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Arbeitsgemeinschaft: Zimmer's Conjecture

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Abstract. The aim of this Arbeitsgemeinschaft was to understand the recent progress on Zimmer's conjecture in [1, 2]. The week focuses on the cocompact case from [1].

Mathematics Subject Classification (2010): 22F05, 22E40, 37C85.

Introduction by the Organizers

The Arbeitsgemeinschaft on Zimmer's Conjecture was attended by 51 participants. One third of the participants were PhD students, one third were PostDoc and the last third were mathematicians with a permanent position. Many of the participants were working on a topic related to the conference. They came from various countries: Germany, France, England, Luxembourg, Russia, Poland, Chile, Switzerland, Mexico, United States, Israel, China, India, Korea,... It is our pleasure to thank the Oberwolfach Institute for providing us wonderful working and living conditions, to thank the speakers for the precision of their talks, and to thank the participants for making this week so lively.

We first recall the main theme of this Arbeitsgemeinschaft as explained in the scheduled program. A lattice Γ in a Lie group G is a discrete subgroup of finite covolume. A special class of Lie groups are the semisimple groups and a smaller special class are the simple groups of real rank at least 2. Lattices in semisimple groups and particularly lattices in simple groups of real rank at least 2 are known to be rigid in variety of ways. We will refer here to lattices in simple groups of rank at least 2 as *higher rank lattices.* The program concerned recent progress on conjectures of Zimmer that a higher rank lattice has only finite image homomorphisms

$\rho\colon \Gamma \to \mathrm{Diff}(M)$ where M is a compact manifold. The recent breakthrough both dramatically improves the state of knowledge and involves many novel ideas and contributions from various areas of mathematics. The main sources of techniques and ideas are:

(1) rigidity theory,
(2) smooth dynamics, particularly hyperbolic dynamics,
(3) homogeneous dynamics, particularly the study of invariant measures,
(4) operator algebras, particularly Lafforgue's strong property (T).

The lectures were organized to introduce participants to the elements of this wide variety of topics. The lectures were organized so as to give an essentially complete proof of the following

Theorem 1. *Let Γ be a lattice in $\mathrm{SL}(n, \mathbb{R})$ with $n > 2$, let M be a compact manifold and let $\rho\colon \Gamma \to \mathrm{Diff}(M)$ be a homomorphism. Then if $\dim(M) < n - 1$, the image of ρ is finite.*

The lectures followed carefully the scheduled program. In addition, we held three evening discussion sessions where participants could ask and answer basic questions, discuss examples, and work through computations and details.

Lectures 1 to 4 (by Shi Wang, Lifan Guan and Itamar Vigdovich) discussed background from Lie theory. In particular, lectures 1 and 2 provided an introduction to symmetric spaces, semisimple groups, and lattices. Lectures 3 and 4 provided an introduction to actions of these groups, suspension of Γ actions and some elementary properties of the suspension.

Lectures 5 to 9 (by Nguyen-Thi Dang, Vladimir Finkelstein, Cagri Sert, Minju Lee and René Ruhr) were an introduction to non-uniformly hyperbolic dynamics, actions of higher rank abelian groups and entropy theory. The concepts introduced here, particularly entropy and Lyapunov exponents, play a key role in both building invariant measures and in controlling growth of derivatives.

Lectures 10 to 13 (by Elyasheev Leitbag, Vincent Pecastaing, Thang Nguyen and Michele Triestino) introduced key superrigidity theorems of Margulis and Zimmer and sketched some ideas of the proofs of these classical results. These results both motivated the original conjecture of Zimmer and play a key role in its resolution.

Lectures 14 to 16 (by Pengyu Yang, Keivan Mallahi-Karai and Manuel Luethi) gave a brief introduction to key results from homogeneous dynamics used in the proof of Theorem 1. This includes results on classification of invariant measures and orbit closures by Ratner, finer results on equidistribution by Shah and Dani-Margulis, and results on epimorphic subgroups by Mozes. These all play a key role in improving invariant measures under averaging in the proof of Theorem 1

Lectures 17 to 20 (by Isabella Scott, Sang-hyun Kim, Federico Vigolo and Ping Ngai (Brian) Chung) gave a brief introduction to property (T) and strong property (T). This included some context and motivation, some proofs, and an indication of how strong property (T) is used in the proof of Theorem 1.

The final lecture (by Homin Lee) combined all the ingredients assembled throughout the program to give a detailed outline of the proof of Theorem 1.

References

[1] Aaron Brown, David Fisher, Sebastian Hurtado. Zimmer's conjecture: Subexponential growth, measure rigidity, and strong property (T). arXiv:1608.04995

[2] Aaron Brown, David Fisher, Sebastian Hurtado. Zimmer's conjecture for actions of $\mathrm{SL}(m, \mathbb{Z})$. arXiv:1710.02735

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Arbeitsgemeinschaft: Zimmer's Conjecture

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Abstracts

Symmetric spaces and lattices

Shi Wang

In this talk, we give definitions and basic examples of symmetric spaces and lattices. We take the references from

(1) Morris–Introduction to arithmetic groups [3],
(2) Eberlein–Geometry of nonpositively curved manifolds [1],
(3) Knapp–Lie group: Beyond an introduction [2],
(4) Steffen Kionke's notes on arithmetic groups [4].

1. Symmetric spaces

1.1. **Definition.** We begin with a geometric definition of symmetric space.

Definition 1. *A Riemannian manifold X is a symmetric space if*

(1) X is connected,
(2) X is homogeneous, that is, the isometry group $Isom(X)$ acts transitively on X,
(3) there is an isometric involution ϕ such that ϕ has at least one isolated fixed point.

Remark 1. *ϕ is called an involution if $\phi^2 = Id$, and p is called an isolated fixed point of ϕ if there exists an neighborhood U of p such that p is the only fixed point of ϕ in U.*

Remark 2. *If p is a fixed point of ϕ (in which case we write $\phi = \phi_p$), we can pick a neighborhood U of p in which p is the only fixed point. $\phi^2 = Id$ implies $(d\phi \circ d\phi)|_{T_pX} = Id$. Since ϕ is an isometry, we see that $d\phi|_{T_pX}$ has only ± 1 eigenvalues. If $d\phi$ had a $+1$ eigenvalue with a unit eigenvector v, then ϕ will send $\gamma_{p,v}$–the geodesic ray at p in the direction v–to itself because an isometry sends geodesics to geodesics. This contradicts with the fact that p is the only fixed point in U. Thus $d\phi|_{T_pX} = -Id$.*

Remark 3. *The above remark shows that geometrically ϕ_p is just the geodesic reversion at p. In particular, the involution that fixes p must be unique, one can also take this as the definition of a symmetric space (See* [1]*). On the other hand, one involution determines all involutions. If $g \in Isom(X)$ sends p to q, then $\phi_q = g \circ \phi_p \circ g^{-1}$ is the unique involution that fixes q.*

1.2. **Examples of symmetric spaces.** Model spaces are symmetric spaces.

- $\mathbb{R}^n$: The group of all translations on $\mathbb{R}^n$ is a subgroup of $Isom(\mathbb{R}^n)$ that acts transitively on $\mathbb{R}^n$, so $\mathbb{R}^n$ is a homogeneous space. Any point $p \in \mathbb{R}^n$, the involution ϕ_p is given by the point reflection at p (which might not be orientation preserving).

- $\mathbb{S}^n$: Take the standard model $\mathbb{S}^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + ... + x_n^2 = 1\}$. $SO(n+1)$ acts isometrically and transitively on $\mathbb{S}^n$, so it is a homogeneous space. Let $p = (1, 0, ..., 0) \in \mathbb{S}^n$, the involution at p is given by $\phi_p(x_0, x_1, ..., x_n) = (x_0, -x_1, ..., -x_n)$.
- $\mathbb{H}^2$: Take the upper half plane model $\mathbb{H}^2 = \{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ with the Riemannian metric $g = \frac{1}{y^2}(dx^2 + dy^2)$. It is also convenient to identify it with the complex coordinates, and we can write $\mathbb{H}^2 = \{z \in \mathbb{C} \mid Im(z) > 0\}$. We can check the action of $SL_2(\mathbb{R})$ on $\mathbb{H}^2$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} z = \frac{az+b}{cz+d}$$

is an isometric and transitive action. Hence $\mathbb{H}^2$ is homogeneous. Let $p = i \in \mathbb{H}^2$, the involution at p is given by $\phi_p(z) = -1/z$.

1.3. **Classification.** Symmetric spaces can be viewed as generalizations of model spaces, they can be constructed explicitly by Lie groups.

Universal construction:

- Let G be a connected Lie group, K be a compact subgroup of G, and $\sigma : G \to G$ is an involutive automorphism such that $K \subset G^\sigma$ is open, where G^σ is the set of fixed points of σ in G. Then G/K, with a G-left invariant metric, is a symmetric space with an involution $\phi(gK) = \sigma(g)K$, and eK is an isolated fixed point of ϕ.
- Conversely, any symmetric space can be constructed this way: If X is a symmetric space with an involution ϕ_p, then take $G = Isom^0(X)$–the connected component containing the identity–and $K = Stab_G(p)$, and $\sigma : G \to G$ given by $\sigma(g) = \phi \circ g \circ \phi$, we can realize X as G/K.

This gives the following explicit examples of symmetric spaces.

G	K	$\sigma : G \to G$ (called the Cartan involution)
$SO(n+1)$	$SO(n)$	$A \mapsto \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix} A \begin{bmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{bmatrix}$
$SL(n, \mathbb{R})$	$SO(n)$	$A \mapsto (A^T)^{-1}$
$SL(n, \mathbb{C})$	$SU(n)$	$A \mapsto (A^*)^{-1}$
$SO^0(p, q)$	$SO(p) \times SO(q)$	$A \mapsto (A^T)^{-1}$
$Sp(2n, \mathbb{R})$	$U(n)$	$A \mapsto (A^T)^{-1}$
$\cdots$	$\cdots$	$\cdots$

Symmetric spaces are classified by Cartan. We say a symmetric space X is *irreducible* if its universal cover does not split nontrivially as an isometric product of two symmetric spaces. There are 3 types of irreducible symmetric spaces: compact type, non-compact type, and Euclidean. The corresponding sectional curvatures satisfy $K \geq 0$, $K \leq 0$ and $K = 0$ respectively. In this series of talks, we mostly focus on non-compact type symmetric spaces, in which case, the above constructions can be simplified to the following.

Construction of irreducible, non-compact type symmetric spaces:

- Let G be a connected, simple, non-compact Lie group with finite center, and K be a maximal compact subgroup of G (unique up to conjugate). Then G/K is a simply connected, non-compact, non-flat, irreducible symmetric space.
- Conversely, any non-compact, non-flat, irreducible symmetric space can be constructed this way, and one can further take G to have trivial center.

Thus, the classification of irreducible, non-compact type symmetric spaces simply follows from the classification theorem of simple, non-compact, real Lie algebra. One can either use the restricted root system (Dynkin diagram with multiplicity), or complexified root system with certain decorations (Vogan/Satake diagrams), see [2].

1.4. **Rank.** There are two notions of rank: the rank of a symmetric space, and the real rank of a Lie group.

Definition 2. *Let X be a symmetric space, the rank of X, denoted by $rk(X)$, is defined to be the maximal integer r such that X has a totally geodesic, r-dimensional flat submanifold.*

Definition 3. *Let G be a connected real Lie group, the real rank of G, denoted by $rk_{\mathbb{R}}(G)$, is defined to be the maximal integer r such that X has a totally geodesic, r-dimensional, (topologically) closed, simply connected, flat submanifold, where X is an associated symmetric space of G.*

Remark 4. *By definition, $rk_{\mathbb{R}}(G) \leq rk(X)$ when $X = G/K$. However, if G has no compact factors, then $rk_{\mathbb{R}}(G) = rk(X)$. When G is compact, $rk_{\mathbb{R}}(G) = 0$, so compact factors does not contribute to the real rank. For this reason, the ambiguity of which X to take in the above definition does not matter.*

The following table shows examples of the two notions of rank.

G	K	$rk(G/K)$	$rk_{\mathbb{R}}(G)$
$SO(n+1)$	$SO(n)$	1	0
$SL(n,\mathbb{R})$	$SO(n)$	$n-1$	$n-1$
$SL(n,\mathbb{C})$	$SU(n)$	$n-1$	$n-1$
$SO^0(p,q)$	$SO(p) \times SO(q)$	$\min\{p,q\}$	$\min\{p,q\}$
$Sp(2n,\mathbb{R})$	$U(n)$	n	n
$\cdots$	$\cdots$	$\cdots$	$\cdots$

Rank one symmetric spaces: There are only 4 rank one symmetric spaces of non-compact type, namely, the real hyperbolic space $\mathbb{H}^n = SO^0(n,1)/SO(n)$, the complex hyperbolic space $\mathbb{CH}^n = SU(n,1)/S(U(n) \times U(1))$, the quaternionic hyperbolic space $H\mathbb{H}^n = Sp(n,1)/(Sp(n) \times Sp(1))$, and the Cayley plane $Ca\mathbb{H}^2 = F_4^{-20}/spin(9)$. The above F_4^{-20} indicates the unique Lie group whose complexified Dynkin diagram is of type F_4 and the quantity $\dim(X) - \dim(K)$ is -20.

2. Lattices

2.1. **Definition.** We state the definition of a lattice.

Definition 4. *A subgroup* Γ *in* G *is a lattice if*

(1) Γ *is discrete,*
(2) G/Γ *has finite volume with respect to the Haar measure on* G.

Definition 5. *A lattice is uniform (cocompact) if* G/Γ *is compact.*

Since G acts by left translation on the symmetric space $X = G/K$, Γ has a discrete left action on X. It follows that G/Γ has finite volume if and only if $\Gamma\backslash X$ has finite volume, and that G/Γ is compact if and only if $\Gamma\backslash X$ is compact. It is not clear just from the definition why (cocompact/non-cocompact) lattices exist, but actually there are quite a lot, most of which arise naturally from arithmetic constructions.

2.2. **Examples.**

- $SL(2,\mathbb{Z}) < SL(2,\mathbb{R})$ is a (non-compact) lattice. We give a brief argument of this fact. First, it is clear $SL(2,\mathbb{Z})$ is discrete. We just need to show $SL(2,\mathbb{R})/SL(2,\mathbb{Z})$, or equivalently $SL(2,\mathbb{Z})\backslash\mathbb{H}^2$ has finite volume. We find a fundamental domain $\mathfrak{F}$ of this action. (See the Figure 2.1 below)

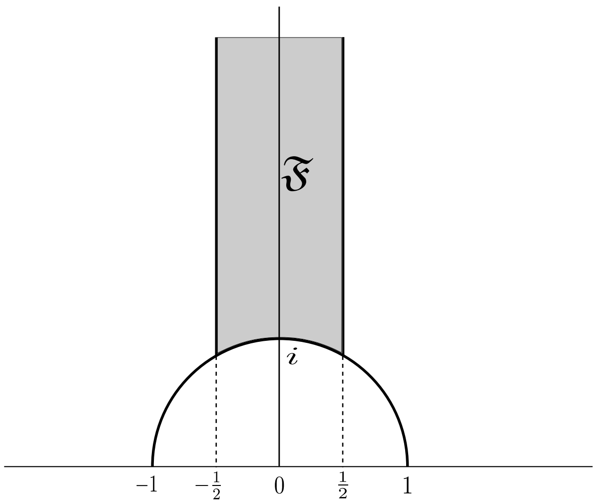

FIGURE 2.1. A fundamental domain $\mathfrak{F}$ for $SL(2,\mathbb{Z})$ action on $\mathbb{H}^2$

We see that $\mathfrak{F}$ is non-compact, and we can compute the area of $\mathfrak{F}$ as a double integral:

$$vol(\mathfrak{F}) = \int_{-1/2}^{1/2} \int_{\sqrt{1-x^2}}^{\infty} \frac{1}{y^2} dy dx = \frac{\pi}{3} < \infty$$

- $SL(n, \mathbb{Z}) < SL(n, \mathbb{R})$ is a lattice.
- $SL(n, \mathbb{Z}[i]) < SL(n, \mathbb{C})$ is a lattice.
- $SO(p, q; \mathbb{Z}) = SO(p, q) \cap SL(p + q, \mathbb{Z}) < SO(p, q)$ is a lattice.
- $Sp(2n, \mathbb{Z}) = Sp(2n, \mathbb{R}) \cap SL(2n, \mathbb{Z}) < Sp(2n, \mathbb{R})$ is a lattice.
- G be a classical semisimple real Lie group, the integer points $G_{\mathbb{Z}} < G$ is a lattice.
- If G is semisimple and defined over $\mathbb{Q}$, then the integer points $G_{\mathbb{Z}} < G$ is a lattice.

2.3. **Integer points via different embeddings.** This following example shows that different embeddings of $G_{\mathbb{Q}} \to G_{\mathbb{R}}$ may give rise to different kinds of lattices.

- $SO(2, 1; \mathbb{Z}) < SO(2, 1)$ is a non-cocompact lattice. This corresponds to the usual embedding $SO(2, 1; \mathbb{Q}) \subset SO(2, 1)$.
- Let $G = SO(7x_1^2 - x_2^2 - x_3^2; \mathbb{R}) \simeq SO(2, 1)$, it turns out that the integer points $G_{\mathbb{Z}} < G$ is a cocompact lattice. (This corresponds to the fact that $7x_1^2 - x_2^2 - x_3^2 = 0$ has no non-trivial integer solutions.)

References

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Structure of Lie groups

Lifan Guan

In this talk, we give a brief review of some basic structure theory of Lie groups, which will be used in the proof of Zimmer's conjecture [2]. The main references will be [1, Section 6.7] and [3, Section 8.2]

1. Algebraic Lie groups.

Let k be a field of characteristic 0. Let $\mathbf{G}$ be an algebraic group defiend over k, i.e., $\mathbf{G}$ is a scheme of finite type over k and there exist morphisms

$$\text{mult} : \mathbf{G} \times \mathbf{G} \to \mathbf{G}, \quad \text{inv} : \mathbf{G} \to \mathbf{G}, \quad \text{ and id} : \text{Spec} k \to \mathbf{G}$$

sastisfying the group laws. $\mathbf{G}$ is called a *linear algebraic group* if $\mathbf{G}$ is affine, or equivalently, $\mathbf{G}$ admits an embedding to $\mathbf{GL}_n$ for some n. A linear algebraic group is *semisimple* if the solvable radical is trivial.

A Lie group G is called *algebraic Lie group* (resp. *semisimple algebraic Lie group*) if there exists an algebraic group (resp. semisimple algebraic group) $\mathbf{G}$ defiend over $\mathbb{R}$ such that $G = \mathbf{G}(\mathbb{R})$. An algebraic Lie group is called *connected* if $\mathbf{G}$ is connected. Important examples for connected semisimple algebraic Lie group are $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SO}(p,q)(\mathbb{R})$, etc. Note that $\mathrm{SO}(p,q)(\mathbb{R})$ is a connected algebraic Lie group but is not connected as a Lie group.

Remark 1. *By definition, a Lie group is called semisimple if it Lie algebra is semisimple, so a semisimple algebraic Lie group is naturally a semisimple Lie group, but not vice visa. For example, the universal cover of* $\mathrm{SL}_2(\mathbb{R})$ *is a semisimple Lie group, but not a semisimple algebraic Lie group. Indeed, a semisimple Lie group is semisimple algebraic if and only if it admits an embedding into* $\mathrm{GL}_n(\mathbb{R})$*.*

2. Maximal compact subgroup

From now on, we always let G to be a connected semisimple algebraic Lie group. A subgroup $K \subset G$ is called a *maximal compact* if it is compact and maximal among all the compact subgroups. A maximal compact subgroup is unique up to conjugation. So let us fix a maximal compact subgroup K from now on. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ be the Lie algebra of K. The Killing form $\mathrm{Tr}(\mathrm{ad}X\mathrm{ad}Y)$, which is easily seen to be invariant under conjugation, is negative definite on $\mathfrak{k}$ and positive definite on its orthogonal complement $\mathfrak{s}$. Clearly, we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$, and moreover the map

$$K \times \mathfrak{s} \to G, \quad (k, X) \mapsto k \exp(X)$$

is a heomomorphism that defines the Cartan decomposition.

When $G = \mathrm{SL}_n(\mathbb{R})$, we can take K to be $\mathrm{SO}_n(\mathbb{R})$. Thus we have,

$$\mathfrak{k} = \{X \in \mathfrak{sl}_n : X + X^T = 0\}, \quad \mathfrak{s} = \{X \in \mathfrak{sl}_n : X = X^T\}.$$

3. Cartan subalgebra

Let notations be as above. A *Cartan subalgebra* $\mathfrak{a}$ of $\mathfrak{g}$ is defined to be a subalgebra that is conjugate (under G) to a maximal abelian subalgebra that is contained in $\mathfrak{s}$. Since any two maximal abelian subalgebras that are contained in $\mathfrak{s}$ are conjugate under K, a Cartan subalgebra is also unique up to conjugation and it dimension is called the *real rank* of G, denoted as $\mathrm{rank}_{\mathbb{R}} G$. Let us fix a Cartan subalgebra $\mathfrak{a} \subset \mathfrak{s}$ from now on. As any two maximal abelian subalgebra that are contained in $\mathfrak{s}$ are conjugate under K, combined with the Cartan decomposition, we get the following KAK decomposition,

$$G = K \exp(\mathfrak{a}) K.$$

When $G = \mathrm{SL}_n(\mathbb{R})$, A canonical choice of Cartan subalgebra is

$$\mathfrak{a} = \{\mathrm{diag}(a_1, \ldots, a_n) \in \mathfrak{sl}_n : a_i \in \mathbb{R}\}.$$

Then the KAK decomposition boils down to the well-known fact in linear algebra stating that any matrix in $\mathrm{SL}_n(\mathbb{R})$ can be written as products of a orthogonal matrix, a diagonal matrix with positive entries and another orthogonal matrix.

4. Restricted root system

An important property of the Cartan subalgebra is that under the adjoint action, elements from $\mathfrak{a}$ can be simultaneously diagonalized. Hence we can decompose $\mathfrak{g}$ as

$$\mathfrak{g} = \mathfrak{z} \oplus \oplus_{\lambda \in \Sigma} \mathfrak{g}^{\lambda},$$

where

$$\mathfrak{g}^{\lambda} = \{Y \in \mathfrak{g} : \mathrm{ad}(X)Y = \lambda(X)Y\} \quad \text{for all } X \in \mathfrak{a}, \text{ and } \mathfrak{z} = \mathfrak{g}^0$$

with

$$\Sigma = \{\lambda \in \mathfrak{a}^* \setminus \{0\} : \mathfrak{g}^{\lambda} \neq \{0\}\}.$$

It is clear that $\mathfrak{a} \subset \mathfrak{z}$, and G is called $\mathbb{R}$-*split* if $\mathfrak{a} = \mathfrak{z}$. Σ is a (restricted) root system.

An element $X \in \mathfrak{a}$ is called *regular* if $\lambda(X) \neq 0$ for all $\lambda \in \Sigma$. Let X be a regular element, set the corresponding set of positive roots to be

$$\Sigma^{+} = \{\lambda \in \Sigma : \lambda(X) > 0\},$$

and set the corresponding Weyl chamber to be

$$\mathfrak{a}^{+} = \{Y \in \mathfrak{a} : \alpha(Y) > 0 \text{ for all } \alpha \in \Sigma^{+}\}.$$

The corresponding *minimal parabolic subalgebra* $\mathfrak{p}$ is defined to be

$$\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{u} \quad \text{where} \quad \mathfrak{u} = \oplus_{\lambda \in \Sigma^{+}} \mathfrak{g}^{\lambda}.$$

We have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{u}$ and moreover the map

$$K \times \mathfrak{a} \times \mathfrak{u} \to G \quad (k, X, Y) \mapsto k \exp(X) \exp(Y)$$

is a heomomorphism that defines the Iwasawa decomposition.

When $G = \mathrm{SL}_n(\mathbb{R})$ with the Cartan algebra be given as above, the set Σ can be identified with $\{e_i - e_j : 1 \leq i, j \leq n, i \neq j\}$. the subset $\{e_i - e_i : 1 \leq j < i \leq n\}$ is a set positive roots and the corresponding mnimal parabolic subalgebra consists of upper trianglar matrices in $\mathfrak{sl}_n$.

5. KAK decomposition and geometric interpretation

Using Weyl chambers $\mathfrak{a}^{+}$, the KAK decomposition can be further refined as

$$G = K \exp(\mathfrak{a}^{+}) K.$$

The KAK decomposition admits an interpretation that involves geodesic flats on the locally symmetric space G/K, endowed with the metric induced from the Killing form restricted on $\mathfrak{s}$. By definition, a *maximal geodesic flat* is a submanifold of G/K which is totally geodesic with zero curvature and is maximal with respect to these conditions. The maximal geodesic flat can be identified as $g \exp \mathfrak{a} e$ where $g \in G$ and $e = [K] \in G/K$ be the base point. The subsets $g \exp \mathfrak{a}^{+} e$ are called *chambers*. Then it follows from the KAK decomposition that any two points in the locally symmetric space lie in a same chamber.

6. Higher rank groups

Say G is of *higher rank* if $\mathrm{rank}_{\mathbb{R}} G \geq 2$.

Proposition 1. *G is of higher rank if and only if there exists unipotent subgroups $U_1, \dots, U_k$ such that*

(1) *$U_1, \dots, U_k$ generates G, and*
(2) *for any $1 \leq i \leq k-1$, U_i commutes with U_{i+1}.*

It is easy to check that (1) and (2) implies higher rank. Conversely, the proof can be done case by case. We sketch the case of $G = \mathrm{SL}(3, \mathbb{R})$ and left the others as exercises. Indeed, for $1 \leq i, j \leq 3$, let

$$U_{ij} = \{I + tE_{ij} : t \in \mathbb{R}\}.$$

It is clear that $\{U_{ij} : 1 \leq i, j \leq 3\}$ generates G and U_{ij} commutes with $U_{i'j'}$ if $j \neq i'$ and $i \neq j'$. Hence, the sequence of unipotent groups

$$U_{12}, U_{13}, U_{23}, U_{21}, U_{31}, U_{32}$$

verify the conditions (1) and (2).

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Examples of actions of lattices on compact manifolds

Itamar Vigdorovich

Let G be a simple Lie group, and let Γ be a lattice in G. Unless stated otherwise assume that $G = SL_n(\mathbb{R})$, whereas for Γ we will often take Γ to be $SL_n(\mathbb{Z})$ while at other times we will need Γ to be a uniform lattice. We are interested in actions of Γ by diffeomorphisms on compact connected smooth manifolds M, i.e homomorphisms $\alpha : \Gamma \to \mathrm{Diff}(M)$. The goal in this talk is to become familiar with some of the classical actions, as well as more exotic ones, and to appreciate how wild it can be to get a hand on classifying all such actions.

1. Trivial actions

Choose M arbitrarily. The trivial action is given by $\alpha(\gamma) = Id_M$ for all γ. More generally, actions for which $Im(\alpha)$ is finite are called *finite actions.* As the name suggests, we are looking for more interesting examples. There is however a remarkably strong constraint on such actions:

Theorem 1 (Zimmer's Conjecture, A. Brown, D. Fisher, S. Hurtado 16'). *Assume $n \geq 3$. Then any action $\Gamma \to$ Diff(M) for which* $\dim M < n-1$*, is finite.*

2. Algebraic actions

2.1. **Action on projective space.** The following example shows that the dimension bound in the theorem above is tight. G acts on $\mathbb{R}^n$ linearly, and it transfers lines to lines. Hence G, and thus also Γ, act on $\mathbb{R}P^{n-1}$. Let $e_1, ..., e_n$ be the standard basis in $\mathbb{R}^n$. The stabilizer of the line $\mathbb{R}e_1$ is the subgroups of G consisting of matrices (in G) of the form $P_1 = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$. Hence we have that $M = \mathbb{R}P^{n-1} \cong G/P_1$.

2.2. **Actions on Grassmannians and flag manifolds.** The above example can be generalized by introducing the following definition which is central in the theory of algebraic groups.

Definition 1. *A closed algebraic subgroup* $\boldsymbol{P}$ *of an algebraic group* $\boldsymbol{G}$ *is called* parabolic *if* $\boldsymbol{G}/\boldsymbol{P}$ *is a complete (equiv. projective) variety.*

P_1 is obviously an example of a parabolic subgroup of G, and it can easily be seen that it is a maximal one (among proper subgroups). More generally, we may choose a different maximal parabolic subgroup $P_k = \begin{pmatrix} *_{k\times k} & *_{(n-k)\times k} \\ 0 & *_{(n-k)\times(n-k)} \end{pmatrix} \leq G$. Observe that P_k is precisely the stabilizer of the subspace $Sp\{e_1, ..., e_k\}$ understood as a point in the Grassmannian $Gr(n, k)$ and so $Gr(n, k) \cong G/P_j$. Even more generally, we may choose

$$P_{k_1,...,k_r} = \begin{pmatrix} (*)_{k_1\times k_1} & * & * & * \\ 0 & (*)_{k_2\times k_2} & * & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & (*)_{k_r\times k_r} \end{pmatrix}$$

which is the stabilizer of the tuple

$$(Sp\{e_1, ..., e_{k_1}\}, Sp\{e_1, ..., e_{k_1+k_2}\}, ..., Sp\{e_1..., e_n\})$$

as a point in the flag variety

$$F_{k_1,....,k_r} := \{F_1 \subseteq ... \subseteq F_r = \mathbb{R}^n \mid \dim F_i \backslash F_{i-1} = k_i\}$$

A special case which stands out is obtained by setting all k_i to be 1, in which case $F_{k_1,....,k_r}$ is called the *full flag* variety. Note that $P_{k_1,...,k_n}$is a minimal parabolic subgroup.

Fact 1. *Up to conjugation, all parabolic subgroups of* G *are of the form* $P_{k_1,...,k_r}$ *(with* $\sum k_i = n$*)*

3. Affine actions

3.1. **Linear actions.** G acts on $\mathbb{R}^n$ linearly. The subset $\mathbb{Z}^n \subseteq \mathbb{R}^n$ is stable under the action of $\Gamma = SL_n(\mathbb{Z})$. This gives a well-defined action of Γ on the torus $\mathbb{T}^n := \mathbb{R}^n/\mathbb{Z}^n$. Observe that as opposed to the previous examples, this action is not the restriction of any G action, and moreover it is volume preserving. In fact, by including the requirement that Γ must preserve a volume form, the bound in Theorem 1 can be strengthened from $n-1$ to n.

3.2. **Actions by automorphisms.** In the previous example we had that $G \leq GL_n(\mathbb{R}) = Aut(\mathbb{R}^n)$ and that $\Gamma.\mathbb{Z}^n = \mathbb{Z}^n$. More generally, whenever we have a Lie group H, a uniform lattice $\Lambda \leq H$ and a representation $\rho : G \to Aut(H)$ for which $\rho(\Gamma).\Lambda = \Lambda$, we obtain an action of Γ on the compact manifold H/Λ.

3.3. **Homogeneous actions.** Given a Lie group H, a uniform lattice $\Lambda \leq H$, and a map $\Gamma \to H$ we get an action of Γ on H/Λ.

3.4. **Affine actions.** We may combine the previous two example to obtain the following:

Definition 2. *An action α of Γ of H/Λ is called* affine *if for any $\gamma \in \Gamma$ there exists $h \in H$ and $\theta \in Aut(H)$ with $A(\Lambda) = \Lambda$ such that $\alpha(\gamma) = A \circ \tau_h$ there τ is left regular action on H.*

These example can even be further generalized by introducing *generalized affine actions* and *quasi-affine actions*. See [5] for more details.

4. Isometric actions

Can M admit a smooth Riemannian metric for which Γ acts (non-trivially) by isometries on M? If $\Gamma = SL_n(\mathbb{Z})$, then a the answer is no by Margulis super-rigidity- indeed $Isom(M)$ is compact Lie group and any homomorphism $SL_n(\mathbb{Z})$ into a compact Lie group must be finite. However, if Γ is cocompact, such action may exists. For the sake of simplicity, we will assume in this example that $G = SO(p, q)$ (similar but slightly more complicated mehtods work also for $SL_n(\mathbb{R})$)

4.1. **Restriction of scalars.** Recall that $SO(p, q)$ is the group of all matrices in $SL_n(\mathbb{R})$ which preserve the bilinear form $x_1^2 + ... + x_p^2 - x_{p+1}^2 - ... - x_n^2$. Instead, let's consider the bilinear form $f(x_1, ..., x_n) = \sum_{i=1}^p x_i^2 - \sqrt{2} \cdot \sum_{i=p+1}^{p+q} x_i^2$ and let $SO(f)$ denote the group matrices in $SL_n(\mathbb{R})$ preserving this form. Let $\sigma : \mathbb{Q}(\sqrt{2}) \to \mathbb{Q}(\sqrt{2})$ the non-trivial Galois automorphism, and let f^σ be the bilinear form obtained by applying σ on the coefficients of f, i.e $f^\sigma(x_1, ..., x_n) = \sum_{i=1}^p x_i^2 + \sqrt{2} \cdot \sum_{i=p+1}^{p+q} x_i^2$. Let $SO(f)_{\mathbb{Z}[\sqrt{2}]}$ denote the matrices in $SO(f)$ with coefficients in $\mathbb{Z}[\sqrt{2}]$. Observe that if we apply σ to an element in $SO(f)_{\mathbb{Z}[\sqrt{2}]}$ we obtain a matrix which is not necessarily in $SO(f)$ but in fact in $SO(f^\sigma)$. We thus have that injective map

$$Id \times \sigma : SO(f)_{\mathbb{Z}[\sqrt{2}]} \to SO(f) \times SO(f^\sigma)$$

and it is not hard (but not trivial) to see that the image of this map is an irreducible lattice in its range. We thus have the following diagram:

$$\begin{array}{ccccccc}
SO(f)_{\mathbb{Z}[\sqrt{2}]} & \xrightarrow{Id\times\sigma} & SO(f)\times SO(f^{\sigma}) & = = = & SO(p,q)\times SO(p+q) & \longrightarrow & SO(p+q) \curvearrowright S^{p+q-1} \\
\Vert & & \downarrow & & \downarrow & & \\
\Gamma & \longrightarrow & SO(f) & = = = = = = = & SO(p,q) & &
\end{array}$$

The bottom row shows that Γ embeds as a lattice in $SO(p,q)$ whereas the top row shows how Γ acts on a sphere. Using Godement Compactness Criterion it can be shown that Γ embeds as a cocompact lattice. See [2, Ch. 5,6] for more details and for a similar construction for $G = SL_n(\mathbb{R})$.

5. Induction (suspension)

Suppose that $H \leq G$ is any closed subgroup. Clearly, we can restrict any G action to an H action. Induction is in some sense a counter construction. If H acts on M, then we can extend it to an action of H on $G \times M$ where H acts on G be multiplication on the right $h.(g,m) = \left(gh^{-1}, h.m\right)$. The action of G on $G \times M$ by multiplication on the left coordinate clearly commutes with the H action above. We thus obtained a well-defined action of G on the space $\tilde{M} := Ind_H^G(M) := (G \times M)/H$. We say that the action $G \curvearrowright \tilde{M}$ is *induced* from the $H \curvearrowright M$, and we also refer to $\tilde{M}$ as *suspension space*, especially in the context of dynamics.. This construction comes equipped with the map $p : \tilde{M} \to M$ - the projection on the first coordinate. This makes $\tilde{M}$ a fiber-bundle over G/H with fibers diffeomorphic to M. Note that $\tilde{M}$ is compact if H is cocompact.

5.1. **Tautological bundles.** In order to become more comfortable with this construction, let us consider a very concrete example. Recall that we have an action of $\Gamma = SL_n(\mathbb{Z})$ of $\mathbb{T}^n$. Thus by the above we get an action of G of $\tilde{\mathbb{T}}^n = (G \times \mathbb{T}^n)/\Gamma$, a space which we interpret as following. G/Γ is identified with the collection of unimodular lattices in $\mathbb{R}^n$ indeed, G acts on the collection of unimodular lattices transitively, and the stabilizer of the standard lattice $\mathbb{Z}^n$ is just Γ. Putting it in fancier terms, G/Γ is the moduli space of all lattices in $\mathbb{R}^n$, or alternatively, it is the moduli space of all flat metrics of a torus $\mathbb{T}^n$. Thus $\tilde{\mathbb{T}}^n$ it is a fiber bundle over this moduli space for which the fiber of each point is (quite tautologically) the object (namely the torus with the respective flat metric) which that point represents.

5.2. **Actions can get quite wild.** Let H be the stabilizer of the action of G on $\mathbb{R}P^{n-1}$. We have that map $\varphi : H \to \mathbb{R}$ given by $\varphi : h \mapsto \log|h_{11}|$ arising as the composition of the following surjective maps:

$$H \overset{\epsilon_{11}}{\to} \mathbb{R}^* \overset{|\cdot|}{\to} \mathbb{R}^*_{\geq 0} \overset{\log}{\to} \mathbb{R}$$

where ϵ_{11} returns the top-left entry of the given matrix. Thus using induction we see that:

$$\mathbb{R}\text{-Actions} \overset{\varphi^*}{\to} H\text{-Actions} \overset{Ind_H^G}{\to} G\text{-Actions} \overset{Res_\Gamma^G}{\to} \Gamma\text{-Actions}$$

Classifying $\mathbb{R}$-actions is the same as classifying vector fields on smooth manifolds which is absolutely hopeless. This puts a damper on the hope to classify G or Γ actions.

5.3. **Even wilder.** Let $\Lambda \leq PSL_2(\mathbb{R})$ a uniform lattice which surjects onto a non-abelian free group F. As F subjects onto $\mathbb{Z}$, classifying all Λ-actions must include (for the least) a classification of $\mathbb{Z}$-actions which is in turn boils down to a classification of diffeomorphisms on compact manifolds which is ludicrous. Using induction, we may show that this problem arises also when $n > 2$. Indeed, consider the map θ from $P = P_2 = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$ onto $PSL_2(\mathbb{R})$ which keeps the left 2×2 corner. We thus have:

$$F\text{-Actions} \overset{\varphi^*}{\to} \Lambda\text{-Actions} \overset{Ind_\Lambda^{PSL_2(\mathbb{R})}}{\to} PSL_2(\mathbb{R})\text{-Actions} \overset{\theta^*}{\to}$$

$$\overset{\theta^*}{\to} P\text{-Actions} \overset{Ind_P^G}{\to} G\text{-Actions} \overset{Res_\Gamma^G}{\to} \Gamma\text{-Actions}$$

6. Blow-ups

All of the example we have presented here include some additional structure (algebraic, homogeneous, Riemannian,). Do all actions admit a *rigid geometric structure*? We present the construction by Katok, Lewis [3], which later Benveniste and Fisher [4] used to answer this question negatively.

Recall that the blow-up of $\mathbb{R}^n$ at the origin $\bar{0}$, is by definition the variety:

$$B = \left\{(x, l) \in \mathbb{R}^n \times \mathbb{R}P^{n-1} \mid x \in l\right\} \subseteq \mathbb{R}^n \times \mathbb{R}P^{n-1}$$

where $\mathbb{R}P^{n-1}$ is naturally identified with the set of lines passing through $\bar{0}$. This construction comes equipped with the projection to the first coordinate $B \twoheadrightarrow \mathbb{R}^n$. Since any point $x \in \mathbb{R}^n \backslash \{0\}$ defines a unique line $l \in \mathbb{R}P^{n-1}$ we see that the fiber of every point in $\mathbb{R}^n$ with respect to this map is just a single point. at the origin however, the fiber is identified with $\mathbb{R}P^{n-1}$ which is called in this context the *exceptional divisor*. The way to think of this construction is that B is the same as the original space $\mathbb{R}^n$ except that we now add the information of all directions from which one can approach the origin in such a way that we do no longer have one origin, but many - one for each approachable direction. It is thus no wonder that this construction can be used to resolve singularities of algebraic varieties, or singularities of group actions.

6.1. **A hybrid action.** Recall from above the action of $\Gamma = SL_n(\mathbb{Z})$ on $\mathbb{T}^n$. This action has one fixed point, however it has an index-2 subgroup which acts on $\mathbb{T}^n$ with multiple fixed points. Indeed, if Γ_2 denotes the 2'nd principal congruence subgroup, namely the kernel of the mod-2 map $SL_n(\mathbb{Z}) \to SL_n(\mathbb{Z}/2\mathbb{Z})$, then the points in the torus $(\frac{1}{2}, 0, ..., 0)$ and $(0, \frac{1}{2}, ..., 0)$ are fixed by Γ_2. The tangent spaces at those points are identified with $\mathbb{R}^n$, and so we can blow-up each those two tangent spaces at their respective origins and glue them together. Γ_2 now acts on this glued exceptional divisors by the projective action which we've discussed in Section 2. We have thus obtained a new manifold with a hybrid action of Γ_2 which is at some part is volume preserving while at an other part it is not.

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Suspension space

David Fisher

This lecture was given by David Fisher on behalf of a participant who could not come to Oberwolfach.

In this talk, we will see construction of suspension space and their basic properties. Here we will always use $\rho : \Gamma \to \mathrm{Diff}(M)$ be a Γ action on compact manifold M. Here Γ be a lattice in G. (For the proof of main theorem, one may assume that $G = \mathrm{SL}(n, \mathbb{R})$.) The details can be found in [1]

1. Suspension space.

1.1. **Construction.** We will define the suspension space. We can define left G action on $G \times M$ as left multiplication only on G.

$$g.(h, x) = (gh, x).$$

On the other hand, we can define right Γ action on $G \times M$ as twisted way.

$$(h, x).\gamma = (h\gamma, \rho(\gamma^{-1})(x)).$$

Definition 1. *We can define*

$$M^\rho = (G \times M)/\Gamma$$

using right Γ *action. This is called* suspension space*. In addition, left* G *action commutes with right* Γ *action on* $G \times M$ *so that we can define left* G *action on* $(G \times M)/\Gamma$*. Let* $\tilde{\rho}$ *be a induced* G*-action on* M^ρ*. We also have natural projection* $\pi : M^\rho \to G/\Gamma$*.*

Remark 1. *The natural projection* $\pi : M^\rho \to G/\Gamma$ *is* G*-equivariant, that is*

$$\pi(\tilde{\rho}(g)(x)) = g.\pi(x)$$

for all $g \in G$ *and* $x \in M^\rho$*. This will be used frequently in order to lift properties on* G/Γ *to* M^ρ*, vice versa.*

We can think M^ρ as fiber bundle over G/Γ with fiber M. In here G action is twisted in some sense. Roughly, we can think M^ρ as $G/\Gamma \times M$. Fix fundamental domain $X \subset G$ for G/Γ in G and identify X with G/Γ. Then G action on $X \times M$ as

$$g.([h], x) = (g.[h], \alpha(g, [h])(x))$$

where

$$\alpha : G \times G/\Gamma \to \Gamma$$
$$(g, [h]) \mapsto \gamma \iff gh\gamma^{-1} \in X.$$

and $g.[h] = [gh\alpha(g, [h])^{-1}]$. Note that $gh\alpha(g, [h])^{-1} \in X$.

We really need to investigate G action "fiberwise" in order to see original Γ action ρ. So it is natural to define following definition.

Definition 2. *Let* $F = \ker(D\pi)$ *where* $\pi : M^\rho \to G/\Gamma$*. We will call* F *is* fiberwise tangent bundle*. For* $x \in M^\rho$*, we can also define* fiberwise tangent space $F(x) \subset T_x M^\rho$*.*

Remark 2. *Note that* F *is* G*-invariant subbundle of* TM^ρ*.*

1.2. **Motivation.** The main motivation of the suspension space is following. First of all, We have plenty of informations about structure of G. We can use this advantage. For example, the growth in G is determined by A and $A < G$ can be identified with $\mathbb{R}^{n-1}$ so that we can use Multiplicative ergodic theorem on suspension space. One more advantage is that A is amenable, so that we can always find invariant measure. This will give us information about exponential growth.

In the hight rank abelian dynamics setting, similar things happen. The kernel of linear functional defined on $\mathbb{Z}^k$ may not see integer points. However, if we induce action so that the linear functional is defined over $\mathbb{R}^k$ then we have every informations about kernel of it.

2. Fiberwise Lyapunov exponent.

One of key ingredients of the main theorem is that using suspension space and study *fiberwise Lyapunov exponents.* Let me recall the Oseledet's multiplicative ergodic theorem (higher rank version) for vector bundle. Recall that we have KAK decomposition and A can be identified with $\mathbb{R}^{n-1}$.

Theorem 1. *For any A-invariant ergodic probability measure on M^ρ, we have*

1. *There is A-invariant measurable set $\Lambda \subset M^\rho$ such that $\mu(\Lambda) = 1$.*
2. *There are linear functionals $\lambda^F_{1,\mu}, \dots, \lambda^F_{p,\mu} : A \to \mathbb{R}$.*
3. *There is A-invariant decomposition of F as*

$$F(x) = E_1(x) \oplus \cdots \oplus E_p(x)$$

defined over $x \in \Lambda$ such that

$$\forall a \in A, \forall v \in E_j(x) \setminus \{0\}, \lim_{|a| \to \infty} \frac{\log ||D_x a(v)|| - \lambda^F_{j,\mu}(a)}{|a|} = 0$$

Here we fix norm on $\mathbb{R}^{n-1}$ so that on A. The last assertion especially shows that for fixed $a \in A$, for any $v \in E_j(x) \setminus \{0\}$ we have

$$\lim_{m \to \infty} \frac{1}{m} \log ||D_x a^m(v)|| = \lambda^F_{j,\mu}(a).$$

The above theorem give us a tool to produce A-invariant probability measure M^ρ that see the failure of the uniform subexponential growth of derivative fiberwise when we assume that $\tilde{\rho}$ fails to have the uniform subexponential growth of derivative fiberwise.

3. Properties and applications of Suspension space

3.1. Invariant measures.

For suspension space, we have following two lemmas.

Lemma 1. *The Γ action ρ on M preserves Borel probability measure if and only if the induced G action $\tilde{\rho}$ on M^ρ preserves Borel probability measure.*

Indeed, if $\tilde{\mu}$ is G via $\tilde{\rho}$ invariant probability measure on M^ρ then, we can disintegrate $\tilde{\mu}$ so that we can find Γ-invariant measure μ on M. On the other hand, if μ is a Γ via ρ invariant probability measure on M then the probability measure $\mu \times \text{Haar}$ on M^ρ is G- invariant.

Let $G = \mathrm{SL}(n, \mathbb{R})$ and Γ be a lattice in G. The above lemma can be used to find ρ-invariant probability measure on M provided that $\dim M < n - 1$. Later, we will see following theorem coming from invariance principle.

Theorem 2. *Let $\rho : \Gamma \to \mathrm{Diff}(M)$ be a Γ action on a compact manifold M. Assume $d = \dim M < n - 1$. Then there is a Γ-invariant probability measure on M.*

Note that Γ is not an amenable group, there may not invariant probability measure a priori. However, the above theorem shows that, under some dimension condition, we can find invariant probability measure. The proof of above theorem is that making suspension space and find G-invariant probability measure on M^ρ using invariance principle. Here we can see the advantage of suspension space. Although Γ is not amenable, $A < G$ is amenable subgroup so that we can find easily A-invariant probability measure on M^ρ. After then, invariance principle helps us to find additional invariance.

3.2. **Fiberwise uniform subexponential growth of derivative.** In this section, we will see one lemma that is used in the proof of the main theorem. That is the notion of the uniform subexponential growth of derivative also can be induced. First, recall the definition of the uniform subexponential growth of derivative.

Definition 3. *Let $\rho : \Gamma \to \mathrm{Diff}(M)$ be a Γ action on compact manifold M. We say ρ has uniform subexponential growth of derivative if for any $\epsilon > 0$ there is $C = C_\epsilon > 0$ such that*

$$\sup_{x \in M} ||D_x \gamma|| < Ce^{\epsilon l(\gamma)}.$$

Here we fix Riemannian metric on M and $l(\gamma)$ denotes the word length of γ in Γ for fixed finite generating set.

Note that the above definition does not depend on choice of Riemannian metric on M and of finite generating set of Γ.

Now we can introduce main lemma in this section.

Lemma 2. *The ρ has uniform subexponential growth of derivative if and only if the induced action of G on M^ρ has uniform subexponential growth of derivative fiberwise., i.e. for any $\epsilon > 0$ there is $C_\epsilon = C > 0$ such that*

$$\sup_{x \in M^\rho} ||D_x g|_F|| \leq Ce^{\epsilon d(e,g)}$$

for any $g \in G$.

The above lemma tells us that we can detect uniform subexponential growth of derivative in the induced action. That means, in order to prove uniform subexponential growth of derivative, we can use induced action, especially we can use structure theorems of G.

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The top Lyapunov exponent

Nguyen-Thi Dang

In this talk, M is a compact smooth manifold of finite dimension m and V a finite dimensionnal real vector space and G is a topological group or semigroup acting on M (by diffeomorphism).

In the first paragraph, we give the general setting and give examples of cocycles. In the second paragraph, we define the first Lyapunov exponent.

In the third paragraph, we give a proof that the $\mathbb{Z}$-cocycle induced by c and f has *uniform subexponential growth* if and only if for all (f-invariant) probability measure $\mu \in \mathscr{P}_f(M)$, both first Lyapunov exponents $\lambda_{top}(c, f, \mu)$ and $\lambda_{top}(c, f^{-1}, \mu)$

cancel. Most importantly, when the cocycle does not grow uniformly in a subexponential way, we detail the construction of an f-invariant measure for which the Lyapunov exponents is positive.

Finally, we state some regularity conditions for the Lyapunov exponent with respect to the average of measures along Følner sequences of an amenable group.

1. Setting

Definition 1. *The measurable map $c : G \times M \to \mathrm{GL}(V)$ is a* cocycle *if it satisfies the* cocycle relation*: for all $f_1, f_2 \in G$ and $x \in M$*

$$c(f_1 f_2, x) = c(f_1, f_2(x))\, c(f_2, x).$$

It is a continuous cocycle *when c is continuous. Let μ be a probability measure on M. The cocycle is $\mu-$*integrable *if for all $f \in G$,*

$$\log^+ \|c(f, .)\| \in L^1(M, \mu)$$

where $\log^+ := \sup(0, \log)$.

Remark that because M is compact, continuous cocycles are integrable for every probability measure.

Definition 2. *Let f be a diffeomorphism. A probability measure μ is f-*invariant *if $f_*\mu = \mu$ i.e. $\mu(f^{-1}(A)) = \mu(A)$ for all Borel subset $A \subset M$. It is* ergodic *if any $f-$invariant Borel subset is of measure* 0 *or* 1*. Denote by $\mathscr{P}_f(M)$ the space of f-invariant probability measures in M.*

By a theorem of Krylov-Bogolyubov, the space $\mathscr{P}_f(M)$ is not empty. Indeed, any weak-$*$ limit of sequence of probability measures

$$\frac{1}{n}\sum_{k=0}^{n-1} f_*^k \mu$$

where $\mu \in \mathscr{P}(M)$ is a f-invariant probability measure on M, by compacity of $\mathscr{P}(M)$.

1.1. Examples of cocycles.

In my talk I only covered examples ♠ and ♡. Example ♢ will play a crucial role in the proof of Theorem 2.8 in [BFH16].

♠ Example (cf. chapter 3 [Via14]): M is a compact smooth manifold. Fix $f \in \mathrm{Diff}^1(M)$ and consider a continuous function $A : M \to \mathrm{GL}(V)$. Set $G := (f^n)_{n\in\mathbb{Z}}$ and for all $x \in M$,

$$c(f, x) := A(x).$$

The continuous cocycle $c : G \times M \to \mathrm{GL}(V)$ is then defined by induction using the cocycle relation, i.e. for all $n \geq 1$,

$$c(f^n, x) = A(f^{n-1}(x))\, c(f^{n-1}, x).$$

♡ Example (cf. paragraph 4.1 [Bro18]): $G = \mathrm{Diff}(M)$, consider the derivative $c(f,x) := D_x f$. It is continuous and in trivializations, $D_x f$ takes value in $\mathrm{GL}(\mathbb{R}^m)$. Furthermore, it satisfies

$$D_x(f_1 \circ f_2) = D_{f_2(x)} f_1 \;\; D_x f_2.$$

♢ Example (suspension space, paragraph 10.1 [Bro18]): Let Γ be a cocompact lattice of $G := \mathrm{SL}(n, \mathbb{R})$ and consider $\alpha : \Gamma \to \mathrm{Diff}^2(M)$ an action. We define a right action of Γ on $G \times M$ by setting for all $(h,x) \in \Gamma \times M$ and $\gamma \in \Gamma$,

$$(h,x).\gamma = (h\gamma^{-1}, \alpha(\gamma)x).$$

The left action of G is defined as follows, for all $(h,x) \in G \times M$ and $f \in G$,

$$f(h,x) = (fh, x).$$

Then the suspension space is the quotient space $M^\alpha := \big(G \times M\big)/\Gamma$. It projects onto the finite volume homogeneous space G/Γ. Denote by $\pi : M^\alpha \to G/\Gamma$ the projection and by F the fibers of this map. For any $f \in G$, denote by $\tilde{L}_f$ (resp. L_f) the left multiplication of f on M^α (resp. G/Γ). In (local) coordinates we write $\tilde{L}_f = (L_f, \tilde{L}_f \restriction_F)$, abusing notations for the first coordinate. By derivating the actions by multiplications, we deduce the following commutative diagram

$$D\pi : D\tilde{L}_f \curvearrowright TM^\alpha \xrightarrow{TF} DL_f \curvearrowright T(G/\Gamma).$$

In suitable trivializations for all $(x,h) \in M^\alpha$, we read

$$D_{(h,x)}\tilde{L}_f = \begin{pmatrix} D_h L_f & 0 \\ (*) & D_x \tilde{L}_f \restriction_F \end{pmatrix}.$$

Hence $c(f,x) := D_x \tilde{L}_f \restriction_F$ is a linear invertible map of $T_x(F)$ and continuous. Since it is constructed using a group action, the cocycle relation is automatic.

2. The top Lyapunov exponent

We define the first Lyapunov exponent in the setting of examples ♠, ♡, ♢. Then we give a proof of Furstenberg-Kesten Theorem using Kingman's subadditive ergodic Theorem.

Definition 3 (First Lyapunov exponent)**.** *Let $c : G \times M \to \mathrm{GL}(V)$ be a continuous cocycle where V is a finite dimensional vector space.*

Fix $f \in G$ and consider an f-invariant probability measure $\mu \in \mathscr{P}_f(M)$. Then the first Lyapunov exponent *of the cocycle c on f with respect to μ is defined by*

$$\lambda_{top}(c, f, \mu) := \inf_{n+\infty} \frac{1}{n} \int_M \log \|c(f^n, x)\| \, \mathrm{d}\mu(x).$$

When the cocycle is the derivative as in ♡, it is called the first Lyapunov exponent of f for the measure μ and denoted by $\lambda_{top}(f, \mu)$.

For the suspension space as in ♢, it is called the first fiberwise Lyapunov exponent *and denoted by $\lambda_{top}(f, \mu) \restriction_F$.*

A function $\varphi : M \to \mathbb{R}$ is *f-invariant* if for μ almost every $x \in M$,

$$\varphi(f(x)) = \varphi(x).$$

Theorem 1 (Kingman). *Let M be a compact space, $f : M \to M$ a continuous function and μ an f-invariant probability measure. Let $(\varphi_n)_{n\geq 1}$ be a sequence of subadditive functions defined over M i.e. such that $\varphi_{n+m} \leq \varphi_n \circ f^m + \varphi_m$ for all integers $n, m \geq 1$, and taking value in $[-\infty, +\infty)$. Assume that* $\sup(0, \varphi_1) \in L^1(M, \mu)$.

Then there exists an f-invariant function $\varphi : M \to [-\infty, +\infty)$ such that the sequence $(\frac{\varphi_n}{n})_{n\geq 1}$ converges μ-almost everywhere towards φ. Furthermore, its positive part is integrable and

$$\int_M \varphi \, \mathrm{d}\mu = \lim_{n+\infty} \frac{1}{n} \int_M \varphi_n \, \mathrm{d}\mu = \inf_{n\geq 1} \frac{1}{n} \int_M \varphi_n \, \mathrm{d}\mu.$$

One can find this statement (as Theorem 3.3) and a proof in the chapter 3 of Viana's book [Via14].

Theorem 2 (Furstenberg-Kesten[FK60]). *Let M be a smooth compact manifold, G a topological group or semigroup acting on M. Let $c : G \times M \to \mathrm{GL}(V)$ be a continuous cocycle where V is a finite dimensional vector space.*

Then for all $f \in G$ and f-invariant probability measure μ,

$$\lambda_{top}(c, f, \mu) = \lim_{n+\infty} \frac{1}{n} \int_M \log \|c(f^n, x)\| \, \mathrm{d}\mu(x).$$

Proof using Theorem 1. Under the hypothesis of Furstenberg-Kesten Theorem, consider the sequence of continuous functions $\varphi_n := \log \|c(f^n, .)\|$. It is subadditive and the positive part of φ_1 satisfies the μ-integrability assumption. By Kingman's Theorem 1, we deduce the convergence. □

3. Uniform subexponential growth of finitely generated actions

Definition 4 (cf. paragraph 6.2 [Bro18]). *Let $c : G \times M \to \mathrm{GL}(V)$ be a continuous cocycle where V is a finite dimensional vector space.*

Then for all $f \in G$ the $\mathbb{Z}$-cocycle $c(f^n, .)$ has uniform subexponential growth *if for all $\varepsilon > 0$, there exists a positive number $C_\varepsilon > 0$ such that for every $n \in \mathbb{Z}$,*

$$\sup_{x\in M} \|c(f^n, x)\| \leq C_\varepsilon e^{\varepsilon|n|}.$$

If the cocycle is the derivative as in example ♡, it is called uniform subexponential growth of derivatives. *In the case of the fiberwise derivative as in example ♢, it is called* fiberwise uniform subexponential growth of derivatives

Proposition 1 (Cf. Proposition 6.3 [Bro18]). *Let M be a compact smooth manifold, G a compactly generated topological group and $c : G \times M \to \mathrm{GL}(V)$ be a* continuous cocycle *where V is a finite dimensionnal real vector space. Fix $f \in G$.*

Then the $\mathbb{Z}$-cocycle induced by c and f has uniform subexponential growth *if and only if for all (f-invariant) probability measure $\mu \in \mathscr{P}_f(M)$, both first Lyapunov exponents $\lambda_{top}(c, f, \mu)$ and $\lambda_{top}(c, f^{-1}, \mu)$ cancel.*

Proof. Suppose first that the growth of the $\mathbb{Z}$-cocycle is uniformly subexponential. Fix an f-invariant probability measure μ. Then for all $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that, for all $x \in M$ and $n \in \mathbb{Z}$,

$$\log \|c(f^n, x)\| \leq \log C_\varepsilon + |n|\varepsilon.$$

By the cocycle relation, for every $x \in M$, we write $c(f^{-n}, f^n(x))^{-1} = c(f^n, x)$. Hence for any $n \geq 1$ and $x \in M$,

$$-\log C_\varepsilon - n\varepsilon \leq -\log \|c(f^{-n}, f^n(x))\| = \log \|c(f^n, x)\| \leq \log C_\varepsilon + n\varepsilon.$$

This allows us to conclude that $\lambda_{top}(c, f, \mu) = \lambda_{top}(c, f^{-1}, \mu) = 0$.

For the converse, we prove the contraposition i.e. that if the $\mathbb{Z}$-cocycle does not have uniform subexponential growth then there exits $\varepsilon > 0$ and an f-invariant probability measure μ such that $\lambda_{top}(c, f, \mu) > \varepsilon$ or $\lambda_{top}(c, f^{-1}, \mu) > \varepsilon$.

Assume that the growth of the $\mathbb{Z}$-cocycle is not uniformly subexponential, meaning that there exists $\varepsilon > 0$, a injective sequence of integers $(n_j)_{j\geq 1} \subset \mathbb{Z}$ such that for all $j \geq 1$,

$$\sup_{x\in M} \|c(f^{n_j}, x)\| > e^{\varepsilon |n_j|}.$$

Fix such an $\varepsilon > 0$ and sequence $(n_j)_{j\geq 1}$. Without loss of generality, we can assume that up to a subsequence, $n_j \to +\infty$ by working either with f or f^{-1}.
Sketch of the construction of μ : Denote by $\mathbb{S}(V)$ the unit sphere of V, by $\mathcal{S}M$ the associated sphere bundle over M and p the projection. The crux of the construction is to find a *Birkhof sum*. The idea is that for all $(x, v) \in \mathcal{S}M$ and $n \geq 1$, the term $\frac{1}{n} \log \|c(f^n, x)v\|$ is a Birkhof sum for the test function

$$\begin{aligned} \Phi : \mathcal{S}M &\longrightarrow \mathbb{R} \\ (x, v) &\longmapsto \log \|c(f, x)v\| \end{aligned}$$

and the dynamical system

$$\begin{aligned} Uf : \mathcal{S}M &\longrightarrow \mathcal{S}M \\ (x, v) &\longmapsto \left(f(x), \frac{c(f, x)v}{\|c(f, x)v\|}\right), \end{aligned}$$

meaning that

$$\frac{1}{n} \log \|c(f^n, x)v\| = \frac{1}{n} \sum_{k=0}^{n-1} \Phi\big(Uf^k(x, v)\big).$$

In the second step, we take any weak-$*$ limit ν of a sequence of probability measures $(\nu_j)_{j\geq 1}$ supported on well chosen orbits of Uf and check that it is a Uf-invariant measure of the sphere bundle $\mathcal{S}M$. Indeed, by the non uniform subexponential growth assumption, we choose for every $j \geq 1$ an element $(x_j, v_j) \in \mathcal{S}M$ such that

$$\|c(f^{n_j}, x_j)v_j\| > e^{\varepsilon n_j}. \tag{3.1}$$

Consider the sequence orbital probability measure of $\mathcal{S}M$ defined for every $j \geq 1$ by

$$\nu_j := \frac{1}{n_j} \sum_{k=0}^{n_j-1} (Uf^k)_* \mathrm{Dirac}_{(x_j,v_j)}.$$

Remark that for every $j \geq 1$, because of the non uniform subexponential growth assumption (3.1) and the Birkhof identity of the first step then $\int_{\mathcal{S}M} \Phi \, \mathrm{d}\nu_j = \frac{1}{n_j} \log \|c(f^{n_j}, x_j)v_j\| > \varepsilon$. Since M is compact, so is its sphere bundle $\mathcal{S}M$. Hence any weak-$*$ limit ν of $(\nu_j)_{j\geq 1}$ is a Uf- invariant probability measure of $\mathcal{S}M$. Furthermore

$$\int_{\mathcal{S}M} \Phi \, \mathrm{d}\nu \geq \varepsilon.$$

In the third step, we choose an ergodic component ν' of ν such that the inequality above holds and denote by $\mu := p_*\nu'$ its push-forward to M by the projection p. It is an f-invariant and ergodic probability measure of M. Now by Uf-invariance of ν' and using the Birkhof identity of the first step, for every $n \geq 1$, we deduce

$$\varepsilon \leq \int_{\mathcal{S}M} \Phi \, \mathrm{d}\nu' = \frac{1}{n} \int_{\mathcal{S}M} \sum_{k=0}^{n-1} \Phi \circ Uf^k \ \mathrm{d}\nu' = \frac{1}{n} \int_{\mathcal{S}M} \log \|c(f^n, x)v\| \, \mathrm{d}\nu'(x,v).$$

By taking the supremum of $\|c(f^n, x)v\|$ when v varies in the spherical fibers $p^{-1}(x)$, we deduce

$$\varepsilon \leq \frac{1}{n} \int_{\mathcal{S}M} \log \|c(f^n, x)v\| \, \mathrm{d}\nu'(x,v) \leq \frac{1}{n} \int_{\mathcal{S}M} \log \|c(f^n, x)\| \, \mathrm{d}\nu'(x,v).$$

Since the disintegration of ν' along each fiber $p^{-1}(x)$ are probability measures, we obtain

$$\varepsilon \leq \frac{1}{n} \int_{\mathcal{S}M} \log \|c(f^n, x)\| \, \mathrm{d}\nu'(x,v) = \frac{1}{n} \int_{M} \log \|c(f^n, x)\| \, \mathrm{d}\mu(x).$$

Finally, taking the limit when n goes to $+\infty$, we obtain a lower bound for the first Lyapunov exponent

$$0 < \varepsilon \leq \lambda_{top}(c, f, \mu).$$

□

4. Regularity of the top Lyapunov exponent

Let H be a locally compact topological group, denote by m_H its Haar measure. Assume that H is *amenable* meaning that it admits a *Følner sequence* i.e. an exhaustion by an increasing sequence of subsets $(F_j)_{j\geq 1}$ of finite Haar measure satisfying the following condition: for all compact subset $Q \subset H$,

$$\limsup_{j+\infty} \sup_{h\in Q} \frac{m_H\big(F_j \triangle hF_j\big)}{m_H(F_j)} = 0,$$

where $F_j \triangle hF_j = \big(F_j \cup hF_j\big) \setminus \big(F_j \cap hF_j\big)$ denotes the symmetric difference of F_j and hF_j.

Examples of amenable groups, $\mathbb{Z}^n, \mathbb{R}^n$. The free group generated by two elements is not amenable.

Assume now that H acts on the compact manifold M. Then for all probability measure $\mu \in \mathscr{P}(M)$, denote by $F_j * \mu$ the averaging measure

$$F_j * \mu := \frac{1}{m_H(F_j)} \int_{F_j} h_* \mu \, \mathrm{d}m_H(h).$$

The first Lyapunov exponent satisfies the following regularity conditions along the averaging measures of a Følner sequence.

Fact 1 (Lemma 9.1 [Via14] and Claim 13.1, Remark 13.2 [Bro18])**.** *Fix $f \in G$ and consider an amenable group $H \subset Z_f(G)$, denote by m_H its Haar measure and fix a Følner sequence $(F_j)_{j\geq 1}$ in H.*

Then the map $\mu \in \mathscr{P}_f(M) \mapsto \lambda_{top}(c, f, \mu)$ is upper semi-continuous and for all f-invariant probability measure μ, the following holds.

- (i) *For all $n \geq 1$, the averaging measure $F_j * \mu$ is f-invariant.*
- (ii) *Every weak-$*$ limit of $(F_j * \mu)_{n\geq 1}$ is an f-invariant and H-invariant probability measure.*
- (iii) *For all $n \geq 1$, the first Lyapunov exponent of the averaging measure $F_j * \mu$ satisfies $\lambda_{top}(c, f, F_j * \mu) = \lambda_{top}(c, f, \mu)$.*
- (iv) *The first Lyapunov exponent of any weak-$*$ limit μ' of $(F_j * \mu)_{j\geq 1}$ has lower bound*
  $\lambda_{top}(c, f, \mu') \geq \lambda_{top}(c, f, \mu)$.

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Oseledec's theorem, Pesin manifolds, metric entropy.

Vladimir Finkelshtein

Let M be a compact Riemannian manifold without boundary and $f : M \to M$ a C^1-diffeomorphism (in §2, 3 we will assume C^2). The main goal of this talk is to discuss the contribution of expansion to the metric entropy of a dynamical system and, in particular, to define all the relevant notions. This talk also serves as a background for the talk on results of Ledrappier–Young. In §1 we will define Lyapunov exponents and Oseledec spaces and see how they measure exponential expansion, in §2 we will partition M into Pesin submanifolds which capture this expansion. Finally, in §3, we will define metric entropy, and explain, using the former notions, how entropy is calculated from the expansion. Sketches of proofs will be given.

1. Oseledec's Theorem

Definition 1. *A point $x \in M$ is called (Oseledec)-regular if there exist $r(x) \in \mathbb{N}$, real numbers $\lambda_1(x) > ... > \lambda_r(x)$, vector subspaces $E_i(x) \subset T_x M$, such that*

(1) $T_x M = E_1(x) \bigoplus ... \bigoplus E_{r(x)}(x)$.
(2) $\lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n v\| \to \lambda_i(x)$ *for all* $v \in E_i(x) \setminus 0$.
(3) $\lim_{n\to\pm\infty} \frac{1}{n} \log |Jac(D_x f^n)| \to \sum_{i=1}^{r(x)} \lambda_i(x) \dim(E_i(x))$.

We denote by $\Gamma' \subset M$ the set of regular points in M.

Theorem 1 (Oseledec, [5])**.** *Let $f : M \to M$ a C^1-diffeomorpshism of a compact Riemmanian manifold. Assume m is an ergodic f-invariant probability measure. Then,*

(1) $m(\Gamma') = 1$.
(2) *The maps $r(x), \lambda_i(x), m_i(x)$ are measurable $f-$invariant functions (in particular, constant m-a.e.). Moreover, the maps E_i are measurable and f-equivariant, i.e.*
$$E_i(f(x)) = D_x f E_i(x).$$

The well-defined numbers $\lambda_i = \lambda_i(x)$ are called *Lyapunov exponents*, $m_i = \dim(E_i(x))$ - their *multiplicities*, $E_i(x)$ - the corresponding *Oseledec spaces.*

Proof. (Sketch from [7, 2, 4]) For $t \in \mathbb{R} \cup \{-\infty\}$, define
$$V_x(t) = \{v \in T_x M : \limsup \frac{1}{n} \log \|D_x f^n v\| \leq t\}.$$

Following observations are straihgtforward:

(1) For each t, $V_x(t)$ is a vector space.
(2) $V_x(t) \subset V_x(s)$ for $s \leq t$.
(3) $V_x(t)$ is f-equivariant.

Consider the graph of $\dim(V_x(t))$ as a function of t. By (1) dimension is well-defined, and by (2) the graph is monotone. We define $\lambda_i(x)$ to be the ordered points of discontinuity of this graph. By (3), the graph is the same for x and $f(x)$, in particular, $r(x)$ and $\lambda_i(x)$ are measurable and f-invariant, hence, constant m-a.e. Setting $V_i(x) := V_x(\lambda_i(x))$ we obtain
$$V_r(x) \subset V_{r-1}(x) \subset ... \subset V_1(x).$$
where V_i is the space of directions that get asymptotically expanded in n steps by less than $e^{\lambda_i n}$ and have a vector that is expanded by $e^{\lambda_i n}$. It is a fact that lim sup can be replaced by lim, and that $V_i(x)$ can be further decomposed in f-equivariant way into sum of $E_j(x)$, as desired in the theorem. □

2. Pesin manifolds

For $x \in M$, Oseledec spaces provide us with local directions in T_xM in which the asymptotic expansion/contraction of Dxf^n happens, namely, the *contracting space*: $E^c(x) = \bigoplus_{\lambda_i<0} E_i(x)$, the *expanding space*: $E^c(x) = \bigoplus_{\lambda_i>0} E_i(x)$. We wish to "integrate" those to a submanifold.

Definition 2. *Pesin stable manifold passing through $x \in M$ is*

$$W^s(x) = \left\{ y \in M : \limsup \frac{1}{n} \log d(f^n(x), f^n(y)) < 0 \right\}.$$

Pesin unstable manifold $W^u(x)$ is defined as the stable manifold of f^{-1}.

Theorem 2 (Stable manifold theorem, [1])**.** *$W^s(x)$ is an injectively immersed submanifold of M with $T_xW^s(x) = E^c(x)$.*

Proof. For simplicity, we assume that f is hyperbolic, i.e. does not admit 0-Lyapunov exponent. We introduce *Lyapunov charts.* We split $\mathbb{R}^{\dim(M)} = \mathbb{R}^u \bigoplus \mathbb{R}^s$, where $s = \dim(E^s), u = \dim(E^u)$.

Lyapunov charts are maps $\Psi_x : O(x) \subset \mathbb{R}^{\dim(\mathbb{M})} \to M$, with $\Psi_x(0) = x, D_0\Psi_x : \mathbb{R}^u, \mathbb{R}^s \to E^u(x), E^c(x)$ with some extra properties. The neighborhoods $O(x)$ at which maps are defined must have controlled radius. Roughly speaking, in local charts of Ψ_x and $\Psi_{f(x)}$, the map f should behave, up to controlled error, like its derivative D_xf. Moreover, it can be arranged that the contraction and expansion happen after one application of D_xf, rather than asymptotically.

We consider

$$\begin{aligned} \mathcal{W}_x &= \{\text{admissible local manifolds at } x\} \\ &= \{\text{graph}(\Psi_x(\mathcal{F})) : \mathcal{F} : \mathbb{R}^s \to \mathbb{R}^u, D_0\mathcal{F} = 0, \|D_0\mathcal{F}\| \text{ is small}\} \end{aligned}$$

In other words, admissible local manifolds are lifts of graphs that are tangent to the stable space at x and look flat-ish. (we omit quantitative details, which are quite technical). Moreover, $\mathcal{W}_x$ can be endowed with a distance: for two local admissible manifolds which come from graphs of functions $\mathcal{F}_1, \mathcal{F}_2$, the distance is $\|\mathcal{F}_1 - \mathcal{F}_2\|_{\max}$

We define the graph transform

$$\begin{aligned} \mathcal{G} : \mathcal{W}_{f(x)} &\to \mathcal{W}_x \\ \mathcal{M} &\mapsto f^{-1}(\mathcal{M}) \end{aligned}$$

It is straightforward observation that $\mathcal{G}$ is a well-defined map, that is also a contraction. Hence, if we take any sequence $\mathcal{V}_n \in \mathcal{W}_{f^n(x)}$ of admissible local manifolds, then $\mathcal{G}^n(\mathcal{V}_n)$ is a Cauchy sequence in $\mathcal{W}_x$. Its limit exists and is unique, regardless of the choices, and is the local stable manifold. Gluing such pieces results in a global Pesin stable manifold. □

We further introduce *fast stable/unstable manifolds* (manifolds along which the convergence/divergence of orbits of points is uniformly exponentially fast, similarly to the case of Pesin manifolds). Clearly, same argument with minor modifications proves their existence. We discuss how fast (un)stable filtration varies Lipschits continuously inside (un)stable Pesin manifold (seen locally as a graph of smooth function).

3. Metric entropy

Let m be a f-invariant proability measure. Given two measurable partitions η, ξ of M, the *mean condtional entropy* is given by

$$H_m(\xi|\eta) = \int -\log(m_x^\xi(\eta(x))\mathrm{d}m(x),$$

where $\eta(x)$ is the atom of η containing x, and m_x^ξ is the conditional probability measure supported on $\xi(x)$.

For a measurable partition η of M, we define the *entropy of f given a partition* η as

$$H_m(f,\eta) = H_m(\eta, \bigvee_{i=1}^{\infty} f^i\eta).$$

The *metric entropy* of f is given by

$$h_m(f) := \sup_\eta\{H_m(f,\eta)\}.$$

The main result of this section is that the sup in the above definition of entropy is achieved by a concrete partition, that "sees" the expansion of f.

Definition 3. *A partition ξ is subordinate to a foliation W^u if for m-a.e. x we have $\xi(x) \in W^u(x)$ and $\xi(x)$ contains an open neighborhood of x in submanifold topology.*

Theorem 3 (Ledrappier–Young, [3])**.** *Let η be any measurable partition that is increasing ($f\eta \prec \eta$) and subordinate to the foliation into unstable Pesin manifolds W^u. Assume that $\bigvee_{i=0}^{\infty} f^i\xi$ is a partition into points. Then*

$$h_m(f) = H_m(f,\eta).$$

Moreover, one can easily construct such partitions.

Finally, we state the relation between entropy and expansion.

Theorem 4 (Margulis-Ruelle inequality, [6])**.**

$$h_m(f) \leq \sum_{\lambda_i>0} \lambda_i \dim(E_i).$$

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Ledrappier–Young Rigidity

Cagri Sert

The talk consisted of three parts: in the first part, after having presented the context and some previous theorems, we stated Ledrappier–Young's breakthrough results in [10, 11, 12]. In the second part, following [3], we reformulated these results in an algebraic setting as algebraic rigidity statements. In the final part, we gave a proof of one of the main implications in the algebraic rigidity statement.

1. Background and statement of results

We start by presenting some fundamental results on the relation between entropy and Lyapunov exponents, give a brief overview of SRB measures in the uniformly hyperbolic setting and finally state the two main results of Ledrappier and Young.

1.1. **Margulis–Ruelle inequality and Pesin's entropy formula.** Metric entropy, introduced in dynamical setting by Kolmogorov [9], yields a numerical measure of complexity of a measurable dynamical system (M, f, μ). Here M denotes a probability space, $f : M \to M$ a measurable map preserving a probability measure μ. In the more special case where M has a differentiable structure (e.g. a compact smooth manifold) and f is a differentiable map, thanks to Oseledets' theorem [16, 18], one has a more geometric measure of complexity provided by Lyapunov exponents. These indicate the rate at which μ-typical nearby points are separated in different directions.

The relation between these two measures of complexity has been studied broadly and it constitutes the context of Ledrappier–Young's works. One of the first results on the relation between entropy and Lyapunov exponents is the Margulis–Ruelle inequality. To state this inequality, let us make our notation precise. Let M be a compact smooth manifold of dimension d, $f : M \to M$ a diffeomorphism preserving a Borel probability measure μ on M. For simplicity, suppose μ to be ergodic. We denote the corresponding Lyapunov exponents by $\lambda_1 > \lambda_2 > \ldots > \lambda_p$ and for μ-a.s. $x \in M$, the Lyapunov spaces by $E_1(x), \ldots, E_p(x)$. We have $\oplus E_i(x) = T_x M \simeq \mathbb{R}^d$ and $v \in T_x M \setminus \{0\}$, $v \in E_i(x)$ if and only if

$$\lim_{n \to \pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \lambda_i,$$

where $D_x f^n : T_x M \to T_{f^n(x)} M$ denotes the derivative of f^n. Let $m_i \in \mathbb{N}$ be the (μ-a.s. constant) dimension of $E_i(x)$.

The Margulis–Ruelle inequality says that for $f \in C^1(M)$, the metric entropy $h_\mu(f)$ is bounded above by the weighted sum of positive Lyapunov exponents, namely

$$h_\mu(f) \leq \sum_{\lambda_i > 0} \lambda_i m_i. \tag{1.1}$$

One is immediately led to wonder what kind probability measures μ (and f) can have a "maximal entropy", in the sense that we have an equality in (1.1). One natural obstruction for the metric entropy to reach the weighted sum of local separation rates is that the probability measure μ sees very little of the space[1]. The following result, due to Pesin [17] (see also Mañe [14]), tells that for a natural class of measures without this obstruction, the equality indeed takes place in (1.1). More precisely, given $f \in C^{1+\alpha}(M)$ and an invariant probability μ that is absolutely continuous with respect to the Riemannian measure on M, *Pesin's entropy formula* states that

$$h_\mu(f) = \sum_{\lambda_i > 0} \lambda_i m_i. \tag{1.2}$$

1.2. **SRB measures.** Here, we partly borrow from [22], which the reader can consult for a good exposition of this topic.

To motivate further the results of Ledrappier–Young and to provide more context, let us take a step back (both in generality and history) and recall some fundamental results in uniformly hyperbolic dynamics. In this context, Anosov diffeomorphisms and Axiom A systems were studied by Anosov, Bowen, Sinai, Ruelle and others yielding a deep understanding of the dynamical properties of such systems. In particular, it follows from [2, 21, 19] that given a C^2 Anosov diffeomorphism f of a smooth compact manifold M, there exists a unique f-invariant and ergodic Borel probability measure μ on M that is characterized by each of the following conditions:

(1) μ has absolutely continuous conditionals along unstable manifolds of f;

(2) $h_\mu(f) = \int |\det D_x f_{|E^u(x)}| d\mu(x)$, where $E^u(x)$ denotes the unstable subspace of $T_x M$;

(3) There is a set $V \subset M$ of full Riemannian measure such that for every continuous function $\phi : M \to \mathbb{R}$ and $x \in V$, we have

$$\frac{1}{n} \sum_{k=0}^{n-1} \phi \circ f^k(x) \to \int \phi d\mu \quad \text{as } n \to \infty.$$

In this setting, the unique such measure μ is called a Sinaï–Ruelle–Bowen measure (an SRB measure for short).

[1] as we shall see, this is clearly expressed by Ledrappier–Young's Theorem 2.

1.3. **Statement of results.** Notice that the implication (1) $\implies$ (2) is a particular case of Pesin's entropy formula, which can be expressed in terms of Lyapunov exponents in the much wider setting of an arbitrary $C^{1+\alpha}$-diffeomorphism of a compact manifold, thanks to Oseledets' theorem. One natural question is whether the equivalence of (1) and (2) above stays valid in this much wider generality, in other words, whether there is a converse to Pesin's result. This is addressed by the following rigidity result of Ledrappier–Young:

Theorem 1 (Ledrappier–Young [11])**.** *Let M be a compact smooth manifold, $f \in C^2(M)$ a diffeomorphism and μ an f-invariant ergodic Borel probability measure. If μ satisfies Pesin's entropy formula, i.e. $h_\mu(f) = \sum_{\lambda_i>0} \lambda_i m_i$, then μ has absolutely continuous conditional measures along unstable manifolds of f.*

Remark 1.4. A major step in the proof of the previous result is to establish that $h_\mu(f, \mathcal{F}_u) = h_\mu(f)$, where the first quantity denotes the entropy of f with respect to the unstable foliation. This equality as well as the above result were previously proved by Ledrappier [10] under the additional assumption that zero does not appear as a Lyapunov exponent.

In a subsequent work [12], Ledrappier–Young clarified to a great extent the aforementioned relation among entropy, Lyapunov exponent and the "size" of the probability measure. To state their result, recall that given a Borel probability measure μ on a metric space (M, d), the pointwise dimension of μ is defined as

$$\lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}$$

whenever the latter exists and is constant μ-a.s. Here $B(x, r)$ stands for the ball of radius r around $x \in M$. Such a probability measure with a well-defined pointwise dimension is sometimes called exact dimensional. Finally, given a system (M, f, μ) as before, for $j = 1, \dots, p$ such that $\lambda_j > 0$, the j^{th} unstable manifold W^j of f is the Pesin submanifold with tangent space at μ a.e. $x \in M$ given by $\oplus_{i=1}^{j} E_i(x)$. In these terms, the result of Ledrappier–Young reads

Theorem 2 (Ledrappier–Young [12])**.** *Let f be a C^2 diffeomorphism of a compact smooth manifold M and let μ be a f-invariant ergodic Borel probability measure on M. Denoting by δ_j the pointwise dimension of the conditional measure of μ along the j^{th} unstable manifold W^j, we have*

$$h_\mu(f) = \sum_{i=1}^{u} \lambda_i(\delta_i - \delta_{i-1}),$$

where u is maximal such that $\lambda_u > 0$ and $\delta_0 := 0$.

Remark 1.5. 1. It is part of the statement that the conditional measure of μ along W^j is exact dimensional. The question of whether the probability measure μ itself is exact dimensional was known as Eckmann–Ruelle conjecture and it was confirmed later by Barreira–Pesin–Schmeling [1].

2. Both of the previous theorems are valid more generally for $f \in C^{1+\alpha}(M)$ (see Brown [5]).
3. Note that both Margulis–Ruelle inequality and Pesin's entropy formula are directly implied by the previous theorem.

2. An algebraic reformulation

Here, approaching to the setting of the Zimmer's conjecture and its solution [6], we present an adaptation of the Ledrappier–Young's rigidity result (Theorem 1) for smooth actions of Lie groups on compact manifolds. We mostly borrow from [3].

2.1. **The Lie group action setting.** Let G be a unimodular Lie group acting smoothly and locally freely on a compact smooth manifold M. Let $g \in G$ and $H \leq G$ be a closed subgroup such that g belongs to the normalizer $N_G(H)$. Denote by $f : M \to M$ the diffeomorphism corresponding to g: $f(x) = g.x$. Let μ be a f-invariant and ergodic Borel probability measure on M such that $H.x \subseteq W^u(x)$ for μ a.e. $x \in M$, where $W^u(x)$ denotes the unstable manifold of f containing x.

Since g normalizes H, f sends H-orbits to H-orbits and since μ-a.s. $H.x \subseteq W^u(x)$, the partition of M into H-orbits is finer than the partition into unstable manifolds. We denote by $\mathcal{F}_H$ the foliation given by H-orbits and as in Remark 1.4, by $h_\mu(f, \mathcal{F}_H)$ the entropy of f with respect to the foliation $\mathcal{F}_H$[2]. Let $m_{i,H}$ denote the multiplicity of λ_i relative to H, i.e. the μ-a.s. constant value of $m_{i,H} = \dim(E^i(x) \cap T_x(H.x))$.

Before stating the version of Theorem 1 in this setting, to fix ideas, let us pause to give two examples that fit in the general framework explained above.

Example 2.2. Let M be the d-dimensional torus $\mathbb{T}^d$, $G = \mathrm{SL}_d(\mathbb{Z}) \ltimes \mathbb{R}^d$ with its natural affine action on M. Let $g \in \mathrm{SL}_d(\mathbb{Z}) < G$ and f the corresponding diffeomorphism of M. The unstable foliation of f is given by the orbits of $U_f < \mathbb{R}^d$ where U_f is the sum of generalized eigenspaces of g corresponding to eigenvalues with modulus greater than one. Therefore, as a subgroup H of G whose orbits refine the unstable foliation of f and normalized by g, we can take any g-invariant subspace of U_f.

Example 2.3. Let M be the unit tangent bundle of a compact hyperbolic surface $\mathrm{PSL}_2(\mathbb{R})/\Gamma$, where Γ denotes a surface group. The group $G = \mathrm{PSL}_2(\mathbb{R})$ acts smoothly and locally freely on its quotient M by left-multiplication. Let $g \in G$ be given by $\begin{pmatrix} e^{1/2} & \\ & e^{-1/2} \end{pmatrix}$ and f be the corresponding diffeomorphism. In this special setting, the action of g corresponds to the time-one action of the geodesic flow on the unit tangent bundle. Let $H < G$ be the upper-triangular unipotent

[2]We remind that this is the entropy $h_\mu(f, \xi_H)$ of f with respect to a measurable partition ξ_H subordinate (for definition, see [3, §7.3]) to $\mathcal{F}_H$ and increasing with respect to f (i.e. $f^{-1}\xi_H$ is finer than ξ_H). It does not depend on the choice of an increasing subordinate partition (see [11, Lemma 3.1.2]). For the existence of such partitions, see Ledrappier-Strelcyn [13]

subgroup. In this case, the foliation $\mathcal{F}_H$ given by H-orbits corresponds precisely to the unstable foliation $\mathcal{F}_u$ of f, which also coincides with the orbits of the horocycle flow on M.

More generally, we can have M given by a quotient G/Γ of a Lie group G by a cocompact lattice $\Gamma < G$ and f a diffeomorphism of M given by the left-multiplication action of any $g \in G$ such that $\mathrm{Ad}(g)$ has eigenvalues of modulus greater than one (e.g. a regular element in a semisimple Lie group G). As a subgroup H as above, one can then take any Lie subgroup of G whose Lie algebra is stabilized by $\mathrm{Ad}(g)$ and is contained in the $\mathrm{Ad}(g)$-expanding subspace in $\mathfrak{g}$ (e.g. if g is a regular element as before, then H is the corresponding horospherical subgroup).

2.2. **A Lie group action version of Ledrappier–Young's rigidity result.** We now state the version of Theorem 1 in the above setting (see [4, Theorem 9.5] and [3, Theorem 8.5]):

Theorem 3. *With the setting of §2.1, the followings are equivalent:*

1. $h_\mu(f, \mathcal{F}_H) = \sum_{\lambda_i>0} \lambda_i m_{i,H}$;

2. for any measurable partition ξ subordinate to the partition into H-orbits and μ almost every x, the conditional measure μ_x^ξ is absolutely continuous with respect to the Riemannian volume on the H-orbit $H.x$;

3. for any measurable partition ξ subordinate to the partition into H-orbits and μ almost every x, the conditional measure μ_x^ξ is equal, up to normalization, to the Haar measure on the H orbit $H.x$;

4. μ is H-invariant.

The equivalence of (3) and (4) is a rather standard fact. To avoid repetition, we will skip the proof of the other implications in this result for which we refer the reader to [3, §8.2] and [4, §9.1] and the references therein.

We mention in passing that in a particular algebraic setting similar to above, the entropy rigidity phenomenon expressed by (1) $\implies$ (4) in the previous theorem is proven by Einsiedler–Katok–Lindenstrauss [8] to be of a far-reaching extent. This also gave striking applications to classical diophantine approximation problems in number theory. See also the more recent work of Einsiedler–Lindenstrauss [7] where this algebraic entropy rigidity phenomenon is taken even further.

2.3. **An application.** We end this note by a simple application of the above theorem to obtain a measure rigidity result in the setting of Example 2.3. This is in the spirit of Margulis–Tomanov's treatment [15] of Ratner's measure classification theorems.

Example 2.3 (continued). Consider a f-invariant Borel probability measure that is also invariant by H [3]. Note that f has a single positive Lyapunov exponent

[3] In the classification of unipotent invariant measures on homogeneous spaces, typically, starting from a H-invariant probability measure on M that does not live on a closed H-orbit, one uses Ratner's polynomial shearing argument to deduce an additional f-invariance as assumed here.

and it is equal to 1. Clearly, we also have that the dimension $m_{1,H} = 1$. Now using (4) $\implies$ (1) in Theorem 3, we get that $h_\mu(f, \mathcal{F}_H) = \lambda_1 m_{1,H} = 1.1 = 1$. Now using the facts that $\mathcal{F}_u = \mathcal{F}_H$ and $h_\mu(f) = h_\mu(f, \mathcal{F}_u)$ (see Remark 1.4), this implies that $h_\mu(f) = 1$. By the general property of metric entropy, this gives $1 = h_\mu(f^{-1})$ and again by Ledrappier–Young (Remark 1.4), we get

$$h_\mu(a^{-1}, \mathcal{F}_u(a^{-1})) = 1. \tag{2.1}$$

For f^{-1} the unstable foliation is given by the orbits of the lower triangular unipotent group H'. Clearly, the Lyapunov exponent and its multiplicity is the same for a^{-1} as for a so that by (2.1) and Remark 1.4, we are in a position to apply (1) $\implies$ (4) to deduce that μ is also invariant with respect to H'. Since the closed group generated by H and H' is $\mathrm{PSL}_2(\mathbb{R})$ itself, we deduce that μ is the Haar probability measure on M.

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Higher Rank Dynamics

Minju Lee

This is a summary of the talk given by the author at MFO, October 16th of 2019, as a part of the program "Arbeitsgemeinschaft: Zimmer's Conjecture". The audiences are expected to have learnt backgrounds such as Oseledet's theorem, and the theorem of Ledrappier-Young[5], [6], from the previous lectures. The goal of the lecture is to provide an example of a higher rank abelian actions, introduce concepts and theorems generalizing those of the rank one actions.

By a Higher rank abelian action, we mean a homomorphism

$$\alpha : \mathbb{Z}^d \to \mathrm{Diff}^1(M)$$

where M is a closed manifold, and $\mathrm{Diff}^1(M)$ is a C^1-diffeomorphism group of M. The basic example here is when $M = \mathbb{T}^k$ and each $\alpha(n)$ is given by a Toral automorphism for all $n \in \mathbb{Z}^d$. Equivalently, $\alpha(\mathbb{Z}^d)$ is a commuting matrices in $\mathrm{SL}(k, \mathbb{Z})$. We briefly explain how such actions can be obtained. Recall the unit theorem of Dirichlet:

Theorem 1 (Dirichlet)**.** *The set of algebraic units $\mathcal{U}_K$ of a number field K is isomorphic to $\mathbb{Z}^{r_1+r_2-1} \times F$, where r_1(respectively $2r_2$) is the number of real (respectively complex) embeddings of K into $\overline{\mathbb{Q}}$.*

Now the following observation is an exercise in linear algebra: If $A \in \mathrm{GL}(k, \mathbb{Z})$, and the characteristic polynomial $f \in \mathbb{Z}[x]$ is irreducible over $\mathbb{Q}$, then

$$\mathrm{C}(A) := \{B \in \mathrm{M}(k, \mathbb{Q}) : BA = AB\}$$

can be identified with a number field K. Moreover, denoting $\mathcal{O}_K$ to be the ring of integers in K, and λ to be a root of f, we have

$$\mathbb{Z}[\lambda] \subset \mathrm{C}(A) \cap \mathrm{M}(k, \mathbb{Z}) \subset \mathcal{O}_K,$$
$$\mathrm{C}(A) \cap \mathrm{GL}(k, \mathbb{Z}) \subset \mathcal{U}_K.$$

By extracting further data(rather than just a number field K), we can say conversely, that the action is determined by these data up to a weak equivalence, that is $\alpha \sim \alpha'$ if they are conjugated by rational matrices. A detailed explanation of this together with examples can be found in [4].

We now state Oseledet's theorem for a higher rank abelian actions:

Theorem 2 (Oseledet). *Let μ be an α-invariant, ergodic probability measure on M. Then there exists a co-null set $\Lambda \subset M$, linear functions $\lambda_1, \cdots, \lambda_p : \mathbb{Z}^d \to \mathbb{R}$, and an α-invariant measurable splitting*

$$\mathrm{T}_x M = E_1(x) \oplus \cdots E_p(x)$$

for all $x \in \Lambda$ such that

(1) $\lim_{n\to\infty} \frac{1}{\|n\|} (\log \|D_x\alpha(n)v\| - \lambda_i(n)) = 0$ *for all $v \in E_i(x) - \{0\}$,*

(2) $\lim_{n\to\infty} \frac{1}{\|n\|} (\log |\mathrm{Jac}_x\alpha(n)| - \sum_{i=1}^{p} \lambda_i(n)\mathrm{dim}E_i) = 0.$

$\lambda_1, \cdots, \lambda_p$ are called Lyapunov functionals. Unlike rank one action, each λ_i is a linear function, rather than a number.

The splitting for $\mathbb{Z}^d$-action can be obtained by repetitive application of the theorem for $\mathbb{Z}$-action. For simplicity, let us explain the construction for $d = 2$, with $\mathbb{Z}^2 = \langle e_1, e_2 \rangle$. We first apply Oseledet's theorem with respect to $\alpha(e_1)$ to get a conull set Λ and an invariant splitting

$$\mathrm{T}_x M = F_1(x) \oplus \cdots \oplus F_{p(x)}(x)$$

with Lyapunov exponents $\lambda_i(x) (i = 1, \cdots, p(x))$ for all $x \in \Lambda$ where for all $v \in F_i(x) - \{0\}$,

$$\lim_{k\to\infty} \frac{1}{k} \log\|D_x\alpha(ke_1)v\| = \lambda_i(x).$$

Since $\alpha(e_1)$ and $\alpha(e_2)$ commute, it follows that $\cup_x F_i(x) (i = 1, \cdots, p(x))$ are $\alpha(e_2)$-invariant. Hence, we may apply Oseledet's theorem on each $\cup_x F_i(x)$ to get a splitting

$$F_i(x) = \oplus_{j=1}^{p_i(x)} F_{ij}(x).$$

with Lyapunov exponents $\lambda_{ij}(x) (j = 1, \cdots, p_i(x))$ such that for all $v \in F_{ij}(x) - \{0\}$, we have

$$\lim_{k\to\infty} \frac{1}{k} \log\|D_x\alpha(ke_2)v\| = \lambda_{ij}(x).$$

Define Lyapunov functionals $\tilde{\lambda}_{ij}(x) : \mathbb{Z}^2 \to \mathbb{R}$ by $(a, b) \mapsto a\lambda_i(x) + b\lambda_j(x)$. Since $\alpha(e_1)$ and $\alpha(e_2)$ commutes, each $\cup_x F_{ij}(x)$ are invariant under $\alpha(e_1)$ as well. For the same reason, $p(x), p_i(x), \lambda(x), \lambda_{ij}(x)$ are all α-invariant measurable functions. And by the α-invariance and ergodicity of μ, they are μ-a.e. constant. Now, one can verify that the α-invariant splitting

$$\mathrm{T}_x M = \oplus_{i,j} F_{ij}(x)$$

with the corresponding Lyapunov functional $\tilde{\lambda}_{i,j}$'s satisfy the condition (1) of Theorem 2 at least when the limit $n \to \infty$ is taken along a fixed direction. This finishes the construction of the splitting, and we refer the reader [2] for the complete proof of Theorem 2.

Let us define an equivalence relation on the set of Lyapunov functionals by $\lambda_i \sim \lambda_j$ if $\lambda_i = c\lambda_j$ for some $c > 0$. The equivalence class will be called a coarse Lyapunov functionals. Given Lyapunov functionals, we consider the complement of its kernels in $\mathbb{R}^d$, and will call each connected component a Weyl chamber. For $n \in \mathbb{Z}^d$, assume that

$$\lambda_1(n) > \lambda_2(n) > \cdots > \lambda_p(n).$$

Note that this order will be preserved for a different choice of $m \in \mathbb{Z}^d$, only when m and n belong to the same Weyl chamber. Since $\alpha(n)$ generates a $\mathbb{Z}$-action, when $\lambda_i(n) > 0$, one can define the i-th unstable manifold through almost every point x in M:

$$W_n^i(x) := \{y \in M : \limsup_{k\to\infty} \frac{1}{k} \log d(\alpha(-kn)x, \alpha(-kn)y) \leq \lambda_i(n)\}.$$

Recall that this is a manifold, which is tangent to

$$T_x W_n^i(x) = \oplus_{j\leq i} E_{\lambda_j}(x),$$

and that the largest one will be called the unstable manifold with respect to $\alpha(n)$, denoted by W_n^u. Next, for each coarse Lyapunov functionals χ, we define

$$W^\chi(x) = \bigcap_{\chi(n)>0} \{y \in M : \limsup_{k\to\infty} \frac{1}{k} \log d(\alpha(-kn)x, \alpha(-kn)y) < 0\} = \bigcap_{\chi(n)>0} W_n^u.$$

This will be called a coarse Lyapunov manifold, and its tangent space is given by

$$T_x W^\chi(x) = \oplus_{\lambda\in\chi} E_\lambda(x).$$

Entropy product formula. We now explain the product formula. The primary reference is [1], [2], and [3]. The product formula says the following:

Theorem 3 (Brown, Rodriguez Hertz, Wang)**.** *Let μ be an α-invariant, ergodic measure on M. Then for each $n \in \mathbb{Z}^d$, we have*

$$h_\mu(n) = \sum_{\chi(n)>0} h_\mu(n \,|\, W^\chi).$$

If (f, M, μ) and (g, N, ν) are measure preserving and ergodic system, they generate $\mathbb{Z}^2$ action on the product space $(M \times N, \mu \times \nu)$, and the formula can be easily verified, because the space itself has a product structure. The theorem says, although when this is not the case, we have a product structure at the level of entropy.

We will only explain the proof for the easy case, using the Ledrappier-Young theorem from the previous lecture. There are two more inputs that we will take for granted, which will be explained along the proof. Let us consider the case $d = 2$, and further assume there are 3 Lyapunov functionals, where none of them are proportional to each other. Choose $n \in \mathbb{Z}^2$, and assume

$$\lambda_1(n) > \lambda_2(n) > 0 \geq \lambda_3(n).$$

Then the first unstable manifold $W_n^1(x)$ is tangent to $E_{\lambda_1}(x)$, while the second unstable manifold $W_n^2(x)$ is tangent to $E_{\lambda_1}(x) \oplus E_{\lambda_2}(x)$. Ledrappier-Young says that $h_\mu(n)$ is determined by the geometic data. Namely,

$$h_\mu(n) = \delta^1(n)\lambda_1(n) + (\delta^2(n) - \delta^1(n))\lambda_2(n)$$

where $\delta^1(n)$ stands for the dimension of μ along the first unstable manifold, and $\delta^2(n) - \delta^1(n)$ can be interpreted as a transversal dimension of μ inside the second unstable manifold.

Now we choose a different $m \in \mathbb{Z}^2$, but that belongs to the same Weyl chamber as n, satisfying

$$\lambda_2(m) > \lambda_1(m) > 0 \geq \lambda_3(m).$$

Note that this time, $W_m^1(x)$ is tangent to $E_{\lambda_2}(x)$ and $W_m^2(x)$ is tangent to $E_{\lambda_1}(x) \oplus E_{\lambda_2}(x)$. Applying Ledrappier-Young theorem again, we obtain

$$h_\mu(m) = \delta^1(m)\lambda_2(m) + (\delta^2(m) - \delta^1(m))\lambda_1(m).$$

Let us write the dimension of μ along the manifold tangent to $E_{\lambda_1}(x)$, $E_{\lambda_2}(x)$, and $E_{\lambda_1}(x) \oplus E_{\lambda_2}(x)$ by d_1, d_2 and d respectively. Then the above two relations can be rewritten as

$$h_\mu(n) = d_1\lambda_1(n) + (d - d_1)\lambda_2(n) \tag{0.1}$$

$$h_\mu(m) = d_2\lambda_2(m) + (d - d_2)\lambda_1(m). \tag{0.2}$$

Here comes one input that we use without proof; the linearity of $h_\mu(\cdot) : \mathbb{Z}^2 \to \mathbb{R}$ on each Weyl chamber. Note that all we needed to obtain (0.1) and (0.2) was the specific order of $\lambda_1(\cdot), \lambda_2(\cdot)$, and $\lambda_3(\cdot)$. Since there are plenty of such elements m, n in the given Weyl chamber, combined with the linearity, we conclude the coefficients of λ_1 and λ_2 in each equation should match. That is, $d_1 = d - d_2$.

The second and final input we use without proof is the following fact:

$$d_1 = h_\mu(n \mid W^{[\lambda_1]}) \text{ and } d_2 = h_\mu(n \mid W^{[\lambda_2]}),$$

where $[\lambda_1], [\lambda_2]$ denotes the coarse Lyapunov functionals. Note that this gives

$$\begin{aligned} h_\mu(n) &= d_1\lambda_1(n) + d_2\lambda_2(n) \\ &= h_\mu(n \mid W^{[\lambda_1]})\lambda_1(n) + h_\mu(n \mid W^{[\lambda_2]})\lambda_2(n), \end{aligned}$$

as desired. We finish the note by commenting on two inputs. The linearity of entropy function on Weyl Chamber can be proved once we have a special type of partition subordinate to the Coarse Lyapunov manifold. The second input is called generalized Ledrappier-Young theorem, which is also due to Brown, Rodriguez Hertz, and Wang. The complete proof can be found in [3].

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Invariance principles

Rene Rühr

1. Additional invariance using Ledrappier-Young

The purpose of this first section is to state and proof the following theorem.

Theorem 1 (see Theorem 11.1 [2]). *Let $\Gamma < G = \mathrm{SL}_n(\mathbb{R})$ be a lattice and let A denote the subgroup of diagonal matrices in G. Let $\alpha : \Gamma \to \mathit{Diff}^{1+\beta}(M)$ be an action and let $M^\alpha = G \times M/\Gamma$ denote the suspension space with induced G-action. Let μ be an ergodic, A-invariant Borel probability measure on M^α whose projection to $\mathrm{SL}_n(\mathbb{R})/\Gamma$ is Haar measure.*

Then, if dim$(M) \leq n-2$ the measure is G-invariant. Moreover, if α preserves a volume form vol and if dim$(M) \leq n-1$ then the measure μ is G-invariant.

As we have seen earlier, G-invariant measures on M^α correspond to Γ-invariant measures on M. Since A-invariant measures projecting to Haar on the base G/Γ always exist, we deduce the existence of a Γ-invariant Borel probability measure on M with the above dimension restrictions.).

We will have to recall some notation. The suspension space M^α fiber bundle over G/Γ (with projection measurable π) with fibers modeled over M,

$$M^\alpha = G \times M/\Gamma \qquad (g,x)\cdot\gamma = (g\gamma, \alpha(\gamma^{-1})x)$$

Denote the action by G on M_α, $\widehat{\alpha}(g)([h],x) = ([gh],x)$.L et us assume now that μ A-ergodic on M^α. We shall denote by $F = \ker(D\pi)$ the subbundle of TM^α when restricting to the tangent space over M. Further, introduce

- fiberwise Lyapunov functionals $\lambda_i^F : \mathrm{Lie}(A) \to \mathbb{R}$, $1 \leq i \leq k$
- maximal fiberwise Lyapunov exponent for $a \in A$ is

$$\lambda_+^F(a,\mu) = \inf_{n\to\infty} \log \|D\widehat{\alpha}(a^n)\Big| E^F(x)\| d\mu(x)$$

- $G_a^+ = \{g : \lim_{n\to-\infty} a^n g a^{-n} = e\}$, $G_a^- = \{g : \lim_{n\to\infty} a^n g a^{-n} = e\}$ with Lie algebras $\mathfrak{g}_a^\pm$

The main point (besides Ledrappier-Young [7] of course) of the following proof is to find an element for which all fiberwise exponents vanish. This observation is due to Hurtado and used in the Bourbaki notes of Cantat.

Proof. Let us assume now that μ A-ergodic on M^α. The proof of the theorem breaks down in the following steps.

- Since $A \simeq \mathbb{R}^{n-1}$ and $\#\{\text{fiberwise Lyapunov exponents } \lambda^F_{i,\mu}\} \leq \dim M \leq n-2$
$$\dim\left(\bigcap \ker(\lambda^F_{i,\mu})\right) \geq 1$$
- Hence $\exists\, a \in A$ such that $\lambda^F_{i,\mu}(a) = 0$ for all i and $F(x) < E^c_a(x)$ for μ-a.e. x
- Hence the (un)-stable manifolds of a are $E^\pm_a(x)$ correspond to $\mathfrak{g}^\pm_a$ over $G^\pm_a.x$
- We deduce that, writing β for the roots of G,
$$h_\mu(a \mid G^+_a) = \sum_{\beta(a)>0} \beta(a)$$
since

$$\textbf{RHS} =_{\textbf{Pesin}} h_{\text{Haar}}(a) \leq_{\textbf{factor}} h_\mu(a) =_{\textbf{Led-You}} \textbf{LHS} = h_\mu(a) \leq_{\textbf{Mar-Rue}} \textbf{RHS}$$

- By Ledrappier-Young, μ is G^+_a-invariant. Similarly G^-_a-invariant.
- The generated algebra $\mathfrak{f} = \langle \mathfrak{g}^+_a, \mathfrak{g}^-_a \rangle$ is the Auslander **ideal** associated to a and equal to $\mathfrak{g}$ since $\mathfrak{g}$ is simple. Hence μ is G-invariant.

□

2. Higher-rank invariance principle: Non-resonance implies invariance

We now give a stronger invariance principle using the concept of resonant roots that has been used in [3].

Definition 1. *A root β is* ***resonant with a fiberwise Lyapunov exponent*** *λ^F_μ if*
$$\beta \in \mathbb{R}^*_+ \lambda^F_\mu,$$
that is, they are coarsely equivalent. A root β is ***non-resonant*** *if it is not resonant with any fiberwise Lyapunov exponent λ^F_μ of μ.*

Let U^β denote the image the root subspace associated to a root β under the exponential map.

We have the following theorem from [5].

Theorem 2 (see Proposition 11.5 [2])**.** *Suppose μ is an ergodic, A-invariant measure on M^α projecting to the Haar measure on G/Γ. Then, for every non-resonant root β, the measure μ is U^β-invariant.*

We again have to recall some notation.

- Let $\beta = \beta$ root of $G = \mathrm{SL}_n(\mathbb{R})$. Let χ^β denote the coarse Lyapunov exponent for the A-action on (M^α, μ) coarsely equivalent to β and let $W^{\chi^\beta}(x)$ be the associated coarse Lyapunov manifold through x

- $\chi^\beta = \{\beta\}$ if β is not resonant with any λ_μ^F (equivalence class now written as set) and $W^{\chi^\beta}(x) = U^\beta(x)$ (otherwise intersects fibers non-trivially)
- $\mathcal{F}^{\chi^\beta}$ partition into $W^{\chi^\beta}(x)$ orbits with $C^{1+\beta}$-leaves
- If β resonant with some fiberwise Lyapunov exponent, let $\chi^{\beta,F}$ be the coarse fiberwise Lyapunov exponent. For β non-resonant, $\chi^{\beta,F} = 0$.
- $W^{\chi^{\beta,F}}(x)$ coarse fiberwise Lyapunov manifold contained in fiber of x. W^{χ^β} is U^β-orbit of $W^{\chi^{\beta,F}}(x)$
- For χ^β and $a \in A$ with $\beta(a) > 0$ define conditional entropy of a conditioned on χ^β-manifolds
$$h_\mu(a \mid \chi^\beta) = h_\mu(a \mid \mathcal{F}^{\chi^\beta}) = h_\mu(a \mid \xi)$$
where ξ measurable a-increasing partition subordinate to $\mathcal{F}^{\chi^\beta}$ and analogously define $h_\mu(a \mid \chi^{\beta,F})$ for $\chi^{\beta,F}$.

We shall also make fact of the following refinement of the Abramov-Rokhlin formula using the product formula of coarse Lyapunov exponents provided in a previous talk.

Theorem 3. *Let μ be an ergodic A-invariant measure on M^β that projects to the Haar measure on $\mathrm{SL}_n(\mathbb{R})/\Gamma$. For any $a \in A$ with $\beta(a) > 0$*
$$h_\mu(a \mid \chi^\beta) = h_{Haar}(a \mid \beta) + h_\mu(a \mid \chi^{\beta,F})$$
where $h_{Haar}(a \mid \beta) = h_{Haar}(a \mid U^\beta)$ the conditional entropy of a in $\mathrm{SL}_n(\mathbb{R})/\Gamma$ conditioned along U-orbits.

Using this formula, Theorem 2 follows by the same type of argument as above.

Proof of Theorem 2. Let β be a root and $a \in A$ such that $\beta(a) > 0$. Let $h_\mu(a|\beta)$ be the conditional entropy along U^β-orbits. By Margulis-Ruelle,
$$h_\mu(a|\beta) \leq \beta(a).$$
If β is non-resonant, $\chi^{\beta,F}$ vanishes identically, so that $W^{\chi^\beta}(x)$ is U^β-orbit of x. Hence, in the refined Abramov-Rohlin formula, $h_\mu(a|\chi^{\beta,F})$ vanishes. So
$$h_\mu(a|\beta) = h_{\mathrm{Haar}}(a|\beta) = \beta(a)$$
and $h_\mu(a|\beta)$ attains its maximal possible value. By the invariance principle of Ledrappier-Young, μ is U^β-invariant. □

3. A Measurability Criterion for Invariance

In [4], the proof of Zimmer's conjecture is extended to C^1-actions, using the following theorem.

Theorem 4. *Let $a \in A$ be $\mathbb{R}$-semisimple. Suppose μ is an a-invariant a-ergodic probability measure on M^α such that*

(1) *$\pi_*\mu$ is Haar*
(2) *all fiberwise Lyapunov exponents of Da are non-positive.*

The μ is G_a^+-invariant.

We start by taking a measurable a-increasing partition ξ on G/Γ subordinate to foliation of G_a^+. The first step is to construct a Borel trivialization which is adapted to ξ, in the sense of the following proposition.

Proposition 1. *The following holds*

- *There exists a Borel isomorphism* $\Psi : M^\alpha \to G/\Gamma \times M$ *and a map* F *on* $G/\Gamma \times M$ *such that* $F \circ \Psi = \Psi \circ \widehat{\alpha}(a^{-1})$*. Push* μ *by* Ψ*, call it* μ^**.*
- $x \mapsto F_x$ *is* ξ*-measurable*
- *The fiberwise Lyapunov exponents for* Da *w.r.t.* μ *are all non-positive if and only if all fiberwise Lyapunov exponents of* F *are are non-negative*
- μ *is* G_a^+*-invariant if and only if* $x \mapsto \mu_x^*$ *is* ξ*-measurable*

The theorem will then follow by applying the following theorem of [1], generalizing a result of Ledrappier [6] from linear to smooth cocycles.

Theorem 5. *Let* $\widehat{F} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ *be a smooth cocycle over* $\hat{f} : (\widehat{\Omega}, \widehat{m}) \to (\widehat{\Omega}, \widehat{m})$ *with fibers modeled over a Riemannian manifold* M *and fiberwise maps* $\widehat{F}_{\widehat{x}} : M \to M$*. Let* $\mathcal{B}_0 \subset \widehat{\mathcal{B}}_{\widehat{\Omega}}$ *be a generating* σ*-algebra such that both* $\widehat{f}$ *and* $x \mapsto \widehat{F}_x$ *are* $\mathcal{B}_0$*-measurable. Let* $\widehat{\mu}$ *be an* $\widehat{F}$*-invariant probability measure projecting down to* $\widehat{m}$*. If* $\lambda_-(\widehat{F}, \widehat{x}, \widehat{\xi}) \geq 0$ *for* $\widehat{\mu}$*-almost every* $(\widehat{x}, \widehat{\xi}) \in \widehat{\mathcal{E}}$ *then the disintegration* $\widehat{x} \mapsto \widehat{\mu}_{\widehat{x}}$ *of* $\widehat{\mu}$ *is* $\mathcal{B}_0$*-measurable.*

Here $\lambda_-(\widehat{F}, \widehat{x}, \widehat{\xi}) = \lim_{n\to\infty} \frac{1}{n} \log \|D\widehat{F}^n_{\widehat{x}}(\widehat{\xi})^{-1}\|^{-1}$ denotes the bottom extremal Lyapunov exponent. For precise assumptions on the fiber bundle and the cocycle, we have to defer to [1],

We note that for the application at hand $\mathcal{B}_0$ is generated by $\Psi\xi$, and it suffices to assume that $\mathcal{B}_0$ is $\hat{f} = a^{-1}$-decreasing (i.e. $\mathcal{B}_0$ refines $\widehat{f}^{-1}\mathcal{B}_0$) to get a well defined map $f : \Omega \to \Omega$, where $(\Omega, \mathcal{A}, m)$ is a Borel space for which there is a measurable projection $P : (\widehat{\Omega}, \widehat{m}) \to (\Omega, m)$ such that $P^{-1}\mathcal{A} = \mathcal{B}_0$, $P_*\widehat{m} = m$ and $f = P \circ \hat{f}$.

4. Ledrappier/Avila-Viana invariance principle

To digest Theorem 5, we shall elaborate on the disintegration part and state the key entropy estimate, following essentially Ledrappier's original argument [6]. Hence assume notation as in the Theorem 5 and the construction $P : \widehat{\Omega} \to \Omega$ and $f = P \circ \widehat{f}$ given right after it.

We have to introduce some further notation. To the cocycle $\widehat{F} : \widehat{\mathcal{E}} \to \widehat{\mathcal{E}}$ with fiberwise maps $\widehat{F}_{\widehat{x}} : M \to M$ we can associate a new cocycle F over f on $\mathcal{E} = \Omega \times M$ with fiberwise maps $F_x : M \to M$ that agree with $\widehat{F}_{\widehat{x}}$ which by assumption is $\mathcal{B}_0$-measurable, hence descends from $\widehat{\Omega}$ to Ω. We let $\mu = (P \times \mathrm{id})_*\widehat{\mu}$ be the F-invariant measure projecting to m under the projection $\pi : \mathcal{E} \to \Omega$. We also introduce $\widehat{\pi} : \widehat{\mathcal{E}} \to \widehat{\Omega}$. Let $\widehat{\mu}_{\widehat{x}}$ and μ_x be the corresponding disintegrations of $\widehat{\mu}$ and μ. The following lemma relates these conditional measures.

Lemma 1. *Write* $x(n) = P(\widehat{f}^{-n})(\widehat{x})$ *and* $F_x^n = F_{f^n(x)} \circ \cdots \circ F_x$*. Then for* $\widehat{m}$*-a.e.* $\widehat{x}$ *we have*

$$\widehat{\mu}_{\widehat{x}} = \lim_{n\to\infty} (F^n_{x(n)})_* \mu_{x(n)}$$

Proof. Let Ψ be a test function on $\widehat{\mathcal{E}}$, in fact we may assume it to only depend in the M-variable and let

$$\mathscr{E}_{\widehat{m}}(\widehat{\mu}_{\widehat{x}}(\Psi)|\hat{f}^n\mathcal{B}_0)$$

denote the conditional expectation of the function $\widehat{x} \to \widehat{\mu}_{\widehat{x}}(\Psi)$ with respect to the σ-algebra $\hat{f}^n\mathcal{B}_0$ on $\widehat{\Omega}$ with respect to $\widehat{m}$. Since $\hat{f}^n\mathcal{B}_0$ is an increasing sequence of σ-algebras by assumption, the conditional expectations define an increasing martingale and hence converge to $\mathscr{E}_{\widehat{m}}(\widehat{\mu}_{\widehat{y}}(\Psi)|\widehat{\mathcal{B}}_{\widehat{\Omega}})(\widehat{x}) = \widehat{\mu}_{\widehat{x}}(\Psi)$ for $\widehat{m}$-a.e. $\widehat{x}$.

Thus it suffices to show that $\mathscr{E}_{\widehat{m}}(\widehat{\mu}_{\widehat{y}}(\Psi)|\hat{f}^n\mathcal{B}_0)(\widehat{x})$ agrees with

$$(F^n_{x(n)})_*\mu_{x(n)}(\Psi) = \mathscr{E}_{\widehat{m}}((\widehat{F}^n_{\widehat{y}(n)})_*\widehat{\mu}_{\widehat{y}(n)}(\Psi)|\mathcal{B}_0)(\widehat{x})$$

where the last identity follows from the definitions, and we have put $\widehat{x}(n) = \hat{f}^{-n}(\widehat{x})$ and $\widehat{F}^n_{\widehat{x}}$ analogously defined as F^n_x. We also identified (by choice of Ψ), $\Psi \circ (P \times \mathrm{id})$ with Ψ).

This claim in turn will follow from $\widehat{F}$-invariance of $\widehat{\mu}$ and uniqueness of conditional measures, for example following along the lines

$$\int_A \mathscr{E}_{\widehat{m}}(\widehat{\mu}_{\widehat{y}}(\Psi)|\hat{f}\mathcal{B}_0)(\hat{f}(\widehat{x}))d\widehat{m}(\widehat{x}) = \int_{\hat{f}A} \mathscr{E}_{\widehat{m}}(\widehat{\mu}_{\widehat{y}}(\Psi)|\hat{f}\mathcal{B}_0)(\widehat{x})d\widehat{m}(\widehat{x}) = \int_{\hat{f}A} \widehat{\mu}_{\widehat{x}}(\Psi)d\widehat{m}(\widehat{x})$$

$$= \int_A \widehat{\mu}_{\hat{f}(\widehat{x})}(\Psi)d\widehat{m}(\widehat{x}) = \int_A (\widehat{F}_{\widehat{x}})_*\widehat{\mu}_{\widehat{x}}(\Psi)d\widehat{m}(\widehat{x}) = \int_A \mathscr{E}_{\widehat{m}}((\widehat{F}_{\widehat{y}})_*\widehat{\mu}_{\widehat{y}}(\Psi)|\mathcal{B}_0)(\widehat{x})d\widehat{m}(\widehat{x})$$

for $A \in \mathcal{B}_0$, where the second to last equality follows from $\widehat{F}$-invariance of $\widehat{\mu}$. □

The conclusion of the theorem will follow if we can show that $(F^n_{x(n)})_*\mu_{x(n)}$ is independent on n, i.e. equal to $\mu_{P(\widehat{x})}$, since this expression is $\mathcal{B}_0$-measurable. Equivalently, we wish to show that $(F_x)_*\mu_x = \mu_{f(x)}$. To capture this property, Ledrappier introduced

$$h = \int -\log \frac{(F_x^{-1})_*\mu_{f(x)}}{d\mu_x}(\xi)d\mu(x,\xi).$$

Lemma 2. *$h = 0$ iff $(F_x)_*\mu_x = \mu_{f(x)}$ for m-a.e. x.*

Proof. This follows from the equality case of Jensen inequality. □

Finally, the principle estimate is the following theorem of Ledrappier in the case of a linear cocycle $F_x \in \mathrm{GL}_d(\mathbb{R})$ with $M = \mathbb{P}(\mathbb{R}^d)$ and extremal Lyapunov exponents $\lambda_\pm$:

Theorem 6.

$$h \leq (d-1)(\lambda_+ - \lambda_-)$$

In his version of Theorem 5, he assumes $\lambda_+ = \lambda_-$ (instead of $\lambda_- \geq 0$) to conclude $\mathcal{B}_0$-measurability. We close this report in noting that the proof relies on geometric considerations using Besicovitch's covering lemma.

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Margulis super rigidity theorem

Elyasheev Leibtag

In this talk we state the Margulis super rigidity theorem (MSR), we give a brief introduction to the theory of linear algebraic groups which play a role in the statement of the theorem, and show how to use super rigidity in order to obtain dimension bounds for isometric actions.

The Margulis super rigidity theorem assures us that indeed in many cases lattices resemble their ambient group in the sense that any representation of the lattice extends to a group representation. i.e. let $\Gamma \leq G$ be a lattice, and let $\pi\colon \Gamma \to GL_n(V)$ be a homomorphism, then π extends to G.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & GL_n(V) \\ \big\downarrow & \nearrow_{\hat{\pi}} & \\ G & & \end{array}$$

The MSR is stated for Lie groups G that are connected components of real points of a linear algebraic group. We will provide a brief introduction of these groups.

Definition 1. *Let K be a algebraically closed field, a subgroup $\underline{H} \leq GL_n(K)$ is called a K-linear algebraic group if $\underline{H}$ is defined by polynomials in $K[x_{11}, ..., x_{ij}, ..., x_{nn}]$. We think of x_{ij} as a matrix entry and the elements of the groups are matrices where these polynomials are satisfied.*

Example 1. $\underline{SL_n} := SL_n(\mathbb{C}) = \{A \in GL_n(\mathbb{C}) \ : \ Det(A) = 1\}$.

Convention 1. *We add an under line notation for the algebraic group, this make it easy to take point in a sub-ring of the field as follows.*

Definition 2. *Let $\underline{H} \leq GL_n(K)$ be a K-linear algebraic group, let $R \leq K$ be a sub-ring, then we define the R-points of $\underline{H}$ to be $\underline{H}(R) := \underline{H} \cap GL_n(R)$.*
Notice that $\underline{H} = \underline{H}(K)$.

Plugging in points of a sub-ring $R \leq K$ gives us a topology on $\underline{H}(R)$ as a subset topology of $GL_n(R)$. we call it the Hausdorff topology of the R-points.

Example 2. *$\underline{SL_n}(\mathbb{Z})$, $\underline{SL_n}(\mathbb{R})$.*

It will be helpful to look at the $\mathbb{R}$-point of a $\mathbb{C}$-linear algebraic group. In order for $\underline{H}(\mathbb{R})$ to be useful we will need the group to be defined over $\mathbb{R}$.

Definition 3. *Let $\underline{H}$ be a K algebraic group, if the coefficients of the defining polynomials are in a field $k \subset K$ we say the group $\underline{H}$ is defined over k.*

The Lie groups we will consider will actually be real points of $\mathbb{C}$ algebraic groups defined over $\mathbb{R}$. Many classes of real Lie groups have an algebraic group structure, for example:

Example 3.
- *If G is connected semi-simple with trivial center, then there is a connected $\mathbb{C}$ algebraic group $\underline{G}$ defined over $\mathbb{R}$ (actually over $\mathbb{Q}$) such that $G \cong \underline{G}(\mathbb{R})^0$.*
- *(Chevalley,46) Every compact subgroup of $GL_n(\mathbb{C})$ is the group of real points of an algebraic group defined over $\mathbb{R}$.*

Regarding lattices in Lie groups there are a few famous results which are good to know and will be useful for our talk:

- (Borel-Harish Chandra) If $\underline{G}$ is a semi-simple algebraic group defined over $\mathbb{Q}$ then $\underline{G}(\mathbb{Z})$ is a lattice in $\underline{G}(\mathbb{R})$.
- (Borel density) Lattices in semi-simple Lie groups are Zariski dense.
- (Kashdan) Lattices are finitely generated.

The theory of linear algebraic groups lets us use maps between fields in order to obtain maps between groups in the following way.
Let $\underline{H}$ be a K linear algebraic group defined over k and let $\varphi\colon k \to k'$ be a field morphism, then we may assume φ maps into $\overline{k'}$ the algebraic closure, and we get a map

$$\varphi_* : H(k) \to GL_n(\overline{k'})$$

Definition 4. *With the data above denote by $\underline{H}_{\varphi}$ the Zariski closure of $\varphi_*\underline{H}(k)$ in $GL_n(\overline{k'})$ and denote by $\underline{H}_{\varphi}(k')$ its k' points.*

Remark 1. *The polynomials defining the group $\underline{H}_{\varphi}(k')$ are obtained from the ones defining $\underline{H}$ by action of φ on the coefficients.*

A useful lemma regarding maps between fields is the following:

Lemma 1. [2] *Let S be a finitely generated integral domain, and $I \subset S$ an infinite set. Then there exist a local field k and a map $\varphi\colon S^{-1}S \hookrightarrow k$ such that the image of I is unbounded*

Now that we have the language of algebraic groups we may ask questions about algebraic properties of lattices.

Theorem 1. [1, 3.1.8] *Let $\underline{H} \leq GL_n(K)$ be an algebraic group, such that $\underline{H}(k)$ is Zariski-dense, then $\underline{H}$ is defined over k.*

A useful corollary of this theorem is one that lets us use the fact that lattices are Zariski dense and finitely generated to get that a semi-simple algebraic group is "defined over its lattice".

Corollary 1. *If $\Lambda \leq \underline{G}$ is finitely generated and Zariski-dense, then the matrix elements of Λ span a finitely generated ring R_Λ such that the algebraic group is defined over its field of fractions F_Λ.*

We are now ready to state the Margulis super rigidity theorem.

Theorem 2. [1, 5.1.2] *Let $\underline{G}$ a algebraic group defined over $\mathbb{R}$ with $Rank_{\mathbb{R}}(\underline{G}) > 1$. Assume $\underline{G}$ is connected semi-simple and $G = \underline{G}(\mathbb{R})$ has no compact factors. Let $\Gamma \leq G^0(\mathbb{R})$ be an irreducible lattice. Let k be a local field of characteristic 0 and let $\underline{H}$ be a connected simple algebraic group defined over k.*
Assume $\pi : \Gamma \to \underline{H}(k)$ is a homomorphism with $\pi(\Gamma)$ Zariski-dense.
Then either $\pi(\Gamma)$ is bounded in $\underline{H}(k)$ (for the Hausdorff topology) or there exist an extension $\hat{\pi} : \underline{G} \to \underline{H}$ defined over k.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \pi(\Gamma) \\ \big\downarrow & & \big\downarrow \\ G & \xrightarrow{\hat{\pi}} & H(k) \end{array}$$

Remark 2. *If k is totally disconnected, this extension can not exist and thus the theorem implies that the image is always bounded.*

A straightforward corollary is a special case of the Mostow rigidity theorem for higher rank Lie groups.

Theorem 3 (Higher rank Mostow)**.** [1, 5.1.1] *Let G and G' be semi-simple Lie groups with trivial center and no compact factors, suppose Γ, Γ' are lattices in G, G' respectively.*
Assume $Rank_{\mathbb{R}}(G) > 1$ and that Γ is an irreducible lattice, then any isomorphism $\pi : \gamma \to \gamma'$ extends to an isomorphism $G \to G'$.

Let M be a compact manifold. It is well known that non-compact simple groups can not act on M by isometries since $Iso(M)$ is compact.
This following corollary of MSR which is used in the proof the the Zimmer conjecture sates that for actions of lattices in simple lie groups there exist dimension bounds for isometric actions on M. For example we get that if we have a homomorphism $\Phi : SL_n(\mathbb{Z}) \to Iso(M)$ and $\dim(M) < n$ then $|\Phi(SL_n(\mathbb{Z}))| < \infty$ (for $n > 2$).

Theorem 4. *Let G be a connected simple non compact group with higher rank, if Γ is a lattice action on a compact n dimensional manifold M, with $\Gamma \to Iso(M)$ infinite image then* $\dim(G) \leq \frac{n(n+1)}{2}$

Proof. By fact stated above since G is connected and simple there exist a $\mathbb{C}$-linear algebraic group $\underline{G}$ with $G = \underline{G}(\mathbb{R})^0$, let $\pi : \Gamma \to Iso(M)$ be the homomorphism

induced by the action map. By Chevalley [3] $Iso(M)$ is real points of an algebraic group so we may denote $\underline{H}(\mathbb{R}) := Iso(M)$ and $\Lambda := \pi(\Gamma)$. As there can not exist an extension of π to G the set Λ it is bounded, also note that it is a finitely generated Zariski-dense subgroup. By 1 $\underline{H}$ is defined over F_Λ which is the field generated by $I_\Lambda := \{\lambda_{ij} \mid \lambda \in \Lambda\}$. Following 1 there exist a field morphism $\varphi : F_\Lambda \to k$ into a local field k such that the image of I_Λ is not bounded.
Hence we get a map $\varphi_* : \underline{H}(F_\Lambda) \to \underline{H}_\varphi(k)$ such that $\varphi_*(\Lambda)$ is unbounded. Assume w.l.o.g that $\underline{H}_\varphi(k)$ is simple (we can project to a simple factor), hence by MSR 2 we get a map from G to $\underline{H}_\varphi(k)$ i.e.

$$\begin{array}{ccc} \Gamma & \xrightarrow{\pi} & \underline{H}(F_\Lambda) \\ \big\downarrow & \searrow^{\rho} & \big\downarrow \varphi_* \\ G & \xrightarrow{\hat{\rho}} & \underline{H}_\varphi(k) \end{array}$$

By simplicity of G the map $\hat{\rho}$ is injective, and so by theory of dimensions of linear algebraic groups we get that

$$\dim_{\mathbb{R}}(G) = \dim_{alg}(\underline{G}) \leq \dim_{alg}(\underline{H}_\varphi) = \dim_{alg}(\underline{H}) = \dim_{\mathbb{R}} Iso(M)$$

It is well known that for a n dimensional manifold $\dim\left(Iso(M)\right) \leq \frac{n(n+1)}{2}$ thus we get that $\dim_{\mathbb{R}}(G) \leq \frac{n(n+1)}{2}$. □

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Super-rigidity for cocycles

Vincent Pecastaing

The aim of this lecture is to state Zimmer's generalization of Margulis' super-rigidity theorem and to explain how it can be used to understand the dynamics of probability measure preserving differentiable actions of higher-rank lattices.

Even though super-rigidity results work in greater generality, we will only consider real Lie groups and real algebraic groups, which is what we need in the proof of Zimmer's conjecture.

1. Cocycles over a measurable group action

Let G be a real semi-simple Lie group all of whose simple factors are of real-rank at least 2. Let H be a real algebraic group. Let (X, μ) be a Borel probability space on which G acts measurably. An *H-valued cocyle* over the action $G \curvearrowright X$ is a measurable map $c : G \times X \to H$ such that for all $g_1, g_2 \in G$ and for μ-almost every $x \in X$,

$$c(g_1 g_2, x) = c(g_1, g_2.x)c(g_2, x). \tag{1.1}$$

For instance, if X is a smooth manifold, μ is the Lebesgue measure and $G \curvearrowright X$ is differentiable, then, given a measurable trivialization $TX \simeq X \times \mathbf{R}^n$, $n = \dim X$, the derivative $D_x g$ of an element $g \in G$ is identified with a linear map $c(g, x) \in \mathrm{GL}(n, \mathbf{R})$. Thus, $c : G \times X \to \mathrm{GL}(n, \mathbf{R})$ is a cocycle - (1.1) being the translation of the chain rule -, called the derivative cocyle. More generally, if G acts linearly on a vector bundle $E \to X$ of rank n, any measurable trivialization of E yields a measurable cocycle $G \times X \to \mathrm{GL}(n, \mathbf{R})$.

If c is an H-valued cocycle over a G-action on (X, μ), then we build from c an action of G on the trivial H-principal bundle $B = X \times H$ via $g.(x, h) = (g.x, c(g, x)h)$. This action is by bundle automorphisms in the sense that it commutes with the right-action of H on B. Conversely, any action of G on B by automorphisms arises in this way, showing that cocycles are the same as G-actions on the principal bundle B. Given a fixed G-action on B, another trivialization $B \simeq X \times H$ gives rise to a new cocycle c' related to c via

$$c'(g, x) = \varphi(g.x)^{-1} c(g, x)\varphi(x) \ \mu - \text{a.e}, \tag{1.2}$$

where $\varphi : X \to H$. Two cocyles satisfying (1.2) for some φ are said to be cohomologous.

2. Statements of super-rigidity for cocycles

A cocycle c is said to be ρ-constant if there exists a Lie group homomorphism $\rho : G \to H$ such that μ-almost everywhere, $c(g, x) = \rho(g)$. A cocycle $c : G \times X \to H$ is said to be G-integrable if for all compact subset $K \subset G$, the map $x \mapsto \sup_{g \in K} \log^+ \|c(g, x)\|$ is in $L^1(X, \mu)$. We can now state the following form of cocycle super-rigidity theorem ([3], Theorem 1.4).

Theorem 1. *Let G be a simply-connected semi-simple Lie group all of whose simple factors are of real-rank at least 2. Let H be a real algebraic group. Let $G \curvearrowright (X, \mu)$ be a measure preserving, ergodic action of G on a probability space (X, μ). Let $c : G \times X \to H$ be a G-integrable cocycle.*

Then, there exists $\rho : G \to H$ a Lie group homomorphism, $K < H$ a compact Lie subgroup centralizing $\rho(G)$ and $c_K : G \times X \to K$ a K-valued cocycle such that c is cohomologous to ρc_K, i.e. there exists $\varphi : X \to H$ mesurable such that for all $g \in G$ and for μ-almost every $x \in X$,

$$c(g, x) = \varphi(g.x)^{-1} \rho(g) c_K(g, x)\varphi(x).$$

The super-rigidity theorem for cocycles says that if G acts on X by preserving a finite measure μ, then c is cohomologous to a ρ-constant cocycle, up to a "compact noise". We stress that the important hypothesis here is the existence of a G-invariant measure μ. This is not guaranteed in general. For instance, $X = G/P$ with P a non-trivial parabolic subgroup has no finite G-invariant measure.

Remark 1. *We note that this version, which applies naturally to any cocycle over a probability measure preserving action of G, is not the same as the initial formulation of Zimmer. Analogously to Margulis' super-rigidity theorem (for instance Theorem 5.1.2 in* [6]*), for which the image of a homomorphism $\rho : \Gamma \to H$ is assumed to be Zariski-dense in a non-compact simple algebraic group H, a standard version of cocycle super-rigidity - Theorem 5.2.5 in* [6] *- makes the assumption that H is simple and that the algebraic hull of the G action on $B = X \times H$ equals H. This means that there is no proper algebraic subgroup $H' < H$ such that G preserves a measurable H'-reduction of B. With this additional assumption, there is no compact perturbation c_K in the conclusion. However, it is a priori difficult to prove that a given action satisfies this assumption.*

An analogous version of Margulis' super-rigidity theorem follows from this result, without the Zariski density assumption. Indeed, if $\Gamma < G$ is a lattice and $\rho : \Gamma \to H$ is a homomorphism into $H = \mathrm{GL}(d, \mathbf{R})$, then we can form the associated principal H-bundle $P^\rho \to X := G/\Gamma$ defined by $P^\rho = (G \times H)/\Gamma$ where Γ acts via $(g, h).\gamma = (g\gamma, \rho(\gamma^{-1})h)$. The natural (left) action of G on P^ρ is by bundle automorphisms, and projects to the action $G \curvearrowright G/\Gamma$ which is volume-preserving. By Theorem 1, there exists $\sigma : G/\Gamma \to P^\rho$ a measurable section such that for all $g \in G$ and for μ-almost every $x \in G/\Gamma$

$$g.\sigma(x) = \sigma(g.x).\bar{\rho}(g)c_K(g, x).$$

If $x_0 = g_0\Gamma$ is in the set of full measure where the above relation holds, we can consider the "isotropy representation" of $g_0\Gamma g_0^{-1} \to H$. First, the cocycle identity says that $\rho_K : \gamma \mapsto c_K(g_0\gamma g_0^{-1}, x_0)$ is a K-valued homomorphism. Then, if $\sigma(x_0)$ is of the form $[(g_0, h_0)]$, on the one hand $g_0\gamma g_0^{-1}\sigma(x_0) = [(g_0\gamma, h_0)] = [(g_0, \rho(\gamma)h_0)]$, and on the other hand $g_0\gamma g_0^{-1}\sigma(x_0) = \sigma(x_0)\bar{\rho}(g_0\gamma g_0^{-1})\rho_K(\gamma) = [(g_0, h_0\bar{\rho}(g_0\gamma g_0^{-1})\rho_K(\gamma))]$. If we note $\bar{\rho}_0(g) = h_0\bar{\rho}(g_0 g g_0^{-1})h_0^{-1}$ and $\rho_C = h_0\rho_K h_0^{-1}$, we obtain that $\rho(\gamma) = \bar{\rho}_0(\gamma)\rho_C(\gamma)$ for all $\gamma \in \Gamma$, with ρ_C compact-valued, centralized by $\bar{\rho}_0(G)$.

We also have a similar version of cocycle super-rigidity for lattices (Theorem 1.5 of [3]).

Theorem 2. *Let Γ be a lattice in a simply-connected semi-simple Lie group G all of whose simple factors are of real-rank at least 2. Let H be a real algebraic group. Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving, ergodic action of Γ on a probability space (X, μ). Let $c : \Gamma \times X \to H$ be a Γ-integrable cocycle.*

Then, there exists $\rho : G \to H$ a Lie group homomorphism, $K < H$ a compact Lie subgroup centralizing $\rho(G)$ and $c_K : \Gamma \times X \to K$ a K-valued cocycle such that c is cohomologous to $\rho|_\Gamma c_K$.

3. Some applications

3.1. **Measurable invariant Riemannian metrics.** From Theorem 2, we can deduce an interesting observation which was one of the main motivations for Zimmer's conjectures (see Section 4 of [5]). Let Γ be as in this last result, let M^n be a compact manifold, endowed with a volume form ω and let $\Gamma \to \mathrm{Diff}(M,\omega)$ be a volume-preserving action. Let $\mathcal{F}^1(M) \to M$ denote the linear frame bundle of M. As M is compact, ω defines a finite Γ-invariant measure on M. Considering an ergodic component, we can apply super-rigidity.

Assume moreover that the Lie algebra $\mathfrak{g}$ does not embed into $\mathfrak{h} = \mathfrak{gl}(n,\mathbf{R})$, for instance $G = \widetilde{\mathrm{SL}}(m,\mathbf{R})$ with $m > n$. Then, the homomorphism ρ in the conclusion of Theorem 2 has to be trivial. Thus, super-rigidity says that the derivative cocycle of the action of Γ is cohomologous to a compact valued cocycle. It means that there exist a compact subgroup $K < \mathrm{GL}(n,\mathbf{R})$ and a *measurable* section $\sigma : M \to \mathcal{F}^1(M)$ such that Γ preserves the measurable reduction $\sigma(M).K \subset \mathcal{F}^1(M)$. In general, an $\mathrm{O}(n)$-reduction of $\mathcal{F}^1(M)$ is the same as a Riemannian metric g on M (an $\mathrm{O}(n)$-reduction is the same as a $\mathrm{GL}(n,\mathbf{R})$-equivariant map $\mathcal{F}^1(M) \to \mathrm{GL}(n,\mathbf{R})/\mathrm{O}(n)$, and the latter is the space of positive-definite inner products on $\mathbf{R}^n$). As Γ preserves the reduction $\sigma(M).K_{max}$ where K_{max} is maximal compact containing K, we deduce that Γ acts on M by preserving a *measurable* Riemannian metric g.

If g is regular enough, say smooth, then Myers-Steenrod theorem implies that its isometry group is a compact Lie group. In this situation, we get that the action factorizes through $\Gamma \to \mathrm{Is}(M,g) \to \mathrm{Diff}(M,\omega)$, with $\dim \mathrm{Is}(M,g) \leq n(n+1)/2$. It follows that the action factorizes through a finite group.

So, Zimmer's conjecture for volume-preserving actions reduces to proving that this metric is regular enough. It turned out that improving directly the regularity of g is very difficult.

3.2. **Lyapunov spectrum.** In the last example, we have applied super-rigidity to deduce that the derivative cocycle is contained in a compact group when there is no homomorphism $\mathfrak{g} \to \mathfrak{sl}(n,\mathbf{R})$. We finish by giving another nice illustration of cocycle super-rigidity (see Section 10.7 of [2]).

Let Γ act linearly on a vector bundle $E \to M$ of rank d, for instance $E = TM$, by preserving an ergodic probability measure μ on M. Then, there is $\rho : G \to \mathrm{GL}(d,\mathbf{R})$ such that for any γ, μ-almost everywhere, the Lyapunov spectrum of γ with respect to μ coincides with the logarithm of the (matrix) spectrum of the $\mathbf{R}$-split component $\rho(\gamma)_h$ in the Jordan decomposition of $\rho(\gamma)$.

To see it, we apply cocycle super-rigidity to the Γ-action on the ($\mathrm{GL}(d,\mathbf{R})$-principal) frame bundle $\mathcal{F}(E) \to M$. We get $\sigma : M \to \mathcal{F}(E)$ a measurable frame field, $\rho : G \to \mathrm{GL}(d,\mathbf{R})$, $K < \mathrm{GL}(d,\mathbf{R})$ compact centralizing $\rho(G)$ and $c_K : \Gamma \times M \to K$ such that for all $\gamma \in \Gamma$ and for μ-almost every $x \in M$,

$$\gamma.\sigma(x) = \sigma(g.x).\rho(\gamma)c_K(\gamma, x).$$

For all x, $\sigma(x) : \mathbf{R}^n \to E(x)$. Fixing an arbitrary Riemannian metric on E, if $\rho(\gamma)_h v = \lambda v$, then we have $\|\gamma^k \sigma(x) v\| = |\lambda|^k \|\sigma(\gamma^k.x)\rho(\gamma)_u^k v_k\|$ where $\rho(\gamma)_u$ is the unipotent Jordan component and v_k lies in a compact subset of $\mathbf{R}^n \setminus \{0\}$. If we had that $\|\sigma(x)\|$ and $\|\sigma(x)^{-1}\|$ are bounded over M, then we would conclude because $\frac{1}{k}\log|\rho(\gamma)_u^k v_k| \to 0$. If the frame field has no reason to be bounded in general, we can nonetheless conclude by using the fact that $\mu\{x : \|\sigma(x)\| \leq n \text{ and } \|\sigma(x)^{-1}\| \leq n\}$ goes to 1 and using Poincaré's recurrence theorem.

In particular, if d is such that there is no non-trivial $\rho : G \to \mathrm{GL}(d, \mathbf{R})$, then all Lyapunov exponents of any element γ with respect to any Γ-invariant measure on M are 0. This is used in [1] to prove by contradiction that in low dimension, a differentiable action of Γ always has uniform subexponential growth of derivatives.

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Proof of Zimmer's Cocycle Superrigidity: ergodicity and Lyapunov exponents

Thang Nguyen

Zimmer's cocycle superrigidity is a generalization of Margulis superrigidity. In the proof of Zimmer's conjecture, it is used to deduce the property uniform subexponential growth derivatives for action of lattices on manifolds. The proof of Zimmer's conjecture also use Margulis superrigidity to obtain the finiteness of the action once we know the lattices act by isometry. The purpose of this talk and the next one is prove the Zimmer's cocycle superrigidity theorem.

We have the Zimmer's cocycle superrigidity theorem as follows.

Theorem 1. [Zim84, FM03] *Let G be a connected semisimple Lie group with no compact factors and all simple factors of real rank at least two. Further, assume G is simply connected as a Lie group or simply connected as an algebraic group. Let V be a vector space over a local field k. Let (X, μ) be a standard probability measure space. Assume that G acts ergodically on X preserving μ. Let $\alpha : G \times X \to GL(V)$ be a G-integrable Borel cocycle. Then α is cohomologous to a cocycle β where $\beta(g, x) = \pi(g)c(g, x)$. Here $\pi : G \to GL(V)$ is a continuous homomorphism and $c : G \times X \to GL(V)$ is a cocycle taking values in a compact group centralizing $\pi(G)$.*

The strategy to prove Theorem 1 is as follows. We consider the space of measurable section $F(S, End(V))$. There is a natural induced action of G on

$F(S, End(V))$. Theorem 1 will follows if there is a proper finite dimensional G-invariant sections. Thus, the proof is divided into two main steps. The first one is showing the existence of a proper finite dimensional A-invariant sections, where A is the Cartan subgroup of G. The second main step is constructing G-invariant sections from A-invariant sections. In this talk, we present the proof of the first main step. On the other hand, in the proof second step, the ergodicity of subgroup actions is needed. Hence we also present the Howe-Moore's ergodicity Theorem in this talk.

The organization of this talk is as follows. First, we state and prove the Howe-Moore's ergodicity Theorem. We prove in full details in some special cases and sketch ideas in more general cases. After that, we prove the first main step in proof of cocycle superrigidity.

1. Ergodicity Theorem

Now, we have the statement the Howe-Moore's ergodicity Theorem.

Theorem 2 (Howe-Moore's ergodicity Theorem). [HM79] *Let G be a connected semisimple Lie group with no compact factors and all simple factors have finite centers. Let (X, μ) be a standard probability measure space that G acts on preserving μ. Assume that G-action is irreducibly ergodic. If H is a closed non-compact subgroup of G then H acts ergodically on S.*

Remark 1. *We note that the ergodicity of an action of a group H on (X, μ) is equivalent with the only invariant vectors in the unitary representation of H on $L^2(X, \mu)$ are constant functions.*

We consider some special cases first.

Proof of Theorem 2 for $G = \mathrm{SL}(2, \mathbb{R})$ and $H = A = \left\{ \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} : \lambda \in \mathbb{R} \right\}$. For every $t, \lambda \in \mathbb{R}$, we have the following relation

$$\begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix} = \begin{pmatrix} 1 & ue^{2\lambda} \\ 0 & 1 \end{pmatrix}.$$

We write $a_\lambda = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}$, and $u_t = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$. The the above relation can be rewritten as $a_\lambda u_t a_{-\lambda} = u_{te^{2\lambda}}$. Thus $a_\lambda u_t a_{-\lambda} \to 1$ as $\lambda \to -\infty$. Suppose $v \in \mathcal{H}$ is an A-invariant vector. Let $U = \{u_t : t \in \mathbb{R}\}$ be the upper triangular unipotent group. Then the fact that v is U-invariant follows from the following lemma.

Lemma 1 (Mautner phenomenon). *Let $(\rho, \mathcal{H})$ is a continuous unitary representation of a topological group G. Suppose $x, y \in G$ be the elements such that $x^n y x^{-n} \to 1$ as $n \to \infty$. If $v \in \mathcal{H}$ such that $\rho(x)v = v$, then $\rho(y)v = v$.*

Proof. Since ρ is continuous and unitary,

$$\|\rho(y)v - v\| = \|\rho(yx^{-n}v - \rho(x^{-n})v\| = \|\rho(x^n yx^{-n}v - v\| \to 0,$$

as $n \to \infty$. It follows that $\rho(y)v = v$. □

By using a similar relation for lower triangular unipotent subgroup, we deduce that v is also invariant under lower triangular unipotent subgroup. Since A, U and the lower triangular unipotent subgroup generate $\mathrm{SL}(2,\mathbb{R})$, the vector v must be G-invariant. The ergodicity now follows from Remark 1. $\square$

Proof of Theorem 2 for $G = \mathrm{SL}(2,\mathbb{R})$ and $H = U = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} : t \in \mathbb{R} \right\}$.

For every $\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R})$ with $b \neq 0$, we have the following relation

$$\begin{pmatrix} 1 & b^{-1}(1-a) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \begin{pmatrix} 1 & b^{-1}(1-a^{-1}) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}.$$

Let $(\rho, \mathcal{H})$ is a continuous unitary representation of G. Suppose $v \in \mathcal{H}$ is an U-invariant vector. It follows that 33

$$\langle \rho \left(\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} \right) v, v \rangle = \langle \rho \left(\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right) v, v \rangle.$$

Letting $b \to 0$, we get that v is A-invariant. Hence, from Mautner phenomenon, v is G-invariant. By Remark 1, H-action is ergodic. $\square$

In general, Theorem 2 follows from the following vanishing matrix coefficient theorem.

Theorem 3. *Let G be as is Theorem 2. Let $(\rho, \mathcal{H})$ be a continuous unitary representation of G such that all factors of G have no invariant vector. Then for every non-zero vectors $v, w \in \mathcal{H}$, the matrix coefficients vanish at infinity, i.e. $\langle \rho(g)v, w \rangle \to 0$ as g leaves compact subsets of G.*

The idea to prove Theorem 3 is using Cartan decomposition to reduce the problem to proving vanishing of matrix coefficient along Cartan subgroups.

2. Non-vanishing top Lyapunov exponent and A-invariant section

Now we move to the first main step in the proof of Theorem 1. The idea of this main step, after a reduction making cocycle taking value in $\mathrm{SL}(V)$, is showing that the top Lyapunov exponent of a element in the Cartan subgroup is non-zero. From there, we can construct a section that are invariant under action of the Cartan subgroup from projections onto the top Lyapunov spaces.

For simplicity, we can assume the cocycle takes values in $\mathrm{SL}(V)$ instead of in $\mathrm{GL}(V)$.

Definition 1. *Let G act on (S, μ) preserving μ. Suppose $\alpha : G \times X \to \mathrm{SL}(V)$ is a cocycle. An algebraic k-group $H < \mathrm{SL}(V)$ is called the algebraic hull of the cocycle α if α is cohomologous to a cocycle taking values in H but not cohomologous to any cocycle taking values in proper algebraic subgroup of H.*

We remark that algebraic hull of a cocycle is unique up to conjugate.

If the cocycle α takes values in H, then there is an induced representation $(\rho, L^2(X, L^2(H)))$, defined as follows,

$$\rho(g)v(x)(h) = v(g^{-1} \cdot x)(\alpha(g^{-1}, x)h).$$

This is a unitary representation of G.

We have the following key lemma regarding this induced representation.

Lemma 2. [Zim78, FM03] *Assume G acts on S preserving probability measure μ. Let $c : G \times X \to \mathrm{SL}(V)$ be an integrable cocycle that is not cohomologous to a compact valued cocycle. Let H be the algebraic hull of the cocycle c. If H is semisimple then the unitary representation of G on $L^2(S, L^2(H))$ does not almost have invariant vectors.*

From now on, for the sake of expository, we always assume that the algebraic hull of the cocycle is seimisimple. We have the following key proposition.

Proposition 1. *Let G, X, V and α be as in Theorem 1. Assume that the cocycle is not cohomologous to one taking compact values. Then there is an element $a \in A$ such that $\langle a \rangle$ acts ergodically on X and the top Lyapunov exponent $\lambda_1(a, \mu) > 0$.*

We recall that the top Lyapunov exponent is defined as follows

$$\lambda_1(a, \mu) = \lim_{n\to\infty} \frac{1}{n} \int_X \log \|\alpha(a^n, x)\| d\mu(x).$$

Proof. The ergodicity conclusion follows immediately from Howe-Moore's ergodicity Theorem. So we only need to prove the claim about non-zero top Lyapunov exponents.

First we show that there is a sequence $g_n \in G$ with $|g_n| = \mathcal{O}(n)$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \int_X \log \|\alpha(g_n, x)\| d\mu(x) > 0.$$

Indeed, by Lemma 2, there is a compact support probability measure f on G with $supp(f)$ generates G and there exists $0 < \delta < 1$ such that for every $v \in L^2(S, L^2(H))$, we have

$$\langle \rho(f)v, v\rangle < \delta \langle v, v\rangle.$$

It follows that

$$\langle \rho(f^{*n})v, v\rangle < \delta^n \|v\|^2,$$

for every $n \in \mathbb{N}$. Since $\rho(f^{*n})$ is an averaging, we deduce that

$$\inf_{g\in supp(f^{*n})} \langle \rho(g)v, v\rangle < \delta^n \|v\|^2,$$

for every $n \in \mathbb{N}$. Hence,

$$\limsup_{n\to\infty} \frac{1}{n} \inf_{g\in supp(f^{*n})} \log\langle \rho(g)v, v\rangle \leq \log \delta < 0.$$

By [Mar91, Lemma V.1.3], there is $d \in \mathbb{N}$ such that the function $b(h) = \|h\|^{-d}$, $h \in H$, belongs to $L^2(H)$, where the norm here is a word norm associated to a compact generating set of H. We consider $v \in L^2(X, L^2(H))$ defined by $v(x) = b$ for a.e. $x \in X$. We have the following estimate

$$\begin{aligned}
\langle \rho(g)v, v\rangle &= \int_X \langle (\rho(g)v)(x), v(x)\rangle_{L^2(H)} d\mu(x) \\
&= \int_X \int_H (\rho(g)v)(x)(h) \cdot v(x)(h) dh d\mu(x) \\
&= \int_X \int_H \|\alpha(g^{-1}, x)h\|^{-d} \cdot \|h\|^{-d} dh d\mu(x) \\
&\geq \int_X \int_H \|\alpha(g^{-1}, x)\|^{-d} \|h\|^{-d} \cdot \|h\|^{-d} dh d\mu(x) \\
&= \int_X \|\alpha(g^{-1}, x)\|^{-d} \|b\|^2_{L^2(H)} d\mu(x).
\end{aligned}$$

Thus, taking logarithm on both sides, we get

$$\begin{aligned}
\log\langle \rho(g)v, v\rangle &\geq \log\left(\int_X \|\alpha(g^{-1}, x)\|^{-d} \|b\|^2_{L^2(H)} d\mu(x)\right) \\
&\geq -d \int_X \log \|\alpha(g^{-1}, x)\| d\mu(x) + \log \|b\|^2_{L^2(H)}.
\end{aligned}$$

Therefore,

$$\limsup_{n\to\infty} \frac{1}{n} \inf_{g\in supp(f^{*n})} \left(-d \int_X \log \|\alpha(g^{-1}, x)\| d\mu(x) + \log \|b\|^2_{L^2(H)} \right) \leq \log \delta < 0.$$

It follows that

$$\limsup_{n\to\infty} \frac{1}{n} \inf_{g\in supp(f^{*n})} - \int_X \log \|\alpha(g^{-1}, x)\| d\mu(x) \leq \frac{1}{d} \log \delta < 0.$$

Hence there is a sequence of $g_n \in supp(f^{*n})$ such that

$$\limsup_{n\to\infty} \frac{-1}{n} \int_X \log \|\alpha(g_n^{-1}, x)\| d\mu(x) \leq \frac{1}{2d} \log \delta < 0.$$

We can assume that $supp(f^{*n})$ is symmetric, and thus can interchange between g_n and g_n^{-1}. As a result, we get

$$\limsup_{n\to\infty} \frac{1}{n} \int_X \log \|\alpha(g_n, x)\| d\mu(x) > 0.$$

Now we refine this claim using Cartan decomposition $G = KAK$. We write $g_n = k_n a_n k_n'$ for every $n \in \mathbb{N}$. Thus,

$$\begin{aligned}
\frac{1}{n} \int_X \log \|\alpha(g_n, x)\| d\mu(x) <& \frac{1}{n}\Bigg(\int_X \log \|\alpha(g_n, x)\| d\mu(x) \\
&+ \int_X \log \|\alpha(k_n, x)\| d\mu(x) + \int_X \log \|\alpha(k_n', x)\| d\mu(x)\Bigg).
\end{aligned}$$

Because $\frac{1}{n}\left(\int_X \log \|\alpha(k_n, x)\| d\mu(x) + \int_X \log \|\alpha(k'_n, x)\| d\mu(x) \right) \to 0$ as $n \to \infty$, we have that

$$\limsup_{n\to\infty} \frac{1}{n} \int_X \log \|\alpha(a_n, x)\| d\mu(x) > 0,$$

where $a_n \in A$ and $\|a_n\| = \mathcal{O}(n)$.

Since A is isomorphic to $\mathbb{R}^m$ for some m, we can pick a basis $\{b_1, \ldots, b_m\}$for A. Any element in A now can be written as a linear combination of vectors $b_1, \ldots, b_m$ with integer coefficients, up to compact errors. Hence, it follows that there exists $1 \le i \le m$ and $n_i = \mathcal{O}(n)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \int_X \log \|\alpha(b_i^{n_i}, x)\| d\mu(x) > 0.$$

Therefore, letting $a = b_i$, the proof is complete. □

As a consequence of the proposition, we can deduce the main goal of the first half of proof of Theorem 1 as follows.

Corollary 1. *Let G, X, V and α be as in Theorem 1. Assume that the cocycle is not cohomologous to one taking compact values. Then there is a measurable A-invariant of the End(V)-bundle over X.*

Proof. Let $a \in A$ is the element in the conclusion of Proposition 1. We note that since α takes values in $\mathrm{SL}(V)$, the fact that $\lambda_1(a, \mu) > 0$ implies the top Lyapunov space of a is a proper subspace $V_1(x)$ of V for a.e. $x \in X$.

We recall that given a cocycle $\alpha : G \times X \to \mathrm{SL}(V)$, we can associate a bundle $X \times End(V)$. The group G acts the space of sections of this bundle as follows

$$(g \cdot f)(x) = \alpha(g^{-1}, x) f(g^{-1}x),$$

for every measurable section f of bundle $X \times End(V)$.

Now we consider the section f defined as $f(x) = proj_{V_1(x)} : V \to V_1(x)$ for a.e. $x \in X$. This section is invariant under $\langle a \rangle$-action. Since A is abelian, this section is also A-invariant. □

This complete the first half of the proof of Theorem 1. In the next talk by Michele Triestino, we will show the invariant subspace of sections is of finite-dimensional, and construct a finite-dimensional subspace that is G-invariant from A-invariant one.

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Proof of Zimmer's Cocycle Superrigidity: centralizers and finite dimensional invariant subspaces

Michele Triestino

Following the strategy of [3], a key step in the proof of Zimmer's Cocycle Superrigidity is to produce a finite dimensional representation of the Lie group G, from which the desired morphism $G \to H$ is obtained.

1. Finite-dimensional invariant subspaces

1.1. **Dimension bound from ergodicity.** Let k be a local field and let W be a vector space over k. Let H be a group acting on a W-vector bundle over a standard Borel space (X, μ), such that the induced action on the base (X, μ) is ergodic. Taking a measurable trivialization of the bundle, we can assume it is defined by the projection $X \times W \to X$, and the action of the group H is given by a measurable cocycle $\alpha : H \times X \to \mathrm{GL}(W)$, so that

$$h.(x, w) = (hx, \alpha(h, x)w).$$

Recall that α is a cocycle if it satisfies the relation $\alpha(gh, x) = \alpha(g, hx)\,\alpha(h, x)$, and $\alpha(e, x) = \mathrm{id}$, where e denotes the trivial element of the group H.

We let $\mathrm{F}(X, W) = \{f : (X, \mu) \to W\}/_{=_{a.e.}}$ denote the k-vector space of measurable sections of the W-vector bundle, modulo the relation of being the same μ-almost everywhere. (In the following we will not distinguish a measurable function from its image in $\mathrm{F}(X, W)$.) The action of G on the vector bundle induces a linear representation $\rho^\alpha : H \to \mathrm{GL}(V)$ on the space of sections $\mathrm{F}(X, W) =: V$, by

$$(\rho^\alpha(h)f)(x) = \alpha(h, h^{-1}x)f(h^{-1}x). \tag{1.1}$$

Therefore a section $f \in \mathrm{F}(X, W)$ is h-invariant if

$$f(hx) = \alpha(h, x)f(x) \quad \text{for } \mu\text{-a.e. } x \in X. \tag{1.2}$$

In the following, we say that a section is *α-invariant* if it is h-invariant for every $h \in H$.

Lemma 1. *Assume H is a group acting ergodically on a standard Borel space (X, μ) and let $\alpha : H \times X \to \mathrm{GL}_n(k)$ be a measurable cocycle over the action. Then the dimension of the subspace $Z^\alpha \subset \mathrm{F}(X, k^n)$ of α-invariant sections is finite (at most n).*

Proof. Given a finite collection of α-invariant sections $f_1, \dots, f_i \in Z^\alpha$, consider the measurable function $m : X \to \{0, \dots, n\}$ defined by $m(x) = \dim\langle f_1(x), \dots, f_i(x)\rangle_k$. Given $h \in H$, we get from (1.2) the equality for μ-a.e. $x \in X$:

$$\begin{aligned} m(hx) &= \dim\langle \alpha(h,x)f_1(x), \dots, \alpha(h,x)f_i(x)\rangle_k \\ &= \dim \alpha(h,x)\langle f_1(x), \dots, f_i(x)\rangle_k = m(x). \end{aligned}$$

Ergodicity implies that $m \equiv m_0 \in \{0, \dots, n\}$ is constant μ-almost everywhere. Consider a family $f_1, \dots, f_i \in Z^\alpha$ maximizing this constant m_0, and we can assume $i = m_0$. Given any α-invariant section $f \in Z^\alpha$, there exist measurable functions $c_j : X \to k$ $(j = 1, \dots, m_0)$ such that for μ-a.e. $x \in X$ one has

$$f(x) = \sum_{j=1}^{m_0} c_j(x) f_j(x).$$

Using α-invariance again, for every $h \in H$ and μ-a.e. $x \in X$ we have

$$f(hx) = \alpha(h,x)f(x) = \sum_{j=1}^{m_0} c_j(x)\alpha(h,x)f_j(x) = \sum_{j=1}^{m_0} c_j(x)f_j(hx).$$

Thus all functions c_j are H-invariant and, by ergodicity, μ-a.e. constant. This implies that the dimension of Z_α is $m_0 \le n$. □

1.2. **Intertwining operators.** Let k be a local field, V_1 and V_2 vector spaces over k, and let H be a group admitting linear representations

$$\rho_\nu : H \to \mathrm{GL}(V_\nu), \quad \nu = 1, 2.$$

A linear map $\phi \in \mathrm{Hom}(V_1, V_2)$ is an *intertwining operator* if it semiconjugates ρ_1 to ρ_2:

$$\phi\, \rho_1(h) = \rho_2(h)\, \phi \quad \text{for every } h \in H. \tag{1.3}$$

We denote by $\mathcal{L}(\rho_1, \rho_2) \subset \mathrm{Hom}(V_1, V_2)$ the collection of intertwining operators, which defines a k-linear subspace of $\mathrm{Hom}(V_1, V_2)$. We record the following basic fact.

Remark 1. *Assume that the vector spaces V_1 and $\mathcal{L} = \mathcal{L}(\rho_1, \rho_2)$ are finite dimensional. Then the subspace $\langle \mathcal{L}V_1\rangle_k \subset V_2$ is also finite dimensional (the dimension is at most* $\dim \mathcal{L} \dim V_1$*).*

We will work in the following setting. We assume H is a subgroup of some larger group G. We let V be a k-vector space and $\rho : G \to \mathrm{GL}(V)$ a linear representation. We assume $B \subset V$ is an H-invariant subspace and we let $\rho_1 : H \to \mathrm{GL}(B)$ denote the induced representation. We also write $\rho_2 : H \to \mathrm{GL}(V)$ for the restriction of ρ to H.

Remark 1. For any z in the centralizer $Z_G(H)$, the restriction $\rho(z)|_B \in \mathrm{Hom}(B, V)$ intertwines ρ_1 and ρ_2:

$$\rho(Z_G(H)) \subset \{\phi \in \mathrm{GL}(V) \mid \phi|_B \in \mathcal{L}(\rho_1, \rho_2)\}.$$

Given a measurable cocycle $\alpha : G \times X \to \mathrm{GL}(W)$, we have a representation $\rho^\alpha : G \to \mathrm{GL}(V)$ where $V := \mathrm{F}(X, W)$ is defined as in (1.1). If $B \subset V$ is a subspace which is H-invariant, then there is an induced action of H on the measurable $\mathrm{Hom}(B, W)$-vector bundle, defined by the measurable cocycle $\beta : H \times X \to \mathrm{GL}(\mathrm{Hom}(B, W))$,

$$\beta(h, x) : \varphi \mapsto \alpha(h, x)\, \varphi\, \rho^\alpha(h)^{-1}. \tag{1.4}$$

We have a natural identification of k-vector spaces

$$\mathrm{F}(X, \mathrm{Hom}(B, W)) \cong \mathrm{Hom}(B, \mathrm{F}(X, W)), \tag{1.5}$$

and we denote by $\Phi : (x, f) \in X \times B \mapsto \Phi(x, f) \in W$ a generic element of one of these spaces.

Considering the representations $\rho_1 : H \to \mathrm{GL}(B)$ and $\rho_2 : H \to \mathrm{GL}(V)$ defined by the restriction of ρ^α to $H \leq G$, we have the following identification:

Lemma 2. *The isomorphism* (1.5) *identifies the subspace of β-invariant sections*

$$Z^\beta \subset \mathrm{F}(X, \mathrm{Hom}(B, W))$$

with the space of intertwining operators $\mathcal{L}(\rho_1, \rho_2) \subset \mathrm{Hom}(B, \mathrm{F}(X, W))$.

Proof. We claim that a measurable function $\Phi : X \times B \to W$, which is linear in the second variable, defines an element of Z^β or $\mathcal{L}(\rho_1, \rho_2)$ if and only if it verifies

$$\Phi\left(hx, \rho^\alpha(h) f\right) = \alpha(h, x)\, \Phi(x, f)$$

for μ-a.e. point $x \in X$, section $f \in B$ and element $h \in H$.

Indeed, assume first that $\Phi \in \mathrm{F}(X, \mathrm{Hom}(B, W))$ is a β-invariant section, then by (1.2) and (1.4) we have

$$\Phi(hx, f) = \beta(h, x)\, \Phi(x, f) = \alpha(h, x)\, \Phi(x, \rho^\alpha(h)^{-1} f).$$

Next, assume $\Phi \in \mathrm{Hom}(B, \mathrm{F}(X, W))$ is an intertwining operator, then by (1.3) and the definition of ρ_1 and ρ_2, we have

$$\Phi(x, \rho^\alpha(h) f) = (\rho^\alpha(h)(\Phi f))(x) = \alpha(h, h^{-1}x)\, \Phi(h^{-1}x, f). \qquad \square$$

1.3. **Extension of a finite-dimensional invariant subspace.**

Proposition 1. *With the notations as above, assume that W is finite dimensional, and $H \leq G$ is a subgroup whose action on (X, μ) is ergodic and admits a finite-dimensional invariant subspace $B \subset \mathrm{F}(X, W)$ of measurable sections. Let K be a subgroup of the centralizer $Z_G(H)$. Then the vector space $B' \subset \mathrm{F}(X, W)$ spanned by $\rho^\alpha(K)B$ is K-invariant and finite dimensional.*

Proof. It is clear from the definition $B' = \langle \rho^\alpha(K)B \rangle_k$ that B' is K-invariant. By Remark 1, the image $\rho^\alpha(K)B$ is contained in the image $\mathcal{L}B$, where $\mathcal{L} = \mathcal{L}(\rho_1, \rho_2)$ is the space of intertwining operators as in the previous section. By Lemma 2, the space $\mathcal{L}$ identifies with the subspace Z^β of β-invariant sections of the $\mathrm{Hom}(B, W)$-vector bundle. As B and W are finite dimensional, we can identify $\mathrm{Hom}(B, W)$ with k^n for some appropriate $n \in \mathbb{N}$. Then Lemma 1 implies that Z^β is finite-dimensional and therefore so is $\mathcal{L}$. After Remark 1, $\mathcal{L}B$ spans a finite dimensional subspace of $\mathrm{F}(X, W)$, and so does $\rho^\alpha(K)B \subset \mathcal{L}B$. $\square$

By a repeated application of Proposition 1 we get the following:

Corollary 1. *Let k be a local field and W a finite dimensional k-vector space. Let G be a group acting on a measurable W-vector bundle over (X, μ).*

Assume there exist subgroups $H_1, \dots, H_n \leq G$ such that:

(1) $G = H_n \cdots H_1$,
(2) $H_{i+1} \leq Z_G(H_i)$ for every $i = 1, \dots, n-1$,
(3) the action of H_i on (X, μ) is ergodic for every $i = 1, \dots, n$,
(4) there exists an H_1-invariant measurable section $s \in \mathrm{F}(X, W)$.

Then the subspace $\langle \rho^\alpha(G)s \rangle_k \subset \mathrm{F}(X, W)$ is G-invariant.

Remark 2. Under the assumptions of Corollary 1, if moreover G is a *locally compact second countable* group, and the section s is not G-invariant, we obtain a non-trivial representation $\rho^\alpha : G \to \mathrm{GL}(E)$, where $E = \langle \rho^\alpha(G)s \rangle_k$ is a finite-dimensional topological space (recall that k is a local field). Therefore ρ^α is a measurable homomorphism of locally compact second countable groups, and hence automatically continuous. In the case G is a connected (real) Lie group, continuity of ρ^α implies that the local field k is necessarily $\mathbb{R}$ or $\mathbb{C}$.

2. Cocycle superrigidity

2.1. **Higher-rank Lie groups.** The first two assumptions in Corollary 1 are satisfied in the case of semisimple connected Lie groups of real rank at least 2.

Lemma 3. *Let G be a semisimple connected real Lie group of higher rank ($\mathrm{rk}_{\mathbb{R}} G \geq 2$) and $A \leq G$ a Cartan subgroup. Then there exist finitely many non-trivial closed subgroups $H_1, \dots, H_n \leq G$ such that*

(1) $G = H_n \cdots H_1$,
(2) $H_1 \leq A$ and $H_{i+1} \leq Z_G(H_i)$ for every $i = 1, \dots, n-1$.

Example 1. *In the case $G = \mathrm{SL}_d(\mathbb{R})$ ($d \geq 3$), we have that Cartan subgroups are conjugate to the group A of diagonal matrices with positive entries. The groups $H_1, \dots, H_n$ can be obtained as follows:*

$$H_1 = \begin{pmatrix} e^s & & & & \\ & e^s & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix} (s \in \mathbb{R}), \quad H_2 = \begin{pmatrix} * & * & & & \\ * & * & & & \\ & & * & & \\ & & & * & \\ & & & & * \end{pmatrix}, \quad H_3 = H_1,$$

$$H_4 = \begin{pmatrix} * & & & & \\ & e^s & & & \\ & & e^s & & \\ & & & * & \\ & & & & * \end{pmatrix} (s \in \mathbb{R}), \quad H_5 = \begin{pmatrix} * & & & & \\ & * & * & & \\ & * & * & & \\ & & & * & \\ & & & & * \end{pmatrix}, \quad H_6 = H_4, \quad \dots$$

The ergodicity assumption in Corollary 1 reduces to the ergodicity of the action of G [4].

Theorem 1 (Howe–Moore)**.** *Let G be a connected, non-compact, simple Lie group with finite center, acting ergodically on a standard Borel space (X, μ). Then every closed non-compact subgroup $H \leq G$ acts ergodically on (X, μ).*

The fourth assumption requires a longer discussion, and was the object of the talk by Thang Nguyen. The fundamental hypothesis is that the cocyle $\alpha : G \times X \to \mathrm{GL}(W)$ comes from a cocycle $c : G \times X \to H$, where

(1) G has no rank 1 factors,
(2) H is a simple algebraic group,
(3) c is G-integrable,
(4) c is not cohomologous to a compact-valued cocycle,
(5) W is an irreducible representation of H.

2.2. **Conclusion.** Assuming the cocycle $\alpha : G \times X \to \mathrm{GL}(W)$ comes from a cocycle $c : G \times X \to H$ as above, we can then apply Corollary 1 and get a finite-dimensional invariant subspace $E \subset \mathrm{F}(X, W)$. Then one can argue (but this requires a careful verification), that the evaluation map from $E \subset \mathrm{F}(X, W)$ to W at a density point $x_0 \in (X, \mu)$ provides a linear isomorphism $ev : E \xrightarrow{\sim} W$, and this does not depend on the point x_0 (ergodicity is used again). Then from the representation $\rho^\alpha : G \to \mathrm{GL}(E)$, we can consider the conjugate representation $\rho : G \to \mathrm{GL}(W)$. At this point one argues that $\rho(G)$ is contained in the image of H (remember that W is an irreducible representation of H), and simplicity of H allows to lift the morphism $\rho : G \to \mathrm{GL}(W)$ to a morphism $\bar{\rho} : G \to H$. This is essentially the desired conclusion for Zimmer's Cocycle Superrigidity [2].

3. Margulis Superrigidity as particular case

The main reference for this part is [1]

3.1. **Suspension bundle.** Here we explain how the argument sketched in the previous paragraph can be slightly simplified in the case of measurable bundles, obtained from the suspension of a representation $\pi : \Gamma \to H$ of an irreducible lattice Γ in a semisimple connected Lie group G of higher rank ($\mathrm{rk}_{\mathbb{R}} G \geq 2$). Let us denote by μ the probability Haar measure on the quotient $X = G/\Gamma$. The group G acts on X by right multiplication, and the Howe–Moore Theorem ensures that this action is ergodic. Choosing an irreducible representation $H \to \mathrm{GL}(W)$, we have an induced representation $\Gamma \to \mathrm{GL}(W)$.

We consider the W-vector bundle over (X, μ) whose total space is $(G \times W)/\Gamma$, where the Γ action is given by $(x, v).\gamma = (x\gamma, \pi(\gamma^{-1})v)$. The right multiplication by elements of G defines a G-action on this bundle, with trivial cocycle $\alpha \equiv \mathrm{id}$. Thus by (1.1), the Lie group G acts on the space of sections $\mathrm{F}(X, W)$ by the left regular representation

$$(\rho^{\mathrm{id}}(g)f)(x) = f(g^{-1}x).$$

We remark that there is a natural identification of sections $f \in \mathrm{F}(X, W)$ with Γ-equivariant measurable functions from G to W. We write $\mathrm{F}_\Gamma(G, W)$ for the collection of such functions.

3.2. **Continuity.** As a consequence of the results discussed in the previous sections, we have a continuous finite-dimensional representation $\rho^{\mathrm{id}} : G \to \mathrm{GL}(E)$, where $E \subset \mathrm{F}_\Gamma(G, W)$. As a consequence of continuity of ρ^{id}, every function $f \in E$ has actually a *continuous* representative. To see this, let us fix $f_1, \dots, f_m$ measurable representatives of a basis of E, and for a fixed f, measurable representative of a section in E, let $c_j : G \to k$ be such that $\rho^{\mathrm{id}}(g)f = \sum_{j=1}^m c_j(g) f_j$. By continuity of ρ^{id}, the functions c_j are also continuous. Using Fubini's Theorem (applied to the convolution), we have that for μ-a.e. every $x_0 \in X$ and a.e. $g \in G$, one has

$$f(g^{-1}x_0) = \sum_{j=1}^{m} c_j(g) f(x_0).$$

The right hand side being continuous with respect to g, then also the left hand side is. By making a change of variables, we see that f is continuous.

3.3. **Evaluation map.** From now on, we can identify E with a finite-dimensional subspace of the space of continuous Γ-equivariant sections $C_\Gamma(G, W)$. We can then consider the map $ev : E \to W$ defined by $ev(f) = f(e)$ (that is, evaluation at the trivial element $e \in G$). By Γ-equivariance, the image $ev(E)$ is a $\pi(\Gamma)$-invariant subspace of W. If π is irreducible, then we must have $ev(E) = W$. Similarly, also $\ker(ev) \subset E$ is Γ-invariant. As $\Gamma < G$ is Zariski-dense, we deduce that $\ker(ev)$ is G-invariant. But the action of G on E is the left regular representation, and this implies that if $f \in \ker(ev)$, that is, $f(e) = 0$, then $f(x) = ev(\rho^{\mathrm{id}}(x^{-1})f) = 0$ at every point $x \in G$. Using the isomorphism ev, we thus obtain a morphism $\hat{\pi} : G \to \mathrm{GL}(W)$, which extends $\pi : \Gamma \to \mathrm{GL}(W)$, as wanted.

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Ratner's measure classification theorem and equidistribution

Pengyu Yang

In this lecture, we present Ratner's measure classification theorem and Dani-Margulis equidistribution theorem for unipotent flows on homogeneous spaces. We also sketch a proof of the equidistribution theorem in the cocompact case.

Let G be a connected real Lie group, and Γ a discrete subgroup of G. We say that Γ is a lattice in G if G/Γ has a G-invariant Borel probability measure. For $g \in G$, let Ad_g denote the adjoint action of g on the Lie algebra $\mathfrak{g}$ of G. An element

$u \in G$ is unipotent if $\mathrm{Ad}_u - \mathrm{Id}$ is nilpotent in $\mathrm{GL}(\mathfrak{g})$. A group U is unipotent if each $u \in U$ is unipotent.

A Borel probability measure μ on G/Γ is algebraic if there exists $x \in G/\Gamma$ and a subgroup H of G such that Hx is closed and is the support of μ, and μ is H-invariant.

We now state Ratner's measure classification theorem, which was conjectured by Dani in the 1980s. The topological version was conjectured earlier in the 1970s by Raghunathan, and was also proved by Ratner. It will be presented in the next lecture.

Theorem 1 (Ratner)**.** *Let G be a connected Lie group and U a unipotent subgroup of G. Then given any discrete subgroup Γ of G, every ergodic U-invariant Borel probability measure on G/Γ is algebraic.*

Now and below we assume that Γ is a lattice in G. For any subgroup S of G, we define the associated set of singular points:

$$\begin{aligned}\mathcal{S}(S) = \{x \in G/\Gamma \mid \exists H \subset G \text{ proper closed subgroup containing } S \text{ such that}\\ Hx \text{ is closed and has an } H\text{-invariant probability measure}\},\end{aligned}$$

and the set of generic points

$$\mathcal{G}(S) = G/\Gamma - \mathcal{S}(S).$$

The following theorem due to Dani and Margulis shows that unipotent trajectories do not spent much time near singular subsets.

Theorem 2. *Let S be any subgroup of G which is generated by the one-parameter unipotent subgroups contained in it. Let F be a compact subset contained in $\mathcal{G}(S)$. Then for any $\epsilon > 0$, there exists a neighborhood Ω of $\mathcal{S}(S)$ such that for any unipotent one-parameter subgroup $\{u_t\}$, any $x \in F$ and any $T \geq 0$, one has*

$$l(\{t \in [0, T] \mid u_t x \in \Omega\}) \leq \epsilon T,$$

where l denotes the Lebesgue measure.

The following equidistribution theorem is also due to Dani and Margulis. We only state the theorem for one single unipotent flow, and one single base point. For their original statement, see [1].

Theorem 3. *Let $\{u_t\}$ be a unipotent one-parameter subgroup, and $x \in \mathcal{G}(\{u_t\})$. For any $T_i \to \infty$, and any $\varphi \in C_c(G/\Gamma)$,*

$$\lim_{i\to\infty} \frac{1}{T_i} \int_0^{T_i} \varphi(u_t x)\, dt = \int_{G/\Gamma} \varphi \, d\mu,$$

where μ is the Haar measure on G/Γ.

We remark that in the case that Γ is cocompact, Theorem 3 follows from Theorem 1 and Theorem 2 via a routine argument. For the general case, one needs the non-escape-of-mass property of unipotent flows, which was established by Dani.

Proof of Theorem 3 in the cocompact case. Let μ_i denote the probability measure on G/Γ such that

$$\int_{G/\Gamma} \varphi \, d\mu_i = \frac{1}{T_i} \int_0^{T_i} \varphi(u_t x)\, dt, \quad \forall \varphi \in C_c(G/\Gamma).$$

Since G/Γ is compact, by passing to a subsequence we may assume that

$$\mu_i \to \lambda, \quad i \to \infty,$$

where λ is a probability measure on G/Γ, and the convergence is with respect to the weak-* topology. Now it suffices to show that the limit measure λ is the Haar measure on G/Γ.

We claim that λ is $\{u_t\}$ invariant. Indeed, fix any $t_0 \in \mathbb{R}$, and for any $\epsilon > 0$, any $\varphi \in C_c(G/\Gamma)$, one has

$$\left| \frac{1}{T_i} \int_0^{T_i} \varphi(u_t x)\, dt - \frac{1}{T_i} \int_0^{T_i} \varphi(u_{t+t_0} x)\, dt \right| < \epsilon,$$

for i large enough. It follows that for i large enough, one has

$$\left| \int_{G/\Gamma} \varphi \, d\lambda - \int_{G/\Gamma} \varphi \, d(u_{t_0 *}\lambda) \right| < 3\epsilon.$$

Since ϵ and t_0 are arbitary, we conclude that λ is $\{u_t\}$-invariant.

Hence it follows from Theorem 1 that each ergodic component of λ is algebraic. Suppose that λ is not G-invariant, then $\lambda(\mathcal{S}(\{u_t\})) = c > 0$. Since λ is the limit of μ_i, for any neighborhood Ω of $\mathcal{S}(\{u_t\})$ we have

$$l(\{t \in [0, T_i] \mid u_t x \in \Omega\}) > \frac{c}{2} T_i,$$

for all i large enough. But this contradicts Theorem 2, taking $\epsilon = \frac{c}{2}$.

Therefore, we conclude that λ is the Haar measure on G/Γ. □

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Ratner's orbit closure theorem and generalized equidistribution

Keivan Mallahi-Karai

In this extended abstract, we will discuss the connection between Ratner's measure classification/equidistribution and orbit closure theorems, and give a proof for the latter based on the former. The proof we present is based on a number of pertinent results in Dani and Margulis's work [1]. We will also discuss a generalization of Ratner's equidistribution theorem due to Shah. First we will set up some notation and formulate the standing assumptions of this note.

Assumptions: Let G be a real Lie group and Γ be a discrete subgroup of G. The projection map $\pi : G \to G/\Gamma$ is defined by $\pi(g) = g\Gamma$. The Lie group G acts on G/Γ by left translation. The (left) Haar measure on G will be denoted by μ_G. If there exists a G-invariant probability measure $\nu_{G/\Gamma}$ on G/Γ, then we say that Γ is a lattice in G. Much work in homogenous dynamics is inspired by the following three questions:

(1) If U is a connected subgroup of G, then classify the (ergodic) U-invariant probability measures on G/Γ.
(2) If U is a connected subgroup of G, classify the orbit closures $\overline{Ux}$ for $x \in G/\Gamma$. In particular, are these orbit closures submanifolds? Are they orbits of possibly larger subgroups?
(3) How is the U-orbit of a point $x \in G/\Gamma$ distributed inside $\overline{Ux}$?

Two remarks are in order. First note that if U is a non-compact subgroup of G and, say, G is simple, then it follows from Howe-Moore theorem that for $\nu_{G/\Gamma}$-almost every $x \in G/\Gamma$ the orbit Ux is dense in G. In view of this fact, Question (1) above is at its core about understanding non-generic behavior. Second, as every invariant measure can be decomposed into ergodic invariant measures, an answer to Question (1) above completely classifies all U-invariant measure on G/Γ.

Definition 1. *A subset $A \subseteq G/\Gamma$ is called homogenous if there exists a closed subgroup $H \leq G$ and a point $x \in G/\Gamma$ such that $A = Hx$ and Hx is the support of an H-invariant Borel probability measure ν_H. Similarly, a Borel probability measure μ is called algebraic if there exists $x \in G/\Gamma$ and a closed subgroup H such that Hx is homogenous and $\mu = \nu_H$.*

When U is an arbitrary closed subgroup, the orbit closure $\overline{Ux}$ can have a rather complicated structure. In her groundbreaking work [2, 3, 4, 5], Ratner provided complete answers to Questions (1)-(3) above when U is a connected unipotent subgroup of G. In order to formulate these results precisely, we will need a few more definitions. Denote by $\mathfrak{g}$ the Lie algebra of G. The adjoint representation of G is the group homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ defined for $g \in G$ and $v \in \mathfrak{g}$ via

$$\mathrm{Ad}(g)(v) = \frac{d}{dt}(g^{-1}\exp(tv)g)\bigg|_{t=0}$$

Here and below $\exp : \mathfrak{g} \to G$ denotes the exponential map. A subgroup U is called *Ad-unipotent*, if for all $g \in U$, all the eigenvalues of $\mathrm{Ad}(g)$ are equal to 1.

Ratner's measure classification theorem provides a complete answer to the first question.

Theorem 1 (Ratner's measure classification theorem for U-actions)**.** *Let G and Γ be as above and U be a closed subgroup of G which is generated by unipotent elements. Then every ergodic U-invariant measure on G/Γ is algebraic.*

Remark 1. *For Ratner's theorem to hold, it suffices that Γ is a discrete subgroup of G.*

Using this theorem, Ratner proved her orbit closure theorem:

Theorem 2 (Ratner's orbit closure theorem for U-actions)**.** *Let G be a connected Lie group and U be an Ad-unipotent subgroup of G. Then for every lattice Γ in G and every $x \in G/\Gamma$, the orbit closure $\overline{Ux}$ is homogenous.*

This theorem affirmatively answers Raghunathan's topological conjecture. Ratner's proof of Theorem 2 is based on her measure classification theorem–Theorem 1 above. This approach is in stark contrast to previous attempts toward the proof of Raghunathan's topological conjecture. For instance, prior to [3], Dani and Margulis [6] had given a complete classification of all possible $\mathrm{SO}(2,1)$-orbit closures in $\mathrm{SL}_3(\mathbb{R})/\mathrm{SL}_3(\mathbb{Z})$ using purely topological methods. The classification of orbit closures of $\mathrm{SO}(2,1)$ is closely related to the Oppenheim conjecture about values of indefinite quadratic forms that was solved by Margulis [7].

Ratner also proved a theorem regarding the distribution of U-orbits in their closure:

Theorem 3 (Ratner's uniform distribution of unipotent orbits)**.** *Let G be a connected Lie group and $U = \{u_t\}_{t\in\mathbb{R}}$ a one-parameter unipotent subgroup of G. Then for every $x \in G/\Gamma$ there exists a closed subgroup H such that Hx is homogeneous, $\overline{Ux} = Hx$, and for every bounded continuous function f on G/Γ, as $t \to \infty$ we have*

$$\frac{1}{T}\int_0^T f(u_t x)\ dt = \int_{Hx} f(y)d\nu_H(y).$$

We will give a proof of Theorem 2 for one-parameter unipotent subgroups which is based on the classification of invariant measures. The proof provided here is not the original proof of Ratner and uses a number of other ingredients from [1].

1. Structure of singular sets

In this section we will gather a number of facts about the behavior of the unipotent orbits that will be needed in the proof of Theorem 2. As a motivation, let us discuss the rough sketch of the proof of Theorem 2. Fix a point $x \in G/\Gamma$, and a one-dimensional unipotent subgroup U for which the structure of $\overline{Ux}$ is to be understood. In order to exploit Theorem 1, one needs to find a U-invariant measure whose support carries information about $\overline{Ux}$. Such a measure can be constructed as a weak limit of integration along a long portion $[0,T]$ of the U-orbit as $T \to \infty$. However, in order to guarantee the existence of a limit (along some subsequence) one needs to show that there is no *escape of mass*, that is, the U-orbit cannot spend too much time in the cusp. This is obvious when G/Γ is compact, but requires a proof for general Γ. We will assume that $\overline{Ux}$ is not included inside any orbit Hx for a proper closed subgroup H and aim to show that $\overline{Ux} = G/\Gamma$. To prove this using a measure classification theorem, we will need to show that the orbit $\overline{Ux}$ cannot be close to the union of proper closed orbits. Theorems discussed in this section can be used to deal with these two obstacles.

Theorem 4 ([1]). *Let G and Γ be as above, $F \subseteq G/\Gamma$ be compact, and $\epsilon > 0$ be given. Then there exists a compact set $K \subseteq G/\Gamma$ such that for every Ad-unipotent one-parameter subgroup $\{u_t\}$ of G and every $x \in F$ and $T \geq 0$ we have*

$$\mathrm{Leb}\{t \in [0,T] : u_t x \in K\} \geq (1-\epsilon)T.$$

Here Leb denotes the Lebesgue measure on the line. In [1], Dani and Margulis define the singular set in the following way

Definition 2. *Let G and Γ be as above. For a proper closed subgroup $H \leq G$, we denote by $\mathcal{S}(H)$ the set of all $x \in G/\Gamma$ for which there exists a proper subgroup $H \leq L < G$ such that Lx carries a finite L-invariant measure. The set $\mathcal{S}(H)$ is called the singular set relative to H. The complement of $\mathcal{S}(H)$ is the set of generic points and will be denoted by $\mathcal{G}(H)$.*

Remark 2. *Let L be the subgroup in the definition of the singular set. Then one can show that the orbit $Lx \subseteq G/\Gamma$ is a closed set. This can be seen using the following divergence criterion: for a sequence $g_n \in G$, the images $\pi(g_n)$ is divergent (that it, it does not have a convergent subsequence) iff there exists $\gamma_n \in \Gamma \setminus \{e\}$ such that $g_n \gamma_n g_n^{-1} \to e$ as $n \to \infty$. For details see Proposition 1.13 in* [8].

One of the results in [1] provides the fine structure of the singular set. In order to discuss this, we will need to define a class of subgroups.

Definition 3. *Denote by $\mathcal{H}$ the class of all proper closed subgroups L of G such that $L \cap \Gamma$ is a lattice in L and $\mathrm{Ad}(L \cap \Gamma) < \mathrm{Ad}L$ is Zariski dense.*

Proposition 1. *The class $\mathcal{H}$ defined above is countable.*

For closed subgroups H, L of a Lie group G define

$$X(L,H) = \{g \in G : Hg \subseteq gL\}.$$

The set $X(L,H)$ is sometimes called the *transporter* since its defining condition is equivalent to $g^{-1}Hg \subseteq L$. We make a number of simple observations about transporters:

(1) If $L \cap \Gamma$ is a lattice in L, then the set $X(L,H)\Gamma/\Gamma$ is included in the singular $\mathcal{S}(H)$.
(2) If $L_1 \subseteq L_2$ then $X(L_1,H) \subseteq X(L_2,H)$.
(3) If $g_1 \in N_G(L)$ and $g_2 \in N_G(L)$ then $g_1 X(L,H) g_2 = X(L,H)$.

The following results ([1], Proposition 2.3) provides the structure of the singular set in terms of transporters:

Proposition 2. *Assume that H is generated by Ad-unipotent elements. With $\mathcal{H}$ as in Definition 3, we have*

$$\mathcal{S}(H) = \bigcup_{L \in \mathcal{H}} X(L, H)\Gamma/\Gamma.$$

The following theorem asserts that the U-orbit of a generic point for H does not "cling" too much to the singular set of H.

Theorem 5 (DM2)**.** *Let G and Γ be as above, and assume that H is a closed connected subgroup of G which is generated by Ad-unipotent elements. Let $F \subseteq \mathcal{G}(H)$ be compact and let $\epsilon > 0$. Then there exists an open neighborhood Ω of $\mathcal{S}(H)$ such that for every Ad-unipotent subgroup $\{u_t\}_{t\in\mathbb{R}}$ of G, every $x \in F$, and every $T \geq 0$ we have*

$$\mathrm{Leb}\{t \in [0, T] : u_t x \in \Omega\} \leq \epsilon T.$$

We will now give a proof of Theorem 2 for a one-parameter unipotent subgroup $U = \{u_t\}_{t\in\mathbb{R}}$. The proof is based on the exposition in [10].

Proof. Fix a point $x \in G/\Gamma$ and for each $T > 0$, consider the measure μ_T on G/Γ that is defined by

$$\int_{G/\Gamma} f(y) d\mu_T(y) = \frac{1}{T}\int_0^T f(u_t x)\, dt.$$

We will show that the measures μ_T converge weakly to the Haar measure. Let us first show that there is no escape of mass: when applied to $F = \{x\}$ and $\epsilon > 0$, Theorem 4 yields a compact set $K \subseteq G/\Gamma$ with $\mu_T(K) \geq 1 - \epsilon$ for all $T \geq 0$. This implies that there exists a subsequence $T_i \to \infty$ along which the sequence of measures μ_{T_i} converge weakly to a *probability* measure μ. It is easy to see that the limiting measure is U-invariant. Using Ratner's measure classification theorem (Theorem 1), we deduce that every ergodic component of μ is algebraic. Decompose μ as $\mu = \alpha\nu_G + (1-\alpha)\nu$, where ν_G is the Haar measure on G/Γ and ν is a linear combination of (countably many) algebraic probability measures that are supported on (closed) orbits of proper closed L containing U. Note that ν is a probability measure which is supported on $\mathcal{S}(U)$. Since x is a generic point, it follows from Theorem 5 that for every $\epsilon > 0$ there exists an open set Ω containing $\mathcal{S}(U)$ such that $\mu_T(\Omega) \leq \epsilon$ for all $T \geq 0$. This implies that the limiting measure μ also assigns a measure less than ϵ to Ω, hence $1 - \alpha \leq \epsilon$. Since $\epsilon > 0$ is arbitrary, we have $\mu = \nu_G$, which finishes the proof.

□

2. A theorem of Shah

Another ingredient in the proof of Zimmer's conjecture is a theorem of Shah which generalizes Ratner's equidistribution theorem. In order to state this theorem we will need some defintions. Let U be a simply connected unipotent subgroup of G. Suppose $B = \{b_1, \ldots, b_k\}$ is a basis for the Lie algebra of U. Denote by $\{\lambda_1, \ldots, \lambda_k\}$ the associated dual basis. We say that B is *triangular* if for every $i, j \in \{1, 2, \ldots, k\}$ such that $k \leq \max(i, j)$ we have $\lambda_k([b_i, b_j]) = 0$. Any permutation of a triangular basis is called a regular basis. Associated to the basis B and given a vector $m = (m_1, \dots, m_k) \in [0, \infty)^k$, we will define the box

$$F_m = \{\exp(t_1 b_1) \cdots \exp(t_k b_k) : t_j \in [0, m_j], j = 1, \dots, k\}.$$

The Haar measure on U is denoted by μ_U. The following theorem is proven in [9]

Theorem 6. *Let G and Γ be as above, and let U be a simply connected unipotent subgroup of G and B is a regular basis for the Lie algebra of U. Then for every $x \in G/\Gamma$, there exists a closed subgroup L of G containing U such that the orbit Lx is closed and admits an L-invariant measure ν_L such that for every compactly supported continuous real-valued function f on G/Γ we have*

$$\lim_{m \to \infty} \frac{1}{\mu_U(B_m)} \int_{B_m} f(hx)\ d\mu(x) = \int_L f(y) d\nu_L(y).$$

Note that when U is a one-parameter subgroup, this reduces to Theorem 3. The proof of Theorem 6 is an inductive argument that relies on this special case as well as ideas from [1].

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Epimorphic subgroups and invariant measures

Manuel Luethi

Throughout this note, we let G be a real Lie group and $\mathfrak{g}$ its Lie algebra. We denote by $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ the adjoint representation. We say that an element $g \in G$ is Ad-diagonalizable or Ad-unipotent if $\mathrm{Ad}(g)$ is diagonalizable or unipotent.

A subgroup $U \leq G$ is Ad-unipotent if all its elements are Ad-unipotent. Ad-diagonalizable subgroups are defined similarly.

For any $k \in \mathbb{N}$ the action on $\mathfrak{g}$ of G and of its subgroups given by the adjoint representation induces a natural representation $\mathrm{Ad}^k : G \to \mathrm{GL}(\bigwedge^k \mathfrak{g})$.

The goal of the talk was to present a simplified proof and an application of the following theorem by Mozes.

Theorem 1 ([M95, Thm. 1]). *Let G be a Lie group, $H, L \leq G$ closed connected subgroups such that $H \leq L$. Assume that the following are true.*

(1) For every $k \in \mathbb{N}$, every H-invariant vector in $\bigwedge^k \mathfrak{g}$ is L-invariant.
(2) H is generated by Ad-*unipotent and* Ad-*diagonalizable one-parameter subgroups.*
(3) Let $V \leq H$ be the subgroup generated by the Ad-*unipotent one-parameter subgroups contained in H. Then the only normal subgroup of L containing V is L.*

If $\Gamma \leq G$ is a discrete subgroup and μ an H-invariant Borel probability measure on $\Gamma \setminus G$, then μ is L-invariant.

1. A sketch of the proof and the role of the assumptions

In order to give an illustration of the role of the assumptions, we will give a sketch of the proof of Theorem 1. We will assume throughout that the measure in question is ergodic for the action of H.

The proof heavily relies on Ratner's measure classification theorems for unipotent flows. In order to state the theorem used, we introduce the following notation. Let X be a manifold and μ a Borel probability measure on X. Let G be a Lie group and assume that G acts on X by diffeomorphisms. Then G acts on the set of Borel probability measures on X by push-forward. We denote by $\Lambda_\mu \leq G$ the stabilizer of μ. Note that this is a closed subgroup. We are now set up to formulate the following crucial result by Ratner used in the proof.

Theorem 2 ([Ra, Thm. 3]). *Let G be a Lie group, $V \leq G$ a subgroup generated by* Ad-*unipotent one-parameter subgroups. Let $\Gamma \leq G$ be a discrete subgroup and assume that μ is a V-invariant ergodic Borel probability measure on G / Γ. Then μ is algebraic, i.e. there exists some $x \in G / \Gamma$ such that $\mu(\Lambda_\mu . x) = 1$.*

In particular, every V-ergodic component of μ is algebraic by Theorem 2.

Given subgroups $V, F \leq G$, we denote $X(F, V) = \{g \in G : Vg \subseteq gF\}$ and we call a tube (for V) a subset $X(F, V)x \subseteq G / \Gamma$, where $x \in G / \Gamma$ and $V \leq F \leq G$ is an intermediate closed subgroup such that Fx is a closed orbit of finite volume and V acts ergodically on Fx. Note that all V-invariant ergodic measures are supported on a single tube for some minimal F. By the work of Ratner [Ra] and Dani-Margulis [DM93], we know that the set of tubes is countable. As $V \lhd H$ is a normal subgroup, any such tube is H-invariant and thus ergodicity of μ implies that there is a minimal tube $X(F, V)x$ which is assigned full measure by μ.

Let $\mathfrak{f}$ denote the Lie algebra of F, let $k = \dim \mathfrak{f}$, and let $f \in \bigwedge^k \mathfrak{g}$ be the image of $\mathfrak{f}$. Then f is invariant under V.

Now note the following simple fact. Let E be a finite dimensional real vector space and let $(a_n)_{n\in\mathbb{N}}$ be a diverging—with respect to any induced operator norm—sequence of jointly diagonalizable automorphisms of E. Let $v \in E \setminus \{0\}$ and assume that the line defined by $a_n v$ converges to the line defined by v as $n \to \infty$ in projective space. Then a_n fixes $\mathbb{R}v$ for all $n \in \mathbb{N}$.

This elementary fact combined with a strong form of Poincaré recurrence allows us to show that the line defined by f is invariant under all Ad-diagonalizable one-parameter subgroups in H. In fact, we can deduce that f is fixed by all Ad-diagonalizable one-parameter subgroups contained in H. Hence assumptions (1) and (2) imply that L normalizes F and hence assumption (3) yields $L \cap F = L$. This in combination with the strong form of Poincaré recurrence mentioned earlier implies that almost every V-ergodic component of μ is L-invariant. This then concludes the proof of the theorem.

2. Examples of groups with the desired properties

We will give a short description of examples of groups with the properties in Theorem 1. In fact, we will mostly talk about property (1). Property (2) will be relatively immediate in the cases we mention, whereas property (3) is a maximality assumption on H relative to L which might or might not be satisfied. In the latter case, one is forced to consider a different group L.

Before we discuss more intricate examples, let us note that clearly any Zariski dense subgroup $H \leq L$ has property (1).

2.1. The Borel subgroup in $\mathrm{SL}_2(\mathbb{R})$.

In fact, this case has been treated already by Ratner and it served as an input to the measure classification, cf. [Ra, Prop. 2.1]. Let

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} : a \in (0,\infty), b \in \mathbb{R} \right\} \leq \mathrm{SL}_2(\mathbb{R}).$$

Proposition 1. *Let E be a finite dimensional continuous representation of $\mathrm{SL}_2(\mathbb{R})$ and let $v \in E$ be B-invariant. Then v is $\mathrm{SL}_2(\mathbb{R})$-invariant.*

Proof. Let $\mathfrak{sl}_2(\mathbb{R}) \subseteq \mathrm{Mat}_2(\mathbb{R})$ denote the traceless matrices, i.e. the Lie algebra of $\mathfrak{sl}_2(\mathbb{R})$ and let $\mathfrak{b} \subseteq \mathfrak{sl}_2(\mathbb{R})$ denote the Lie algebra of B. Then $\mathfrak{sl}_2(\mathbb{R})$ is spanned by the triple

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

and $\mathfrak{b}$ is spanned by $\{e, h\}$.

It is a standard fact that every continuous, finite-dimensional representation of a Lie group is smooth. Note that taking derivatives, it suffices to show that for every vector in E with $\mathfrak{b}.v = \{0\}$ we have $\mathfrak{sl}_2(\mathbb{R}).v = \{0\}$. Furthermore, as every representation of $\mathfrak{sl}_2(\mathbb{R})$ decomposes into a direct sum of irreducible representations, we can assume without loss of generality that E is irreducible.

Assume that $\dim E > 0$. Using the proof of the classification of irreducible finite dimensional representations of $\mathfrak{sl}_2(\mathbb{R})$ [FH91, §11.1], we know that the action of h on E is diagonalisable with one-dimensional eigenspaces and that for any eigenvector $w \in E$ of h for eigenvalue λ the vector ew is an eigenvector of h for eigenvalue $\lambda + 2$.

Thus, if now $v \in E \setminus \{0\}$ and $\mathfrak{b}.v = 0$, then v is an eigenvector of h for eigenvalue 0. Hence ev is an eigenvector for eigenvalue 2. This is absurd. It follows that $E = \{0\}$ and hence the proposition. □

2.2. **A class of groups defined dynamically.** Let us give a slight generalization of the above example. A much more general version of this is due to Shah [S96]. Let A be the positive diagonal subgroup of $\mathrm{SL}_3(\mathbb{R})$. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in A such that $(a_n)_{11}/(a_n)_{22} \to \infty$ and $(a_n)_{22}/(a_n)_{33} \to \infty$ as $n \to \infty$. Let

$$U^+ = \{g \in \mathrm{SL}_3(\mathbb{R}) : a_n^{-1} g a_n \to 1 \text{ as } n \to \infty\}.$$

An elementary calculation shows that U^+ is the full lower triangular unipotent subgroup of $\mathrm{SL}_3(\mathbb{R})$, i.e.

$$U^+ = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ z & y & 1 \end{pmatrix} : x, y, z \in \mathbb{R} \right\}.$$

Let $M \le \mathrm{SL}_3(\mathbb{R})$ be the group generated by the sequence $(a_n)_{n\in\mathbb{N}}$. Then M normalizes U^+.

Proposition 2 (Cf. [S96, Lem. 5.2])**.** *Any $MU^+ \le \mathrm{SL}_3(\mathbb{R})$-invariant vector in a finite dimensional continuous representation of $\mathrm{SL}_3(\mathbb{R})$ is invariant under $\mathrm{SL}_3(\mathbb{R})$.*

Proof. In what follows, we assume that E is a non-trivial representation of $\mathrm{SL}_3(\mathbb{R})$. Similarly to the previous case, any finite-dimensional representation E of $\mathrm{SL}_3(\mathbb{R})$ has a basis consisting of eigenvectors of the subgroup A [FH91, §12]. In particular, there exists a finite set Λ of homomorphisms $A \to (0, \infty)$ and a decomposition $E = \bigoplus_{\lambda\in\Lambda} E_\lambda$ such that

$$E_\lambda = \{v \in E : \forall a \in A\, a.v = \lambda(a)v\} \ne \{0\}$$

Given $v \in E$, we write $v = \sum_{\lambda\in\Lambda} v_\lambda$, where $v_\lambda \in E_\lambda$. After passing to a subsequence, we can decompose $\Lambda = \Lambda_+ \sqcup \Lambda_0 \sqcup \Lambda_-$, where

$$\Lambda_+ = \{\lambda \in \Lambda : \lambda(a_n) \to \infty \text{ as } n \to \infty\}$$
$$\Lambda_- = \{\lambda \in \Lambda : \lambda(a_n) \to 0 \text{ as } n \to \infty\}$$
$$\Lambda_0 = \{\lambda \in \Lambda : \exists c \in (0, \infty)\, \lambda(a_n) \to c \text{ as } n \to \infty\}$$

Assume now that $v \in E \setminus \{0\}$ satisfies $MU^+.v = \{v\}$, and in particular assume that $F = \{v \in E : MU^+.v = \{v\}\}$ is non-trivial. Note that, as A normalizes MU^+, F is an A-invariant subspace of E.

Let

$$U^- = \left\{ \begin{pmatrix} 1 & r & s \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} : r, s, t \in \mathbb{R} \right\}.$$

We first note that $U^- = \{g \in \mathrm{SL}_3(\mathbb{R}) : a_n g a_n^{-1} \to 1 \text{ as } n \to \infty\}$. We will show that every vector in F is U^--invariant. As $\mathrm{SL}_3(\mathbb{R})$ is generated by $U^+ \cup U^{-1}$, this then implies the proposition.

Let now $v \in F$ arbitrary and let $g \in U^-$, then

$$a_n(g.v) = (a_n g a_n^{-1}).(a_n.v) = (a_n g a_n^{-1}).v \to v$$

as $n \to \infty$. On the other hand we have

$$a_n(g.v) = \sum_{\lambda \in \Lambda} \lambda(a_n)(g.v)_\lambda$$

In particular, it follows that $g.v \in \bigoplus_{\lambda \in \Lambda_0 \cup \Lambda_-} E_\lambda$, and we denote the latter by E_-. As $AU^+.F = F$ and as v was arbitrary, it follows that $U^-AU^+.F \subseteq E_-$. For what follows, we note that U^-AU^+ contains a neighborhood of the identity in $\mathrm{SL}_3(\mathbb{R})$. This can be shown by checking that the Lie algebras of U^-, U^+, and A span the Lie algebra of $\mathrm{SL}_3(\mathbb{R})$.

Let $\mathfrak{h} \subseteq \mathfrak{gl}(E)$ denote the image of the Lie algebra of $\mathrm{SL}_3(\mathbb{R})$ under the differential of the representation at the identity. Then, similarly to what we said above, $\mathfrak{h} \subseteq \mathfrak{gl}(E)$ is semisimple and in particular algebraic [Bo91, Ch. II, Cor. 7.9]. As by the previous discussion an open neighbourhood of 0 in $\mathfrak{h}$ maps v into E_-, and thus Zariski density implies that all of $\mathfrak{h}$ maps v into E_-. This shows $\mathrm{SL}_3(\mathbb{R}).v \subseteq E_-$. The linear hull of $\mathrm{SL}_3(\mathbb{R})$ in E_- is a subrepresentation of E and thus by the irreducibilty assumption we get $E = E_-$. As $\mathrm{SL}_3(\mathbb{R})$ does not have any non-trivial characters, we obtain $E = E_0$, where $E_0 = \bigoplus_{\lambda \in \Lambda_0} E_\lambda$.

In order to finish the proof set

$$\Lambda_1 = \{\lambda \in \Lambda : \forall n \in \mathbb{N}\, \lambda(a_n) = 1\} \subseteq \Lambda_0$$

and note that by definition of F and by A-invariance we have

$$F = \bigoplus_{\lambda \in \Lambda_1} (F \cap E_\lambda).$$

Let $v \in F \cap E_\lambda$ for some $\lambda \in \Lambda_1$ and $g \in U^-$. Then by the same argument as above we get

$$v = \lim_{n\to\infty} a_n.(g.v) = \lim_{n\to\infty} \sum_{\lambda' \in \Lambda_0} \lambda'(a_n)(g.v)_{\lambda'}$$

and hence $(g.v)_{\lambda'} = 0$ whenever $\lambda' \neq \lambda$. Hence $g.v = v$. In particular, v is fixed by both U^+ and U^-, which together generate all of $\mathrm{SL}_3(\mathbb{R})$, i.e. v is $\mathrm{SL}_3(\mathbb{R})$-invariant. □

2.3. **Small groups.** In both examples above, the subgroups were relatively large. In the first case, the subgroup was (up to finite index) a maximal (and also minimal) parabolic subgroup of $\mathrm{SL}_2(\mathbb{R})$. In the second case, the unipotent radical of the group generated is U^+, which is a full horospherical subgroup.

In fact, it can be shown that every simple $\mathbb{R}$-algebraic group admits a solvable subgroup B of dimension at most three with property (1). We cite the following explicit example for $\mathrm{SL}_n(\mathbb{R})$ from [M95] and refer the reader to the original article for an outline of the proof. Let $n \in \mathbb{N}$ at least 3. Then B can be chosen to be the closed subgroup with Lie algebra $\mathfrak{b} \subseteq \mathfrak{sl}_n(\mathbb{R})$ generated by the elements

$$h = \begin{pmatrix} n-1 & & & \\ & n-3 & & \\ & & \ddots & \\ & & & 1-n \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 & 0 & & & 0 \\ & 0 & 1 & 0 & & \\ & & \ddots & & & \\ & & & 0 & 1 & 0 \\ & & & & 0 & 1 \\ 0 & & & & & 0 \end{pmatrix},$$

$$u = \begin{pmatrix} 0 & 0 & & & 1 \\ & 0 & & & 0 \\ & & \ddots & & \\ 0 & & & 0 & 0 \\ 0 & & & & 0 \end{pmatrix}.$$

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Property (T)

Isabella Scott

1. Definition and Examples

Throughout this abstract, $\mathcal{H}$ will denote a Hilbert space and G a topological group. Write $U(\mathcal{H})$ for the unitary group of $\mathcal{H}$ and let π be a unitary representation of G.

Then for Q a subset of G, and $\epsilon > 0$, we say that ξ is (Q, ϵ)-*invariant* for π if and only if

$$\sup_{\xi \in Q} ||\pi(x)\xi - \xi|| < \epsilon.$$

Moreover, π has *almost-invariant vectors* if for every compact $Q \subseteq G$, and every $\epsilon > 0$, π has (Q, ϵ)-invariant vectors. This sets us up for the central definition of the talk, Kazhdan's definition of Property (T).

Definition 1. *$Q \subseteq G$ is a* Kazhdan set *for G if there is some $\epsilon > 0$ such that every unitary representation π of G with a (Q, ϵ)-invariant vector has a nontrivial invariant vector.*

G has Property (T) *if it has a compact Kazhdan set.*

In other words, Property (T) says that the existence of a nontrivial invariant vector for a unitary representation is ensured by there being "approximately invariant vectors" over a "small" set.

Example 1. *Every compact group G has Property (T) with Kazhdan set G and $\epsilon = \sqrt{2}$.*

Example 2. *$\mathbb{R}^n$ and $\mathbb{Z}^n$ do not have Property (T). This follows from them being amenable and the following fact, which states that essentially, being amenable is orthogonal to having Property (T).*

Proposition 1. *If G is a locally compact group, the following are equivalent:*

(i) G is amenable and has Property (T).
(ii) G is compact.

Example 3. *$SL_n(\mathbb{R})$, $n \geq 2$, is an example of a noncompact group with Property (T). In fact, any simple real Lie group of real rank at least 2 has Property (T).*

2. First Consequences

The results in this section are largely due to Kazhdan ([2]) and illustrate his motivation for defining Property (T).

The first result of this section gives us information about the hereditary properties of Property (T).

Theorem 1. *Let G be a locally compact group with a closed subgroup H. Suppose G/H has a finite, invariant, regular Borel measure. Then:*

$$G \text{ has Property (T) if and only if } H \text{ does.}$$

In particular, for any lattice $\Gamma \leq G$, G has Property (T) if and only if Γ does.

Thus, any lattice in a simple real Lie group of real rank at least 2 has Property (T). In particular, $SL_n(\mathbb{Z})$, $n \geq 2$, is the first example of an infinite discrete Property (T) group we have encountered.

On the other hand, F_2 embeds as a lattice in $SL_2(\mathbb{R})$, so $SL_2(\mathbb{R})$ and hence $SL_2(\mathbb{Z})$ do not have Property (T).

The next result in this section concerns the existence of compact generating sets for groups with Property (T).

Theorem 2. *Let G be a group with Property (T). Then G is compactly generated.*

Corollary 1. *Any lattice with Property (T) is finitely generated, so in particular any lattice of a simple real Lie group of real rank at least 2 is finitely generated.*

3. Property (FH) and Actions on Trees

In this section we introduce another property of topological groups, Property (FH) and investigate its relation of Property (T) and actions of G on trees. In this section $\mathcal{H}$ will always be assumed to be a real Hilbert space. Because every isometry of a Hilbert space is affine, we will be interested in affine representations of G.

Definition 2. *An* affine isometric action*, or just* isometric action *of G on $\mathcal{H}$ is a strongly continuous group homomorphism $\alpha : G \to \mathrm{Isom}(\mathcal{H})$.*

Definition 3. *G has* Property (FH) *iff every affine isometric action of G on a real Hilbert space has a fixed point.*

Equivalently, G has Property (FH) iff $H^1(G, \pi) = 0$ for every orthogonal representation π of G.

We now show that Property (FH) implies there is no action of G on a tree that does not fix a vertex or geometric edge. In order to state this result, we remind the reader of Serre's convention for graphs.

A graph $X = (V, \mathbb{E})$ is a pair consisting of a set of vertices, V, and a set of oriented edges $\mathbb{E}$; that is, each $e \in \mathbb{E}$ has a "starting vertex" $s(e)$ and a "terminal vertex" $t(e)$. Moreover, since we want to consider unoriented graphs, we stipulate there is a fixed-point free involution $e \mapsto \overline{e}$ where $s(e) = t(\overline{e})$ and $t(e) = s(\overline{e})$. We call the set $\{e, \overline{e}\}$ a *geometric edge* of X.

The main result of this section is the following.

Theorem 3. *Let G be a group with Property (FH). Then every automorphic action of G on X fixes either some vertex or some geometric edge.*

We will give an overview of the proof here.

Lemma 1. *Let G be a group acting on a tree X. If G has a bounded orbit then G fixes either a vertex or a geometric edge.*

Proof. Let O be the bounded orbit and X_0 the convex hull of O. Then X_0 is a G-invariant bounded subtree. Define X_1 to be the subtree of X_0 obtained by deleting all the leaves of X_0 and all edges adjacent to them. This is still a G-invariant bounded subtree. Continue with this process and let N be minimal so that X_N is empty. Then X_{N-1} is either a single vertex or a pair of vertices with a geometric edge between them. Because X_{N-1} is G-invariant, G fixes a vertex or a geometric edeg respectively. □

Before we state the next lemma, we need some background. If G acts on X by automorphism, we may define a Hilbert space $\mathcal{H}$ by

$$\mathcal{H} = \{\xi : \mathbb{E} \to \mathbb{R} \mid \xi(e) = -\xi(\overline{e}) \text{ and } \sum_{e \in \mathbb{E}} |\xi(e)|^2 < \infty\}$$

with inner product $\langle \xi, \eta \rangle = \frac{1}{2} \sum_{e \in \mathbb{E}} \xi(e)\eta(e)$. Thus, from the action of G on $\mathbb{E}$, we obtain an orthogonal representation $\pi_X : G \to \mathcal{H}$.

Lemma 2. *Let G be a group acting on a tree X by automorphisms fixing no vertex or geometric edge. Then $H^1(G, \pi_X) \neq 0$.*

Proof sketch. It can be shown that, in this setting, a cocycle is coboundary if and only if it's bounded. However, assuming that $H^1(G, \pi_X) = 0$ and hence every cocycle is bounded, one can construct a bounded orbit, hence contradicting the previous lemma. □

Thus, since G having Property (FH) is equivalent to $H^1(G, \pi_X) = 0$, we obtain that any action of G on a tree fixes either a vertex or a geometric edge.

Finally, the following result, due independently to Delorme and Guichardet, (Theorem 2.12.4 in [1]) ties this last section back into Property (T).

Theorem 4. *(1) If G has Property (T), then G has Property (FH)*
(2) If G is a σ-compact locally-compact group with Property (FH), then G has Property (T).

In this way, we obtain that any automorphic action of a group G with Property (T) on a tree X with must either fix a vertex or geometric edge.

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Property (T) groups acting on the circle

Sang-hyun Kim

Recall a countable group Γ has property (T) if and only if it has the property (FH). This means that whenever one is given with

- a real Hilbert space $\mathcal{H}$;
- a representation $\pi : \Gamma \to O(\mathcal{H})$;
- a 1-cocycle $c : \Gamma \to \mathcal{H}$,

one can find some $\xi \in \mathcal{H}$ such that the following holds for all $g \in \Gamma$:

$$c(g) = \xi - \pi(g)\xi.$$

For $\tau \in (0, 1)$, we denote by $\mathrm{Diff}_+^{1+\tau}(S^1)$ the set of orientation-preserving $C^{1+\tau}$ diffeomorphisms on S^1. It is a simple exercise to see that $\mathrm{Diff}_+^{1+\tau}(S^1)$ is actually a group where the binary operation is given by the function composition.

In this talk, we prove the following result originally due to A. Navas.

Theorem 1. [5] *Let $\tau > 1/2$, and let Γ be a countable group with property (T). Then every homomorphism $\Gamma \to \mathrm{Diff}_+^{1+\tau}(S^1)$ has a finite image.*

One can consider the above theorem as a generalization of the following result. This result is in a sense a resolution of Zimmer's conjecture in dimension one, independently proved by Burger–Monod and by Ghys;

Theorem 2. [2, 3] *If* Γ *is a lattice in a higher-rank simple Lie group, then every representation*

$$\Phi : \Gamma \to \mathrm{Homeo}_+(S^1)$$

admits a finite orbit. Furthermore, every representation

$$\Psi : \Gamma \to \mathrm{Diff}^1_+(S^1)$$

has a finite image.

In order to prove Theorem 1, we let τ and Γ be as in the hypothesis, and fix a representation

$$\Phi : \Gamma \to \mathrm{Diff}^{1+\tau}_+(S^1).$$

We use a "recipe" of $\mathcal{H}, \pi$ and c to apply the property (T) of Γ, and then will find some Γ–invariant measure on $S^1 \times S^1$, and finally deduce that $\Phi(\Gamma)$ is finite.

We use the notation as below.

- $S^1 = \mathbb{R}/2\pi\mathbb{Z}$;
- $\Delta = \{(x,x) \mid x \in S^1$;

We have a diagonal action of Γ on $S^1 \times S^1$ induced from Φ.

Let us define a Liouville measure μ on $S^1 \times S^1 \setminus \Delta$, by specifying its value for $a < b < c < d < a + 2\pi$ as:

$$\mu((a,b) \times (c,d)) := \log[e^{ia}, e^{ib}, e^{ic}, e^{id}].$$

Here, $[\cdot]$ denotes the cross ratio so that

$$[x,y,z,w] := \frac{(w-y)(z-x)}{(z-y)(w-x)}.$$

For each $g \in \Gamma$, we have the pull-back measure

$$g\mu(A) := \mu(g^{-1}A).$$

The following Radon–Nikodym derivative is defined almost every where:

$$\frac{dg\mu}{d\mu}(x).$$

Moreover, we have that

$$\frac{dg\mu}{d\mu} d\mu = dg\mu,$$

and that

$$\frac{dgh\mu}{d\mu} = \frac{dg\mu}{d\mu} \cdot \frac{dh\mu}{d\mu} \circ g^{-1}.$$

We will let

$$\mathcal{H} := \{K \in L^2(S^1 \times S^1 \setminus \Delta, \mu) \mid K(x,y) = K(y,x) \text{ almost everywhere}\}.$$

A *natural* unitary representation $\pi : \Gamma \to O(\mathcal{H})$ can be defined for $K \in \mathcal{H}$ by

$$\pi(g)K := \sqrt{\frac{dg\mu}{d\mu}} \cdot K \circ g^{-1}.$$

We have the following.

Lemma 1. *(1) $\pi(gh)K = \pi(g)\pi(h)K$ a.e. for all $g, h \in \Gamma$ and for all symmetric map $K : S^1 \times S^1 \setminus \Delta \to \mathbb{R}$.*
(2) $\|\pi(g)K\| = \|K\|$ for all $K \in \mathcal{H}$.

As a final ingredient of our desired recipe, we define a 1-cocycle, which may be regarded "formally" a 1-coboundary as below:

$$c(g) := 1 - \pi(g)1 = 1 - \sqrt{dg\mu/d\mu}.$$

Lemma 2. *We have $c \in Z^1(\Gamma, \pi)$. That is, we have*

- *$c(gh) = c(g) + \pi(g)c(h)$ for all $g, h \in \Gamma$;*
- *$\|c(g)\| < \infty$ for all $g \in \Gamma$.*

Remark 1. *An alternative approach is to consider*

- *$\mathcal{H}' := \{K \in L^2(S^1 \times S^1 \setminus \Delta, \mathrm{Leb}) \mid K(x, y) = K(y, x) \text{ a.e.}\}$;*
- *$\pi(g)K(x, y) := \sqrt{(g^{-1})'(x)(g^{-1})'(y)}K \circ g^{-1}$;*
- *$\phi := (2 \tan \frac{x-y}{2})^{-1}$;*
- *$c(g) := \phi - \pi(g)\phi$.*

Remark 2. *Note that Reznikov considered a Hilbert transform*

$$(Hf)(x) = \frac{1}{2\pi} \int_{S^1} \frac{f(y)dy}{\tan \frac{x-y}{2}},$$

and defined the fundamental cocycle *ℓ from $\mathrm{Diff}_+^{1+\tau}(S^1)$ to some subspace J of the space of bounded operators on $L^2(S^1, \mathrm{Leb})$, where J can be given with some Hilbert structure. More specifically, he defined*

$$\ell(g) = \pi(g)^{-1} H \pi(g) - H,$$

so that

$$\ell(gh) = \ell(g)^{\pi(h)} + \ell(h).$$

Reznikov also mentioned that $[\ell] \neq 0$ unless the action is "completely pathological".

To finish the proof, we apply the property (T) to see that there exists $F \in \mathcal{H}$ satisfying

$$1 - \sqrt{\frac{dg\mu}{d\mu}} = c(g) = F - \pi(g)F = F - \sqrt{\frac{dg\mu}{d\mu}} F \circ g^{-1}$$

for all $g \in \Gamma$. So, we have a Γ–invariant measure

$$d\lambda := d\mu(1 - F)^2 = dg\mu(1 - F \circ g^{-1})^2.$$

Moreover, we note that

- $\lambda([a,a]\times[b,c])=0$;
- $\lambda((a,b)\times(b,c))=\infty$;

It is then easy to deduce the following intermediate result.

Lemma 3. *Let $\tau>1/2$, let Γ be a countable group with property (T), and let*

$$\Phi:\Gamma\to \mathrm{Diff}_+^{1+\tau}(S^1)$$

be a homomorphism. If $g\in\Gamma$ satisfies $\#\,\mathrm{Fix}\,\Phi(g)\geq 3$, then $\Phi(g)=1$.

Let us consider a 3–fold cover

$$\mathrm{Homeo}_+^{(3)}(S^1):=\{f\in\mathrm{Homeo}_+(S^1)\mid \mathrm{Rot}(2\pi/3)f=f\,\mathrm{Rot}(2\pi/3)\}\to\mathrm{Homeo}_+(S^1).$$

We have a universal central extension:

$$1\to\mathbb{Z}/3\mathbb{Z}\to\mathrm{Homeo}_+^{(3)}(S^1)\to\mathrm{Homeo}_+(S^1)\to 1.$$

A classical result due to Hölder asserts that every group acting faithfully and freely on S^1 is abelian. The proof of the main theorem can then be deduced from the above lemma, combined with Hölder's theorem and the above central extension.

Let us remark that Fisher and Margulis (documented in [1]) that for each countable group Γ with property (T) there exists $\epsilon_0=\epsilon_0(\Gamma)>0$ such that Γ has property F_{L^p} for all $p\in(1,2+\epsilon_0)$. So, instead of $\mathcal{H}$ we may consider

$$\mathcal{B}:=L^p_{\mathrm{symmetric}}(S^1\times S^1\setminus\Delta,\mu)$$

in the above proof. Then we obtain the following strengthened result.

Theorem 3. [1] *For each a countable group Γ with property (T), there exists $\tau_0=\tau_0(\Gamma)<1/2$ such that every representation $\Gamma\to\mathrm{Diff}_+^{1+\tau_0}(S^1)$ has a finite image. In particular, every representation $\Gamma\to\mathrm{Diff}_+^{1.5}(S^1)$ has a finite image.*

Lastly, we note the following result due to Y. Lodha, N. Matte Bon and M. Triestino.

Theorem 4. [4] *For each a countable group Γ with property (T), every representation*

$$\Gamma\to\mathrm{PL}_+(S^1)$$

has a finite image.

One can replace $\mathrm{PL}_+(S^1)$ by $\mathrm{PDiff}_+^{1.5}(S^1)$, the group of *piecewise* $C^{1.5}$ diffeomorphisms. It is not currently known whether there exists an embedding of Γ into $\mathrm{Diff}_+^1(S^1)$, or even $\mathrm{Homeo}_+(S^1)$.

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Strong property (T) in the proof of Zimmer's conjecture

Federico Vigolo

The aim of this talk is to introduce Lafforgue's strong property (T) and show how it is used in the proof of the Zimmer Conjecture.

1. Strong Property (T)

Recall that a finitely generated group Γ has property (T) if every unitary representation $\pi\colon \Gamma \to U(\mathcal{H})$ with no non-zero fixed point cannot have almost invariant vectors. Given a probability measure μ on Γ, the operator $\pi(\mu)\colon \mathcal{H} \to \mathcal{H}$ is defined as $\pi(\mu)(v) = \int \pi(g)d\mu(g)$. In particular, if S is a symmetric finite generating set of Γ and $\mu_S = \frac{1}{|S|}\sum_{s\in S}\delta_S$ is the sum of delta measures, then $\pi(\mu_S)(v) = \frac{1}{|S|}\sum_{s\in S}\pi(s)(v)$.

It is an exercise to show that Γ has property (T) if and only if for every unitary representation π with no non-zero invariant vectors the operator $\pi(\mu_S)$ has norm strictly smaller than 1. In turn, this implies that Γ has property (T) if and only if for *every* unitary representation π the operators $\pi(\mu_S)^n = \pi(\mu_S^{*n})$ converge exponentially fast to the orthogonal projection to the subspace of Γ-invariant vectors $\mathcal{H} \to \mathcal{H}^\Gamma$.

Strong property (T) can be seens as a strengthening of the latter characterisation. The main difference from the usual property (T) is that it also applies to representations that are not unitary:

Definition 1. *A group* Γ *has* strong property (T) *if there exists a* $t > 0$ *and a sequence of measures* μ_n *supported on* $B(n) \subset \Gamma$ *that satisfy the following condition: for every linear representation* $\pi\colon \Gamma \to GL(\mathcal{H})$ *such that* $\|\pi(g)\| < Le^{t\ell(g)}$ *for some* $L \geq 1$*—here* $\ell(g)$ *is the* word length*—there exists a projection* $P\colon \mathcal{H} \to \mathcal{H}^\Gamma$ *such that* $\pi(\mu_n)$ *converges to* P *exponentially fast.*

Remark 1. *(1) This is the definition of strong property (T) given in* [1] *(it is modelled on the definition given in* [2]*). The original definition is given by Lafforgue in terms of idempotent elements in group algebras.*

(2) *It is easy to show that strong (T) implies (T). On the other hand, Lafforgue showed that hyperbolic groups cannot have strong property (T). Since there are hyperbolic groups with (T), it follows that strong property (T) really is a stronger notion.*

Strong property (T) is used to prove the following:

Theorem 1 ([1, Theorem 2.9])**.** *If Γ has strong property (T) and $\alpha\colon \Gamma \curvearrowright M$ is a C^∞ action with uniform subexponential growth of derivatives (USEGOD) on a compact manifold, then α preserves a Riemmanian metric on M.*

Remark 2. (1) *This is a key step in the proof of the Zimmer conjecture: knowing that α preserves a Riemannian metric means that α is a homomomorphism of Γ into the compact Lie group* Isom(M)*. The Zimmer conjecture will then follow by Margulis's superrigidity.*

(2) *It is possible to prove a version of Theorem 1 for actions of class C^k. This requires the usage of a Banach-valued strong property (T).*

2. Idea of the proof of Theorem 1

To illustrate the main ideas, we sketch the proof of Theorem 1 for a smooth action on the unit circle S^1. In this context, the space of Riemannian manifolds on S^1 can be naturally identified with the space of positive valued C^∞ functions on S^1. This is a subset of the vector space of all smooth functions on S^1:

$$\{\text{Riemannian metrics}\} \longleftrightarrow \{\sigma\colon S^1 \to \mathbb{R}_{>0} \mid \sigma \text{ smooth}\} \subset C^\infty(S^1;\mathbb{R}).$$

The action $\alpha\colon \Gamma \curvearrowright S^1$ induces an action on the space of Riemannian metrics. When identifying Riemannian metrics with smooth functions, this action is given by $\gamma\cdot\sigma(x) = |\frac{d\gamma}{dx}|(x)\sigma(\gamma\cdot x)$. Therefore, α extends to a linear action on $C^\infty(S^1;\mathbb{R})$. Now that we have a linear representation, we wish to use strong property (T) to find a fixed point in $C^\infty(S^1;\mathbb{R})$ and then show that that fixed point is actually a Riemannian metric.

To begin with, it is necessary to put a Hilbert norm on $C^\infty(S^1;\mathbb{R})$. More precisely, we need to chose a norm so that:

(1) the completion of $C^\infty(S^1;\mathbb{R})$ is a Hilbert space;
(2) the linear action $\Gamma \curvearrowright C^\infty(S^1;\mathbb{R})$ extends by continuity to an action on the completion so that we can apply strong property (T);
(3) there is enough control on the completion so that Γ-fixed points in the completion can be assumed to strictly positive functions in $C^\infty(S^1;\mathbb{R})$.

Definition 2. *For $k \in \mathbb{N}$, the k-*th Sobolev norm *of $\sigma \in C^\infty(S^1;\mathbb{R})$ is defined as*

$$\|\sigma\|_{2,k} = \left(\sum_{i=0}^{k} \|\frac{d^i\sigma}{dx^i}\|_2^2\right)^{\frac{1}{2}}.$$

*The k-*th Sobolev space *$W^{2,k}$ is the completion of $C^\infty(S^1;\mathbb{R})$ with respect to the norm $\|\cdot\|_{2,k}$. Note that $W^{2,k}$ is can be identified with a closed subspace of $L^2(S^1)$.*

The Sobolev space $W^{2,k}$ is a Hilbert space and it is easy to show that the action of Γ on $C^\infty(S^1;\mathbb{R})$ extends to a continuous action on $W^{2,k}$, *i.e.* a representation $\pi_k\colon \Gamma \to GL(W^{2,k})$ by continuous operators. The following fact is much less obvious and it depends on a (non-trivial) estimate on the norms of higher order derivatives in terms of the norm of the first derivatives:

Proposition 1. *If $\Gamma \curvearrowright S_1$ has subexponential growth of derivatives, then for every $\epsilon > 0$ there exists an L_ϵ such that $\|\pi_k(g)\| < L_\epsilon e^{\epsilon \ell(g)}$ for every $g \in \Gamma$.*

Proposition 1 implies that the representations π_k easily satisfy the requirements necessary to apply strong property (T). That is, if μ_n is the sequence of measures as by Definition 1, then the operators $\pi_k(\mu_n)$ converge to a projection to the set of Γ-fixed vectors in $W^{2,k}$ in the operator norm. In particular, if σ_0 is any fixed Riemannian metric on S^1, the limit

$$\widetilde{\sigma} = \lim_{n\to\infty} \pi_k(\mu_n)(\sigma_0)$$

is Γ-invariant. Importantly, when identifying $W^{2,k}$ with a subspace of $L^2(S^1)$, the element $\pi_k(\mu_n)(\sigma_0)$ does not depend on k. In particular, the function $\widetilde{\sigma}$ does not depend on k and it belongs on $W^{2,k}$ for every $k \in \mathbb{N}$.

The following holds true:

Theorem 2 (Sobolev Embedding). *Every function σ in the Sobolev space $W^{2,k}$ is of class C^r for $r = k - \frac{1}{2}$. Furthermore, there is a constant K such that $|\sigma(x)| \leq K\|\sigma\|_{2,k}$ for every $x \in S^1$.*

It follows from the Sobolev Embedding Theorem that the Γ-fixed point $\widetilde{\sigma}$ is of class C^∞. All that it remains to do to prove that $\widetilde{\sigma}$ is a Γ-invariant Riemannian metric is to show that $\widetilde{\sigma}(x) > 0$ for every $x \in S^1$. In order to do this it is necessary to use the USEGOD condition a second time.

In fact, it follows from USEGOD that for every ϵ there is a L_ϵ such that

$$g\cdot \sigma_0(x) \geq \frac{1}{L_\epsilon} e^{-\epsilon \ell(g)} \sigma_0(g\cdot x).$$

On the other hand, the convergence of $\pi_k(\mu_n)(\sigma_0)$ to $\widetilde{\sigma}$ is exponentially fast. Combining this with the second part of the Sobolev Embedding Theorem implies that

$$|\pi_k(\mu_n)(\sigma_0)(x) - \widetilde{\sigma}(x)| \leq e^{s\cdot n}$$

for some fixed $s > 0$. Since μ_n is supported on the ball of radius n, $\pi_k(\mu_n)(\sigma_0)(x)$ is at least as large as $\frac{1}{L_\epsilon} e^{-\epsilon\cdot n} \min\{\sigma_0(y) \mid y \in S^1\}$. Choosing $\epsilon < s$ shows that $\widetilde{\sigma}(x)$ must indeed be positive. This concludes the proof of Theorem 1 in the case of actions on S^1. □

This proof can be adapted to work for actions on higher dimensional manifolds as well. The complications are for the most part technical and the main difficulty lies in finding the correct analogue of the Sobolev norm. The strategy adopted in [1] was to note that Riemannian metrics can be seen as smooth sections of a vector bundle over M (this is a vector space with pointwise operations). In turn, the 'derivatives' of such sections can be seen as elements of appropriate *jet spaces*.

Fixing a Riemannian metric, one can define natural norms on the jet spaces. These norms can be used to define an analogue of the Sobolev norm and the Sobolev Embedding Theorem remains true. All the other elements of the proof can be made to work as well.

In order to prove Theorem 1 for actions of class C^k, it is necessary to use the Sobolev spaces $W^{p,k}$ and let $p \to \infty$. In fact, a more general version of the Sobolev Embedding Theorem would then imply that the limit $\widetilde{\sigma}$ is of class C^r for $r = k - \frac{n}{p}$. These Sobolev spaces are not Hilbert spaces, it is thus necessary to also use a Banach version of strong property (T).

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Strong Property (T): Ideas of proof for $G = \mathbf{SL_3}(\mathbb{R})$

Ping Ngai (Brian) Chung

1. Introduction

Lafforgue's strong property (T) plays an important role in the proof of Zimmer's conjecture by A. Brown, D. Fisher and S. Hurtado [BFH16]. The goal of this talk is to understand the ideas of Lafforgue's proof that $\mathrm{SL}_3(\mathbb{R})$ has strong property (T) [Laf08, Sect. 2]. To simplify the exposition we only prove that $\mathrm{SL}_3(\mathbb{R})$ has Kazhdan's property (T) using Lafforgue's ideas. We mostly follow the exposition in [del16, Ch. 1]

2. Statement

Recall the following definitions:

Definition 1. *Let G be a Lie group and $\mathcal{H}$ be a Hilbert space. A unitary representation $G \to \mathcal{U}(\mathcal{H})$ has* **almost invariant vectors** *if:*
for all $\varepsilon > 0$, and compact subset $Q \subset G$, there exists a unit vector $v \in \mathcal{H}$ such that

$$\|\pi(g)v - v\| < \varepsilon \qquad \text{for all } g \in Q.$$

Definition 2. *A real Lie group G has* **property (T)** *if any unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$ that has almost invariant vectors also has a $\pi(G)$-invariant unit vector.*

We refer the readers to Dave Morris's book [WM15, Ch. 13] for basic properties and motivations of this definition.

The goal of the lecture is to prove the following theorem using ideas of Lafforgue [Laf08]:

Theorem 1. *The group $G = \mathrm{SL}_3(\mathbb{R})$ has property (T).*

The theorem is due to Kazhdan [Kaz67]. See [WM15, Theorem 13.1] for an exposition of a standard proof. The proof we present follows closely the ideas in Lafforgue's proof [Laf08, Thm. 2.1] that $\mathrm{SL}_3(\mathbb{R})$ has strong property (T). The purpose is to understand Lafforgue's proof in this simpler setting. We will comment on the changes needed to prove that $\mathrm{SL}_3(\mathbb{R})$ has strong property (T).

Unless otherwise specified, any constants C, C' below are explicit positive absolute constants. In particular the statements are true for, say, $C = 100$ and $C' = 20$. See [del16, Ch. 1] for the exact constants.

3. Outline of the proof

Let

$$G := \mathrm{SL}_3(\mathbb{R}), \qquad K := \mathrm{SO}_3, \qquad U := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \cap K.$$

The proof follows from a few reductions.

(1) If $\pi : G \to \mathcal{U}(\mathcal{H})$ has almost invariant vectors, then π has a nonzero $\pi(G)$-invariant vector.

(2) If $\pi : G \to \mathcal{U}(\mathcal{H})$ has almost invariant vectors, then there exists $P \in \mathcal{B}(\mathcal{H})$ such that

$$\lim_{g\to\infty} \left\| \int_{K\times K} \pi(kgk')\, dk\, dk' - P \right\|_{\mathcal{B}(\mathcal{H})} = 0.$$

Moreover, the limit $P \in \mathcal{B}(\mathcal{H})$ satisfies $\|P\|_{\mathcal{B}(\mathcal{H})} = 1$ and image$(P) \subset \mathcal{H}^\pi := \{v \in \mathcal{H} \mid \pi(g)v = v \quad \text{for all } g \in G\}$, the space of $\pi(G)$-invariant vectors.

(3) For any unitary representation $\pi : G \to \mathcal{U}(\mathcal{H})$, any $\pi(K)$-invariant unit vectors $v, w \in \mathcal{H}$, and elements $g_1, g_2 \in G$, we have

$$|\langle \pi(g_1)v, w\rangle - \langle \pi(g_2)v, w\rangle| \le C \max(|g_1|^{-1/2}, |g_2|^{-1/2}),$$

where for $g \in G$, $|g| := \max(\|g\|, \|g^{-1}\|)$ for the operator norm $\|\cdot\|$ on the group G.

(4) For any unitary representation $\pi : K \to \mathcal{U}(\mathcal{H})$, any $\pi(U)$-invariant unit vectors $v, w \in \mathcal{H}$ and $\delta \in [-1, 1]$, we have

$$|\langle \pi(k_\delta)v, w\rangle - \langle \pi(k_0)v, w\rangle| \le C' \sqrt{|\delta|},$$

where for each $\delta \in [-1, 1]$,

$$k_\delta := \begin{pmatrix} \delta & -\sqrt{1-\delta^2} & 0 \\ \sqrt{1-\delta^2} & \delta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in K.$$

We further divide (2) into two parts:

(2a) If $\pi : G \to \mathcal{U}(\mathcal{H})$ has almost invariant vectors, then there exists $P \in \mathcal{B}(\mathcal{H})$ such that
$$\lim_{g\to\infty} \left\| \int_{K\times K} \pi(kgk')\, dk\, dk' - P \right\|_{\mathcal{B}(\mathcal{H})} = 0.$$

(2b) The limit $P \in \mathcal{B}(\mathcal{H})$ satisfies $\|P\|_{\mathcal{B}(\mathcal{H})} = 1$ and $\mathrm{image}(P) \subset \mathcal{H}^\pi$.

The proof then follows the order (4) $\Rightarrow$ (3) $\Rightarrow$ (2a) $\Rightarrow$ (2b) $\Rightarrow$ (1). Clearly (1) is the definition of G having property (T).

4. (2) $\Rightarrow$ (1): Existence of the limit as a projection operator implies existence of a nonzero invariant vector

This reduction is straightforward from the properties of P: Since $\|P\|_{\mathcal{B}(\mathcal{H})} = 1$, in particular $P \neq 0$, the image of P contains nonzero vectors. Now (1) follows as $\mathrm{image}(P) \subset \mathcal{H}^\pi$.

5. (3) $\Rightarrow$ (2a): Cauchy criterion on matrix coefficients of G/K implies existence of limit

Let
$$A_g := \int_{K\times K} \pi(kgk')\, dk\, dk' \in \mathcal{B}(\mathcal{H}).$$
Then it suffices to show that there exists $P \in \mathcal{B}(\mathcal{H})$ such that
$$\lim_{g\to\infty} \|A_g - P\|_{\mathcal{B}(\mathcal{H})} = 0.$$
This shall follow from the Cauchy criterion and the completeness of $\mathcal{B}(\mathcal{H})$. Recall that for $\phi \in \mathcal{B}(\mathcal{H})$,
$$\|\phi\|_{\mathcal{B}(\mathcal{H})} = \sup_{v\in\mathcal{H}, \|v\|=1} \|\phi v\| = \sup_{v,w\in\mathcal{H}, \|v\|=\|w\|=1} \langle \phi v, w\rangle.$$
Then (3) implies that for all $g_1, g_2 \in G$,
$$\|A_{g_1} - A_{g_2}\|_{\mathcal{B}(\mathcal{H})} \leq C \max(|g_1|^{-1/2}, |g_2|^{-1/2}).$$
This implies that the family $\{A_g\}_{g\in G} \subset \mathcal{B}(\mathcal{H})$ satisfies the Cauchy criterion. By completeness of $\mathcal{B}(\mathcal{H})$, there exists an operator $P \in \mathcal{B}(\mathcal{H})$ such that
$$\|A_g - P\|_{\mathcal{B}(\mathcal{H})} \to 0 \qquad \text{as } g \to \infty.$$

6. (2a) $\Rightarrow$ (2b): Properties of the limit operator P

We prove the two assertions $\|P\|_{\mathcal{B}(\mathcal{H})} = 1$ and $\mathrm{image}(P) \subset \mathcal{H}^\pi$ separately.

$\|P\|_{\mathcal{B}(\mathcal{H})} = 1$: Since π has almost invariant vectors, for each $g \in G$, by taking $Q = KgK$, we have that for all $\varepsilon > 0$, there exists a nonzero vector $v \in \mathcal{H}$ such that
$$\|A_g v\| \geq (1-\varepsilon)\|v\|.$$

Thus $\|A_g\|_{\mathcal{B}(\mathcal{H})} \geq 1$. On the other hand, as the average of unitary operators, $\|A_g\|_{\mathcal{B}(\mathcal{H})} \leq 1$. Thus $\|A_g\|_{\mathcal{B}(\mathcal{H})} = 1$. Since $P = \lim_{g\to\infty} A_g$, we also have $\|P\|_{\mathcal{B}(\mathcal{H})} = 1$.

image$(P) \subset \mathcal{H}^\pi$: For each $v \in \mathcal{H}$, we need to show that $\pi(g)Pv = Pv$ for all $g \in G$. We consider the average of $\pi(k)\pi(g)Pv$ over $k \in K$,

$$\begin{aligned}\int_K \pi(k)\pi(g)Pv\,dk &= \lim_{h\to\infty} \int_{K\times K\times K} \pi(kgk'hk'')v\,dk\,dk'\,dk'' \\ &= \lim_{h\to\infty} \int_K A_{gk'h} v\,dk' = Pv.\end{aligned}$$

Here the first equality comes from the fact that $P = \lim_{g\to\infty} A_g$ and the definition of A_g, the second equality follows from the definition of $A_{gk'h}$, and the last follows since P is the limit of A_{g_i} along any sequence g_i in G that goes to infinity.

Since π acts **unitarily** on $\mathcal{H}$, $\|\pi(kg)Pv\| = \|Pv\|$ for all $k \in K$. The previous equality shows that Pv is the average of a family of vectors $\{\pi(kg)Pv\}_{k\in K} \subset \mathcal{H}$ with the same norm. By strict convexity of $\mathcal{H}$, this implies $\pi(kg)Pv = Pv$ for all $k \in K$ and $g \in G$. In particular Pv is $\pi(G)$-invariant.

We remark that combining these two properties and the obvious observation that P acts as the identity on $\mathcal{H}^\pi$, we can in fact conclude that P is the orthogonal projection onto $\mathcal{H}^\pi$.

7. (4) ⇒ (3): Estimates on matrix coefficients of K/U implies estimates on matrix coefficients of G/K

By the KAK decomposition of $G = \mathrm{SL}_3(\mathbb{R})$, we know that there is a bijection

$$\begin{array}{ccc} K\backslash G/K & \leftrightarrow & \text{Weyl chamber } \Lambda \\ K\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}K & \leftarrow\!\shortmid & (r,s,t) \end{array}$$

Recall that the Weyl chamber of G can be written as

$$\Lambda := \{(r,s,t) \in \mathbb{R}^3 \mid r \geq s \geq t, \quad r+s+t=0\}.$$

Fix two $\pi(K)$-invariant unit vectors $v, w \in K$. Then the matrix coefficient $g \mapsto \langle \pi(g)v, w\rangle$, which *a priori* is a function of G, is indeed a function of $K\backslash G/K \leftrightarrow \Lambda$. Let

$$c(r,s,t) := \langle \pi(g)v, w\rangle \qquad \text{for all } g \in K\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}K.$$

Note that

$$\|g\| = e^r, \qquad \|g^{-1}\| = e^{-t} \qquad \text{for } g \in K\begin{pmatrix} e^r & 0 & 0 \\ 0 & e^s & 0 \\ 0 & 0 & e^t \end{pmatrix}K.$$

Then it suffices to show that

$$(*)\qquad |c(r,s,t)-c(r',s',t')| \le C\max(\min(e^{-r/2},e^{t/2}),\min(e^{-r'/2},e^{t'/2})).$$

The strategy is then the following:

Step 1 By considering suitable embeddings of $U\backslash K/U \to K\backslash G/K$, we show that (4) implies the upper bound

$$|c(r,s,t)-c(r',s',t')| \le C'e^{t/2} \qquad \text{for all } t=t',\quad s,s'\ge -1.$$

Step 2 Apply the previous step to the representation $g\mapsto \pi((g^t)^{-1})$, (4) implies the upper bound

$$|c(r,s,t)-c(r',s',t')| \le C'e^{-r/2} \qquad \text{for all } r=r',\quad s,s'\le 1.$$

Step 3 Bootstrap the estimates from **Step 1, 2** to other pairs of points (r,s,t), $(r',s',t')\in\Lambda$ by connecting them suitably with finitely many segments where either the t-coordinates are equal or r-coordinates are equal (with $s,s'\in[-1,1]$ for all except the first and last segment).

Without loss of generality, assume that $\max(r,-t)\le\max(r',-t')$. By insisting that, if $r\ge -t$, we start with a segment from (r,s,t) fixing the r-coordinate, and a segment fixing the t-coordinate otherwise, we can obtain the desired estimate (*). The key point here is that since the difference on each segment is bounded by an exponential function of the distance from the origin of Λ (given by $\max(r,-t)$), the sum of the total differences on this path is bounded by a geometric series with leading term equal to $\min(e^{-r/2},e^{t/2})$. We refer to [del16, Ch. 1] for the precise choice of paths.

More precisely, in **Step 1**, for each $t\le 0$, we consider the map

$$\begin{array}{cccc} f_t: & U\backslash K/U & \to & K\backslash G/K \\ & k & \mapsto & D_t k D_t \end{array}$$

where $k\in K$, and

$$D_t := \begin{pmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t/2} & 0 \\ 0 & 0 & e^{t/2} \end{pmatrix}.$$

Note that D_t commutes with U, therefore the map is well-defined. To use (4), we use the bijection

$$\begin{array}{ccc} [-1,1] & \leftrightarrow & U\backslash K/U \\ \delta & \mapsto & k_\delta. \end{array}$$

We can then prove **Step 1** by applying (4) and the embedding f_t for each $t\le 0$.

We remark that the exact argument would go through if the bound $C'\sqrt{|\delta|}$ in (4) is replaced by $C'|\delta|^\theta$ for some $\theta>0$ (but not for $\theta=0$), with suitable adjustments on the exponents in the upper bound of (3).

8. Idea of (4): Explicit calculation on the irreducible unitary representations of SO_3

By the Peter-Weyl theorem, it suffices to show (4) for all **irreducible** unitary representations π of $K = SO_3$. In fact we know that the irreducible unitary representations of SO_3 are parametrized by positive integers n, as described by the classical theory of spherical harmonics. Moreover, the unitary representation $SO_3 \to \mathcal{U}(L^2(S^2))$ induced by the natural action of SO_3 on the 2-sphere S^2 contains each irreducible unitary representation with multiplicity 1. Therefore it suffices to show (4) for this unitary representation $\pi_{L^2(S^2)} : K \to \mathcal{U}(L^2(S^2))$.

With respect to this representation $\pi_{L^2(S^2)}$, we can describe (4) more geometrically. For $\delta \in [-1, 1]$ and $f \in L^2(SO_3)$, let

$$T_\delta f(k) := \int_{U \times U} f(uk_\delta^{-1}u'k)\, du\, du'.$$

Clearly $T_\delta f$ is invariant under the left multiplication by U. Therefore we can identify $S^2 \leftrightarrow SO_2\backslash SO_3 = U\backslash K$, and consider T_δ as a unitary representation of K on $L^2(S^2)$. (4) can then be rephrased as

$$\|T_\delta - T_0\|_{\mathcal{B}(L^2(S^2))} \leq C'\sqrt{|\delta|}.$$

Geometrically, $T_\delta f(x)$ is the average of f on the circle $\{y \in S^2 \mid \langle x, y\rangle = \delta\}$. From here this is verified by an explicit calculation (see [Laf08, Lem. 2.2]).

It is worth noting that there is no analogous inequality for $L^2(S^1)$ (the best bound is $C'|\delta|^0$). Therefore one cannot use the same strategy to (incorrectly) show that $SL_2(\mathbb{R})$ has property (T).

9. Modification to prove strong property (T)

To prove strong property (T) using these ideas, we need to relax the assumption that π is a unitary representation to a representation with small exponential growth. For instance, the equality

$$\langle \pi(g)v, \pi(h)w\rangle = \langle \pi(h^{-1}g)v, w\rangle,$$

which we used multiple times, is now only true up to a multiple bounded by $\|\pi(h)\|^2$. If $h \in K$, then this can easily be fixed by modifying the norm $\|\cdot\|$ on $\mathcal{H}$ so that K still preserves the norm (and therefore $\pi|_K$ is a unitary representation of K). However, if $h \notin K$, then one only has a small exponential bound on $\|\pi(h)\|$ with respect to $\|h\|$. In this section, we list two main modifications that are necessary in the proof.

(i) In (2a) $\Rightarrow$ (2b), to prove image$(P) \subset \mathcal{H}^\pi$, we can no longer use the fact that $\|\pi(kg)Pv\| = \|Pv\|$ to prove

$$\int_K \pi(kg)Pv\, dk = Pv \qquad \Rightarrow \qquad \pi(kg)Pv = Pv \quad \text{for all } k \in K.$$

Instead, Lafforgue proved a version of exponential decay of matrix coefficients: suppose π has small exponential growth, for all $g \in G$, and unit vectors $v, w \in \mathcal{H}$ such that v is $\pi(K)$-invariant and w has nontrivial K-type (i.e. w is in an irreducible K-subrepresentation of $\mathcal{H}$ that is not trivial),

$$|\langle \pi(g)v, w\rangle| \leq C\|g\|^{-\theta}$$

for some $\theta > 0$. This is proved in [Laf08, Prop. 2.4].

(ii) In **Step 1** of (4) $\Rightarrow$ (3), when applying (4) to the embedding f_t with the $\pi(U)$-invariant vectors $\pi(D_t)v$ and $\pi(D_t^{-1})w$, we need to use the fact that for a unitary representation π,

$$\langle \pi(D_t k_\delta D_t)v, w\rangle = \langle \pi(k_\delta)(\pi(D_t)v), \pi(D_t^{-1})w\rangle.$$

Without the unitary assumption, we only have that

$$|\langle \pi(D_t k_\delta D_t)v, w\rangle| \leq |\pi(D_t)|^2 |\langle \pi(k_\delta)(\pi(D_t)v), \pi(D_t^{-1})w\rangle\rangle|$$

(in fact this inequality is applied to $|\langle \pi(D_t k_\delta D_t)v, w\rangle - \langle \pi(D_t k_0 D_t)v, w\rangle|$ on the left hand side.) Note that $\|D_t\| = e^{-t}$ for $t \leq 0$. If π has sufficiently slow exponential growth, say of rate $\varepsilon < 1/4$, then we can still obtain the conclusion of **Step 1** with a bound of $C' e^{(1/2-2\varepsilon)t}$. Since $1/2 - 2\varepsilon > 0$, this is still sufficient to obtain the conclusion of (3).

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Proof of Zimmer's conjecture for cocompact lattice in $\mathbf{SL}(n, \mathbb{R})$.

Homin Lee

The goal of the lecture is giving a full proof of following theorem based on previous lectures. Especially, we will assemble every ingredients including, subexponential growth, entropy and strong property (T).

Theorem 1. *Let Γ be a lattice in $G = \mathrm{SL}(n, \mathbb{R})$ with $n > 2$, let M be a compact manifold and let $\rho\colon \Gamma \to \mathrm{Diff}(M)$ be a homomorphism. Then if $d = \dim(M) < n - 1$, the image of ρ is finite.*

Indeed, we can use general invariance principle so that the homomorphism $\rho\colon \Gamma \to \mathrm{Diff}_{\mathrm{vol}}(M)$ is finite provided by $\dim(M) \leq n-1$. The main arguments, that are Proposition 1 and 2, do not use dimension conditions and volume preserving conditions.

For detailed proof, one can see [1].

1. Introduction

I will start with giving outlines and steps in the proof. Eventually, based on previous lectures, we will see that Proposition 1 and 2 are enough to show the Theorem 1. After then, we will see proofs of Proposition 1 and 2.

Indeed, the philosophy of the proof can be written as "averaging gives invariant one.". We already saw this philosophy when we find invariant Riemannian metric on M using strong property (T). We will use *"averaging"* arguments repeatedly on proofs.

2. Outline of the proof

We will see that proving Proposition 1 and 2 is enough. We always assume conditions and notations on Theorem 1. The first step is making suspension space. We will follow notations for suspension spaces as previous lectures. Morally, the following propositions say that if action fails to have subexponential growth, then there should exist measure that can see the failure of subexponential growth.

Proposition 1. *If ρ fails to have uniform subexponential growth of derivative, then there is A-invariant ergodic probability measure μ' on suspension space M^ρ such that $\lambda^F_{top}(s,\mu') > 0$ for some $s \in A$. This implies that there is a nonzero fiberwise Lyapunov exponent $\lambda^F_{j',\mu'} : A \to \mathbb{R}$.*

Proposition 2. *Suppose that there is A-invariant ergodic probability measure μ' on suspension space M^ρ such that there is a nonzero fiberwise Lyapunov exponent $\lambda^F_{j',\mu'} : A \to \mathbb{R}$. Then there is A-invariant ergodic probability measure μ on M^ρ such that*

(1) there is a nonzero fiberwise Lyapunov exponent $\lambda^F_{j,\mu} : A \to \mathbb{R}$,
(2) $\pi_*\mu$ is Haar measure on G/Γ where $\pi : M^\rho \to G/\Gamma$ is natural projection.

Before going further, let me remark a few things.

Remark 1. *The above propositions hold without assumption on relation between dimension and rank. On the other hand, cocompactness of Γ is only used in the above propositions. In* [3]*, they overcome this issues.*

Step 1. Prove that ρ has uniform subexponential growth of derivative. Here we will use Proposition 1 and 2. If ρ fails to have uniform subexponential growth of derivative, then the above propositions show that there is A-invariant ergodic probability measure μ on M^ρ such that there is nonzero fiberwise Lyapunov exponent and projects to Haar measure on G/Γ. This gives us to G-invariant probability measure on M^ρ with nonzero fiberwise Lyapunov exponent. However, this gives contradiction due to Zimmer's cocycle superrigidity theorem and dimension condition. More precisely, there is no nontrivial group homomorphism from $\mathrm{SL}(n,\mathbb{R})$ to $\mathrm{SL}(d,\mathbb{R})$ for $d < n-1$, fiberwise derivative cocycle is cohomologous to compact group valued cocycle. This contradicts to have nonzero fiberwise Lyapunov exponent.

Step 2. Using strong property (T) and uniform subexponential growth of derivative, we can find invariant Riemannian metric on M with certain regularity. For simplicity, assume that ρ is C^∞ action. Then we can find smooth Γ-invariant Riemannian metric on M. Therefore, the problem is reduced to group homomorphism $\rho : \Gamma \to \mathrm{Isom}(M)$ where $\dim \mathrm{Isom}(M) \leq \frac{d(d+1)}{2}$ where d is dimension of M.

Step 3. Finally, Margulis superrigidity shows that every group homomorphism from Γ to $\mathrm{Isom}(M)$ has finite image provided that $\dim \mathrm{Isom}(M) < n^2 - 1$. This implies that $\rho(\Gamma)$ should be finite since $\frac{d(d+1)}{2} < n^2 - 1$ when $d < n-1$.

Now it is enough to show Proposition 1 and 2.

3. Some facts about averaging

In order to prove Proposition 1 and 2, we need some facts from averaging on general manifold and homogeneous space. The facts for homogeneous space requires Ratner's theorems.

Proposition 3 (Averaging on M^ρ.)**.** *Let $s \in A$ and let μ be a s-invariant probability measure on M^ρ. Let H be an amenable group that commutes with s. We can average μ along H via Følner sequence on H.*

(1) For any weak limit of averaging μ along H is s and H invariant.*

(2) (Upper semi-continuity) $\lambda^F_{top}(s, \mu') \geq \lambda^F_{top}(s, \mu)$ for any weak limit μ' of averaging μ along H.*

Note that in the above Proposition, M^ρ does not role that means it is true for any space that G acts.

Theorem 2 (Averaging on G/Γ.)**.** *Let $\widehat{\mu}$ be a probability measure on $\mathrm{SL}(n, \mathbb{R})/\Gamma$. For unipotent subgroup U in $\mathrm{SL}(n, \mathbb{R})$,*

(1) there is a unique weak limit of averaging μ along U denoted by $U * \widehat{\mu}$.*

(2) if $\widehat{\mu}$ is A invariant, then $U * \widehat{\mu}$ is also A-invariant.

(3) if $\widehat{\mu}$ is A invariant and A ergodic, then $U * \widehat{\mu}$ is also A-ergodic.

(4) if $\widehat{\mu}$ is A invariant and U^{ij} invariant, then $\widehat{\mu}$ is also U^{ji} invariant. where U^{ij} is one-parameter unipotent subgroup $\{I + tE_{ij} : t \in \mathbb{R}\}$ and $(E_{ij})_{kl} = \delta_{i,k}\delta_{j,l}$.

4. Proof of the Proposition 1.

The proof of the Proposition 1 is similar for one diffeomorphism. Recall following lemma.

Lemma 1. *The ρ has uniform subexponential growth of derivative if and only if the induced action of G on M^ρ has uniform subexponential growth of derivative fiberwise., i.e. for any $\epsilon > 0$ there is $C_\epsilon = C > 0$ such that*

$$\sup_{x \in M^\rho} ||D_x g|_F|| \leq C e^{\epsilon d(e,g)}$$

for any $g \in G$.

Now we assume that ρ fails to have uniform subexponential growth of derivative. Then we may assume that G action on M^ρ fails to have uniform subexponential growth of derivative fiberwise. Then we can find witness of failure of uniform subexponential growth that is $g_n \in G$, $g_n \to \infty$ and (x_n, v_n) in UF where UF is unit sphere bundle in fibers. In other words, there is $\epsilon > 0$ such that

$$||D_{x_n} g_n v_n|| > e^{\epsilon d(e, g_n)}.$$

Using KAK decomposition, we may replace g_n to $a_n \in A$ after some calculation. Now we can think $a_n = \exp(t_n X_n)$ for some $X_n \in \mathrm{Lie}(G)$. Note that since unit sphere in $\mathrm{Lie}(G)$ is compact, we can find accumulation point of $\{X_n\}$ in the unit sphere. We can define action of G on UF as in the previous lecture and define the probability measure

$$\nu_n = \frac{1}{[t_n]} \sum_{j=0}^{[t_n]-1} \exp(jX_n)_* \delta_{(x_n, v_n)}$$

on UF. Now, using compactness, up to subsequence there is a nonzero X that is accumulation point of $\{X_n\}$ such that weak* limit ν of ν_n is invariant under $s = \exp(X) \in A$. Define $\overline{\mu}$ be a projection of ν to M^ρ with respect to $UF \to M^\rho$. Then, due to our construction, $\overline{\mu}$ is s-invariant and we have $\lambda^F_{top}(s, \overline{\mu}) \geq \epsilon > 0$. Since A is abelian, especially commute with s and amenable, we can average $\overline{\mu}$ along A so that we can find μ' that is A-invariant ergodic probability measure on M^ρ such that $\lambda^F_{top}(s, \mu') \geq \epsilon > 0$ after using facts from averaging and ergodic decomposition (with respect to A). This proves Proposition 1.

5. Proof of the Proposition 2.

For simplicity, I will give a sketch of proof when $G = \mathrm{SL}(3, \mathbb{R})$ case. Let μ' be the measure in the condition. Note that the projection $\pi : M^\rho \to G/\Gamma$ is equivariant under actions.

5.1. **First step.** We can find two elements s_1 and s_2 that is linearly independent when we identify A with $\mathbb{R}^2$. For example, $s_1 = \mathrm{diag}(1/2, 1/2, 4)$ and $s_2 = \mathrm{diag}(1/2, 4, 1/2)$. Then we can find a fiberwise Lyapunov exponent $\lambda^F_{j,\mu'}$ such that $\lambda^F_{j',\mu'}(s_1) \neq 0$. Without loss of generality, we may assume that it is positive. Then we can find one parameter unipotent subgroup U^{ij} that is commute with s_1. In our case, U^{12}. Define probability measures on M^ρ as

(1) $\mu_0 = \mu'$.
(2) μ_1 be the measure that is averaging μ' along U^{12}. That is s_1 invariant and U^{12} invariant.
(3) μ_2 be the measure that is averaging μ_1 along A. That is A-invariant but may not be U^{12} invariant.
(4) $\widehat{\mu_i} = \pi_* \mu_i$, $i = 0, 1, 2$, where $\pi : M^\rho \to G/\Gamma$ be the natural projection.

Then $\widehat{\mu_1} = U^{12} * \widehat{\mu_0}$ from the facts about averaging on G/Γ. Especially, $\widehat{\mu_1}$ is U^{12} invariant. Also, $\widehat{\mu_1}$ is A-invariant since μ_0 so that $\widehat{\mu_0}$ is A-invariant. Here we used facts from averaging on G/Γ (mainly due to Ratner's theroem). Since

$\widehat{\mu_1}$ is already A-invariant and μ_0 is A-ergodic, we can see that $\widehat{\mu_1} = \widehat{\mu_2}$ and $\widehat{\mu_2}$ is A-ergodic and still U^{12} invariant. Moreover, due to upper-semi continuity of top averaging fiberwise Lyapunov exponents, after using ergodic decomposition, we may find μ_2' such that

(1) μ_2' is A-ergodic;
(2) $\lambda^F_{top}(s_1, \mu_2') > 0$ and;
(3) $\pi_* \mu_2' = \widehat{\mu_2}$ is A and U^{12} invariant.

Now we can deduce that $\pi_* \mu_2' = \widehat{\mu_2}$ is A, U^{12} and U^{21} invariant due to Ratner's theorem.

5.2. **Second Step.** We need to improve A, U^{12} and U^{21} invariance to G invariance. We will play same game as before. Since μ_2' is A-ergodic, we can find nonzero Lyapunov linear functional $\lambda^F_{i,\mu_2'} : A \to \mathbb{R}$. Here we need to divide two cases. Let $s_2 = \text{diag}(1/2, 4, 1/2)$ and $s_3 = \text{diag}(4, 1/2, 1/2)$. Then $\lambda^F_{i,\mu_2'}(s_2) \neq 0$ or $\lambda^F_{i,\mu_2'}(s_3) \neq 0$. Since both cases are similar, we only focus on $\lambda^F_{i,\mu_2'}(s_2) \neq 0$. As before, up to inverse, we may assume it is positive. Again we define probability measures on M^ρ as

(1) $\mu_3 = \mu_2'$ that is A-ergodic.
(2) μ_4 be the measure that is averaging μ_3 along U^{13}. That is s_2 invariant and U^{13} invariant.
(3) μ_5 be the measure that is averaging μ_4 along A. That is A invariant.
(4) $\widehat{\mu_i} = \pi_* \mu_i$, i=3,4,5.
(5) $\widehat{\mu_3} = \widehat{\mu_2}$ is A-ergodic, U^{12} and U^{21} invariant.
(6) $\widehat{\mu_4}$ is A-ergodic U^{13} and U^{21} invariant (here we use U^{13} and U^{21} commutes.) so that U^{31} and U^{12} invariant due to Ratner's theorem
(7) $\widehat{\mu_5} = \widehat{\mu_4}$ is $A, U^{12}, U^{21}, U^{13}$ and U^{31} invariant so that G invariant.

We can see that μ_5 is A-invariant measure on M^ρ such that projects to Haar measure on G/Γ. As before upper-semi continuity of top averaging fiberwise Lyapunov exponents shows that $\lambda^F_{top}(s_2, \mu_5) > 0$. This proves Proposition 2.

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