Mathematisches Forschungsinstitut Oberwolfach

Report No. 31/2019

DOI: 10.4171/OWR/2019/31

Dynamische Systeme

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7 July – 13 July 2019

Abstract. This workshop continued the biannual series at Oberwolfach on Dynamical Systems that started as the "Moser-Zehnder meeting" in 1981. The main themes of the workshop are the new results and developments in the area of dynamical systems, in particular in Hamiltonian systems and symplectic geometry. This year special emphasis where laid on different kinds of spectra (in contact geometry, in Riemannian geometry, in dynamical systems and in symplectic topology).

Mathematics Subject Classification (2010): 37, 53D.

Introduction by the Organizers

The workshop *Dynamische Systeme*, organised by M.-C. Arnaud (Avignon), H. Eliasson (Paris), H. Hofer (Princeton) and V. Kaloshin (Maryland) was well attended with over 50 participants with broad geographic representation from 13 countries. The workshop covered a large area of dynamical systems mainly in a symplectic or Hamiltonian setting with a special focus on rigidity problems: spectrum theory of dynamical systems (length spectrum, PDE spectrum, spectral invariants of symplectic topology), some aspects of limits theorems in ergodic theory, horseshoes and chaos, celestial mechanics.

Striking results about spectral rigidity of billiards were presented by M. Leguil for dispersing billiards, J. de Simoi for axis symmetric domains and A. Sorrentino for convex billiards. Ergodic results for the billiard flow, a flow that is hard to study because of its discontinuities, were presented by V. Baladi (existence of a measure of maximal entropy for a Sinaï billiard) and I. Mebourne (decay of correlations).

Other results in ergodic theory were presented: the growth of nomalizing sequences was explained by S. Gouezel and results on infinite Lebesgue spectrum for area preserving toral flows were explained by G. Forni.

Recent results concerning the spectrum in symplectic toplogy, as action selectors, were presented: D. Cristofaro-Gardiner presented results on the spectral recognition of rank one contact forms on closed three-manifolds, L. Polterovich dealt with quantum footprints of symplectic rigidity, S. Seyfaddini presented Floer homology and Hamiltonian homeomorphisms, F. Schlenk explained a simple construction of an action selector on aspherical symplectic manifolds, C. Viterbo presented results on barcode and small eigenvalues of the Witten Laplacian

Several results concerning topological entropy were presented: P. Le Calvez proved that a smooth generic area preserving diffeomorphism of a closed surface has an horseshoe and then positive topological entropy, Barney Bramham proved that a Reeb flow has a global section or has a horseshoe, S. Crovisier explained the structure of the periods of periodic orbits for dissipative diffeomorphisms of the disc with zero entropy.

Results concerning Arnol'd diffusion were presented: M. Gidea dealt with Energy Drift and Diffusion Process in the Three-Body Problem and T. Seara spoke about recent results in geometric methods for Arnol'd diffusion,

Several other topics in dynamics where discussed in different talks: P. Berger presented results on the emergence of wandering Fatou components among polynomial automorphisms of the plane, J. Chaika presented results on horocycle orbits in strata of translations surfaces, A. Knauf dealt with asymptotic completeness in celestial mechanics, T. Jäger presented some aspects of topological dynamics and aperiodic ordert, D. Turaev spoke about stable multiparticle choreographies in repelling potential, L.S. Young presented results on the dynamics of the brain

The meeting was held in an informal and stimulating atmosphere. The traditional walk was organized by F. Schlenk on Wednesday afternoon.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Viktor L. Ginzburg in the "Simons Visiting Professors" program at the MFO.

Workshop: Dynamische Systeme

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Abstracts

Homoclinic orbits for area preserving diffeomorphisms of surfaces

Patrice Le Calvez
(joint work with Martín Sambarino)

Let S be a smooth closed orientable surface of genus g, furnished with a smooth area form ω. For $1 \leq r \leq \infty$, denote $\mathrm{Diff}^r_\omega(S)$ the set of C^r diffeomorphisms of S preserving ω, endowed with the C^r-topology. We have :

Theorem. *For $1 \leq r \leq \infty$, there exists a residual set $\mathcal{R} \subset \mathrm{Diff}^r_\omega(S)$ such that if $f \in \mathcal{R}$, then*

- *there exist hyperbolic periodic points,*
- *every hyperbolic periodic point has a transverse homoclinic intersection.*

Using the fact that the existence of a hyperbolic periodic point with a transverse homoclinic intersection is an open property that implies the positiveness of the entropy, we immediately deduce:

Corollary. *For $1 \leq r \leq \infty$, there exists a dense open set $\mathcal{O} \subset \mathrm{Diff}^r_\omega(S)$ such that the topological entropy of every element of $f \in \mathcal{O}$ is positive.*

Let us precise the theorem. We denote $\mathcal{G}^r_\omega(S) \subset \mathrm{Diff}^r_\omega(S)$ the (residual) set of diffeomorphisms satisfying the following conditions.

- Every periodic point is either elliptic or hyperbolic. Moreover, if z is an elliptic periodic point of period q, then the eigenvalues of $Df^q(z)$ are not roots of unity.
- Stable and unstable branches of hyperbolic points that intersect must also intersect transversally (in particular there is no saddle connection).
- If U is a neighborhood of an elliptic periodic point z, then there is a topological closed disk D containing z, contained in U, and bordered by finitely many pieces of stable and unstable manifolds of some hyperbolic periodic point z'.

Denote $\mathrm{Fix}_h(f)$ the set of hyperbolic fixed points of $f \in \mathrm{Diff}^r_\omega(S_g)$ and $\mathrm{Per}_h(f)$ the set of hyperbolic periodic points. Let us recall the following folklore result, consequence of Lefschetz formula:

Proposition. *If $f \in \mathcal{G}^r_\omega(S)$, then $\#\mathrm{Per}_h(f) \geq \max(0, 2g-2)$.*

Our theorem will be divided in two parts.

Theorem A. *If $f \in \mathcal{G}^r_\omega(S)$ and $\#\mathrm{Per}_h(f) > \max(0, 2g-2)$, then every hyperbolic periodic point of f has a transverse homoclinic intersection.*

Theorem B. *The set of $f \in \mathcal{G}^r_\omega(S)$ such that $\#\mathrm{Per}_h(f) > \max(0, 2g-2)$ is dense in $\mathcal{G}^r_\omega(S)$.*

The proof of Theorem A is based on a theorem of Mather. We can prove that the four branches of $z \in \mathrm{Per}_h(f)$ accumulate on z and have the same closure $K(z)$ in S. So, we can define an equivalence relation on $\mathrm{Per}_h(f)$ writing

$$z \sim z' \Leftrightarrow K(z) = K(z').$$

In that case, z has homoclinic intersection if it is the case of z' and we say that the class κ is *homoclinic*. Using some homological arguments, we begin to prove that if $\#\mathrm{Fix}_h(f) > 2g-2$, there is a point with a homoclinic intersection. Consequently, if there is a unique class κ, then κ is homoclinic. Using improvements of Mather's theory due to Koropecki-Le Calvez-Nassiri, we can prove in the case where there is at least two classes, that every class κ is contained in a connected open set $V \neq S$ whose genus g' satisfies $\#\kappa > 2g'-2$ and use the same argument as before.

To prove Theorem B, we must begin to understand what are the elements $f \in \mathcal{G}^r_\omega(S)$ such that $\#\mathrm{Per}_h(f) = 2g-2$. This is given by the following result:

Theorem C. *If $f \in \mathcal{G}^r_\omega(S)$ and $\#\mathrm{Per}_h(f) = 2g-2$, then:*

- *if $g = 1$, then f is isotopic to the identity or to a power of a Dehn twist;*
- *if $g > 1$, $\exists q \geq 1$ such that f^q is isotopic to the identity. Moreover there is no non trivial periodic continua and consequently f is transitive and every stable or unstable branch is dense.*

So, to prove Theorem B, it is sufficient to show that one can approximate a map $f \in \mathcal{G}^r_\omega(S)$, isotopic to the identity, such that $\#\mathrm{Per}_h(f) = \sharp\mathrm{Fix}_h(f) = 2g-2$. with a map having a supplementary periodic point. The rotation vector $\rho_f(\mu_\omega) \in H_1(S,\mathbb{R})$ of the measure induced by ω is not zero, because f is not Hamiltonian. So, there exists a simple loop $\lambda \subset S \setminus \mathrm{Fix}_h(f)$ such that $[\lambda] \wedge \rho_f(\mu_\omega) \neq 0$. Fix a small annular neighborhood A of λ and a C^∞ divergence free vector field X supported on A such that $\rho_{\varphi^s_X}(\mu_\omega) = s[\lambda]$ if $t \in \mathbb{R}$, and set $f^s = \varphi^s_X \circ f$. One has

$$\rho_{f^s}(\mu_\omega) \wedge \rho_f(\mu_\omega) = s\,[\lambda] \wedge \rho_f(\mu_\omega) \neq 0 \ \text{ if } \ s \neq 0.$$

The following proposition easily implies theorem B and can be proven using the Forcing theory introduced by Le Calvez-Tal.

Proposition. $\forall \varepsilon > 0, \exists s \in (0,\varepsilon)$, *$f^s \notin \mathcal{G}^r_\omega(S)$ or $h_{\mathrm{top}}(f^s) > 0$.*

Note that the the main theorem was proven by Takens in case $r = 1$ using the C^1-closing Lemma. It was also proved by Robinson and Pixton in the case of the sphere and by Oliveira in the case of the torus. Moreover it was announced by Xia in the case of Hamiltonian diffeomorphisms.

On the Integrability of Birkhoff Billiards

Alfonso Sorrentino
(joint work with Guan Huang, Vadim Kaloshin)

A *Birkhoff billiard* is a dynamical model describing the motion of a billiard ball inside a strictly convex domain $\Omega \subset \mathbb{R}^2$ with smooth boundary $\partial\Omega$. The massless ball moves with unit velocity and without friction following a rectilinear path; when it hits the boundary it reflects elastically according to the standard *reflection law*: the angle of reflection is equal to the angle of incidence.

This conceptually simple model, yet dynamically very rich, has been proposed by G. D. Birkhoff as a mathematical playground where "[...]*the formal side, usually so formidable in dynamics, almost completely disappears, and only interesting qualitative questions need to be considered*" [3, pp. 155-156].

Since then, billiards have captured the attention of many researchers in various areas of mathematics. Whereas it is clear how the geometry (*i.e.*, the shape) of the domain determines the billiard dynamics, a more subtle and intriguing question is to which extent dynamical information can be used to reconstruct the shape of the billiard domain. This translates into compelling inverse problems and rigidity questions, that provide the ground for some of the foremost conjectures in dynamical systems.

In this talk I shall focus on the so-called *Birkhoff conjecture*, namely the possibility of classifying billiard domains which admit an integrable dynamics.

The easiest example of billiard is given by a billiard in a disc: in this case it is easy to check that the angle of reflection remains constant at each reflection, hence it is an *integral of motion*, which makes the circular billiard an *integrable* dynamical system.

Integrability is one of the most important issue in the study of dynamical systems. In the case of billiards, it translates into a very peculiar geometric property: the existence of so-called *caustics*. For circular billiards, for example, the fact that the angle of reflection remains constant implies that each trajectory is tangent to a concentric circle, which is an example of a caustic. The family of all these caustics foliates the whole circular billiard domain.

More precisely, we say that a curve Γ is a *caustic* for a billiard, if every time a trajectory is tangent to Γ, then it remains tangent after each reflection.

Whereas the mere existence of caustics does not provide significant information on the shape of the domain[1], the presence of a foliation of the billiard table by caustics seems to be a more peculiar property.

Billiards in an ellipse have a similar dynamical picture: trajectories not passing through a focal point are tangent to a confocal conic section, either a confocal ellipse or the two branches of a confocal hyperbola. Thus confocal ellipses are convex caustics, and they foliate the whole domain with the exception of the segment between the foci.

[1]A striking result by Lazutkin [9] shows that all Birkhoff billiards with sufficiently smooth boundary admit a positive measure set of caustics, accumulating to the boundary of the billiard.

Are there other billiards admitting an integrable dynamics? This appearantly naïve question has given rise to one of the most famous (and impenetrable) problems in dynamical systems:

Conjecture (Birkhoff[2]). *Integrable Birkhoff billiards correspond to ellipses.*

Despite its long history and the amount of attention that it has captured, this conjecture is still open. Some of the most relevant contributions are:

- Bialy [2] proved that the only Birkhoff billiard fully foliated by caustics is in the disc. This result was also proved by Wojtkowski [12], by an integral-geometric approach.
- Innami [7] proved, using Aubry-Mather theory, that the existence of caustics with rotation numbers accumulating to 1/2 implies that the billiard domain must be an ellipse.
- The analogue of this conjecture under the assumption that there exists an integral of motion polynomial in the velocity (*Algebraic Birkhoff conjecture*), has been recently proved by Glutsyuk [5].

Instead of considering all possible Birkhoff billiards, one could restrict the analysis to domains that are sufficiently close to ellipses and study the same question in this context (*Perturbative Birkhoff Conjecture*):

- Levallois & Tabanov [10] proved the non-integrability of algebraic perturbations of ellipses.
- Delshams & Ramírez-Ros [4] showed the non-integrability of entire symmetric perturbations of ellipses.

In this talk I shall describe a recent development obtained in collaboration with Vadim Kaloshin, proving that the Perturbartive Birkhoff Conjecture holds true. For nearly circular domains, this result was firstly proved in [1].

Theorem (Kaloshin, S. [8]). *Let $\mathcal{E}_0$ be an ellipse of eccentricity $0 \le e_0 < 1$ and semi-focal distance c; let $k \ge 39$. For every $K > 0$, there exists $\varepsilon = \varepsilon(e_0, c, K)$ such that if Ω is C^k-smooth domain and*

i) *the billiard map in Ω admits invariant curves/caustics foliated by periodic points for all rotation numbers $\frac{1}{q}$, $q \ge 3$,*
ii) *$\partial\Omega$ is K-close to $\mathcal{E}_0$, with respect to the C^k-norm,*
iii) *$\partial\Omega$ is ε-close to $\mathcal{E}_0$, with respect to the C^1-norm,*

then Ω is an ellipse.

Notice that he notion of integrability i) that we require is very weak. A natural question is what happens if only a small neighbourhood of the boundary is foliated by caustics, or in another words there are invariant curves/caustics corresponding to rotation numbers in $(0, \delta)$, for some $0 < \delta < \frac{1}{3}$.

A partial answer to this question was recently provided in collaboration with Guan Huang and Vadim Kaloshin in [6] for domains that are a sufficiently smooth

[2]Although some vague indications of this question can be found in [3], its first appearance was in a paper by Poritsky [11], so sometimes it is referred to as *Birkhoff-Poritsky conjecture.*

perturbation of ellipses of small eccentricities, under the assumption that for a given $q_0 \geq 3$ there exist invariant curves foliated by periodic points, for all rotation numbers $\frac{j}{q}$, with $q \geq q_0$ and $j = 1, 2, 3$ such that $\gcd(j, q) = 1$. The upper bound on the eccentricity, the smallness condition on the perturbation and the smoothness requirements, depend all on the choice of q_0.

Acknowledgement. The author acknowledges the support of the project H2020-MSCA-ITN-ETN "STARDUST-R" and the Italian MIUR Excellence Department Project awarded to the Department of Mathematics, University of Rome Tor Vergata, CUP E83C18000100006.

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On the measure of maximal entropy of Sinai billiards

Viviane Baladi
(joint work with M. Demers)

Sinai billiards maps and flows are uniformly hyperbolic — however grazing orbits give rise to singularities. Most existing works on the ergodic properties of billiards are about the SRB measure (i.e. the Liouville measure in the case of flows), for which exponential mixing is known (both in discrete [6] and continuous time [2]). Another natural equilibrium state is the measure of maximal entropy. Since the discrete-time billiard map T is discontinuous, the mere existence of this measure is not granted a priori. The results of [1] presented in this talk are the following:

Assuming finite horizon, we propose a definition h_* for the topological entropy of T. We prove that h_* is not smaller than the value given by the variational principle, and that it is compatible with the definitions of Bowen using spanning or separating sets. To get more, we need an additional condition. Letting (r, φ) be the billiard coordinates, fix an angle φ_0 close to $\pi/2$ and $n_0 \in \mathbb{N}$ large. Let $s_0 \in (0, 1)$ be the smallest number such that any orbit of length n_0 has at most $s_0 n_0$ collisions with $|\varphi| > \varphi_0$. (Due to the finite horizon condition, we can choose φ_0 and n_0 such that $s_0 < 1$. If in addition there are no triple tangencies on the table — a generic condition — then $s_0 \leq 2/3$.) Assume that $h_* > s_0 \log 2$. Then, using a transfer operator acting on a space of anisotropic distributions (adapting the arguments of Demers and Zhang [4] to our setting), we construct an invariant probability measure μ_* of maximal entropy for T (i.e., $h_{\mu_*}(T) = h_*$), we show that μ_* has full support and is Bernoulli, and we prove that μ_* is different from the SRB measure except if all non grazing periodic orbits have multiplier equal to h_*. (A key step to carry out the Hopf argument is to show absolute continuity of the unstable foliation with respect to μ_*.) Next, h_* is compatible with the Bowen–Pesin–Pitskel topological entropy of the restriction of T to a non-compact domain of continuity. Last, applying results of Lima and Matheus [5], and Buzzi [3], the map T has at least Ce^{nh_*} periodic points of period n, for all n.

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Asymptotic completeness in celestial mechanics

Andreas Knauf

(joint work with Stefan Fleischer)

The general setting is the one of a smooth (non-compact) manifold P of dimension d with a volume form Ω and a C^1-vector field X so that the Lie derivative $\mathcal{L}_X\Omega$ vanishes. Then the initial value problem for the differential equation $\dot{x} = X(x)$ has a maximal solution of the form

$$\Phi \in C^1(D, P) \quad \text{on} \quad D = \left\{ (t, x) \in \mathbb{R} \times P \;\middle|\; T^-(x) < t < T^+(x) \right\},$$

with the *escape times* $T := T^+ : P \to (0, +\infty]$ and $T^- : P \to [-\infty, 0)$, and Φ preserves the volume form Ω.

Definition 1. *The* **wandering set** *of* Φ *is given by*

$$\mathrm{Wand} := \big\{x \in P \mid \textit{for some neighborhood } U_x \textit{ of } x \textit{ and time } t_x : \\ U_x \cap \Phi\big(\,((t_x, T(x)) \times U_x) \cap D\big) = \emptyset\big\}.$$

The **singular set** *of* Φ *is given by* $\mathrm{Sing} := \{x \in P \mid T(x) < \infty\}$.

Lemma 2.

(1) Sing *is a Borel set*
(2) $\mathrm{Sing} \subseteq \mathrm{Wand}$.

Let $\bar{\iota}_m : \overline{\mathcal{H}}_m \to P \quad (m \in \mathbb{N})$ be a sequence of codimension one closed ∂-submanifolds of P, which we call *Poincaré surfaces.*

Assumptions:

(1) The vector field X is transversal to their relative interior $\iota_m : \mathcal{H}_m \to P$. Thus ($\boldsymbol{i}$ being the inner product) the $(d-1)$-form

$$\mathcal{V} := \boldsymbol{i}_X\,\Omega$$

on P induces volume forms $\mathcal{V}_m := \iota_m^* \mathcal{V}$ on $\mathcal{H}_m$.
(2) We assume that $\lim_{m\to\infty} \int_{\mathcal{H}_m} \mathcal{V}_m = 0$.

Definition 3. *The set of* **transition points** *is given by*

$$\mathrm{Trans} := \{x \in P \mid \exists\, m_0 \in \mathbb{N}\,\forall\, m \geq m_0 : \mathcal{O}^+(x) \cap \overline{\mathcal{H}}_m \neq \emptyset\},$$

$\mathcal{O}^+$ *being the forward orbit.*

Our main result in [FK19a] is the following.

Theorem 4. *From the assumptions it follows that* $\Omega(\mathrm{Trans} \cap \mathrm{Wand}) = 0$.

We applied this to scattering by n particles on

(1) joint configuration space $\widehat{M} := \mathbb{R}^{dn} \setminus \Delta$, with

$$\Delta := \{q \in \mathbb{R}^{dn} \mid \text{there exist } 1 \leq i < j \leq n : \; q_i = q_j\}$$

(2) phase space $\hat{P} := T^*\widehat{M}$
(3) Hamiltonian function $H \in C^2(\hat{P}, \mathbb{R})$, $H(q,p) := K(p) + V(q)$ with

$$K(p) := \textstyle\sum_{k=1}^{n} \frac{\|p_k\|^2}{2m_k} \quad \text{and} \quad V(q) := \sum_{1\leq i<j\leq n} V_{i,j}(q_i - q_j).$$

Definition 5. V *is* **admissible**, *if* $\lim_{\|q\|\to\infty} V_{i,j}(q) = 0$, *there exists an* $\alpha \in (0,2)$ *such that* $D^2 V_{i,j}(q) = \mathcal{O}(\|q\|^{-\alpha-2}) \quad (\|q\| \leq 1)$, *and for some* $C_V > 0$ *either*

(1) for suitable $Z_{i,j} \in \mathbb{R}$, $\big|\big\langle \frac{q}{\|q\|}, \nabla V_{i,j}(q)\big\rangle + \alpha \frac{Z_{i,j}}{\|q\|^{\alpha+1}}\big| \leq C_V \qquad (\|q\| \leq 1)$
(2) or the $V_{i,j}$ *are bounded above, and, with* $W_-(q) := \max(-W(q), 0)$,

$$\langle q, \nabla V_{i,j}(q)\rangle \leq C_V + \alpha\,(V_{i,j})_-(q) \qquad (\|q\| \leq 1).$$

Concerning the set $\mathrm{Coll} := \big\{x \in \mathrm{Sing} \;\big|\; \lim_{t \nearrow T^+(x)} q(t,x) \text{ exists}\big\}$, we have

Theorem 6. [FK19b]
For all $n \in \mathbb{N}$, $d \geq 2$ and $E \in \mathbb{R}$ the set $\mathrm{Coll} \cap H^{-1}(E)$ *of phase space points leading to a collision has Liouville measure zero, provided V is admissible.*
Additionally $\int_0^{T(x)} K(p(t,x))\, dt < \infty \qquad (x \in \mathrm{Coll})$.

Definition 7. *We call the potential V*

- **long-ranged**, *if for some $\epsilon > 0$, $\|\nabla V_{i,j}(q)\| = \mathcal{O}(\|q\|^{-1-\epsilon}) \quad (\|q\| \geq 1)$,*
- **moderated**, *if for some $\alpha \in (0,2)$, $\|\nabla V_{i,j}(q)\| = \mathcal{O}(\|q\|^{-\alpha-1})$ $(\|q\| \leq 1)$.*

Note that α-homogeneous potentials are long-ranged and moderated.

Definition 8. *For an initial condition $x_0 \in P \setminus \mathrm{Sing}$ the* **asymptotic velocities** *are*

$$\overline{v}^{\pm}(x_0) := \lim_{t \to \pm\infty} \frac{q(t, x_0)}{t} \in \mathbb{R}^{dn}.$$

Concerning these Cesàro limits, in [Kn18] we proved

Theorem 9. *For $n = 4$, $d \geq 3$ and long-ranged moderated central potentials the set of x_0 for which $\overline{v}^{\pm}(x_0)$ does not exist, has Liouville measure zero.*
The Liouville measure of Sing *is zero, too.*

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Growth of normalizing sequences in limit theorems

Sébastien Gouëzel

Definition 1. *Let (X, P) be a probability space, $T : X \to X$ a measurable map and $f : X \to \mathbb{R}$ a measurable function. We say that (X, P, T, f) satisfies a limit theorem, for the normalizing sequence $(B_n) \in (0, +\infty)^{\mathbb{N}}$, if there exists a real random variable Z which is not almost surely 0 such that $S_n f / B_n$ converges in distribution with respect to P towards Z, where $S_n f = \sum_{k=0}^{n-1} f \circ T^k$ is the Birkhoff sum of f for T.*

There are many examples of such limit theorems. Let us give a few classical ones:

(1) If T preserves P and f is integrable, then Birkhoff theorem states that $S_n f / n$ converges to $E(f \mid \mathcal{I})$ where $\mathcal{I}$ is the σ-algebra of invariant subsets. This is a limit theorem with normalizing sequence $B_n = n$, when $E(f \mid \mathcal{I})$ is not uniformly zero. When P is ergodic, this reduces to the fact that $S_n f / n$ converges to $\int f$.

(2) If T is an Anosov map and P is a Gibbs measure for a Hölder potential, one can get a limit theorem even when $\int f = 0$: if f is Hölder continuous, then the central limit theorem holds. This means that $S_n f/\sqrt{n}$ converges to a Gaussian random variable $\mathcal{N}(0, \sigma^2)$. This is a limit theorem when $\sigma^2 > 0$ or, equivalently, when f can not be written as $g - g \circ T$ for some measurable function g. The normalizing sequence is $B_n = \sqrt{n}$.

(3) If T is mixing and $f = g - g \circ T$, then $S_n f$ converges in distribution to $Z - Z'$, where Z and Z' are independent random variables distributed like g. This is a limit theorem if g is not constant, i.e., if f is not almost everywhere zero. The normalizing sequence is $B_n = 1$.

(4) Starting from a sequence of independent identically distributed random variables whose renormalized partial sums converge to a stable law, one gets an ergodic probability preserving system (X, T, P) and a function f such that $S_n f/n^{1/\alpha}$ satisfies a limit theorem, for any $\alpha \in (0, 2]$.

Our goal in [2] is to investigate the possible shape of limit theorems for general systems. The limit Z can be arbitrary, as was proved by Aaronson and Weiss in [1]:

Theorem 1.1. *For any real random variable Z and any probability preserving non-atomic system (X, T, P), there exists a measurable function f and a sequence $B_n \to \infty$ such that $S_n f/B_n$ converges in distribution to Z.*

On the other hand, it is easy to show that the sequence B_n can not be arbitrary. We prove in [2] that it can not grow more than polynomially:

Proposition 1. *Let (X, T, P) be a probability preserving map. Assume that, for some function f, the sequence $S_n f/B_n$ satisfies a limit theorem. Then there exists $C > 0$ such that $B_n = O(n^C)$.*

The argument for this proposition is easy. Our main interest, however, is in systems which do not preserve P. If one removes all assumptions, then it is easy to create stupid examples in which B_n can grow arbitrarily fast, by using for instance the left shift on $\mathbb{Z}$. A form of rigidity comes from assuming *conservativity*, i.e., that almost every point of a set A comes back infinitely often to A under the iteration of the dynamics. This means that the values of f seen through the dynamics will exhibit some weak kind of recurrence, preventing the Birkhoff sums from growing too quickly. Our main result in this direction is the following theorem.

Theorem 1.2. *Let (X, T, m) be a conservative map, and P a probability measure which is absolutely continuous with respect to m. Suppose that, for some measurable function f, the renormalized Birkhoff sums $S_n f/B_n$ satisfy a limit theorem with respect to P. Then B_n can not grow exponentially: for any $\delta > 0$, one has $B_n = o(e^{\delta n})$.*

This is considerably harder than Proposition 1. It turns out that the result in this theorem is also optimal: in [2], we exhibit for each $\gamma < 1$ a conservative map and a measurable function exhibiting a limit theorem for $B_n = e^{n^\gamma}$. This shows that the possible behaviors in conservative maps are much wilder than in probability preserving systems. We also construct examples in which $\limsup B_{n+1}/B_n = +\infty$ and $\liminf B_{n+1}/B_n = 0$, in striking contrast to probability preserving maps, for which $B_{n+1}/B_n \to 1$.

The proofs of Proposition 1 and Theorem 1.2 have been formalized in the proof assistant Isabelle/HOL, based on the ergodic theory library we had developed for a previous article [3]. This means that these statements are certified, and can be trusted with a degree of confidence which is much higher than anything that could be achieved by the most careful authors and referees.

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Infinite Lebesgue spectrum for conservative flows on the torus

Giovanni Forni

(joint work with B. Fayad, A. Kanigowski)

We prove that a class of smooth (real-analytic) full-measure Diophantine locally Hamiltonian flows on the 2-dimensional torus with a single, sufficiently degenerate, rest point have Lebesgue spectrum of infinite (countable) multiplicity.

The proof that the spectrum is absolutely continuous is based on estimate on decay of correlations for smooth coboundaries, which allow to prove that such correlations are square-integrable as functions of time.

The proof that the spectrum is Lebesgue with infinite multiplicity is based on a new criterion, which is well-adapted to smooth dynamical systems with square-integrable correlations on a set of sufficiently rich (dense) subset of smooth observables.

As a consequence of our criterion we derive that smooth time-changes of horocycle flows also have Lebesgue spectrum of infinite multiplicity, thereby completing the proof of a conjecture by A. Katok and J.-P.-Thouvenot [5], Conjecture 6.8.

The speaker had proved that such flows have Lebesgue maximal spectral type in joint work with C. Ulcigrai [4].

We describe the class of flows for which our result holds. These are flows often called Kochergin flows, after A. V. Kochergin who proved that they are mixing. There are very few results on the rate of mixing for flows on surfaces. B. Fayad [1] proved polynomial decay of correlations for a class of Kochergin flows on the 2-torus. The decay rate in this work is however not sufficient to derive that the spectrum is absolutely continuous. B. Fayad and A. Kanigowski [3] recently proved that a subset of full Hasudorff dimension of flows in this class have a generalized Ratner property and are mixing of all orders.

Our class of Kochergin flows consists of flows on the torus with a single rest point locally modeled on a Hamiltonian flow of Hamiltonian

$$H(x,y) = y(x^2+y^2)^l, \qquad \text{for } (x,y) \text{ near } (0,0) \in \mathbb{R}^2\,,$$

with l a sufficiently large integer, and an orbit foliations which coincides with the foliation of a non-singular Diophantine flow, with the exception of two singular orbits ending at the rest point in the future or in the past. It is easy to realize these flows as infinitely differentiable locally Hamiltonian flows on the torus by a partition of unity argument. By a more involved approximation argument it is possible to construct real analytic examples.

Our Kochergin flows have a representation as special flows above an irrational rotation of rotation number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ under a roof function $\varphi : \mathbb{T} \to \mathbb{R}^+$ everywhere of class C^2 except for a singularity at the origin. More specifically we assume that the rotation number α satisfies a full measure Diophantine condition $DC_{\log,\xi}$ of the form: there exist $C > 0$ such that

$$|\alpha - \frac{p}{q}| \geq \frac{C}{q^2(\log q)^{1+\xi}}\,, \qquad \text{for all } (p,q) \in \mathbb{Z}\times\mathbb{Z}\setminus\{0\}\,.$$

We also assume that the roof function has a singularity of the following form: there exist constants $M_1, N_1, R_1 > 0$ and $\eta \in (0,1)$ such that

$$\lim_{\theta\to 0^+} \frac{\varphi(\theta)}{\theta^{-(1-\eta)}} = M_1 \quad \text{and} \quad \lim_{\theta\to 0^-} \frac{\varphi(\theta)}{\theta^{-(1-\eta)}} = M_1 \tag{1}$$

$$\lim_{\theta\to 0^+} \frac{\varphi'(\theta)}{\theta^{-(2-\eta)}} = -N_1 \quad \text{and} \quad \lim_{\theta\to 0^-} \frac{\varphi'(\theta)}{\theta^{-(2-\eta)}} = N_1 \tag{2}$$

$$\lim_{\theta\to 0^+} \frac{\varphi''(\theta)}{\theta^{-(3-\eta)}} = R_1 \quad \text{and} \quad \lim_{\theta\to 0^-} \frac{\varphi''(\theta)}{\theta^{-(3-\eta)}} = R_1\,. \tag{3}$$

We now recall the definition of a spectral type of a flow and in particular the definition of Lebesgue spectral type with countable multiplicity. The Koopman group of a flow $\phi_{\mathbb{R}}$, preserving a probability measure μ on a space X, is the strongly continuous group $U^{\phi}_{\mathbb{R}}$ of unitary operators on $L^2(X, d\mu)$ defined as follows:

$$U^{\phi}_t(f) = f\circ\phi_t\,, \qquad \text{for all } f \in L^2(X,d\mu) \text{ and for all } t\in\mathbb{R}\,.$$

By the spectral theorem for strongly continuous unitary groups, there exists a sequence of probability measures

$$\nu_1 \gg \nu_2 \gg \cdots \gg \nu_k \gg$$

such that $U^{\phi}_{\mathbb{R}}$ on $L^2(X,d\mu)$ is unitarily equivalent to the unitary group $\hat{U}_{\mathbb{R}}$ defined as

$$\hat{U}_t : \hat{f}(\xi) \to e^{it\xi}\hat{f}(\xi)\,, \qquad \text{for all } \hat{f} \in \bigoplus_{k=1}^{\infty} L^2(\mathbb{R}, d\nu_k(\xi)) \text{ and for all } t\in\mathbb{R}\,.$$

The measure class of the measure ν_1 is called the *maximal spectral type* of the unitary group. By the spectral theorem all spectral measures are absolutely continuous with respect to the maximal spectral type.

The sequence $\nu_1 \gg \nu_2 \gg \cdots \gg \nu_k \gg$ is called the *spectral type* of the unitary operator. The spectrum is said to be *absolutely continuous* if all the measures ν_k are absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, or equivalently, if the maximal spectral type is.

An operator with absolutely continuous spectrum can fail to have Lebesgue maximal spectral type if there exists a set $C \subset \mathbb{R}$ of positive Lebesgue measure such that $\nu_1(C) = 0$, hence $\nu_k(C) = 0$ for all $k \in \mathbb{N}$. The multiplicity of the spectrum is the cardinality of the decomposition in the above definition.

Thus the spectrum is *homogeneous Lebesgue* if all measures ν_i are equivalent to the Lebesgue measure and it is *homogeneous Lebesgue of countable multiplicity* if the sequence $\nu_1 \gg \nu_2 \gg \cdots \gg \nu_k \gg \ldots$ is infinite. In particular, an operator with absolutely continuous spectral type can fail to have homogeneous Lebesgue spectrum of multiplicity $k \in \mathbb{N} \setminus \{0\}$ if its spectral type consists of at most $k-1$ spectral measures or if there exists a set $C \subset \mathbb{R}$ such that $\nu_k(C) = 0$.

We are now ready to state our main theorem on the spectral type of Kochergin flows.

Theorem 1. *The Koopman group of a Kochergin flow described above above has Lebesgue spectrum of countable multiplicity if its rotation number $\alpha \in DC_{\log,\xi}$ with $\xi < 1/10$ and if the roof function $\varphi : \mathbb{T} \to \mathbb{R}^+$ has a power singularity at the origin (as in the above formulas) with $0 < \eta < 1/1000$.*

A similar result holds for smooht time-changes (reparametrizations) of classical horocycle flows for compact hyperbolic surfaces. This theorem confirms the Katok–Thouvenot conjecture ([5], Conjecture 6.8).

Theorem 2. *The Koopman group of a smooth time-change of the classical horocycle flow for a compact hyperbolic surface has Lebesgue spectrum of countable multiplicity.*

We conclude by stating our main abstract criterion for countable Lebesgue spectrum. Let $\mathcal{F} : L^2(\mathbb{R}, dt) \to L^2(\mathbb{R}, d\tau)$ denote the Fourier transform, given by the formula

$$\mathcal{F}(f)(\tau) = \int_{\mathbb{R}} f(t) e^{-2\pi \imath t \tau} dt\,, \qquad \text{for all } f \in L^2(\mathbb{R}, dt)\,.$$

Theorem 3. *Let $\{U_{\mathbb{R}}\}$ be a strongly continuous one-parameter unitary group on a Hilbert space H with absolutely continuous spectrum. For a fixed $n \in \mathbb{N}$, let us assume that for every compact set $C \subset \mathbb{R} \setminus \{0\}$ of positive Lebesgue measure there exists $\epsilon_{n,C} > 0$ such that the following holds. For every $\epsilon \in (0, \epsilon_{n,C})$ there exist vectors $f_1, \ldots, f_{n+1} \in H$ such that*

$$\|\langle U_t(f_i), f_j\rangle\|_{L^2(\mathbb{R},dt)} \leq \delta_{ij} + \epsilon\,, \qquad \textit{for all } i,j \in 1,\dots,n+1\,;$$
$$\|\prod_{i=1}^{n+1} \mathcal{F}(\langle U_t(f_i), f_i\rangle)\|_{L^{\frac{2}{n+1}}(C)} > (n+1)!(1+\epsilon)^n\epsilon\,.$$

Then the spectral type of $\{U_\mathbb{R}\}$ is Lebesgue with multiplicity at least $n+1$.

The above criterion is applied through the following corollary, which makes clear that it is enough to realize sufficiently arbitrary correlations. The derivative on the convolutions appears since we only prove square-integrable decay of correlations for smooth coboundaries, which are derivattives along the flow direction.

Corollary 4. *Let us assume that, for every $n \in \mathbb{N}$ and any system of even functions $\omega_1,\dots,\omega_{n+1} \in \mathcal{S}(\mathbb{R})$ (the Schwartz space), and for any any $\epsilon > 0$, there exists vectors $f_1,\dots,f_{n+1} \in H$ such that, for all $i,j \in \{1,\dots,n+1\}$, we have*

$$\|\langle U_t(f_i), f_j\rangle - \frac{d^2}{dt^2}\omega_i * \omega_i(t)\delta_{ij}\|_{L^2(\mathbb{R})} \leq \epsilon\,.$$

Then the spectral type of the strongly continuous one-parameter unitary group $U_\mathbb{R}$ is Lebesgue with countable multiplicity.

The construction of the functions $f_1,\dots,f_{n+1}$, for an arbitrary $n \in \mathbb{N}$, for the applications to the Koopman group of a Kochergin flow and to time-changes of horocycle flows is based on the generalization of a construction found in [4] which consists in defining functions supported in long and thin flow boxes or “towers” (of transverse area converging to zero and diverging height). The functions are in fact supported on a subset of fixed height for a sequence of longer and longer (thinner and thinner) flow boxes. Their self correlations can be made arbitrary on an interval of fixed size with a small square integrable error coming from the correlations for times longer than the height of the flow box, while the mutual correlations can be made small by taking the functions orthogonal on their common domain of definition (this orthogonality property is in turn realized by taking the horizontal factors of the functions to be orthogonal).

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Dynamical spectral rigidity of convex planar domains

Jacopo De Simoi
(joint work with Vadim Kaloshin, Qiaoling Wei)

Let $\Omega \subset \mathbb{R}^2$ be a convex planar domain; is it possible to deform Ω in such a way that the length of every periodic orbit of the billiard system inside Ω is preserved? Isometric deformations trivially satisfy this prescription; we say that Ω is *dynamically spectrally rigid* if no other deformation satisfies this prescription. It has been conjectured by Sarnak in the early 1990's that every (convex) domain with smooth boundary should be spectrally rigid.

In this talk we will see the proof that any sufficiently (finitely) smooth $\mathbb{Z}_2$ symmetric strictly convex domain sufficiently close to a circle is dynamically spectrally rigid (within $\mathbb{Z}_2$ symmetric domains).

Our strategy associates to each domain Ω a corresponding Linearized Isospectral Operator. Studying functional properties (injectivity) of this operator gives information about spectral rigidity of the associated domain. We show that this property holds for every domain sufficiently close to the circle.

The construction is explicit and generalizations are expected in further work in progress with other collaborators. Moreover, thanks to the concrete nature of the functional-analytic problem, numerical explorations and computer-assisted approaches are feasible; reports in these directions will be available at the end of the summer.

Renormalization of Hénon maps with zero entropy

Sylvain Crovisier
(joint work with Enrique Pujals, Charles Tresser)

For C^2-diffeomorphisms on compact surfaces, a positive topological entropy is associated [7] with the existence of "horseshoes": up to taking an iterate, these are subsets where the dynamics is conjugate to a shift. On the contrary the dynamics of systems with vanishing entropy seem very constrained and lead to the following questions: *To what extend can one describe the dynamics of surface diffeomorphisms with zero topological entropy? How do they bifurcate to positive entropy systems?*

In the case of conservative diffeomorphisms of the sphere, Franks and Handel have answered [5] to the first question, showing that the dynamics resemble the dynamics of the time-one maps of hamiltonian flows. More generally, Le Calvez and Tal have proved [8] that, for homeomorphisms, the transitive subsets have a factor which is a periodic orbit, an irrational rotation or an odometers (Rees' surgery produces large classes of exotic examples [2] but they are generally not differentiable). The work we present here deals with C^2 diffeomorphisms of the disc $\mathbb{D}$ which contract the area. In particular, we discuss a conjecture made by one of us in the early 80's and based on numerical experiments. This conjecture appeared and has been discussed in [6, 1]. We say that an integer $n \geq 1$ is a period of f if there exists a point which is fixed by f^n and not by a smaller iterate.

Conjecture (Tresser)**.** *For any dissipative diffeomorphism of the disc, there exists n_0 such that the set of periods is contained in $\{n.2^k,\ n \leq n_0, k \in \mathbb{N}\}$. When infinite, it contains a subset of the form $\{n.2^k, k \in \mathbb{N}\}$.*

This contrasts from diffeomorphisms with positive entropy, whose set of periods contains a subset of the form $n.\mathbb{N}$. The conjecture partially extends to surfaces Sharkovsky's theorem [9]: continuous interval maps have zero topological entropy exactly when the set of periods is finite or has the form $\{2^k, k \in \mathbb{N}\}$.

Two of us have defined [3] the class of mild dissipative diffeomorphisms. These are the C^2-diffeomorphisms which send the closed disc $\mathbb{D}$ into its interior, contract the area and whose ergodic measures μ not supported on a sink satisfy: for μ-almost every point x, both stable branches of x intersects the boundary of the disc. For instance any real Hénon map $(x, y) \mapsto (1 - ax^2 + y, bx)$ with jacobian $|b|$ less than $1/4$ induces a diffeomorphism in this class. Gambaudo-Tresser's conjecture holds for more general mild dissipative diffeomorphisms:

Theorem 1. *For any mild dissipative diffeomorphism f of the disc whose topological entropy vanishes, the set of period is the union of a finite set with finitely many sets of the form $\{n.2^k, k \in \mathbb{N}\}$.*

A diffeomorphism f is *renormalizable* if there exist $D \subset \mathbb{D}$ homeomorphic to the disc and $k > 1$ such that $f^k(D) \subset D$ and $f^i(D) \cap D = \emptyset$ for each $1 \leq i < k$.

Theorem 2. *For any mild dissipative diffeomorphism of the disc whose topological entropy vanishes,*

- *either f is renormalizable,*
- *or any forward orbit of f converges to a fixed point.*

These two statements were already known for Hénon maps which are strongly dissipative (i.e. whose jacobian is very close to 0): De Carvalho, Lyubich and Martens have even shown [4] by a perturbative method that the statement of Sharkovsky's theorem for interval maps extends then.

In our proof we analyze in details the dynamics of these systems:

Theorem 3. *For any mild dissipative diffeomorphism of the disc whose topological entropy vanishes, any orbit accumulates*

- *either on a periodic orbit,*
- *or on an invariant compact set K which is a generalized odometer Λ.*

By *generalized odometer*, we mean that there exists a continuous semi-conjugacy $\pi\colon (K, f) \to (\Lambda, h)$ between K and a dynamics on the Cantor set such that:

– (Λ, h) is an odometer: for any $\varepsilon > 0$ there exists $m \geq 1$ and a partition $K = A_1 \cup A_2 \cup \cdots \cup A_m$ into compact sets with diameter smaller than ε, satisfying $h(A_i) = A_{i+1}$ for $1 \leq i < m$ and $h(A_m) = A_1$. In particular (Λ, h) has a unique invariant probability measure ν.
– ν-almost every point in Λ has a unique preimage by p.

In particular (K, f) is uniquely ergodic.

The class of mild dissipative diffeomorphisms of the disc with a finite set of periods is C^1-open and defines a natural generalization of *Morse-Smale diffeomorphisms*; those with an infinite set of periods exhibit a generalized odometer and are *infinitely renormalizable.* In particular the boundary of the set of systems with zero entropy (in the class of mild dissipative diffeomorphisms of the disc) is included in the set of infinitely renormalizable systems with zero entropy; we conjecture that this inclusion is an equality.

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A simple construction of an action selector on aspherical symplectic manifolds

Felix Schlenk
(joint work with Alberto Abbondandolo and Carsten Haug)

1. Introduction

Hamiltonian systems on symplectic manifolds tend to have many periodic orbits. The "actions" of these orbits form an invariant for the Hamiltonian system. The set of actions can be very large, however. To get useful invariants, one selects for each Hamiltonian function just one action value by some minimax procedure: A so-called action selector associates to every time-periodic Hamiltonian function on a symplectic manifold the action of a periodic orbit of its flow in a continuous way. For this one needs compactness assumptions on either the symplectic manifold or the support of the Hamiltonian vector field. The mere existence of an action selector has many applications to Hamiltonian dynamics and symplectic topology: It readily yields a symplectic capacity and thus implies Gromov's non-squeezing theorem, implies the almost existence of closed characteristics on displaceable hypersurfaces and in particular the Weinstein conjecture for displaceable energy surfaces of contact type, often proves the non-degeneracy of Hofer's metric and its unboundedness, etc., see for instance [1, 2, 3, 6, 7, 9].

Action selectors were first constructed for the standard symplectic vector space $(\mathbb{R}^{2n}, \omega_0)$ by Viterbo [9] and Hofer–Zehnder [3]. For more general symplectic manifolds (M, ω), action selectors were obtained, up until now, only by means of Floer homology: For symplectically aspherical symplectic manifolds (namely those for which $[\omega]|_{\pi_2(M)} = 0$), Schwarz [7] constructed the so-called PSS selector when M is closed, and his construction was adapted to convex symplectic manifolds in [2]. Examples of convex symplectic manifolds are cotangent bundles and their fiberwise starshaped subdomains, on which most of classical mechanics takes place. We refer to Appendix A of [1] for a short description of these selectors. For some further classes of symplectic manifolds and Hamiltonian functions, the PSS selector was constructed in [4, 5, 8].

In this work we give a more elementary construction of an action selector for closed or convex symplectically aspherical manifolds. Our construction uses only results from Chapter 6.4 of the text book [3] by Hofer and Zehnder, that rely on Gromov compactness and rudimentary Fredholm theory, but on none of the more advanced tools in the construction of Floer homology (such as exponential decay, the spectral flow, unique continuation, gluing, or transversality). In this way, the three basic properties of an action selector (spectrality, continuity and local non-triviality) are readily established by rather straightforward proofs, since the only tool at our hands is the compactness property of certain spaces of holomorphic cylinders.

2. Idea of the construction

In the rest of this note I outline the construction of our action selector on a closed symplectically aspherical manifold (M, ω). Denote by $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the circle of length 1. Recall that the Hamiltonian action functional on the space of contractible loops $C^\infty_{\text{contr}}(\mathbb{T}, M)$ associated to a Hamiltonian function $H \in C^\infty(\mathbb{T} \times M, \mathbb{R}) =: \mathscr{H}(M)$ is given by

$$\mathbb{A}_H(x) := \int_{\mathbb{D}} \bar{x}^*(\omega) + \int_{\mathbb{T}} H(t, x(t))\, dt,$$

where $\bar{x} \in C^\infty(\mathbb{D}, M)$ is such that $\bar{x}|_{\partial\mathbb{D}} = x$. The critical points of $\mathbb{A}_H$ are the contractible 1-periodic solutions of the Hamiltonian equation

$$\dot{x}(t) = X_H(t, x(t)),$$

where the vector field X_H is defined by $\omega(X_H, \cdot) = dH$, and the set of critical values of $\mathbb{A}_H$ is called the action spectrum of H and denoted by $\operatorname{spec}(H)$. An action selector is a map $\sigma\colon \mathscr{H}(M) \to \mathbb{R}$ with the following three basic properties.

A1 (Spectrality) $\sigma(H) \in \operatorname{spec}(H)$ for all $H \in \mathscr{H}(M)$.

A2 (C^∞-continuity) σ is continuous with respect to the C^∞-topology on $\mathscr{H}(M)$.

A3 (Local non-triviality) There exists $H \in \mathscr{H}(M)$ with $H \leq 0$ and support in a symplectically embedded ball in M such that $\sigma(H) < 0$.

A first idea for defining an action selector is to boldly take the smallest action value of a 1-periodic orbit,

$$\sigma(H) := \min \operatorname{spec}(H).$$

Since $\operatorname{spec}(H)$ is a compact subset of $\mathbb{R}$, this definition makes sense, and yields an invariant with the spectral property. However, this invariant is not very useful, since it fails to be continuous and monotone, two crucial properties for applications. To see why, consider radial functions

$$H_f(z) := f(\pi|z|^2) \quad \text{on } \mathbb{R}^{2n},$$

where $f\colon \mathbb{R} \to \mathbb{R}$ is a smooth function with compact support. For an arbitrary symplectic manifold, such functions can be constructed in a Darboux chart and then be extended by zero to the whole manifold. The critical points of $\mathbb{A}_H$ are the origin and the (Hopf-)circles on those spheres that have radius r with $s = \pi r^2$ and $f'(s) \in \mathbb{Z}$; at such a critical point x the value of the action is

$$\mathbb{A}_{H_f}(x) = f(s) - s\,f'(s), \tag{1}$$

see the left drawing in Figure 1. Now take the profile functions f, f_+, f_- as in the right drawing: $f' \in [0,1]$ and $f'(s) = 1$ for a unique s, while f_-, f_+ are C^∞-close to f and satisfy $f_- \leq f \leq f_+$ and $f'_-, f'_+ \in [0,1)$. Then the formula (1) shows that $\sigma(H_f)$ is much smaller than $\sigma(H_{f_-}) \approx \sigma(H_{f_+})$, whence σ is neither continuous nor monotone. Or take g with $|g|$ very small and very steep. Then $\sigma(H_g)$ is much smaller than $\sigma(H_f)$, whence monotonicity fails drastically.

The above discussion shows that the continuous, or monotone, selection of an action from $\operatorname{spec}(H)$ must be done by some kind of minimax procedure for the action functional. This was done for the Hofer–Zehnder selector by minimax over a uniform minimax family, and for the Viterbo selector and the PSS selector by a homological minimax. Our minimax will be over certain spaces of perturbed holomorphic cylinders.

To introduce our construction, we first look at a toy model: Consider the quadratic form $q(x,y) = x^2 - y^2$ on $\mathbb{R}^2$ and its perturbations

$$q_h = q + h$$

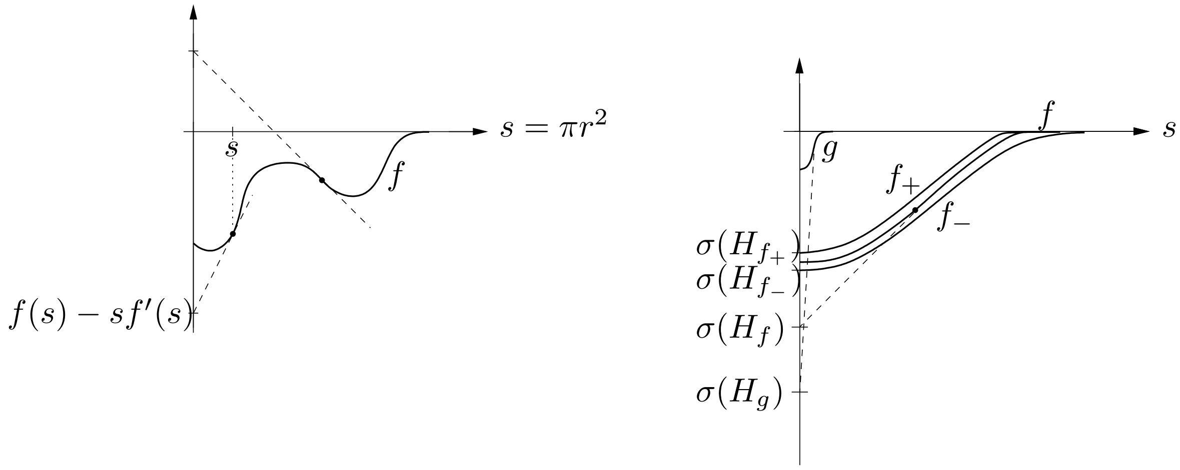

FIGURE 1. Radial functions and their minimal spectral values

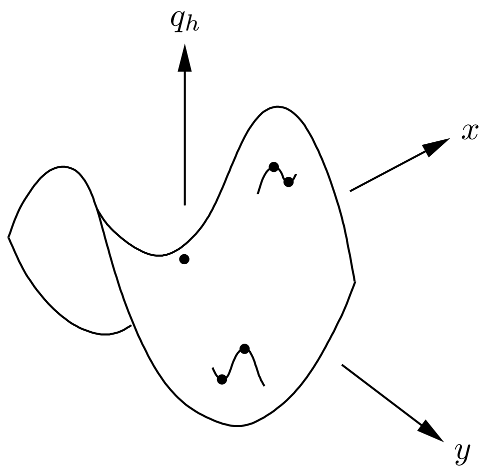

FIGURE 2. A perturbed quadratic form q_h

where h is a compactly supported function on $\mathbb{R}^2$. Here, the indefinite quadratic form q models the symplectic action and the compactly supported function h models the Hamiltonian term in $\mathbb{A}_H$, cf. [3, §3.3]. If $h = 0$, the only critical point of q_h is the origin, with critical value 0. If h consists, for instance, of two little positive bumps, one centered at $(1, 0)$ and one at $(0, 1)$, then the graph of q_h looks as in Figure 2. A continuous selection of critical values $h \mapsto \sigma(h)$ should, in our example, choose again 0, by somehow discarding the four new critical values.

In this finite dimensional example, one could define an action selector by the minimax formula

$$\sigma(h) = \inf \max_Y q_h,$$

where the infimum is over the space of all images Y of continuous maps $\mathbb{R} \to \mathbb{R}^2$ that are compactly supported perturbations of the embedding $y \mapsto (0, y)$. Monotonicity in h is clear from the definition, and spectrality can be proved by standard deformation arguments using the negative gradient flow of q_h. The definition of the Hofer–Zehnder action selector (see [3, Section 5.3]) is based on a similar idea and uses the fact that the Hamiltonian action functional for loops in $\mathbb{R}^{2n}$ has a nice negative gradient flow.

Alternatively, one can fix a very large number c such that the sublevel $\{q_h < -c\}$ coincides with the sublevel $\{q < -c\}$ and define the same critical value $\sigma(h)$ as

$$\inf\left\{a \in \mathbb{R} \mid \text{the image of } i^a_*\colon H_1(\{q_h < a\}, \{q < -c\}) \to H_1(\mathbb{R}^2, \{q < -c\}) \text{ is non-zero}\right\},$$

where the map i^a is the inclusion

$$i^a\colon\ (\{q_h < a\}, \{q < -c\}) \hookrightarrow \big(\mathbb{R}^2, \{q < -c\}\big)$$

and we are using the fact that

$$H_1(\mathbb{R}^2, \{q < -c\}) \cong \mathbb{Z}.$$

Viterbo's definition of an action selector for compactly supported Hamiltonians on $\mathbb{R}^{2n}$ uses a similar construction, which is applied to suitable generating functions, see [9]. The Floer homological translation of this second definition is, in turn, at the basis of Schwarz's construction of an action selector for symplectically aspherical manifolds, see [7], and of all its subsequent generalizations.

Here, we would like to define an action selector $\sigma(h)$ using only spaces of bounded negative gradient flow lines: In the case of the Hamiltonian action functional $\mathbb{A}_H$, these will correspond to finite energy solutions of the Floer equation, which have good compactness properties. A first observation is that the knowledge of the space of all bounded negative gradient flow lines of q_h is not enough for defining an action selector. Indeed, it is easy to perturb q on a small disc disjoint from the origin in such a way that the negative gradient flow lines of q_h look like in Figure 3: A new degenerate critical point z is created, and the constant orbits at $(0,0)$ and at z are the only bounded negative gradient flow lines. But since $q_h(z)$ could be either positive or negative, the set $\{(0,0), z\}$ contains too little information for us to conclude that the value of the action selector should be $q_h(0,0) = 0$.

If, however, we are allowed to deform the function q_h, we can use bounded gradient flow lines to define an action selector that identifies the lowest critical value that "cannot be shaken off". More precisely, take a family $\{h^s\}_{s\in\mathbb{R}}$ of compactly supported functions such that $h^s = h$ for s small and $h^s = 0$ for s large, and look at the space $\mathscr{U}(h^s)$ of bounded solutions of the non-autonomous gradient equation

$$\dot{u}(s) \ = \ -\nabla q_{h^s}(u(s)), \qquad s \in \mathbb{R}.$$

The boundedness of u is equivalent to bounded energy

$$E(u) := \int_{\mathbb{R}} |\nabla q_{h^s}(u(s))|^2\, ds \ = \ \lim_{s\to-\infty} q_{h^s}(u(s)) - \lim_{s\to+\infty} q_{h^s}(u(s)) + \int_{\mathbb{R}} \frac{\partial h^s}{\partial s}(u(s))\, ds \ < \ \infty,$$

or, since $h^s = h$ in the first limit and $h^s = 0$ in the second limit, to the fact that $u(s)$ is asymptotic for $s \to -\infty$ to the following critical level of q_h

$$q_h^-(u) \ := \ \lim_{s\to-\infty} q_h(u(s))$$

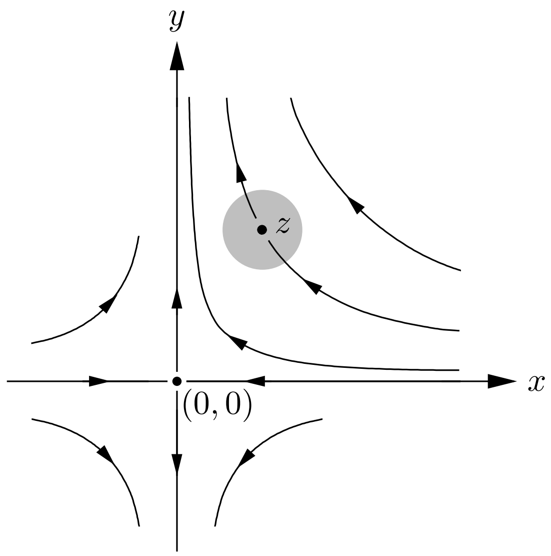

FIGURE 3. The only bounded gradient flow lines are the constant orbits at $(0,0)$ and z.

and for all s large lies on the x-axis and converges for $s \to +\infty$ to the origin (the only critical point of q). The number

$$\min_{u \in \mathscr{U}(h^s)} q_h^-(u)$$

is the lowest critical value of q_h from which a bounded h^s-negative gradient flow line starts.

In our example from Figure 2, if we take $h^s = \beta(s)\, h$ with a cut-off function β, then $\mathscr{U}(h^s)$ contains no flow line u emanating from the two low critical points p_1 or p_2 near $(0,1)$. On the other hand, it is easy to construct a family h^s that has a negative-gradient line $u(s)$ that converges to p_1 for $s \to -\infty$ and to the origin for $s \to +\infty$. To be sure that we discard all inessential critical values, we therefore set

$$\sigma(h) := \sup_{h^s} \min_{u \in \mathscr{U}(h^s)} q_h^-(u).$$

In the example, it is quite clear that for every deformation h^s there exists a flow line in $\mathscr{U}(h^s)$ emanating from the critical point $(0,0)$, that is, $\sigma(h) = 0$ as it should be. In general, it is not hard to see that $\sigma(h)$ is a critical value of q_h that depends continuously and in a monotone way on h.

The number $\sigma(h)$ is the lowest critical value c of q_h such that for every deformation h^s of h there exists a bounded flow line $u \in \mathscr{U}(h^s)$ starting at a critical level not exceeding c. Equivalently, $\sigma(h)$ is the highest critical value c of q_h such that for every critical level $c' < c$ there exists a deformation h^s of h such that all flow lines of q_{h^s} starting at level c' are unbounded. That is: the whole critical set strictly below c can be shaken off.

Imitating the above construction, and inspired by the proof of the degenerate Arnol'd conjecture in [3, §6.4], we can define an action selector for 1-periodic

Hamiltonians on a closed symplectically aspherical manifold (M,ω) in the following way. Given $H \in C^\infty(\mathbb{T}\times M)$ we consider s-dependent Hamiltonians K in $C^\infty(\mathbb{R}\times\mathbb{T}\times M)$ such that $K(s,\cdot,\cdot)=H$ for s small and $K(s,\cdot,\cdot)=0$ for s large. Following Floer's interpretation of the L^2-gradient flow of the action functional, we consider the space $\mathscr{U}(K)$ of solutions $u\in C^\infty(\mathbb{R}\times\mathbb{T},M)$ of Floer's equation

$$\partial_s u + J(u)\big(\partial_t u - X_K(s,t,u)\big) = 0 \tag{2}$$

that have finite energy

$$E(u) = \int_{\mathbb{R}\times\mathbb{T}} |\partial_s u|_J^2 < \infty.$$

Here, J is a fixed ω-compatible almost complex structure on TM and $|\cdot|_J$ is the induced Riemannian norm. The space $\mathscr{U}(K)$ is C^∞_{loc}-compact by Gromov's compactness theorem. Now define the function

$$a_H^-\colon \mathscr{U}(K)\to\mathbb{R}, \qquad a_H^-(u) := \lim_{s\to-\infty}\mathbb{A}_H(u(s))$$

and finally define the action selector of H by

$$A_J(H) := \sup_K \min_{u\in\mathscr{U}(K)} a_H^-(u),$$

where the supremum is taken over all deformations K of H as above. The number $A_J(H)$ is the smallest essential action of H in the following sense: It is the lowest critical value c of $\mathbb{A}_H$ (that is, the lowest action of a contractible 1-periodic orbit of H) such that for every deformation K of H there exists a finite energy solution of Floer's equation for K and J that starts at a critical level $\leq c$.

In our finite dimensional model, we could have allowed for a larger class of deformations of the gradient flow of q_h, by looking at families h^s that for s large do not depend on s but are not necessarily zero, and by taking the gradient with respect to any family g_s of Riemannian metrics that depend on s on a compact interval. In the symplectic setting, the role of Riemannian metrics is played by ω-compatible almost complex structures. We may thus modify the above definition by looking at functions K with $K(s,\cdot,\cdot)=H$ for s small and $K(s,\cdot,\cdot)$ independent of s for s large, and at families J^s of ω-compatible almost complex structures that depend on s on a compact interval. By using these larger families of deformations we also obtain an action selector, $A(H)$. This has the advantage that $A(H)$ is manifestly independent of the choice of J.

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Quantum footprints of symplectic rigidity

Leonid Polterovich
(joint work with Laurent Charles)

According to the quantum-classical correspondence, quantum mechanics contains classical mechanics as the limiting case when the Planck constant tends to 0. In the talk, I have discussed quantum footprints of symplectic topology of the phase space, focusing on rigidity phenomena.

First, I presented a link found in [2] between symplectic displacement energy, a fundamental notion of symplectic dynamics introduced by Hofer [3], and the quantum speed limit, a universal constraint on the speed of quantum-mechanical processes discovered by Margolus and Levitin in [4]. In particular, positivity of displacement energy of open subsets implies that on scales larger than the quantum one, i.e., of the order $\hbar^\epsilon$ with $\epsilon < 1/2$, the speed limit for semiclassical processes involving semiclassical states is more restrictive than the universal one.

Second, I explained a connection between the Poisson bracket invariant of a finite open cover of a closed symplectic manifold and the noise-localization uncertainty relation [5]. Recall that this invariant measures, roughly speaking, the minimal possible magnitude of Poisson non-commutativity of a partition of unity subordinated to the cover. In dimension two, optimal bounds on the Poisson bracket invariant were recently found by Buhovsky, Logonov and Tanny [1]. In higher dimensions, they are still out of reach.

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The horocycle flow on the moduli space of translation surfaces

Jon Chaika
(joint work with John Smillie, Barak Weiss)

A translation surface is given by a collection of polygons $P_1, ..., P_j$ in the plane so that the sides can be grouped in pairs that are parallel and of equal length. Identifying these paired sides by translation we obtain a translation surface, which is a Riemann surface equipped with a singular flat metric. The singular points of the metric are cone points whose cone angles are in $2\pi\mathbb{Z}$. Two translation surfaces are equivalent if there is a diffeomorphism between them whose derivative is identically 1 between them. Such objects can be stratified by the orders of the cone points of the surface to obtain a *strata* which we denote as $\mathcal{H}$. (Note we are suppressing the data that determines the strata in our notation.)

A trend in the study of strata of translation surfaces has been to use techniques inspired by the study of homogeneous spaces. A major collection of results on homogeneous spaces are rigidity results for the unipotent flows, like Ratner's Theorems:

Theorem. (Ratner) Let G be a connected Lie group, Γ a lattice in G, $X = G/\Gamma$, and $U = \{u_s : s \in \mathbb{R}\}$ a one-parameter Ad-unipotent subgroup of G.

(1) For any $x \in X$, $\overline{Ux} = Hx$ is the orbit of a group H satisfying $U \subset H \subset G$, and Hx is the support of an H-invariant probability measure μ_x.
(2) For any $x \in X$ there exists μ_x so that $\text{supp}(\mu_x) = \overline{Ux}$ and

$$\forall f \in C_c(X), \quad \lim_{T\to\infty} \frac{1}{T}\int_0^T f(u_s x)ds = \int_X f d\mu_x.$$

The group $SL(2,\mathbb{R})$ has a unipotent subgroup $h_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and the previous theorem is false for the action of h_t on the strata of translation surfaces that have two cone points each of cone angle 4π. We denote this stratum $\mathcal{H}$ and we have:

Theorem 1. There exists a translation surface x and a measure μ so that $x \notin \text{supp}(\mu)$ (and so $\overline{\{h_t x\}_{t\in\mathbb{R}}} \neq \text{supp}(\mu)$) but for all $f \in C_c(\mathcal{H})$ we have

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(h_t x)dt = \int f d\mu.$$

Theorem 2. There is a dense G_δ subset of $\mathcal{H}$, B, so that for all $x \in B$ there exists $f \in C_c(\mathcal{H})$ so that

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T f(h_t x)dt \text{ does not exist.}$$

Theorem 3. There is a translation surface $x \in \mathcal{H}$ so that $\overline{\{h_t x\}_{t\in\mathbb{R}}}$ has non-integer Hausdorff dimension.

Theorem 4. There is a one parameter family of translation surfaces $\{x_s\}_{s\in\mathbb{R}^+}$ so that whenever $\alpha < \beta$ we have that $\overline{\{h_t x_\alpha\}_{t\in\mathbb{R}}}$ is a proper subset of $\overline{\{h_t x_\beta\}_{t\in\mathbb{R}}}$.

The main object in the proof of these results is the *tremor* of a translation surface. Given ν a transverse invariant measure to the horizontal foliation on a translation surface $\cup P_i/\sim$ we want to obtain a new translation surface $\cup P_i'/\sim$. We build each P_i' by taking each side γ of P_i, which is a vector (h, v) in coordinates and replacing it with the vector $(h + s\nu(\gamma), v)$ in coordinates where $s \in \mathbb{R}$. The resulting translation surface is the time s tremor of $\cup P_i/\sim$. Note when our transverse measure comes from (the disintegration of) Lebesgue this agrees with the horocycle flow. The surfaces in Theorems 1, 3 and 4 are all tremors of surfaces that have extra symmetries. A key tool in our study is the fact that tremors commute with the horocycle flow, and the horocycle flow of these surfaces with extra symmetries can be understood.

Energy Drift and Diffusion Process in the Three-Body Problem

Marian Gidea
(joint work with Maciej Capiński)

1. Introduction

In the context of perturbed Hamiltonian systems, we develop a general method to show the existence of orbits that drift in energy, as well as of orbits whose energy exhibits symbolic dynamics. This method allows one to obtain quantitative information on such orbits – estimates on the range of the perturbation parameter for which such orbits exist, on the speed of these orbits, and on the Hausdorff dimension of their initial conditions –, as well as to obtain a description of the stochastic process that governs the time-evolution of such orbits.

Our method can be applied to concrete models with realistic parameters, under explicit conditions on the system. These conditions are of topological nature, and can be verified either analytically or numerically via computer assisted proofs.

We apply our method to the planar elliptic restricted three-body problem, viewed as a perturbation of the planar circular restricted three-body problem, with the perturbation parameter ε being the eccentricity of the orbits of the primaries. We prove that, for all suitably small (non-zero) values of ε, there are orbits whose energy drifts by $O(1)$, at a rate of $O(\epsilon)$. We also show the existence of orbits whose energy exhibits symbolic dynamics, and we estimate that the Hausdorff dimension of such orbits is at least 4 in the 5-dimensional extended phase space. In addition, we show that for any given diffusion process, there exists a set of initial condition whose time-evolution in energy approximately follows that process.

Our results address some conjectures by Arnold and Chirikov.

2. Main result

In the planar circular restricted three-body problem (PCR3BP), two primary masses m_1, m_2 move on circular orbits about their center of mass, and a third infinitesimal particle, i.e., $m_3 = 0$, moves under the gravitational fields of m_1, m_2

without affecting their orbits. The motion of m_3 relative to a co-rotating system of coordinates, which places m_1 at $(\mu, 0)$ and m_2 at $(-1+\mu, 0)$, where $\mu = m_2/(m_1+m_2)$, is given by the autonomous Hamiltonian

$$H_0(z) = \frac{1}{2}((p_1+q_2)^2 + (p_2-q_1)^2) - \omega(q_1, q_2).$$

Here $z = (p_1, p_2, q_1, q_2)$, and $\omega(q_1, q_2) = \frac{1}{2}(q_1^2+q_2^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$, where $r_1 = d(m_3, m_1)$, $r_2 = d(m_3, m_2)$. There are 5 equilibrium points for this problem. We focus on one of them, referred to as L_1, which is of saddle-center type, and is surrounded by (Lyapunov) periodic orbits.
Since the underlying Hamiltonian system is autonomous, the energy function H_0 is preserved along trajectories.

In the planar elliptic restricted three body problem (PER3BP), the masses m_1, m_2 move on elliptic orbits of eccentricity ε around the center of mass, while the infinitesimal mass $m_3 = 0$ still moves under the gravitational fields of m_1, m_2 without affecting their motion. Relative to a rotating-pulsating coordinate system, which fixes m_1, m_2 at $(\mu, 0)$ and $(-1+\mu, 0)$, respectively, the motion of m_3 is given by the non-autonomous Hamiltonian

$$H_\varepsilon(z, \theta) = \frac{1}{2}((p_1+q_2)^2 + (p_2-q_1)^2) - \frac{1}{1+\varepsilon\cos\theta}\omega(q_1, q_2).$$

Here θ is the true anomaly, and is taken as the 'new time' parameter.

The Hamiltonian of the PER3BP can be written as a small perturbation of the one for the PCR3BP, i.e.,

$$H_\varepsilon(z, \theta) = H_0(z) + \varepsilon H_1(z, \theta; \varepsilon).$$

The energy function H_ε is not preserved along trajectories.

We consider a concrete model for the PER3BP, namely the the Neptune-Triton system. In this case the normalized mass is $\mu = 0.0002089$, and the eccentricity is $\varepsilon_1 = 1.6 \cdot 10^{-5}$.

Theorem 1. *Consider the PER3BP model with the parameters from the Neptune-Triton system. We have the following results:*

(1) *(Diffusing orbits) For every $\varepsilon \in (0, \varepsilon_1]$, there exists a point $z(\varepsilon)$ and $t(\varepsilon) \in (0, T/\epsilon)$, such that*

$$H_0\left(\Phi^\varepsilon_{t(\varepsilon)}(z(\varepsilon))\right) - H_0(z(\varepsilon)) > C,$$

where $C = 2\cdot 10^{-9}$ and $T = 5.7 \times 10^{-4}$.

(2) *(Symbolic dynamics) Let $\varepsilon_0 = 10^{-8} < \varepsilon_1$. For any $\varepsilon \in (0, \varepsilon_0]$ and any sequence $\{I^\sigma\}_{\sigma\in\mathbb{N}}$, $I^\sigma \in [2\eta, C-2\eta]$ such that $\left|I^{\sigma+1} - I^\sigma\right| > 2\eta$ there exists a point z and an increasing sequence of times $t^\sigma > 0$ such that*

$$|(H_0(\Phi^\varepsilon_{t^\sigma}(z)) - H_0(z)) - I^\sigma| < \eta \qquad \text{for all } \sigma \in \mathbb{N},$$

where $\eta = 10^{-10}$.

(3) *(Hausdorff dimension) The Hausdorff dimension of the set of points z which exhibit symbolic dynamics as in (2) is greater or equal to* 4 *(in the* 5 *dimensional extended phase space).*

(4) *(Stochastic behavior) Let* $\mu, \sigma \in \mathbb{R}$, $Y_0 \in (0, C)$, *and* $\gamma > \frac{3}{2}$. *Denote by* f_ε *the first return time to* $\{q_2 = 0\}$. *Consider the stochastic processes*

$$Y_t := Y_0 + \mu t + \sigma W_t, \qquad \textit{for } t \in [0,1],$$

where W_t *is the standard Browian motion.*

Then for each $0 < \varepsilon < \varepsilon_0$ *there exists a set* $\Omega_\varepsilon \subseteq \{q_2 = 0\}$, *endowed with the probability measure* $\mathbb{P}_\varepsilon$ *equal to the normalized Lebesgue measure on* Ω_ε, *so that the stochastic process* $X_t^\varepsilon : \Omega_\varepsilon \to \mathbb{R}$ *defined by*

$$X_t^\varepsilon(z) := H_0\left((f_\varepsilon)^{\lceil t\varepsilon^{-\gamma}\rceil}(z)\right), \qquad \textit{for } t \in [0,1],$$

satisfies

$$\lim_{\varepsilon \to 0} X^\varepsilon_{t\wedge\tau} \overset{d}{=} Y_{t\wedge\tau}.$$

Above, $\tau_X := \inf\{t : X_t^\varepsilon \geq C \textit{ or } X_t^\varepsilon \leq 0\}$, $\tau_Y := \inf\{t : Y_t \geq C \textit{ or } Y_t \leq 0\}$ *are stopping times, and the convergence is in distribution.*

3. Methodology

The main geometric mechanisms relies on following several homoclinic orbits associated to a family of Lyapunov orbits around L_1, which exist in the PCRT3BP for $\varepsilon = 0$. In the PER3BP, for $\varepsilon > 0$ small, as we follow the homoclinics, the return map to a neighborhood (in the extended phase space) of the family of Lyapunov orbits is either increasing or decreasing the energy.

To obtain orbits that drift in energy, we identify a 'strip' in the Poincaré section $\{q_2 = 0\}$, corresponding to some range of θ-values, where the return map yields an increase in energy by $O(\varepsilon)$. By repeatedly returning to this strip for $O(1/\varepsilon)$-times, one can obtain a growth of energy by $O(1)$.

To obtain symbolic dynamics, we identify two 'strips' in the Poincaré section $\{q_2 = 0\}$, corresponding to two disjoint ranges of θ-values, such that the return map to one strip yields an increase in energy by $O(\varepsilon)$, and the return map to the other strip yields a decrease in energy by $O(\varepsilon)$.

To show that the set of initial conditions that yield symbolic dynamics has Hausdorff dimension at least 3 in the 4-dimensional Poincaré section (hence at least 4 in the 5-dimensional extended phase space), we show that this set of initial conditions projects, relative to some suitable coordinate system, onto a certain 3-dimensional rectangle.

To prove the statement on stochastic behavior, we first construct a random walk Y_t^ε which approaches the chosen Brownian motion with drift $Y_t = Y_0 + \mu t + \sigma W_t$ as $\varepsilon \to 0$, and then use symbolic dynamics to obtain orbits whose energy 'shadow' the values of Y_t^ε.

To obtain Theorem 1, we first prove some general results on perturbed Hamiltonian systems, which show that, if certain topological conditions are satisfied, then there exist orbits with the desired properties.

To apply those general results to the PER3BP, we verify that the underlying system satisfies the appropriate topological conditions. This verification is done via a computer assisted proof. This amounts to performing all algebraic operations in interval arithmetic, to integrate the differential equations with interval-based ODE solvers, and to obtain all numerical solution with rigorous bounds.

See [1] for details.

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Acknowledgement: The research of M.G. was partially supported by NSF grant DMS-1814543.

Emergence of wandering stable open components

Pierre Berger
(joint work with Sebastien Biebler)

Given a holomorphic endomorphism f of a complex manifold X, the *Fatou set* consists of the set of points $x \in X$ which have a neighborhood U such that $(f^n|U)_n$ is normal. In particular the connected components of the Fatou set, called *Fatou components*, are mapped into each other under the dynamics. The understanding of complexity of the dynamics on the Fatou set is a problem of fundamental interest.

When $X = \mathbb{P}^1(\mathbb{C})$, a celebrated result of Sullivan [1] shows that any rational function does not have any wandering Fatou component. In higher dimension, the problem of the existence of a wandering Fatou component was first studied in 1991 in the work of Bedford and Smillie [2] in the context of polynomial automorphisms of $\mathbb{C}^2$. Our first main result is an answer to this problem:

Theorem. *There exists a locally dense set of real polynomial automorphisms f of $\mathbb{C}^2$ which display a* wandering *Fatou component $\mathcal{C}$ satisfying:*

(1) the real trace $\mathcal{C} \cap \mathbb{R}^2$ of $\mathcal{C}$ is non-empty,
(2) for every compact set $K_0 \subset \mathcal{C}$, the union $\bigcup_{n\geq 0} f^n(K_0)$ is bounded and the diameter of $f^n(K_0)$ converges to 0 as $n \to \infty$.

The proof relies on a robust geometric model on parameter family of dynamics. It implies the existence of a real wandering open stable components at a dense set of parameters. By open stable component, we mean a maximal connected, open set of asymptotic points. We prove that this component has a historical behaviour, and that this model occurs densely among families inside the dissipative Newhouse domain $\mathcal{N}^r$. This allows us to complement the solution of Kiriki-Soma [3] on the last Taken's problem from the finitely regular case to the C^∞-case:

Theorem. *For every $r \in [2, \infty]$, there is a dense subset of $\mathcal{N}^r$ formed by dynamics f which display a wandering stable open component C_0 satisfying:*

(1) for every $x \in C_0$, the limit set of the orbit of x intersects a horseshoe Λ,

(2) every $x \in C_0$ has its sequence $(\mathsf{e}_n(x))_{n\geq 0}$ of empirical measures $\mathsf{e}_n(x) := \frac{1}{n}\sum_{i=0}^{n-1} \delta_{f_p^i(x)}$ which diverges.

Moreover, in the two latter results, we show that for every $x \in \mathcal{C}$ (resp. $x \in C_0$), the set of accumulation points of $(\mathsf{e}_n(x))_{n\geq 0}$ has its covering number $\mathcal{N}$ satisfying:

$$\liminf_{\eta \to 0} \frac{\log \log \mathcal{N}(\eta)}{-\log \eta} > 0$$

for the set of probability measures endowed with the Wasserstein distance.

This indicates that the statistical complexity of the dynamics is high. To quantify the complexity of the statistical behavior of typical orbits for differentiable dynamical systems, the notion of emergence has been introduced in [4]. The latter inequality confirms the main conjecture of [4] saying that super polynomial emergence is typical in many senses and in many categories of dynamical systems.

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The spectral recognition of rank one contact forms on closed three-manifolds

Daniel Cristofaro-Gardiner

(joint work with Marco Mazzucchelli)

A *contact form* on a $(2n+1)$-dimensional manifold is a differential one-form λ such that $\lambda \wedge (d\lambda)^n$ is a volume form. A contact form determines a canonical vector field R, called the *Reeb vector field*, defined by the equations

$$d\lambda(R, \cdot) = 0, \quad \lambda(R) = 1.$$

For example, there is a natural contact form on the unit cotangent bundle of any Riemannian manifold, such that the integral curves of R project to geodesics.

Closed orbits of the Reeb vector field are called *Reeb orbits*. The *spectrum* $\mathcal{A}(Y, \lambda) \subset \mathbb{R}_{>0}$ of the pair (Y, λ) is the set of periods of Reeb orbits; the *simple spectrum* $\mathcal{A}_{simp}(Y, \lambda)$ is the set of periods of simple Reeb orbits, in other words those orbits that are not multiple covers. It is natural to ask to what degree we can reconstruct λ from $\mathcal{A}$ and $\mathcal{A}_{simp}$; the analogous question in Riemannian geometry, called *length spectrum rigidity*, is much studied, see for example [8].

Define the *rank* of $\mathcal{A}(Y, \lambda)$ to be the rank of the $\mathbb{Z}$-submodule of $\mathbb{R}$ that it generates. Our results address spectral recognition in the rank one case, where in dimension 3 it turns out that much can be said:

Theorem 1. [2, 7] *Let (Y, λ) be a closed three-manifold with a contact form. Then the following are equivalent:*

- *The action spectrum has rank* 1*.*
- *Every orbit of the Reeb flow is closed.*
- *Every Reeb orbit has a common period. (In other words, there exists a positive real number T such that for any $\ell \in \mathcal{A}_{simp}(Y, \lambda)$, ℓ divides T.)*

Theorem 2. [2]
Let (Y, λ) be a closed three-manifold with a rank one contact form. Then, λ can be recovered from $\mathcal{A}_{simp}$ and Y: that is, if λ_1 and λ_2 are two rank one contact forms on Y such that $\mathcal{A}_{simp}(Y, \lambda_1) = \mathcal{A}_{simp}(Y, \lambda_2)$, then there is a diffeomorphism $\Psi : Y \to Y$ such that $\Psi^* \lambda_2 = \lambda_1$.

We remark that the fact that the second bullet point in Theorem 1 implies the third is classical, due to Wadsley [7].

The proofs of Theorem 1 and Theorem 2 are quite different. Theorem 1 uses a Floer homology for closed three-manifolds with a contact form, called *embedded contact homology* (ECH), see [6]. ECH is defined in terms of Reeb orbits and pseudoholomorphic curves, but is canonically isomorphic to Seiberg-Witten Floer cohomology, and so connects gauge theory and low-dimensional contact and symplectic topology. It can be used to define a series of spectral invariants which recover the volume $\int_Y \lambda \wedge d\lambda$ via a kind of Weyl law, called the "volume property", proved in [5]; this is the key fact that allows us to prove Theorem 1, for other applications of the volume property see [3, 4, 1].

The proof of Theorem 2 is more elementary, and uses the classification of Seifert fibrations, as well as a Moser trick for recovering contact forms on closed three-manifolds from their Reeb vector fields.

It is natural to ask to what degree results like the above might hold in higher dimensions. At present, this certainly seems out of reach, at least in the level of generality above. Indeed, in dimensions above three, it is not even currently known whether or not the Reeb vector field associated with a contact form on a closed manifold must always have a closed orbit; conjecturally this is true, via the famous "Weinstein conjecture", but a proof is generally considered far off. In dimension 3, however, other questions are perhaps more tractable. Here are two such questions:

Question. *Let (Y, λ) be a closed three-manifold with a contact form. What ranks are possible? In the rank* 2 *case, must it be the case that there are exactly two geometrically distinct Reeb orbits?*

Concerning the first question here, M. Hutchings has asked whether or not in particular the only possible ranks are one, two, or infinity, in analogy with [4].

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Stable multi-particle choreographies in repelling potentials.

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(joint work with Vered Rom-Kedar)

The classical approach to the ergodicity problem has been founded in the works of Sinai. In modern terms, the idea going back to Boltzmann is that the gas of hard spheres is a universal model for many-particle systems in the limit of high temperature, so proving the ergodicity for this particular system will justify the ergodicity assumption for a large class of physical systems that are modeled by it. Sinai's program for proving the ergodicity of the Boltzmann gas is based on exploring the special hyperbolic structure in the phase space, which stems from the convex shape of the colliding spheres. However, we show that practically any smooth approximation to the Boltzmann gas, with any number of particles and at arbitrarily high energies, can exhibit a positive measure set of stable motions (analogues of the so-called choreographies from the Celestial Mechanics) for which no particle ever comes close to a collision with others. Thus, the main argument (instability due to collisions) behind the possible ergodicity of the Boltzmann gas becomes invalid once the interaction potential gets smoothed.

Let a single-particle motion be described by a Hamiltonian $H_0(p, q)$. One can consider a system of any number N of identical particles and introduce an interaction potential V, so the multi-particle Hamiltonian will be $\sum_{j=1,\dots,,N} H_0(p_j, q_j) + \varepsilon \sum_{k\neq j} V(q_k - q_j)$, where ε is small. We require V to be bounded from below but allow it to be infinite at $q_k = q_j$, i.e., the interaction is weak except for the moments of collision where the repulsion forces become strong. This is a general model for the motion of particles in a repelling potential when the kinetic energy is sufficiently high. Let a single particle system have a stable (elliptic) periodic orbit $(q^(t), p^(t))$. We show that for practically every choice of the interaction potential V the multi-particle system has a positive measure set of quasi-periodic orbits close to $\{(q_j, p_j) = (q^(t + \theta_j), p^(t + \theta_j))\}$, $j = 1, \dots, N$, where the phases $(\theta_1, \dots, \theta_N)$

minimise a certain averaged potential. We prove a similar result at all sufficiently high energies for a system of interacting particles confined in a bounded domain D, provided the billiard in D is integrable or has a KAM-torus. This will show the non-ergodicity of systems of any number of interacting particles in a rectangular box at arbitrarily high energies. To establish the same for dispersive domains, we use our old result that elliptic periodic orbits can be created by smoothing the billiard potential. The measure of the set of these particular choreographic orbits goes to zero very fast as N grows, so the idea here is just to show, as a matter of principle, that multi-particle systems can avoid collisions (hence, avoid the ergodisation) with positive probability at any energy and for any number of particles. On the other hand, the probability to stay near the choreographic regimes for times longer than the mean free-flight time does not automatically need to be very small at large N – this question deserves a further study.

Floer homology and Hamiltonian homeomorphisms

Sobhan Seyfaddini
(joint work with Buhovsky-Humilière and Le Roux-Viterbo)

1. Main Result

Let (M, ω) be a closed and connected symplectic manifold and let $\mathrm{Ham}(M, \omega)$ be the group of Hamiltonian diffeomorphisms of (M, ω). We denote by $\overline{\mathrm{Ham}}(M, \omega)$ the C^0 closure of Hamiltonian diffeomorphisms of (M, ω); this is often referred to as the group of Hamiltonian homeomorphisms of (M, ω).

Our goal is to show that using barcodes and persistence homology one can indirectly define (filtered) Floer homology for Hamiltonian homeomorphisms, in the case where (M, ω) is symplectically aspherical which means that ω and the first Chern class c_1 vanish on $\pi_2(M)$. Examples of aspherical symplectic manifolds include tori and surfaces of positive genus.

Barcodes: A barcode $\mathcal{B} = \{I_j\}_{1 \leq j \leq N}$ is a finite collection of intervals (or bars) $I_j = (a_j, b_j]$, $a_j \in \mathbb{R}$, $b_j \in \mathbb{R} \cup \{+\infty\}$. The space of barcodes can be equipped with the so-called bottleneck distance which will be denoted by d_{bottle}. These notions, which first appeared in topological data analysis (see for example [9, 7, 11, 17, 6, 33, 3, 10, 27, 8]), have found many interesting applications in symplectic dynamics, following their introduction to the field, by Polterovich and Shelukhin [28].

As explained in the articles of Polterovich-Shelukhin [28] and Usher-Zhang [32], using Hamiltonian Floer homology one can associate a canonical barcode $\mathcal{B}(H)$ to every Hamiltonian H. The barcode $\mathcal{B}(H)$ encodes a significant amount of information about the Floer homology of H: it completely characterizes the filtered Floer complex of H up to quasi-isomorphism, and hence it subsumes all of the previously constructed filtered Floer theoretic invariants. For example, the spectral invariants of H correspond to the endpoints of the half-infinite bars in $\mathcal{B}(H)$.

Given a barcode $\mathcal{B} = \{I_j\}_{1\leq j\leq N}$ and $c \in \mathbb{R}$ define $\mathcal{B}+c = \{I_j + c\}_{1\leq j\leq N}$, where $I_j + c$ is the interval obtained by adding c to the endpoints of I_j. Let $\sim$ denote the equivalence relation on the space of barcodes given by $\mathcal{B} \sim \mathcal{C}$ if $\mathcal{C} = \mathcal{B} + c$ for some $c \in \mathbb{R}$; we will denote the quotient space by $\widehat{Barcodes}$. Now the bottleneck distance descends to a distance on $\widehat{Barcodes}$ which we will continue to denote by d_{bottle}. If H, G are two Hamiltonians the time–1 maps of whose flows coincide, then $\mathcal{B}(H) = \mathcal{B}(G)$ in $\widehat{Barcodes}$. Hence, we obtain a map $\mathcal{B} : (\mathrm{Ham}(M,\omega), d_{C^0}) \to (\widehat{Barcodes}, d_{\text{bottle}})$. The question of continuity of the mapping $\mathcal{B}$ was first addressed by Le Roux, Viterbo, and the third author in [25] where it is proven that $\mathcal{B}$ is continuous and extends to $\overline{Ham}(M,\omega)$ when M is a surface. Our next result states that the same is true for any closed and symplectically aspherical manifold.

Theorem 1.1. *Let (M,ω) be closed, connected, and symplectically aspherical. The mapping*

$$\mathcal{B} : (\mathrm{Ham}(M,\omega), d_{C^0}) \to (\widehat{Barcodes}, d_{\text{bottle}})$$

is continuous and extends continuously to $\overline{\mathrm{Ham}}(M,\omega)$.

The above result has been extended to certain non-aspherical manifolds such as $\mathbb{C}P^n$, by Shelukhin [31], and negatively monotone manifolds by Kawamoto [22].

2. Dynamical Applications

We will now list some dynamical applications of Theorem 1.1.

2.1. **The Arnold conjecture.**

We will now explain how Theorem 1.1 allows us to present a generalization of the Arnold conjecture which continues to hold for Hamiltonian homeomorphisms.

The (homological) Arnold conjecture states that a Hamiltonian *diffeomorphism* of a closed and connected symplectic manifold (M,ω) must have at least as many fixed points as the *cup length* of M. Cup length, denoted by $\mathrm{cl}(M)$, is a topological invariant of M which is defined as follows:[1]

$$\mathrm{cl}(M) := \max\{k+1 \,:\, \exists\, a_1,\dots,a_k \in H_*(M),\ \forall i, \deg(a_i) \neq \dim(M) \text{ and } a_1 \cap \cdots \cap a_k \neq 0\}.$$

This version[2] of the Arnold conjecture was proven, for Hamiltonian diffeomorphisms, on $\mathbb{C}P^n$ [14, 15], negatively monotone manifolds [23], and symplectically aspherical manifolds [13, 20, 29].

It was proven by Matsumoto [26] that Hamiltonian *homeomorphisms* of surfaces satisfy the Arnold conjecture; see also [16, 24]. However, we showed in [5] that

[1] Here, $\cap$ refers to the intersection product in homology. Cup length can be equivalently defined in terms of the cup product in cohomology.

[2] The original version of the Arnold conjecture, in which the lower bound for the number of fixed points is predicted to be the minimal number of critial points of a smooth function on M, has also been established on ashperical manifolds; see [29].

every closed and connected symplectic manifold of dimension at least 4 **admits a Hamiltonian homeomorphism with a single fixed point.**

This is where Theorem 1.1 enters the scene: the result allows us to define the action spectrum of a Hamiltonian homeomorphism (upto a shift). In particular, we can now make sense of *the total number of distinct values in the action spectrum of a Hamiltonian homeomorphism ϕ: This is simply the total number of distinct endpoints in the barcode $\mathcal{B}(\phi)$.* The theorem below tells us that, in spite of the counter-example from [5], the cup length estimate from the homological Arnold conjecture survives if we include in the count $\#\mathrm{Endpoints}(\mathcal{B}(\phi))$, which denotes the total number of (distinct) endpoints of the bars in $\mathcal{B}(\phi)$.

We need the following notion before stating the result: A subset $A \subset M$ is homologically non-trivial if for every open neighborhood U of A the map $i_* : H_j(U) \to H_j(M)$, induced by the inclusion $i : U \hookrightarrow M$, is non-trivial for some $j > 0$. Clearly, homologically non-trivial sets are infinite.

Theorem 2.1. *Let (M, ω) denote a closed, connected and symplectically aspherical manifold. Let $\phi \in \overline{\mathrm{Ham}}(M, \omega)$ be a Hamiltonian homeomorphism. If $\#\mathrm{Endpoints}(\mathcal{B}(\phi))$ is smaller than $\mathrm{cl}(M)$, then the set of fixed points of ϕ is homologically non-trivial, hence is infinite.*

In the smooth case, Theorem 2.1 was established by Howard [21], and our proof is inspired by his. For a smooth Hamiltonian diffeomorphism, endpoints in $\mathcal{B}(\phi)$ correspond to actions of certain fixed points. Therefore, Theorem 2.1 is a generalization of the Arnold conjecture in the smooth setting.

A version of the above theorem has been established on more general symplectic manifolds by Kawamoto [22].

2.2. **The displaced disks problem.** The displaced disks[3] problem, posed by F. Béguin, S. Crovisier, and F. Le Roux, asks if a C^0 small Hamiltonian homeomorphism can displace a *large* symplectic ball. We will show that the answer is negative on all symplectically aspherical manifolds. The case of closed surfaces was resolved in [30].

By a symplectic ball we mean the image of a symplectic embedding $i : (B, \omega_0) \to (M, \omega)$, where (B, ω_0) denotes a closed Euclidean ball equipped with the standard symplectic structure. If we know that B has radius r, we then refer to its image as a symplectic ball of radius r.

Theorem 2.2. *Let (M, ω) be closed, connected and symplectically aspherical. For every $r > 0$, there exists $\epsilon > 0$ with the following property: if $\phi \in \overline{\mathrm{Ham}}(M, \omega)$ displaces a symplectically embedded ball of radius r, then $d_{C^0}(\mathrm{Id}, \phi) > \epsilon$.*

The above result has been extended to certain non-aspherical manifolds such as $\mathbb{C}P^n$, by Shelukhin [31], and negatively monotone manifolds by Kawamoto [22].

[3]The original question was posed in the two-dimensional setting, whence the use of the word "disk".

2.3. **Rokhlin groups and almost conjugacy.** We will be addressing the following question of Béguin, Crovisier, and Le Roux: Does $\overline{\mathrm{Ham}}(M, \omega)$ possess a dense conjugacy class? The fact that the answer to this question is negative is a consequence of Theorem 2.2. The case of surfaces was resolved in [12, 30]. The question of existence of topological groups which possess dense conjugacy classes is of interest in ergodic theory; see [18, 19]. Glasner and Weiss refer to such groups as *Rokhlin* groups. An interesting example of a Rokhlin group is the identity component of the group of homeomorphisms of any even dimensional sphere equipped with the topology of uniform convergence. For further examples see [18, 19].

Theorem 2.2 has the following corollary.

Corollary 2.3. *$\overline{\mathrm{Ham}}(M, \omega)$ is not a Rokhlin group when M is symplectically aspherical.*

The above result has been extended to certain non-aspherical manifolds such as $\mathbb{C}P^n$, by Shelukhin [31], and negatively monotone manifolds by Kawamoto [22].

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Symbolic dynamics for Reeb flows

Barney Bramham

(joint work with Umberto Hryniewicz, Gerhard Knieper)

Suppose that a 3-dimensional Reeb flow admits the existence of a so called transversal foliation. What can one conclude about the dynamics of the Reeb flow from this additional structure? This talk reports on work in progress in which we use this structure to obtain symbolic dynamics. As a consequence we conclude that when the periodic orbits are non-degenerate then zero topological entropy implies the existence of a global surface of section, on the complement of a finite collection of invariant tori, with a well defined smooth return map.

The definition of a transversal foliation was introduced in [10] by Hofer-Wysocki-Zehnder as a generalization of a Poincaré surface: Let M be a closed three-manifold with a non-vanishing vector field X.

Definition 1. *A* transversal foliation *for* (M, X) *is a pair* $(\mathcal{F}, \mathcal{P})$*, where* $\mathcal{P}$ *is a finite collection of periodic orbits of the flow of* X*, called the* spanning orbits*, and* $\mathcal{F}$ *is a smooth foliation of the complement* $M \backslash \mathcal{P}$ *by embedded punctured Riemann surfaces transverse to* X*, such that the closure of each leaf in* M *is embedded and each boundary circle is in* $\mathcal{P}$*.*

When can we assume existence of a transversal foliation? There is no reason to expect existence in general, but when X is the Reeb vector field of a contact form λ on M then Hofer-Wysocki-Zehnder showed that transversal foliations arise as projections of pseudoholomorphic curves in $\mathbb{R} \times M$ down to M and they used this to prove remarkable existence results. For example, any star-shaped energy surface in $\mathbb{R}^4$ for which the periodic orbits are non-degenerate admits a transversal foliation [10] to which our results apply. In the restricted, circular, planar, three body problem for energies just above the first Lagrange point (i.e. lowest energies for which there can exist trajectories between neighborhoods of the two primary bodies) the regularized energy surfaces are of Reeb type [1] and there is theoretical evidence [5, 6] and numerical evidence [9] that transversal foliations exist.

In any case, assuming a transversal foliation exists, there is considerable structure: The foliation of $M \backslash \mathcal{P}$ has a skeleton of isolated (so called rigid) leaves in $\mathcal{F}$ that enclose a finite number of three dimensional chambers

$$\Omega_1, \ldots, \Omega_r.$$

Each chamber $\Omega = \Omega_i$ has two incoming A, B and two outgoing C, D boundary surfaces. The incoming, respectively outgoing, surfaces meet at a hyperbolic spanning orbit h_{in}, respectively h_{out}. The flow through the chamber Ω thus gives one a map φ from the incoming surface $A \cup B$ to the outgoing surface $C \cup D$, which is defined except at some points which never exit the chamber and are asymptotic to h_{out}. Thus when the flow exits a chamber it gets split up in two directions; some portion of the trajectories exit through C and some through D, and then again through the next chambers, and so on. This can lead to a complicated mixing process (not necessarily in the technical sense). The basic possibilities can be conveniently captured by forming a graph $\mathcal{G}$ whose vertices are the rigid leaves and whose edges correspond to the existence of a single connecting trajectory. There is an associated shift of finite type corresponding to permissable bi-infinite paths in $\mathcal{G}$. If one does this correctly then positivity of the entropy of the shift space is equivalent to some invariant manifolds between some hyperbolic spanning orbits meeting without coinciding exactly.

All of this has essentially been known for some time. The new insight is the following. It turns out that one does not need to treat separately the different cases where invariant manifolds have non-transverse intersections of finite or infinite order, or one-sided intersections, rather than transverse intersections. Indeed all cases lead to a continuous surjective map, the 'coding map', from a compact invariant subset of the Reeb flow to the shift space of finite type associated to $\mathcal{G}$. This allows to conclude positive entropy of some time-reparameterization of the Reeb flow whenever there is not a globally defined return map to the leaves, and also positive entropy for the Reeb flow itself:

Theorem 2. *Suppose a Reeb flow on a closed three-manifold M admits a regular transversal foliation with non-degenerate spanning periodic orbits. Then either:*

(i) There is a global return map (on the complement of a finite collection of invariant tori).

(ii) Or there are trajectories which visit the chambers in all permissable ways determined by the graph $\mathcal{G}$ and the topological entropy of the Reeb flow is strictly positive.

The first option occurs precisely when the topological entropy of the shift space of finite type associated to the graph $\mathcal{G}$ is vanishing.

Here a periodic orbit is called non-degenerate if 1 is not an eigenvalue for the linearized first return map. The transversal foliation in Theorem 2 does not have to arise from pseudo-holomorphic curves.
The mechanism that ensures surjectivity of the coding map is that the map φ from the incoming boundary to the outgoing boundary of a chamber is either very nicely behaved, corresponding to conclusion (i), or there is an infinite amount of "stretching" going on, which corresponds to conclusion (ii). This is illustrated in Figure 1.

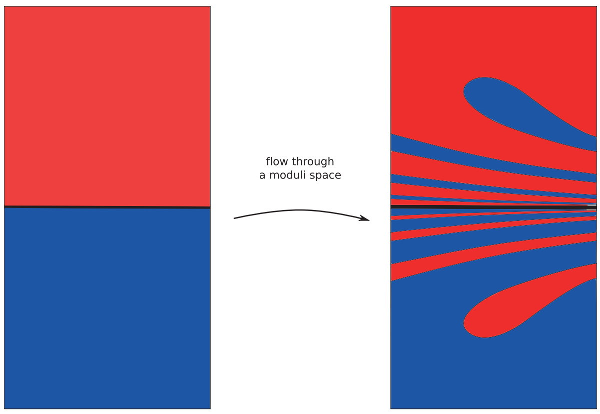

FIGURE 1. On the left the two incoming rigid leaves A and B. On the right the two outgoing leaves, indicating how the points from A and B get "mixed" up. This phenomenon appears even if stable and unstable manifolds have only a topological crossing, and allows us to construct trajectories with arbitrary permissable prescribed orbit itinaries.

Let us close with some implications for Reeb flows on the tight three-sphere. The symplectic methods in [10] establish the existence of a regular transversal foliation, whose leaves have genus zero, for the Reeb flow for a C^∞-generic class of tight contact forms on S^3, namely all those contact forms whose periodic orbits are non-degenerate. From this one obtains:

Corollary 3. *Suppose that λ is a tight contact form on S^3 all of whose periodic orbits are non-degenerate. If the topological entropy of the Reeb flow vanishes then the Reeb flow admits a global surface of section, with a smooth return map, on the complement of a finite number of invariant tori.*

This points in the direction of a structure theorem for three dimensional Reeb flows in a similar spirit to the results of Franks-Handel [8] for zero entropy Hamiltonian surface diffeomorphisms, see also LeCalvez-Tal [13], to conclude finer information. In some sense a special case is:

Corollary 4. *Suppose that λ is a tight contact form on S^3 all of whose periodic orbits are non-degenerate and at least 3 distinct simple periodic orbits. Then a dense orbit implies positive topological entropy.*

Note that the famous Katok examples [11] yield non-degenerate Reeb flows on S^3 with zero entropy and dense orbits and exactly two simple periodic orbits. Thus the assumption of three periodic orbits cannot be reduced to two.

Corollary 5. *Suppose that λ is a tight contact form on S^3 all of whose periodic orbits are non-degenerate. Then the Reeb flow has either 2 or infinitely many simple periodic orbits.*

This is not a new statement: When the flow is Morse-Smale this statement was proven by Hofer-Wysocki-Zehnder in [10], also using finite energy foliations. For a very large class of contact manifolds, including all contact structures on S^3, Cristofaro-Gardner-Hutchings-Pomerleano [4] recently proved the above "two or infinite" statement without the Morse-Smale assumption using embedded contact homology. The degenerate case is still an open question.

The idea of the proof of Corollary 5 using Theorem 2 goes as follows. If there are finitely many periodic orbits then by classical results of Katok [12] the topological entropy vanishes, so that by Theorem 2 we obtain a global surface of section S with genus zero. It turns out that this surface of section either contains a disk, in which case Franks' theorem [7] applies, or some component of S is an annulus and the return map is a twist map, or some component has negative Euler characteristic, in either case there are infinitely many periodic points.

A difficult question is whether the non-degeneracy assumptions in Corollaries 3, 4 and 5 can be weakened or removed. There is a history of works showing that flows with certain hyperbolicity assumptions can be related to symbolic dynamics. For example under global hyperbolicity assumptions [16, 2, 17, 3, 15] a conjugacy rather than semi-conjugacy to a shift of finite type is established, and in three dimensions [14] where the hyperbolicity assumption is only positivity of the topological entropy, see [14] for more references. Our assumption that the spanning orbits of the transversal foliation are non-degenerate can perhaps be considered as our "hyperbolicity assumption". This is comparitively weak, and the conclusions are correspondingly weaker, namely that our coding map will in general be very non-injective.

It remains to see how the resulting algebraic information can be further exploited. A natural, but presumably very difficult, question is whether it is possible to encode also the *measure* of the set of points entering a given chamber which then exit through a particular part of the boundary and to understand how this extends to arbitrary finite sequences of chambers. This might be a step towards saying something about the metric entropy.

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Spectral determination of a class of open dispersing billiards

Martin Leguil
(joint work with Péter Bálint, Jacopo De Simoi, Vadim Kaloshin)

We consider billiard tables $\mathcal{D} \subset \mathbb{R}^2$ given by $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i$, for some integer $m \geq 3$, where each $\mathcal{O}_i$ is a convex domain with sufficiently smooth boundary $\partial\mathcal{O}_i$ (at least of class C^3; in some places, we will actually assume the boundary to be analytic). We refer to each of the $\mathcal{O}_i$'s as *obstacle*, and parametrize $\partial\mathcal{O}_i$ in arclength. We assume that the *non-eclipse condition* holds, i.e., that the convex hull of any two obstacles is disjoint from the other $m-2$ obstacles. The set of all billiard tables obtained by removing from the plane m strictly convex obstacles with C^3, resp. analytic boundary satisfying the non-eclipse condition will be denoted by $\mathbf{B}(m)$, resp. $\mathbf{B}^\omega(m) \subset \mathbf{B}(m)$.

Fix $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}(m)$. We denote the collision space by

$$\mathcal{M} = \cup_i \mathcal{M}_i, \qquad \mathcal{M}_i = \big\{(q,v),\ q \in \partial\mathcal{O}_i,\ v \in \mathbb{R}^2,\ \|v\| = 1,\ \langle v, n\rangle \geq 0\big\},$$

where n is the unit normal vector to $\partial\mathcal{O}_i$ pointing inside $\mathcal{D}$. Each $x = (q,v) \in \mathcal{M}$ can be identified with a pair $(s,r) \in \mathbb{R} \times [-1,1]$, where s is the associated arclength parameter, φ is the oriented angle between n and v, and $r := \sin(\varphi)$. Whenever it is well-defined, the image by the billiard map $\mathcal{F}$ of a pair (s,r) of parameters is the new pair (s',r') associated to the next collision of the billiard trajectory with $\partial\mathcal{D}$; the map $\mathcal{F}$ is symplectic for the form $ds \wedge dr$ (in fact, exact symplectic).

It is clear that a lot of trajectories will escape to infinity. In fact, due to the convexity of the obstacles, the set of points $x = (s,r)$ whose iterates $\mathcal{F}^n(x)$ under the billiard map are well-defined for any $n \in \mathbb{Z}$ is homeomorphic to a Cantor set $\mathcal{NE}$ (see e.g. [4, 7]). The restriction of the dynamics to $\mathcal{NE}$ is conjugated to a subshift of finite type associated to the transition matrix $A := (1-\delta_{ij})_{1\leq i,j\leq m} \in \mathfrak{M}_m(\mathbb{R})$. In other words, any *admissible* word $(\varsigma_j)_j \in \{1,\cdots,m\}^{\mathbb{Z}}$ – i.e., such that $\varsigma_{j+1} \neq \varsigma_j$ for all $j \in \mathbb{Z}$ – can be realized by a unique orbit.[1] In particular, any periodic orbit of period $p \geq 2$ can be represented by an admissible word $\sigma^\infty := \ldots\sigma\sigma\sigma\ldots$ for some *finite admissible word* $\sigma = (\sigma_1\sigma_2\ldots\sigma_p) \in \{1,\cdots,m\}^p$. We denote by Adm the set of finite admissible words $\sigma \in \cup_{p\geq 2}\{1,\cdots,m\}^p$.

The *Marked Length Spectrum* $\mathcal{MLS}(\mathcal{D})$ of $\mathcal{D}$ is defined as the function

$$\mathcal{L}\colon \mathrm{Adm} \to \mathbb{R}_+, \quad \sigma \mapsto \mathcal{L}(\sigma),$$

where $\mathcal{L}(\sigma)$ is the length of the closed trajectory labeled by σ. In the following, an object is said to be a $\mathcal{MLS}$*-invariant* if it can be obtained by the sole knowledge of the Marked Length Spectrum. We are interested in the following *inverse problem*:

$$\mathcal{MLS}(\mathcal{D}) \quad \rightsquigarrow \quad \text{“Geometry” of } \mathcal{D}?$$

Note that in finite regularity, the information given by $\mathcal{MLS}(\mathcal{D})$ is insufficient to reconstruct the geometry of the whole table; at best, we can hope to describe the geometry near points associated to an arclength parameter s such that $(s,r) \in \mathcal{NE}$

[1] Each symbol $\varsigma_j \in \{1,\cdots,m\}$ corresponds to the obstacle $\mathcal{O}_{\varsigma_j}$ where the bounce happens.

for some $r \in [-1,1]$. In a first work [1], we show that it is indeed possible to extract some information from $\mathcal{MLS}(\mathcal{D})$ on the local geometry near very specific points. In [3], we assume that the boundary of the obstacles is analytic, i.e. $\mathcal{D} \in \mathbf{B}^\omega(m)$, in such a way that local geometric information may determine the whole table; in this case, under some symmetry and (mild) non-degeneracy assumptions, we show that $\mathcal{MLS}(\mathcal{D})$ does determine $\mathcal{D}$ up to isometries.

1.1. **Results in finite regularity.** Let us fix $m \geq 3$ and $\mathcal{D} = \mathbb{R}^2 \setminus \bigcup_{i=1}^m \mathcal{O}_i \in \mathbf{B}(m)$. Given a periodic point $x = (s,r) \in \mathcal{NE}$, the basic idea is to combine the information given by a sequence $(x_n)_{n\geq 0}$ of periodic points $x_n \in \mathcal{NE}$ accumulating x in order to extract some geometric quantities at the point of arclength parameter s. Thanks to the symbolic coding recalled above, this amounts to considering periodic orbits encoded by longer and longer finite admissible words obtained by truncating the coding of x.

One major issue is that *a priori*, the information obtained in this way is "averaged" over the different points in the orbit of x; yet, in [1], we found out some mechanism which allows us to distinguish between the two points in 2-periodic orbits. More precisely, let us consider a 2-periodic orbit encoded by a word $\sigma = (\sigma_1\sigma_0) \in \{1,\cdots,m\}^2$, $\sigma_0 \neq \sigma_1$. Let $\tau_1 \in \{1,\cdots,m\} \setminus \{\sigma_0,\sigma_1\}$, and set $\tau := (\tau_1\sigma_0)$. We consider the sequence of periodic orbits encoded by the words $h_n := \tau\sigma^n \in \mathrm{Adm}$, $n \geq 0$; as $n \to +\infty$, their points accumulate the points of some orbit h_∞ that is homoclinic to the orbit encoded by σ.

Theorem 1 (Bálint-De Simoi-Kaloshin-Leguil [1])**.** *We denote by $R_0, R_1 > 0$ the respective radii of curvature at the points with symbols σ_0, σ_1, and let $\lambda < 1$ be the smallest eigenvalue of $D\mathcal{F}^2$ at the points of σ. For $n \gg 1$, it holds:*

(1) $\mathcal{L}(\tau\sigma^n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C \cdot \mathcal{Q}\left(\frac{2R_0}{\mathcal{L}(\sigma)}, \frac{2R_1}{\mathcal{L}(\sigma)}\right)\lambda^n + O(\lambda^{\frac{3n}{2}})$, *$n$ even,*

(2) $\mathcal{L}(\tau\sigma^n) - (n+1)\mathcal{L}(\sigma) - \mathcal{L}^\infty = -C \cdot \mathcal{Q}\left(\frac{2R_1}{\mathcal{L}(\sigma)}, \frac{2R_0}{\mathcal{L}(\sigma)}\right)\lambda^n + O(\lambda^{\frac{3n}{2}})$, *$n$ odd,*

for some real number $\mathcal{L}^\infty = \mathcal{L}^\infty(\sigma,\tau) \in \mathbb{R}$, some constant $C = C(\sigma,\tau) > 0$, and some explicit quadratic form $\mathcal{Q}\colon \mathbb{R}\times\mathbb{R} \to \mathbb{R}$.

The reason why the parity of n affects the estimates is due to the "palindromic" symmetry of h_n: indeed, each point in $\partial\mathcal{D}$ with arclength parameter s such that (s,r) belongs to the orbit h_n for some $r \in [-1,1]$ is seen twice – as $(s,-r)$ also belongs to h_n – except when $r = 0$; this is the case for exactly two points in the orbit h_n, associated to perpendicular bounces. Among those two points, only one contributes to the first order term in the above estimates, and it is either on the boundary of the obstacle $\mathcal{O}_{\sigma_0}$ or of the obstacle $\mathcal{O}_{\sigma_1}$ depending on the parity of n. Theorem 1 has the following geometric consequence:

Corollary 2 (Bálint-De Simoi-Kaloshin-Leguil [1])**.** *The radii of curvature at the bouncing points of periodic orbits of period two are $\mathcal{MLS}$-invariants.*

Moreover, by Theorem 1, the *Lyapunov exponent* $-\frac{1}{2}\log\lambda$ of σ is also a $\mathcal{MLS}$-invariant. More generally, let us consider a periodic orbit of period $p \geq 2$, encoded by some finite admissible word $\widetilde{\sigma} \in \{1,\cdots,m\}^p$. The *Lyapunov exponent* $\mathrm{LE}(\widetilde{\sigma})$ of

$\widetilde{\sigma}$ is equal to $-\frac{1}{p}\log\widetilde{\lambda}$, where $\widetilde{\lambda} = \widetilde{\lambda}(\widetilde{\sigma}) < 1$ is the smallest eigenvalue of $D\mathcal{F}^p$ at the points in the orbit. By adapting the construction explained above, we get:

Theorem 3 (Bálint-De Simoi-Kaloshin-Leguil [1]). *The Lyapunov exponent of each periodic orbit is a $\mathcal{MLS}$-invariant.*

1.2. **$\mathcal{MLS}$-determination of analytic billiard tables.** Let us now consider the case where the boundary of the table is analytic. Fix $m \geq 3$, and let $\mathbf{B}^{\omega}_{\mathrm{sym}}(m) \subset \mathbf{B}^{\omega}(m)$ be the subset of all billiard tables $\mathcal{D} \in \mathbf{B}^{\omega}(m)$ such that:

- the jets of the curvature function $\mathcal{K}$ are the same at the endpoints of the 2-periodic orbit (12);
- the jets of $\mathcal{K}|_{\partial\mathcal{O}_1}$, $\mathcal{K}|_{\partial\mathcal{O}_2}$ are even, assuming that $0_1 \in \partial\mathcal{O}_1$, $0_2 \in \partial\mathcal{O}_2$ are the arclength parameters of the endpoints of the orbit (12).

In the analytic setting, and modulo the partial $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry assumption introduced above, we can show:

Theorem 4 (De Simoi-Kaloshin-Leguil [3]). *There exists an open and dense set of billiard tables $\mathbf{B}^*_{\mathrm{sym}}(m) \subset \mathbf{B}^{\omega}_{\mathrm{sym}}(m)$ so that if $\mathcal{D} \in \mathbf{B}^*_{\mathrm{sym}}(m)$, then the geometry of $\mathcal{D}$ is entirely determined (modulo isometries) by $\mathcal{MLS}(\mathcal{D})$.*

The open and dense condition we require is actually a *non-degeneracy condition*: roughly speaking, it means that after a change of coordinates, the first coefficient in the expansion of the dynamics does not vanish.[2]

It is a standard fact that any continuous deformation of smooth domains which preserves the (unmarked) Length Spectrum $\mathcal{LS}(\mathcal{D})$ automatically preserves $\mathcal{MLS}(\mathcal{D})$ (see e.g. [8, Proposition 3.2.2]). A family $(\mathcal{D}_t)_{t\in(-1,1)}$ is an *iso-length-spectral family of billiards in* $\mathbf{B}^*_{\mathrm{sym}}(m)$ if each $\mathcal{D}_t$ is in $\mathbf{B}^*_{\mathrm{sym}}(m)$, the map $(-1,1) \ni t \mapsto \mathcal{D}_t$ is continuous, and $\mathcal{LS}(\mathcal{D}_t) = \mathcal{LS}(\mathcal{D}_0)$, for all $t \in (-1,1)$. Therefore, we obtain:

Corollary 5 ([3]). *Any iso-length-spectral deformation in $\mathbf{B}^*_{\mathrm{sym}}(m)$ is isometric.*

Our results are an analog of the result of Colin de Verdière [2] for the class of chaotic billiards under consideration, or an analog in terms of the Marked Length Spectrum of the results of Zelditch [9, 10, 11] (see also [5]).

Let us give some ideas of the proof. Fix $\mathcal{D} \in \mathbf{B}^*_{\mathrm{sym}}(m)$. For the 2-periodic $\sigma = (12)$, we consider the same sequence $(h_n)_{n\geq 0}$ of periodic orbits accumulating some orbit h_∞ homoclinic to σ. In a first time, we show that after a *canonical*[3] symplectic change of coordinates, the dynamics of the square $\mathcal{F}^2$ of the billiard map in a neighbourhood of the trajectory h_∞ can be replaced with two maps: the *Birkhoff Normal Form* $N = N(\sigma)$ of $\mathcal{F}^2$ associated to σ, and some gluing map $\mathcal{G} = \mathcal{G}(\sigma,\tau)$. Working with this new system of coordinates, we show that for each integer $n \geq 0$, the Lyapunov exponent of h_n can be expanded as a series

[2]More precisely, we ask that the first coefficient of a certain Birkhoff Normal Form is non-zero.

[3]i.e., such that the change of coordinates respects the billiard symmetry.

(reminiscent of the asymptotic expansion of the lengths obtained in [6]):

$$2\lambda^n \cosh(2(n+1)\mathrm{LE}(h_n)) = \sum_{p=0}^{+\infty}\sum_{q=0}^{p} L_{q,p} n^q \lambda^{np},$$

for some sequence $(L_{q,p})_{\substack{p=0,\cdots,+\infty\\ q=0,\cdots,p}}$, and where $\lambda = \lambda(\sigma) < 1$. In particular, each coefficient $L_{q,p}$ is a $\mathcal{MLS}$-invariant. Then, we show that modulo the non-degeneracy condition mentioned previously, it is possible to extract enough information from $(L_{q,p})_{p,q}$ to recover N and the differential $D\mathcal{G}$ at some points in h_n. In fact, the $\mathcal{MLS}$-determination of N does not require any symmetry assumption, and the same procedure can be carried out for more general palindromic periodic orbits. Following [2], and thanks to the $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetry of $\{\mathcal{O}_1, \mathcal{O}_2\}$, we then show that the jet of $\mathcal{K}$ can be read off from the coefficients of N, which by analyticity determines entirely the geometry of $\mathcal{O}_1, \mathcal{O}_2$. Finally, we explain how the information given by the differential of the gluing map $\mathcal{G}$ can be utilized in order to recover the geometry of the other obstacles (note that no symmetry assumption is needed for this last step).

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Barcodes and small eigenvalues of the Witten Laplacian

Claude Viterbo
(joint work with D. Le Peutrec, F. Nier)

Let f be a smooth function on a closed Riemannian manifold (M, g). On one hand, we define the Barcode or Barannikov complex of this function. When f is Morse, the Barannikov complex is chain equivalent to the Morse complex, but has the property that the boundary operator sends each generator to an other generator or to 0, and each generator has at most one preimage. Thus the Barannikov complex yields a matching between pairs of critical values, and connecting a matched pairs by a bar yields the barcode of f. This can be shown to extend to any continuous function. On the other hand if $\Delta_{f,h}$ is the Witten Laplacian on differential forms, obtained by setting $d_{f,h} = e^{-\frac{f}{h}} h d e^{-\frac{f}{h}}$, $d^*_{f,h}$ its adjoint and

$$\Delta_{f,h} = (d_{f,h} + d^*_{f,h})^2 = d_{f,h} d^*_{f,h} + d^*_{f,h} d_{f,h}$$

As h goes to 0, the small eigenvalues of $\Delta_{f,h}$ have been studied by many authors, starting from Witten (in fact even earlier in the case of functions). We refer to [1] for a complete bibliography. The asymptotic value of the eignevalues is for a Morse function, of the type $P(h)e^{C/h}$ where $P(h)$ depends on the Hessian of f. In [1] we proved that C is the length of the bar corresponding for p-forms, to critical points of index p. This implies that if $\lambda_j(h)$ is asmall eigenvalue, we have

$$\lim_{h \to 0} h \log \lambda_j(h) = f(y) - f(x)$$

where $f(y), f(x)$ are critical values. We prove that this last result extends to the general case, where f does not need to be Morse, but has only finitely many critical values. In [2] we conjecture that this should still hold in general as follows : a barcode for a general smooth function will have only finitely many bars of size greater than ε. As a result, eigenvalues $\lambda(h)$ of the order $O(e^{-\frac{\varepsilon}{h}})$ should satisfy

$$\lim_{h \to 0} h \log \lambda(h) = b$$

where b is a bar of size larger than ε

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Some aspects of topological dynamics and aperiodic order

Tobias Jäger

(joint work with M. Baake, G. Fuhrmann, F. García-Ramos, E. Glasner, D. Lenz and others.)

A topological dynamical system is the continuous action of a topological group G on a topological space X by homeomorphisms. For simplicity, we restrict here to the case $G = \mathbb{Z}$, so that the action is given by the iterates of a single homeomorphism T. Moreover, we assume throughout that X is compact metric. Then (X, T) is called *strictly ergodic* if it is uniquely ergodic (i.e. admits a unique T-invariant probability measure on X) and *minimal* (i.e. every orbit is dense). Strictly ergodic systems appear naturally in a variety of contexts. For instance, the Jewitt-Krieger-Theorem asserts that every ergodic measure-preserving dynamical system has a strictly ergodic model (via isomorphism, that is, a measurable change of coordinates). In the case of discrete spectrum, the Halmos-Von Neumann Theorem states that this model can be chosen to be a rotation on a compact group.

However, from the topological viewpoint, a strictly ergodic action with discrete spectrum may be very different from a group rotation (e.g. [1, 2, 3, 4, 5, 6]). In fact, a natural hierarchy with different levels of complexity for such systems can be formulated with respect to the *maximal equicontinuous factor (MEF)* of the system. The latter is the largest possible topological factor of the system that is equicontinuous [7, 8]. In the following, we always denote the MEF of (X, T) by (X_{eq}, T_{eq}) and the corresponding factor map by $\pi_{eq} : X \to X_{eq}$. If (X, T) is either minimal or uniquely ergodic (or both), then (X_{eq}, T_{eq}) is both minimal and uniquely ergodic, and we denote its unique invariant measure by μ_{eq}. When (X, T) is uniquely ergodic with invariant measure μ and (X, T, μ) and $(X_{eq}, T_{eq}, \mu_{eq})$ are measure-theoretically isomorphic, we say π_{eq} is *isomorphic*. Based on the invertibility properties of the map π_{eq}, strictly ergodic systems with discrete spectrum may be classified as follows.

π_{eq} is a conjugacy (1-1)
$\Rightarrow \pi_{eq}$ is regular (almost surely 1-1, that is, 1-1 on a set of full measure)
$\Rightarrow \pi_{eq}$ is isomorphic and almost 1-1 (that is, 1-1 on a residual subset)
$\Rightarrow \pi_{eq}$ is isomorphic
$\Rightarrow (X, \mu, T)$ has discrete spectrum.

Recent work by various authors has aimed to provide equivalent dynamical characterisations for the different levels of this hierarchy. Thereby, discrete spectrum is known to be equivalent to the existence of a dense set of L^2-eigenfunctions by the classical result von Halmos-Von Neumann. More recently, results of Li, Tu and Ye [1] and of Downarowicz and Glasner [3] showed that π_{eq} is isomorphic if and only if the system is *mean equicontinuous* (as introduced by Fomin in 1951 [9]), that is, for all $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies

$$d_B(x, y) \ = \ \limsup_{n\to\infty} \sum_{i=0}^{n-1} d(T^i x, T^i y) \ < \ \epsilon \ .$$

Here, d_B is called the *Besicovitch pseudometric* on X, which is a metric if and only if (X,T) is equicontinuous. In similar spirit, García-Ramos characterized discrete spectrum by using the related weaker notion of *μ-mean equicontinuity* [2].

In order to characterise the second and third level of the above hierarchy, we introduce the following notions. We say (X,T) is *diam mean equicontinuous* if for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in X$

$$\limsup_{n\to\infty} \sum_{i=0}^{n-1} \operatorname{diam}(T^i(B_\delta(x))) \ < \ \epsilon \ .$$

Further, we call (X,T) *frequently stable* if for every $\epsilon > 0$ and $x \in X$ there exists $\delta > 0$ such that

$$\overline{D}\left\{i \in \mathbb{Z}_+ : \operatorname{diam}(T^i B_\delta(x)) > \epsilon\right\} < 1 \ ,$$

where $\overline{D}(N) = \limsup_{n\to\infty}(\sharp N \cap [1,n])/n$ denotes the upper asymptotic density of a subset $n \subseteq \mathbb{N}$.

Theorem 1 (García-Ramos/Ye/J., in preparation)**.** *Suppose (X,T) is a minimal topological dynamical system. Then the following are true:*

(i) *(X,T) is regular if and only if it is diam mean equicontinuous.*
(ii) *(X,T) is almost 1-1 and isomorphic if and only it is mean equicontinuous and frequently stable.*

Further dynamical properties can be related to the above hierarchy. Here, we concentrate on the notion of tameness, as studied for instance by Glasner in [5, 10]. A pair of subsets $U_0, U_1 \subseteq X$ is called an *independence pair* (for the system (X,T)) if $d(U_0,U_1) > 0$ and there exists an infinite set $S \subseteq \mathbb{Z}$ with the property that for all $a \in \{0,1\}^S$ there exists $x_a \in X$ such that

$$T^s(x_a) \in U_{a_s} \quad \text{ for all } s \in S \ .$$

(X,T) is called *tame* if no such independence pair exists, and *non-tame* otherwise [11]. By the results of Glasner in [5, 10], it was known that any tame minimal system is almost 1-1. We provide the following improvement.

Theorem 2 (Fuhrmann/Glasner/Oertel/J., [4])**.** *If (X,T) is minimal and tame, then it is regular.*

Altoghether, we obtain the following dynamical classification of strictly ergodic systems with discrete spectrum:

Equicontinuity ($\Leftrightarrow$ topological discrete spectrum)
$\Rightarrow$tame ($\Leftrightarrow$ no infinite independence)
$\Rightarrow$diam-mean equicontinuous ($\Leftrightarrow$ π_{eq} regular)
$\Rightarrow$mean equicontinous and frequently stable ($\Leftrightarrow$ π_{eq} almost 1-1 and isomorphic)
$\Rightarrow$mean equicontinuous ($\Leftrightarrow$ π_{eq} isomorphic)
$\Rightarrow$ $\mu-$mean equicontinuous. ($\Leftrightarrow$ discrete spectrum)

We note that there exist counter-examples showing every implication is strict [12, Section 5], [11, Section 11] or [13], [10, Remark 5.8], [14, Example 5.1], and [3, Theorem 3.1].

The above classification finds an application in the context of aperiodic order and mathematical quasicrystals, where different levels of the hierarchy correspond to different degrees of complexity of aperiodic structures with long-range order. In order to make this more precise, we concentrate on so-called *model sets*, which are aperiodic structures constructed via the cut and project method (introduced by Yves Meyer in [15]. A *cut and project scheme (CPS)* consists of a triple $(G, H, \mathcal{L})$ of two locally compact abelian groups G (called *external group*) and H (*internal group*) and a co-compact discrete subgroup $\mathcal{L} \subseteq G \times H$ such that the natural projections $\pi_G : G \times H \to G$ and $\pi_H : G \times H \to H$ satisfy the following:

(i) the restriction $\pi_G|_{\mathcal{L}}$ is injective;
(ii) the image $\pi_H(\mathcal{L})$ is dense.

If (i) and (ii) hold, we call $\mathcal{L}$ an *irrational lattice*. Given a subset $W \subseteq H$ (referred to as *window*), one defines a point set

$$\Lambda(W) \;=\; \pi_G\left(\mathcal{L} \cap (G \times W)\right) \;=\; \{l \in L \mid l^* \in W\}\ .$$

If W is compact, $\Lambda(W)$ is uniformly discrete and if W has non-empty interior, then $\Lambda(W)$ is relatively dense. Hence, if W is *proper* (that is, compact and $\overline{\mathrm{int}(W)} = W$), $\Lambda(W)$ is a Delone set. In this case, we call $\Lambda(W)$ a *model set*. It is called a *regular model set* if the Haar measure of the boundary of W is zero, and an *irregular model set* otherwise. The dynamical hull $\Omega(\Lambda(W))$ of a model set is defined as the closure of the translation orbit,

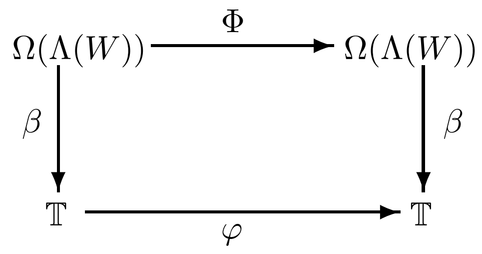
$$\Omega(\Lambda(W)) \;=\; \mathrm{cl}(\{\Lambda(W) - g \mid g \in G\})$$

where the closure is taken with respect to some suitable topology (e.g. [16, 17]). Due to results of Schlottmann [18] and Moody [19] (see also [20]), it is known that the dynamics given by the translation flow on the hull fit into the above framework (which can be generalised to actions of amenable groups for this purpose).

Theorem 3 (Torus parametrisation, [18, 21])**.** *Let $\mathbb{T} = G \times H/\mathcal{L}$ and*

$$\varphi : G \times \mathbb{T} \quad , \quad (t,(g,h)) \mapsto \varphi^t(g,h) = (g+t,h)\ .$$

Then there exists a continuous onto map $\beta : \Omega(\Lambda(W)) \to \mathbb{T}$ such that the following diagram commutes.



The map β is almost one-to-one. Moreover, β is regular if and only if the Haar measure of ∂W is zero, and irregular otherwise.

The dynamics of regular models sets are fairly well-understood (e.g. [21, 22]). In particular, by regularity, these are always uniquely ergodic and their topological entropy is zero. In contrast to this, the case of irregular models sets still remains largely unexplored. Recent results show that these often exhibit positive entropy and multiple ergodic measures [23], but this is not always the case [4]. However, as irregular models sets are also not regular in the sense of the above hierarchy, we obtain the following.

Theorem 4 (Fuhrmann/Glasner/Oertel/J., [4])**.** *The dynamics on the hull of an irregular model set are always non-tame.*

Hence, non-tameness provides a 'lower bound' for the complexity of such systems.

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Decay of correlations for the infinite horizon planar periodic Lorentz gas

Ian Melbourne
(joint work with Péter Bálint, Henk Bruin, Oliver Butterley, Dalia Terhesiu)

In this talk, we describe results on mixing rates for nonuniformly hyperbolic maps and flows. A key example is the infinite horizon planar periodic Lorentz gas where the mixing rate is $1/t$ and we obtain sharp upper bounds [2] and lower bounds [4].

First, we consider an example that is easier to explain, namely a semidispersing billiard with billiard table $Q \subset \mathbb{R}^2$ given by $Q = R \setminus \Omega$ where R is a rectangle and $\Omega \subset R$ is a scatterer (convex with C^3 boundary of nonvanishing curvature). The corresponding Lorentz flow $f_t : Q \times S^1 \to Q \times S^1$ is defined by particles starting at $p \in Q$ and moving in direction $\theta \in S^1$ with unit speed until colliding with ∂Q, whereupon θ changes to $\pi - \theta$ (so collisions are specular). The Poincaré map $f : \partial Q \times [-\pi/2, \pi/2] \to \partial Q \times [-\pi/2, \pi/2]$ is called the billiard map.

By [6, 7], the semidispersing billiard map has mixing rate $1/n$ for Hölder observables. More precisely, let μ denote the Liouville measure $d\mu = \cos\theta \, dr \, d\theta$ on $\partial Q \times [-\pi/2, \pi/2]$. This is the natural invariant measure. Then for all v, w Hölder, the correlation function

$$\rho_{v,w}(n) = \int v \, w \circ f^n \, d\mu - \int v \, d\mu \int w \, d\mu$$

satisfies $\rho_{v,w}(n) = O(1/n)$. In [3], we show that this decay rate is sharp. Indeed, there is a constant $c > 0$ (explicitly computable) such that

$$\rho_{v,w}(n) \sim c\frac{1}{n} \int v \, d\mu \int w \, d\mu$$

for all v, w Hölder with nonzero mean and supported on $\partial\Omega \times [-\pi/2, \pi/2]$.

The corresponding result for the Lorentz flow is more challenging; results for such flows is the main topic of the talk. The natural invariant measure on $Q \times S^1$ is Lebesgue. It is now necessary to consider observables v and w that are "sufficiently smooth in the flow direction" in addition to being Hölder. Modulo this caveat, the results are the same as for the billiard map. Define the correlation function

$$\rho_{v,w}(t) = \int v \, w \circ f_t \, d\text{Leb} - \int v \, d\text{Leb} \int w \, d\text{Leb}.$$

Then in [2] we show that $\rho_{v,w}(t) = O(1/t)$ while in [4] we show that

$$\rho_{v,w}(t) \sim c\frac{1}{t}\int v\,d\mu \int w\,d\mu$$

for observables with nonzero mean and supported on $\Omega \times S^1$.

The assumption that observables are sufficiently smooth in the flow direction is a significant restriction since the flow itself is not smooth. In particular, physically relevant observables such as velocity are excluded. As discussed in the talk, we expect to be able to show that the results go through for all Hölder (and dynamically Hölder) observables if the scatterer Ω is typical (for a C^2 open and C^∞ dense set of C^3 boundaries), and also for certain specific choices of Ω. However for a result that holds for *all* choices of Ω, the restriction to sufficiently smooth observables seems hard to remove.

The results mentioned above for semidispersing billiards hold equally for Bunimovich stadia (where Q is bounded by a rectangle with semicircles adjoined at each end) and Sinai billiards with cusps (nonvanishing curvature). In addition, results with different decay rates than $1/n$ and $1/t$ are also obtained in [2, 3, 4]. The methods in [2, 3, 4] are not restricted to billiards or to low-dimensional examples, and essentially optimal upper and lower bounds on decay of correlations are obtained for multidimensional (not necessarily Markovian) intermittent maps and the corresponding intermittent solenoidal flows.

Returning to the example mentioned in the title of the talk, recall that the planar periodic Lorentz gas is the Lorentz flow with $Q = \mathbb{T}^2 - \Omega$ where Ω is a finite union of scatterers. The billiard map has exponential decay of correlations by [8] and [5]. In the finite horizon case (bounded collision times), it was recently shown that the flow also has exponential decay of correlations [1]. In [2], we show that the decay rate for the flow with infinite horizon is $O(1/t)$. This result is shown to be sharp in [4].

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Recent results in geometric methods for Arnol'd diffusion

Tere M-Seara
(joint work with Marian Gidea, Rafael de la Llave)

The problem of Arnold diffusion consists in studying in which Hamiltonian systems the effects of perturbations can accumulate over time to produce effects much larger than the size of the perturbations. Specially in integrable systems.

We will describe a recent progress in the so-called geometric methods. The main idea in the geometric program is to find simple geometric structures whose presence implies a rich orbit structure. The mechanism presented here is based on the presence of Normally Hyperbolic Invariant Manifolds (NHIM). The mechanism is rather robust. It does not need that the perturbations are Hamiltonian (applies to small dissipation problems or for space craft maneuvers that involve burns), can be applied to to concrete problems and enjoys remarkable genericity properties since it does not require non-generic assumptions such as convexity.

We first present a general shadowing result. Assume we have a map $f : M \to M$, C^r-smooth, $r \geq r_0$, $m = \dim M$, having a Normally Hyperbolic Invariant Manifold Λ. Assume moreover that $W^u(\Lambda)$ intersects transversally $W^s(\Lambda)$ along a homoclinic manifold Γ satisfying certain extra transversality conditions (Γ is transverse to the foliation). We call Γ an homoclinic channel.

Under these conditions one can define the Scattering map: $s : \Omega_-(\Gamma) \subset \Lambda \to \Lambda$ given by: $s(x_-) = x_+$ if there exists $x \in \Gamma$ such that $d(f^{-m}(x), f^{-m}(x_-)) \to 0$, $d(f^n(x), f^n(x_+)) \to 0$, as $m, n \to \infty$. Then we have the general shadowing results:

Theorem 1. *Assume that Λ and Γ are compact. Then, for every $\delta > 0$ there exists $n^* \in \mathbb{N}$ depending on δ, and a family of functions $m_i^* : \mathbb{N}^{2i+1} \to \mathbb{N}$, $i \geq 0$, depending on δ, such that, for every pseudo-orbit $\{y_i\}_{i\geq 0}$ in Λ of the form*

$$y_{i+1} = f^{m_i} \circ \sigma^{\Gamma} \circ f^{n_i}(y_i), \tag{1}$$

for all $i \geq 0$, with $n_i \geq n^*$ and $m_i \geq m_i^*(n_0, \ldots, n_{i-1}, n_i, m_0, \ldots, m_{i-1})$, there exists an orbit $\{z_i\}_{i\geq 0}$ of f in M such that, for all $i \geq 0$,

$$z_{i+1} = f^{m_i+n_i}(z_i),$$

and $d(z_i, y_i) < \delta$.

The above result can be immediately extended to the case of countably many scattering maps.

Theorem 2 (Shadowing Lemma for Orbits of the Scattering Map)**.** *Assume that $f : M \to M$ is a sufficiently smooth map, $\Lambda \subseteq M$ is a normally hyperbolic invariant manifold with stable and unstable manifolds which intersect transversally along an homoclinic channel $\Gamma \subseteq M$, and σ is the scattering map associated to Γ.*

Assume that f preserves a measure absolutely continuous with respect to the Lebesgue measure on Λ, and that σ sends positive measure sets to positive measure sets.

Let $\{x_i\}_{i=0,\dots,n}$ *be a finite pseudo-orbit of the scattering map in* Λ*, i.e.,* $x_{i+1} = \sigma(x_i)$*,* $i = 0, \dots, n-1$*,* $n \geq 1$*, which is contained in some open set* $\mathcal{U} \subseteq \Lambda$ *with almost every point of* $\mathcal{U}$ *recurrent for* $f_{|\Lambda}$*.*

Then, for every $\delta > 0$ *there exists an orbit* $\{z_i\}_{i=0,\dots,n}$ *of* f *in* M*, with* $z_{i+1} = f^{k_i}(z_i)$ *for some* $k_i > 0$*, such that* $d(z_i, x_i) < \delta$ *for all* $i = 0, \dots, n$*.*

This result can also be extended to the case of countably many scattering maps. This theorem tells us that, if the system has recurrence, we can follow any heteroclinic connexion between points in Λ, without knowing the dynamics of the points. Therefore, we don't need the classical approaches that use invariant tori, periodic orbits, Aubry-Mather sets etc, and the result can be applied even if f does not satisfy a twist condition. The only thing to verify is that the system has a NHIM with stable and unstable manifolds which intersect transversaly. Now we will give conditions (easy to verify and generic) to ensure that, in the perturbative setting, a System satifies the hypotheses required in the previous theorem.

We consider the perturbative setting where $f_\varepsilon : M \to M$ is a symplectic map, $\Lambda_\varepsilon \subseteq M$ is a normally hyperbolic invariant manifold and Γ_ε is a homoclinic channel for f_ε, and $\sigma_\varepsilon : \Omega^-(\Gamma_\varepsilon) \to \Omega^+(\Gamma_\varepsilon)$ is the corresponding scattering map. We assume that Λ_ε is described via a parametrization $k_\varepsilon : \Lambda_0 \to \Lambda_\varepsilon$, and let $(\tilde{f}_\varepsilon)_{|\Lambda_0} = k_\varepsilon^{-1} \circ (f_\varepsilon)_{|\Lambda_\varepsilon} \circ k_\varepsilon$, $\tilde{\sigma}_\varepsilon = k_\varepsilon^{-1} \circ \sigma_\varepsilon \circ k_\varepsilon$.

Theorem 3. *Assume that for all* $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$*, there exists a scattering map* σ_ε*, defined in a domain* $U := k_\varepsilon^{-1}(\Omega^-(\Gamma_\varepsilon)) \subset \Lambda_0$*, such that*

$$\tilde{\sigma}_\varepsilon = \mathrm{Id} + \varepsilon J \nabla S + O(\varepsilon^2), \tag{2}$$

where S *is some real valued* C^ℓ*-function on* $U \subset \Lambda_0$*.*

Suppose that $J\nabla S(\tilde{x}_0) \neq 0$ *at some point* $\tilde{x}_0 \in U \subset \Lambda_0$*.*

Let $\tilde{\gamma} : [0,1] \to \Lambda_0$ *be an integral curve through* $\tilde{x}_0$ *for the vector field* $J\nabla S$*. Suppose that there exists a neighborhood* $\mathcal{U}_{\tilde{\gamma}} \subset U$ *of* $\tilde{\gamma}([0,1])$ *in* Λ_0 *such that a.e. point in* $\mathcal{U}_{\tilde{\gamma}}$ *is recurrent for* $\tilde{f}_{\varepsilon|\Lambda_0}$*. Let* $\gamma_\varepsilon = k_\varepsilon \circ \tilde{\gamma}$ *be the corresponding curve in* Λ_ε*.*

There exists $\varepsilon_1 > 0$ *sufficiently small, and a constant* $K > 0$*, such that for every* $\varepsilon \in (-\varepsilon_1, \varepsilon_1)$*,* $\varepsilon \neq 0$*, and every* $\delta > 0$*, there exists an orbit* $\{z_i\}_{i=0,\dots,n}$ *of* f_ε *in* M*, with* $n = O((\varepsilon)^{-1})$*, such that for all* $i = 0, \dots, n-1$*,*

$$z_{i+1} = f_\varepsilon^{k_i}(z_i), \quad \text{for some } k_i > 0,$$

and for all $i = 0, \dots, n$*, we have*

$$d(z_i, \gamma_\varepsilon(t_i)) < \delta + K\varepsilon, \ \text{for } t_i = i \cdot \varepsilon,$$

In applications, it is often the case that $\Lambda_0 = B^d \times \mathbb{T}^d$, and we have a system of action-angle coordinates (I, ϕ) on Λ_0 with $I \in B^d$ and $\phi \in \mathbb{T}^d$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and $B^d \subseteq \mathbb{R}^d$ is a disk in $\mathbb{R}^d$ or $B^d = \mathbb{R}^d$. Since one can typically find a scattering path for which the action variable changes by some positive distance independent of ε, implicitly one can find a true orbit for which the action variable changes by $O(1)$; this is stated precisely in the following corollary.

There exists a sufficiently small neighborhood V_{Λ_ε} of Λ_ε in M such that for every point $z \in V_{\Lambda_\varepsilon}$ there exists a unique point $z' \in \Lambda_\varepsilon$ which is the closest point to z. The point z' is the image of some unique point $\tilde{z} \in \Lambda_0$ via k_ε, i.e., $z' = k_\varepsilon(\tilde{z})$. We denote by $I(z)$ the I-coordinate of the corresponding point $\tilde{z} \in \Lambda_0$, i.e., $I(z) := I(\tilde{z})$.

Corollary 4. *Assume that a scattering map σ_ε as in Theorem 3 is given. If $J\nabla S$ is transverse to some level set $\{I = I_*\}$ in Λ_0 at some point $(I_*, \phi_*) \subset U$, then there exist $0 < \varepsilon_1 < \varepsilon_0$ and $\rho > 0$, such that for every $0 < \varepsilon < \varepsilon_1$ there exists an orbit $\{z_i\}_{i=0,\dots,n}$ of f_ε, such that*

$$\|I(z_n) - I(z_0)\| > \rho.$$

This Corollary gives diffusion but we don't have control on the "size" (the constant $\rho > 0$) of this diffusion. Next proposition (and corollary) give big changes in action assuming we have more than one sccatering map satisfying generic hypothesis of transversality.

Proposition 5. *Let be $\mathcal{U}$ a connected, relatively compact, open subset of Λ_ε with the property that almost every point $x \in \mathcal{U}$ is recurrent for the restriction $(f_\varepsilon)_{|\Lambda_\varepsilon}$. Assume that the vector fields $X^l = X_{S^l} := J\nabla S^l$, $l = 1, \dots, L$, satisfy the Chow-Hörmander condition $\forall x \in \mathcal{U}$, and consider p and q two arbitrary points in $\mathcal{U}$.*

Then, for every $\delta > 0$ there exists $\varepsilon_0 > 0$ s.t. $\forall\, 0 < |\varepsilon| < \varepsilon_0$, there exists an orbit z_i, $i = 0, \dots m$ with $z_{i+1} = f_\varepsilon^{k_i}(z_i)$ for some $k_i > 0$, $i = 0, \dots, m-1$, s.t. $|z_0 - p| \le \delta$, $\quad |z_m - q| \le \delta$

Moreover, given $\eta : [0,1] \to \mathcal{U}$ a smooth path in $\mathcal{U} \subseteq \Lambda_\varepsilon$ such that the δ-neighborhood of η is contained in $\mathcal{U}$, and choosing $0 = t_0 < t_1 < \dots < t_m = 1$ such that $d(\eta(t_i), \eta(t_{i+1})) < \delta$, $i = 0, \dots, m-1$, then there exists $\varepsilon_0 > 0$ s.t. $\forall\, 0 < |\varepsilon| < \varepsilon_0$, $\exists \{z_i\}_{i=0,\dots m}$ with $z_{i+1} = f_\varepsilon^{k_i}(z_i)$ for some $k_i > 0$, $i = 0, \dots, m-1$ such that
$d(z_i, \eta(t_i)) < \delta$ for all $i = 0, \dots, m$

Corollary 6. *Let $\eta : [0,1] \to \Lambda_\varepsilon$ be a smooth path in Λ_ε. Then we have the following dichotomy:*

- *Either there exists a neighborhood $\mathcal{V}$ of η in Λ_ε with the property that the measure of $cl\left(\bigcup_{i\in\mathbb{Z}} f_\varepsilon^i(\mathcal{V})\right)$ is finite, in which case for every $\delta > 0$ sufficiently small there is an orbit of f_ε which is δ-shadowing η, for all $0 < |\varepsilon| < \varepsilon_0$, for some ε_0. — In this case we have controllability of the dynamics.*
- *Or, there is no neighborhood $\mathcal{V}$ of η as above, in which case there exist orbits that start arbitrarily close to η and travel under $(f_\varepsilon)_{|\Lambda_\varepsilon}$ some distance $O(1)$ – independent of ε – away from η. — In this case we have escape orbits under $(f_\varepsilon)_{|\Lambda_\varepsilon}$.*

We end this extended abstract recalling that all the conditions required in the theorem are satisfied by generic perturbations of a priori-unstable Hamiltonian systems in any dimension.

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