécureux–de la Salle–W.)**.** *If Γ is an $\tilde{A}_2$-lattice then Γ has strong property (T) with respect to Hilbert spaces and all ℓ^p-spaces with $1 < p < \infty$.*

Using Theorem 10 one obtains

Corollary 13. *For every uniformly bounded representation π of Γ on an ℓ^p-space we have $H^1(\Gamma, \pi) = 0$.*

As a consequence ℓ^p-cohomology vanishes for the building on which Γ acts. We show this also for buildings whose automorphism group is not cocompact:

Theorem 14 (Lécureux–de la Salle–W.)**.** *If X is a locally finite $\tilde{A}_2$-buildings then $\ell^p H^1(X) = 0$.*

4. Buildings

Before being able to say something about the proof of Theorems 12 and 14, we need to establish a fact about $\tilde{A}_2$-buildings.

Definition 15. A *projective plane* consists of points, lines, and an incidence relation between points and lines such that any two distinct points are incident with a unique line and vice versa. It has order q if every point is incident with $q+1$ lines and vice versa.

Definition 16. A building of type $\tilde{A}_2$ is a 1-connected simplicial complex in which every vertex link is the incidence graph of a projective plane.

Any three points in a projective plane form a triangle. As a consequence vertex links in $\tilde{A}_2$-buildings contain many 6-cycles and the buildings themselves contain many *apartments*: subspaces that are isomorphic to the Euclidean plane tiled by regular triangles. Fixing a vertex x in the building, every apartment containing that vertex can be folded, essentially uniquely, onto a *sector*: the points y such that the geodesics $[x, y]$ start off in the same (closed) triangle. Note that the vertices of the sector can be canonically identified with $\Lambda := \mathbb{N} \times \mathbb{N}$. The general fact we need is that these maps on apartments fit together to give a well defined map on the whole building:

Fact 17. *Let X be the set of vertices of a building of type $\tilde{A}_2$ and let $x \in X$. There is a canonical map $\omega_x \colon X \to \Lambda$ that on each apartment restricts to the above folding.*

This fact allows, in particular, to introduce the *sphere* $S_\lambda(x) = \omega_x^{-1}(\lambda)$ for $\lambda \in \Lambda$ as well as the *vectorial distance* $\sigma(v, w) = \omega_v(w)$.

Example 18. If $x \in X$ is arbitrary then, by definition $\mathrm{lk}(x)$ is the incidence graph of a projective plane. The sets $S_{(1,0)}(x)$ and $S_{(0,1)}(x)$ are the sets of points and lines of that projective plane.

5. Proof sketch

A general problem when adapting results for lattices in reductive groups over local fields to deal with general building lattices is of course that there is no ambient non-discrete locally compact group to work with. A somewhat surprising but well-known phenomenon is that the building itself can serve as a substitute to some extent. With this in mind one defines, given a representation $\pi \colon \Gamma \to \mathcal{B}(E)$, the *induced space*

$$\tilde{E} = \{f \colon X \to E \mid f(gx) = \pi(g)f(x)\}.$$

Note that this would define the induced representation if X was replaced by a supergroup of Γ.

Next we define $A_\lambda \colon \tilde{E} \to \tilde{E}$ to be the operator that averages over the λ-sphere:

$$A_\lambda f(x) = \frac{1}{|S_\lambda(x)|} \sum_{y \in S_\lambda(x)} f(y).$$

At this point, in order to avoid technicalities, we will assume that Γ acts transitively on *types*, i.e. that X does not admit a Γ-invariant coloring. Then the key steps in the proof are the following propositions. In both statements appropriate assumptions on π are implicitly made.

Definition 19. An element $f \in \tilde{E}$ is *harmonic* if $A_\lambda f = f$ for all $\lambda \in \Lambda$.

Proposition 20. *The net $(A_\lambda)_{\lambda\in\Lambda}$ converges in norm to a projection P onto harmonic functions.*

Proposition 21. *Harmonic functions are constant.*

Note that harmonicity can be expressed as vanishing of the Laplace-operator, which characterizes first ℓ^p-cohomology, while the statement that harmonic functions are constant translates into vanishing of first ℓ^p-cohomology. This indicates the connection of Proposition 21 with Theorem 14.

Let us verify that the propositions indeed give Theorem 12 at least in the case of a lattice Γ that acts freely and transitively on the vertex set X. In this case E and $\tilde{E}$ are in bijection: once some $x_0 \in X$ is fixed, every $\xi \in E$ gives rise to $f_\xi \in \tilde{E}$ by setting $f_\xi(gx_0) = \pi(g)\xi$; the inverse map is just evaluation at x_0. In this correspondence a constant function in $\tilde{E}$ corresponds to an invariant vector in E. We can now define the measure μ_λ on Γ by

$$\mu_\lambda = \frac{1}{|S_\lambda(x_0)|} \sum_{g.x_o \in S_\lambda(x_0)} \delta_g$$

so that A_λ is integration over m_λ (up to the identification of E and $\tilde{E}$). Then the sequence μ_{λ_n}, where $\lambda_n \in \Lambda$ is an arbitrary sequence tending to infinity, satisfies the condition of Definition 6.

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Computing fibrings

Giles Gardam
(joint work with Dawid Kielak)

A basic mathematical instinct is to break an object into simpler pieces; we then hope to understand the pieces and how they are assembled back together. One way a 3-manifold could break into simpler pieces is if it fibres over the circle, so that it is assembled by taking the mapping torus of a homeomorphism of a surface. The usefulness of this approach has been proved by the spectacular resolution of Thurston's Virtual Fibring Conjecture by Agol [1] building on work of Wise, Kahn–Markovic and others: every closed hyperbolic 3-manifold has a finite cover that fibres over the circle.

A 3-manifold M may well of course fibre in multiple essentially different ways, corresponding to different 'characters' $\pi_1 M \to \mathbb{Z}$, and a remarkable theorem of Thurston controls exactly which characters come from fibrings [10]. Specifically, what is now known as the Thurston norm is a semi-norm $H^1(M, \mathbb{R}) \to [0, \infty)$ whose unit ball is an integral polytope P. If a character $\phi\colon \pi_1 M \to \mathbb{Z}$ corresponds to a fibring then it lies in the cone over an open maximal face of P, and the other characters in the same cone are also fibred, so we call the face fibred and think of this as a marking of the polytope.

The fundamental group of a fibred 3-manifold with boundary is *free-by-cyclic*, that is, of the form $F_n \rtimes_\alpha \mathbb{Z}$ where F_n is a finite rank free group and $\alpha \in \mathrm{Aut}(F_n)$. The classes of 3-manifold groups and free-by-cyclic groups have substantial overlap but also their own particular behaviour; there is an emerging picture that the similarities run deep, as is the case for the mapping class group of a surface and $\mathrm{Out}(F_n)$. Kielak showed that, analogously to the situation for 3-manifolds, the ways in which we can fibre a free-by-cyclic group (or more generally an ascending

HNN extension of a finitely generated free group) are controlled by a polytope [8]. We make this effective.

Theorem 1. *There is an algorithm that takes input a free-by-cyclic* $G = F_n \rtimes_\alpha \mathbb{Z}$ *(given say by* $\alpha \in \mathrm{Aut}(F_n)$*) and computes the associated polytope.*

One application is that we have a new computable conjugacy class invariant for $\mathrm{Out}(F_n)$. An implementation of this algorithm is work in progress. A key ingredient is the word problem in a certain skew field containing the integral group ring $\mathbb{Z}F_n$ [4]; recently this problem has been revisited and a solution implemented [9].

It has long been known that the Thurston polytope is computable [11]; our methods however are algebraic yet actually extend to recover much of the 3-manifold picture. The Thurston norm provides a measure of the complexity of a character: Poincaré duality gives a corresponding class in $H_2(M)$ and we take (roughly speaking) the minimal negative Euler characteristic of surfaces representing it (for a fibred character this is just the negative Euler characteristic of the fibre). For a free-by-cyclic group the polytope determines the L^2-Euler characteristic of the kernel of a character (which makes sense even when the kernel is infinitely generated) and we conjecture that we can, similarly as for 3-manifolds, interpret this as the minimal complexity achievable for corresponding splittings.

Conjecture. *Let* G *be a free-by-cyclic group. For a character* $\phi\colon G \to \mathbb{Z}$*, the negative* L^2*-Euler characteristic* $-\chi^{(2)}(\ker \phi)$ *coincides with the minimal value of* $-\chi(A)$ *such that* G *splits as an HNN extension* $A{*}_B$*, with both* A *and* B *finitely generated, inducing the character* ϕ*.*

The conjecture is true if G is moreover a one-relator group [7, Theorem 6.4]. This conjecture would allow us to overcome, in certain cases, the lack of effectiveness in Feighn and Handel's theorem [6] that free-by-cyclic groups are coherent, that is, that their finitely generated subgroups are finitely presented. This would then enable us to determine which faces of the polytope are marked. Note that unlike for 3-manifolds, the polytope and its marking need not be symmetric, so the marking actually describes which characters are in the BNS invariant [2], which one could call being 'semi-fibred'; the BNS invariant is not computable for general finitely presented groups [3, Theorem 6.4].

Theorem 2. *Assuming the conjecture, there is an algorithm that furthermore computes the marking of the polytope associated to a free-by-cyclic group.*

This would allow us to reduce the isomorphism problem for free-by-cyclic groups to the conjugacy problem in $\mathrm{Out}(F_n)$. Note that neither problem has a solution in full generality in the literature; Dahmani has a partial reduction in the other direction for atoroidal automorphisms [5].

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Arithmetic groups and Reidemeister classes

Yuri Santos Rego
(joint work with Paula M. Lins de Araujo)

Given a group Γ and an automorphism $\varphi \in \mathrm{Aut}(\Gamma)$ we say that two elements $g, h \in \Gamma$ are φ-conjugate if there exists $k \in \Gamma$ such that $g = kh\varphi(k)^{-1}$. This defines an equivalence relation generalizing the usual notion of conjugacy, and the class of the element g is denoted by $[g]_\varphi$. With this notation, the set of *Reidemeister classes* of φ is just $\mathcal{R}(\varphi) = \{[g]_\varphi \mid g \in \Gamma\}$ and the *Reidemeister number* of φ is the number of elements $R(\varphi) = |\mathcal{R}(\varphi)| \in \mathbb{N} \cup \{\infty\}$.

1. Background on twisted conjugacy

The study of Reidemeister classes arose from two major topics. On the one hand we have fixed-point theory in algebraic topology; see e.g. [20] for a well-documented historical account and introduction to the topic. As an example, the induced automorphism $f_* \in \mathrm{Aut}(\pi_1(M))$ of a self-homeomorphism f of a compact, connected nilmanifold M has $R(f_*) = \infty$ if and only if the Lefschetz number of f is zero [5], which in turn implies that f is fixed-point-free up to homotopy.

On the other hand, counting problems related to (twisted) conjugacy classes of groups are also far from new. To name a few results in this direction, the first example of infinite group with $R(\mathrm{id}) < \infty$ was given in the seminal paper by Higmann–Neumann–Neumann [9]. In the succeeding decades, G. Zappa [21] studied the relationship between the structure of polycyclic groups and the existence of automorphisms φ with $R(\varphi) = 1$. In the linear algebraic setting, R. Steinberg [18] showed that $R(\varphi) = 1$ if φ is a (rational) automorphism with finitely many fixed points of a connected linear algebraic group over an algebraically closed field. Moving over to more recent results, E. Jabara [10] established, in particular, that residually finite groups admitting automorphisms φ with $R(\varphi) = p$ prime must be virtually nilpotent.

In contrast, a plethora of groups 'that occur in nature', with interesting geometric features, turn out to have $R(\varphi) = \infty$ for *any* automorphism φ. The oldest documented result is perhaps that of the infinite dihedral group [21], whereas famous modern examples include the family of Gromov hyperbolic groups [3]. This motivates the following.

Definition 1. We say that Γ has *property* R_∞ if $R(\varphi) = \infty$ for all $\varphi \in \mathrm{Aut}(\Gamma)$.

Historically, the quest to determine which groups have R_∞ was sparked by the work of Fel'shtyn and Hill in the 1990s [4]. This has been an active area ever since.

2. The arithmetic case

As mentioned, many natural groups are known to exhibit R_∞. A good number of the groups investigated so far in fact belong to the family of S-arithmetic groups [13]. For the purpose of this note, we think of an S-arithmetic group as a concrete subgroup of matrices in $\mathbf{G} \cap \mathrm{GL}_n(\mathcal{O}_S)$, where $\mathbf{G} \leq \mathrm{GL}_n$ is a linear algebraic group defined over a global field $\mathbb{K}$ and $\mathcal{O}_S \subset \mathbb{K}$ is a ring of S-integers (also known as Dedekind domain of arithmetic type). Typical examples of $\mathcal{O}_S$ are: rings of integers $\mathcal{O}$ of algebraic number fields, polynomials (and Laurent polynomials) on one variable over finite fields $\mathbb{F}_q[t], \mathbb{F}_q[t, t^{-1}]$, and rational numbers whose denominators lie in a finite set of primes $\mathbb{Z}[\frac{1}{p_1 \cdots p_r}]$.

Summarizing the state of knowledge in the arithmetic set-up, the literature indicates that *non-amenable* S-arithmetic groups typically have property R_∞. Some of the key examples are $\mathrm{GL}_n(\mathcal{O})$ with $|\mathcal{O}^\times| < \infty$ [15], $\mathrm{GL}_n(\mathbb{F}_q[t])$ and $\mathrm{GL}_n(\mathbb{F}_q[t, t^{-1}])$ [12], and lattices in semisimple Lie groups (with finitely many connected components and finite center) [14]. Taking the Tits alternative into account, we are led to the following motivating question.

Problem. *Determine which soluble S-arithmetic groups have property R_∞.*

As it turns out, the above classification problem is rather challenging. Indeed, soluble groups have a rather erratic behavior regarding Reidemeister classes. To begin with, the simplest example of soluble arithmetic lattice is $\mathbb{Z} = \mathbf{U}_2(\mathbb{Z}) = \left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right) \leq \mathrm{GL}_2(\mathbb{Z})$, which does not have R_∞ because $R(-\mathrm{id}) = 2$. On the other hand, going up a few dimensions, the group of integral upper unitriangular matrices $\mathbf{U}_n(\mathbb{Z})$ does have R_∞ whenever $n \geq 5$ — a similar situation happens for other rings of integers [16]. Further families of (virtually) polycyclic groups contain members that do and members that do not have R_∞ [1, 2, 20]. Varying pattern is also seen for the metabelian lamplighter groups $C_p \wr \mathbb{Z}$ — which have property R_∞ if and only if $p \in \{2, 3\}$ [8] — and for a family of metabelian polycyclic groups [7] which includes the *Gardam group* $\mathcal{G} = \mathbb{Z}^2 \rtimes_A \mathbb{Z}$, $A = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$. In contrast, the metabelian Baumslag–Solitar groups $\mathrm{BS}(1, p)$ [19] always have R_∞.

3. Checking for R_∞ in groups of triangular matrices

The previously mentioned subclasses of soluble groups can be put into the broader framework of S-arithmetic subgroups in Borel groups of type $\mathtt{A}$. We need a bit more notation for this. Throughout R means a commutative unital ring.

Write $\mathbf{B}_n(R)$ for the subgroup of upper triangular matrices of $\mathrm{GL}_n(R)$. By $\mathbb{P}\mathbf{B}_n(R)$ we mean its projective variant, i.e. $\mathbf{B}_n(R)$ modulo multiples of the identity. Both $\mathbf{B}_n(R)$ and $\mathbb{P}\mathbf{B}_n(R)$ are the semi-direct product of the subgroup of upper unitriangular matrices $\mathbf{U}_n(R)$ and the respective diagonal parts. In the metabelian case, $\mathbb{P}\mathbf{B}_2(R)$ is isomorphic to the group of affine transformations $\mathbb{A}\mathrm{ff}(R) = (R,+) \rtimes (R^\times,\cdot) = \left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right) \leq \mathrm{GL}_2(R)$. Lastly, $B_n^+(R)$, $\mathbb{P}B_n^+(R)$ and $\mathbb{A}\mathrm{ff}^+(R)$ denote the variants of our groups without torsion on the diagonal.

With the above notation, we have e.g. $\mathrm{BS}(1,p) \cong \mathbb{A}\mathrm{ff}^+(\mathbb{Z}[1/p])$, $C_p \wr \mathbb{Z} = \mathbb{A}\mathrm{ff}^+(\mathbb{F}_p[t,t^{-1}])$, and $\mathcal{G} = \mathbb{A}\mathrm{ff}^+(\mathbb{Z}[\sqrt{2}])$. We also have $\mathbf{U}_n(\mathcal{O})$ as a characteristic subgroup of $\mathbb{P}\mathbf{B}_n(\mathcal{O})$. Our main results give sufficient conditions for our groups of interest to have property R_∞, not only over S-arithmetic rings but rather a much wider class of rings.

Theorem 1 ([11]). *Let R be an integral domain. Given $\psi \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}(R))$ and $\psi^+ \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}^+(R))$, denote by $\overline{\psi}$ (resp. $\overline{\psi}^+$) the induced automorphism on $\mathbb{A}\mathrm{ff}(R)/\mathbf{U}_2(R)$ (resp. on $\mathbb{A}\mathrm{ff}^+(R)/\mathbf{U}_2(R)$). The following hold for all $n \geq 2$.*

1. *If $R(\overline{\psi}) = \infty$ for all $\psi \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}(R))$, then $\mathbb{A}\mathrm{ff}(R)$, $\mathbb{P}\mathbf{B}_n(R)$ and $\mathbf{B}_n(R)$ have property R_∞.*
2. *If $R(\overline{\psi}^+) = \infty$ for all $\psi^+ \in \mathrm{Aut}(\mathbb{A}\mathrm{ff}^+(R))$, then $\mathbb{A}\mathrm{ff}^+(R)$, $\mathbb{P}B_n^+(R)$ and $B_n^+(R)$ have property R_∞.*

Theorem 2 ([11]). *Let R be an integral domain such that $(R^\times,\cdot)$ is finitely generated. Given a ring automorphism $\alpha \in \mathrm{Aut}_{\mathrm{ring}}(R)$, let α_{add} denote the same map viewed as automorphism of $(R,+)$. Suppose both α_{add} and the map $(r,s) \mapsto (\alpha_{\mathrm{add}}(s), \alpha_{\mathrm{add}}(r))$ have infinite Reidemeister number for all $\alpha \in \mathrm{Aut}_{\mathrm{ring}}(R)$. Then the groups $\mathbf{B}_n(R)$ and $\mathbb{P}\mathbf{B}_n(R)$ have property R_∞ for all $n \geq 5$.*

Applying our results to the arithmetic case and working out new metabelian examples we also obtain the following list.

Theorem 3 ([11]). *The matrix groups $\Gamma = G(\mathcal{O}_S)$ have R_∞ in the following cases.*

1. *$G \in \{\mathbf{B}_n, \mathbb{P}\mathbf{B}_n, \mathbb{A}\mathrm{ff}, B_n^+, \mathbb{P}B_n^+, \mathbb{A}\mathrm{ff}^+ \mid n \geq 2\}$ and $\mathcal{O}_S = \mathbb{Z}[1/m]$;*
2. *$G \in \{B_n^+, \mathbb{P}B_n^+, \mathbb{A}\mathrm{ff}^+ \mid n \in \mathbb{N}_{\geq 2}\}$ and $\mathcal{O}_S = \mathbb{F}_q[t, t^{-1}, f(t)^{-1}]$, where q is a power of any prime p, the polynomial $f(t) \in \mathbb{F}_p[t] \subseteq \mathbb{F}_q[t]$ is non-constant monic and irreducible over $\mathbb{F}_q \supseteq \mathbb{F}_p$, and $f(t) \neq t-1$ in the case $p = 2$;*
3. *$G \in \{\mathbf{B}_n, \mathbb{P}\mathbf{B}_n \mid n \geq 5\}$ and $\mathcal{O}_S \in \{\mathbb{F}_p[t], \mathbb{F}_p[t,t^{-1}], \mathcal{O}\}$.*

On the other hand, $\Gamma = G(\mathcal{O}_S)$ does not *have property R_∞ when*

4. *$G \in \{\mathbf{B}_2, \mathbb{A}\mathrm{ff}\}$ and $\mathcal{O}_S = \mathbb{F}_q[t]$;*
5. *$G \in \{B_2^+, \mathbb{A}\mathrm{ff}^+\}$ and $\mathcal{O}_S = \mathbb{F}_q[t,t^{-1}]$ with $q \geq 4$.*

We point out that Gonçalves and Kochloukova [6] discovered via homological finiteness properties a connection between Reidemeister numbers and the

Σ-invariants of Bieri–Neumann–Strebel. Here we remark that the soluble S-arithmetic groups we consider are distinguished by their finiteness properties [17]. A careful inspection of the list from our previous theorem and other cases from the literature naturally indicate the following.

Conjecture. *Let $\Gamma \leq \mathbf{B}$ be an S-arithmetic subgroup of a Borel subgroup of a split, connected, reductive, non-commutative linear algebraic group $\mathbf{G}$ defined over a global field. If Γ is finitely presented and* not *virtually polycyclic, then it has R_∞.*

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Cubulations determined by their length function

Elia Fioravanti
(joint work with Jonas Beyrer and Mark Hagen)

Beginning with the foundational work of Sageev [11], the theory of group actions on CAT(0) cube complexes has exerted a strong influence on geometric group theory and low-dimensional topology in the last two decades. On the one hand, knowing that a group G acts "nicely" on a CAT(0) cube complex reveals a lot of its algebraic structure (as exemplified by the work of Agol, Haglund, Wise [1, 10] and many others). On the other hand, an impressive number of classically studied groups happens to admit such nice actions on cube complexes.

In this talk, we are rather interested in understanding — for a fixed finitely generated group G — all the different ways that G can act by isometries on a CAT(0) cube complex. A good notion of "space of all such actions" would provide a useful tool in the study of outer automorphisms of G in a fairly general setting (e.g. whenever G is a *special* group in the sense of [10]).

Definition 1.

(1) Let G be a group. A *cubulation* of G is a properly discontinuous, cocompact G–action on a CAT(0) cube complex X by cubical automorphisms.

(2) Two cubulations $G \curvearrowright X$ and $G \curvearrowright Y$ are *equivalent* if there exists a G–equivariant isomorphism of cube complexes $\Phi\colon X \to Y$.

One quickly realises that, whenever a group admits at least one cubulation, it actually admits infinitely many pairwise non-equivalent ones. In most cases, such abundance is due to "interesting reasons", i.e. it tells us something about the specific "geometry" of the group.

Example 2.

(0) Let $G \curvearrowright X$ be a cubulation given by a homomorphism $\rho\colon G \to \mathrm{Aut}(X)$. Every $\varphi \in \mathrm{Aut}(G)$ induces a twisted cubulation $\rho\circ\varphi$. If $\mathrm{Out}(G)$ is infinite, this results in infinitely many non-equivalent cubulations of G.

(1) Let $G = F_n$ with $n \geq 2$. Although G admits cubulations of arbitrarily high dimension, it is particularly interesting to restrict to those of dimension 1. These are proper cocompact actions on simplicial trees. They correspond to the vertices in the spine of the Culler–Vogtmann Outer Space [6].

(2) Let $G = \pi_1\Sigma_g$ for a closed oriented surface Σ_g of genus $g \geq 2$. Every finite filling collection of closed curves on Σ_g gives rise to a cubulation of G via Sageev's construction [11]. Non-isotopic collections of curves yield non-equivalent cubulations. Even fixing the genus g, these cubulations can have arbitrarily high dimension.

(3) Let $G = \pi_1 M$ for a closed, hyperbolic 3–manifold M. Sageev's construction can be applied to any family of quasi-convex, immersed surfaces in M. From the work of Kahn and Markovic [7], M always contains many such surfaces. In particular, G admits cubulations of arbitrarily high dimension.

It should be apparent from these examples that, in general, the space of all cubulations of G is terribly large and poorly understood. It can be endowed with the topology of equivariant Gromov–Hausdorff convergence, but this is somewhat unwieldy in such generality.

It is reasonable to try and make sense of the space of cubulations of G by embedding into some more concrete object with a nice topology, for instance a locally convex topological vector space. A natural attempt is by pairing each cubulation $G \curvearrowright X$ with its length function $\ell_X \in \mathbb{R}^G$, i.e. the function that associates to every $g \in G$ its *translation length* in X. As customary, we compute the latter with respect to the ℓ^1 metric on X, rather than the CAT(0) metric.

Unfortunately, there are many silly ways of deforming a cubulation without altering its length function (e.g. attaching loose edges, or inflating edges to squares). These procedures can be performed in absolute generality – on any cubulation of any group – and therefore cause the space of cubulations to be *uninterestingly* large, i.e. for reasons independent of the specific geometry of the group.

This leads us to introduce the following requirements.

Definition 3 ([4, 9])**.** A CAT(0) cube complex X is said to be:

(1) *essential* if no halfspace is at finite Hausdorff distance from its hyperplane;
(2) *hyperplane-essential* if every hyperplane is an essential cube complex;
(3) *EHE* if it is both essential and hyperplane-essential.

By the work of Hagen–Touikan [9], cubulations can always be “shaved” into being EHE. In particular, the class of groups admitting an EHE cubulation coincides with the class of cubulated groups. It should also be noted that all cubulations mentioned in parts (1)–(3) of Example 2 are EHE.

We propose the space of EHE cubulations as a more interesting object to study. The rationale supporting this restriction is provided by the following results:

Theorem 4 ([3])**.** *Let G be a hyperbolic group. Two EHE cubulations of G are equivalent if and only if they have the same length function.*

Theorem 5 ([8])**.**

(1) *Every cubulated hyperbolic admits infinitely many EHE cubulations.*
(2) *Burger–Mozes groups admit a unique EHE cubulation.*

By analogy with the work of Culler and Morgan on actions on trees [5], one may wish for an analogue of Theorem 4 under weaker assumptions on the groups and the actions. We provide this under stronger assumptions on the cube complex:

Theorem 6 ([2])**.** *Let G be any group. Two irreducible cubulations of G without free faces are equivalent if and only if they have the same length function.*

This holds more generally for all minimal, non-elementary G–actions (not necessarily proper or cocompact) on irreducible cube complexes with no free faces.

Theorems 4 and 6 can be used to compactify the space of G–actions satisfying either of their sets of hypotheses, provided some upper bound is set to the dimensions of the cube complexes considered. Boundary points will be projectivised length functions of G–actions on finite-rank median spaces [2].

This can be viewed as an analogue of Thurston's compactification of Teichmüller space in the cubical setting.

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Boundary amenability and measure equivalence rigidity among Artin groups

Camille Horbez
(joint work with Jingyin Huang)

Let Γ be a finite labeled graph, with no edge loop and no multiple edges, where every edge st is labeled by an integer $m_{st} \geq 2$. The *Artin group* G_Γ with underlying graph Γ is defined by the following presentation: it has one generator per vertex of Γ, and one relation of the form

$$\underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m}$$

whenever a and b span an edge labeled by m. While Artin groups remain very mysterious in general, much more is known in restriction to certain specific classes.

We restrict our attention to Artin groups of cohomological dimension at most 2 whose associated Coxeter group (defined by adding an extra relation $s^2 = 1$ for every standard generator s of G_Γ) is hyperbolic. In terms of the defining graph of Γ, this amounts to imposing that whenever Γ contains a triangle str, one has

$$\frac{1}{m_{st}} + \frac{1}{m_{tr}} + \frac{1}{m_{rs}} < 1,$$

and Γ does not contain any diagonal-free square with all labels equal to 2 [4]. Every 2-dimensional Artin group G_Γ acts on its *modified Deligne complex*, a 2-dimensional simplicial complex introduced by Charney and Davis [3]. When G_Γ is of hyperbolic type, this carries a natural metric which is globally CAT(-1), with finitely isometry types of simplices isometric to simplices in the hyperbolic plane $\mathbb{H}^2$. Recently, Martin and Przytycki built a cone-off of the modified Deligne complex with the extra property that the action of G_Γ is acylindrical [14].

Boundary amenability. Our first main result concerns the asymptotic geometry of these groups. Recall that a countable group G is *boundary amenable* if it admits an action on a compact topological space X which is *Borel amenable*, i.e. such that there exists a sequence of Borel maps $\nu_n : X \to \mathrm{Prob}(G)$ (the space of probability measures on G, equipped with the topology of pointwise convergence) which is asymptotically equivariant in the sense that for every $x \in X$ and every $g \in G$, one has $||\nu_n(gx) - g \cdot \nu_n(x)||_1 \to 0$ as n goes to $+\infty$.

Boundary amenability is known to hold for many groups having negative curvature features. In the context of Artin groups, Guentner and Niblo proved in [10] that all spherical Artin groups (those whose associated Coxeter group is finite) and more generally all Artin groups of FC type, are boundary amenable: these are precisely the Artin groups whose modified Deligne complex has a natural structure of a CAT(0) cube complex. We prove that 2-dimensional Artin groups of hyperbolic type are boundary amenable, as a consequence of the following theorem.

Theorem 1. *Let X be a connected piecewise hyperbolic* CAT(-1) *simplicial complex with countably many simplices that belong to finitely many isometry types. Let G be a countable group acting discretely on X by isometries.*

If the G-stabilizer of every vertex of X is boundary amenable, then G is boundary amenable.

The proof goes by associating to every point of the horofunction compactification of X, an asymptotically equivariant sequence ν_n of probability measures on the countable set of simplices of X. When the horofunction is naturally associated to a point ξ in the Gromov boudary $\partial_\infty X$, the probability measures ν_n are defined by considering the geodesic ray r from a fixed base point $*$ to ξ, and choosing for ν_n a carefully weighted probability measure on the simplices close to $r_{|[0,n]}$.

Corollary 1. *Every 2-dimensional Artin group of hyperbolic type is boundary amenable.*

This implies that their reduced C^*-algebra is exact [2, 17] and that they satisfy the Novikov conjecture on higher signatures [18, 11]. As explained below, this also has applications in measured group theory.

Measure equivalence rigidity. Measure equivalence of countable groups was introduced by Gromov [9] as a measure-theoretic analogue of quasi-isometry of finitely generated groups. Two countable groups G_1 and G_2 are *measure equivalent* if there exists a standard measure space Σ on which $G_1 \times G_2$ acts by measure-preserving Borel automorphisms, so that the actions of G_1 and G_2 are esentially free and have finite-measure fundamental domains. Typically, two lattices in the same locally compact second countable group are always measure equivalent.

A first striking theorem in the theory, due to Ornstein and Weiss [16], building on earlier work of Dye [5, 6], states that countably infinite amenable groups are all measure equivalent. In contrast, strong rigidity results were obtained for lattices in higher rank simple Lie groups (Furman [7]), for some products of negatively curved groups (Monod and Shalom [15]), and for mapping class groups of finite-type surfaces (Kida [13]). We obtain superrigidity results for infinitely many 2-dimensional Artin groups of hyperbolic type, among which the following.

Theorem 2. *Let G_Γ be an Artin group whose underlying graph is an n-cycle with $n \geq 4$, with all labels at least* 3.

Then G_Γ is superrigid for measure equivalence: every countable group H which is measure equivalent to G_Γ, is almost isomorphic to G_Γ.

In the statement, we say that two groups G_1 and G_2 are *almost isomorphic* if there exist finite-index subgroups $G_i^0 \subseteq G_i$ and finite normal subgroups $F_i \trianglelefteq G_i^0$ such that G_1^0/F_1 is isomorphic to G_2^0/F_2.

Our theorem is actually more general: sufficient conditions on Γ ensuring the superrigidity of G_Γ, coming from earlier work of Crisp [4], are that Γ is connected, triangle-free, with all labels at least 3, with no separating vertex or edge, and every label-preserving automorphism of Γ that fixes the neighborhood of a vertex is the identity. All these groups have finite outer automorphism group (Crisp [4]) and are also superrigid for quasi-isometry (Huang and Osajda [12]).

Through work of Furman [8], our theorem can be rephrased in terms of orbit equivalence rigidity: if G_Γ and H have (stably) orbit equivalent free ergodic actions on standard probability spaces, then G_Γ and H are almost isomorphic, and in fact the actions are almost conjugate.

Our general proof strategy is inspired by Kida's work on mapping class groups. Tools include earlier work of Crisp giving an analogue of the curve graph, all of whose automorphisms come (essentially) from the G_Γ-action. We also use the horofunction compactification $\overline{X}$ of the coned-off Deligne complex, notably the amenability of the G_Γ-action on the Gromov boundary, and a *barycenter map* associating a point in X to every triple of pairwise distinct points of the Gromov boundary, in order to apply an argument originating from work of Adams [1].

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$\mathcal{AH}-$accessibility of groups

Sahana Balasubramanya

The poset of acylindrically hyperbolic structures on a group G, denoted $\mathcal{AH}(G)$, was introduced in [1]. It aims at understanding all of the cobounded, acylindrical actions that a group can admit on a hyperbolic space. More precisely, the poset contains equivalence classes of (not necessarily finite) generating sets X such that the Cayley graph $\Gamma(G, X)$ is hyperbolic and the action of G on $\Gamma(G, X)$ is acylindrical. Two generating sets are equivalent if their word metrics are bi-Lipschitz equivalent, and the ordering reflects the amount of information the action retains about the group.

Some properties of this poset that were studied in [1] are discussed in this talk. Firstly, the cardinality of this poset can either be $1, 2$ or $\geq c$. The case when $|\mathcal{AH}(G)| = 2$ occurs exactly when G is virtually cyclic and the case when $|\mathcal{AH}(G)| \geq c$ occurs exactly when G is an acylindrically hyperbolic group. This is a large class of groups that contains many interesting examples.

Actions of an acylindrically hyperbolic group G are built by inducing actions from its hyperbolically embedded subgroups. (These are a generalization of peripheral subgroups of relatively hyperbolic groups.) Indeed, we can embed $\mathcal{AH}(H)$ into $\mathcal{AH}(G)$ when H is a hyperbolically embedded subgroup of G. A consequence of the proof is that $\mathcal{AH}(G)$ contains chains and anti-chains of cardinality c.

It is easy to see that equivalent generating sets yield the same set of loxodromic elements. One can therefore ask if an action is characterized by the set of its loxodromics. The authors construct infinitely many incomparable actions which have the same set of loxodromic elements, answering the question in the negative.

The existence of largest actions for certain acylindrically hyperbolic groups is discussed. A group G is said to be $\mathcal{AH}-$accessible if the poset $\mathcal{AH}(G)$ contains the largest element. The following acylindrically hyperbolic groups are known to be $\mathcal{AH}-$accessible: non-elementary hyperbolic groups, mapping class groups, RAAGs, fundamental groups of compact orientable 3-manifolds. The largest actions in all these cases are well-studied actions of these groups.

Lastly, a recent result from [2] is discussed, which shows that being $\mathcal{AH}-$accessible is preserved under finite extensions. This also shows that being acylindrically hyperbolic is preserved in this specific case of a quasi-isometry, which relates to the larger open question of the quasi-isometry rigidity of this class of groups.

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Small hyperbolic groups with property (T)

Marek Kaluba

(joint work with Pierre-Emmanuel Caprace, Marston Conder and Stefan Witzel)

The goal of our research is to provide explicit presentations of infinite hyperbolic groups that enjoy peculiar properties. Our initial motivation revolves around the following problem dating back to a remark [2, Remark 5.3.B] of M. Gromov.

Is every hyperbolic group residually finite?

While looking for a possible counter-example though we need to exclude families known to be residually finite. The groundbreaking work of Agol, Haglund and Wise implies that all cocompactly cubulated hyperbolic groups are residually finite (see [1] and references therein). On the other hand, Kazhdan's property (T) is an obstruction to cocompact cubulations [7]. Therefore to identify a potential candidate to negative answer for the question, property (T) is a precious ally.

Since the advent of random methods in the theory of finitely presended groups, we know that a "generic" random group is hyperbolic and (with the correct choice of parameters for the random model) it enjoys property (T), see e.g. [9]. And yet, very few explicit examples of such groups are known. We are interested in finding small, tractable presentations of such groups.

What is the smallest total length of a presentation of an infinite hyperbolic group with Kazhdan's property (T)?

Generalized n-fold triangle groups. We call a group *generalized n-fold triangle group* if it admits a structure of a triangle of groups i.e. is the fundamental group of a poset of groups over a 2-dimensional simplex. Moreover we requre that

- (the unique) face group is trivial
- the groups on the edges are are C_k, cyclic of order k.

In other words, a group G is a generalized n-fold triangle group when it admits a presentation

$$G = \langle x_1, x_2, x_3 \mid x_1^n, x_2^n, x_3^n, r_1, \ldots, r_k \rangle,$$

where each relation r_i involves at most two of the generators x_1, x_2, x_3. The term generalized refers to the fact that traditionally a triangle group is formed by intersecting such G and the kernel of the map $G \to C_k$ sending each generator of G to the generator of C_k, i.e. be adding additional relation $x_1x_2x_3$ to the presentation of G.

Note. *Coxeter group given by*

$$\Delta(l, m, r) = \langle a, b, c \mid a^2, b^2, c^2, (ab)^l, (bc)^m, (ac)^r \rangle$$

is an example of generalized 2*-fold triangle group.*

Fix $G = \langle x_1, x_2, x_3 \mid x_1^k = x_2^k = x_3^k = 1 \rangle$ to be the free product of three copies of C_k. In general, let $P_1 = \langle x_2, x_3 \rangle$, $P_2 = \langle x_1, x_3 \rangle$ and $P_3 = \langle x_1, x_2 \rangle$. Fix $L_i \triangleleft P_i$, three different normal, finite index subgroups.

Definition (due to Lubotzky-Manning-Wilton [4])**.** A generalized n-fold triangle group associated to (L_1, L_2, L_3) is defined as the quotient group

$$G(L_1, L_2, L_3) = G/\langle L_1, L_2, L_3 \rangle.$$

In the case of Coxeter groups we have e.g. $L_1 = \langle (bc)^m \rangle$.

A. Lubotzky, J. Manning, and H. Wilton [4] were able to prove the existence of hyperbolic groups with property (T) among generalized 18-fold triangle groups. Moreover they rised the following question.

> *What is the smallest k such that there exist a generalized k-fold triangle group that is hyperbolic and has Kazhdan's property (T)?*

Since Coxeter groups do not have property (T), such k satisfies $3 \leq k \leq 18$. The aim of this note is to sketch a proof of the following theorem.

Theorem. *In the question above $k \leq 5$. More precisely, let*

$$G = C_k * C_k * C_k = \langle x_1, x_2, x_3 \rangle.$$

Fix subgroups $L_i < P_i$ such that

- $P_1/L_1 = C_5 \times C_5$*, i.e.* $L_1 = \langle [x_2, x_3] \rangle$,
- $P_2/L_2 = H(\mathbb{F}_5)$ *(the Heisenberg group over the field of* 5 *elements), i.e.* $L_2 = \langle [x_1, x_3, x_1], [x_1, x_3, x_3] \rangle$*, and*

- $P_3/L_3 = PSL_2(\mathbb{F}_{109})$, *i.e.*

$$\begin{aligned} L_3 =\langle & x_1x_2x_1x_2^{-1}x_1^{-1}x_2x_1x_2x_1^{-1}x_2^{-1}x_1^{-1}x_2x_1x_2^{-1}x_1^{-1}x_2^{-1}, \\ & x_2x_1x_2x_1x_2^2x_1^{-1}x_2x_1^2x_2^{-2}x_1^{-1}x_2x_1^{-1}x_2^{-1}x_1^2, \\ & x_2x_1^{-1}x_2x_1x_2^{-1}x_1x_2^2x_1^{-1}x_2x_1x_2^{-1}x_1x_2x_1^{-1}x_2^{-1}x_1^2, \\ & x_2x_1x_2^{-1}x_1x_2x_1^{-1}x_2x_1^{-2}x_2^{-1}x_1^{-1}x_2x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1^2, \\ & x_2x_1^{-1}x_2x_1^{-1}x_2^{-2}x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1^{-1}x_2x_1^{-2}x_2^{-2}x_1^2, \\ & x_1x_2x_1^{-2}x_2^{-1}x_1^{-1}x_2^{-1}x_1^{-1}x_2^{-2}x_1x_2^{-1}x_1^{-2}x_2^2x_1x_2^{-1}, \\ & x_1^{-2}x_2^{-1}x_1^{-2}x_2x_1x_2^{-1}x_1x_2^{-1}x_1^2x_2^{-1}x_1x_2x_1^{-2}x_2^2\rangle. \end{aligned}$$

Then

$$\mathcal{H} = G/\langle L_1, L_2, L_3\rangle$$

is a generalized 5*-fold triangle group which is hyperbolic and has property (T).*

$\mathcal{H}$ is hyperbolic. The proof follows from analyzing the links of Cayley graphs of $G_i = P_i/L_i$. It was shown in [5, Theorem II.12.28] that coset graphs $\Gamma_{G_i}(\langle x_j\rangle, \langle x_k\rangle)$ can be used to determine the geometry upon which $G = G(L_1, L_2, L_3)$ acts. In particular, if r_H denotes the half of girth of Γ_H, then

$$\frac{1}{r_{G_1}} + \frac{1}{r_{G_2}} + \frac{1}{r_{G_3}} < 1$$

implies that G is (non-elementary) hyperbolic. Since $r_{C_5\times C_5} = 2$, $r_{H(\mathbb{F}_5)} = 3$ and one can show (e.g. by computer calculations) that $r_{PSL_2(\mathbb{F}_{109})} = 7$, the sum of reciprocals is equal to $\frac{41}{42}$, i.e. G is hyperbolic.

$\mathcal{H}$ has property (T). The proof relies on a criterion of M. Ershov and A. Jaikin-Zapirain [3], relating angles between groups to property (T).

Let $A, B < G$ be subgroups and let (π, V) be a unitary representation of G without non-zero G-fixed vectors. We define

$$\varepsilon_G(A, B; \pi) = \sup_{\|a\|=\|b\|=1} \left\{\langle a, b\rangle \colon a \in V^A, b \in V^B\right\}$$

which can be thought as the cosine of angles between subspaces of V fixed by A and B. The (cosine of) angle between groups is defined as

$$\varepsilon_G(A, B) = \sup_{\pi\in\widehat{G}} \varepsilon_G(A, B; \pi)$$

and serves the role of the idealized (cosine) of angles between subspaces V^A and V^B. The intuition behind the definition is that if the angle between subgroups is bounded from 0 by a positive constant, then a vector almost fixed by one of the subgroups must me moved significantly by the other.

Theorem ([3], Corrolary to Theorem 5.9). *Let $G = \langle A_1, A_2, A_3\rangle$, and for three distinct indices $i, j, k \in \{1, 2, 3\}$ let $X_i = \langle A_j, A_k\rangle$. If each X_i is finite and*

$$\varepsilon_{X_1}^2 + \varepsilon_{X_2}^2 + \varepsilon_{X_3}^2 + 2\varepsilon_{X_1}\varepsilon_{X_2}\varepsilon_{X_3} < 1,$$

then G has property (T).

It is easy to compute that $\varepsilon_{C_5\times C_5}(\langle x_2\rangle, \langle x_3\rangle) = 0$. Moreover the angle

$$\varepsilon_{H(\mathbb{F}_5)}(\langle x_1\rangle, \langle x_3\rangle) = \frac{1}{\sqrt{5}}$$

has been computed in [3]. Let $X = PSL_2(\mathbb{F}_{109})$. To compute $\varepsilon_X(\langle x_1\rangle, \langle x_2\rangle)$, the angle in X between $\langle x_1\rangle$ and $\langle x_2\rangle$ we make use of the following result.

Theorem. *Let $X = \langle A, B\rangle$ be a group generated by two finite, proper subgroups A, B, and consider $S = (A \cup B) \setminus \{e\}$ as its generating set. Let $\Delta = 1 - \frac{1}{|S|}\sum_{s\in S} s$ denote the Laplacian of X associated to the generating set. Then $1 - \varepsilon_X(A, B)$ is the smallest positive eigenvalue of Δ.*

A numerical estimate of

$$\varepsilon_X \approx 0.877825171$$

was obtained by applying standard ARPACK eigenvalue routines to the adjacency matrix of the full Cayley graph of (X, S). However, since $X = PSL_2(\mathbb{F}_{109})$ has about half a million elements, we do not know how to compute the eigenvalue in a certified way directly.

In order to obtain the required certification, we have computed the largest eigenvalue of the hermitian operator

$$\varrho(\Delta) = \sum_{i=1}^{4} \rho(x_1)^i + \rho(x_2)^i$$

for each non-trivial irreducible representation ρ of X individually. An implementation of those representations is available at [8]. The certification, including provably correct bounds, is provided by the Arblib library [6] and eigenvalue routines therein. The smallest eigenvalues for all non-trivial irreducible representations of X have been computed up to 67 decimal places and allow to certify that

$$|\varepsilon_X - 0.87782517106| < 10^{-10}.$$

Thus we can bound the quantity in the Ershow–Jaikin-Zapirain

$$\varepsilon_{C_5\times C_5}^2 + \varepsilon_{H(\mathbb{F}_5)}^2 + \varepsilon_X^2 + 2\varepsilon_{C_5\times C_5}\varepsilon_{H(\mathbb{F}_5)}\varepsilon_X \in [0.77975134287 \pm 10^{-10}] < 1,$$

which proves that $\mathcal{H}$ has property (T).

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Profinite invariants of (non-)arithmetic groups

Roman Sauer
(joint work with Holger Kammeyer, Steffen Kionke, Jean Raimbault)

A finitely generated, residually finite group Γ is called *profinitely rigid* if any other such group Λ with the same set of finite quotients as Γ is isomorphic to Γ; this can be expressed in terms of profinite completions: if $\widehat{\Lambda} \cong \widehat{\Gamma}$, then $\Lambda \cong \Gamma$. While all finitely generated abelian groups have this property, there are already virtually cyclic groups which are not profinitely rigid. In general, profinite rigidity is extremely difficult to characterize. Recent work of Bridson–McReynolds–Reid–Spitler [2] shows that profinite rigidity holds for certain Kleinian groups, including the Weeks manifold group. On the other hand we note that profinite rigidity of free groups, surface groups or $SL_n(\mathbb{Z})$ is still open.

Two related questions seem more accessible: (i) to establish profinite rigidity among a certain class of groups and (ii) to find *profinite invariants*. A group invariant is *profinite* if it takes the same value on finitely generated, residually finite groups whose profinite completions are isomorphic. In this talk we discuss the profinite invariance of the sign of the Euler characteristic within a most relevant subclass of finitely generated, residually finite groups: *arithmetic groups with the congruence subgroup property*. In particular, this (conjecturally) includes all irreducible lattices in higher-rank semisimple Lie groups. The following is our main result.

Main Theorem. Let $\underline{G_1}$ and $\underline{G_2}$ be linear algebraic groups defined over number fields k_1 and k_2, and let $\Gamma_1 \leq \underline{G_1}(k_1)$ and $\Gamma_2 \leq \underline{G_2}(k_2)$ be arithmetic subgroups. Suppose that $\underline{G_1}$ and $\underline{G_2}$ have a finite congruence kernel and that Γ_1 is profinitely commensurable with Γ_2. Then $\operatorname{sign}\chi(\Gamma_1) = \operatorname{sign}\chi(\Gamma_2)$.

The following theorem shows that we cannot extend the main result from the sign of the Euler characteristic to its actual value.

Theorem. For positive integers m and n, let $\Gamma_{m,n}$ be the level four principal congruence subgroup of $\mathrm{Spin}(m,n)(\mathbb{Z})$. Then $\widehat{\Gamma_{8,2}} \cong \widehat{\Gamma_{4,6}}$ but

$$\chi(\Gamma_{8,2}) = 2^{89} \cdot 5^2 \cdot 17 \quad \text{whereas} \quad \chi(\Gamma_{4,6}) = 2^{90} \cdot 5^2 \cdot 17.$$

The spinor groups $\mathrm{Spin}(m,n)(\mathbb{Z})$ arise from the $(m+n)$-ary integral diagonal quadratic form with m coefficients "+1" and n coefficients "-1".

The existence of the above examples can be used to show that one cannot broaden the conclusion of the Main Theorem from arithmetic to residually finite groups that admit a finite classifying space. The latter is referred to as being of type (F).

Theorem. There are three residually finite groups Γ_1, Γ_2, and Γ_3 of type (F) which have isomorphic profinite completions such that

$$\chi(\Gamma_1) < 0, \qquad \chi(\Gamma_2) = 0, \qquad \chi(\Gamma_3) > 0.$$

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Reporter: Lara Bessmann

Participants

Prof. Dr. Goulnara N. Arzhantseva
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1090 Wien
AUSTRIA

Lara Bessmann
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Martin R. Bridson
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Benjamin Brück
Department of Mathematics
University of Copenhagen
Universitetsparken 5
2100 København
DENMARK

Prof. Dr. Marc Burger
Departement Mathematik
ETH-Zürich
ETH Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Prof. Dr. Ruth Charney
Department of Mathematics
Brandeis University
415 South St
Waltham, MA 02453
UNITED STATES

Prof. Dr. Tim de Laat
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Mikael de la Salle
UMPA, UMR 5669, CNRS
École Normale Supérieure de Lyon
Office: Nord 495
46, allée d'Italie
69364 Lyon Cedex 07
FRANCE

Prof. Dr. Thomas Delzant
Institut de Recherche
Mathématique Avancee
Université de Strasbourg et CNRS
7, rue Rene Descartes
67084 Strasbourg Cedex Cedex
FRANCE

Prof. Dr. Cornelia Drutu Badea
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Elia Fioravanti
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Dr. Giles Gardam
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Mark Hagen
Department of Mathematics
University of Bristol
Woodland Road
Bristol BS8 1TW
UNITED KINGDOM

Dr. Sahana H Balasubramanya
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Alessandra Iozzi
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Dr. Marek Kaluba
Institut für Mathematik
Technische Universität Berlin
Straße des 17. Juni 135
10623 Berlin
GERMANY

Daniel Keppeler
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Dr. Dawid Kielak
Fakultät für Mathematik
Universität Bielefeld
Postfach 100131
33501 Bielefeld
GERMANY

Prof. Dr. Dessislava H. Kochloukova
Departamento de Matematicas
IMECC - UNICAMP
Rua Sérgio Buarque de Holanda, 651
13083-970 Campinas -SP
BRAZIL

Prof. Dr. Linus Kramer
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Gilbert Levitt
Département de Mathématiques
Université de Caen
LMNO
P.O. Box 5186
14032 Caen Cedex
FRANCE

Prof. Dr. Ashot Minasyan
School of Mathematical Sciences,
Highfield
University of Southampton
University Road
Southampton SO17 1BJ
UNITED KINGDOM

Philip Möller
Mathematisches Institut
Universität Münster
Einsteinstrasse 62
48149 Münster
GERMANY

Prof. Dr. Narutaka Ozawa
Research Institute for Mathematical
Sciences
Kyoto University
Kitashirakawa, Sakyo-ku
Kyoto 606-8502
JAPAN

Prof. Dr. Maria Beatrice Pozzetti
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Piotr Przytycki
Department of Mathematics and Statistics
McGill University
805, Sherbrooke Street West
Montréal QC H3A 0B9
CANADA

Prof. Dr. Bertrand Rémy
Centre de Mathématiques Laurent Schwartz
École polytechnique
Route de Saclay
91128 Palaiseau Cedex
FRANCE

Dr. Yuri Santos Rego
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg
G03-220a
Universitätsplatz 2
39106 Magdeburg
GERMANY

Prof. Dr. Roman Sauer
Institut für Algebra und Geometrie
Fakultät für Mathematik (KIT)
Englerstrasse 2
76131 Karlsruhe
GERMANY

Prof. Dr. Petra Schwer
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität Magdeburg
Gebäude 03, Rm. 206a
Universitätsplatz 2
39106 Magdeburg
GERMANY

Dr. Olga Varghese
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Federico Vigolo
Department of Mathematics
The Weizmann Institute of Science
POB 26
234 Herzl St.
Rehovot 7610001
ISRAEL

Prof. Dr. Anna Katharina Wienhard
Mathematisches Institut
Ruprecht-Karls-Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. George A. Willis
School of Mathematical and Physical Sciences
University of Newcastle
Callaghan, Newcastle NSW 2308
AUSTRALIA

PD Dr. Stefan Witzel
Mathematisches Institut
Justus-Liebig-Universität Giessen
Arndtstrasse 2
35392 Gießen
GERMANY