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Unexpected properties of the Klein configuration of 60 points in $\mathbb{P}^3$

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Abstract

Felix Klein in course of his study of the regular icosahedron and its symmetries encountered a highly symmetric configuration of 60 points in $\mathbb{P}^3$. This configuration has appeared in various guises, perhaps post notably as the configuration of points dual to the 60 reflection planes in the group G_{31} in the Shephard-Todd list.

In the present note we show that the 60 points exhibit interesting properties relevant from the point of view of two paths of research initiated recently. Firstly, they give rise to two completely different unexpected surfaces of degree 6. Unexpected hypersurfaces have been introduced by Cook II, Harbourne, Migliore, Nagel in 2018. One of unexpected surfaces associated to the configuration of 60 points is a cone with a single singularity of multiplicity 6 and the other has three singular points of multiplicities $4, 2$ and 2. Secondly, Chiantini and Migliore observed in 2020 that there are non-trivial sets of points in $\mathbb{P}^3$ with the surprising property that their general projection to $\mathbb{P}^2$ is a complete intersection. They found a family of such sets, which they called grids. An appendix to their paper describes an exotic configuration of 24 points in $\mathbb{P}^3$ which is not a grid but has the remarkable property that its general projection is a complete intersection. We show that the Klein configuration is also not a grid and it projects to a complete intersections. We identify also its proper subsets, which enjoy the same property.

1 Introduction

Arrangements of hyperplanes defined by finite reflection groups and sets of points corresponding to the duals of the hyperplanes are a rich source of examples and a testing ground for various conjectures in commutative algebra and algebraic geometry. It is not surprising that they find their place in new paths of research developed within these two branches of mathematics.

In the present note we study the configuration Z_{60} of 60 points in $\mathbb{P}^3$ determined by the reflection group G_{31} in the Shephard Todd list [11]. This configuration has been known for long, its origins go back to Felix Klein's doctoral thesis [8]. We recall basic properties, relevant for our study in Section 2. In section 3 we discuss another group acting on the configuration, discovered only recently by Cheltsov and Shramov [2]. In fact, their article was a departure point for this work.

Our main results are presented in the two subsequent sections. The first results concern unexpected surfaces admitted by the set Z_{60}. The concept of *unexpected hypersurfaces* has

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been introduced first for curves in the ground breaking article [4] by Cook II, Harbourne, Migliore and Nagel and generalized to arbitrary dimensions in the subsequent article [6] by Harbourne, Migliore, Nagel and Teitler. Loosely speaking, a finite set of points Z in a projective space admits an unexpected hypersurface of degree d if a general fat point or a finite number of general fat points impose less conditions on forms of degree d vanishing at Z than naively expected, see Definition 4.1 for precise statement. In Theorems 4.3 and 4.4 we show that Z_{60} admits two different types of unexpected surfaces. One is a cone with a single singular point of multiplicity 6 and the other is a surface of degree 6 with three singular points of multiplicities $4, 2$ and 2. It seems to be the first case where a fixed set of points admits unexpected hypersurfaces in two distinct ways. This is a new and quite unexpected phenomenon.

Theorem 5.5 goes in different direction. Chiantini and Migliore realized that there are sets of points spanning the whole $\mathbb{P}^3$ with the striking property that their general projections to a plane are complete intersections. We say that the set has the *geproci* property, see Definition 5.1. In their recent work [3] they construct a series of examples of such sets, which they call grids, as they result as intersection points of two families of lines in $\mathbb{P}^3$ organized so that lines in both families are disjoint but each line from one family intersects all lines from the other family. They showed that for a small number of points, grids are the only sets with the *geproci* property. On the other hand, in the appendix to their work there is an example of a set of 24 points which has the *geproci* property but is not a grid. We show that the set Z_{60} behaves in the same way. It is not a grid but it has the *geproci* property. More precisely its projection from a general point in $\mathbb{P}^3$ to $\mathbb{P}^2$ is a complete intersection of curves of degree 6 and 10. This provides in particular a positive answer to Question 7.1 in [3].

We work over the field of complex numbers.

2 Klein's arrangement (60_{15}) – a historical outline

We will follow a classical construction by F. Klein with a modern glimpse. Let $\mathrm{Gr}(2,4) = Q$ be the Grassmannian of lines in $\mathbb{P}^3$ embedded via the Plücker embedding as a smooth quadric hypersurface in $\mathbb{P}^5$. Let $L = H \cap Q$ with H being a hyperplane in $\mathbb{P}^5$, then L is called as a *linear line complex*. This object was studied by F. Klein in his PhD thesis, see for instance [8]. In particular, using Klein's language, there are six fundamental linear complexes corresponding to the choice of the coordinate planes $H_i = \{x_i = 0\}$ for homogeneous coordinated $x_0, \dots, x_5$ on $\mathbb{P}^5$. The intersections

$$Q \cap H_{i_1} \cap H_{i_2} \cap H_{i_3} \cap H_{i_4}$$

with $0 \le i_1 < i_2 < i_3 < i_4 \le 5$ consist of exactly two points each. These lines corresponding to the points form a configuration of

$$2 \cdot \binom{6}{4} = 30 \text{ lines in } \mathbb{P}^3.$$

We denote their union by $\mathbb{L}_{30}$. These lines intersect by 3 in 60 points. Klein showed that these points can be divided into 15 subsets of 4 points. Each subset determines vertices of a fundamental tetrahedron. The edges of these tetrahedra are contained in the 30 lines. The faces and the vertices are all mutually distinct, this gives an (60_{15}) configuration of 60 planes and 60 vertices: through each vertex there pass exactly 15 planes, and each plane contains exactly 15 vertices. Moreover, through each of the 30 edges 6 planes pass, and each edge contains 6 vertices. It is well-known by a result due to Shephard and Todd [11] that this arrangement is defined by the unitary reflection group, denoted there by $G_{11,520}$. In the Orlik-Solomon notation the symbol $\mathcal{A}_2^3(60)$ is used for Klein's (60_{15})-arrangement. Following Hunt's notation for arrangements of planes in $\mathbb{P}^3$ [7], we denote by t_i the number of points where exactly $i \geq 3$ of planes from the arrangement intersect, and by $t_j(1)$ the number of lines contained in exactly $j \geq 2$ planes in the arrangement. For $\mathcal{A}_2^3(60)$ we have:

$$t_{15} = 60, \quad t_6 = 480, \quad t_4 = 960,$$

$$t_6(1) = 30, \quad t_3(1) = 320, \quad t_2(1) = 360.$$

It is worth noticing that the set Z_{60} and $\mathbb{L}_{30}$ build a $(60_3, 30_6)$ configuration of points and lines.

To the arrangement $\mathcal{A}_2^3(60)$ we can associate an interesting arrangement of 10 quadrics – these are called in literature *Klein's fundamental quadrics*. It is worth emphasizing that through each of 30 lines described above there are 4 fundamental quadrics containing that line. We refer to [2] for the table of incidences between the 30 lines and 10 Klein's fundamental quadrics.

3 A finite Heisenberg group and a group of order 80

Let $H_{2,2}$ be the subgroup of $\mathrm{SL}_4(\mathbb{C})$ generated by the following four matrices:

$$S_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; S_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \; T_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

This group has order 32 (note that all pairs of generators commute except of $S_1T_1 = -T_1S_1$) and the center and the commutator of $H_{2,2}$ are both equal to $\{\mathbb{I}, -\mathbb{I}\}$, where $\mathbb{I}$ denotes the identity matrix of size 4. We call $H_{2,2}$ the (finite) *Heisenberg group.*

Consider the natural projection $\phi : \mathrm{SL}_4(\mathbb{C}) \to \mathrm{PGL}_4(\mathbb{C})$ and for every group $G \subset \mathrm{SL}_4(\mathbb{C})$ we define $\overline{G} = \phi(G)$. Denote by G_{80} the subgroup in $\mathrm{SL}_4(\mathbb{C})$ generated by $H_{2,2}$ and the following matrix

$$T = \frac{1+i}{2} \begin{pmatrix} -i & 0 & 0 & i \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & -i & i & 0 \end{pmatrix},$$

and by $\overline{G_{80}}$ the image of G_{80} in $\mathrm{PGL}_4(\mathbb{C})$.

With the notation fixed, we are now in the position to endow the configurations described in Section 2 with coordinates. Checking all properties of Klein's arrangement listed in Section 2 boils thus to elementary linear algebra. First, one can show that Klein's fundamental quadrics are invariant under the action of the group $\overline{G_{80}}$, and this group splits them into two orbits (note that the quadrics are $H_{2,2}$ invariant):

$$\begin{aligned} \mathcal{Q}_1 &= x^2+y^2+z^2+w^2, \\ \mathcal{Q}_2 = T(\mathcal{Q}_1) &= xw+zy, \\ \mathcal{Q}_3 = T^2(\mathcal{Q}_1) &= xz+yw, \\ \mathcal{Q}_4 = T^3(\mathcal{Q}_1) &= x^2+y^2-z^2-w^2, \\ \mathcal{Q}_5 = T^4(\mathcal{Q}_1) &= x^2-y^2-z^2+w^2, \\ \mathcal{Q}_6 &= x^2-y^2+z^2-w^2, \\ \mathcal{Q}_7 = T(\mathcal{Q}_6) &= xw-yz, \\ \mathcal{Q}_8 = T^2(\mathcal{Q}_6) &= xy+zw, \\ \mathcal{Q}_9 = T^3(\mathcal{Q}_1) &= xy-zw, \\ \mathcal{Q}_{10} = T^4(\mathcal{Q}_1) &= xz-yw. \end{aligned}$$

Using equations of quadrics and taking into account that each pair of them intersects in 4 of 30 Klein's lines, we identify the equations of lines

$$\begin{aligned}
&\ell_1 = V(x,y), && \ell_2 = V(z-w,x-y), \\
&\ell_3 = V(z-w,x+y), && \ell_4 = V(z+i\cdot w,x+i\cdot y), \\
&\ell_5 = V(z+i\cdot w,x-i\cdot y), && \ell_6 = V(z+w,x-y), \\
&\ell_7 = V(z+w,x+y), && \ell_8 = V(z-i\cdot w,x+i\cdot y), \\
&\ell_9 = V(z-i\cdot w,x-i\cdot y), && \ell_{10} = V(z,x), \\
&\ell_{11} = V(y-w,x-z), && \ell_{12} = V(y-w,x+z), \\
&\ell_{13} = V(y+i\cdot w,x+i\cdot z), && \ell_{14} = V(y+i\cdot w,x-i\cdot z), \\
&\ell_{15} = V(y+w,x-z), && \ell_{16} = V(y+w,x+z), \\
&\ell_{17} = V(y-i\cdot w,x+i\cdot z), && \ell_{18} = V(y-i\cdot w,x-i\cdot z), \\
&\ell_{19} = V(w,x), && \ell_{20} = V(y-z,x-w), \\
&\ell_{21} = V(y-z,x+w), && \ell_{22} = V(y+i\cdot z,x+i\cdot w), \\
&\ell_{23} = V(y+i\cdot z,x-i\cdot w), && \ell_{24} = V(y+z,x-w), \\
&\ell_{25} = V(y+z,x+w), && \ell_{26} = V(y-i\cdot z,x+i\cdot w), \\
&\ell_{27} = V(y-i\cdot z,x-i\cdot w), && \ell_{28} = V(z,y), \\
&\ell_{29} = V(w,y), && \ell_{30} = V(w,z).
\end{aligned}$$

Finally, taking intersection points of lines, we identify coordinates of points in the Z_{60} set

$$\begin{array}{lll}
P_1 = [0:0:1:1] & P_2 = [0:0:1:i] & P_3 = [0:0:1:-1] \\
P_4 = [0:0:1:-i] & P_5 = [0:1:0:1] & P_6 = [0:1:0:i] \\
P_7 = [0:1:0:-1] & P_8 = [0:1:0:-i] & P_9 = [0:1:1:0] \\
P_{10} = [0:1:i:0] & P_{11} = [0:1:-1:0] & P_{12} = [0:1:-i:0] \\
P_{13} = [1:0:0:1] & P_{14} = [1:0:0:i] & P_{15} = [1:0:0:-1] \\
P_{16} = [1:0:0:-i] & P_{17} = [1:0:1:0] & P_{18} = [1:0:i:0] \\
P_{19} = [1:0:-1:0] & P_{20} = [1:0:-i:0] & P_{21} = [1:1:0:0] \\
P_{22} = [1:i:0:0] & P_{23} = [1:-1:0:0] & P_{24} = [1:-i:0:0] \\
P_{25} = [1:0:0:0] & P_{26} = [0:1:0:0] & P_{27} = [0:0:1:0] \\
P_{28} = [0:0:0:1] & P_{29} = [1:1:1:1] & P_{30} = [1:1:1:-1] \\
P_{31} = [1:1:-1:1] & P_{32} = [1:1:-1:-1] & P_{33} = [1:-1:1:1] \\
P_{34} = [1:-1:1:-1] & P_{35} = [1:-1:-1:1] & P_{36} = [1:-1:-1:-1] \\
P_{37} = [1:1:i:i] & P_{38} = [1:1:i:-i] & P_{39} = [1:1:-i:i] \\
P_{40} = [1:1:-i:-i] & P_{41} = [1:-1:i:i] & P_{42} = [1:-1:i:-i] \\
P_{43} = [1:-1:-i:1] & P_{44} = [1:-1:-i:-i] & P_{45} = [1:i:1:i] \\
P_{46} = [1:i:1:-i] & P_{47} = [1:-i:1:i] & P_{48} = [1:-i:1:-i] \\
P_{49} = [1:i:-1:i] & P_{50} = [1:i:-1:-i] & P_{51} = [1:-i:-1:i] \\
P_{52} = [1:-i:-1:-i] & P_{53} = [1:i:i:1] & P_{54} = [1:i:-i:1] \\
P_{55} = [1:-i:i:1] & P_{56} = [1:-i:-i:1] & P_{57} = [1:i:i:-1] \\
P_{58} = [1:i:-i:-1] & P_{59} = [1:-i:i:-1] & P_{60} = [1:-i:-i:-1].
\end{array}$$

The following lemma proved in [2, Lemma 3.18] gives useful geometric information about Z_{60} and $\mathbb{L}_{30}$.

Lemma 3.1 (Line-point incidences)**.** *The set $Z_{60} \subset \mathbb{P}^3$ contains all the intersection points of lines in $\mathbb{L}_{30}$. Moreover, for every point $P \in Z_{60}$ there are exactly three lines from $\mathbb{L}_{30}$ passing through P.*

Now we turn to algebraic properties of the ideal of points in Z_{60}.

Lemma 3.2 (Generators of $I(Z_{60})$)**.** *The ideal $J = I(Z_{60})$ of points in Z_{60} is generated by 24 forms of degree 6.*

Proof. We want to show that the following, particularly nice forms, generate J, namely

$$\begin{array}{llll}
xy(x^4-y^4) & xz(z^4-x^4) & xw(x^4-w^4) & yz(y^4-z^4) \\
yw(w^4-y^4) & zw(z^4-w^4) & xy(z^4-w^4) & xz(y^4-w^4) \\
xw(y^4-z^4) & yz(x^4-w^4) & yw(x^4-z^4) & zw(x^4-y^4) \\
yw(x^2y^2-z^2w^2) & xw(x^2y^2-z^2w^2) & yz(x^2y^2-z^2w^2) & xz(x^2y^2-z^2w^2) \\
zw(x^2z^2-y^2w^2) & xw(x^2z^2-y^2w^2) & yz(x^2z^2-y^2w^2) & xy(x^2z^2-y^2w^2) \\
zw(y^2z^2-x^2w^2) & yw(y^2z^2-x^2w^2) & xz(y^2z^2-x^2w^2) & xy(y^2z^2-x^2w^2).
\end{array} \tag{1}$$

To this end, we study first the diminished set $W = Z_{60} \setminus \{P_{25}, P_{26}, P_{27}, P_{28}\}$, i.e., the set Z_{60} without the 4 coordinate points in $\mathbb{P}^3$.

We claim that there is no form of degree 5 vanishing along W. Note that the coordinate points in $\mathbb{P}^3$ lie in pairs on lines ℓ_1, ℓ_{10}, ℓ_{19}, ℓ_{28}, ℓ_{29} and ℓ_{30} from the set $\mathbb{L}_{30}$. In particular, the remaining 24 lines $\mathbb{L}_{24}$ contain each still 6 points from the set W. Note that the coordinate points in $\mathbb{P}^3$ are not contained in quadrics Q_1, Q_4, Q_5, Q_6. Hence each quadric contains 12 lines from $\mathbb{L}_{30}$. Suppose now that there is a surface Ω of degree 5 vanishing along W. Then, by Bézout Theorem, Ω contains all lines in $\mathbb{L}_{24}$. Taking the intersection of Ω with irreducible quadric Q_1, we identify 12 lines contained in Q_1 as lying in Ω. Again, by Bézout Theorem, this is possible only if Q_1 is a component of Ω. By the same token, quadrics Q_4, Q_5 and Q_6 are also components of Ω. As this is clearly not possible, we conclude that Ω does not exist.

Hence the points in W impose independent conditions on forms of degree 5 in $\mathbb{P}^3$, i.e., we have

$$H^1(\mathbb{P}^3; \mathcal{O}_{\mathbb{P}^3}(5) \otimes I(W)) = 0.$$

By [9, Theorem 1.8.3], this gives

$$\operatorname{reg}(I(W)) = 6.$$

It follows that W imposes independent conditions on forms of degree 6 as well, hence

$$h^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(6) \otimes I(W)) = \binom{9}{3} - 56 = 28.$$

In addition to generators listed in (1) we have the following 4 generators:

$$\begin{aligned}
g_1 &= 2x^2y^2z^2 - x^4w^2 - y^4w^2 - z^4w^2 + w^6,\\
g_2 &= 2x^2y^2w^2 - x^4z^2 - y^4z^2 - w^4z^2 + z^6,\\
g_3 &= 2x^2z^2w^2 - x^4y^2 - z^4y^2 - w^4y^2 + y^6,\\
g_4 &= 2y^2z^2w^2 - y^4x^2 - z^4x^2 - w^4x^2 + x^6.
\end{aligned}$$

Now, it is easy to see that requiring vanishing at the 4 coordinate points kills the above additional generators. □

4 Unexpected hypersurfaces associated with Z_{60}

In the ground-breaking work [4] by Cook II, Harbourne, Migliore and Nagel introduced the concept of unexpected curves. This notion was generalized to arbitrary hypersurfaces in the subsequent article [6] by Harbourne, Migliore, Nagel and Teitler.

Definition 4.1. We say that a reduced set of points $Z \subset \mathbb{P}^N$ *admits an unexpected hypersurface* of degree d if there exists a sequence of non-negative integers $m_1, \ldots, m_s$ such that for general points $P_1, \ldots, P_s$ the zero-dimensional subscheme $P = m_1P_1 + \ldots + m_sP_s$ fails to impose independent conditions on forms of degree d vanishing along Z and the set of such forms is non-empty. In other words, we have

$$h^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z) \otimes I(P)) > \max\left\{0, h^0(\mathbb{P}^N; \mathcal{O}_{\mathbb{P}^N}(d) \otimes I(Z)) - \sum_{i=1}^{s} \binom{N + m_s - 1}{N}\right\}.$$

Following [3, Definition 2.5] we introduce also the following notion.

Definition 4.2 (Unexpected cone property)**.** Let Z be a finite set of points in $\mathbb{P}^N$ and let d be a positive integer. We say that Z has the *unexpected cone property* $\mathcal{C}(d)$, if for a general point $P \in \mathbb{P}^3$, there exists an unexpected (in the sense of Definition 4.1) surface S_P of degree d and multiplicity d at P passing through all points in Z.

Theorem 4.3 (Unexpected cone property of Z_{60})**.** *The set Z_{60} has the $\mathcal{C}(6)$ property. Moreover, the unexpected cone of degree 6 is unique.*

Proof. Let $P = (a : b : c : d)$ be a general point in $\mathbb{P}^3$. Then

$$\begin{aligned}
F =& xy(x^4-y^4)cd(c^4-d^4) + xz(z^4-x^4)bd(b^4-d^4) + xw(x^4-w^4)bc(b^4-c^4)\\
&+yz(y^4-z^4)ad(a^4-d^4) + yw(w^4-y^4)ac(a^4-c^4) + zw(z^4-w^4)ab(a^4-b^4)\\
&+5xy(z^4-w^4)cd(a^4-b^4) + 5xz(y^4-w^4)bd(c^4-a^4) + 5xw(y^4-z^4)bc(a^4-d^4)\\
&+5yz(x^4-w^4)ad(b^4-c^4) + 5yw(x^4-z^4)ac(d^4-b^4) + 5zw(x^4-y^4)ab(c^4-d^4)\\
&+10yw(x^2y^2-z^2w^2)ac(c^2d^2-a^2b^2) + 10xw(x^2y^2-z^2w^2)bc(a^2b^2-c^2d^2)\\
&+10yz(x^2y^2-z^2w^2)ad(a^2b^2-c^2d^2) + 10xz(x^2y^2-z^2w^2)bd(c^2d^2-a^2b^2)\\
&+10zw(x^2z^2-y^2w^2)ab(a^2c^2-b^2d^2) + 10xw(x^2z^2-y^2w^2)bc(b^2d^2-a^2c^2)\\
&+10yz(x^2z^2-y^2w^2)ad(b^2d^2-a^2c^2) + 10xy(x^2z^2-y^2w^2)cd(a^2c^2-b^2d^2)\\
&+10zw(y^2z^2-x^2w^2)ab(a^2d^2-b^2c^2) + 10yw(y^2z^2-x^2w^2)ac(b^2c^2-a^2d^2)\\
&+10xz(y^2z^2-x^2w^2)bd(b^2c^2-a^2d^2) + 10xy(y^2z^2-x^2w^2)cd(a^2d^2-b^2c^2)
\end{aligned}$$

defines a cone of degree 6 with the vertex at P. Being unexpected cone for $\mathcal{C}(6)$ follows immediately from Lemma 3.2 since a point of multiplicity 6 is expected to impose 56 conditions.

The equation of F has been found by Singular [5] and can be verified by the script [10] accompanying our manuscript. Once the equation is there, the claimed properties can be checked, at least in principle, by hand. However, the highly symmetric form of F, with respect to the sets of variables $\{x, y, z, w\}$ and $\{a, b, c, d\}$ is not a coincidence. It was established in [6] that the BMSS-duality, observed first in [1], implies that the equation of F, considered as a polynomial in variables $\{a, b, c, d\}$, describes the tangent cone at P of the surface defined by F in variables $\{x, y, z, w\}$. Since the set of zeroes of F is a cone, it is the same cone in both sets of variables.

The property that F is unique follows easily from the fact that F is irreducible (it is a cone over a smooth curve of degree 6) and any other cone of degree 6 with vertex at P and multiplicity 6 would intersect F in 36 lines. These lines, being general (since P is general) would contain each at most one point of Z_{60}, thus not covering the whole set Z_{60}. □

Theorem 4.4 (Unexpected surface with 3 general points)**.** *Let P, Q_1, Q_2 be general points in $\mathbb{P}^3$. Then there exists a unique surface of degree 6 vanishing in all points of Z_{60} with a point of multiplicity 4 at P and multiplicity 2 at Q_1 and Q_2.*

Proof. Let $P = (a : b : c : d)$. We consider first sextics vanishing at all the points in Z_{60} and at a general point P to order 4. We are not able to write them explicitly down because the equations are too complex. In fact, the coefficients in front of monomials of degree 6 in variables x, y, z, w are polynomials of degree 45 in variables a, b, c, d. This is quite surprising when confronted with the proof of Theorem 4.3.

Our approach is quite standard. We outline it here and refer to our script [10] for details. We build an interpolation matrix whose columns are the 24 generators of $J = I(Z_{60})$. In the rows we write down one by one all 20 differentials of order 4 of the generators and evaluate them at P. This gives a 20×24 matrix. Even though many of its coefficients are 0, the matrix is still too large to reproduce here. Nevertheless it is simple enough, so that Singular can compute its rank, which is 15. That means that vanishing at P to order 4 imposes only 15 instead of expected 20 conditions on generators of J. With this fact established, the remaining part of the proof is easy. We have linear system of sextics of dimension $24 - 15 = 9$, so it allows 2 singularities Q_1 and Q_2 anywhere. Interestingly, no symbolic algebra program we asked, was able to determine the coefficients of the unique sextic vanishing at P to order 4 and at Q_1, Q_2 to order 2. We expect that their coefficients in coordinates of the singular points are huge. □

Remark 4.5. Note that unexpected hypersurfaces with multiple singular points seem to be quite rare. The sextics described in Theorem 4.4 are the only example, we are aware of, apart of a series of examples related to Fermat-type arrangements constructed by the third author in [12].

5 Projections of Z_{60}

In this section we study general projections of Z_{60}. Our motivation comes from a recent work of Chiantini and Migliore [3].

Definition 5.1 (*geproci* property)**.** We say that a finite set $Z \subset \mathbb{P}^3$ has a *general projection complete intersection* property (*geproci* in short), if its projection from a general point in $\mathbb{P}^3$ to $\mathbb{P}^2$ is a complete intersection.

Obvious examples of *geproci* sets are complete intersections in $\mathbb{P}^3$ with the property that one of the intersecting surfaces is a plane. Thus it is interesting to study non-degenerate (i.e. not contained in a hyperplane) sets Z with the *geproci* property.

Chiantini and Migliore observed in [3] that such sets exist. They distinguished grids as a wide class of sets which enjoy the *geproci* property.

Definition 5.2 ((a, b)-grid)**.** Let a, b be a positive integers. A set Z of ab points in $\mathbb{P}^3$ is an (a, b) - grid if there exists a set of pairwise skew lines $\ell_1, ..., \ell_a$ and a set of b skew lines $\ell'_1, ..., \ell'_b$, such that

$$Z = \{\ell_i \cap \ell'_j \ : \ i \in \{1, ..., a\}, j \in \{1, ..., b\}\}.$$

This means in particular that for all i, j the lines ℓ_i, ℓ'_j are different and incident.

Indeed, projecting to $\mathbb{P}^2$ lines in $\mathbb{P}^3$ from a general point in $\mathbb{P}^3$ results again in lines in $\mathbb{P}^2$. Let $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ be such a projection. Then $\pi(Z)$ is the complete intersection of curves $C = \pi(\ell_1) + \ldots + \pi(\ell_a)$ and $D = \pi(\ell'_1) + \ldots + \pi(\ell'_b)$.

Appendix to [3] contains examples of sets of points in $\mathbb{P}^3$, which are not (a, b)-grids, but which have the *geproci* property. Such sets seem to be extremely rare, which motivated the following problem.

Question 5.3 (Chiantini, Migliore, [3] Question 7.1)**.** Are there any examples of ab points in $\mathbb{P}^3$, other than $ab = 12, 16, 20$ or 24, that are not (a, b)-grids but that have a general projection that is a complete intersection in $\mathbb{P}^2$ of type (a, b)?

It turns out that the set Z_{60} is not a grid and yet it has the *geproci* property.

Proposition 5.4. *The set Z_{60} is not an (a, b)-grid for any choice of a, b.*

Proof. The only possibilities for the numbers a and b up to their order are:

$$(2, 30), \ \ (3, 20), \ \ (4, 15), \ \ (5, 12) \ \text{ and } \ (6, 10).$$

It is easy to check that each of distinguished lines $\ell_1, \ldots, \ell_{30}$ contains 6 points from Z_{60} and this is the highest number of collinear points in Z_{60}. There are additional 320 lines meeting Z_{60} in 3 points. In any case, the number of collinear points in Z_{60} is too small to allow the grid structure. □

Theorem 5.5 (Z_{60} is *geproci*)**.** *The set $\pi(Z_{60})$ has the* geproci *property. More precisely, its general projection to $\mathbb{P}^2$ is a complete intersection of curves of degree* 6 *and* 10.

Proof. Let $P = (a : b : c : d)$ be a general point in $\mathbb{P}^3$ and let π be the rational map

$$\mathbb{P}^3 \ni (x : y : z : w) \dashrightarrow (ay - bx : bz - cy : cw - dz) \in \mathbb{P}^2.$$

Then, the cone F associated to P in the proof of Theorem 4.3 projects to the following curve of degree 6 in variables $(s : t : u)$

$$\begin{aligned} C_6 =& b(a^4 - b^4)tu(t^4 - u^4) + c(a^4 - c^4)su(u^4 - s^4) + d(a^4 - d^4)st(s^4 - t^4) \\ &+ 5b(d^4 - c^4)s^4tu + 5c(b^4 - d^4)st^4u + 5d(c^4 - b^4)stu^4 \\ &+ 10b(a^2d^2 - b^2c^2)s^2t^3u + 10c(a^2d^2 - b^2c^2)s^3t^2u + 10d(a^2c^2 - b^2d^2)s^3tu^2 \\ &+ 10b(b^2d^2 - a^2c^2)s^2tu^3 + 10c(a^2b^2 - c^2d^2)st^2u^3 + 10d(c^2d^2 - a^2b^2)st^3u^2. \end{aligned}$$

By construction, the projection of Z_{60} is contained in C_6. Somewhat surprisingly, there is a certain ambiguity in the choice of a curve of degree 10 cutting out on C_6 precisely the set Z_{60}. The most appealing way comes from the geometry of the arrangement of lines $\mathbb{L}_{30}$. Using explicit equations of lines ℓ_i and coordinates of points P_i, it is easy to check that there are 6 ways of choosing 10 disjoint lines among $\{\ell_1, \ldots, \ell_{30}\}$ covering the set Z_{60}. These selections are indicated in Table 1.

Table 1: The division of 60 lines in 6 groups of 10 disjoint lines.

	A	B	C	D	E	F
ℓ_1	+	+				
ℓ_2					+	+
ℓ_3			+	+		
ℓ_4			+		+	
ℓ_5				+		+
ℓ_6			+	+		
ℓ_7					+	+
ℓ_8				+		+
ℓ_9			+		+	
ℓ_{10}			+			+
ℓ_{11}		+		+		
ℓ_{12}	+				+	
ℓ_{13}	+			+		
ℓ_{14}		+			+	
ℓ_{15}	+				+	
ℓ_{16}		+		+		
ℓ_{17}		+			+	
ℓ_{18}	+			+		
ℓ_{19}				+	+	
ℓ_{20}	+		+			
ℓ_{21}		+				+
ℓ_{22}		+	+			
ℓ_{23}	+					+
ℓ_{24}		+				+
ℓ_{25}	+		+			
ℓ_{26}	+					+

ℓ_{27}		+	+			
ℓ_{28}				+	+	
ℓ_{29}			+			+
ℓ_{30}	+	+				

Since the curve C_6 is irreducible, the image under the projection of any selection of 10 disjoint lines out of $\mathbb{L}_{30}$, cuts C_6 in exactly 60 distinct points. It follows that C_6 and lines intersect transversally and thus the intersection is scheme theoretic.

We complete our considerations providing explicit equations of images ℓ'_i of the 30 lines from $\mathbb{L}_{30}$.

$$\begin{aligned}
\ell'_1 &= s,\\
\ell'_2 &= (c^2 - cd)s + (ac - ad - bc + bd)t + (-ab + b^2)u,\\
\ell'_3 &= (c^2 - cd)s + (ac - ad + bc - bd)t + (-ab - b^2)u,\\
\ell'_4 &= (i \cdot cd + c^2)s + (i \cdot (ad + bc) + ac - bd)t + (i \cdot ab - b^2)u,\\
\ell'_5 &= (i \cdot cd + c^2)s + (i \cdot (ad - bc) + ac + bd)t + (i \cdot ab + b^2)u\\
\ell'_6 &= (c^2 + cd)s + (ac + ad - bc - bd)t + (ab - b^2)u,\\
\ell'_7 &= (c^2 + cd)s + (ac + ad + bc + bd)t + (ab + b^2)u,\\
\ell'_8 &= (-i \cdot cd + c^2)s + (-i \cdot (ad - bc) + ac + bd)t + (-i \cdot ab + b^2)u,\\
\ell'_9 &= (-i \cdot cd + c^2)s + (-i \cdot (ad + bc) + ac - bd)t + (-i \cdot ab - b^2)u,\\
\ell'_{10} &= cs + at,\\
\ell'_{11} &= (bc - cd)s + (-ad + bc)t + (-ab + bc)u,\\
\ell'_{12} &= (bc - cd)s + (-ad - bc)t + (-ab - bc)u,\\
\ell'_{13} &= (i \cdot cd + bc)s + i \cdot (ad - bc)t + (i \cdot ab - bc)u,\\
\ell'_{14} &= (i \cdot cd + bc)s + i \cdot (ad + bc)t + (i \cdot ab + bc)u,\\
\ell'_{15} &= (bc + cd)s + (ad + bc)t + (ab - bc)u,\\
\ell'_{16} &= (bc + cd)s + (ad - bc)t + (ab + bc)u,\\
\ell'_{17} &= (-i \cdot cd + bc)s - i \cdot (ad + bc)t + (-i \cdot ab + bc)u,\\
\ell'_{18} &= (-i \cdot cd + bc)s - i \cdot (ad - bc)t + (-i \cdot ab - bc)u,\\
\ell'_{19} &= cds + adt + abu,\\
\ell'_{20} &= (bc - c^2)s + (-ac + bd)t + (b^2 - bc)u,\\
\ell'_{21} &= (bc - c^2)s + (-ac - bd)t + (-b^2 + bc)u,\\
\ell'_{22} &= (i \cdot c^2 + bc)s + i \cdot (ac - bd)t + (-i \cdot b^2 + bc)u,\\
\ell'_{23} &= (i \cdot c^2 + bc)s + i \cdot (ac + bd)t + (i \cdot b^2 - bc)u,\\
\ell'_{24} &= (bc + c^2)s + (ac + bd)t + (b^2 + bc)u,\\
\ell'_{25} &= (bc + c^2)s + (ac - bd)t + (-b^2 - bc)u,\\
\ell'_{26} &= (-i \cdot c^2 + bc)s - i \cdot (ac + bd)t + (-i \cdot b^2 - bc)u,\\
\ell'_{27} &= (-i \cdot c^2 + bc)s - i \cdot (ac - bd)t + (i \cdot b^2 + bc)u,\\
\ell'_{28} &= t,\\
\ell'_{29} &= dt + bu,\\
\ell'_{30} &= u.
\end{aligned}$$

□

Remark 5.6. One might expect that the projection of 10 disjoint lines in $\mathbb{L}_{30}$ is somehow special. However, it can be checked that the contrary situation holds since the lines form the star configuration – they intersect only in pairs producing 45 double intersection points. The intersections take place away from the C_6 curve.

We derive for completeness the following corollary to Theorem 5.5.

Corollary 5.7 (Subsets of Z_{60} with the *geproci* property)**.** *Removing* 6 *collinear points from* Z_{60} *produces a set* Z *of* 54 *points with the* geproci *property. Their projection is a* *complete intersection of the curve of degree* 6 *and now the remaining* 9 *lines covering* Z*. This procedure can be repeated with remaining sets of* 6 *collinear points. Thus we get sets of* $60, 54, 48, 42, 36, 30, 24, 18$ *and* 12 *points, which are not* (a, b)*-grids, with the* geproci *property.*

On the other hand, it is interesting to note that whereas the curve C_6 of degree 6 vanishing along $\pi(Z_{60})$ is unique, there are six ways to choose a completely reducible (i.e. splitting in lines) degree 10 curve cutting out $\pi(Z_{60})$ on C_6. The example studied in this work motivates the following definition.

Definition 5.8 (Half grid)**.** Let a, d be positive integers. A set Z of ad points in $\mathbb{P}^3$ is an (a, d) - half grid if there exists a set of mutually skew lines $\ell_1, \ldots, \ell_a$ covering Z and a general projection of Z to a plane is a complete intersection of images of a lines with a (possibly reducible) curve of degree d.

It is clear that Z_{60} is a half grid. Moreover, any grid is a half grid. It is also clear that the point in Z are equidistributed over the lines. Taking 6 collinear points out of Z_{60} results also in a half grid.

Problem 5.9. Are there any half grids in $\mathbb{P}^3$ other than Z_{60} and its subgrids?

6 F_4 root system and the associated set of 24 points

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References

[1] T. Bauer, G. Malara, T. Szemberg, and J. Szpond. Quartic unexpected curves and surfaces. *Manuscripta Math.*, 161(3-4):283–292, 2020.

[2] I. Cheltsov and C. Shramov. Finite collineation groups and birational rigidity. *Sel. Math., New Ser.*, 25(5):68, 2019. Id/No 71.

[3] L. Chiantini and J. Migliore. Sets of points which project to complete intersections. *Trans. AMS*, to appear. With an appendix by the authors and A. Bernardi, G. Denham, G. Favacchio, B. Harbourne, T. Szemberg and J. Szpond.

[4] D. Cook II, B. Harbourne, J. Migliore, and U. Nagel. Line arrangements and configurations of points with an unexpected geometric property. *Compos. Math.*, 154(10):2150–2194, 2018.

[5] W. Decker, G.-M. Greuel, G. Pfister, and H. Schönemann. Singular 4-1-1 — A computer algebra system for polynomial computations. http://www.singular.uni-kl.de, 2018.

[6] B. Harbourne, J. Migliore, U. Nagel, and Z. Teitler. Unexpected hypersurfaces and where to find them. *Michigan Math. J.*, to appear.

[7] B. Hunt. Coverings and ball quotients with special emphasis on the 3-dimensional case. Bonn. Math. Schr. 174, 87 p. (1986)., 1986.

[8] F. Klein. Gesammelte mathematische Abhandlungen. Erster Band: Liniengeometrie. Grundlegung der Geometrie. Zum Erlanger Programm. Herausgegeben von R. Fricke und A. Ostrowski. (Von F. Klein mit ergänzenden Zusätzen versehen.) Mit einem Bildnis Kleins. Berlin: J. Springer, XII u. 612 S. 8° (1922) (1921)., 1921.

[9] R. Lazarsfeld. *Positivity in algebraic geometry.*, volume 48, 49 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[10] P. Pokora, T. Szemberg, and J. Szpond. Computations preformed with SINGULAR. www.szpond.up.krakow.pl/mfo2020.txt.

[11] G. C. Shephard and J. A. Todd. Finite unitary reflection groups. *Canadian J. Math.*, 6:274–304, 1954.

[12] J. Szpond. Unexpected hypersurfaces with multiple fat points, In: Stamate D., Szemberg T. (eds) Combinatorial Structures in Algebra and Geometry. NSA 2018. Springer Proceedings in Mathematics & Statistics, vol 331. p. 161 – 182, Springer, Cham, 2020.

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