Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2020

DOI: 10.4171/OWR/2020/33

Mini-Workshop: Almost Complex Geometry (online meeting)

Organized by
Daniele Angella, Firenze
Joana Cirici, Barcelona
Jean-Pierre Demailly, Gières
Scott Wilson, New York

4 October – 10 October 2020

Abstract. The mini-workshop focused on very recent developments of analytic and algebraic techniques for studying almost complex structures which are not necessarily integrable. It provided a forum to discuss and compare techniques from PDE's, elliptic theory, deep algebraic structures, as well as geometric flows, and new topological invariants, towards attacking several longstanding open problems in the field of complex and almost complex geometry.

Mathematics Subject Classification (2020): 32Qxx, 53Cxx, 58Jxx, 32J27.

Introduction by the Organizers

The workshop *Almost Complex Geometry*, organised by Daniele Angella (Università di Firenze), Joana Cirici (Universitat de Barcelona), Jean-Pierre Demailly (Institut Fourier), and Scott Wilson (Queens College, CUNY) took place in hybrid format due to the pandemic emergency. Thirteen participants attended the workshop, one of which was in person, coming from eight different countries and sharing a diverse and complimentary background. Other than the eight contributed talks, the workshop consisted of five discussion sessions, a concluding problem session, and asynchronous discussions held on an online platform. The aim of the workshop was to allow all the participants to share their recent advances in several interdisciplinary fields, and to provide a forum to discuss and compare different and newly proposed techniques for addressing longstanding foundational open problems in complex and almost complex geometry.

Shortly after a 1953 conference at Cornell University on *Fiber Bundles and Differential Geometry*, Hirzebruch [Hir54] published a list of fundamental problems related to the topology and geometry of manifolds. Many of these problems were solved in the following years, especially those centering on the Hirzebruch-Riemann-Roch Theorem. Some were solved only decades later through major advances in the area. Five problems have remained essentially open to this day [Kot13], two of which concern almost complex geometry: Problem 13 on complex structures on $\mathbb{CP}^3$ and on S^6, related to topological obstructions to integrability of almost complex structures, and Problem 20 on $\bar{\partial}$-harmonic forms and Dolbeault theory for almost complex manifolds. In the workshop, we discussed very recent developments concerning these two open problems, relating them to new algebraic, cohomological, and analytical approaches to the study of the topology of complex and almost complex manifolds.

Topology of complex and almost complex manifolds. A very relevant problem, continuously pursued by geometers and topologists for decades, is to find topological obstructions to the integrability of almost complex structures. In real dimension four there are many examples of almost complex manifolds not admitting integrable structures. In this case, obstructions are attained using a variety of tools, ranging from the Enriques-Kodaira classification of complex surfaces and Chern number inequalities, to gauge theory via Seiberg-Witten invariants. In real dimensions ≥ 6, it is not known whether there exists any almost complex manifold not admitting an integrable structure. Although Yau speculated that such examples do not exist, we are still in search for evidence of the conjecture.

This conjecture is related to Problem 13 in [Hir54], which concerns the existence and classification of complex manifolds. It asks whether there is a complex structure on $\mathbb{CP}^3$ with vanishing second Chern class. Hirzebruch noted that such a structure would not admit a Kähler metric and that a negative answer would imply that the six-sphere does not carry a complex structure. The problem of deciding whether the six-sphere admits the structure of a complex manifold goes back to Heinz Hopf and is still one of the most prominent open problems in complex geometry (see the talk by **Michael Albanese**). Many solutions have been proposed in both directions, but so far these have always turned out to be faulty (see [ABG$^+$18] for a record).

Related to this, a wide open problem is to understand what are the rational homotopy types of complex manifolds. One knows there are strong restrictions on the rational homotopy types of Kähler manifolds, as they are all formal. Forthcoming work [Mil20] suggests there are no rational homotopy obstructions to almost complex structures other than the necessary conditions required of Chern numbers and the congruence relations coming from index theory (see the talk by **Aleksandar Milivojevic**). Sullivan has asked: *Are there any additional rational homotopy theoretic obstructions to integrable complex structures*? There appears to be no progress at all on this problem. Nevertheless, there is important recent progress in understanding the topology of almost complex manifolds when considering compatible special metrics, such as the recently established *almost formality* for Sasaki and Vaisman manifolds [OV19].

Another approach to these problems is suggested by a conjecture formulated by Bogomolov, asserting the transverse embeddability of arbitrary compact complex manifolds into foliated algebraic varieties. In this direction, a universal embedding space for compact almost complex manifolds was recently introduced and studied in [DG17, Cle19, Shi19]. It can be used to investigate the existence of topological obstructions to integrability, as the obstructions to transverse embeddability. Another very different approach is through the calculus of variations, by the parabolic flows proposed in [Yau93] and [KT20]. The latter work extends Ricci flow in a way that is compatible with almost complex structures. Establishing long-time existence of such flows could prove to be as equally effective as the Ricci flow program, and have significant impact on the problems above.

Cohomology of complex and almost complex manifolds. Problem 20 in [Hir54], attributed to Kodaira and Spencer, concerns almost-Hermitian structures and a potential generalization of Dolbeault numbers to the category of almost complex manifolds. The problem includes two questions. The first one asks whether the Hodge-theoretic numbers defined via $\bar{\partial}$-harmonic forms are metric independent. Very recently, Holt and Zhang [HZ20] have answered this question negatively by constructing examples on the Kodaira-Thurston surface using methods from PDE theory, harmonic analysis, and number theory (see the talk by **Thomas Holt**). The second part of the question asks for a definition of metric-independent numbers generalizing Dolbeault cohomology to the non-integrable case. In [CW18a], the authors naturally extend the definition of Dolbeault cohomology groups, introducing a Frölicher-type spectral sequence for all almost complex manifolds. The two works together thus offer a full solution to Problem 20, via completely different techniques.

Understanding cohomological invariants for (almost) complex manifolds, such as Kodaira dimension, or Bott-Chern and Aeppli cohomologies, as well as their relations with the underlying topological structure is a common issue for the two problems above (see the talk by **Jonas Stelzig**). These cohomologies are also related to the existence of further geometric structures on the manifolds, and to issues of formality in the sense of rational homotopy [AT13]. There is a surprising number of new cohomological approaches for almost complex, symplectic, and generalized complex manifolds which are receiving widespread attention. Examples are the cohomological decompositions for almost complex manifolds of Li and Zhang and subsequent developments [LZ15] (see the talk by **Adriano Tomassini**), the Hodge theory of Tseng and Yau, in the corresponding setting of symplectic manifolds [TTY16], and further extensions [LV15, AOT18], and Cavalcanti's Dolbeault cohomology in the unifying generalized complex setting [Cav09]. In this context, locally homogeneous manifolds of nilpotent Lie groups play a useful role as test examples: compare [BMn12, LU19] for some results on the classification of low-dimensional nilmanifolds with complex or generalized complex structures in terms of the real homotopy type.

Special metrics on complex manifolds. We paid special attention to recent powerful results arising with the presence of special metrics, such as almost Kähler, strong Kähler with torsion, and balanced metrics. These metrics are also important because of their role in the geometry of compactification of heterotic superstrings with torsion to 4-dimensional Minkowski spacetime (see the talk by **Anna Fino**).

Additional cohomological properties under non-Kähler metric assumptions were obtained earlier in Verbitsky's work on harmonic forms for nearly Kähler manifolds [Ver11]. This has strongly influenced the works of Wilson on Hermitian Hodge theory [Wil19], and of Cirici-Wilson [CW18b] and Tardini-Tomassini [TT19] on almost Kähler Hodge theory (see the talk by **Nicoletta Tardini**), with strong connections with symplectic geometry and topology.

Progress is being made on many interesting open problems related to the integrability of restricted geometries, such as Donaldson's question on almost complex structures tamed by a symplectic form on 4-dimensional manifolds [TWY08, TW11]. The analogue of Donaldson's tamed-to-compatible question in higher dimension forces the almost complex structure to be integrable: it asks whether there exist non-Kähler Hermitian-symplectic complex manifolds (see the talk by **Dan Popovici**). Hermitian-symplectic structures are closely related to SKT metrics, which in the generalized geometry approach, allows one to develop a Hodge theory for SKT metrics (see the talk by **Gil Cavalcanti**).

Open problems. The following problems were discussed.

- The foundational open problems are of course the Hopf and Yau problems [Hop48, Hir54, Yau93, ABG$^+$18], concerning the existence of a higher-dimensional almost complex manifold admitting no integrable almost complex structures. The six-dimensional sphere is clearly an interesting, potential example; on the other side, one may hopefully think about more interesting examples from the rational homotopy point of view.
- Related to the above, we recall the Sullivan question: *Are there any additional rational homotopy theoretic obstructions to integrable complex structures?*
- There are four-manifolds which admit an almost complex structure but no complex structures. The statement follows by the Enriques–Kodaira classification and Chern number inequalities, but it would be useful to have other arguments for this that avoid classification and the special properties of compact complex Kähler surfaces [Lam99, Buc99].
- One important unsolved problem in the theory of almost complex manifolds is to find good criteria ensuring the existence of rational pseudo-holomorphic curves. The reader is referred to the survey book [MDS04]

for basic information on the subject. Consider a compact symplectic $2n$-dimensional smooth differentiable manifold (M, ω), and a compatible almost complex structure J on M. Then the first Chern class $c_1(M, J)$ is well defined as the first Chern class of the complex tangent bundle (T_M, J), and this class is independent of J. Assuming that $\int_M c_1(M, J) \wedge \omega^{n-1} > 0$, it is expected that M possesses a covering family $(C_t)_{t \in S}$ of rational pseudo-holomorphic curves. A positive solution to this question, raised in [BDPP04], would provide a vast generalization to the symplectic case of Mori's results on the existence of rational curves, which are currently only known for projective algebraic varieties. In fact, the only known proofs involve characteristic p techniques. Finding alternative more geometric or analytic arguments would probably yield a lot of consequences in symplectic geometry. The general theory of moduli spaces of pseudo-holomorphic curves, the study of their compactifications, along with cohomology theories for almost complex manifolds, seem instrumental in such problems.

- It was pointed how properties like finite-dimensionality of the Dolbeault cohomology and metric independence of the kernel of the Dolbeault Laplacian in the integrable case should be appreciated, in view of finding obstructions to integrability. In other words, developing a deeper understanding of the harmonic theory on almost complex manifolds, and clarifying what particular properties are special to integrable setting would be interesting. The same must also be asked of the newly developed Dolbeault theory as well. A related question is the spectral theory of non-integrable complex vector bundles, as initiated e.g. in [Lae02] and [Pop13], which, even in the case of complex manifolds, is connected to important unsolved problems such as transcendental Morse inequalities and Kähler invariance of plurigenera.
- A natural question is to understand an analogue of Lamari's [Lam99] and Buchdahl's [Buc99] result for complex manifolds of complex dimension greater that two. It was suggested to investigate further non-linear algebraic structures in the cohomology ring.
- In analogy with the Barge-Sullivan theorem, one may ask about the realization of bi-differential bi-graded algebras as Dolbeault double-complex of a complex manifold.
- De Rham theory is powerful in that it can be computed in several ways, and has a dual theory with evaluation map, via homology (which too can be represented in several ways, *e.g.* cycles, currents etc.) It would be advantageous to push the theory of Dolbeault cohomology in these ways as well. In particular, there may be connections with the theory of pseudo-holomorphic curves, or generalizations thereof.
- The Streets-Tian question concerning the (non-)existence of non-Kähler Hermitian-symplectic manifolds is another fundamental problem, related to the Donaldson tamed-to-compatible question for almost complex four-manifolds. More in general, Popovici asks: *Do there exist a Kähler metric*

in the Aeppli cohomology class of any Hermitian-symplectic structure?. Special cases are well-understood, including dimension two, nilmanifolds and solvmanifolds with special complex structures, twistor spaces. A problem is to determine other conditions implying that any manifold admitting a Hermitian-symplectic metric also necessarily admits a Kähler metric.

- A question that arose is the relation between the existence of SKT metrics and the degeneration of the Frölicher spectral sequence. In [Pop13], it is conjectured that SKT metric forces $E_2 = E_\infty$.
- A natural question is how to extend the definition of $\partial\overline{\partial}$-Lemma for almost complex manifolds, and understand its relationship to Dolbeault, Bott-Chern and Aeppli cohomologies, and the duality of the latter two.

References

[ABG+18] I. Agricola, G. Bazzoni, O. Goertsches, P. Konstantis, and S. Rollenske, *On the history of the Hopf problem*, Differential Geom. Appl. **57** (2018), 1–9.

[AOT18] D. Angella, A. Otiman, and N. Tardini, *Cohomologies of locally conformally symplectic manifolds and solvmanifolds*, Ann. Global Anal. Geom. **53** (2018), no. 1, 67–96.

[AT13] D. Angella and A. Tomassini, *On the $\partial\overline{\partial}$-lemma and Bott-Chern cohomology*, Invent. Math. **192** (2013), no. 1, 71–81.

[BMn12] G. Bazzoni and V. Muñoz, *Classification of minimal algebras over any field up to dimension 6*, Trans. Amer. Math. Soc. **364** (2012), no. 2, 1007–1028.

[BDPP04] S. Boucksom, J.-P. Demailly, M. Păun, Th. Peternell, *The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension*, arXiv:math/0405285 and J. Algebraic Geom. **22** (2013), 201–248.

[Buc99] N. Buchdahl, *On compact Kähler surfaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 1, vii, xi, 287–302.

[Cav09] G. R. Cavalcanti, *Computations of generalized Dolbeault cohomology*, Special metrics and supersymmetry, AIP Conf. Proc., vol. 1093, Amer. Inst. Phys., Melville, NY, 2009, pp. 57–69.

[Cle19] G. Clemente, *Geometry of universal embedding spaces for almost complex manifolds*, arXiv e-prints (2019), arXiv:1905.06016.

[CW18a] J. Cirici and S. O. Wilson, *Dolbeault cohomology for almost complex manifolds*, Preprint arXiv:1809.01416 (2018).

[CW18b] J. Cirici and S. O. Wilson, *Topological and geometric aspects of almost Kähler manifolds via harmonic theory*, Preprint arXiv:1809.1414 (2018).

[DG17] J.-P. Demailly and H. Gaussier, *Algebraic embeddings of smooth almost complex structures*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 11, 3391–3419.

[DGZ16] S. Dinew, V. Guedj, A. Zeriahi, *Open problems in pluripotential theory*, Complex Var. Elliptic Equ. **61** (2016), no. 7, 902–930.

[Hir54] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) **60** (1954), 213–236.

[HZ20] T. Holt and W. Zhang, *Harmonic Forms on the Kodaira-Thurston Manifold*, arXiv e-prints (2020), arXiv:2001.10962.

[Hop48] H. Hopf, *Zur Topologie der komplexen Mannigfaltigkeiten*, In Studies and Essays Presented to R. Courant on his 60th Birthday, January 8, 1948, pages 167–185, Interscience Publishers, Inc., New York, 1948.

[KT20] C. L. Kelleher and G. Tian, *Almost Hermitian Ricci flow*, arXiv e-prints (2020), arXiv:2001.06670.

[Kot13] D. Kotschick, *Updates on Hirzebruch's 1954 Problem List*, arXiv e-prints (2013), arXiv:1305.4623.

[Lae02] L. Laeng, *Estimations spectrales asymptotiques en géométrie hermitienne*, Theses, Université Joseph-Fourier - Grenoble I, October 2002.

[Lam99] A. Lamari, *Courants kählériens et surfaces compactes*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 1, vii, x, 263–285.

[LU19] A. Latorre and R. Ugarte, L.and Villacampa, *On the real homotopy type of generalized complex nilmanifolds*, arXiv e-prints (2019), arXiv:1905.11111.

[LV15] H. V. Lê and J. Vanžura, *Cohomology theories on locally conformal symplectic manifolds*, Asian J. Math. **19** (2015), no. 1, 45–82.

[LZ15] T.-J. Li and W. Zhang, *Almost Kähler forms on rational 4-manifolds*, Amer. J. Math. **137** (2015), no. 5, 1209–1256.

[MDS04] D. McDuff and D. Salamon, *J-Holomorphic Curves and Symplectic Topology*, Colloquium Publ., Vol. **52**, Amer. Math. Soc, 2004, 669 pp.

[Mil20] A. Milivojevic, *The rational topology of closed almost complex manifolds*, Ph.D thesis, to appear (2020).

[OV19] L. Ornea and M. Verbitsky, *Supersymmetry and Hodge theory on Sasakian and Vaisman manifolds*, Preprint arXiv:1910.01621 (2019).

[Pop13] D. Popovici, *Transcendental Kähler cohomology classes*, Publ. Res. Inst. Math. Sci. **49** (2013), no. 2, 313–360.

[Pop13] D. Popovici, *Degeneration at E_2 of certain spectral sequences*, Internat. J. Math. **27** (2016), no. 14, 1650111, 31 pp.

[Shi19] T. Shin, *Almost complex manifolds are (almost) complex*, Preprint arXiv:1903.10043 (2019).

[TT19] N. Tardini and A. Tomassini, *Differential operators on almost-Hermitian manifolds and harmonic forms*, Preprint arXiv:1909.06569 (2019).

[TTY16] C.-J. Tsai, L.-S. Tseng, and S.-T. Yau, *Cohomology and Hodge theory on symplectic manifolds: III*, J. Differential Geom. **103** (2016), no. 1, 83–143.

[TW11] V. Tosatti and B. Weinkove, *The Calabi-Yau equation, symplectic forms and almost complex structures*, Geometry and analysis. No. 1, Adv. Lect. Math. (ALM), vol. 17, 2011, pp. 475–493.

[TWY08] V. Tosatti, B. Weinkove, and S.-T. Yau, *Taming symplectic forms and the Calabi-Yau equation*, Proc. Lond. Math. Soc. (3) **97** (2008), no. 2, 401–424.

[Ver11] M. Verbitsky, *Hodge theory on nearly Kähler manifolds*, Geom. Topol. **15** (2011), no. 4, 2111–2133.

[Wil19] S. O. Wilson, *Harmonic symmetries for Hermitian manifolds*, Preprint arXiv:1906.02952, to appear in Proc. AMS. (2019).

[Yau93] S.-T. Yau, *Open problems in geometry*, Differential geometry: partial differential equations on manifolds (Los Angeles, CA, 1990), Proc. Sympos. Pure Math., vol. 54, Amer. Math. Soc., Providence, RI, 1993, pp. 1–28.

Mini-Workshop (online meeting): Almost Complex Geometry

Table of Contents

Abstracts

A Generalised Volume Invariant for Aeppli Cohomology Classes of Hermitian-Symplectic Metrics

Dan Popovici
(joint work with Sławomir Dinew)

This is a report on the very recent joint work [DP20] of the author with S. Dinew.

(I) Because it lies at the sweet spot between symplectic and complex Hermitian geometries, the following notion has emerged as a cornerstone of complex analytic geometry.

Definition 1. ([Sul76], [HL83], [ST10, Definition 1.5]) *Let X be a compact complex manifold. A* **Hermitian-symplectic (H-S)** *metric on X is a C^∞ positive definite $(1, 1)$-form ω on X such that ω is the component of bidegree $(1,1)$ of a real C^∞ d-closed 2-form $\widetilde{\omega}$ on X.*

If X carries such a metric, X is said to be a **Hermitian-symplectic (H-S) manifold**.

The analogous notion on almost-complex manifolds X is the notion of symplectic form ω compatible with the almost-complex structure J of X. In 2006, Donaldson asked the following question that has come to be referred to as *Donaldson's tamed-to-compatible conjecture.*

Question 2. ([Don06, Question 2]) *If J is an almost-complex structure on a compact 4-manifold which is tamed by a symplectic form, is there a symplectic form compatible with J?*

Later on, Streets and Tian asked the following complementary question in arbitrary dimension but supposing J to be *integrable.*

Question 3. ([ST10, Question 1.7]) *Do there exist non-Kähler Hermitian-symplectic complex manifolds X with $dim_{\mathbb{C}}X \geq 3$?*

So far, the only case where the answer is known lies at the intersection of Questions 2 and 3. This is the case of compact complex surfaces, for which it has been proved that the existence of an H-S metric implies the existence of a Kähler metric.

(II) In [DP20], we investigated Question 3 by introducing and studying a functional F on the open convex subset $\mathcal{S}_{\{\omega_0\}} \subset \{\omega_0\}_A \cap C^\infty_{1,1}(X, \mathbb{R})$ of all the Hermitian-symplectic metrics ω lying in the Aeppli cohomology class $\{\omega_0\}_A \in H^{1,1}_A(X, \mathbb{R})$ of an arbitrary Hermitian-symplectic metric ω_0.

Lemma and Definition 4. ([DP20, Lemma and Definition 3.1]) *For every Hermitian-symplectic metric ω on X, there exists a unique smooth $(2, 0)$-form $\rho^{2,0}_\omega$ on X such that*

$$(1)\quad (i)\ \partial\rho^{2,0}_\omega = 0 \quad \text{and} \quad (ii)\ \bar\partial\rho^{2,0}_\omega = -\partial\omega \quad \text{and} \quad (iii)\ \rho^{2,0}_\omega \in Im\,\partial^\star_\omega + Im\,\bar\partial^\star_\omega.$$

Moreover, property (iii) ensures that $\rho_\omega^{2,\,0}$ *has* **minimal** L^2_ω **norm** *among all the* $(2,\,0)$*-forms satisfying properties (i) and (ii).*

We call $\rho_\omega^{2,\,0}$ *the* $(2,\,0)$**-torsion form** *and its conjugate* $\rho_\omega^{0,\,2}$ *the* $(0,\,2)$**-torsion form** *of the Hermitian-symplectic metric* ω*.*

With this preliminary notion in place, we set

Definition 5. ([DP20, Definition 3.3]) *Let* X *be a compact complex Hermitian-symplectic manifold with* $dim_{\mathbb{C}}X = n$*. For the Aeppli cohomology class* $\{\omega_0\}_A \in \mathcal{HS}_X$ *of any Hermitian-symplectic metric* ω_0*, we define the following* **energy functional***:*

$$F : \mathcal{S}_{\{\omega_0\}} \to [0,\,+\infty), \qquad F(\omega) = \int\limits_X |\rho_\omega^{2,\,0}|^2_\omega\,dV_\omega = ||\rho_\omega^{2,\,0}||^2_\omega, \tag{2}$$

where $\rho_\omega^{2,\,0}$ *is the* $(2,\,0)$*-torsion form of the Hermitian-symplectic metric* $\omega \in \mathcal{S}_{\{\omega_0\}}$ *defined in Lemma and Definition 4, while* $|\ |_\omega$ *is the pointwise norm and* $||\ ||_\omega$ *is the* L^2 *norm induced by* ω*.*

The immediate observation that justifies the introduction of this functional is the following equivalence:

$$\omega \text{ is Kähler} \iff F(\omega) = 0.$$

By studying the first variation of our functional, we obtained the following results.

Theorem 6. ([DP20, Corollary 3.6]) *Suppose* $n = 3$*. Then a Hermitian-symplectic metric* ω *on a compact complex manifold* X *of dimension* 3 *is a* **critical point** *of the energy functional* F **if and only if** ω *is* **Kähler***.*

Proposition 7. ([DP20, Corollary 3.7]) *Let* X *be a compact complex manifold of dimension* $n = 3$ *admitting Hermitian-symplectic metrics. Then, for any Aeppli-cohomologous Hermitian-symplectic metrics* ω *and* ω_η*, the respective* $(2,\,0)$*-torsion forms* $\rho_\omega^{2,\,0}$ *and* $\rho_\eta^{2,\,0} := \rho_{\omega_\eta}^{2,\,0}$ *satisfy the identity:*

$$F(\omega_\eta) + Vol_{\omega_\eta}(X) = F(\omega) + Vol_\omega(X),$$

where $Vol_\omega(X) := \int_X \omega^3/3!$*, and are related by*

$$\rho_\eta^{2,\,0} = \rho_\omega^{2,\,0} + \partial\eta.$$

Consequently, we get the following invariant attached to any Aeppli class of Hermitian-symplectic metrics that generalises the classical volume of a Kähler class.

Definition 8. ([DP20, Definition 3.8]) *Let* X *be a* 3*-dimensional compact complex manifold supposed to carry Hermitian-symplectic metrics. For any such metric* ω *on* X*, the constant*

$$A = A_{\{\omega\}_A} := F(\omega) + Vol_\omega(X) > 0 \tag{3}$$

depending only on $\{\omega\}_A$ *is called the* **generalised volume** *of the Hermitian-symplectic Aeppli class* $\{\omega\}_A$*.*

In [DP20], we also obtained two cohomological interpretations of this invariant.

(III) We further identified an obstruction to the Aeppli cohomology class of a given Hermitian-symplectic metric containing a Kähler metric.

Lemma and Definition 9. ([DP20, Lemma and Definition 4.1]) *Suppose that ω is a* **Hermitian-symplectic metric** *on a compact complex n-dimensional manifold X.*

(i) The $(0, 2)$-torsion form $\rho_\omega^{0,\,2} \in C^\infty_{0,\,2}(X, \mathbb{C})$ of ω represents an E_2-cohomology class $\{\rho_\omega^{0,\,2}\}_{E_2} \in E_2^{0,\,2}(X)$ on the second page of the Frölicher spectral sequence of X. Moreover, $\{\rho_\omega^{0,\,2}\}_{E_2} \in \ker(d_2 : E_2^{0,\,2}(X) \to E_2^{2,\,1}(X))$.

(ii) Suppose that $n = 3$. Then, the class $\{\rho_\omega^{0,\,2}\}_{E_2} \in E_2^{0,\,2}(X)$ is constant when the Hermitian-symplectic metric ω varies in a fixed Aeppli cohomology class.

The class $\{\rho_\omega^{0,\,2}\}_{E_2} \in E_2^{0,\,2}(X)$ is called the E_2**-torsion class** *of the Hermitian-symplectic Aeppli class $\{\omega\}_A$.*

Since ω is Kähler if and only if $\rho_\omega^{0,\,2} = 0$, we get the following necessary condition for a given Hermitian-symplectic Aeppli class $\{\omega\}_A$ to contain a Kähler metric.

Corollary 10. ([DP20, Corollary 4.2]) *Suppose that $n = 3$. If a given Hermitian-symplectic Aeppli class $\{\omega\}_A$ contains a Kähler metric, then its E_2-torsion class $\{\rho_\omega^{0,\,2}\}_{E_2} \in E_2^{0,\,2}(X)$ vanishes.*

Moreover, the condition $\{\rho_\omega^{0,\,2}\}_{E_2} = 0$ in $E_2^{0,\,2}(X)$ is equivalent to $\rho_\omega^{0,\,2} \in Im\,\bar\partial$ for some (hence every) metric ω lying in $\{\omega\}_A$.

This corollary shows that, if a Hermitian-symplectic manifold X exists such that the E_2-torsion class associated with every Hermitian-symplectic metric ω is non-zero, then X carries no Kähler metric. The existence of such manifolds will answer affirmatively the Streets-Tian Question 3.

It is likely that the methods of [DP20] can be extended to the case of **SKT manifolds**, namely compact complex manifolds X carrying metrics ω such that $\partial\bar\partial\omega = 0$. There are many non-Kähler SKT manifolds, but the SKT property is likely to have weaker and still interesting cohomological consequences.

References

[DP20] S. Dinew, D. Popovici, *A Generalised Volume Invariant for Aeppli Cohomology Classes of Hermitian-Symplectic Metrics*, arXiv e-print DG 2007.10647v1.

[Don06] S. K. Donaldson, *Two-forms on Four-manifolds and Elliptic Equations* Inspired by S. S. Chern, 153–172, Nankai Tracts Math., **11**, World Sci.Publ., Hackensack, NJ, 2006.

[HL83] R. Harvey, H. B. Lawson, *An intrinsic characterization of Kähler manifolds*, Invent. Math. **74**, (1983), 169-198.

[ST10] J. Streets, G. Tian, *A Parabolic Flow of Pluriclosed Metrics*, Int. Math. Res. Notices, **16** (2010), 3101-3133.

[Sul76] D. Sullivan, *Cycles for the dynamical study of foliated manifolds and complex manifolds*, Invent. Math. **36** (1976), 225-255.

Bott-Chern and Appli cohomology, the Borel spectral sequence and maximally non-integrable structures

Jonas Stelzig

The talk presented some suggestions for definitions and partial results that were inteded to serve as a starting point for further discussion during the workshop and were mostly related to the recent definition of Dolbeault Cohomology for almost complex manifolds [4] by Cirici and Wilson. Three topics were discussed:

Bott-Chern and Aeppli cohomology:

On a complex manifold X, one has a diagram:

$$\begin{array}{ccccc} & & H_{BC}(X) & & \\ & \swarrow & \downarrow & \searrow & \\ H_{\overline{\partial}}(X) & \Longrightarrow & (H_{dR}(X),F,\bar{F}) & \Longleftarrow & H_{\partial}(X) \\ & \searrow & \downarrow & \swarrow & \\ & & H_A(X) & & \end{array}$$

and X satisfies the $\partial\overline{\partial}$-Lemma iff all arrows in this diagram are isomorphisms and the spectral sequences degenerate.[1] If X is merely known to be an almost complex manifold, by the work of Cirici and Wilson [4], one still has an analogue of the horizontal strip of this diagram:

$$H_{Dol}(X) \Longrightarrow H_{dR}(X) \Longleftarrow H_{Dolbar}(X)$$

where, if $(A_X, \overline{\mu}, \overline{\partial}, \partial, \mu)$ denotes the complex of $\mathbb{C}$-valued forms, $H_{Dol}(X) = H_{\overline{\partial}}(H_{\overline{\mu}}(A_X))$ and $H_{Dolbar}(X) = H_{\partial}(H_{\mu}(A_X))$. What can we put in the vertical row? As some minimal requirements, one would like such tentative H_{BC} and H_A to ...

(1) ... coincide with the classical definitions for integrable complex manifolds
(2) ... fit in the diagram, i.e. map to (resp. receive maps from) H_{dR}, H_{Dol} and H_{Dolbar} (and all higher pages of the spectral sequences).
(3) ... be equipped with a real structure.
(4) ... be an algebra (H_{BC}), resp. an H_{BC}-module (H_A).

Definition 1. *Given a 4-complex $(A, \overline{\mu}, \overline{\partial}, \partial, \mu)$, define the sub- resp. quotient double complex to be:*

$$A_s := \ker \overline{\mu} \cap \ker \overline{\partial}^2 \cap \ker \partial^2 \cap \mu \qquad A_q := A/(\operatorname{im} \overline{\mu} + \operatorname{im} \overline{\partial}^2 + \operatorname{im} \partial^2 + \operatorname{im} \mu)$$

With this, define

$$H_{BC}(X) := H_{BC}((A_X)_s) \quad H_A(X) := H_A((A_X)_q)$$

for any almost complex manifold X.

[1] A priori weaker statements like the top central arrow being injective are actually enough.

Theorem 2. *For an almost complex manifold X, set $H_{BC}(X) := H_{BC}((A_X)_s)$ and $H_A(X) := H_A((A_X)_q)$. These satisfy four minimal requirements listed above.*

In particular, this yields candidates for a definition of the $\partial\overline{\partial}$-property in the almost-complex setting.

Remark 3. *Many results in the study of cohomologies associated with double complexes have very simple proofs using the fact that there is a discrete classification of all indecomposable double complexes* [5]. *It is therefore a natural question whether there is a similar classification for 4-complexes. However, this is not the case. In fact, one can show that even in the simplest geometrically relevant case of 4-complexes concentrated in bidegrees (p,q) with $0 \leq p, q \leq 2$, the classification problem is wild.*

Borel spectral sequence:

For a fibre bundle $F \longrightarrow E \xrightarrow{\pi} B$ of complex manifolds, Borel constructed, a spectral sequence

$$E_0 = \operatorname{gr}_L A_E \Longrightarrow H_{\overline{\partial}}(E)$$

Arising from the filtered complex $(A_E, \overline{\partial}, L)$ where L^k consists of the forms which, locally, have at least k components which are pullbacks from forms on the base B. Under some mild conditions, this satisfies $E_2 = H_{\overline{\partial}}(B) \otimes_{\mathbb{C}} H_{\overline{\partial}}(F)$.

It is straightforward to check that, even if $F \to E \to B$ is just a fibre bundle of almost complex manifolds, all four components $\overline{\mu}, \overline{\partial}, \partial, \mu$ respect the filtration L. In particular, $(H_{\overline{\mu}}(E), \overline{\partial}, L)$ and $(A_E, \overline{\mu}, L)$ are filtered complexes and one obtains an analogue of Borel's spectral sequence

$${}_{\overline{\mu}}E_0 = \operatorname{gr}_L H_{\overline{\mu}}(E) \Longrightarrow (H_{Dol}(E), L)$$

and the E_0 of this can be computed via another spectral sequence, the $\overline{\mu}$-spectral sequence:

$$E_0 = \operatorname{gr}_L A_E \Longrightarrow (H_{\overline{\mu}}(E), L)$$

To have a chance a good description for the first of these, one should understand the second one and this section was focussed on this more modest goal. From the short exact sequence

$$(*) \qquad 0 \longrightarrow A_B^{1,0} \xrightarrow{\pi^*} A_E^{1,0} \longrightarrow A_{E/B}^{1,0} \longrightarrow 0$$

one obtains a bigraded complex $(A_{E/B}^{p,q}, \overline{\mu}_{E/B})$ of forms along the fibres and one has:

Theorem 4. *There are isomorphisms*

$${}_{\overline{\mu}}E_0 \cong \pi^* A_B \otimes_{A_E^0} A_{E/B} \text{ and } {}_{\overline{\mu}}E_1 \cong \pi^* A_B \otimes_{A_E^0} H_{\overline{\mu}}(A_{E/B}).$$

Any hermitian metric induces a splitting of $(*)$, which in turn induces an isomorphism of spaces of total forms $A_E \cong A_B \otimes A_{E/B}$. If such a metric can be chosen such that the splitting is compatible with the $\overline{\mu}$'s on both factors, then there are canonical identifications ${}_{\overline{\mu}}E_2 \cong {}_{\overline{\mu}}E_\infty \cong H_{\overline{\mu}}(E) \cong \pi^* H_{\overline{\mu}}(A_B) \otimes_{A_E^0} H_{\overline{\mu}}(A_{E/B})$.

The assumptions for the E_2-generation happen for a product, yielding in particular a Künneth-formula for $H_{\overline{\mu}}$, but also non-product examples can be constructed.

Complex structures with $\overline{\mu}$ of (locally) constant rank:

A complex structure is integrable if and only if $\overline{\mu} = 0$. This can be seen as the requirement for $\overline{\mu}$ to have constant rank 0. One can study more generally almost complex J structures for which $\overline{\mu}$ has (locally) constant rank. Since for nonzero locally constant rank $\overline{\mu}$ is a nontrivial bundle homomorphism $A_X^{1,0} \to A_X^{0,2}$, one has restrictions on the Chern classes of J. The extreme case is that of maximal rank, so-called maximally non-integrable structures. In low dimensions, these are:[2]

real dimension	condition
4	$5\chi + 6\sigma = 0$
6	$3c_1 = c_1^2 = c_1c_2 = 0$
8	$c_3 - 7c_1^3 = c_2^2 + 5c_1^4 - 5c_4 - 6c_2c_1^2 = 0$

We raised the question whether it might be possible to deform any almost-complex structure satisfying the necessary Chern class conditions to a maximally non-integrable one and presented the following result:

Theorem 5. *All complex surfaces which are non-torus solvmanifolds and $S^1 \times S^3$ admit left-invariant maximally non-integrable almost complex structures. The real $2n$-dimensional torus admits (non-left-invariant) complex structures for which $\overline{\mu}$ is of any constant (necessarily even) rank between 0 and $2n$.*

All of these structures have vanishing Chern classes and are homotopic to the standard one. They are given explicitly and one can show:

Theorem 6. *There exists a maximally non-integrable complex structure on the real 4-torus T for which $H_{Dol}(T)$, $H_{BC}(T)$ and $H_A(T)$ are infinite-dimensional.*

References

[1] J. Armstrong, *On four-dimensional almost-Kähler manifolds*, Quart. J. Oxford Ser. (2), **48**, 1997.

[2] A. Borel, *A spectral sequence for complex analytic bundles*, Appendix Two in: F. Hirzebruch *Topological Methods in Algebraic Geometry*, vol. 131 of *Grundlehren Der Mathematischen Wissenschaften*. Springer, Berlin [et al.], 1978.

[3] R.L. Bryant, *On the geometry of almost complex 6-manifolds*, *Asian J. Math.* 10:3, 2006.

[4] J. Cirici, S. Wilson, *Dolbeault cohomology for almost complex manifolds.* arXiv:1809.01416 [math], 2018.

[5] J. Stelzig, *On the structure of double complexes.* arXiv:1812.00865 [math], 2018.

[2] c.f. [1], resp. [3] for the 4-, resp. 6-dimensional case.

Almost complex manifolds with a small sum of Betti numbers

Michael Albanese
(joint work with Aleksandar Milivojević)

A smooth manifold admits an almost complex structure J if and only if it admits a non-degenerate two-form ω. There is a notion of integrability for each structure; namely, the manifold admits coordinate charts in which the structure takes its standard form. By the Newlander-Nirenberg theorem, J is integrable if and only if the Nijenhuis tensor N_J vanishes, while by Darboux's theorem, ω is integrable if and only if it is closed. A manifold equipped with an integrable almost complex structure is equivalent to a complex manifold, while a manifold equipped with an integrable non-degenerate two-form is a symplectic manifold. As there are complex manifolds which are not symplectic, and symplectic manifolds which are not complex, the integrability conditions $N_J = 0$ and $d\omega = 0$ are independent. The existence of an almost complex structure (equivalently, a non-degenerate two-form) on a smooth manifold imposes some topological restrictions. What if we further require integrability of J or ω?

On a closed manifold M, the integrability condition $d\omega = 0$ implies that ω^k defines a non-zero element of $H^{2k}_{\text{dR}}(M)$ for $k = 1, \dots, \frac{1}{2}\dim M$; in particular, the even Betti numbers are non-zero, a restriction that does not apply to almost complex manifolds. This is in stark contrast to the case of open manifolds where it follows from Gromov's work on the h-principle that an open manifold admits a symplectic form if and only if it admits an almost complex structure, see [5].

What if we instead impose integrability of J? The situation is far less understood. In the closed case, the only known distinction between the class of almost complex manifolds and the class of complex manifolds occurs in real dimension four. In particular, there are closed four-manifolds which admit almost complex structures, none of which are integrable, e.g. $(S^1 \times S^3)\#(S^2 \times S^2)\#(S^2 \times S^2)$. It has been suggested by Yau that no such examples exist in higher dimensions, see [6, Problem 52]. In the open case, it again follows from Gromov's work on the h-principle that if M admits an almost complex structure and $\dim M \leq 6$, then M admits an integrable almost complex structure [4, page 103]. The reason the condition $N_J = 0$ is less understood than the condition $d\omega = 0$ is that we do not know the answer to the following fundamental question:

Question. *What topological restrictions does integrability of J impose?*

Note that if a closed $2n$-dimensional manifold is symplectic, we have $b(M) \geq n + 1$ where $b(M)$ denotes the sum of Betti numbers of M. Complex manifolds do not satisfy this inequality. For example, Hopf manifolds and Calabi-Eckmann manifolds are diffeomorphic to the product of two odd-dimensional spheres and hence have $b(M) = 4$. Motivated by this, Dennis Sullivan made the following conjecture:

Conjecture (Sullivan). *Let M be a compact complex manifold with* $\dim_{\mathbb{C}} M \geq 3$. *Then $b(M) \geq 4$.*

If the conjecture were true, then S^6 would provide an example of a manifold of dimension greater than four which admits almost complex structures, none of which are integrable (contradicting Yau's suggestion). Would there be any other such examples? That is, are there other closed manifolds with $b(M) < 4$ which admit almost complex structures? Note that $b(M) \geq 2$ for a closed orientable manifold, so there are two cases to consider, namely $b(M) = 2$ and $b(M) = 3$.

If $b(M) = 2$, then M is a rational homology sphere, i.e. M has the same rational homology as a sphere. A famous result of Borel and Serre [3] states that the only spheres which admit almost complex structures are S^2 and S^6. The following generalises this result to the case of rational homology spheres:

Theorem. [1, Theorem 2.2] *Let M be a rational homology sphere. If M admits an almost complex structure, then* $\dim M = 2$ *or* 6*.*

There are infinitely many six-dimensional rational homology spheres. At least one of them admits an almost complex structure, namely S^6. As the following theorem demonstrates, this property is not shared by all six-dimensional rational homology spheres.

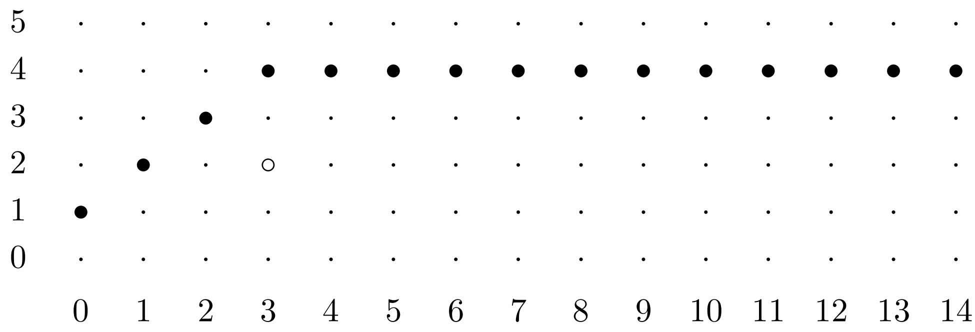
Theorem. [1, Corollary 2.5] *There are infinitely many six-dimensional rational homology spheres which admit almost complex structures, and infinitely many which do not.*

This result was established by proving that the spin^c property is preserved under a process known as spinning. In particular, repeatedly spinning the Wu manifold $SU(3)/SO(3)$ produces an orientable non-spin^c rational homology sphere S in any dimension greater than or equal to 5.

Corollary. *Let M be a compact manifold of even dimension greater than* 4*. The existence of an almost complex structure on M cannot be detected by $H^*(M;\mathbb{Q})$ alone.*

In particular, for any such manifold M, the manifold $M' = M\#S$ satisfies $H^*(M;\mathbb{Q}) \cong H^*(M;\mathbb{Q})$ and does not admit an almost complex structure. A natural question to ask then is which rational cohomology rings can be realised by manifolds which admit almost complex structures? This is known as the realisation problem, see Aleksandar Milivojević's talk.

If a closed manifold with $b(M) = 3$ admits an almost complex structure then $\dim M = 4$ and $H^*(M;\mathbb{Q}) \cong H^*(\mathbb{CP}^2;\mathbb{Q})$; this follows from [1, Theorem 3.3] and unpublished work of Jiahao Hu and Zhixu Su. Therefore, the only potential counterexamples to Sullivan's conjecture are six-dimensional rational homology spheres. This is summarised in the following figure where the full circles represent the minimal sum of Betti numbers among known compact complex manifolds in the given dimension and the empty circle indicates the position of a potential complex manifold diffeomorphic to a six-dimensional rational homology sphere.



It is also interesting to investigate the boundary case, i.e. $b(M) = 4$. In this case, the manifold M has two possible rational cohomology rings, either the rational cohomology ring of a product of two spheres, or $\mathbb{Q}[x]/(x^4)$. The study of which of these rings can be realised by a manifold admitting an almost complex structure was initiated in [2]. See Aleksandar Milivojević's talk for further results in this direction.

References

[1] M. Albanese and A. Milivojević, 2019. *On the minimal sum of Betti numbers of an almost complex manifold.* Differential Geometry and its Applications, **62**, pp.101-108.

[2] M. Albanese and A. Milivojević, 2019. *Connected sums of almost complex manifolds, products of rational homology spheres, and the twisted spin^c Dirac operator.* Topology and its Applications, **267**, p.106890.

[3] A. Borel and J-P. Serre, 1953. *Groupes de Lie et puissances réduites de Steenrod.* American Journal of Mathematics, **75** (3), pp.409-448.

[4] M. Gromov, 1986. *Partial differential relations.* Ergebnisse der Mathematik und ihrer Grenzgebiete, (3), pp.Vol-9.

[5] A. Haefliger, 1971. *Lectures on the theorem of Gromov.* In Proceedings of Liverpool Singularities Symposium II (pp. 128-141). Springer, Berlin, Heidelberg.

[6] S.T. Yau, 1993. *Open problems in geometry.* In Proc. Symp. Pure Math, **54**, No. 1, pp. 1-28.

Realization of rational spaces by almost complex manifolds

Aleksandar Milivojević

A general goal one could hope for is to describe the homotopy types of closed smooth manifolds admitting almost complex structures. This is a relatively difficult problem, and not solved in full generality in dimensions ≥ 10. More tangibly, we can aim to describe the *rational* homotopy types of closed smooth manifolds admitting almost complex structures. Two spaces (we will take all spaces to be connected degree-wise finite dimensional cell complexes whose fundamental group is trivial or at least nilpotent with nilpotent action on the higher homotopy groups) are *rationally* homotopy equivalent if there is a map between them inducing an isomorphism on rationalized homotopy groups, or equivalently on rational homology. Every space X has its associated *rationalization* $X \to X_{\mathbb{Q}}$, a rational homotopy equivalence where $X_{\mathbb{Q}}$ is a *rational* space in the sense that the ordinary homotopy groups of $X_{\mathbb{Q}}$ are isomorphic to rational vector spaces as abelian groups. The rationalization is unique up to homotopy equivalence; rational spaces are homotopy equivalent if and only if they are rationally homotopy equivalent. To build the rationalization of a space one can either "tensor the Postnikov tower of X by $\mathbb{Q}$" or take a cellular decomposition and replace the maps of spheres and cones on them with maps of rationalized spheres and their cones. The Postnikov tower picture informs one that the information of a rational space can be encoded in a free graded–commutative differential graded algebra; the categories of nilpotent rational spaces and nilpotent such algebras are equivalent in a suitable sense [1].

Now, given a rational space $X_{\mathbb{Q}}$, we can ask if there is a closed almost complex manifold (M, J) with a rational homotopy equivalence $M \xrightarrow{f} X_{\mathbb{Q}}$. If there is to be a positive answer, $X_{\mathbb{Q}}$ must satisfy Poincaré duality on its rational cohomology and hence there must be an index n (the real dimension of M) such that $H_n(X_{\mathbb{Q}}; \mathbb{Q}) \cong \mathbb{Q}$ and $H_{>n}(X_{\mathbb{Q}}; \mathbb{Q}) = 0$; there also must be rational cohomology classes in $X_{\mathbb{Q}}$ that pull back to the rational Chern classes of (M, J). So let us restrict our question to: given a rational Poincaré duality space $X_{\mathbb{Q}}$ with a choice of non-zero element $[X_{\mathbb{Q}}] \in H_n(X_{\mathbb{Q}}; \mathbb{Q})$ (its "fundamental class") and a choice of $c_1, c_2, \ldots$ in $H^{2*}(X_{\mathbb{Q}}; \mathbb{Q})$, is there a closed almost complex manifold (M, J) and a rational homotopy equivalence $M \xrightarrow{f} X_{\mathbb{Q}}$ such that $f_*[M] = [X_{\mathbb{Q}}]$ and $f^*c_i = c_i(M, J)$? (Varying the choice of fundamental class and c_i we retrieve our original question.)

If there were such an M, notice that the "Chern numbers" $\langle c_I, [X_{\mathbb{Q}}]\rangle$ satisfy

$$\langle c_I, [X_{\mathbb{Q}}]\rangle = \langle c_I(X_{\mathbb{Q}}), f_*[M]\rangle = \langle f^*c_I(X_{\mathbb{Q}}), [M] = \langle c_I(M), [M]\rangle = \int_M c_I(M),$$

and the latter must be integers that satisfy all the (finitely many) congruence conditions coming from index theory, i.e.

$$\int ch(E)td(TM) \in \mathbb{Z} \text{ for any complex vector bundle } E \to M.$$

This condition can be recast purely in terms of $X_{\mathbb{Q}}, [X_{\mathbb{Q}}], c_i$. Furthermore, since $\int_M c_n$ is the Euler characteristic of the putative almost complex manifold M, we must have $\langle c_n, [X_{\mathbb{Q}}]\rangle$ be the Euler characteristic of X.

If the dimension n is divisible by 4, we furthermore have:

- The non-degenerate symmetric bilinear pairing on $H^{\frac{n}{2}}(X_{\mathbb{Q}}; \mathbb{Q})$ must be the rationalization of a unimodular pairing over the integers, i.e. in some basis the pairing on $H^{\frac{n}{2}}(X_{\mathbb{Q}}; \mathbb{Q})$ is of the form $\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}$.
- If we form the "Pontryagin classes" via $1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots)$, the Hirzebruch L–polynomial evaluated on these classes and paired with $[X_{\mathbb{Q}}]$ must calculate the signature of the pairing on $H^{\frac{n}{2}}(X_{\mathbb{Q}}; \mathbb{Q})$ correctly, $\langle L(p_i), [X_{\mathbb{Q}}]\rangle = \sigma(X_{\mathbb{Q}})$.

As for sufficiency, we have:

Theorem. *Let $X_{\mathbb{Q}}$ be a simply connected rational space, with $H_*(X_{\mathbb{Q}};\mathbb{Q})$ finite–dimensional, satisfying rational Poincaré duality, of even dimension $n \geq 6$, with a choice of non-zero $[X_{\mathbb{Q}}] \in H_n(X_{\mathbb{Q}};\mathbb{Q})$ and a choice of $c_i \in H^{2i}(X_{\mathbb{Q}};\mathbb{Q})$. Then, if $n \not\equiv 4 \bmod 8$, the above necessary conditions (Chern number integrality and congruences, top Chern class calculating Euler characteristic, intersection pairing being diagonal with ± 1, and signature being calculated correctly from the Chern classes) are sufficient for the existence of a closed simply connected almost complex manifold (M, J) and a map $M \xrightarrow{f} X$ inducing an isomorphism on rational homology, such that $f_*[M] = [X]$ and $f^*(c_i) = c_i(M, J)$. If $n \equiv 4 \bmod 8$ and $c_1 \neq 0$, the same holds; if $c_1 = 0$, imposing a further set of congruence conditions on the Chern numbers yields the desired result.*

The further congruence in the case of $n \equiv 4 \bmod 8$ and $c_1 = 0$ stems from the fact that for an almost complex manifold M in these dimensions with $c_1 = 0$ *integrally*, we have $\int_M ch(E \otimes \mathbb{C})\hat{A}(TM) \in 2\mathbb{Z}$ for any real vector bundle $E \to M$.

The proof mimics the rational realization result of Sullivan for smooth manifolds [1, Theorem 13.2], utilizing the techniques of normal surgery, the Pontryagin–Thom construction, rational Poincaré duality, Stong's description [3] of the integer lattice in the rational homology of the classifying space BU corresponding to the image of the map from complex bordism to BU sending a manifold to the pushforward of its fundamental class via the classifying map for the tangent bundle, and the freedom to ignore torsion information as we are working rationally.

From the proof, one sees that if the conditions of the theorem are satisfied, the answer to the question of realizability of X by an almost complex manifold depends *only on the rational cohomology algebra* of X. In particular, taking a simply connected commutative differential graded algebra over $\mathbb{Q}$ whose corresponding rational space is realizable by an almost complex manifold by the above theorem, one can deform this algebra at will within the category of rational Poincaré duality algebras and obtain corresponding almost complex manifolds. Since the rational homotopy type of a simply connected space X is uniquely determined by the induced C_∞ structure on its rational cohomology algebra $(H^*(X), \{m_i\}_{i\geq 2})$, we have the following:

Corollary. *In dimensions $n \not\equiv 4 \bmod 8$, the realizability of a simply connected C_∞-algebra $(H, \{m_i\}_{i\geq 2})$ by a closed almost complex manifold depends only on the multiplication m_2.*

One would like to remove the assumption in the case of $n \equiv 4 \bmod 8$ that either $c_1 \neq 0$ or that the Chern numbers satisfy a stronger set of congruences than those of almost complex manifolds, in order to conclude that *in all even dimensions the realizability of a simply connected C_∞-algebra $(H, \{m_i\}_{i\geq 2})$ by a closed almost complex manifold depends only on the multiplication m_2*; this possibility remains to be studied.

Further corollaries are: (1) An even-dimensional simply connected rational Poincaré duality space with vanishing Euler characteristic, and vanishing signature if the dimension is divisible by four, is realized by a closed almost complex manifold. (2) In odd complex dimensions, realizability is guaranteed if the Euler characteristic is divisible by a certain positive integer $d(n)$ depending on the dimension ($d(6) = 2, d(10) = 24$ and likely, but not yet verified, $d(2n) = (n-1)!$). In particular every simply connected six–dimensional rational Poincaré duality space is realized by a closed almost complex manifold. In fact, the almost complex structure in the six–dimensional case can be taken to have $c_1 = 0$ integrally, and hence the obstructions to admitting a maximally non-integrable almost complex structure from Jonas Stelzig's talk ($3c_1 = 0, c_1^2 = 0, c_1c_2 = 0$) vanish; we do not know if these are the only obstructions to admitting such a structure.

As an example, one can calculate that there is a closed almost complex manifold with the same rational homotopy type as quaternionic three–space $\mathbb{HP}^3$; this leads one to generally ask when the cohomology algebra $\mathbb{Q}[x]/(x^k)$ is realizable in terms of k and $\deg(x)$. For $k = 3$ only $\deg(x) = 2$ is realized, corresponding to the rational homotopy type of $\mathbb{CP}^2$ (unpublished work of Zhixu Su and independently Jiahao Hu, and [6]); compare this with the general (not necessarily almost complex) smooth manifold case for which there are unexpected solutions for $\deg(x)$ when $k = 3$ [4], [5].

References

[1] D. Sullivan, 1977. *Infinitesimal computations in topology.* Publications Mathématiques de l'Institut des Hautes Études Scientifiques, 47(1), pp.269-331.

[2] W. Browder, 2012. *Surgery on simply-connected manifolds* (Vol. 65). Springer Science & Business Media.

[3] R.E. Stong, 1965. *Relations among characteristic numbers–I.* Topology, 4(3), pp.267-281.

[4] Su, Z., 2014. Rational analogs of projective planes. Algebraic & Geometric Topology, 14(1), pp.421-438.

[5] L. Kennard, and Z. Su, 2019. *On dimensions supporting a rational projective plane.* Journal of Topology and Analysis, 11(03), pp.535-555.

[6] M. Albanese, and A. Milivojević, 2019. *On the minimal sum of Betti numbers of an almost complex manifold.* Differential Geometry and its Applications, 62, pp.101-108.

Harmonic forms on almost-Hermitian manifolds

Nicoletta Tardini

(joint work with Adriano Tomassini)

On a smooth manifold X endowed with an almost-complex structure J, namely a $(1,1)$-tensor such that $J^2 = -\mathrm{Id}$, the exterior derivative d splits as the sum of four operators. More precisely, if $A^{p,q}(X)$ denotes the space of (p,q)-forms on X, one has that

$$d : A^{p,q}(X) \to A^{p+2,q-1}(X) \oplus A^{p+1,q}(X) \oplus A^{p,q+1}(X) \oplus A^{p-1,q+2}(X)$$

$$d = \mu + \partial + \overline{\partial} + \bar{\mu}\,.$$

The operators μ and $\bar{\mu}$ characterize the integrability of J, indeed J is integrable, that means that (X, J) is a complex manifold, if and only if $\mu = \bar{\mu} = 0$.

Therefore, in the integrable case, the vanishing of d^2 implies that $\partial^2 = 0$, $\overline{\partial}^2 = 0$ and $\partial\overline{\partial} + \overline{\partial}\partial = 0$. In particular, one can define several cohomology groups on a complex manifold: the de Rham, Dolbeault and conjugate Dolbeault cohomologies respectively

$$H^{\bullet}_{dR}(X;\mathbb{C}) := \frac{\operatorname{Ker} d}{\operatorname{Im} d}, \qquad H^{\bullet,\bullet}_{\overline{\partial}}(X) := \frac{\operatorname{Ker} \overline{\partial}}{\operatorname{Im} \overline{\partial}}, \qquad H^{\bullet,\bullet}_{\partial}(X) := \frac{\operatorname{Ker} \partial}{\operatorname{Im} \partial}$$

and the Bott-Chern and Aeppli cohomologies respectively

$$H^{\bullet,\bullet}_{BC}(X) := \frac{\operatorname{Ker}\partial \cap \operatorname{Ker}\overline{\partial}}{\operatorname{Im}\partial\overline{\partial}}, \qquad H^{\bullet,\bullet}_{A}(X) := \frac{\operatorname{Ker}\partial\overline{\partial}}{\operatorname{Im}\partial + \operatorname{Im}\overline{\partial}}.$$

On a compact complex manifold an effective way to compute them is to fix a Hermitian metric and compute the associated spaces of harmonic forms, since all these cohomologies are isomorphic to the kernel of suitable elliptic differential operators.

In the non integrable case the situation is very different, indeed, the vanishing of d^2 implies the following relations

$$\begin{cases} \mu^2 & = & 0 \\ \mu\partial + \partial\mu & = & 0 \\ \partial^2 + \mu\overline{\partial} + \overline{\partial}\mu & = & 0 \\ \partial\overline{\partial} + \overline{\partial}\partial + \mu\bar{\mu} + \bar{\mu}\mu & = & 0 \\ \overline{\partial}^2 + \bar{\mu}\partial + \partial\bar{\mu} & = & 0 \\ \bar{\mu}\overline{\partial} + \overline{\partial}\bar{\mu} & = & 0 \\ \bar{\mu}^2 & = & 0 \end{cases}.$$

In particular, the Dolbeault, Bott-Chern and Aeppli cohomologies are not well-defined but their spaces of harmonic forms still are. Namely, if we fix a Hermitian metric g on an almost-complex manifold (X, J) the operator $\Delta_{\overline{\partial}} := \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ is still elliptic. When J is integrable, $\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X) := \operatorname{Ker}\Delta_{\overline{\partial}}$ is isomorphic to the Dolbeault cohomology so it is a holomorphic invariant. When J is non integrable $\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X)$ a priori depends on the metric. This motivated a question by Kodaira and Spencer, listed as Problem 20 in a paper by Hirzebruch [4], regarding whether, on almost-complex manifolds, the dimensions of $\mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X)$ depend on the choice of the Hermitian metric. In [5] Holt and Zhang answered positively to this question, giving an explicit construction on the Kodaira-Thurston manifold of an almost-complex structure with $\dim \mathcal{H}^{0,1}_{\overline{\partial}}(X)$ varying with different choices of Hermitian metrics. However, very recently Cirici and Wilson in [1, 2] proposed a generalization of the Dolbeault cohomology for almost-complex manifolds and studied several spaces of harmonic forms which inject in them.

In [6] we introduced several spaces of harmonic forms on almost-Hermitian manifolds, recovering, in the integrable setting, the spaces of Dolbeault, Bott-Chern and Aeppli harmonic forms and we gave also applications to symplectic geometry.

Let (X, J) be an almost-complex manifold, we considered the operators, introduced in [3]

$$\delta := \partial + \bar{\mu} : A^\bullet \to A^{\bullet+1}\,, \qquad \bar{\delta} := \overline{\partial} + \mu : A^\bullet \to A^{\bullet+1}$$

These two operators anticommute but their squares are zero if and only if J is integrable. If g is a Hermitian metric on (X, J) one can define

$$\Delta_{\bar{\delta}} := \bar{\delta}\bar{\delta}^* + \bar{\delta}^*\bar{\delta}\,, \quad \Delta_\delta := \delta\delta^* + \delta^*\delta\,,$$

$$\Delta_{BC(\delta,\bar{\delta})} := (\delta\bar{\delta})(\delta\bar{\delta})^* + (\delta\bar{\delta})^*(\delta\bar{\delta}) + (\bar{\delta}^*\delta)(\bar{\delta}^*\delta)^* + (\bar{\delta}^*\delta)^*(\bar{\delta}^*\delta) + \bar{\delta}^*\bar{\delta} + \delta^*\delta\,,$$

$$\Delta_{A(\delta,\bar{\delta})} := \delta\delta^* + \bar{\delta}\bar{\delta}^* + (\delta\bar{\delta})^*(\delta\bar{\delta}) + (\delta\bar{\delta})(\delta\bar{\delta})^* + (\bar{\delta}\delta^*)^*(\bar{\delta}\delta^*) + (\bar{\delta}\delta^*)(\bar{\delta}\delta^*)^*\,.$$

If J is integrable, these differential operators coincide with the classical Laplacian operators on complex manifolds and, in that case, their kernels has a cohomological interpretation. We studied Hodge theory for these operators on almost-Hermitian manifolds, in particular $\Delta_{\bar{\delta}}$ and Δ_δ are elliptic differential operators of the second order and $\Delta_{BC(\delta,\bar{\delta})}$ and $\Delta_{A(\delta,\bar{\delta})}$ are elliptic differential operators of the fourth order, therefore their kernels are finite-dimensional as soon as X is compact. Moreover, such operators also induce Hodge decompositions on the space of differential forms and they respect all the natural dualities and conjugation symmetries.

In particular, on bi-graded forms we are just reinterpreting the spaces $\mathrm{Ker}\,(\Delta_{\overline{\partial}} + \Delta_\mu) = \mathcal{H}^{\bullet,\bullet}_{\overline{\partial}}(X) \cap \mathcal{H}^{\bullet,\bullet}_{\mu}(X) = \mathcal{H}^{\bullet,\bullet}_{\bar{\delta}}(X) := \mathrm{Ker}\,(\Delta_{\bar{\delta}}) \cap A^{\bullet,\bullet}(X)$ considered by Cirici and Wilson in [1], where $\Delta_\mu := \mu\mu^* + \mu^*\mu$ and $\mathcal{H}^{\bullet,\bullet}_{\mu}(X) := \mathrm{Ker}\,\Delta_\mu$. However, on total degree forms we have $\mathcal{H}^{\bullet}_{\overline{\partial}}(X) \cap \mathcal{H}^{\bullet}_{\mu}(X) \subseteq \mathcal{H}^{\bullet}_{\bar{\delta}}(X) := \mathrm{Ker}\,(\Delta_{\bar{\delta}}) \cap A^{\bullet}(X)$ and this inclusion can be strict.

In fact, this theory turns out to be helpful in detecting the existence of compatible symplectic structures. Indeed, suppose now that (X, J) admits a compatible symplectic structure, that is a non-degenerate d-closed 2-form ω. Hence (X, J, ω) is called almost-Kähler manifold. We can use the almost-Kähler identities in order to improve our knowledge of the differential operators defined above and the associated harmonic forms. In particular, if (X, J, ω) is an almost-Kähler manifold, then by [2]

$$\Delta_{\overline{\partial}} + \Delta_\mu = \Delta_\partial + \Delta_{\bar{\mu}} \quad \text{and} \quad \Delta_d = 2(\Delta_{\overline{\partial}} + \Delta_\mu + [\bar{\mu}, \partial^*] + [\mu, \overline{\partial}^*] + [\partial, \overline{\partial}^*] + [\overline{\partial}, \partial^*])\,.$$

and we proved in [6] the following

Theorem. *Let (X, J, ω) be an almost-Kähler manifold, then*

$$\Delta_{\bar{\delta}} = \Delta_\delta \quad \textit{and} \quad \Delta_d = 2\Delta_\delta + [\delta, \bar{\delta}^*] + [\bar{\delta}, \delta^*]\,.$$

Both these results generalize the classical Hodge relations on compact Kähler manifolds $\Delta_{\overline{\partial}} = \Delta_\partial$ and $\Delta_d = 2\Delta_{\overline{\partial}}$. In fact, as a consequence, we also proved that on a compact almost Kähler manifold

$$\Delta_d = 2\Delta_\delta$$

if and only if the manifold is Kähler. Moreover, a second application of our results is the following

Corollary. *Let (X, J, ω) be a compact almost-Kähler manifold, then*

$$\mathcal{H}^{\bullet}_{\bar{\delta}}(X) \subseteq \mathcal{H}^{\bullet}_{dR}(X) \simeq H^{\bullet}_{dR}(X)\,.$$

In particular, for the respective dimensions there is a topological upper bound,

$$dim\,\mathcal{H}^{\bullet}_{\bar{\delta}}(X) \leq b_{\bullet}(X)\,,$$

where $b_{\bullet}(X)$ denotes the Betti numbers of X.

These results represent obstructions to the existence of symplectic structures compatible with a given fixed almost-complex structure.

We also proved that on bi-graded forms all the spaces of $\bar{\delta}-$, $\delta-$, Bott-Chern and Aeppli (and actually more) harmonic forms coincide. Notice that this result, in the integrable case, implies that on compact Kähler manifolds all the cohomology groups are isomorphic, hence showing that compact Kähler manifolds satisfy the $\partial\overline{\partial}$-lemma. However, in the non integrable case, we cannot conclude this since we do not have a cohomological counterpart for the spaces of harmonic forms. In fact, it is not clear what should be a natural notion of $\partial\overline{\partial}$-lemma in the non integrable setting. Moreover, generalized Hard-Lefschetz conditions can be proved for these spaces of harmonic forms, indeed in [6] we showed that on a compact almost-Kähler $2n$-dimensional manifold, for any k, the maps

$$\omega^k \wedge - : \mathcal{H}^{n-k}_{BC(\delta,\bar{\delta})}(X) \to \mathcal{H}^{n+k}_{BC(\delta,\bar{\delta})}(X)$$

are isomorphisms.

References

[1] J. Cirici, S. O. Wilson, Dolbeault cohomology for almost complex manifolds, `arXiv:1809.01416 [math.DG]`, 2020.

[2] J. Cirici, S. O. Wilson, Topological and geometric aspects of almost Kähler manifolds via harmonic theory, *Sel. Math. New Ser.*, **26**, no. 35 (2020).

[3] P. de Bartolomeis, A. Tomassini, On formality of some symplectic manifolds. *Internat. Math. Res. Notices* **2001**, no. 24, (2001) 1287–1314.

[4] F. Hirzebruch, Some problems on differentiable and complex manifolds, *Ann. of Math. (2)* **60**, (1954). 213–236.

[5] T. Holt, W. Zhang, Harmonic Forms on the Kodaira-Thurston Manifold, `arXiv:2001.10962 [math.DG]`, 2020.

[6] N. Tardini, A. Tomassini, Differential operators on almost-Hermitian manifolds and harmonic forms, *Complex Manifolds*, **7**, no. 1, (2020) 106–128.

When is $\dim \mathcal{H}^{p,q}_{\bar\partial}$ an almost complex invariant?

Tom Holt

(joint work with Weiyi Zhang)

Let M be a compact, almost complex manifold of real dimension $2n$, with some chosen almost Hermitian metric. We consider the space $\mathcal{H}^{p,q}_{\bar\partial}$, defined to be the kernel of $\Delta_{\bar\partial}$ in the space of (p,q)-forms. For complex structures, $h^{p,q} := \dim \mathcal{H}^{p,q}_{\bar\partial}$ does not depend on the choice of almost Hermitian metric since $\mathcal{H}^{p,q}_{\bar\partial}$ is isomorphic to the Dolbeault cohomology group. In the almost complex setting Kodaira and Spencer asked the following question, which appeared as Problem 20 in Hirzebruch's 1954 problem list [1].

Question (Kodaira-Spencer)**.** *Let M be an almost complex manifold and consider the numbers $h^{p,q}$. Are they independent of the Hermitian structure on M?*

This question has already been answered positively in the cases of $h^{p,0}$ and $h^{p,n}$, for any $p = 0, 1, \ldots, n$ [2]. Furthermore when M has dimension 4, $h^{1,1}$ is at least an almost Kähler metric invariant [3] and may well also be an almost Hermitian metric invariant. In this talk, however, we shall demonstrate that the answer to the Kodaira-Spencer question is in general negative, in particular there exists a fixed almost complex structure on the Kodaira-Thurston manifold KT^4 along with a family of compatible metrics over which $h^{0,1}$ takes multiple distinct values. In fact, in proving this we shall use a family of almost Kähler metrics, thus showing $h^{p,q}$ is not even guaranteed to be an almost Kähler metric invariant. See [4] for the full proof of this.

The Kodaira-Thurston manifold is defined to be $\Gamma\backslash G$, where G is the group $(\mathbb{R}^4, \circ)$ with group operation

$$\begin{pmatrix} t_0 \\ x_0 \\ y_0 \\ z_0 \end{pmatrix} \circ \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} t + t_0 \\ x + x_0 \\ y + y_0 \\ z + z_0 + x_0 y \end{pmatrix}$$

and Γ is the subgroup $\mathbb{Z}^4$ acting on G by left multiplication. It turns out the calculation of $\mathcal{H}^{0,1}_{\bar\partial}$ amounts to finding pairs of functions $f, g \in C^\infty(KT^4)$ satisfying a particular system of PDEs. The trick to solving these is to use classical Fourier analysis to decompose f uniquely into a sum of simpler functions

$$f(t,x,y,z) = \sum_{k\in\mathbb{Z}} \Bigg(\sum_{l,m\in\mathbb{Z}} f_{k,l,m,0} e^{2\pi i(kt+lx+my)} + \sum_{\substack{n\in\mathbb{Z}\backslash\{0\} \\ m\in\{0,1,\ldots,|n|-1\}}} \sum_{\xi\in\mathbb{Z}} f_{k,m,n}(x+\xi) e^{2\pi i(kt+(m+n\xi)y+nz)} \Bigg)$$

where

$$f_{k,l,m,0} = \int_{[0,1]^4} f(t,x,y,z)e^{-2\pi i(kt+lx+my)}dt\,dx\,dy\,dz \in \mathbb{C}$$

$$f_{k,m,n}(x+\xi) = \int_{[0,1]^3} f(t,x,y,z)e^{-2\pi i(kt+(m+n\xi)y+nz)}dt\,dy\,dz \in \mathcal{S}(\mathbb{R})$$

and likewise for g. Equipped with this decomposition we can split $h^{0,1}$ into two parts $h^{0,1} = h'_{0,1} + h''_{0,1}$ with $h'_{0,1}$ being the number of independent solutions where f, g take the form of

$$f_{k,l,m,0}e^{2\pi i(kt+lx+my)}.$$

and $h''_{0,1}$ being the number of independent solutions where f, g take the form of

$$\sum_{\xi\in\mathbb{Z}} f_{k,m,n}(x+\xi)e^{2\pi i(kt+(m+n\xi)y+nz)}.$$

In the case of $h'_{0,1}$ the system of PDEs simplifies to a system of number theoretic equations. Looking for solutions here amounts to answering a generalised Gauss circle problem, that is to say the number of solutions is equal to the number of points in a lattice that are intersected by a circle with some radius and centre depending on the chosen metric and almost complex structure. Indeed, by varying the metric we can change the value of $h'_{0,1}$ and by varying the almost complex structure it can even be made arbitrarily large.

In the case of $h''_{0,1}$ the system of PDEs on the Kodaira-Thurston manifold simplifies to a system of ODEs on $\mathbb{R}$. Importantly, we are only looking for Schwartz solutions to these ODEs, which is a consequence of requiring the sum over ξ to converge to a smooth function. We can find such solutions by considering the Stokes phenomenon of the ODEs, in particular we ask whether a solution which decays as $x \to +\infty$ will also decay as $x \to -\infty$. It turns out this is rarely the case and we can easily find a family of almost Kähler metrics for which $h''_{0,1}$ is always zero.

Bringing together our considerations of $h'_{0,1}$ and $h''_{0,1}$, we can find a family of almost Kähler metrics for which $h^{0,1}$ takes more than one value. As a result we conclude the following:

Theorem. *There exist almost complex structures on the Kodaira-Thurston manifold such that $h^{0,1}$ varies with different choices of almost Kähler metrics.*

This answers an almost Kähler version of the Kodaira-Spencer question in the negative.

References

[1] F. Hirzebruch, *Some problems on differentiable and complex manifolds*, Ann. of Math. (2) 60, (1954). 213–236.

[2] H. Chen and W. Zhang, *Kodaira Dimensions of Almost Complex Manifolds I*, arXiv:1808.00885, 2018.

[3] T. Holt and W. Zhang, *Harmonic Forms on the Kodaira-Thurston Manifold*, arXiv:2001.10962, 2020.

[4] T. Holt and W. Zhang, *Almost Kähler Kodaira-Spencer problem*, arXiv:2010.12545, 2020.

Hodge theory on SKT manifolds

Gil Cavalcanti

In an SKT manifold $H = d^c\omega$ is a closed form. This is a subtle integrability condition placed on the metric and complex structure. Including this integrability condition into the de Rham complex yields a twisted complex which is more compatible with the geometric structure than the usual Dolbeault complex and for which a version of Hodge's Theorem holds. We use this newfound Hodge theory to investigate the existence of SKT structures on some complex manifolds.

Balanced metrics and the Hull-Strominger system

Anna Fino
(joint work with Gueo Grantcharov and Luigi Vezzoni)

The purpose of the talk was to review some general results about balanced metrics and present new smooth solutions to the Hull-Strominger system [14, 24], showing that the Fu-Yau solution [11, 12] on torus bundles over K3 surfaces can be generalized to torus bundles over K3 orbifolds [9].

A Hermitian metric g on a complex manifold of complex dimension n is balanced if its fundamental form ω is co-closed, or equivalently if $d\omega^{n-1} = 0$. Balanced manifolds were introduced and studied by Michelsohn in [17]. The balanced condition is preserved under proper holomorphic submersions and birational transformations [2]. Compact examples of non-Kähler compact balanced manifolds are given, for instance, by six-dimensional twistor spaces [17], Mishezon manifolds, complex manifolds in the Fujiki class $\mathcal{C}$ and complex parallelizable manifolds [1].

Existence of balanced metrics play a central role in the study of the Hull-Strominger system, which describes the geometry of compactification of heterotic superstrings with torsion to 4-dimensional Minkowski spacetime. Let M be a compact complex manifold of complex dimension 3 with a nowhere vanishing holomorphic $(3,0)$-form Ω. Let E be a complex vector bundle over M with a Hermitian metric H along its fibers and the $\alpha' \in \mathbb{R}$ be a constant (slope parameter). The Hull-Strominger system, for the fundamental form ω of a Hermitian metric g on M, is given by:

$$F_H \wedge \omega^2 = 0,\ F_H^{2,0} = F_H^{0,2} = 0, \tag{1}$$

$$d(\|\psi\|_\omega\, \omega^2) = 0, \tag{2}$$

$$i\partial\bar\partial\omega = \frac{\alpha'}{4}\,\mathrm{tr}\left(R_\nabla \wedge R_\nabla - F_H \wedge F_H\right), \tag{3}$$

where F_H and R_∇ are respectively the curvatures of H and of a metric connection ∇ on TM. The equation (1) describes the Hermitian-Yang-Mills equations for the connection of H. The equation (2) says that ω is conformally balanced and it was originally written as $d^*\omega = i(\overline{\partial} - \partial)\ln(\|\Omega\|_\omega)$ (the equivalence was proved by Li and Yau in [16]). The last equation is the anomaly cancellation equation (or Bianchi identity) and couples the two metrics ω and H.

The first solutions of the Hull-Strominger system on compact non-Kähler manifolds, taking ∇ as the Chern connection of ω, were constructed by Fu and Yau [11, 12]. Up to now many solutions are provided by the choice of ∇ given by the Chern connection [3, 4, 5, 6, 7, 8, 10, 18, 21, 19, 20, 23]. New examples of solutions of the Hull-Strominger system on non-Kähler torus bundles over K3 surfaces originally considered by Fu and Yau, with the property that the connection ∇ is Hermitian-Yang-Mills have been constructed in [13].

In [9] we extended the Fu-Yau ansatz to Hermitian 3-folds foliated by non-singular elliptic curves, showing the following:

Theorem. ([9]) Let X be a compact K3 orbifold with a Ricci-flat Kähler form ω_X and orbifold Euler number $e(X)$. Let ω_1 and ω_2 be anti-self-dual $(1,1)$-forms on X such that $[\omega_1],[\omega_2] \in H^2_{orb}(X,\mathbb{Z})$ and the total space M of the principal T^2 orbifold bundle $\pi : M \to X$ determined by them is smooth. Let E be a stable vector bundle of degree 0 over (X,ω_X) such that

$$\alpha'(e(X) - (c_2(E) - \frac{1}{2}c_1^2(E))) = \frac{1}{4\pi^2}\int_X (\|\omega_1\|^2 + \|\omega_2\|^2)\frac{\omega_X^2}{2}.$$

Then, M admits a Hermitian structure (M,ω_u) and there is a metric h along the fibers of E such that $(V = \pi^*E, H = \pi^*(h), M, \omega_u)$ solves the Strominger system.

Using Seifert S^1-bundles [15], we can show that the smooth manifolds $S^1 \times \sharp_k(S^2 \times S^3)$, $13 \le k \le 22$, and $\sharp_r(S^2 \times S^4)\sharp_{r+1}(S^3 \times S^3)$, $14 \le r \le 22$, have a solution to the Hull-Strominger system via the Fu-Yau ansatz. The cases $k = 22$ and $r = 22$ respectively correspond to the solutions of Fu and Yau. In this way we construct new simply-connected compact non-Kähler 6-manifolds admitting a solution of the Hull-Strominger system.

References

[1] E. Abbena, A. Grassi, *Hermitian left invariant metrics on complex Lie groups and cosymplectic hermitian manifolds*, Boll. Un. Mat. Ital. A **5** (1986) 371–379.

[2] L. Alessandrini, G. Bassanelli, *Modifications of compact balanced manifolds*, C. R. Acad. Sci. Paris Sér. I Math **320** (1995) no. 12, 1517–1522.

[3] J Chu, L. Huang, X. Zhu, *The Fu-Yau equation on compact astheno-Kähler manifolds*, Adv. Math. **346** (2019), 908–945.

[4] J. Chu, L. Huang, X. Zhu, *The Fu-Yau equation in higher dimensions*, Peking Math. J. **2** (2019), no. 1, 71–97.

[5] T. Fei, Z.J. Huang, S. Picard, *A construction of infinitely many solutions to the Strominger system*, `arXiv:1703.10067`, To appear in J. Diff. Geom.

[6] T. Fei, Z.J. Huang, S. Picard, *The Anomaly flow over Riemann surfaces*, `arXiv:1711.08186` to appear in Int. Math. Res. Notices.

[7] T. Fei , S.-T. Yau, *Invariant Solutions to the Strominger System on Complex Lie Groups and Their Quotients*, Comm. Math. Phys. **338** (2015) No. 3, 1183–1195.

[8] M. Fernández, S. Ivanov, L. Ugarte, R. Villacampa, *Non-Kähler heterotic-string compactifications with non-zero fluxes and constant dilaton*, Commun. Math. Phys. **288** (2009), 677–697.

[9] A. Fino, G. Grantcharov, L. Vezzoni, *Solutions to the Hull-Strominger system with torus symmetry*, , `arXiv:1901.10322`.

[10] J.X. Fu, L. S. Tseng, S.T. Yau, *Local heterotic torsional models*, Comm. Math. Phys. **289** (2009) 1151–1169.

[11] J.X. Fu, S.T. Yau, *A Monge-Ampère type equation motivated by string theory*, Comm. Anal. Geom. **15** (2007) 29–76.

[12] J.X. Fu, S.T. Yau, *The theory of superstring with flux on non-Kähler manifolds and the complex Monge-Ampère equation*, J. Differential Geom. **78** (2008), no. 3, 369–428.

[13] M. Garcia-Fernandez, *T-dual solutions of the Hull-Strominger system on non-Kähler threefolds*, J. Reine Angew. Math. **766** (2020), 137–150.

[14] C. Hull, *Superstring compactifications with torsion and space-time supersymmetry*, In Turin 1985 Proceedings "Superunification and Extra Dimensions" (1986), 347–375.

[15] J. Kollár, *Seifert G_m-bundles*, `arXiv: math/0404386`.

[16] J. Li, S.T. Yau, *The existence of supersymmetric string theory with torsion*, J. Differential Geom. **70** (2005) 143–181.

[17] M.L. Michelsohn, *On the existence of special metrics in complex geometry*, Acta Math. **143** (1983), pp. 261–295.

[18] A. Otal, L. Ugarte, R. Villacampa, *Invariant solutions to the Strominger system and the heterotic equations of motion*, Nuclear Phys. B **920** (2017), 442–474.

[19] D.H. Phong, S. Picard, X.W. Zhang, *Anomaly flows*, Comm. Anal. Geom. **26** (2018), no. 4, 955–1008.

[20] D.H. Phong, S. Picard, X.W. Zhang, *The Anomaly flow on unimodular Lie groups*, Advances in complex geometry, Contemp. Math., **735** 217–237, Amer. Math. Soc., Providence, RI, 2019.

[21] D.H. Phong, S. Picard, X.W. Zhang, *The Fu-Yau equation with negative slope parameter*, Invent. Math. **209** (2017), 541–576.

[22] D.H. Phong, S. Picard, X.W. Zhang, *Fu-Yau Hessian equations*, `arXiv:1801.09842`, to appear in J. Diff. Geom.

[23] D.H. Phong, S. Picard, X.W. Zhang, *New curvature flows in complex geometry*, Surveys in Differential Geometry **22** (2017), No. 1, 331–364.

[24] A.E. Strominger, *Superstrings with torsion*, Nuclear Phys. B **274**(2) (1986), 253–284.

On the cohomology of almost complex manifolds

Adriano Tomassini

(joint work with Richard K. Hind)

Cohomological properties provide a connection between analytical and topological features of complex manifolds. Given any complex manifold (M, J), natural complex cohomologies are defined, e.g., *Dolbeault, Bott-Chern* and *Aeppli* cohomologies, defined respectively as

$$H^{\bullet,\bullet}_{\overline{\partial}}(M) = \frac{\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\overline{\partial}}\,,\ H^{\bullet,\bullet}_{BC}(M) = \frac{\operatorname{Ker}\partial\cap\operatorname{Ker}\overline{\partial}}{\operatorname{Im}\partial\overline{\partial}}\,,\ H^{\bullet,\bullet}_{A}(M) = \frac{\operatorname{Ker}\partial\overline{\partial}}{\operatorname{Im}\partial+\operatorname{Im}\overline{\partial}}\,.$$

If (M, J) is compact and it admits a Kähler metric, then the complex de Rham groups are isomorphic to the direct sum of (p, q)-Dolbeault groups. For a $2n$-dimensional almost complex manifold (M, J), the exterior differential d acting on the space of complex valued (p, q)-forms $A^{p,q}(M)$

$$d : A^{p,q}(M) \to A^{p+2,q-1}(M) \oplus A^{p+1,q}(M) \oplus A^{p,q+1}(M) \oplus A^{p-1,q+2}(M)$$

splits as

$$d = \mu + \partial + \overline{\partial} + \bar{\mu},$$

where $\overline{\partial}$, respectively $\bar{\mu}$ are the $(p, p+1)$, respectively, the $(p-1, q+2)$ components of d. Then, the almost complex structure J is integrable if and only if $\bar{\mu} = 0$. Consequently, in the non integrable case $\overline{\partial}$ is not a cohomological operator. Let (M, J) be a $2n$-dimensional almost complex manifold. Then J acts as involution on the space of 2-forms $A^2(M)$ by

$$J\alpha(X, Y) = \alpha(JX, JY),$$

for every pair of vector fields X, Y on M. Then we denote as usual by $\Lambda_J^-(M)$ (respectively $\Lambda_J^+(M)$) the $+1$ (resp. -1)-eigenbundle; then the space of corresponding sections $A_J^-(M)$ (respectively $A_J^+(M)$) are defined to be the spaces of J-*anti-invariant*, (respectively J-*invariant*) forms, i.e.,

$$A_J^{\pm}(M) = \{\alpha \in A^2(M) \mid J\alpha = \pm\alpha\}$$

$$A^{(2,0),(0,2)}(M)_{\mathbb{R}} = A_J^-(M), \quad A^{1,1}(M)_{\mathbb{R}} = A_J^+(M)$$

Let

$$\mathcal{Z}_J^{\pm}(M) = \mathcal{Z}^2(M) \cap A_J^{\pm}(M) = \{\alpha \in A_J^{\pm}(M) \mid d\alpha = 0\}.$$

Then, according to the previous decomposition on forms, T.-J. Li and W. Zhang [9], motivated by the study of comparison of tamed and compatible sympletic cones on an almost complex manifold (M, J), introduced the J-*invariant* and J-*anti-invariant* cohomology groups defined respectively as defined as

$$H_J^{\pm}(X) = \left\{\mathfrak{a} \in H^2_{dR}(X; \mathbb{R}) \mid \exists\, \alpha \in \mathcal{Z}_J^{\pm} \mid \mathfrak{a} = [\alpha]\right\}$$

and they gave the following (see [9, Definition 4.12]). An almost complex structure J on M is said to be

- $\mathcal{C}^\infty$-*pure* if
$$H_J^+(M) \cap H_J^-(M) = \{0\}.$$
- $\mathcal{C}^\infty$-*full* if
$$H^2_{dR}(M; \mathbb{R}) = H_J^+(M) + H_J^-(M).$$
- $\mathcal{C}^\infty$-*pure-and-full* if
$$H^2_{dR}(M; \mathbb{R}) = H_J^+(M) \oplus H_J^-(M).$$

In [3] and [4], the same authors continue the study of the J-anti-invariant cohomology of an almost complex manifold (M, J). Let h_J^- be the dimension of the real vector space of closed anti-invariant 2-forms on (M, J). Note that in the case when the manifold is 4-dimensional every closed anti-invariant form α is Δ_{g_J}-harmonic, where g_J is a Hermitian metric and Δ_{g_J} denotes the Hodge Lapacian. Thus in the compact 4 dimensional case h_J^- is the dimension of the anti-invariant cohomology. The following appear in [3].

Conjecture 2.4. *For generic almost complex structures J on a compact 4-manifold M, $h_J^- = 0$.*

Conjecture 2.5. *On a compact 4-manifold, if $h_J^- \geq 3$, then J is integrable.*

For other results on $\mathcal{C}^\infty$-pure-and-full and J-anti-invariant closed forms see [1, 5, 7]

By starting with a (compact) Kähler surface with holomorphically trivial canonical bundle, Drăghici, Li and Zhang obtain non integrable almost complex structures with $h_J^- = 2$. More precisely, for a given (compact) Kähler surface (M, J) with holomorphically trivial canonical bundle, they take a closed 2-form trivializing the canonical bundle. Then, fixing a conformal class of Hermitian metrics compatible with J, they consider the Gauduchon metric representing such a conformal class and they associate an almost complex structure $J_{f,s,l}$ depending on three smooth functions satisfying some suitable conditions. Then, generically, $h^-_{J_{f,s,l}} = 0$, but cases when $h^-_{J_{f,s,l}} = 1$ and $h^-_{J_{f,s,l}} = 2$ also occur. Therefore, again in [3], as an extension of Conjecture 2.5, the authors asked the following natural

Question 3.23. *Are there (compact, 4-dimensional) examples of non-integrable almost complex structures J with $h_J^- \geq 2$ other than the ones arising from* [3], *Proposition 3.21? In particular, are there any examples with $h_J^- \geq 3$?*

First note that the space of closed anti-invariant forms with respect to the standard integrable complex structure i on $\mathbb{R}^4 \equiv \mathbb{C}^2$ is infinite dimensional: indeed, for every given holomorphic function $h(z_1, z_2)$, the real and imaginary parts of $h(z_1, z_2)dz_1 \wedge dz_2$ are closed and anti-invariant. We show the same can also hold in the non integrable case (see [6]).

Theorem. *There exists a (non integrable) almost complex structure on $\mathbb{R}^4$, such that the space of closed J-anti-invariant forms is infinite dimensional.*

As a consequence, we see that compactness is essential for Conjecture 2.5. In contrast we also show the following

Theorem. *There exists a family of almost complex structures $\{J_f\}$ on $\mathbb{C}^2$, parameterized by smooth functions $f : \mathbb{C}^2 \to \mathbb{R}$, with the following properties.*

- *J_f coincides with the standard complex structure i exactly at points where $f = 0$;*
- *J_f is integrable if and only if the gradient of f in the z_2 direction is 0;*
- *if f has compact support and $f \not\equiv 0$ then $h^-_{J_f} = 1$.*

In particular, an arbitrarily small, compactly supported, perturbation of a complex structure having an infinite dimensional space of anti-invariant forms may admit only a single such form up to scale. This provides supporting evidence for Conjecture 2.5, showing that typically anti-invariant forms do not persist under nonintegrable perturbations.

A similar argument gives the following,

Corollary. *There exist almost complex structures on $\mathbb{C}^2$ which agree with i outside of a compact set and have $h_J^- = 0$.*

We note that integrable complex structures on $\mathbb{C}^2$ which agree with i outside of a compact set are biholomorphic to $\mathbb{C}^2$ itself, and so have $h_J^- = \infty$. This follows from Yau, [11], Theorem 5, since such complex structures can be extended to give complex structures on $\mathbb{CP}^2$.

In the compact case, we construct a 2-parameter family of (non integrable) almost complex structures on the Kodaira-Thurston manifold, depending on two smooth functions, for which the anti-invariant cohomology group has maximum dimension equal to 2. This provides an affirmative answer to Question 3.23. In the last section, we give a simple construction to obtain 6-dimensional compact almost complex manifolds with arbitrary large anti-invariant cohomology. Hence dimension 4 is also an essential part of Conjecture 2.5. For almost-complex structures on a 4-manifold which are tamed by a symplectic form, Drăghici, Li and Zhang show in [2], Theorem 3.3, that $h^-_J \leq b^+ - 1$. Thus any counterexamples to Conjecture 2.5 cannot come from tamed almost-complex structures on symplectic 4-manifolds with $b^+ \leq 3$. Moreover T.-J. Li in [8], Theorem 1.1, shows that symplectic 4-manifolds of Kodaira dimension 0 all have $b^+ \leq 3$.

References

[1] D. Angella, A. Tomassini, W. Zhang, *On decomposablity of almost-Kähler structures*, Proc. Amer. Math. Soc. **142** (2014), 3615–3630.

[2] T. Drăghici, T.-J. Li, W. Zhang, *Symplectic form and cohomology decomposition of almost complex four-manifolds*, Int. Math. Res. Not. **2010** (2010), 1–17.

[3] T. Drăghici, T.-J. Li, W. Zhang, *On the J-anti-invariant cohomology of almost complex 4-manifolds*, Quarterly J. Math. **64** (2013), 83–111.

[4] T. Drăghici, T.-J. Li, W. Zhang, *Geometry of tamed almost complex structures on 4-dimensional manifolds*, Fifth International Congress of Chinese Mathematicians. Part1, 2, 233–251, AMS/IP Stud. Adv. Math., **51**, pt. 1, 2, Amer. Math. Soc., Providence, RI, 2012.

[5] A. Fino, A. Tomassini, *On some cohomolgical properties of almost complex manifolds*, J. Geom. Anal. **20** (2010), 107–131.

[6] R. Hind, A. Tomassini, *On the Anti-invariant Cohomology of Almost Complex Manifolds*, J. Geom. Anal. (2020), https://doi.org/10.1007/s12220-020-00461-8

[7] R. Hind, C. Medori, A. Tomassini, *On taming and compatible symplectic forms*, J. Geom. Anal. **25** (2015), 2360–2374.

[8] T.-J. Li, *Quaternionic bundles and Betti numbers of symplectic 4-manifolds with Kodaira dimension zero*, Int. Math. Res. Not. **2006** (2010), 1–28.

[9] T.-J. Li, W. Zhang, *Comparing tamed and compatible symplectic cones and cohomological properties of almost complex manifolds*, Comm. Anal. and Geom. **17** (2009), 651–684.

[10] C. H. Taubes, *Tamed to compatible: symplectic forms via moduli space integration*, J. Symplectic Geom. **9** (2011), 161–250.

[11] S. T. Yau, *Calabi's conjecture and some new results in algebraic geometry*, Proc. Nat. Acad. Sci. **74** (1977), 1798–1799.

Reporter: Michael Albanese

Participants

Dr. Michael Albanese
CIRGET
Université du Québec a Montréal
201 avenue du Président Kennedy
P.O. Box 8888
Montréal H3C 3P8
CANADA

Prof. Dr. Daniele Angella
Dipartimento di Matematica e
Informatica "Ulisse Dini"
Università degli Studi di Firenze
Viale Morgagni, 67/A
50134 Firenze
ITALY

Prof. Dr. Gil Cavalcanti
Mathematisch Instituut
Universiteit Utrecht
Budapestlaan 6
P. O. Box 80.010
3508 TA Utrecht
NETHERLANDS

Dr. Joana Cirici
Departament de
Matemàtiques/Informatica
Universitat de Barcelona
Gran Via 585
08007 Barcelona, Catalonia
SPAIN

Prof. Dr. Jean-Pierre Demailly
Institut Fourier
Laboratoire de Mathématiques
UMR 5582 du CNRS
100 rue des Maths
38610 Gières
FRANCE

Dr. Anna Maria Fino
Dipartimento di Matematica
Universita degli Studi di Torino
Via Carlo Alberto, 10
10123 Torino
ITALY

Thomas Holt
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Dr. Casey Kelleher
Department of Mathematics
Princeton University
Fine Hall
Washington Road
P.O. Box 08544
1000 Princeton
UNITED STATES

Dr. Aleksandar Milivojevic
Department of Mathematics
Stony Brook University
Math. Tower
Stony Brook, NY 11794-3651
UNITED STATES

Prof. Dr. Dan Popovici
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Dr. Jonas Stelzig
Mathematisches Institut
Ludwig-Maximilians-Universität
München
Theresienstraße 39
80333 München
GERMANY

Dr. Nicoletta Tardini
Dipartimento di Scienze Matematiche,
Fisiche e Informatiche, Plesso di
Matematica
Università degli Studi di Parma
Parco Area delle Scienze, 53/A
43124 Parma
ITALY

Dr. Adriano Tomassini
Dipartimento di Scienze Matematiche,
Fisiche e Informatiche, Plesso di
Matematica
Università degli Studi di Parma
Parco Area delle Scienze, 53/A
43124 Parma
ITALY

Prof. Dr. Scott Wilson
Department of Mathematics
Queens College
CUNY
65-30 Kissena Blvd.
Queens, NY 11367-1597
UNITED STATES

Dr. Weiyi Zhang
Mathematics Institute
University of Warwick
Zeeman Building
Coventry CV4 7AL
UNITED KINGDOM