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Toric Geometry

Organized by
Jürgen Hausen, Tübingen
Diane Maclagan, Coventry
Hal Schenck, Ames

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Abstract. Toric geometry is a subfield of algebraic geometry with rich interactions with geometric combinatorics, and many other fields of mathematics. This workshop brought together a broad range of mathematicians interested in toric matters, and their generalizations and applications.

Mathematics Subject Classification (2010): 14M25.

Introduction by the Organizers

The 2019 Oberwolfach meeting on "Toric Geometry" brought together a broad range of mathematicians interested in toric geometry for a productive exchange of ideas.

A toric variety X is a partial compactification of an algebraic torus T with an action of T on X extending the action of T on itself. In the normal case there is a combinatorial dictionary between the algebraic geometric properties of the variety X and the combinatorics of an associated polyhedral-geometric object, which allows geometric examples and counterexamples to be constructed using geometric combinatorics. Toric varieties also arise as good choices for ambient spaces for interesting varieties (generalizing projective space), and as degenerations of more general varieties. They are now part of the tool kit of most practicing algebraic geometers.

For this reason, experts in toric geometry now work in a diverse range of subareas of algebraic geometry and related fields. An important function of this workshop was to bring a representative selection of these experts, who would not normally attend the same conferences, together, to exchange ideas, techniques, and problems.

Talks at the workshop included extensions of the toric dictionary, applications of toric ideas in related fields, and extensions of the toric dictionary to related settings. Talks in the first category include work of Christine Berkesch, who described explicit constructions of resolutions by vector bundles of sheaves on toric varieties, work of Lev Borisov on when strong exceptional collections of line bundles are full, work of Dave Anderson on operational K-theory of toric varieties, work of Eleonore Faber on non-commutative resolutions of toric varieties, and work of Hiroshi Sato on further developing the toric Mori theory dictionary. The use of toric varieties in broader programmes arose in the talk of Nathan Ilten on Fano schemes for complete intersections in toric varieties, in the talk of Andrea Petracci, on smoothing toric Fano 3-folds, motivated by the classification of Fano manifolds, in the joint talk of Fatemeh Mohammadi, Megumi Harada, and Lara Bossinger on different approaches to toric degenerations of Grassmannians, and in the talk of Zhuang He on the Mori dream space property for combinatorially defined varieties.

An emerging theme over past decade has been the use of toric ideas in related areas. Talks in this direction at the workshop include from Kiumars Kaveh, and from Milena Hering, on toric vector bundles, from Milena Wrobel on using toric techniques to study Fano varieties, from Jarek Wiśniewski on toric techniques in the study of complex contact manifolds, and from Joachim Jelisiejew on generalizations of the Białynicki-Birula decomposition. These were augmented by tropically inspired talks from Oliver Lorscheid, introducing blue schemes, and Konstanze Rietsch on a tropical aspect of mirror symmetry. Finally, an important aspect of toric geometry is the interactions with polyhedral combinatorics. Talks in this vein include from Karim Adiprasito on Lefschetz theorems in rings inspired by the Chow ring of a toric variety, from Mateusz Michalek on a class of polytopes independently discovered in several different areas outside mathematics, and from Sam Payne on triangulations of simplices with vanishing local h-polynomial.

On Tuesday evening we had a lively session of five minute talks, beginning with the junior participants, and followed by some of the most senior mathematicians in attendance. Some of these talks were expanded into full length talks later in the week by popular demand (and indeed, many more would have been if space had allowed, while still preserving the special Oberwolfach tradition of allowing enough time for collaboration). Five minute speakers, with approximate titles, were:

(1) Francesco Galuppi - Tensors
(2) Sara Lamboglia - About tropical Fano Schemes
(3) Lara Bossinger - Universal coefficients for cluster algebras
(4) Leonid Monin - Cohomology ring of $\mathrm{Bl}_1\mathbb{P}^2$
(5) Zhuang He - The Mori dream space property of $\mathrm{Bl}_{\mathrm{pts}}\mathbb{P}^n$
(6) Andriy Regeta - Automorphism groups of toric varieties
(7) Weikun Wang - DT invariants on local Hirzebruch orbifolds

(8) Ivan Arzhantsev - Homogeneous toric varieties
(9) Bernard Teisser - Using toric degenerations to avoid wild ramifications
(10) Christian Haase - News on reflexive dimensions
(11) Jarosław Wiśniewski - Grids: toric methods beyond the toric set up
(12) Victor Batyrev - Rationality criterion / Conjecture

Non-mathematical highlights include the rather damp Wednesday hike (for which the rain did not deter a majority of the participants), and a inclusive musical evening on Thursday night. We are grateful to Oberwolfach for hosting this workshop, and providing such excellent working conditions.

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Workshop: Toric Geometry

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Abstracts

Fano Schemes for Complete Intersections in Projective Toric Varieties

Nathan Ilten
(joint work with Tyler Kelly)

1. Introduction

How many lines does a smooth cubic surface $X \subset \mathbb{P}^3$ contain? Over an algebraically closed field $\mathbb{K}$ of characteristic zero, the famous Cayley-Salmon theorem tells us that the answer is 27. Here is another question of similar flavor: consider a general divisor $X \subset \mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(3,3)$. After embedding $\mathbb{P}^2 \times \mathbb{P}^2$ in $\mathbb{P}^8$ via the Segre embedding, how many lines are contained in X?

To answer this question, we consider the general framework of a complete intersection X in a projective toric variety $Y \subset \mathbb{P}^n$, and study the Fano scheme $\mathbf{F}_k(X)$ parametrizing k-dimensional linear subspaces of $\mathbb{P}^n$ contained in X. The case when $Y = \mathbb{P}^n$ is classical, and has been studied by a variety of authors, see e.g. [1, 2, 5].

2. Approach

Our approach is to split up $\mathbf{F}_k(X)$ into pieces by intersecting with the irreducible components of the Fano scheme $\mathbf{F}_k(Y)$ for the toric variety Y. These irreducible components have been described by Ilten and Zotine [4], see also [3]. We briefly summarize the relevant results.

Given a finite subset $\mathcal{A}$ of $\mathbb{Z}^n$, there is an associated projective toric variety $Y_\mathcal{A} = \mathrm{Proj}\mathbb{K}[S_\mathcal{A}]$, where $S_\mathcal{A}$ is the semigroup generated by $\mathcal{A} \times 1 \subset \mathbb{Z}^n \times \mathbb{Z}$. A *Cayley structure* on $\mathcal{A}$ of length ℓ is a surjective affine-linear map π from a face τ of $\mathcal{A}$ to the set Δ_ℓ of vertices of the standard ℓ-simplex. Here, a face of $\mathcal{A}$ is the intersection of $\mathcal{A}$ with a face of its convex hull. There is a natural partial order on the set of Cayley structures on $\mathcal{A}$, see [4, §3]. We identify any two Cayley structures obtained by permuting the elements of Δ_ℓ.

Example. The set $\mathcal{A} = \{e_1, e_2, e_3\} \times \{e_1, e_2, e_3\} \subset \mathbb{Z}^3 \times \mathbb{Z}^3$ gives the toric variety $Y_\mathcal{A} = \mathbb{P}^2 \times \mathbb{P}^2$ embedded in $\mathbb{P}^8$ via the Segre embedding. The two maximal Cayley structures on $\mathcal{A}$ are obtained by projecting to either the first or second $\mathbb{Z}^3$-factor.

Theorem 1 ([4]). *There is a bijection between maximal Cayley structures on $\mathcal{A}$ of length $\ell \geq k$ and irreducible components $Z_{\pi,k}$ of $\mathbf{F}_k(Y_\mathcal{A})$.*

Given some subvariety X of $Y_\mathcal{A}$ and maximal Cayley structure $\pi : \tau \to \Delta_\ell$, we set $V_{\pi,k} = \mathbf{F}_k(X) \cap Z_{\pi,k}$. By studying the $V_{\pi,k}$, we thus gain information about $\mathbf{F}_k(X)$.

3. Results

Although our results also apply to complete intersections $X \subset Y_{\mathcal{A}}$, we make the simplying assumption here that $X \subset Y_{\mathcal{A}}$ is a nef hypersurface of class $\alpha \in \mathrm{Pic}(Y_{\mathcal{A}})$. We set $\delta = \deg \alpha_{|L}$ for L any linear space corresponding to a point of $Z_{\pi,k}$; this may be calculated combinatorially as the volume of a certain polytope. The *expected dimension* of $V_{\pi,k}$ is

$$\phi(\mathcal{A},\pi,\alpha,k) = \dim\tau - \ell + (k+1)(\ell-k) - \binom{k+\delta}{k}.$$

Theorem 2. *If $V_{\pi,k} \neq \emptyset$, then $\dim V_{\pi,k} = \phi(\mathcal{A},\pi,\alpha,k)$. If furthermore X is general, then equality holds.*

When we assume that $Y_{\mathcal{A}}$ is smooth even more can be said:

Theorem 3. *Assume that $Y_{\mathcal{A}}$ is smooth. Then $\mathbf{F}_k(X)$ is the disjoint union of the $V_{\pi,k}$. If X is general, then $\mathbf{F}_k(X)$ is smooth.*

Theorem 4. *Assume that $Y_{\mathcal{A}}$ is smooth, $\alpha - [P]$ is nef for all prime torus invariant divisors P, $\phi \geq 0$, $\delta \geq 3$, and $\dim\tau \geq 2k+1$. Then $V_{\pi,k}$ is non-empty.*

We also note that if $Y_{\mathcal{A}}$ is smooth, then $V_{\pi,k}$ is the zero locus of a section of the vector bundle $\mathrm{Sym}^{\delta}\mathbb{S}^*$, where $\mathbb{S}$ is the tautological subbundle on a relative Grassmannian $\mathrm{Gr}_Z(k+1,\mathcal{E})$ for a split vector bundle $\mathcal{E}$ over a toric variety Z. The variety Z and bundle $\mathcal{E}$ may be determined explicitly from $\mathcal{A}$ and the Cayley structure π. This allows one to compute the degree of $V_{\pi,k}$ by computing the degree of an explicitly determined Chern class.

Example. We continue the example of a general $\alpha = (3,3)$ hypersurface X in $\mathbb{P}^2 \times \mathbb{P}^2$. For either one of the maximal Cayley structures, $\delta = 3$ and we have

$$\phi(\mathcal{A},\pi,\alpha,1) = 4 - 2 + 2\cdot 1 - 4 = 0$$

so $V_{\pi,1}$ is non-empty and smooth of dimension zero. It turns out to be the zero locus of $\mathrm{Sym}^3\mathbb{S}^*$ for $\mathbb{S}$ the tautological bundle on $\mathrm{Gr}_{\mathbb{P}^2}(2,\mathcal{O}(-1)\oplus\mathcal{O}(-1)\oplus\mathcal{O}(-1))$. The top Chern class has degree 189, and we conclude that X contains exactly $378 = 2\cdot 189$ lines.

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Constructions of virtual resolutions

Christine Berkesch

Let X be a smooth toric variety over a field $\Bbbk$ with $\mathrm{Pic}(X)$-graded Cox ring S and irrelevant ideal B. For example, when X is a product of projective spaces $\mathbb{P}^{\boldsymbol{n}} := \mathbb{P}^{\boldsymbol{n}_1} \times \mathbb{P}^{\boldsymbol{n}_2} \times \cdots \times \mathbb{P}^{\boldsymbol{n}_r}$, then its Cox ring is $\Bbbk[x_{i,j} \mid 1 \le i \le r, 0 \le j \le n_i]$ and its irrelevant ideal is $\bigcap_{i=1}^{r} \langle x_{i,0}, x_{i,1}, \ldots, x_{i,n_i} \rangle$; in this case, we identify $\mathbb{Z}^r \cong \mathrm{Pic}(\mathbb{P}^{\boldsymbol{n}})$ via the standard basis and write $\boldsymbol{n} := (\boldsymbol{n}_1, \ldots, \boldsymbol{n}_r) \in \mathbb{Z}^r$, so $\mathrm{Cox}(\mathbb{P}^{\boldsymbol{n}})$ has a natural $\mathbb{Z}^r$-multigrading.

Definition 1. A free complex $F := [F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots]$ of $\mathrm{Pic}(X)$-graded S-modules is called a *virtual resolution* of a $\mathrm{Pic}(X)$-graded S-module M if the corresponding complex $\widetilde{F}$ of vector bundles on X is a locally free resolution of the sheaf $\widetilde{M}$.

In other words, a virtual resolution is a free complex of S-modules whose higher homology groups are supported on the irrelevant ideal of X.

Our results in [2] for products of projective spaces provide strong evidence that, when seeking to understand geometric properties through a homological lens, virtual resolutions for smooth toric varieties provide are the proper analogue of minimal free resolutions for projective space. Here we survey a number of constructions of virtual resolutions.

Resolution of the diagonal. The first construction arises in the proof of a Hilbert Syzygy type theorem for virtual resolutions over a product of projective spaces.

Theorem 2. [2] *If M is a B-saturated finitely generated graded module over* $\mathrm{Cox}(\mathbb{P}^{\boldsymbol{n}})$*, then there is a virtual resolution of M of length at most* $\dim(\mathbb{P}^{\boldsymbol{n}}) = |\boldsymbol{n}|$.

This proof is based on a minor variation of Beilinson's resolution of the diagonal, which uses $\Omega^{\boldsymbol{a}}_{\mathbb{P}^{\boldsymbol{n}}} := \Omega^{a_1}_{\mathbb{P}^{n_1}} \boxtimes \Omega^{a_2}_{\mathbb{P}^{n_2}} \boxtimes \cdots \boxtimes \Omega^{a_r}_{\mathbb{P}^{n_r}}$, the external tensor product of the exterior powers of the cotangent bundles on the factors of $\mathbb{P}^{\boldsymbol{n}}$. Choose $\boldsymbol{d} \in \mathbb{Z}^r$ such that for any $\boldsymbol{u} \in \mathbb{Z}^r$, $\Omega^{\boldsymbol{u}}_{\mathbb{P}^{\boldsymbol{n}}}(\boldsymbol{u}+\boldsymbol{d}) \otimes \widetilde{M}$ has no higher cohomology; such a $\boldsymbol{d}$ exists by the Fujita Vanishing Theorem [4, Theorem 1]. Then our proof produces a virtual resolution of M, in which the i-th module is

$$\bigoplus_{\substack{\boldsymbol{0} \le \boldsymbol{u} \le \boldsymbol{n} \\ |\boldsymbol{u}| = i}} S(-\boldsymbol{u}) \otimes_{\Bbbk} H^q\big(\mathbb{P}^{\boldsymbol{n}}, \Omega^{\boldsymbol{u}}_{\mathbb{P}^{\boldsymbol{n}}} \otimes \widetilde{M}(\boldsymbol{u}+\boldsymbol{d})\big)\,.$$

While such a virtual resolution has the benefit of being somewhat short in homological length, it also has a large number of multidegrees appearing in each homological degree.

Monomial ideals. For this construction, we return to working over an arbitrary smooth complete toric variety X, and again observe that there is a Hilbert Syzygy type result in the monomial ideal case.

Theorem 3. [9] *If $I \neq S$ is a B-saturated monomial ideal, then there is a virtual resolution of S/I of length at most* $\dim(X)$.

In fact, the construction in the proof of this theorem produces a monomial ideal J in S with $I = J : B^\infty$, so that the projective dimension of S/J is at most $\dim(X)$. As such, the free resolution of S/J is a virtual resolution of S/I with the desired short length.

Returning to products of projective spaces, a recent result provides a case in which a squarefree monomial ideal I is "virtually Cohen–Macaulay."

Theorem 4 ([5])**.** *Let Δ be a simplicial complex that is pure and balanced, meaning that each of the vertices of every facet correspond to a different projective space in $\mathbb{P}^{\boldsymbol{n}}$. If I_Δ is a Stanley–Riesner squarefree monomial ideal in $S = \operatorname{Cox}(\mathbb{P}^{\boldsymbol{n}})$, then there is a virtual resolution of S/I_Δ of length equal to the codimension of S/I_Δ.*

As in the previous result, the virtual resolution constructed in this proof is again a free resolution. This time, it resolves another squarefree monomial ideal $I_{\Delta'}$, where $\Delta' \supseteq \Delta$ is a pure, balanced simplicial complex that is shown to be shellable, and hence Cohen–Macaulay.

Virtual resolution of a pair. The definition of multigraded regularity, as defined in [7], provides an invariant that detects several virtual resolutions that are subcomplexes of a free resolution.

Theorem 5 ([2])**.** *Let $X = \mathbb{P}^{\boldsymbol{n}}$, fix a degree $\boldsymbol{d} \in \mathbb{Z}^r$, and let M be a B-saturated graded finitely generated S-module that is $\boldsymbol{d}$-regular. If $C(M, \boldsymbol{d}+\boldsymbol{n})$ is the subcomplex of a minimal free resolution of M consisting of all free summands of degree at most $\boldsymbol{d}+\boldsymbol{n}$, then $C(M, \boldsymbol{d}+\boldsymbol{n})$ is a virtual resolution for M.*

The complex $C(M, \boldsymbol{d}+\boldsymbol{n})$ is called the *virtual resolution of the pair* $(M, \boldsymbol{d}+\boldsymbol{n})$. Note that it can be computed without first computing an entire minimal free resolution of M; simply dispose of generators not within the degree bound at each homological degree.

Mapping cone construction. Again working over an arbitrary smooth complete toric variety X, let M be a finitely generated graded S-module, and suppose that $F\colon F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_t} F_t \leftarrow 0$ is a virtual resolution of M and $\operatorname{Ext}^t(M,S)^\sim = 0$. If G^* is a free resolution of φ_t^*, that is shifted with indexing reversed, as in the diagram below, then there is an induced map α^* to G^* from $\operatorname{Hom}_S(F,S)$, i.e., from the projective to the acyclic complex.

$$\begin{array}{ccccccccccccccc}
\cdots \longrightarrow & 0 & \longrightarrow & F_0^* & \xrightarrow{\varphi_1^*} & F_1^* & \xrightarrow{\varphi_2^*} & \cdots & \xrightarrow{\varphi_{t-2}^*} & F_{t-2}^* & \xrightarrow{\varphi_{t-1}^*} & F_{t-1}^* & \xrightarrow{\varphi_t^*} & F_t^* & \longrightarrow 0 \\
 & \downarrow{\scriptstyle \alpha_0^*} & & \downarrow{\scriptstyle \alpha_1^*} & & \downarrow{\scriptstyle \alpha_2^*} & & & & \downarrow{\scriptstyle \alpha_{t-2}^*} & & \downarrow{\scriptstyle \alpha_{t-1}^*} & & \downarrow{\scriptstyle \alpha_t^*} & \\
\cdots \longrightarrow & G_{-1}^* & \xrightarrow[\psi_0^*]{} & G_0^* & \xrightarrow[\psi_1^*]{} & G_1^* & \xrightarrow[\psi_2^*]{} & \cdots & \xrightarrow[\psi_{t-2}^*]{} & G_{t-2}^* & \xrightarrow[\psi_{t-1}^*]{} & G_{t-1}^* & \xrightarrow[\psi_t^*=\varphi_t^*]{} & G_t^* & \longrightarrow 0
\end{array}$$

Dualize this diagram to get:

$$\begin{array}{ccccccccccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & F_0 & \xleftarrow{\varphi_1} & F_1 & \xleftarrow{\varphi_2} & \cdots & \xleftarrow{\varphi_{t-2}} & F_{t-2} & \xleftarrow{\varphi_{t-1}} & F_{t-1} & \xleftarrow{\varphi_t} & F_t & \longleftarrow & 0 \\
 & & \uparrow{\scriptstyle \alpha_{-1}} & & \uparrow{\scriptstyle \alpha_0} & & \uparrow{\scriptstyle \alpha_1} & & & & \uparrow{\scriptstyle \alpha_{t-2}} & & \uparrow{\scriptstyle \alpha_{t-1}} & & \uparrow{\scriptstyle \alpha_t} & & \\
\cdots & \xleftarrow[\psi_{-1}]{} & G_{-1} & \xleftarrow[\psi_0]{} & G_0 & \xleftarrow[\psi_1]{} & G_1 & \xleftarrow[\psi_2]{} & \cdots & \xleftarrow[\psi_{t-2}]{} & G_{t-2} & \xleftarrow[\psi_{t-1}]{} & G_{t-1} & \xleftarrow[\psi_t=\varphi_t]{} & G_t & \longleftarrow & 0
\end{array}$$

When $G_{-2} = 0$, then the mapping cone $\operatorname{cone}(\alpha)$ can be partially minimized to the following virtual resolution of M, which is shorter than F:

$$\begin{array}{c} F_0 \\ \oplus \\ G_{-1} \end{array} \xleftarrow{\partial_1} \begin{array}{c} F_1 \\ \oplus \\ G_0 \end{array} \xleftarrow{\partial_2} \begin{array}{c} F_2 \\ \oplus \\ G_1 \end{array} \xleftarrow{\partial_3} \cdots \leftarrow \begin{array}{c} F_{t-2} \\ \oplus \\ G_{t-3} \end{array} \leftarrow G_{t-2} \leftarrow 0.$$

What makes a complex virtual. We conclude with a result that generalizes to virtual resolutions the well-known exactness criterion of Buchsbaum and Eisenbud.

Theorem 6 ([6]). *A graded free chain complex* $F\colon F_0 \xleftarrow{\varphi_1} F_1 \xleftarrow{\varphi_2} \cdots \xleftarrow{\varphi_t} F_t \leftarrow 0$ *over the Cox ring of a smooth complete toric variety is a virtual resolution if and only if both of the following conditions are satisfied:*

(a) $\operatorname{rank}(\varphi_i) + \operatorname{rank}(\varphi_{i+1}) = \operatorname{rank}(F_i)$ *for each* $i = 1, 2, \ldots, t$, *and*
(b) $\operatorname{depth}(I(\varphi_i) : B^\infty) \geq i$.

A number of theorems here have been implemented in the `VirtualResolutions` package of Macaulay2 [8]. See [1] for more details.

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Toric bundles

Kiumars Kaveh
(joint work with Christopher Manon)

The talk covered recent results joint with Christopher Manon (University of Kentucky) regarding classification of torus equivariant bundles on toric varieties (toric bundles for short). It was based on the two preprints [6, 7] which make direct connection between study of toric bundles and theory of buildings on one hand and tropical geometry and valuations on the other hand. One of the goals of the talk was to illustrate that *doing tropical geometry over the semifield of piecewise linear functions is directly related to study of toric bundles.*

For simplicity we work over $\mathbb{C}$ but many of the results hold over other fields as well. Line bundles on toric varieties are quite well-studied and (toric) line bundles on toric varieties can be classified with piecewise linear functions on the fan Σ of the toric variety. This has served as a central idea in many breakthrough results in combinatorial algebraic geometry in recent decades. On the other hand vector bundles and more general bundles (such as principal bundles) on toric varieties are far less understood. There are nice classifications of toric vector bundles and more general sheaves on toric varieties (see for example [5, 8, 9]). Nevertheless many questions on toric vector bundles remain open, partly because, in general, the combinatorics involved is much more complicated to keep track of compared to that of line bundles.

We denote by N the lattice $\mathbb{Z}^n$ with dual lattice $M = N^*$ and $T = T_N$ the torus with N as its cocharacter lattice. Σ is a fan in $N \otimes \mathbb{Q} = \mathbb{Q}^n$ with $|\Sigma| \subset N \otimes \mathbb{Q}$ the union of all cones in Σ.

A *toric vector bundle* $\mathcal{E}$ on X_Σ is a vector bundle on X_Σ with a linear action of T lifting that of X_Σ. Extending the line bundle case, Klyachko [8] gave a classification of toric vector bundles. The Klyachko data of a toric vector bundle can be described as follows. Let $E \cong \mathbb{C}^r$ be an r-dimensional vector space, it represents the fiber of $\mathcal{E}$ over a point x_0 in the open orbit in X_Σ. To each ray $\rho \in \Sigma(1)$ one associates a decreasing filtration of E by vector subspaces:

$$(E_i^\rho)_{i\in\mathbb{Z}} = (\cdots \supset E_{-1}^\rho \supset E_0^\rho \supset E_1^\rho \supset \cdots)$$

Moreover, each cone $\sigma \in \Sigma$ imposes a certain *compatibility condition* between the filtrations $(E_i^\rho)_{i\in\mathbb{Z}}$ for $\rho \in \sigma$. This is analogue of the *linearity* condition of a piecewise linear function on cones of Σ.

Let G be a linear algebraic group. We denote by $\mathbf{B}(G)$ the underlying space of its (spherical Tits) building. It can be regarded as the infinite union of $\mathbb{Q}$-span of cocharacter lattices of all maximal tori in G glued together according to intersection of maximal tori in G.

A *toric principal G-bundle* $\mathcal{P}$ is a principal G-bundle over a toric variety X_Σ with an action of T lifting that of X_Σ and commuting with the G-action. A *frame* on $\mathcal{P}$ is a G-isomorphism between the fiber $\mathcal{P}_{x_0}$ and G.

Definition 1 (Piecewise linear map to a building)**.** We say that a function $\Phi : |\Sigma| \to \mathbf{B}(G)$ is a *piecewise linear map with respect to* Σ if the following hold:

(1) For each cone $\sigma \in \Sigma$, the image $\Phi(\sigma)$ lies in an apartment $A_\sigma = \check{\Lambda}(H_\sigma) \otimes \mathbb{Q}$. Here $H_\sigma \subset G$ is the corresponding maximal torus with cocharacter lattice $\check{\Lambda}(H_\sigma)$.

(2) For each $\sigma \in \Sigma$, the restriction $\Phi_{|\sigma} : \sigma \to A_\sigma$ is a $\mathbb{Q}$-linear map.

We call Φ an *integral* piecewise linear map if for each $\sigma \in \Sigma$, Φ_σ sends $\sigma \cap \mathbb{Z}^n$ to the lattice $\check{\Lambda}(H_\sigma)$.

The next theorem gives a classification of toric principal G-bundles, far extending the classification of toric line bundles by piecewise linear functions (see [6, Section 2.1]).

Theorem 2 (Classification of toric principal bundles)**.** *There is a one-to-one correspondence between framed toric principal G-bundles $\mathcal{P}$ over X_Σ and integral piecewise linear maps $\Phi : |\Sigma| \to \mathbf{B}(G)$. Moreover this extends to an equivalence of categories.*

In the case of toric vector bundles, the piecewise linear map $\Phi : |\Sigma| \to \mathbf{B}(\mathrm{GL}(E))$ encodes the same information as the data of Klyachko filtrations of $\mathcal{E}$.

We should also mention papers [1, 2, 3] that give other classifications of toric principal bundles.

Next we sketch how toric vector bundles can be viewed as tropical points. As far as we know this connection with tropical geometry has not been known before. Recall that a *semifield* $\mathcal{O}$ is a set equipped with addition and multiplication operations which work in the same way as the corresponding operations in a field with the exception that there may not be additive inverses. An important example of a semifield is $\overline{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ equipped with operations of min (in place of addition) and + (in place of multiplication). Tropical geometry is sometimes defined as algebraic geometry over the semifield $(\overline{\mathbb{Q}}, \min, +)$. Extending usual tropical geometry, one can define the notion of tropical variety $\mathrm{Trop}_{\mathcal{O}}(I)$ where I is an ideal in a polynomial ring and $\mathcal{O}$ is an (idempotent) semifield (see [4]).

We are interested in the semifield $\mathcal{O}_N$ of integral piecewise linear functions on a lattice $N \cong \mathbb{Z}^n$. Addition and multiplication in $\mathcal{O}_N$ are pointwise min and + of functions, and we let ∞ be a formal additive identity.

Let E be a finite dimensional $\mathbb{C}$-vector space. Let $E \hookrightarrow \mathbb{C}^n$ be the linear embedding corresponding to a choice of a spanning set $\mathcal{B} \subset E^*$ and let $I \subset \mathbb{C}[x_1, \ldots, x_n]$ be the associated linear ideal.

Theorem 3 (Toric vector bundles as tropical points)**.** *Given a point $\bar{\phi} \in \mathrm{Trop}_{\mathcal{O}_N}(I)$ we can construct a complete polyhedral fan $\Sigma \subset \mathbb{N}$ and a toric vector bundle $\mathcal{E}(\bar{\phi})$ on X_Σ with general fiber E. Conversely, for every toric vector bundle $\mathcal{E}$ with general fiber E over a complete toric variety X_Σ, we can find $\mathcal{B} \subset E^*$ such that $\mathcal{E} = \mathcal{E}(\bar{\phi})$ for a unique point $\bar{\phi} \in \mathrm{Trop}_{\mathcal{O}_N}(I)$.*

Alternatively a toric bundle can be encoded by data of a *valuation.* In commutative algebra one often considers valuations with values in a totally ordered group. We are interested in valuations with values in a semifield, and more specifically the semialgebra $\mathcal{O}_N$ of piecewise linear functions. Let A be a $\mathbb{C}$-algebra. A *valuation* $\mathfrak{v} : A \to \mathcal{O}_N$ is a function which satisfies the following properties for any $e, f \in A$ and $C \in \mathbb{C} \setminus \{0\}$.

(1) $\mathfrak{v}(ef) = \mathfrak{v}(e) + \mathfrak{v}(f)$,
(2) $\mathfrak{v}(e + f) \geq \min(\mathfrak{v}(e), \mathfrak{v}(f))$,
(3) $\mathfrak{v}(Ce) = \mathfrak{v}(e)$,
(4) $\mathfrak{v}(e) = \infty \iff e = 0$.

Here $\geq$ means that the inequality holds pointwise, i.e. $\phi(x) \geq \psi(x)$, $\forall x \in N$. Analogously we have the notion of a prevaluation on a vector space. Let E be a finite dimensional $\mathbb{C}$-vector space. A *prevaluation* $\mathfrak{v} : E \to \mathcal{O}_N$ is a function satisfying (2)-(4). We also require $\mathfrak{v}(E)$ to a finite set. The following gives another classification of toric vector bundles in terms of prevaluations (see [7, Section 3]).

Theorem 4 (Toric vector bundles as prevaluations)**.** *T_N-toric vector bundles $\mathcal{E}$ (up to pull back via toric morphisms) are in one-to-one correspondence with prevaluations $\mathfrak{v}$ on finite dimensional vector spaces E with values in $\mathcal{O}_N$.*

Finally, far extending the above we have the following classification of torus equivariant families (see [7, Section 5]).

Theorem 5 (Classification of toric families)**.** *Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded $\mathbb{C}$-domain. Flat T_N-equivariant families $\mathfrak{X}$ over a T_N-toric variety and with reduced, irreducible fibers and generic fiber isomorphic to* Spec(A) *are in bijection with homogeneous valuations $\mathfrak{v} : A \to \mathcal{O}_N$ which satisfy certain finiteness condition (namely $(A, \mathfrak{v})$ has a finite* Khovanskii basis *for each $\sigma \in \Sigma$).*

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On strong exceptional collections of line bundles of maximal length on Fano toric Deligne-Mumford stacks

Lev Borisov
(joint work with Chengxi Wang)

Let X be a smooth complex projective variety or a Deligne-Mumford stack. An object F of the triangulated category $D(X) = D^b(\mathrm{Coh}/X)$ is called *exceptional* if

$$\mathrm{Hom}(F, F) \cong \mathbb{C},\ \mathrm{Hom}(F, F[k]) \cong 0, k \neq 0.$$

A list of exceptional objects $F_1, \dots, F_n$ is called a strong exceptional collection if the only nonzero homomorphisms between F_i and $F_j[k]$ occur for $i \leq j$ and $k = 0$. If $F_1, \dots, F_n$ generate $D(X)$, then the collection is called a *full strong exceptional* collection. A stereotypical example of such collection is

$$\mathcal{O}(-n), \mathcal{O}(-n+1), ..., \mathcal{O}(-1), \mathcal{O}$$

on $\mathbb{CP}^n$.

It is easy to show that any full strong exceptional collection must give a basis of the Grothendieck group $K_0(X)$ (this implies in particular that this group is finitely generated and free). However, it is a priori possible to have strong exceptional collections which give a basis in $K_0(X)$ but are not full. In fact, such examples would lead to remarkable *phantom categories* [1].

In the joint work with C. Wang [2], we looked at strong exceptional collections of line bundles on Fano toric DM stacks X of Picard rank at most two, where it is known from previous work that full exceptional collections exist. We have shown that in this setting every strong exceptional collection of line bundles of length equal to $\mathrm{rk}\, K_0(X)$ is full.

The method is essentially combinatorial. Here is how it works in the case of the weighted projective space (in the stack sense) $X = \mathbb{CP}(w_0, \dots, w_n)$ for positive integers w_i with no common divisor. The Picard group of X is $\mathbb{Z}$; all line bundles are $\mathcal{O}(k)$ for some integer k.

The rank of $K_0(X)$ is $\sum_i w_i$. The line bundles $\mathcal{O}(k)$ and $\mathcal{O}(l)$ can be a part of a strong exceptional collection if and only if

$$(k-l), (l-k) \notin \sum_{i=0}^{n} \mathbb{Z}_{<0} w_i$$

i.e. difference between integer numbers that encode the collection can not be written as negative integer linear combinations of w_i.

The main idea of the proof is the following. Given a full exceptional collection of line bundles $\mathcal{O}(k)$ of length $\sum_i w_i$, consider the smallest index k among them. We claim that we can replace $\mathcal{O}(k)$ by $\mathcal{O}(k + \sum_i w_i)$ in the following sense.

- New line bundle is not in the collection.
- The replacement leads to another strong exceptional collection.
- New line bundle is in the category generated by the old ones.

This process eventually terminates and leads to a sequence of consecutive $\sum_i w_i$ integers. It is known that the corresponding line bundles generate $D(X)$, which then implies that the original collection does.

The Picard rank two case is similar but more technically challenging.

The talk also discussed possible future directions, such as looking at exceptional rather than strong exceptional collections; higher Picard rank cases of X of dimension two; non-Fano cases; non-line bundles.

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Riemann-Roch theorems in operational K-theory

Dave Anderson

(joint work with Richard Gonzales, Sam Payne)

Let $T \cong (\mathbb{C}^*)^n$ be a torus, with character lattice M and cocharacter lattice $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Let Δ be a (rational polyhedral) fan in the real vector space $N_{\mathbb{R}}$, with corresponding toric variety $X = X(\Delta)$.

When $X = X(\Delta)$ is nonsingular (and complete), there are well-known descriptions of its equivariant cohomology and K-theory rings (e.g., [3, 5, 9]):

$$H_T^* X(\Delta) = PP^* \qquad \text{and} \qquad K_T^\circ(X) = PExp(\Delta),$$

where $PP^*(\Delta)$ and $PExp(\Delta)$ are the rings of *piecewise polynomial* and *piecewise exponential* functions on Δ. These isomorphisms fail, however, in the case when $X(\Delta)$ is singular.

One is led to ask what geometric invariant of $X(\Delta)$ is represented by the combinatorial rings $PP^*(\Delta)$ and $PExp(\Delta)$. For the first, the answer was given in [8]: for any fan Δ, one has

$$PP^*(\Delta) = A_T^* X(\Delta),$$

the equivariant operational Chow cohomology ring. Motivated by this, we constructed an operational K-theory ring and proved an analogous result for exponential functions [1]:

$$PExp(\Delta) = \mathrm{op}K_T^\circ X(\Delta).$$

Operational cohomology theories are part of a general framework of *bivariant theories*, developed by Fulton and MacPherson to unify and extend various types of Riemann-Roch theorems [7]. A bivariant theory associates a group $B(X \to Y)$ to any morphism $X \to Y$ in a given category, along with several operations (modeled on pullback, pushforward, etc.) and subject to compatibility axioms. A natural transformation of bivariant theories, respecting all operations, is called a *Grothendieck transformation*. A main theme of [7] is that Riemann-Roch theorems can be unified under the conceptual framework of Grothendieck transformations.

In this talk, I report on recent work with Gonzales and Payne [2]. In the category of complex varieties with torus action, we construct Grothendieck transformations relating operational K-theory to other bivariant theories, and extract some consequences. First, we have an extension of Grothendieck-Verdier-Riemann-Roch.

Theorem 1. *There are Grothendieck transformations*

$$\mathrm{op}K^\circ_T(X \to Y) \to \mathrm{op}\hat{K}^\circ_T(X \to Y)_{\mathbb{Q}} \xrightarrow{\mathrm{ch}} \hat{A}^*_T(X \to Y)_{\mathbb{Q}},$$

the second of which induces isomorphisms of groups, and both are compatible with the natural restriction maps to T'-equivariant groups, for $T' \subset T$. Furthermore, equivariant lci morphisms have canonical orientations $[f]$, and if f is such a morphism, then

$$\mathrm{ch}([f]_K) = \mathrm{td}(T_f) \cdot [f]_A,$$

where $\mathrm{td}(T_f)$ *is the Todd class of the virtual tangent bundle.*

Here $\mathrm{op}\hat{K}_T$ denotes a certain completion of equivariant K-theory, and likewise for $\hat{A}^*_T$.

Specializing the Riemann-Roch formula to statements for homology and cohomology, we obtain the following.

Corollary. *If $f\colon X \to Y$ is an equivariant lci morphism, then the diagrams*

$$\begin{array}{ccc} \mathrm{op}K^\circ_T(X) & \xrightarrow{\mathrm{ch}} & \hat{A}^*_T(X)_{\mathbb{Q}} \\ {\scriptstyle f_*}\big\downarrow & & \big\downarrow{\scriptstyle f_*(\ \cdot\mathrm{td}(T_f))} \\ \mathrm{op}K^\circ_T(Y) & \xrightarrow{\mathrm{ch}} & \hat{A}^*_T(Y)_{\mathbb{Q}} \end{array} \quad \textit{and} \quad \begin{array}{ccc} K^T_\circ(X) & \xrightarrow{\tau} & \hat{A}^T_*(X)_{\mathbb{Q}} \\ {\scriptstyle f^*}\big\uparrow & & \big\uparrow{\scriptstyle \mathrm{td}(T_f)\cdot f^*} \\ K^T_\circ(Y) & \xrightarrow{\tau} & \hat{A}^T_*(Y)_{\mathbb{Q}} \end{array}$$

commute. For the first diagram, f is assumed proper.

Next, we have an extension of the localization theorem (Lefschetz-Riemann-Roch). Let $S \subseteq R(T) = \mathbb{Z}[M]$ be the multiplicative set generated by $1 - \mathrm{e}^\lambda$ for all nonzero $\lambda \in M$, and let $\bar{S} \subseteq \Lambda_T = \mathrm{Sym}^* M$ be the multiplicative set generated by all nonzero $\lambda \in M$.

Theorem 2. *There are Grothendieck transformations*

$$\mathrm{op}K^\circ_T(X \to Y) \to S^{-1}\mathrm{op}K^\circ_T(X \to Y) \xrightarrow{\mathrm{loc}^K} S^{-1}\mathrm{op}K^\circ_T(X^T \to Y^T) \quad \textit{and}$$

$$A^*_T(X \to Y) \to \bar{S}^{-1}A^*_T(X \to Y) \xrightarrow{\mathrm{loc}^A} \bar{S}^{-1}A^*_T(X^T \to Y^T),$$

inducing isomorphisms of $S^{-1}R(T)$-modules and $\bar{S}^{-1}\Lambda_T$-modules, respectively. Furthermore, if $f\colon X \to Y$ is a flat equivariant map whose restriction to fixed loci $f^T\colon X^T \to Y^T$ is smooth, then there are equivariant multiplicities $\varepsilon^K(f)$ *in $S^{-1}\mathrm{op}K^\circ_T(X^T)$ and $\varepsilon^A(f)$ in $\bar{S}^{-1}A^*_T(X^T)$, so that*

$$\mathrm{loc}^K([f]) = \varepsilon^K(f) \cdot [f^T]$$

and

$$\mathrm{loc}^A([f]) = \varepsilon^A(f) \cdot [f^T].$$

Corollary. *Let $f\colon X \to Y$ be a flat equivariant morphism whose restriction to fixed loci $f^T\colon X^T \to Y^T$ is smooth. Then the diagrams*

$$\begin{CD} \mathrm{op}K_T^\circ(X) @>>> S^{-1}\mathrm{op}K_T^\circ(X^T) \\ @V{f_*}VV @VV{f_*^T(\ \cdot \varepsilon^K(f))}V \\ \mathrm{op}K_T^\circ(Y) @>>> S^{-1}\mathrm{op}K_T^\circ(Y^T) \end{CD} \quad \text{and} \quad \begin{CD} K_\circ^T(X) @>>> S^{-1}K_\circ^T(X^T) \\ @A{f^*}AA @AA{\varepsilon^K(f)\cdot(f^T)^*}A \\ K_\circ^T(Y) @>>> S^{-1}K_\circ^T(Y^T) \end{CD}$$

commute, where f is assumed proper for the first diagram. Under the same conditions, the following diagrams also commute:

$$\begin{CD} A_T^*(X) @>>> \bar{S}^{-1}A_T^*(X^T) \\ @V{f_*}VV @VV{f_*^T(\ \cdot \varepsilon^A(f))}V \\ A_T^*(Y) @>>> \bar{S}^{-1}A_T^*(Y^T) \end{CD} \quad \text{and} \quad \begin{CD} A_*^T(X) @>>> \bar{S}^{-1}A_*^T(X^T) \\ @A{f^*}AA @AA{\varepsilon^A(f)\cdot(f^T)^*}A \\ A_*^T(Y) @>>> \bar{S}^{-1}A_*^T(Y^T). \end{CD}$$

Finally, by exploiting the Riemann-Roch isomorphism $\mathrm{op}\hat{K}_T^\circ(X)_\mathbb{Q} \cong \hat{A}_T^*(X)_\mathbb{Q}$ and appealing to some calculations of Katz and Payne, we obtain an application to the usual K-theory of toric varieties.

Theorem 3. *There are projective toric threefolds X such that the restriction map from the K-theory of T-equivariant vector bundles on X to the ordinary K-theory of vector bundles on X is not surjective.*

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Smoothing toric Fano 3-folds

Andrea Petracci
(joint work with Alessio Corti and Paul Hacking)

There is a well known dictionary for many algebro-geometric properties of toric varieties in terms of combinatorial properties of their associated fans. What can be said about the deformation theory of toric varieties?

K. Altmann has extensively studied the deformation theory of *affine* toric varieties. In particular, he has observed the following [2, 3]: if one has a Minkowski decomposition of lattice polytopes

$$F = F_1 + \cdots + F_k$$

in a lattice $\overline{N}$, then one can construct a deformation of the affine toric variety U_F associated to the cone

$$\mathbb{R}_{\geq 0}(F \times \{1\}) \subseteq \overline{N}_{\mathbb{R}} \oplus \mathbb{R}.$$

In general, different Minkowski decompositions of F will give different deformations of U_F.

The deformation theory of *projective* toric varieties is poorly understood, although the case of smooth varieties has been studied in [7, 8]. The following question is natural, because Fano varieties are the simplest kind of projective varieties.

Question. *Let P be a reflexive polytope and let X_P be the toric Fano variety associated to the face fan of P. What kind of combinatorial datum on P produces deformations or smoothings of X_P?*

If P has dimension 2, then X_P is unobstructed and smoothable [1], therefore the question is void.

Consider the case of dimension 3. First we need the following definition. Define an A-triangle to be either a unitary segment or a triangle which is affinely equivalent to the triangle with vertices $(0,0), (m,0), (0,1)$, for some positive integer m.

Each facet F of P gives a toric affine open subscheme U_F of X_P. Fix a Minkowski decomposition of a facet F of P into A-triangles: the corresponding Altmann deformation of U_F has general fibre with at most cA-singularities.

If we choose a Minkowski decomposition into A-triangles for each facet of P, would it be possible to glue the Altmann deformations of the affine charts U_F's in order to produce a deformation of X_P? The answer is (roughly) yes and the argument uses simultaneous resolutions (following Wahl [14], Kollár–Shepherd-Barron [9] and Matsushita [10]). More precisely one constructs a toric crepant partial resolution $\pi\colon Y \to X_P$ such that Y is unobstructed and smoothable. The construction of Y is related to the Cayley trick and fine mixed subdivisions of facets of P. Since $\mathrm{R}^1\pi_*\mathcal{O}_Y = 0$, *the* general smoothing of Y can be blown down to *a* deformation of X_P.

What can be said about the general fibre X_t of the deformation of X_P constructed above? It could happen that we have obtained the trivial deformation of X_P (see for instance [13]). Problems arise when there is an edge in the dual polytope P° of lattice length 1, or equivalently there is a component of the 1-skeleton of X_P of anticanonical degree 1. For such edge we need to impose an extra condition on Y. This extra condition, which I will name ($\star$), can be phrased in purely combinatorial terms and ensures that the sheaf $\mathcal{E}xt^1(\Omega_{X_P}, \mathcal{O}_{X_P})$ has enough sections; due to lack of space, I will not spell out this condition. In these favourable circumstances (i.e. when the condition ($\star$) is satisfied for every edge of P° of lattice length 1), one can prove that the general fibre X_t of the constructed deformation of X_P has only ordinary double points. Therefore, by Namikawa [11], X_t is smoothable, and consequently X_P is smoothable. To sum up, we have the following theorem.

Theorem 1 ([6]). *Let P be a 3-dimensional reflexive polytope of dimension 3 and let X_P be the toric Fano 3-fold associated to the face fan of P. Fix a Minkowski decomposition of every facet of P into A-triangles in such a way that the condition ($\star$) is satisfied for every unitary edge of the dual polytope of P. Then X_P is smoothable.*

Now some remarks about this theorem:

- One can compute the Hodge numbers of the smoothing of X_P constructed in the theorem.
- The theorem answers the question above only partially. Not only it is valid in dimension 3 only, but it does not say anything for reflexive 3-topes which have at least one facet which is not Minkowski decomposable into A-triangles, e.g. $\mathbb{P}(1, 2, 3, 6)$. Therefore the quest for the combinatorial avatars of deformations/smoothings of X_P is still open.
- The Fano variety X_P has Gorenstein singularities if P is reflexive. What can be said about $\mathbb{Q}$-Gorenstein deformations of toric Fano varieties whose singularities are not Gorenstein?
- The theorem perfectly matches with the Fanosearch programme [4, 5] (see [12] for a survey). Indeed, the choice of a Minkowski decomposition of each facet of P into A-triangles (such that ($\star$) is satisfied for every unitary edge of P°) produces a smoothing V of X_P (by the theorem) and easily induces a Laurent polynomial f in 3 variables which is supported on P. One can see that, in all examples, the Fano manifold V is *mirror* to f; this roughly means that counting rational curves in V is equivalent to taking periods coming from the Hodge theory of the fibration $f\colon \mathbb{G}_\mathrm{m}^3 \to \mathbb{A}^1$. So far we lack a conceptual reason which explains this Mirror Symmetry phenomenon.

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What is a blue scheme?

Oliver Lorscheid

Introduction. The area of $\mathbb{F}_1$-geometry searches for an extension of scheme theory in Grothendieck's sense to make sense of varieties over $\mathbb{F}_1$, the *field with one element*. Blue schemes provide one such approach. Some major applications of this theory are

(1) a formalism of algebraic groups over $\mathbb{F}_1$, which answers a problem of Jacques Tits from the 1950's; cf. [4];
(2) a formulation of tropical scheme theory that subsumes and generalizes previous results by Jeff and Noah Giansiracusa and by Diane Maclagan and Felipe Rincón; cf. [3] and [5];
(3) the construction of the moduli space of matroids, together with applications to the representation theory of matroids; cf. the joint paper [1] with Matthew Baker.

Blue schemes as an extension of toric geometry. In a sense, a blue scheme can be seen as a generalization of a toric variety that reaches toroidal embeddings, logarithmic schemes, $\mathbb{F}_1$-geometry and tropical geometry. In the following, we give a taste of this concept.

A *blueprint* is a (commutative) semiring B^+ (with 0 and 1) together with a multiplicative subset $B^\bullet$ that generates B^+ and that contains 0. A *morphism of* *blueprints* is a homomorphism of semirings that restricts to a map between the respective multiplicative subsets.

The main example of a blueprint that appears in toric geometry is the following: let k be a field and A the intersection of $\mathbb{Z}^n$ with (the dual of) a cone in $\mathbb{R}^n$. We define B^+ as the semigroup algebra $k[A]$ (where the addition of A becomes the multiplication of $k[A]$) and $B^\bullet$ as the subset of all monomial terms $c \cdot a \in k[A]$ where $c \in k$ and $a \in A$, including the zero element $0 \cdot a$ of $k[A]$.

A *prime ideal* of a blueprint $B = (B^\bullet, B^+)$ is a subset $\mathfrak{p}$ of $B^\bullet$ that contains 0 such that $\mathfrak{p} \cdot B^\bullet = \mathfrak{p}$ and such that $S = B^\bullet - \mathfrak{p}$ is a multiplicative subset of $B^\bullet$. It is possible to define localizations of a blueprint B at multiplicative subsets of $B^\bullet$ and to endow the set of all prime ideals of B with a topology and a structure sheaf in the category of blueprints. This provides the *spectrum* Spec$B$ *of* B. By gluing spectra, one finally obtains the notion of a blue scheme.

To continue the example from toric geometry: if $B^+ = k[A]$ and $B^\bullet = \{c \cdot a | c \in k, a \in A\}$, then the prime ideals of B correspond to (the complements of) the faces of the cone spanned by A in $\mathbb{R}^n$ and the topology of Spec$B$ is determined by the inclusion relation of faces. This means that Spec$B$ is homeomorphic to the Kato fan of the affine toric variety defined by $A \subset \mathbb{R}^n$. There is however a subtle difference in the structure sheaves of the Kato fan and the blue scheme, which is important for the mentioned extensions of blue schemes beyond toric geometry.

A refinement of toric sheaf cohomology. In the joint work [2] with Jaret Flores and Matt Szczesny, we have defined sheaf cohomology for blue schemes provided the structure sheaf takes values in blueprints B for which B^+ is a ring. As our main result, we show that this theory refines sheaf cohomology for toric varieties, in the sense that it yields the usual cohomology of toric geometry together with certain families of generators.

Problems. In the following, I list a few problems related to toric geometry that seem interesting and whose answers are not clear to me.

(1) What is the meaning of the distinguished generators of the cohomology groups for toric geometry? In which cases do they form a basis for the cohomology groups? What is the additional structure that is induced by these generators on the Cox ring of a toric variety?
(2) Can the distinguished generators be used for explicit computations of the cohomology of toric varieties?
(3) "Blue sheaf cohomology" applies to blue schemes coming from toroidal embeddings and logarithmic schemes. Is this cohomology a useful tool to study toroidal embeddings and logarithmic schemes?
(4) As it is, blue sheaf cohomology does not yet apply to tropical schemes. How can we extend it? (In this case, tropical Riemann-Roch and Brill-Noether theory would be some striking applications)

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Non-commutative resolutions of toric varieties

Eleonore Faber
(joint work with Greg Muller, Karen E. Smith)

We are interested in rings that can be associated to an affine (toric) variety $X = \operatorname{Spec}(R)$ defined over a field k and how these rings measure the singularities of X: there is of course the (commutative) coordinate ring R of X, but one can also look at endomorphism rings $\operatorname{End}_R(M)$ for M an R-module or the ring of differential operators $D_k(R)$.

The theorem of Auslander–Buchsbaum–Serre states that R is regular if and only if its *global dimension*, that is, the supremum of projective dimensions of R-modules, is finite. For $D_k(R)$ only one implication is known: if R is regular, then the global dimension of $D_k(R)$ is finite (for k of characteristic 0 this is due to Roos and for positive characteristic it is due to P. Smith). If M is a faithful R-module and $\Lambda = \operatorname{End}_R(M)$ is of finite global dimension, then Λ is called a *non-commutative resolution (=NCR)* of singularities of R (or $X = \operatorname{Spec}(R)$), a notion due to Dao–Iyama–Takahashi–Vial. In general, a non-commutative desingularization of R should be a certain non-commutative R-algebra of finite global dimension, that can be viewed as a potential analogue of a resolution of singularities of X. Van den Bergh introduced *non-commutative crepant resolutions (=NCCRs)* to interpret Bridgeland's solution to the conjecture by Bondal and Orlov on the derived invariance of flops in 2004. For Λ to be *crepant*, that is, a NCCR, one needs additionally that M is torsion-free and Λ is a non-singular order.

The purpose of the present work is the study of a certain NCR of an affine normal toric variety: Let R be the coordinate ring of an affine toric variety over a field k of arbitrary characteristic. The module $R^{1/q}$ of q-th roots of R, where q is a positive integer, is then the direct sum of so-called conic modules. We look at $\Lambda = \operatorname{End}_R(R^{1/q})$, in particular its global dimension. If k is a perfect field of prime characteristic p and $q = p^e$, then the ring of differential operators $D_k(R)$ is a direct limit of $\operatorname{End}_R(R^{1/q})$ and this description allows us to make statements about the global dimension of the non-noetherian ring $D_k(R)$.

Let now R be the coordinate ring of an affine normal toric variety, that is, $R = k[C \cap M]$, where $M \cong \mathbb{Z}^d$ is a lattice and C is assumed to be a full dimensional rational polyhedral cone in the vector space $M_{\mathbb{R}} = M \otimes \mathbb{R}$. Let $R^{1/q} = \mathrm{Span}\{x^{m/q} : \frac{m}{q} \in C \cap q^{-1}M\}$. The main result of [4] is

Theorem 1. *The global dimension of* $\mathrm{End}_R(R^{1/q})$*, for q large enough, is equal to the Krull-dimension of R. In particular,* $\mathrm{End}_R(R^{1/q})$ *is a NCR of R.*

The finiteness of the global dimension also follows from Špenko and Van den Bergh [6, 1.3.6], who proved with a much more general machinery the existence of NCRs for reductive quotient singularities, albeit with less explicit bounds on the global dimension.

The combinatorial structure of the conic modules allows us to show

Corollary. *The ring* $\mathrm{End}_R(R^{1/q})$ *is a NCCR if and only if* $\mathrm{Spec}(R)$ *is a simplicial toric variety.*

Finally, if k is perfect of prime characteristic p, then $D_k(R) = \bigcup_{e \in \mathbb{N}} \mathrm{End}_{R^{p^e}}(R)$. The Frobenius map $F^e : R \to R^{p^e}$ induces an isomorphism

$$\mathrm{End}_{R^{p^e}}(R) \cong \mathrm{End}_R(R^{1/p^e}).$$

Combining this fact with Theorem 1 and a result about global dimension of direct limits [1], if follows that in this case

$$\mathrm{gl.\,dim}(D_k(R)) \leq \lim_{e \in \mathbb{N}} (\mathrm{End}_{R^{p^e}}(R)) + 1 = \dim(R) + 1 \ .$$

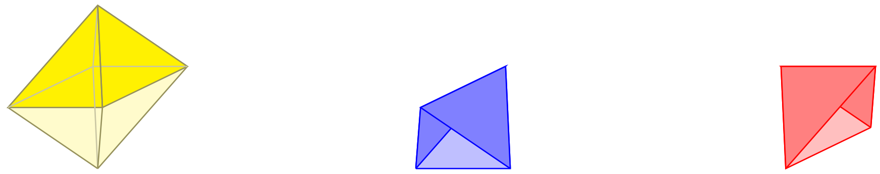
It is not clear whether this bound is sharp.

Our proof of Theorem 1 is combinatorial and constructs minimal projective resolutions of the M-graded simples of an endomorphism ring $\mathrm{End}_R(\mathbb{A})$, where $\mathbb{A}$ is a complete sum of conic modules (definitions see below).

The key observation, which dates back to [2], is that $R^{1/q}$ is a direct sum of *conic* modules A_Δ, which can be described as follows: Let $v \in M_{\mathbb{R}}$, then the corresponding conic module is $A_v := \mathrm{Span}\{x^m : m \in M \cap (C + v)\}$. Conic modules are maximal Cohen–Macaulay R-modules of rank 1 and can be parametrized by *chambers of constancy* $\Delta := \{w \in M_{\mathbb{R}} : A_v = A_w\}$ in $M_{\mathbb{R}}$. Each chamber of constancy Δ is a disjoint union of open polyhedral cells. Together, ranging over all chambers of constancy, these cells define a CW decomposition of $M_{\mathbb{R}}$.
Since there are only finitely many isomorphism classes of conic modules, see [3], we may index them by Δ, that is, $A_v = A_\Delta$ for $v \in \Delta$. For each chamber of constancy Δ, we use the combinatorics of the faces of Δ to construct a chain complex $K^\bullet_\Delta$ of conic modules. We prove an Acyclicity Lemma for these conic complexes: the complex $\mathrm{Hom}_R(A_{\Delta'}, K^\bullet_\Delta)$ is either acyclic or a resolution of the ground field k, depending on whether or not $A_\Delta \cong A_{\Delta'}$.

Further, we call a direct sum $\mathbb{A}$ of conic modules *complete* if every conic R-module is isomorphic to a direct summand of $\mathbb{A}$. An example for this is $R^{1/q}$ for q large enough, see [3] and [5]. The Acyclicity Lemma implies that the complex $\mathrm{Hom}_R(\mathbb{A}, K^\bullet_\Delta)$ is a finite projective resolution of a simple $\mathrm{End}_R(\mathbb{A})$-module. Using standard arguments on global dimension of noetherian rings, we show that every finitely generated $\operatorname{End}_R(\mathbb{A})$-module has finite global dimension, and thus $\operatorname{gl.dim}(\operatorname{End}_R(\mathbb{A}))$ is also finite.

Example. Look at the cone over the unite square in $\mathbb{R}^3$. The corresponding toric algebra $R = k[C \cap \mathbb{Z}^3]$ has presentation $k[x, y, z, w]/(xz - yw)$ and is not simplicial. There are three isomorphism types of conic R-modules, represented by $R = A_0$, $(x, y)R = A_1$ and $(x, w)R = A_2$. Their (representative) chambers are pictured below in Figure 1, with boundary cells in each chamber colored darker, and Figure 2 shows how the chambers fit together.



(A) Chamber of A_0 (B) Chamber of A_1 (C) Chamber of A_2

FIGURE 1. The three types of chambers with shaded boundary cells.

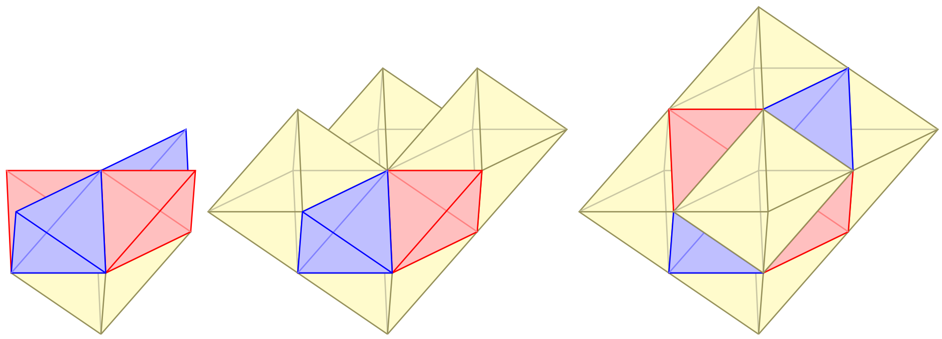


FIGURE 2. Unions of chambers

The complexes $K^\bullet_\Delta$ corresponding to chambers of type (A) will have length three while the length of the complexes corresponding to any other chamber will have length strictly less. The complex $K^\bullet_\Delta$ corresponding to the first type of chamber is

$$A_0 \longrightarrow A_0^{\oplus 4} \longrightarrow A_1^{\oplus 2} \oplus A_2^{\oplus 2} \longrightarrow A_0 \ .$$

The complex of R-modules corresponding to the latter two types of chambers is

$$A_2 \longrightarrow A_0^{\oplus 2} \longrightarrow A_1 \quad \text{and} \quad A_1 \longrightarrow A_0^{\oplus 2} \longrightarrow A_2.$$

Here $\operatorname{End}_R(\mathbb{A})$, for $\mathbb{A} = A_0 \oplus A_1 \oplus A_2$, is of global dimension 3 but has two simples of projective dimension 2, coming from the two tetrahedral chambers of constancy. This shows that $\operatorname{End}_R(\mathbb{A})$ is not an NCCR.

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The length of an extremal ray of a toric variety

Hiroshi Sato

(joint work with Osamu Fujino)

For a $\mathbb{Q}$-Gorenstein projective toric n-fold X over an algebraically closed field and a K_X-negative extremal ray $R \subset \mathrm{NE}(X)$, we define the *length* of R as

$$l(R) = \min_{[C]\in R}(-K_X \cdot C),$$

where $[C]$ stands for the numerical class of a curve $C \subset X$. The length of an extremal ray is an important invariant for the theory of the birational geometry. For a "long" extremal ray $R \subset \mathrm{NE}(X)$, we can determine the structure of the extremal contraction φ_R associated to R. For example, the following holds:

Proposition 1 (see [1, Proposition 2.9] and [2, proposition 2.1])**.** *Let X be a $\mathbb{Q}$-factorial projective toric n-fold with $\rho(X) = 1$. In this case, we have the unique extremal ray $R = \mathrm{NE}(X)$. Then, the following hold:*

(1) *If $l(R) > n$, then X is isomorphic to $\mathbb{P}^n$.*
(2) *If $l(R) \geq n$ and $X \not\simeq \mathbb{P}^n$, then X is isomorphic to the weighted projective space $\mathbb{P}(1, 1, 2, \cdots, 2)$.*

The following is the main theorem of this talk in which we deal with the case where the extremal contraction associated to a long extremal ray is *birational*.

Theorem 2 ([3, Theorem 3.2.1])**.** *Let $f : X \to Y$ be a projective toric morphism with $\dim X = n$. Assume that K_X is $\mathbb{Q}$-Cartier. Let R be a K_X-negative extremal ray of $\mathrm{NE}(X/Y)$ and let $\varphi_R : X \to W$ be the extremal contraction morphism associated to R. Assume that φ_R is birational. Then we obtain*

$$l(R) < d + 1,$$

where

$$d = \max_{w\in W} \dim \varphi_R^{-1}(w) \leq n - 1.$$

When $d = n-1$, we have a sharper inequality

$$l(R) \le d = n-1.$$

In particular, if $l(R) = n-1$, then $\varphi_R : X \to W$ can be described as follows: There exists a torus invariant smooth point $P \in W$ such that $\varphi_R : X \to W$ is a weighted blow-up at P with the weight $(1, a, \cdots, a)$ for some positive integer a. In this case, the exceptional locus E of φ_R is a torus invariant prime divisor and is isomorphic to $\mathbb{P}^{n-1}$. Moreover, X is $\mathbb{Q}$-factorial in a neighborhood of E.

As an application of Theorem 2, we obtain the following. We think that this theorem can be a modified version of Exercise (a) in [5, p.90]

Theorem 3 ([3, Theorem 4.2.3])**.** *Let X be a $\mathbb{Q}$-Gorenstein projective toric n-fold and let D be an ample Cartier divisor on X. Then $K_X + (n-1)D$ is pseudo-effective if and only if $K_X + (n-1)D$ is nef.*

We remark that the following example shows that the estimate for *small* contractions in Theorem 2 is sharp.

Example. We fix $N = \mathbb{Z}^n$ with $n \ge 3$. Let $\{e_1, \cdots, e_n\}$ be the standard basis of N. We put

$$e_{n+1} = (\underbrace{a, \cdots, a}_{n-k+1}, \underbrace{-1, \cdots, -1}_{k-1})$$

with $2 \le k \le n-1$, where a is any positive integer. Let Σ be the fan in $\mathbb{R}^n$ such that the set of maximal cones of Σ is

$$\left\{ \langle \{e_1, \cdots, e_{n+1}\} \setminus \{e_i\} \rangle \mid 1 \le i \le n-k+1 \right\}.$$

Let us consider the smooth toric variety $X = X_\Sigma$ associated to the fan Σ. By construction, we obtain a flipping contraction $\varphi : X \to W$ whose exceptional locus is isomorphic to $\mathbb{P}^{n-k}$. We can directly check that

$$-K_X \cdot C = n-k+1-\frac{k}{a}$$

for every torus invariant curve C in the φ-exceptional locus $\mathbb{P}^{n-k}$ (for the details, see [3, Example 3.3.2]).

Finally, we describe the results for the case where the associated extremal contraction is a *Fano* contraction.

Theorem 4 ([4, Theorem 3.1])**.** *Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric n-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a K_X-negative extremal ray $R \subset \mathrm{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of φ_R. If a general fiber of φ_R is isomorphic to $\mathbb{P}^d$ and*

$$-K_X \cdot C > \frac{d+1}{2}$$

holds for any curve C on X contracted by φ_R, then φ_R is a $\mathbb{P}^d$-bundle over W.

The following example shows that Theorem 4 is sharp.

Example. Let $\{e_1, \dots, e_n\}$ be the standard basis for $N = \mathbb{Z}^n$ and $p : N \to \mathbb{Z}^{n-d}$ be the projection

$$(x_1, \dots, x_d, x_{d+1}, \dots, x_n) \mapsto (x_{d+1}, \dots, x_n)$$

for $1 \le d < n$. Put

$$v_1 := e_1,\ \dots,\ v_d := e_d,\ v_{d+1} := -(e_1 + \cdots + e_d),$$

$$y_1 := e_{d+1},\ \dots,\ y_{n-d-1} := e_{n-1},\ y_{n-d} := e_1 + e_{d+1} + \cdots + e_{n-1} + 2e_n.$$

Let Σ be the fan in N whose maximal cones are generated by

$$\{v_1, \dots, v_{d+1}, y_1, \dots, y_{n-d}\} \setminus \{v_i\}$$

for $1 \le i \le d+1$. In this case, $X = X_\Sigma$ has a Fano contraction whose general fiber is isomorphic to $\mathbb{P}^d$. From this noncomplete variety, one can easily construct a projective toric n-fold which has a Fano contraction associated to an extremal ray of length $\frac{d+1}{2}$ (for the details, see [4, Example 3.2]).

If we make the inequality in Theorem 4 stronger, then the assumption that a general fiber of a Fano contraction is isomorphic to the projective space automatically holds as follows:

Corollary ([3, Theorem 3.2.9] and [4, Corollary 3.3])**.** *Let $X = X_\Sigma$ be a $\mathbb{Q}$-factorial projective toric n-fold. Let $\varphi_R : X \to W$ be a Fano contraction associated to a K_X-negative extremal ray $R \subset \mathrm{NE}(X)$, and $d = n - \dim W$ be the dimension of a fiber of φ_R. If $-K_X \cdot C > d$ holds for any curve C on X contracted by φ_R, then φ_R is a $\mathbb{P}^d$-bundle over W.*

The following example shows that this Corollary is sharp.

Example. Let $F := \mathbb{P}(1, 1, 2, \dots, 2)$ be the d-dimensional weighted projective space and W a $\mathbb{Q}$-factorial projective toric $(n-d)$-fold. Then, the length of the extremal ray corresponding to the first projection $\varphi : X = W \times F \to W$ is d (see [2, Proposition 2.1] and [3, Proposition 3.1.6]).

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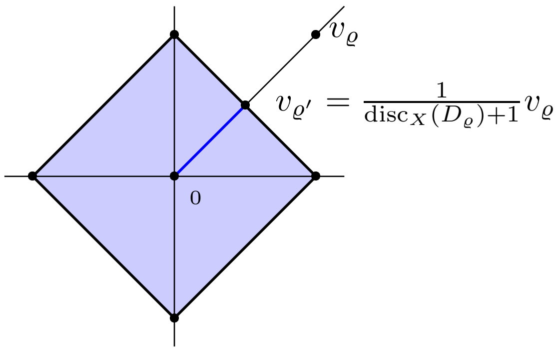
On the anticanonical complex

Milena Wrobel
(joint work with Christoff Hische)

In this talk we develop a combinatorial tool for the study of singular Fano varieties as they pop up in the minimal model programme, that means projective varieties with an ample anticanonical divisor and at most terminal or canonical singularities.

Toric Fano varieties X are in one to one correspondence with certain lattice polytopes, the so called *Fano polytopes* A_X. The Fano polytope A_X is determined by the property that its boundary ∂A_X encodes the discrepancies of any toric resolution of singularities: the discrepancy $\mathrm{disc}_X(D_\varrho)$ of an exceptional divisor D_ϱ corresponding to a ray ϱ equals minus one plus the ratio of the length of the shortest nonzero integer vector of ϱ by the length of the unique vector $v'_\varrho := \varrho \cap \partial A_X$.



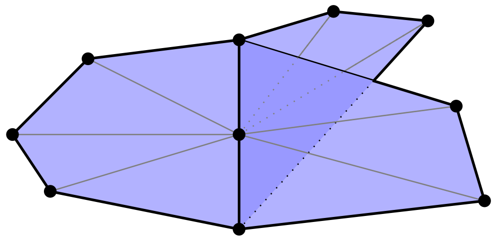
The Fano polytope of $\mathbb{P}^1 \times \mathbb{P}^1$

This allows to decide whether or not X has at most terminal or canonical singularities by looking at the position of the lattice points in A_X and turns the Fano polytope into the central combinatorial tool for the classification of toric Fano varieties [1, 9, 10, 11, 12].

Given an arbitrary Fano variety X, the main problem is to understand the discrepancies of suitable desingularizations in a combinatorial way. For this we use the natural embedding $X \to Z$ into a toric variety Z, constructed via the Cox ring of X; see [7]. In particular, we can assign a tropical variety $\mathrm{trop}(X)$ to X. The idea is to work with desingularizations $X'' \to X$ obtained by a two-step procedure:

$$\begin{array}{ccccc} Z'' & \longrightarrow & Z' & \longrightarrow & Z \\ \uparrow & & \uparrow & & \uparrow \\ X'' & \longrightarrow & X' & \longrightarrow & X \end{array}$$

The first step $X' \to X$ is the *weak tropical resolution*, that means that Z' is the toric variety having as its fan the coarsest common refinement of the fan of Z and the tropical variety $\mathrm{trop}(X)$, and $X' \subseteq Z'$ is the proper transform of $X \subseteq Z$. In the second step $Z'' \to Z'$ is a toric desingularization and $X'' \subseteq Z''$ is the proper transform of $X' \subseteq Z'$. This approach was used in [2] in the case of Fano varieties $X \subseteq Z$ with a torus action of complexity one, i.e. the generic orbit of the torus action is of codimension one in X. In this situation, the toric Fano polytope is replaced with a polyhedral complex supported inside the tropical variety $\mathrm{trop}(X)$, the so called *anticanonical complex* $\mathcal{A}_X$. The variety X is log terminal if and only if $\mathcal{A}_X$ is bounded and in the latter case the boundary of $\mathcal{A}_X$ controls the discrepancies of all toric desingularizations arising in the above way. The anticanonical complex has been used in [2] to classify $\mathbb{Q}$-factorial terminal Fano threefolds with a 2-torus action and Picard number 1.



Anticanonical complex of a Fano $\mathbb{C}^*$-surface

Looking at the Fano polytope in the toric case and at the anticanonical complex in the case of varieties with torus action of complexity one, one makes the following observation: These two combinatorial objects can be, roughly speaking, obtained by "drawing discrepancies". Following this idea, in the article [8], we extend the picture to Fano varieties $X \subseteq Z$ with a torus action of higher complexity. In fact, most of our results also work beyond the Fano case but we make this simplifying assumption here.

An *anticanonical complex for* $X \subseteq Z$ is a polyhedral complex $\mathcal{A}_X$ supported on $\mathrm{trop}(X)$ such that the boundary of $\mathcal{A}_X$ encodes the discrepancies of any toric resolution factoring through the weak tropical resolution $X' \to X$ as described above. Having at least one such toric resolution providing a resolution of singularities of X, we can use the anticanonical complex $\mathcal{A}_X$ to detect the singularity type of X.

Our main result gives a criterion for the existence of an anticanonical complex for Fano varieties X with an action of an algebraic torus $\mathbb{T}$: In this situation we can reduce the problem to a lower dimensional variety Y suitably representing the field of invariant rational functions $\mathbb{C}(X)^{\mathbb{T}}$ and defining a so called *explicit maximal orbit quotient* $X \dashrightarrow Y$.

Theorem 1. *Let $X \subseteq Z$ be a Fano variety with torus action having an explicit maximal orbit quotient $X \dashrightarrow Y$, where Y is complete and admits a locally toric weak tropical resolution. Then X admits an anticanonical complex $\mathcal{A}_X$ and the following statements hold:*

(i) *X has at most log terminal singularities if and only if the anticanonical complex $\mathcal{A}_X$ is bounded.*
(ii) *X has at most canonical singularities if and only if 0 is the only lattice point in the relative interior of $\mathcal{A}_X$.*
(iii) *X has at most terminal singularities if and only if 0 and the primitive generators of the rays of the defining fan of Z are the only lattice points of $\mathcal{A}_X$.*

As an example class we consider the *general arrangement varieties* introduced in [6]. These varieties come with a torus action of arbitrary complexity c having an explicit maximal orbit quotient $X \dashrightarrow \mathbb{P}_c$ and the critical values of the quotient map form a general hyperplane arrangement. All Fano varieties with a torus action of complexity one are contained in this class: the maximal orbit quotient is a line and the set of critical values is a point configuration on this line. Other examples are the *intrinsic quadrics*, i.e. varieties with a Cox ring defined by a single quadratic relation; see [3, 4, 5]. Applying our Theorem 1 to this special situation we arrive at the following:

Corollary. *Every Fano general arrangement variety admits an anticanonical complex.*

In the case of general arrangement varieties it is possible to describe the anticanonical complex explicitly. As a direct consequence of this description we can give first bounds on the defining data of log terminal Fano general arrangement varieties. As a second application we classify threedimensional $\mathbb{Q}$-factorial Fano intrinsic quadrics of complexity two having at most canonical singularities.

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Combinatorial Lefschetz Theorems beyond positivity

Karim Adiprasito

The hard Lefschetz theorem, in almost all cases that we know, is connected to rigid algebro-geometric properties. Most often, it comes with a notion of an ample class, which not only induces the Lefschetz theorem but the induced bilinear form satisfies the Hodge-Riemann relations as well, which give us finer information about its signature (see for instance Voisin, CUP 2002).

Even in the few cases that we have the hard Lefschetz without the Hodge-Riemann relations, they are often at least conjecturally present in some form, as for instance in the case of Grothendieck's standard conjectures and Deligne's proof of the hard Lefschetz standard conjecture. This connection is deep and while we understand Lefschetz theorems even for singular varieties, to this day, we have no way to understand the Lefschetz theorem without such a rigid atmosphere for it to live in.

Our goal and result in [1] is to provide a different criterion for varieties to satisfy the hard Lefschetz theorem that goes beyond positivity, and abandons the Hodge-Riemann relations entirely (but not the associated bilinear form); instead of finding Lefschetz elements in the ample cone of a variety, we give general position criteria for an element in the first cohomology group to be Lefschetz. The price I pay for this achievement is that the variety itself has to be sufficiently "generic".

For the current results I therefore turn to toric varieties, which allow for a sensible notion of genericity without sacrificing all properties of the variety, most importantly, without changing its Betti vector. Specifically, I consider varieties with a fixed equivariant cohomology ring, and allow variation over the Artinian reduction, i.e., the variation over the torus action. The main result can be summarized as follows:

Theorem 1 (A, [1])**.** *Consider a PL $(d-1)$-sphere Σ, or more generally a PL rational homology sphere of that dimension, and the associated graded commutative face ring $\mathbb{R}[\Sigma]$ (see Stanley, Birkhäuser Prog. in Math. 1996). Then there exists an open dense subset of the Artinian reductions $\mathcal{R}$ of $\mathbb{R}[\Sigma]$ and an open dense subset $\mathcal{L} \subset A^1(\Sigma)$, where $A(\Sigma) \in \mathcal{R}$, such that for every $k \leq \frac{d}{2}$, we have:*

(1) Generic Lefschetz theorem: *For every $A(\Sigma) \in \mathcal{R}$ and every $\ell \in \mathcal{L}$, we have an isomorphism*

$$A^k(\Sigma) \xrightarrow{\ \cdot \ell^{d-2k}\ } A^{d-k}(\Sigma).$$

(2) Hall-Laman relations: *The* Hodge-Riemann bilinear form

$$\begin{array}{cccccc} \mathrm{Q}_{\ell,k}: & A^k(\Sigma) & \times & A^k(\Sigma) & \longrightarrow & A^d(\Sigma) \cong \mathbb{R} \\ & a & & b & \longmapsto & \deg(ab\ell^{d-2k}) \end{array}$$

is nondegenerate when restricted to any squarefree monomial ideal in $A(\Sigma)$, as well as the annihilator of any squarefree monomial ideal.

The Lefschetz theorem is therefore as announced valid for generic Artinian reductions. A slightly weaker form applies to general, non PL triangulations. In particular, the more algebrao-geometric reader may consult the following Corollary for easier visualization.

Corollary. *Consider $\mathfrak{F}$ a complete simplicial fan in $\mathbb{R}^d$. Then, after perturbing the rays of $\mathfrak{F}$ to a suitable rational fan $\mathfrak{F}'$, the Chow ring of the toric variety $X_{\mathfrak{F}'}$ satisfies the hard Lefschetz theorem with respect to a generic degree one element, while the equivariant Chow ring remains unchanged from $X_{\mathfrak{F}}$ to $X_{\mathfrak{F}'}$.*

These results have a myriad of consequences, among them:

(1) *g-conjecture, McMullen Isr. J. Math. 1971:* It proves that the f-vector, i.e. the number of vertices, edges, two-dimensional faces etc. of a simplicial sphere is also the f-vector of some simplicial polytope.
(2) *Grünbaum conjecture, J. Comb. Theor. 1970:* It generalizes a result of Déscartes: If Δ is a simplicial complex of dimension d that allows a PL embedding into $\mathbb{R}^{2d}$ then

$$f_d(\Delta) \ \leq \ (d+2)f_{d-1}(\Delta)$$

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The tropical critical point and mirror symmetry

Konstanze Rietsch
(joint work with Jamie Judd)

The abstract below was co-written with Jamie Judd.

This talk was about critical points of positive Laurent polynomials over a field of Puiseaux series and tropicalisation. We begin with three introductions.

Introduction I: Flag varieties and the representation $V(2\rho)$. It is well-known that toric varieties come with a 'preferred' anti-canonical divisor, namely the sum of all of the torus-invariant divisors. More surprisingly, a similar preferred choice can be made for general flag varieties G/P. For a full flag variety G/B for example, this analogue is the union of the Schubert and opposite Schubert divisors. However, a section of the anticanonical bundle of G/B is the same thing as a vector in the 2ρ-representation $V(2\rho)$ of G. This leads to the realisation that there should be a preferred line in the 'middle' of this representation (indeed lying in the 0-weight space). This observation was made by Victor Ginzburg who some years ago asked the co-author *how come* in the context of mirror symmetry there was this special line in $V(2\rho)$. This question was one of the motivations for the work presented in this talk. The first work in this direction was the author's paper [4]. Namely this paper gave a construction of a canonical point in the Gelfand-Zetlin polytope of an arbitrary representation $V(\lambda)$ of $GL_n(\mathbb{C})$ using a version of the mirror Landau-Ginzburg superpotential over a field of generalised Puiseaux series, and for $\lambda = 2\rho$ it proved that this point 'indexes' a special basis vector corresponding to the preferred divisor of GL_n/B described above.

Introduction II: Toric varieties. Let us turn our attention back to toric varieties and the fact that a toric variety X comes with a "preferred" anti-canonical divisor D_{ac}, namely the sum of all of the torus invariant divisors. If X has an ample T-invariant anticanonical divisor D (i.e. X is Fano), then this fact gives rise to a distinguished point in the associated moment polytope P_D. Namely $m \in P_D$ is the unique lattice point where $(\chi^m) = D_{ac} - D$. Here $\chi^m \in \mathbb{C}(X)$ is the character associated to m, viewed as a rational function on X.

One application of our result is to generalize this observation to obtain a distinguished point in the moment polytope P_D for any T-invariant ample divisor D in a projective toric variety X. This distinguished point may or may not be a lattice point, but when it is it defines a "preferred" divisor in the class $[D]$.

Introduction III: Symplectic toric manifolds. Another setting where special points in moment polytopes arise is in symplectic geometry. For example in $\mathbb{C}P^1$ the central fiber $L = \mu^{-1}(\frac{1}{2})$ of the moment map $\mu : \mathbb{C}P^1 \to [0,1]$ is a great circle. It bounds two holomorphic disks which have equal volume and it is not hard to see that it is non-displaceable. This means that the image L' of L under any Hamiltonian isotopy has nontrivial intersection with L. More generally the symplectic manifold $\mathbb{C}P^N$ with moment map $\mu : \mathbb{C}P^N \to \Delta_N$ given by

$$[z_0 : \ldots : z_N] \mapsto \left(\frac{|z_1|^2}{\sum_{i=0}^N |z_i|^2}, \ldots, \frac{|z_N|^2}{\sum_{i=0}^N |z_i|^2} \right)$$

has a "special" Lagrangian torus fiber called the *Clifford torus* which is a non-displaceable Lagrangian generalising the great circle in $\mathbb{C}P^1$, see [1]. It is the fiber of a special point in the moment polytope Δ_N, namely $(\frac{1}{n+1}, \ldots, \frac{1}{n+1})$.

An approach to constructing non-displaceable moment map fibers in toric symplectic manifolds more generally was developed in the work of Fukaya, Oh, Ono and Ota, see [2] and references therein, as well as [6]. It amounts to constructing a mirror superpotential over a field of generalised Puiseaux series (in the Fano case essentially the Laurent polynomial introduced by Givental with one summand for each ray of the fan), and then considering valuations of critical points. We apply our result to obtain a distinguished point in the moment polytope of any symplectic toric manifold X. And thus we can define a "special" Lagrangian torus fiber in X generalising the Clifford torus of $\mathbb{C}P^N$.

Main Results. We first introduce the tropicalisation of a torus following Lusztig [7]. Consider the field of generalised Puiseaux series

$$\mathbf{K} = \left\{ \sum_{i=0}^{\infty} m_i t^{c_i} \mid m_i \in \mathbb{C}, c_i \in \mathbb{R}, c_i \to \infty \text{ monotone} \right\},$$

with the natural valuation map $\mathrm{val}_{\mathbf{K}} : \mathbf{K} \to \mathbb{R} \cup \{\infty\}$. Define the positive part $\mathbf{K}_{>0}$ of $\mathbf{K}$ to be the semifield of those elements with leading coefficient in $\mathbb{R}_{>0}$. Given an algebraic torus T over $\mathbf{K}$ with cocharacter lattice $X_*(T) = N$ and character lattice $X^*(T) = M$, we can also define the positive part of T as

$$T(\mathbf{K}_{>0}) := \{p \in T(\mathbf{K}) \mid \chi^m(p) \in \mathbf{K}_{>0} \text{ for all } m \in M\}.$$

Let $N_\mathbb{R} = X_*(T) \otimes \mathbb{R}$. We use the dual pairing between M and N and the valuation $\mathrm{val}_\mathbf{K}$ on $\mathbf{K}$ to define

$$\mathrm{Val}_T : T(\mathbf{K}_{>0}) \to N_\mathbb{R},$$

namely by setting $\langle m, \mathrm{Val}_T(p)\rangle = \mathrm{val}_\mathbf{K}(\chi^m(p))$. Using $T(\mathbf{K}_{>0})$ and Val_T we may define $\mathrm{Trop}(T)$ as the set of equivalence classes for the equivalence relation on $T(\mathbf{K}_{>0})$ given by setting $p \sim p'$ if $\mathrm{Val}_T(p) = \mathrm{Val}_T(p')$. We thus may identify $\mathrm{Trop}(T)$ with $N_\mathbb{R}$ using the valuation map Val_T.

The following theorem is the analogue over $\mathbf{K}_{>0}$ of a result of Galkin [3]. See also [8, Theorem 10.2] where the same result is proved in an analogous way, but only for certain specific Laurent polynomials (mirror to SL_n/P).

Theorem 1 ([5]). *Let W be a Laurent polynomial on the torus T with coefficients in $\mathbf{K}_{>0}$. Furthermore assume that the Newton polytope of W is full dimensional with 0 in the strict interior. Then W has a unique critical point in the positive part $T(\mathbf{K}_{>0})$ of the torus.*

Note that a Laurent polynomial W as above defines a piecewise linear map

$$\mathrm{Trop}(W) : N_\mathbb{R} \to \mathbb{R}$$

by setting $\mathrm{Trop}(W)(d) := \mathrm{val}_\mathbf{K}(W(t^d))$ where $d \mapsto t^d$ is some lift of Val_T. Namely if $W = \sum_i \gamma_i x^{v_i}$ for some $\gamma_i \in \mathbf{K}_{>0}$ with valuation c_i, and some $v_i \in M$, then

$$\mathrm{Trop}(W)(d) = \min_i (c_i + \langle v_i, d\rangle).$$

We define a polytope $\mathcal{P}_W$ in $N_\mathbb{R}$ by $\mathcal{P}_W = \{d \in N_\mathbb{R} \mid \mathrm{Trop}(W) \geq 0\}$. (Note that $\mathcal{P}_W$ is possibly empty). We also have the following theorem.

Theorem 2 ([5]). *Let W be as in* Theorem 1 *and p_{crit} its unique positive critical point. Define $d_{crit} := \mathrm{Val}_T(p_{crit})$. Then $\mathrm{Trop}(W)(d_{crit})$ is the maximal value of the piecewise linear function $\mathrm{Trop}(W)$. In particular if the polytope $\mathcal{P}_W$ is non-empty, then d_{crit} lies in $\mathcal{P}_W$ and thus gives a distinguished point inside it.*

We call d_{crit} the *tropical critical point of W*. In [5] we also give an algorithmic construction of the tropical critical point, which is explicit and has a piecewise linear flavour, but is beyond the scope of this note.

Applications. The three introductions relate three ways in which the main result can be applied. Firstly, Gelfand-Zetlin polytopes arise as polytopes $\mathcal{P}_W$ for Givental's Laurent polynomial mirrors W of type A flag varieties SL_n/B. In this case the tropical critical point was earlier computed in [4], and has an interpretation as described in Introduction I.

In the context of Introduction II, the moment polytope P_D of a toric variety with ample toric divisor $D = \sum r_i D_i$ also comes with a natural associated W for which $\mathcal{P}_W = P_D$. Namely we can take Givental's mirror of the toric variety, which

has one summand for each primitive ray generator v_i of the fan, with appropriate $\mathbf{K}_{>0}$ coefficients: $W = \sum t^{r_i} x^{v_i}$. Thus the theorems in the previous section give rise to a canonical point d_{crit} in P_D which, if it is a lattice point, can be interpreted as a specifying a preferred divisor in the class $[D]$.
Finally, if P_D is a rational Delzant polytope (corresponding to a toric symplectic manifold), then we are in the setting of Introduction III and the moment map fibre of the special point d_{crit} gets a symplectic interpretation as a non-displaceable Lagrangian via [2]. We note that this application extends to the more general setting of symplectic orbifolds, thanks to an extension of [2] by Woodward [9].

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Toric methods beyond toric varieties

Jarosław Wiśniewski

(joint work with Jarosław Buczyński, Andrzej Weber, Eleonora Romano, Gianluca Occhetta, and Luis Solá Conde)

There are several ways one may try to extend toric methods to situation when torus acting on a variety has rank smaller than the dimension of the variety in question. The present approach has been developed and tested in a joint project, [2], [5] and [6] which aims at providing methods for dealing with a conjecture by LeBrun and Salamon. The original conjecture in differential geometry is about quaternion-Kähler manifolds, [4], while its algebro-geometric counterpart is about projective complex contact manifolds, [1].

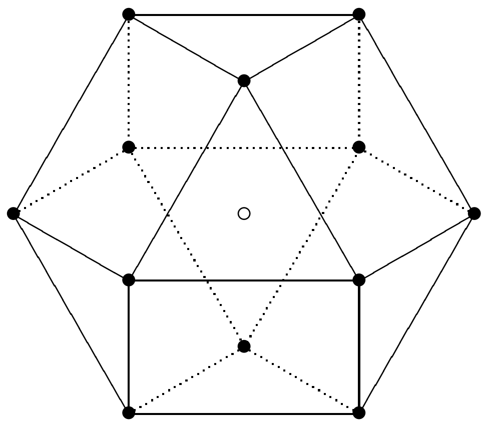
Suppose we are given an algebraic torus H, with the lattice of characters M, acting on a projective manifold X with an ample line bundle L. To the triple (X, L, H) we associate its *grid data* containing the following information:

- the fixed point components Y_i, $X^H = \bigsqcup_i Y_i$, with the action of H on co-normal bundles $N^*_{Y_i/X}$ described by the weight decomposition
$$N^*_{Y_i/X} = \bigoplus_j [N^*_{Y_i/N}]^{\nu_{ij}}$$
where the weights $\nu_{ij} \in M$ are called the *compass* of the action of H at the component Y_i;
- for a given linearization $\mu : H \times L \to L$ consider a function $\mu : \{Y_i\} \to M$ which maps a component Y_i to the weight of the action of H on the fiber L_y for every $y \in Y_i$; changing the linearization adds a fixed character to μ, so up to translations in M the map μ does not depend on the linearization.

The grid data is the input to Lefschetz-Riemann-Roch theorem which provides M-graded Euler characteristic polynomial $\chi_H(X, L^{\otimes m})$.

A *grid* of the triple (X, L, H) is a formal 1-complex in M with vertices at points which are in the image of μ labeled by the respective components Y_i. The edges of a grid at $\mu(Y_i)$ are spanned by vectors ν_{ij} labeled by respective $[N^*_{Y_i/X}]^{\nu_{ij}}$. We note that, possibly, multiple Y_i are mapped by μ to the same point in M. In the situation studied by Goresky, Kottwitz and MacPherson, [3], which includes the assumption that Y_i's are isolated points, the grid is the image under μ of the respective GKM graph. If X is a toric variety under the action of H, then the grid is 1-skeleton of the moment polytope associated to the pair (X, L). If X is a homogeneous variety under the action of a semi-simple group G with a maximal torus $H < G$ then the vertices of the grid are obtained from the dominant weight associated to the representation of G on $\Gamma(X, L)$ via the Weil group action; the edges join points obtained one from another by a reflection. In particular, in this case the edges of the grid are not only the edges of its conves hull.

The picture below presents a two dimensional projection of a grid in $\mathbb{Z}^3$ whose convex hull is a cuboctahedron. The variety X is the closed orbit of dimension 5 in the projectivization of the adjoint representation of SL_4 with $L = \mathcal{O}_X(1)$. The grid describes the action of a 3-dimensional Cartan torus $H < SL_3$. The grid has five adges at each vertex $\bullet$ but the edges joining the pairs of vertices symmetric with respect to the center $\circ$ has been removed from the picture for better clarity.



Suppose we are given a subtorus $H' \hookrightarrow H$ with the quotient $H'' = H/H'$ and related exact sequence of lattices of characters

$$0 \longrightarrow M'' \longrightarrow M \longrightarrow M' \longrightarrow 0$$

A grid of an H-action can be subjected to the following two operations:

- *downgrading the grid* of (X, L, H) to the action of the subtorus $H' \hookrightarrow H$ by projecting the grid data via $M \to M'$;
- *restricting the grid* of (X, L, H) to a fixed point component $Y' \subset X^{H'}$ with the grid of $(Y', L_{Y'}, H'')$ located in M'' via an affine embedding $M'' \hookrightarrow M$.

These procedures can be used to understand the structure of (X, L, H). For example, downgrading associated to a projection of the above 3-dimensional grid along a plane parallel to a triangular face of the cuboctahedron yields a grid of a $\mathbb{C}^*$ acting on X with 3 fixed points components: two copies of $\mathbb{P}^2$ and flags in $\mathbb{P}^2$ or, equivalently $(1,1)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$. On the other hand, downgrading associated to a projection along a quadratic face gives an action of $\mathbb{C}^*$ with four fixed points components: two copies of $\mathbb{P}^1 \times \mathbb{P}^1$ and two copies of $\mathbb{P}^1$.

The method explained above was used to analyse grids associated to low-dimensional contact manifolds and was crucial for proving LeBrun–Salamon conjecture in dimension 7 and 9 in [2].

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Stability of tangent bundles on smooth toric Picard-rank-2 varieties and surfaces

Milena Hering

(joint work with Benjamin Nill, Hendrik Süss)

Let X be a smooth toric variety over a field of characteristic 0, with tangent bundle $\mathcal{T}_X$. Let $\mathcal{O}(D)$ be an ample line bundle. Recall that the slope of a torsion-free sheaf $\mathcal{E}$ on a normal projective variety X with respect to an ample line bundle $\mathcal{O}(D)$ is defined to be

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E}) \cdot D^{n-1}}{\operatorname{rank}(\mathcal{E})},$$

and that $\mathcal{E}$ is *stable* (resp. *semistable*) with respect to $\mathcal{O}(D)$ if for any subsheaf $\mathcal{F}$ of $\mathcal{E}$ of smaller rank, we have $\mu(\mathcal{F}) < \mu(\mathcal{E})$ (resp. $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$).

Theorem 1. *Let X be a smooth toric surface or a smooth toric variety of Picard rank 2. Then there exists an ample line bundle $\mathcal{L}$ on X such that T_X is stable with respect to $\mathcal{L}$ if and only if it is an iterated blow-up of projective space.*

The case of Picard rank 2 has been independently obtained by Dasgupta, Dey, and Khan [1].

The theorem is not true in higher dimensions and higher Picard rank. For example, there exists a smooth toric Fano 3-fold of Picard rank 4 whose tangent bundle is stable with respect to the anticanonical polarisation, but that does not admit a morphism to $\mathbb{P}^3$.

One motivation to study stability of tangent bundles stems from the connection between the existence of a destabilising subbundle $\mathcal{F}$ of the tangent bundle with respect to a certain polarisation and, if the slope of $\mathcal{F}$ is positive, the existence of a rational fibration to a lower-dimensional variety whose relative tangent bundle is $\mathcal{F}$ [2].

Our results are based on the following combinatorial criterion for stability of the tangent bundle $\mathcal{T}_X$ on a toric variety X. Let $\mathcal{O}(D)$ be ample and let P_D be the lattice polytope associated to D. For each ray ρ in the fan Σ, let P_D^ρ denote the facet corresponding to ρ.

Proposition 2. *The tangent bundle on a smooth projective toric variety X of dimension n is (semi)-stable with respect to an ample line bundle $\mathcal{O}(D)$ on X if and only if for every proper subspace $F \subset N \otimes k$ the following inequality holds:*

$$\frac{1}{\dim F} \sum_{v_\rho \in F} \mathrm{vol}(P_D^\rho) \overset{(\leq)}{<} \frac{1}{n} \sum_{\rho} \mathrm{vol}(P_D^\rho) =: \frac{1}{n} \mathrm{vol} \partial P_D. \tag{1}$$

Here, $\mathrm{vol}(P^\rho)$ denotes the lattice volume inside the affine span of P^ρ with respect to the lattice $\mathrm{span}(P^\rho) \cap M$.

A situation of particular interest is when X is Fano, $\mathcal{E} = \mathcal{T}_X$ is the tangent bundle, and $D = -K_X$ the anticanonical divisor, in particular, since the existence of a Kähler-Einstein metric on a Fano variety implies that the tangent bundle is polystable with respect to the anticanonical polarisation, see for example [5]. The recent proof of the Yau-Tian-Donaldson conjecture shows that a Fano manifold has a Kähler-Einstein metric if and only if it is K-polystable. For a general toric Fano variety K-stability is equivalent to the fact that for the polytope corresponding to the anticanonical polarisation the barycenter coincides with the origin [4]. Thus we recover [3, Thm 1.1] in the smooth case.

Corollary. *Let P be a smooth reflexive polytope with barycenter in the origin. Then P satisfies the non-strict inequality (1) for every proper linear subspace $F \subset N_{\mathbb{Q}}$.*

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Symmetric edge polytopes

Mateusz Michałek
(joint work with A. D'Ali, E. Delucchi, A. Higashitani, K. Jochemko, M. Kummer)

It is a common technique in algebraic combinatorics to study an object by associating to it a lattice polytope. The following definition was introduced in [11].

Definition 1 (Symmetric edge polytope)**.** The *symmetric edge polytope* associated to a graph G is:

$$\mathcal{P}_G := \operatorname{conv}\{\mathbf{e}_v - \mathbf{e}_w, \mathbf{e}_w - \mathbf{e}_v : \{v, w\} \in E\} \subset \mathbb{R}^V.$$

It turns out that symmetric edge polytopes play an important role in many branches of mathematics. Among other fields, they appeared in number theory in the study of functions with similar properties to the Riemann ζ function. For trees, complete graphs and some complete bipartite graphs: $K_{2,n}, K_{3,n}$ their Ehrhart polynomials have roots on the complex line $\mathrm{Re}(x) = -\frac{1}{2}$ [1, 10, 9].

Conjecture. [9, Conjecture 4.10] *All roots of the Ehrhart polynomial of any $\mathcal{P}_{K_{a,b}}$ have real part equal to $-\frac{1}{2}$.*

We note that $\mathcal{P}_G$ is always reflexive, centrally symmetric and any of its boundary induced triangulations is unimodular. It is natural to study its h^*-polynomial through the so-called γ-polynomial:

$$h^*(t) = \sum_{i=0}^{\lfloor \frac{n+1}{2} \rfloor} \gamma_i t^i (1+t)^{n+1-2i},$$

where we assume that h^* is of degree $n+1$. The h^*-vector for $\mathcal{P}_G$ is the h-vector for a traingulation of a sphere. For many G we know such flag triangulations exist, which basically correspond to a quadratic Gröbner basis of the associated toric ideal. This relates to important conjectures by: Charney–Davis, Gal and Nevo-Petersen [2, 7, 12]. The strongest one may be stated as follows.

Conjecture. *If there exists a flag triangulation (in general of a homology sphere, in our case of the boundary of the reflexive polytope) then the γ-vector is positive* *and is the f-vector of a balanced simplicial complex (also probably flag simplicial complex).*

The conjecture is known to hold e.g. for $\mathcal{P}_{K_{a,b}}$ [8]. The special cases seem important to study as symmetric edge polytopes were already used to disprove conjectures, e.g. on the size of the real part of roots of the Ehrhart polynomial [13].
Recently, further connections of symmetric edge polytopes to the Kuramoto model were unraveled [5]. Kuramoto model describes the physical model – in terms of differential equations – for interacting oscillators. The oscillators may be represented by vertices of a graph G and which oscillators interact is encoded by the edges. It turns out that the maximal number of the steady states is bounded by the volume of $\mathcal{P}_G$ [4]. Further, if one knows the Gröbner basis/regular triangulation one may apply homotopy continuation techniques to actually compute the steady states [3]. This has been carried out for some families of graphs [14], [5]. One of the very interesting cases left is a join of a few cycles of odd length by an edge.

Further connections to fundamental polytopes associated to finite metric spaces are presented in [5, 6].

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Toric degenerations of tropical Grassmannians

Fatemeh Mohammadi, Megumi Harada, Lara Bossinger
(joint work with Kristin Shaw, Laura Escobar)

This extended abstract is for a non-traditional talk in which 3 speakers shared the 60 minutes to discuss 3 aspects of the study of toric degenerations of tropical Grassmannians, as well their connections.

Let X be an irreducible complex projective variety of dimension d. To study the geometry of X, one strategy that can be used is to study instead the central fiber of a *toric degeneration* $\mathcal{X}$ of X, where a toric degeneration is a flat family of varieties whose central fiber X_0 is a toric variety; the fact that both X and X_0 appear as fibers of the same family $\mathcal{X}$ means that information about X can be read off of X_0. The combinatorial data associated to toric varieties yield powerful tools for computing geometric invariants of toric varieties. Therefore, in the presence of a toric degeneration $\mathcal{X}$, it may be hoped that we can obtain geometric information about X from the combinatorics associated to X_0.

Tropical geometry enters the picture as follows. Given a variety X as above, realized as $\mathrm{Proj}(A) \cong \mathrm{Proj}(\mathbb{C}[x_1,\ldots,x_n]/I)$ where A is its homogeneous coordinate ring and $\mathbb{C}[x_1,\ldots,x_n]/I$ a choice of presentation of A, its tropicalization trop(X) (also denoted $\mathcal{T}(I)$ below) is a subset of $\mathbb{R}^n$ consisting of those (weight) vectors whose corresponding initial ideals $\mathrm{in}_w(I)$ contain no monomials. In fact, trop(X) carries additional combinatorial structure, namely, it is a $(d+1)$-dimensional subfan of the Gröbner fan. A Gröbner degeneration to an initial ideal, namely, degenerating the original ideal I of relations to the initial ideal $\mathrm{in}_w(I)$, yields a toric degeneration when the initial ideal $\mathrm{in}_w(I)$ is prime and binomial. Since primality is impossible if $\mathrm{in}_w(I)$ contains a monomial, the maximal-dimensional cones of the tropicalization trop(X) can be thought of as the set of weight vectors which provide candidates for toric degenerations.

In the context of the tropical Grassmannian, some partial results are known, but much remains open. In the case of the tropicalization $\mathrm{trop}(\mathrm{Gr}(2,n))$, i.e. the tropical Grassmanian of 2-planes in n-space, it is known from work of Speyer and Sturmfels that all the maximal-dimensional cones yield prime ideals. For the Grassmannians $\mathrm{Gr}(k,n)$ for higher values of k, fewer results are known, although some general results show the existence of one maximal prime cone in all $\mathrm{trop}(\mathrm{Gr}(k,n))$ (e.g. the Gelfand-Zeitlin cone). Mohammadi and Shaw have recently used the correspondence between tropical line arrangements and the so-called coherent matching fields from [5] to give a combinatorial characterization of the maximal cones in $\mathrm{trop}(\mathrm{Gr}(3,n))$ which yield prime ideals. To every point w in the top-dimensional cone of the $\mathrm{trop}(\mathrm{Gr}(3,n))$ whose corresponding ideal $\mathrm{in}_w(I)$ is binomial, they associate a toric ideal T_w which leads to a characterization of toric initial ideals. The following figure illustrates the condition presented in the theorem above.

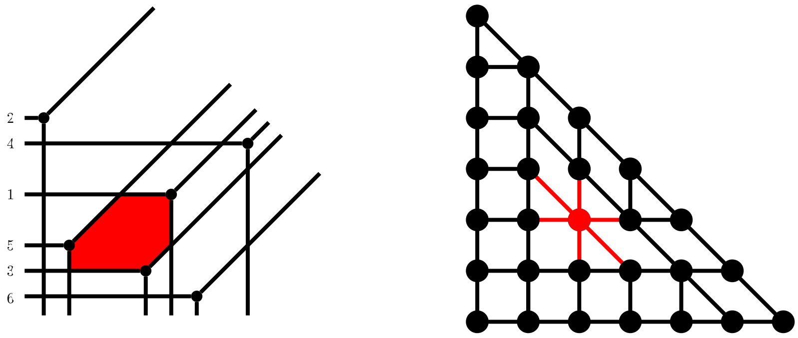

Figure 1. A tropical hyperplane with a hexagon and its dual regular subdivision.

Theorem 1 ([3]). *Suppose that the ideal T_w is quadratically generated. Then the ideal $\mathrm{in}_w(I)$ is prime if and only if the corresponding tropical line arrangement does not contain any hexagon.*

Following up to these known results for the tropical Grassmannian, we may ask: what is the relationship between the different toric degenerations which arise from different maximal prime cones? Recall that the tropicalization of X carries a fan structure (being a subfan of the Gröbner fan); hence it is natural to expect, for intsance, that if two maximal prime cones C_1 and C_2 are adjacent (i.e. they share a codimension-1 facet), then their associated toric degenerations should be in some way related.

It turns out that there is a general theory, not restricted to the case of Grassmannians, which can address this question. Several years ago Kaveh and Manon proved in [2] that the tropical geometry approach to toric degenerations is related to that of Newton-Okounkov bodies. Assuming that C is a maximal prime cone, Kaveh and Manon show that the toric degeneration associated to its initial ideal $\mathrm{in}_C(I)$ can also be obtained from the point of view of Newton-Okounkov bodies. More precisely, they construct – using any choice of rational and linearly independent vectors $u_1, \ldots, u_{d+1}$ contained in the cone C – a valuation $\nu : A \setminus \{0\} \to \mathbb{Q}^{d+1}$ with respect to which the associated graded algebra $gr_\nu(A)$ of A is isomorphic to the coordinate ring $\mathbb{C}[x_1, \ldots, x_n]/\mathrm{in}_C(I)$ obtained through Gröbner theory. From their result, it follows that the value semigroup $S(A, \nu)$ and its associated Newton-Okounkov polytope $\Delta(A, \nu)$ can be easily computed from the vectors $u_1, \ldots, u_{d+1}$.

Based on this work, Escobar and Harada [1] have recently shown that the toric degenerations of adjacent maximal prime cones can be related in the following way. Suppose now that C_1 and C_2 are adjacent maximal prime cones in $\mathrm{trop}(X)$ sharing a codimension-1 facet $C = C_1 \cap C_2$. Escobar and Harada show that there are natural choices of $\{u_1, u_2, \ldots, u_{d+1}\} \in C_1$ and $\{u'_1, u'_2, \ldots, u'_{d+1}\} \in C_2$ such that the corresponding Newton-Okounkov polytopes $\Delta(A, \nu_{M_1})$ and $\Delta(A, \nu_{M_2})$ project to the same polytope $\Delta(A, \nu_M)$ under the linear projection $\pi : \mathbb{R}^{d+1} \to \mathbb{R}^d$ which forgets the last coordinate. Moreover, they show that the fibers are of the same Euclidean length (up to a global constant). It readily follows that there are two natural piecewise-linear maps $\mathsf{F}_{12} : \Delta(A, \nu_{M_1}) \to \Delta(A, \nu_{M_2})$ and $\mathsf{S}_{12} : \Delta(A, \nu_{M_1}) \to \Delta(A, \nu_{M_2})$, the "flip" and "shift" maps respectively, which behave as the identity on the first d coordinates. We call these **(geometric) wall-crossing maps**. In addition to the (convex)-geometric wall-crossing maps discussed above, under the additional technical hypothesis that the two cones C_1 and C_2 lie in the same maximal cone of the Gröbner fan, it is also possible to construct – using a set of standard monomials coming from Gröbner theory – a natural bijection $\Theta : S(A, \nu_{M_1}) \to S(A, \nu_{M_2})$ commuting with the projection π to $S(A, \nu_M)$. We call this the **algebraic** wall-crossing. Since the semigroups $S(A, \nu_{M_i})$ for $i = 1, 2$ are subsets of the respective cones $P(A, \nu_{M_i}) := \mathrm{cone}(\Delta(A, \nu_{M_i}))$ and the maps F_{12} and S_{12} naturally extend to the level of the cones, it is natural to ask whether Θ is simply the restriction to $S(A, \nu_{M_1})$ of either of the geometric wall-crossing maps. In general, this is not true. Moreover, the map Θ is not generally straightforward to compute, but for the case of the tropical Grassmannian of 2-planes in m-space, they are able to show that the algebraic wall-crossing map Θ is the restriction of the "flip" map F_{12}.

It turns out that the above "flip" map F_{12} on $\mathrm{trop}(\mathrm{Gr}(2, n))$ can be related to the "flip" in the sense of cluster mutations, at least on a certain closed subfan called the *totally positive tropical Grassmannian*, see [4]. We denote it by $\mathrm{trop}^+(\mathrm{Gr}(2, n))$.

Proposition 2. *There exists a unique maximal cone C^+ of the Gröbner fan for the Plücker ideal $I_{2,n}$, such that $C^+ \cap \mathrm{trop}(\mathrm{Gr}(2, n)) = \mathrm{trop}^+(\mathrm{Gr}(2, n))$. Furthermore, we have bijections*

$$\left\{\begin{matrix} \textit{maximal cones} \\ \sigma \subset \mathrm{trop}^+(\mathrm{Gr}(2,n)) \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{triangulations} \\ \textit{of the } n\textit{-gon} \end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix} \textit{cluster seeds on the} \\ \textit{cluster algebra } A_{2,n} \end{matrix}\right\}.$$

Every cluster seed, in particular, gives a full-rank valuation $g_\sigma : A_{2,n} \setminus \{0\} \to \mathbb{Z}^{d+1}$, where $d = \dim(\mathrm{Gr}(2, n))$. It has one-dimensional leaves and induces the same toric degeneration of $\mathrm{Gr}(2, n)$ as the maximal prime cone σ. The cluster structure on the Grassmannian corresponds to an atlas of complex tori (one for every cluster seed, or here for every triangulation of the n-gon). The tori cover a dense open set inside $\mathrm{Gr}(2, n)$, a so-called $\mathcal{A}$-cluster variety. The transition maps between two adjacent tori in the atlas are called *mutation*. In this case, mutation is encoded in the combinatorial rule of "flipping" arcs in the triangulations corresponding to the tori.

Moreover, mutation can be *tropicalized* to give piece-wise linear maps $\mu^{\mathrm{trop}} : \mathbb{Z}^{d+1} \to \mathbb{Z}^{d+1}$. Consider two triangulations of the n-gon related by a flip, i.e. cluster mutation. Let σ and $\mu(\sigma)$ be the corresponding maximal (prime) cones in $\mathrm{trop}(\mathrm{Gr}(2, n))$. Then for every element b of the standard monomial basis of C^+ we have $\mu^{\mathrm{trop}}(g_\sigma(b)) = g_{\mu(\sigma)}(b)$. In particular, in this case, the tropical mutation coincides with the algebraic wall-crossing from above.

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Białynicki-Birula decompositions

Joachim Jelisiejew
(joint work with Łukasz Sienkiewicz)

The classical Białynicki-Birula decomposition [1] applies to a smooth complete $\mathbb{G}_m$-variety X. The subvariety $X^{\mathbb{G}_m}$ is also smooth, say with components $F_1, \dots, F_r$ and the Białynicki-Birula decomposition is the disjoint union

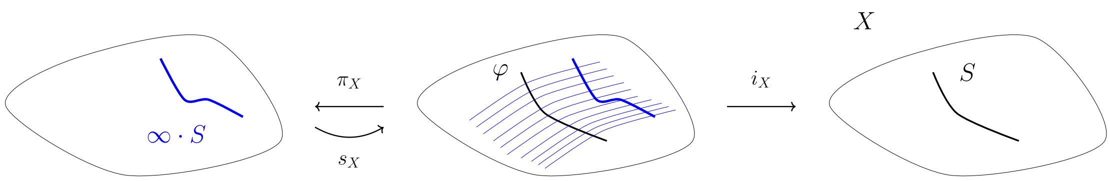
$$X^+ = X_1^+ \sqcup \ldots \sqcup X_r^+$$

where $X_i^+ = \{x \in X \mid \lim_{t\to 0} tx \in F_i\}$ is a smooth locally closed subvariety of X. In particular, the $\mathbb{G}_m$-action on X_i^+ extends to an $\mathbb{A}^1$-action, where $\mathbb{A}^1 = \mathbb{G}_m \cup \{0\}$. The multiplication by zero gives the limit map $\pi_X\colon X^+ \to X^{\mathbb{G}_m}$. The map π_X is an affine space fiber bundle with a natural section $s_X\colon X^{\mathbb{G}_m} \to X^+$. This gives immediate consequences for topology, for example to computing homology.

Generalizing to the singular case. Drinfeld [2] observed that X^+ is defined functorially as

$$X^+(S) = \left\{\varphi\colon \mathbb{A}^1 \times S \to X \mid \varphi \text{ is } \mathbb{G}_m\text{-equivariant}\right\}, \quad (1)$$

that proved that such defined X^+ exists for all varieties and for schemes of finite type. In this setting, X^+ also comes with a map $i_X\colon X^+ \to X$ resembling the embedding of X_i^+ above, and with a limit map $\pi_X\colon X^+ \to X^{\mathbb{G}_m}$ that has a section s_X. The map π_X is affine but not necessarily a bundle. The obtained structure is



The variety X^+ is a bridge between $X^{\mathbb{G}_m}$ and X. Using it we can reduce questions about X to questions about $X^{\mathbb{G}_m}$. As an example far from the smooth case [4], this decomposition is used to study *singularities* of the Hilbert scheme of points and to see there are plenty of them:

Theorem 1 ([4, Main theorem]). *The scheme* $\mathrm{Hilb}_{\mathrm{pts}}(\mathbb{A}^{16})$ *has arbitrary singularities up to retraction. In particular, it is non-reduced and for all primes p there exist finite algebras over* $\mathbb{F}_p$ *nonliftable to characteristic zero.*

Generalizing to groups other than $\mathbb{G}_m$. Jelisiejew and Sienkiewicz [5] generalized the idea of Białynicki-Birula decompositions to groups other than $\mathbb{G}_m$.

Drinfeld's functorial description (1) generalizes readily once we know what to put instead of $\mathbb{A}^1$.

One way of looking on $\mathbb{A}^1$ is that it is a monoid with multiplication and $\mathbb{G}_m$ is its group of units. Consider any connected linearly reductive affine group $\mathbf{G}$ and an affine monoid $\overline{\mathbf{G}}$ with zero that has $\mathbf{G}$ as group of units. For a $\mathbf{G}$-scheme X locally of finite type define X^+ by

$$X^+(S) = \left\{\varphi\colon \overline{\mathbf{G}} \times S \to X \mid \varphi \text{ is } \mathbf{G}\text{-equivariant}\right\}.$$

Intuitively, X^+ parameterizes points whose $\mathbf{G}$-orbit compactifies to a $\overline{\mathbf{G}}$-orbit. The X^+ is represented by a scheme and comes with maps π_X, i_X, s_X defined as above, where $0 \in \overline{\mathbf{G}}$ is used in place of $0 \in \mathbb{A}^1$. For the pair $(\mathbf{G}, \overline{\mathbf{G}}) = (\mathbb{G}_m, \mathbb{A}^1)$ we recover the classical case described above. For all $\overline{\mathbf{G}}$ and for smooth X, the result of Białynicki-Birula generalizes verbatim:

Theorem 2 ([5, Theorem 1.5]). *The schemes* X^+ *and* $X^{\mathbf{G}}$ *are smooth and* π_X *is a bijection between their connected components. The morphism*

$$\pi_X\colon X^+ \to X^{\mathbf{G}}$$

is an affine space fiber bundle with a $\overline{\mathbf{G}}$*-action fiber-wise. Moreover, each component of* X^+ *is a locally closed subvariety of* X *via* i_X*.*

We can easily check whether a given point lies in a dominant cell:

Proposition 3. *If* $x \in X^{\mathbf{G}}$ *is such that the* $\mathbf{G}$*-action on* $T_{X,x}$ *extends to a* $\overline{\mathbf{G}}$*-action, then* $i_X\colon X^+ \to X$ *is an open immersion near* $x \in X^+$*. In this case* $x \in X$ *has a Zariski-open affine* $\mathbf{G}$*-stable neighborhood.*

For larger groups, the map $X^+ \to X$ may be non dominant, but still useful for investigations.

Example. Let X be a smooth complete variety with a $T = \mathbb{G}_m^2$-action. Let $T^{(1)} = \{(*, 1)\} \subset T$ and $T^{(2)} = \{(1, *)\} \subset T$. Assume that $X^{T^{(1)}}$, $X^{T^{(2)}}$ are both finite and take $x \in X^{T^{(1)}} \cap X^{T^{(2)}}$. The classical Białynicki-Birula decomposition for $T^{(i)}$ $(i = 1, 2)$ gives a cell $x \in X_j^{+,(i)} \subset X$ which is an affine space. The Białynicki-Birula decomposition for T and monoid $\mathbb{A}^2$ gives also a cell $x \in X_j^+$ which is an affine space. Moreover, $X_j^+ = X_j^{+,(1)} \cap X_j^{+,(2)}$. The upshot is that the intersection of affine cells $X_j^{+,(1)}$ and $X_j^{+,(2)}$ is an affine space.

In summary, there are three main advantages of the generalized Białynicki-Birula decomposition. First, we may deal with singularities efficiently. Second, we may work with large groups. Third, using the functorial description we may work relatively. The main current research directions are: to find new applications, perhaps to other moduli spaces and to think about a generalization from $\overline{\mathbf{G}}$ to a more flexible type of compactification. There is also a handful of more specific problems, e.g. related to deformations of curves.

An open direction of Białynicki-Birula decompositions is concerned with nonreductive groups. For the additive group $\mathbb{G}_a$ there are no monoids with zero that compactify it, but we can take the canonical compactification $\overline{\mathbf{G}} := \mathbb{P}^1$ interpreted as $\mathbb{G}_a \cup \{\infty\}$. For a projective $\mathbb{G}_a$-variety X we then define

$$X^+(S) = \left\{\varphi\colon \mathbb{P}^1 \times S \to X \mid \varphi \text{ is } \mathbb{G}_a\text{-equivariant}\right\},$$

and show that $X^+ \to X^{\mathbb{G}_a}$ is affine and, in special cases, an affine fiber bundle component-wise [work in progress]. This is even more surprising as $X^{\mathbb{G}_a}$ is connected [3] but still X^+ is very much disconnected.

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The Mori Dream Space property of blow-ups of projective spaces at points and lines

Zhuang He

(joint work with Lei Yang)

We work over $\mathbb{C}$. Let X be the blow-up of $\mathbb{P}^3$ at 6 points in very general position and all the 15 lines passing through pairs of them. Write $\overline{\mathrm{Eff}}(X)$ for the cone of pseudo-effective divisors on X.

Question. *Is $\overline{\mathrm{Eff}}(X)$ rational polyhedral? Is X a Mori Dream Space?*

Mori Dream Spaces are introduced by [9], as natural generalizations of toric varieties. The name hints the fact that Mori's minimal program can be run for any divisor on a Mori Dream Space. The effective cone of a Mori Dream Space is rational polyhedral, with a decomposition into convex polytopes called *Mori Chambers*, which parameterizes birational contractions from this space.

We recall some results on the birational geometry of the blow-ups of $\mathbb{P}^n$ at points. Let X^n_r be the blow-up of $\mathbb{P}^n$ at r very general points. By [4, 13], X^n_r is a Mori Dream Space if and only if

$$\frac{1}{r-n-1} + \frac{1}{n+1} \geq \frac{1}{2}.$$

That is, $r \leq 8$ for $\mathbb{P}^2$ and $\mathbb{P}^4$, $r \leq 7$ for $\mathbb{P}^3$, and $r \leq n+3$ for $n \geq 5$. Furthermore, X_r^n is a Mori Dream Space if and only if $\overline{\mathrm{Eff}}(X_r^n)$ is rational polyhedral. When X_r^n is a Mori Dream Space, X_r^n is in fact log Fano [1], and the effective cone of higher codimension cycles are all rational polyhedral [6].

In another direction, the moduli space $\overline{M}_{0,6}$ is isomorphic to the blow-up of $\mathbb{P}^3$ at 5 general points and all the 10 lines through them, via Kapranov's blow-up construction of $\overline{M}_{0,n}$ [11]. Since $\overline{M}_{0,6}$ is log Fano, it is a Mori Dream Space. For other $\overline{M}_{0,n}$, [5] proved that $\overline{M}_{0,n}$ is not a Mori Dream Space for $n > 133$, which was later improved by [8] and [7] to $n \geq 10$.

We say a birational map $f : X \dashrightarrow X$ is a pseudo-automorphism if both f and f^{-1} contract no divisors. Our main result is the following:

Theorem 1. *There exists an infinite-order pseudo-automorphism ϕ of X, induced by the complete linear system of the following divisor:*

$$\begin{aligned} D := \quad & 13H - 7(E_1 + E_2 + E_5) - 5(E_0 + E_3 + E_4) \\ & - 3(E_{03} + E_{04} + E_{34}) - 4(E_{05} + E_{13} + E_{24}) - (E_{12} + E_{15} + E_{25}). \end{aligned}$$

The effective cone $\overline{\mathrm{Eff}}(X)$ has infinitely many extremal rays. Hence $\overline{\mathrm{Eff}}(X)$ is not rational polyhedral, and X is not Mori Dream.

We discovered this infinite-order pseudo-automorphism ϕ by considering Keum's 192 infinite-order automorphisms of Jacobian K3 Kummer surfaces [12]. In our context, when the six points are very general in $\mathbb{P}^3$, X has a unique anticanonical section S which is a smooth K3 Kummer surface of Picard rank 17, associated to the Jacobian of a smooth curve of genus 2. Keum first constructed 192 infinite-order automorphisms of a Jacobian K3 Kummer surface S of Picard number 17, each associated to one of 192 combinatorial objects called *Weber Hexads.*

Any pseudo-automorphism of X must fix the unique anticanonical section S, hence restricts to an automorphism of S.

Theorem 2. *The restriction $\phi_{|S} : S \to S$ equals to Keum's automorphism associated to the Weber Hexad $\{1, 2, 5, 12, 14, 23\}$.*

The restriction of ϕ to S is a different construction of some of Keum's automorphisms from Keum's original approach. We discovered this pseudo-automorphism ψ by considering the divisor classes on X whose restriction to S are (-2)-curves or the divisor class inducing Keum's automorphisms. We used *Macaulay2* to verify the birationality of ϕ for special six points before we found a formal proof.

We actually proved the following improvement of Theorem 1. Let Y be the blow-up of $\mathbb{P}^3$ at six very general points $p_0, \cdots, p_5$ and the 9 lines $\overline{p_i p_j}$ indexed by

$$(ij) \in I = \{03, 04, 34, 12, 15, 25, 05, 13, 24\}.$$

Then ϕ is in fact a pseudo-automorphism of Y. Hence $\overline{\mathrm{Eff}}(Y)$ is not rational polyhedral, and Y is not Mori Dream.

We next state an application to the blow-ups of $\mathbb{P}^n$ at points and lines.

Question. *Consider the blow-up of $\mathbb{P}^n$ ($n \geq 3$) at ($n+3$) points in general position and certain lines through the ($n+3$) points. For what configurations of the lines is the effective cone of the blow-up rational polyhedral?*

For $n \geq 3$, we define Y_n to be the blow-up of $\mathbb{P}^n$ at $(n+3)$ points in very general position and 9 lines through six of them, such that when the six points are indexed by 0 to 5, the 9 lines are labeled by $I = \{03, 04, 34, 12, 15, 25, 05, 13, 24\}$. In particular $Y_3 = Y$ as defined above.

Theorem 3. *For $n \geq 3$, $\overline{\mathrm{Eff}}(Y_n)$ is not rational polyhedral, and Y_n is not Mori Dream.*

Theorem 3 has an application to $\overline{M}_{0,n}$:

Theorem 4. *For $n = 7, 8$ and 9, the blow-up of $\overline{M}_{0,n}$ at a very general point is not a Mori Dream Space.*

The pseudo-automorphism ϕ induces a birational map $\psi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$, which has a surprising interaction with the structure theory of $\mathrm{Bir}(\mathbb{P}^3)$. By a classical result of Max Noether and Castelnuovo [3], the birational automorphism group $\mathrm{Bir}(\mathbb{P}^2)$ is generated by PGL(3) and the standard cremona $\sigma_2 : [x : y : z] \mapsto [1/x : 1/y : 1/z]$. For $n \geq 3$ the analogue is false, where $\mathrm{Bir}(\mathbb{P}^n)$ is strictly larger than the subgroup $G_n := \langle \mathrm{PGL}(n+1), \sigma_n \rangle$ [10, 14]. A natural question is whether every $\sigma \in \mathrm{Bir}(\mathbb{P}^n)$ which only contracts rational hypersurfaces are in G_n. [2] disproved the equality by examples of monomial birational maps which only contract rational hypersurfaces but not in G_n in odd dimensions. We applied their criterion [2, Thm. 1.4] characterizing elements in G_n and find:

Theorem 5. *The map ψ only contracts rational hypersurfaces but $\psi \notin G_3$.*

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Triangulations of simplices with vanishing local h-polynomial

Sam Payne
(joint work with E. Gunther, A. Moura, J. Schuchardt, and A. Stapledon)

Let Γ be a triangulation of a simplex Δ of dimension $d-1$. The h-polynomial $h(\Gamma;x) = h_0 + h_1x + \cdots + h_dx^d$ is characterized by the equation

$$\sum_{i=0}^{d} h_i(x+1)^{d-i} = \sum_{i=0}^{d} f_{i-1}x^{d-i},$$

where $f_{-1} = 1$ and f_i is the number of i-dimensional faces of Γ, for $i \geq 0$. Its coefficients are non-negative integers.

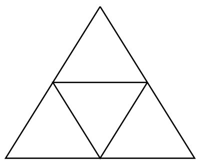
This talk focused on the local h-polynomial $\ell(\Gamma;x) = \ell_0 + \ell_1x + \cdots + \ell_dx^d$, introduced by Stanley in his seminal paper [2], which is a powerful tool for studying $h(\Gamma;x)$, and a fascinating invariant in its own right. Roughly speaking, $\ell(\Gamma;x)$ encodes the portion of $h(\Gamma;x)$ that is not accounted for by the restriction of Γ to the proper faces of Δ. More precisely, it is characterized via Möbius inversion by the equation

$$h(\Gamma;x) = \sum_{F \leq \Delta} \ell(\Gamma_F;x),$$

where Γ_F denotes the restriction of the triangulation Γ to a face F (which may be empty or all of Δ), together with the condition $\ell(\emptyset;x) = 1$.

The coefficients of $\ell(\Gamma;x)$ are nonnegative and satisfy $\ell_i = \ell_{d-i}$. Moreover, if the triangulation is regular, then these coefficients are unimodal. Among other applications, Stanley used local h-polynomials to prove that h-polynomials increase coefficientwise under refinement.

After introducing local h-polynomials and reviewing these basic properties, I discussed recent progress toward understanding the structure of triangulations Γ such that $\ell(\Gamma;x) = 0$, from [1]. One interesting example for $d = 3$, is the following triangulation, which we call the *triforce*[1].



[1] This name reflects the subdivision's realization in a sacred golden relic that is the ultimate source of power in the action-adventure video game series *The Legend of Zelda*.

For $d \leq 4$, we show that all such triangulations are obtained from either the trivial subdivision or the triforce by iterating certain subdivision operations that preserve local h, which may be described as follows.

Roughly speaking, the local h-polynomial is additive for refinements that nontrivially subdivide only one facet, multiplicative for joins, and vanishes on the trivial subdivision. In particular, if Γ' is a refinement of Γ that nontrivially subdivides only one facet, and that subdivision is the cone over a subdivision of a codimension 1 face, then $\ell(\Gamma'; x) = \ell(\Gamma; x)$. We call such subdivisions *conical facet refinements*.

Theorem 1. *For $d = 3$, any triangulation with vanishing local h-polynomial is obtained from either the trivial subdivision or the triforce by a sequence of conical facet refinements.*

Theorem 2. *For $d = 4$, any triangulation with vanishing local h-polynomial is obtained from the trivial subdivision by a sequence of conical facet refinements.*

Although these statements do not generalize naïvely to higher dimensions, the proofs are based on a careful analysis of the graph formed by the union of the edges of Γ that meet the interior of Δ, which we call the *internal edge graph*. Its structure does not change in higher dimensions.

Theorem 3. *Suppose $d \geq 4$ and $\ell_1 = \ell_2 = 0$. Then the internal edge graph of Γ is a union of trees each of which contains exactly one vertex supported on a face of codimension at least 2.*

References

[1] E. Gunther, A. Moura, S. Payne, J. Schuchardt, and A. Stapledon, *Triangulations of simplices with vanishing local h-polynomial*, preprint, 2019.

[2] R. Stanley, *Subdivisions and local h-vectors*, J. Amer. Math. Soc. **5** (1992), no. 4, 805–851.

Reporter: Christoff Hische

Participants

Prof. Dr. Karim Adiprasito
Einstein Institute for Mathematics
The Hebrew University of Jerusalem
Edmond J. Safra Campus
Givat Ram
Jerusalem 9190401
ISRAEL

Prof. Dr. Dave Anderson
Department of Mathematics
The Ohio State University
100 Mathematics Building
231 West 18th Avenue
Columbus, OH 43210-1174
UNITED STATES

Prof. Dr. Ivan Arzhantsev
Department of Computer Science
National Research University
Higher School of Economics
20 Myasnitskaya St.
Moscow 101 000
RUSSIAN FEDERATION

Prof. Dr. Victor V. Batyrev
Fachbereich V - Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Christine Berkesch
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455
UNITED STATES

Prof. Dr. Lev A. Borisov
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Dr. Lara Bossinger
Instituto de Matemáticas UNAM
Unidad Oaxaca
Centro Historico
Antonio de Léon # 2, altos
Oaxaca de Juárez, CP 68000
MEXICO

Dr. Michel Brion
Laboratoire de Mathématiques
Université de Grenoble Alpes
Institut Fourier, Bureau 43C
100 rue des Maths
38610 Gières Cedex
FRANCE

Prof. Dr. Alicia Dickenstein
Departamento de Matemática (FCEN)
Universidad de Buenos Aires
Pabellon 1 - Ciudad Universitaria
1428 Buenos Aires C-EGA
ARGENTINA

Prof. Dr. Sandra Di Rocco
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Dr. Eleonore Faber
School of Mathematics
University of Leeds
Rm. 8.05
Leeds LS2 9JT
UNITED KINGDOM

Dr. Francesco Galuppi
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. Antonella Grassi
Department of Mathematics
University of Pennsylvania
David Rittenhouse Laboratory
209 South 33rd Street
Philadelphia, PA 19104-6395
UNITED STATES

Prof. Dr. Christian Haase
Institut für Mathematik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Megumi Harada
Department of Mathematics and
Statistics
McMaster University
1280 Main Street West
Hamilton ON L8S 4K1
CANADA

Prof. Dr. Jürgen Hausen
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Yang-Hui He
School of Mathematics, Computer
Science
and Engineering
City, University of London
Northampton Square
London EC1V 0HB
UNITED KINGDOM

Zhuang He
Department of Mathematics
Northeastern University
567 Lake Hall
Boston MA 02115-5000
UNITED STATES

Dr. Milena Hering
School of Mathematics
The University of Edinburgh
James Clerk Maxwell Building
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Christoff Hische
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Nathan Ilten
Department of Mathematics and
Statistics
Simon Fraser University
Room SSC K 10517
8888 University Drive
Burnaby BC V5A 1S6
CANADA

Dr. Joachim Jelisiejew
Institute of Mathematics of the
Polish Academy of Sciences
ul. Sniadeckich 8
00-956 Warszawa
POLAND

Prof. Dr. Thomas Kahle
FMA - IAG
Otto-von-Guericke-Universität
Magdeburg
Universitätsplatz 2
39106 Magdeburg
GERMANY

Prof. Dr. Kiumars Kaveh
Department of Mathematics
University of Pittsburgh
301 Thackery Hall
Pittsburgh, PA 15260
UNITED STATES

Prof. Dr. Antonio Laface
Departamento de Matematica
Facultad de Ciencias Fisicas y
Matematicas
Universidad de Concepción
Casilla 160-C
Concepción
CHILE

Dr. Sara Lamboglia
Institut für Mathematik
Goethe-Universität Frankfurt
Robert-Mayer-Straße 6-10
60325 Frankfurt am Main
GERMANY

Dr. Martina Lanini
Dipartimento di Matematica
Università degli Studi di Roma II
"Tor Vergata"
Via della Ricerca Scientifica
00133 Roma
ITALY

Prof. Dr. Oliver Lorscheid
Instituto Nacional de Matematica
Pura e Aplicada - (IMPA), Room 350
Jardim Botanico
Estrada Dona Castorina, 110
22460 Rio de Janeiro, RJ 320
BRAZIL

Prof. Dr. Diane Maclagan
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Christopher Manon
Department of Mathematics
University of Kentucky
715 Patterson Office Tower
Lexington, KY 40506-0027
UNITED STATES

Dr. Mateusz Michalek
Max-Planck-Institut für Mathematik
in den Naturwissenschaften
Inselstrasse 22 - 26
04103 Leipzig
GERMANY

Prof. Dr. Fatemeh Mohammadi
Department of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UNITED KINGDOM

Dr. Leonid Monin
Department of Mathematics
University of Bristol
University Walk
Bristol BS8 1TW
UNITED KINGDOM

Prof. Dr. Benjamin Nill
Fakultät für Mathematik
Institut für Algebra und Geometrie
Otto-von-Guericke-Universität
Magdeburg
Postfach 4120
39016 Magdeburg
GERMANY

Prof. Dr. Sam Payne
Department of Mathematics
The University of Texas at Austin
1 University Station C1200
Austin, TX 78712-1082
UNITED STATES

Dr. Andrea Petracci
Institut für Mathematik
Freie Universität Berlin
Raum 115
Arnimallee 3
14195 Berlin
GERMANY

Andriy Regeta
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. Konstanze Rietsch
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Hiroshi Sato
Department of Applied Mathematics
Fukuoka University
8 Chome-19-1 Nanakuma, Jonan-ku
Fukuoka 814-0180
JAPAN

Dr. Karin Schaller
Fachbereich Mathematik und Informatik
Freie Universität Berlin
Arnimallee 3
14195 Berlin
GERMANY

Prof. Dr. Henry K. Schenck
Department of Mathematics
Iowa State University
Ames, IA 50011
UNITED STATES

Prof. Dr. Gregory G. Smith
Department of Mathematics and
Statistics
Queen's University
Jeffery Hall
Kingston ON K7L 3N6
CANADA

Prof. Dr. Martín Sombra
Facultat de Matematiques
Departament d'Algebra i Geometria
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Catalonia
SPAIN

Prof. Dr. Frank Sottile
Department of Mathematics
Texas A & M University
Blocker 601K
1401 Post Oak Circle
College Station, TX 77843-3368
UNITED STATES

Dr. Spela Spenko
Department of Mathematics
Vrije Universiteit Brussel
Pleinlaan 2
1050 Bruxelles
BELGIUM

Prof. Dr. Michael Stillman
Department of Mathematics
Cornell University
503 Malott Hall
Ithaca, NY 14853-4201
UNITED STATES

Dr. Hendrik Süß
School of Mathematics
The University of Manchester
Alan Turing Building, Room 1.106a
Oxford Road
Manchester M13 9PL
UNITED KINGDOM

Dr. Bernard Teissier
IMJ - PRG
Bâtiment Sophie Germain
Case 7012
8, Place Aurélie Nemours
75205 Paris Cedex 13
FRANCE

Dr. Weikun Wang
Department of Mathematics
University of Maryland
2303 Kirwan Hall
College Park, MD 20742-4015
UNITED STATES

Prof. Dr. Jaroslaw Wisniewski
Instytut Matematyki
Uniwersytet Warszawski
ul. Banacha 2
02-097 Warszawa
POLAND

Dr. Milena Wrobel
Institut für Mathematik
Carl-von-Ossietzky-Universität
Oldenburg
Ammerländer Heerstrasse 114-118
26129 Oldenburg
GERMANY