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ON THE COMPUTATIONAL CONTENT OF THE THEORY OF BOREL EQUIVALENCE RELATIONS

NIKOLAY BAZHENOV, BENOIT MONIN, LUCA SAN MAURO, AND RAFAEL ZAMORA

Abstract. This preprint offers computational insights into the theory of Borel equivalence relations. Specifically, we classify equivalence relations on the Cantor space up to computable reductions, i.e., reductions induced by Turing functionals. The presented results correspond to three main research focuses: (i) the poset of degrees of equivalence relations on reals under computable reducibility; (ii) the complexity of the equivalence relations generated by computability-theoretic reducibilities ($\leqslant_T, \leqslant_{tt}, \leqslant_m, \leqslant_1$); (iii) the effectivization of the notion of hyperfiniteness.

1. Introduction

A *reduction* of an equivalence relation E on a domain X to an equivalence relation F on a domain Y is a (nice) function $f : X \to Y$ such that

$$x \; E \; y \;\Leftrightarrow\; f(x) \; F \; f(y).$$

That is, f pushes down to an injective map on the quotient sets, $X_{/E} \mapsto Y_{/F}$. To assess the relative complexity of E and F, it is natural to impose a bound on the complexity of the reduction f, as otherwise, if the size of $X_{/E}$ is not larger than the size of $X_{/F}$, then the Axiom of Choice alone would guarantee the existence of a reduction from E to F.

In the literature, such a reducibility has two main interpretations:

- In descriptive set theory, one typically assumes that X and Y are Polish spaces and f is Borel;
- In computability theory, one typically assumes that $X = Y = \omega$ and f is computable.

In this preprint, we hybridize these two traditions, by classifying equivalence relations on the Cantor space up to computable reductions, i.e., reductions induced by Turing functionals: for $E, F \subseteq 2^\omega \times 2^\omega$, E is *computably reducible* to F (notation: $E \leqslant_{\mathbf{0}} F$) if there exists a total Turing functional Φ_e so that, for all $X, Y \in 2^\omega$,

$$X \; E \; Y \;\Leftrightarrow\; \Phi_e^X \; F \; \Phi_e^Y.$$

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Similarly, one defines $\mathbf{d}$-computable reductions by allowing the oracle to access to any $D \in \mathbf{d}$.

Computable reductions in Cantor space have been previously explored by Miller [27], who focused on calibrating the complexity of reductions between well-known combinatorial Borel equivalence relations. Here, we introduce three parallel lines of research:

First, in Section 2, we initiate a systematic study of $\mathbf{ER}(2^\omega)$ — the poset of degrees of equivalence relations on reals under computable reducibility. In recent years, degree structures generated by computable reducibility (on ω) attracted a lot of interest (see, e.g., [1, 3, 4, 6]). Here, we want to understand to which extent information available for the countable case can be lifted to the domain of reals. We will show that $\mathbf{ER}(2^\omega)$ is a rich (and maybe wild) structure: we explore its maximal chains and antichains, and we prove that it is neither an upper nor a lower semilattice. We will also offer a comparison between our poset and $\mathbf{ER}(\omega)$, the analogous degree structure for countable equivalence relations [2], and we use this comparison to illustrate the failure of the celebrated Glimm-Effros dichotomy [16] for computable reducibility.

Secondly, in Section 3, we use computable reducibility to compare equivalence relations generated by the main computability-theoretic reducibilities: i.e., Turing reducibility $\leqslant_T$, truth-table reducibility $\leqslant_{tt}$, many-one reducibility $\leqslant_m$, and one-one reducibility $\leqslant_1$. The complexity of such equivalence relations is of primary importance in the theory of countable Borel equivalence relations. For example, it is a long open question of Kechris [22] whether Turing equivalence is a universal countable Borel equivalence relation; the same question is open for 1-equivalence, see Marks [26] for many results in the vicinity of this problem. On the other hand, Turing equivalence and 1-equivalence of c.e. sets are respectively universal Σ^0_4 and universal Σ^0_3 equivalence relations[1] [17, 13]. In this section, we compare these equivalence relations with respect to both $\mathbf{0}$-computable reductions and $\mathbf{0}'$-computable reductions. Finally, we investigate which oracles are able to perform reductions between $\mathrm{Id}(2^\omega)$ (the identity on reals) or E_0 (eventual agreement on reals) to $\equiv_T$.

Finally, in Section 4, we study the effective content of Dougherty-Jackson-Kechris' theorem [10], which states that a Borel equivalence relation is hyperfinite if and only if it is Borel reducible to E_0. We show that the effectivization of this result already fails for Σ^0_2 equivalence relations, as Σ^0_2 equivalence relations computably below E_0 need not to have a hyperfinite Σ^0_2 presentation. On the other hand, we show that each Π^0_1 equivalence relation is smooth via a computable reduction, effectivizing a classical fact.

1.1. **Preliminaries.** We follow the tradition of calling elements of the Cantor space *reals*. Sometimes we also adopt the tradition of calling set of reals

[1]We note that the result on Σ^0_3 universality of 1-equivalence (for c.e. sets) easily follows from Proposition 5 of [31]. We are grateful to Victor Selivanov for communicating this fact.

mass problems. We denote reals with uppercase letters, natural numbers with lowercase letters. We assume that the reader is familiar with basic notation and terminology from computability theory and descriptive set theory (as in [32, 15]).

1.1.1. *Strings.* Let $\sigma, \tau \in 2^{<\omega}$ and $A \in 2^\omega$. The *concatenation* of σ and $\tau \in 2^{<\omega}$ is denoted by $\sigma^\frown\tau$. If σ is an initial segment of τ, we write $\sigma \subseteq \tau$. The notation $\sigma * A$ corresponds to the following real

$$\sigma * A = \begin{cases} \sigma(i) & \text{for } i < |\sigma| \\ A(i) & \text{for } i \geqslant |\sigma|. \end{cases}$$

We say that a set of strings is *prefix-free* if all strings in it are pairwise incomparable with respect to $\subseteq$. Given $\sigma \in 2^{<\omega}$, we denote by $[\sigma]$ the clopen subset of 2^ω generated by σ, i.e., $\{X \in 2^\omega : \sigma \subset X\}$.

1.1.2. *Turing functionals.* We fix an enumeration $\{\Phi_e\}_{e\in\omega}$ of Turing functionals. We write $\Phi_e(X, n)$ for the value of the functional with oracle $X \in 2^\omega$ on integer $n \in \omega$. We write $\Phi_e(X, n)[t]\downarrow$ to emphasize that the computation halts before t steps and $\Phi_e(X, n)[t]\uparrow$ to emphasize the opposite. We sometimes view Turing functionals as partial function from $2^\mathbb{N}$ to $2^\mathbb{N}$ in which case $\Phi(X)$ — if defined — denotes the set $Y \in 2^\mathbb{N}$ such that $\Phi(X, n)\downarrow = Y(n)$.

A total Turing functional $\Phi : 2^\omega \to 2^\omega$ can be *represented* by a computable function $\varphi : 2^{<\omega} \to 2^{<\omega}$ which satisfies the following requirements:

(1) if $\sigma \subseteq \tau$, then $\varphi(\sigma) \subseteq \varphi(\tau)$;
(2) for all n, there is m so that

$$|\varphi(\sigma)| \geqslant n, \text{ for every } |\sigma| \geqslant m;$$

(3) for all $X \in 2^\omega$, $\Phi(X) = \bigcup_{n\in\omega} \varphi(X\restriction_n)$.

1.1.3. *Ordinals.* By O we denote the set of notations of computable ordinals. For $a \in O$ we write $|a|$ for the ordinal coded by a. Given X we denote by O^X the set of codes for ordinals which are computable in X. Given $a \in O^X$ we write $|a|^X$ for the ordinal coded by a, when X is used as oracle for the computation, or sometimes $|a|$ when there is no ambiguity. Let ω_1^X denote the smallest non X-computable ordinal and by ω_1^{ck} the smallest non-computable ordinal. For $\alpha < \omega_1^X$ we write $O^X_{<\alpha}$ the set of codes for ordinals which are computable in X and such that $|a| < \alpha$. For any $\alpha < \omega_1^X$ the set $O^X_{<\alpha}$ is $\Delta^1_1(X)$ uniformly in a code for α. For every X the set O^X is $\Pi^1_1(X)$-complete uniformly in X: It is a $\Pi^1_1(X)$ set and for any $\Pi^1_1(X)$ set $\mathcal{P} \subseteq \omega \times 2^\omega$ there is computable function $f : \omega \to \omega$ such that $(n, Y) \in \mathcal{P}$ iff $f(n) \in O^{X\oplus Y}$.

Given an ordinal $\alpha < \omega_1^{ck}$ we write $X^{(\alpha)}$ for the α-th jump of X.

1.1.4. *Equivalence relations.* Let E, F be equivalence relations. For $x \in \mathrm{dom}(E)$ and $A \subseteq \mathrm{dom}(E)$, we denote by $[x]_E$ and $[A]_E$ respectively the E-class of x and the E-saturation of A. Given a mass problem $\mathcal{A}$, we denote by $E(\mathcal{A})$ the equivalence relation consisting of exactly two classes: $\mathcal{A}$ and $2^\omega \smallsetminus \mathcal{A}$.

The *uniform join* $E \oplus F$ of equivalence relations $E, F \subseteq 2^\omega \times 2^\omega$ is the equivalence relation which encodes E into $[0]$ (i.e., if $X \; E \; Y$, then $0^\frown X \; E \oplus F \; 0^\frown Y$) and F into $[1]$ (i.e., if $X \; F \; Y$, then $1^\frown X \; E \oplus F \; 1^\frown Y$). Fixing a uniformly computable sequence of strings $(\tau_i)_{i\in\omega}$ which is prefix-free (e.g., see $\mathfrak{S}_{0^\infty}$ below for an example), one can define the *generalized uniform join* $\bigoplus_{i\in\omega} E_i$ of countably many equivalence relations by encoding each E_i into $[\tau_i]$. If $X \subseteq 2^\omega \times 2^\omega$, then $E_{/X}$ denotes the equivalence relation generated by the set of pairs $E \cup X$. In analogy with the terminology for equivalence relations on ω, we say that $E_{/X}$ is a *quotient* of E. A quotient G of a uniform join $E \oplus F$ is *pure* if for any $X \in 2^\omega$, we have:

- $[0^\frown X]_G \cap [0] = [0^\frown X]_{E\oplus F}$,
- $[1^\frown X]_G \cap [1] = [1^\frown X]_{E\oplus F}$.

It is easy to see that any pure quotient of $E \oplus F$ is an upper bound of E and F.

We conclude this section with two simple but useful lemmas.

Lemma 1.1. *Let $X \in 2^\omega$, and let R, S be equivalence relations on reals. If $R \leqslant_{\mathbf{0}} S$, then the number of R-classes containing X-computable reals is at most the same as the number of S-classes containing X-computable reals.*

Proof. If Φ is a total Turing operator and $A \leqslant_T X$, then the real Φ^A is also X-computable. □

Lemma 1.2. *If $\Phi : E \leqslant_{\mathbf{0}} F$ and Y is a real in* $\mathrm{range}(\Phi)$*, then $[\Phi^{-1}(Y)]_E$ contains a real X which is Y'-computable.*

Proof. Construct a Y-computable tree $T \subset 2^\omega$ by saying that $\sigma \in T$ if and only if Φ^σ is an initial segment of Y. This induces, computably in Y, a $\Pi^0_1(Y)$ class $\mathcal{C}$. Since $\mathcal{C}$ is Y-computably bounded, it follows that there is $X \in \mathcal{C}$ such that $X \leqslant_T Y'$ and $\Phi(X) = Y$. □

2. A structural view of $\mathbf{ER}(2^\omega)$

Here, we offer a first look into $\mathbf{ER}(2^\omega)$, the poset of degrees of equivalence relations on reals up to computable reducibility: in particular, we compare this poset to known degree structures and we prove that it is not a lattice.

Remark 2.1. *Note that most of the results obtained in this section apply also to the substructure of* $\mathbf{ER}(2^\omega)$ *consisting of* Borel *equivalence relations.*

2.1. **Chains and antichains.** Some information about $\mathbf{ER}(2^\omega)$ is readily inherited by the theory of Borel equivalence relations. For instance, Louveau and Veličković [25] proved that the inclusion of reals modulo finite differences $(2^\omega, \subseteq^*)$ embeds in $(\mathcal{E}, \leqslant_B)$, the poset of Borel equivalence relations under Borel reducibility. It follows that the latter poset has antichains of size $2^{\aleph_0}$, and thus the same holds for $\mathbf{ER}(2^\omega)$, since computable reducibility is a refinement of Borel reducibility. Yet, this result is not optimal; we will ameliorate it below. Before this, let us state some simple properties of $\mathbf{ER}(2^\omega)$.

Proposition 2.2. $\mathbf{ER}(2^\omega)$ *has a least element but no maximal element. Furthermore, each degree of* $\mathbf{ER}(2^\omega)$ *is countable and it has countably many predecessors.*

Proof. Any constant Turing functional (e.g., $X \mapsto 0^\infty$) reduces $\mathrm{Id}_1(2^\omega)$ to any given equivalence relation. Thus, $\mathbf{Id_1}(2^\omega)$ is the least degree of $\mathbf{ER}(2^\omega)$. Next, it is clear that no finite equivalence relation can be maximal, as $F \oplus \mathrm{Id}_1(2^\omega)$ is strictly above F, if F is finite. So, suppose that F is infinite and fix a countable F-transversal $\{u_e\}_{e\in\omega}$. We build a pure quotient $F \oplus F_{/A}$ by diagonalizing against all potential reductions $F \oplus F_{/A}$ to F. More precisely, for all e, let $(0^\frown u_e, 1^\frown u_e)$ be in A if and only if $\Phi_e(0^\frown u_e) \not F\ \Phi_e(1^\frown u_e)$. The quotient so constructed is pure, since 0- and 1-classes of $F \oplus F$ are never collapsed, thus $F \leqslant_{\mathbf{0}} F \oplus F_{/A}$. Moreover, having diagonalized against all Φ_e, we obtain that $F \oplus F_{/A} \not\leqslant_{\mathbf{0}} F$ and so F is not maximal. Finally, each reduction from E to a given F is witnessed by one of countably many Turing functionals. Hence, each degree is at most countable and it has at most countably many predecessors. □

Combining the countable predecessor property and the lack of maximal elements, we immediately obtain a size bound to the chains of $\mathbf{ER}(2^\omega)$.

Proposition 2.3. *Every maximal chain of* $\mathbf{ER}(2^\omega)$ *has size* $\aleph_1$.

Proof. Since every degree in $\mathbf{ER}(2^\omega)$ has at most countably many predecessors, $\mathbf{ER}(2^\omega)$ cannot have chains of size $> \aleph_1$. On the other hand, every countable chain $(E_i)_{i\in\omega}$ inside $\mathbf{ER}(2^\omega)$ is extendable. Indeed, Proposition 2.2 guarantees that there is F strictly above $\bigoplus_{i\in\omega} E_i$, and, for all i, we have $E_i < (\bigoplus_{i\in\omega} E_i) \oplus F$. □

Since the cardinality of $\mathbf{ER}(2^\omega)$ as a poset is $2^{2^{\aleph_0}}$ and its height is $\aleph_1$, it immediately follows that the width of $\mathbf{ER}(2^\omega)$ is $2^{2^{\aleph_0}}$. Examples of antichains of maximum size are obtained as a consequence of the next lemma, recalling that the Medvedev lattice embeds antichains of size $2^{2^{\aleph_0}}$ [29] (the reader is referred to [33] for more information about the Medvedev lattice)

Lemma 2.4. *If* $\mathcal{X}, \mathcal{Y}$ *are mass problems of incomparable Medvedev degrees, then* $E(\mathcal{X})$ *and* $E(\mathcal{Y})$ *are incomparable by computable reducibility.*

Proof. Towards a contradiction, suppose that there are $\mathcal{X} \mid_M \mathcal{Y}$ so that $\Phi : E(\mathcal{X}) <_{\mathbf{0}} E(\mathcal{Y})$. Note that $\mathcal{X} \cup \mathcal{Y}$ contains no computable reals, as otherwise there would be a Medvedev reduction from either to the other. It follows that Φ maps $\mathcal{X}$ to $\mathcal{Y}$ and $\overline{\mathcal{X}}$ to $\overline{\mathcal{Y}}$, as, by Lemma 1.1, a Turing functional necessarily maps classes containing computable reals to classes containing computable reals. But then Φ induces a Medvedev reduction from $\mathcal{Y}$ to $\mathcal{X}$, a contradiction. □

Now, we will show that *every* maximal antichain of $\mathbf{ER}(2^\omega)$ is uncountable. To this end, let us first consider minimal pairs.

Proposition 2.5. *In $\mathbf{ER}(2^\omega)$, there are $2^{\aleph_0}$ degrees which form pairwise a minimal pair.*

Proof. Let X, Y be reals such that $X \not\leqslant_T Y'$. We claim that $E(\{X\})$ and $E(\{Y\})$ is a minimal pair. Towards a contradiction, suppose that there is an equivalence relation on reals $F \neq \mathrm{Id}_1(2^\omega)$ such that $\Phi_X : F \leqslant_{\mathbf{0}} E(\{X\})$ and $\Phi_Y : F \leqslant_{\mathbf{0}} E(\{Y\})$. One can show that F has exactly two equivalence classes: one closed and not open, and the other open and not closed.

By Lemma 1.2, there is $Z_0 \leqslant_T Y'$ such that $\Phi_Y(Z_0) = Y$. By continuity, Z_0 belongs to the closed class of F and therefore $\Phi_X(Z_0) = X$. Thus, $X \leqslant_T Z_0 \leqslant_T Y'$, contradicting our choice of X and Y. So, to obtain $2^{\aleph_0}$ degrees of $\mathbf{ER}(2^\omega)$ which form pairwise a minimal pair, it suffices to use a family of $2^{\aleph_0}$ Turing degrees which are pairwise inequivalent w.r.t. jump-reducibilities. □

Proposition 2.6. *Every maximal antichain of $\mathbf{ER}(2^\omega)$ has size at least $2^{\aleph_0}$.*

Proof. Fix a family $\mathfrak{F}$ of $2^{\aleph_0}$ degrees which form pairwise a minimal pair, as in the previous proposition. Suppose that $\mathfrak{A}_0$ is an antichain of size $< 2^{\aleph_0}$. Let $\mathfrak{A}$ be the downward closure of $\mathfrak{A}_0$ with respect to computable reducibility. Note that, for all equivalence relation $E \neq \mathrm{Id}_1$ in $\mathfrak{A}$, E can be bounded by at most one equivalence relation F in $\mathfrak{F}$. Hence, there are continuum many equivalence relations in $\mathfrak{F} \smallsetminus \mathfrak{A}$ which avoid the upward closure of $\mathfrak{A}$. So $\mathfrak{A}_0$ can be extended and therefore it is not maximal. □

Question 1. *Are there maximal antichains in $\mathbf{ER}(2^\omega)$ of different size? Namely, is there an antichain of exactly continuum size?*

2.2. **Comparing $\mathbf{ER}(\omega)$ and $\mathbf{ER}(2^\omega)$.** We now move to the comparison between $\mathbf{ER}(\omega)$ and $\mathbf{ER}(2^\omega)$, by which we will be able to obtain a further structural result: the poset $\mathbf{ER}(2^\omega)$ is not a lattice.

To this end, it will be often convenient to fix a suitable collection of binary strings as follows. For a real X, let

$$\mathfrak{S}_X = \{\sigma \in 2^{<\omega} : (\forall i < |\sigma| - 1)(\sigma(i) = X(i)) \ \&\ \sigma(|\sigma| - 1) \neq X(|\sigma| - 1)\}.$$

Observe that $\mathfrak{S}_X$ is prefix-free. Note that $[\mathfrak{S}_X]$ covers Cantor space with the exception of X.

Theorem 2.7. $\mathbf{ER}(\omega)$ *embeds into* $\mathbf{ER}(2^\omega)$.

Proof. Let $(\tau_i)_{i\in\omega}$ denote a uniformly computable enumeration of $\mathfrak{S}_{0^\infty}$ (i.e., $\{0^n 1 : n \in \omega\}$. The strategy for encoding a given equivalence relation R on natural numbers by an equivalence relation $\iota(R)$ on reals is straightforward: each number x is encoded by the clopen set $[\tau_x]$, so that, if $u \; R \; v$, we could set $X \; \iota(R) \; Y$ for all $X \in [\tau_u]$ and $Y \in [\tau_v]$. But for technical reasons, due to the fact that 0^∞ is a limit point of $\mathfrak{S}$ and we need to preserve computable reductions, we adopt a slightly more complicated encoding by which we cylindrify $\mathfrak{S}$.

Let $F_0 \subseteq 2^\omega \times 2^\omega$ be the equivalence relation defined as

$$X \; F_0 \; Y \Leftrightarrow (\exists i, n, m)(\tau_{\langle i,n\rangle} \subset X \;\&\; \tau_{\langle i,m\rangle} \subset Y).$$

Next, for an equivalence relation $R \subseteq \omega \times \omega$, define a collection of pairs of reals,

$$A_R := \{(\tau_{\langle i,0\rangle}{}^\frown 0^\infty, \tau_{\langle j,0\rangle}{}^\frown 0^\infty) : i \; R \; j\}.$$

Finally, let $\iota\colon \mathcal{P}(\omega \times \omega) \to \mathcal{P}(2^\omega \times 2^\omega)$ be the map so that $\iota(R)$ coincides with F_{0/A_R}. We claim that ι induces an embedding from $\mathbf{ER}(\omega)$ to $\mathbf{ER}(2^\omega)$.

Claim 2.8. *If* $R \leqslant_{\mathbf{0}} S$, *then* $\iota(R) \leqslant_{\mathbf{0}} \iota(S)$.

Proof. Given $f : R \leqslant_{\mathbf{0}} S$, we represent a Turing functional Φ via the computable function $\varphi : 2^{<\omega} \to 2^{<\omega}$ recursively defined as follows. For $a \in \{0, 1\}$, $\varphi(a) := a$ and

$$\varphi(\sigma^\frown a) := \begin{cases} \tau_{\langle f(i),m\rangle} & \text{for the least } m \\ & \text{so that } \tau_{\langle f(i),m\rangle} \supseteq \varphi(\sigma), \quad \text{if } (\exists i, n)(\sigma^\frown a = \tau_{\langle i,n\rangle}), \\ \varphi(\sigma)^\frown a, & \qquad\qquad\qquad\qquad\qquad \text{otherwise.} \end{cases}$$

The effectiveness of φ is guaranteed by the fact that $\mathfrak{S}$ is computable. For all $X \in 2^\omega$, let $\Phi(X)$ be $\bigcup_{\sigma\subset X} \varphi(\sigma)$.

We shall now prove that $\Phi : \iota(R) \leqslant_{\mathbf{0}} \iota(S)$. By construction, $\Phi^{-1}(0^\infty) = \{0^\infty\}$. So, it suffices to check that Φ is a reduction on the remaining classes. Let X, Y be so that $\tau_{\langle i,n_0\rangle} \subset X$ and $\tau_{\langle j,n_1\rangle} \subset Y$. By definition of φ, we have that there are m_0 and m_1 so that $\Phi(X) = \tau_{\langle f(i),m_0\rangle} * X$ and $\Phi(Y) = \tau_{\langle f(j),m_1\rangle} * Y$, where

$$\sigma * A = \begin{cases} \sigma(i) & \text{for } i < |\sigma| \\ A(i) & \text{for } i \geqslant |\sigma|. \end{cases}$$

The following two chains of implications follow from the definitions of F_0, A_R, and A_S,

(1) $i \; R \; j \Leftrightarrow \tau_{\langle i,0\rangle}{}^\frown 0^\infty \; \iota(R) \; \tau_{\langle j,0\rangle}{}^\frown 0^\infty \Leftrightarrow X \; \iota(R) \; Y$,
(2) $f(i) \; S \; f(j) \Leftrightarrow \tau_{\langle f(i),0\rangle}{}^\frown 0^\infty \; \iota(S) \; \tau_{\langle f(j),0\rangle}{}^\frown 0^\infty \Leftrightarrow \Phi(X) \; \iota(S) \; \Phi(Y)$.

Since f reduces R to S, we conclude that $X \; \iota(R) \; Y$ if and only if $\Phi(X) \; \iota(S) \; \Phi(Y)$, as desired. □

Claim 2.9. *If* $\iota(R) \leqslant_{\mathbf{0}} \iota(S)$, *then* $R \leqslant_{\mathbf{0}} S$.

Proof. First, observe that continuity forces any given reduction $\Phi : \iota(R) \leqslant_{\mathbf{0}} \iota(S)$ to map 0^∞ to 0^∞, as all other S-classes are open and not closed. Next, for all i, find a binary string σ so that $\Phi(\tau_{\langle i,0\rangle}{}^\frown\sigma) \supseteq \tau_{\langle j,m\rangle}$ for some j and m, and define $f(i) = j$. It is not hard to check that $f : R \leqslant_{\mathbf{0}} S$. □

Theorem 2.7 is proved. □

Our next result shows that for the embedding $\iota\colon \mathbf{ER}(\omega) \hookrightarrow \mathbf{ER}(2^\omega)$ constructed in Theorem 2.7, its image range(ι) is *not* an initial segment of the poset $\mathbf{ER}(2^\omega)$.

Proposition 2.10. *There exists an equivalence relation E on reals such that:*

- *E has countably many classes,*
- *$E <_{\mathbf{0}} \iota(\mathrm{Id}(\omega))$, and*
- *for any equivalence relation $R \subseteq \omega \times \omega$, we have $E \not\equiv_{\mathbf{0}} \iota(R)$.*

Proof. We build a Turing operator Ψ, and the desired equivalence relation E will be induced by the following natural condition:

$$X \; E \; Y \;\Leftrightarrow\; (\Psi(X), \Psi(Y)) \in \iota(\mathrm{Id}(\omega)). \tag{1}$$

Consider disjoint c.e. sets

$$A = \{e : \varphi_e(e)\!\downarrow\, = 0\} \text{ and } B = \{e : \varphi_e(e)\!\downarrow\, = 1\}.$$

Fix effective approximations $(A_s)_{s\in\omega}$ and $(B_s)_{s\in\omega}$ of these c.e. sets.

We define our operator Ψ via a computable function $\psi\colon 2^{<\omega} \to 2^{<\omega}$. Put $\psi(\epsilon) := \epsilon$.

Suppose that $\sigma \in 2^{<\omega}$, and for all σ' with $|\sigma'| < |\sigma|$ the value $\psi(\sigma')$ is already defined. Let ξ be the parent of σ. Consider the following three cases.

Case 1. If the string $\psi(\xi)$ contains ones, then put $\psi(\sigma) := \psi(\xi)^\frown 0$.

Case 2. Suppose that $\psi(\xi) = 0^i$ for some $i \in \omega$, and there exists $k < |\sigma|$ such that:

- either $k \in A_{|\sigma|}$ and $\sigma(k) = 1$,
- or $k \in B_{|\sigma|}$ and $\sigma(k) = 0$.

Then choose a fresh number $u \in \omega$ such that $|\tau_{\langle u,0\rangle}| > i + 1$, and put $\psi(\sigma) = \tau_{\langle u,0\rangle}$.

Case 3. Otherwise, set $\psi(\sigma) := \psi(\xi)^\frown 0$.

This concludes the description of the operator Ψ. Note the following properties of Ψ:

(a) If a real X separates the pair (A, B), then $\Psi(X) = 0^\infty$.
(b) Otherwise, there is a number u such that $\Psi(X) = \tau_{\langle u,0\rangle}{}^\frown 0^\infty$.

Consider the relation E defined via equation (1). It is not hard to show that E has countably many equivalence classes.

Fix a real Y_0, which separates the pair (A, B). Since the reduction Ψ is continuous, we deduce that:

- The class $[Y_0]_E = \Psi^{-1}(0^\infty)$ is a closed subset of 2^ω.
- If $X \not\!\!E\, Y_0$, then the class $[X]_E$ is the Ψ-preimage of a clopen set $[\tau_{\langle u,0\rangle}]$; hence, $[X]_E$ is clopen.
- Therefore, the compactness of the Cantor space implies that the set $[Y_0]_E$ is not open.

Towards a contradiction, assume that for some $R \subseteq \omega \times \omega$, we have $E \equiv_{\mathbf{0}} \iota(R)$. It is clear that the relation R must have infinitely many classes.

Fix a computable reduction Φ from $\iota(R)$ to E. Define $Z := \Phi(0^\infty)$. Since the set $\{0^\infty\} = \Phi^{-1}([Z]_E)$ is closed and not open, we deduce that $Z \in [Y_0]_E$. Then the computable real Z separates the pair (A, B), which gives a contradiction. Proposition 2.10 is proved. □

2.3. **Lack of suprema and infima.** We prove that $\mathbf{ER}(2^\omega)$ is neither an upper nor a lower semilattice. To show that the poset is not an upper semilattice, we use mutually dark equivalence relations, defined and investigated in [6]: $R, S \subseteq \omega \times \omega$ are *mutually dark* if neither of the two equivalence relations Turing computes a transversal of the other. Mutually dark equivalence relations exist and they have no least upper bound, see [6].

Theorem 2.11. *$\mathbf{ER}(2^\omega)$ is not an upper semilattice.*

Proof. Our reasoning is similar to the proofs of [1, 6] (we refer to them for additional details). Let $R, S \subseteq \omega \times \omega$ be mutually dark. Suppose that F is the least upper bound of $\iota(R)$ and $\iota(S)$. Observe that Theorem 2.7 guarantees that F cannot fall in range(ι), up to computable reducibility. We will prove that F cannot exist. Specifically, we will build by stages a pure quotient $E := \bigcup_s E_s$ of $\iota(R) \oplus \iota(S)$ to which F does not reduce. This suffices, since any pure quotient of $\iota(R) \oplus \iota(S)$ is an upper bound of $\iota(R), \iota(S)$. During the construction we will restrain some E-classes that we do not want to collapse further on.

Choose a countable set of reals $M = \{u_i : i \in \omega\}$ such that the reals u_i are pairwise not F-equivalent.

Stage 0. Let E_0 be $\iota(R) \oplus \iota(S)$. Do not restrain any equivalence classes.

Stage $i + 1$. We distinguish two cases.

(1) If Φ_i is nontotal, let $E_{i+1} := E_i$.
(2) If Φ_i is total, search for a pair of reals (u, v) from M such that
    (a) either $u \mathrel{F} v \not\Leftrightarrow \Phi_i(u) \mathrel{E_i} \Phi_i(v)$,
    (b) or $\Phi_i(u)$ and $\Phi_i(v)$ are unrestrained, $\Phi_i(u) \in [0]$ and $\Phi_i(v) \in [1]$.
    If the outcome is (a), let $E_{i+1} := E_i$. If the outcome is (b), let $E_{i+1} := E_{i/(\Phi_i(u),\Phi_i(v))}$. In both cases, restrain any real which E_i-equivalent to $\Phi_i(u)$ or $\Phi_i(v)$.

Claim 2.12. *For all i, the action described at stage $i + 1$ terminates.*

Proof. We only need to check that, if we execute action (2), the search terminates. Suppose otherwise. This means that Φ_i is total and it reduces

F to E_i. Furthermore, range(Φ_i) cannot hit infinitely many equivalence classes of both $E_i \upharpoonright [0]$ and $E_i \upharpoonright [1]$, as otherwise we would reach outcome (b). Without loss of generality, assume that range(Φ_i) hits infinitely many $E_i \upharpoonright [0]$-classes and finitely many $E_i \upharpoonright [1]$-classes. Since $E_i \upharpoonright [0]$ encodes $\iota(R)$, we have that the image through F and Φ_i of any transversal of $\iota(S)$ gives, with at most finitely many exceptions, a transversal of $\iota(R)$. But observe that any S-transversal $(x_e)_{e\in\omega}$ computes a $\iota(S)$-transversal (e.g., via the map $x_e \mapsto \tau_{\langle e,0\rangle}{}^\frown 0^\infty$) and any $\iota(R)$-transversal $(X_e)_{e\in\omega}$ computes a R-transversal (e.g., as $\{i : (\exists n, e)(\tau_{\langle i,n\rangle} \subseteq X_e)\}$. Hence, the Turing degree of S (which obviously computes some S-transversal) would compute a R-transversal, contradicting mutual darkness. □

The last claim ensures that, for all i, the diagonalization against Φ_i terminates by disproving that Φ_i is a reduction from F to E_i. The restraints and the fact that E is a quotient of E_i guarantee that $\Phi_i : F \not\leqslant_{\mathbf{0}} E$, showing that F is not a least upper bound of $\iota(R)$ and $\iota(S)$. □

Now we show that not every pair of degrees from $\mathbf{ER}(2^\omega)$ has an infimum.

Theorem 2.13. $\mathbf{ER}(2^\omega)$ *is not a lower semilattice.*

Proof. Let X be a real. Fix an X-computable 1–1 enumeration $(\tau_i)_{i\in\omega}$ of the set $\mathfrak{S}_X$. We define an equivalence relation $R(X)$ as follows: reals Y and Z are $R(X)$-equivalent if and only if either $Y = Z$, or there is $i \in \omega$ such that $Y, Z \in [\tau_i]$.

Let X_0 be a real such that $X_0 \not\leqslant_T \mathbf{0}'$. We will prove that the relations $R(X_0)$ and $R(0^\infty)$ do not have the greatest lower bound with respect to computable reducibility.

Lemma 2.14. *If an equivalence relation E is computably reducible to both $R(X_0)$ and $R(0^\infty)$, then E has only finitely many classes.*

Proof. Towards a contradiction, assume that E has infinitely many classes, and fix a computable reduction Φ from E to $R(0^\infty)$. The continuity of Φ implies that all E-classes are closed subsets of 2^ω. In addition, one of the following two cases holds.

(1) Suppose that all E-classes are clopen. Then the compactness of the Cantor space implies that E has only finitely many classes, which contradicts the choice of E.
(2) There is a class $[Y_0]_E$, which is not open, and all other E-classes are clopen. Notice that $\Phi(Y_0) = 0^\infty$.

In this case, by Lemma 1.2, we obtain that $[Y_0]_E$ contains a $\mathbf{0}'$-computable real. Without loss of generality, we assume that $Y = Y_0$. Consider a computable reduction Ψ from E to $R(X_0)$. Since the class $[Y_0]_E$ is not open, we deduce that $\Psi(Y_0) = X_0$ and $X_0 \leqslant_T Y_0 \leqslant_T \mathbf{0}'$, which contradicts the choice of the real X_0.

Both cases above lead to a contradiction. Lemma 2.14 is proved. □

By Lemma 2.14, any lower bound E of $R(X_0)$ and $R(0^\infty)$ has only finitely many equivalence classes. Suppose that E has precisely N classes, and Φ is a computable reduction from E to $R(X_0)$. Choose a number $i_0 \in \omega$ such that $\mathrm{range}(\Phi) \cap [\tau_{i_0}] = \varnothing$. Then the operator Ψ defined as

$$\Psi(0^\frown Y) = \Phi(Y), \quad \Psi(1^\frown Y) = \tau_{i_0}{}^\frown 0^\infty,$$

provides a computable reduction from $E \oplus \mathrm{Id}_1$ to $R(X_0)$.

Therefore, if E is a lower bound of $R(X_0)$ and $R(0^\infty)$, then the relation $E \oplus \mathrm{Id}_1$ is also a lower bound for these equivalences. We deduce that $R(X_0)$ and $R(0^\infty)$ do not have an infimum with respect to computable reducibility. Theorem 2.13 is proved. □

2.4. **Further comparison with $\mathbf{ER}(\omega)$.** As is clear, $\mathbf{ER}(\omega)$ and $\mathbf{ER}(2^\omega)$ are quite different structures, e.g., the former poset has cardinality $2^{\aleph_0}$, while the latter has cardinality $2^{2^{\aleph_0}}$. By the next proposition, we exhibit a first-order difference between the two posets. Such a difference showcases an interesting property of $\mathbf{ER}(2^\omega)$: the least element is the only one which is comparable with any other element.

Proposition 2.15. *$\mathbf{ER}(\omega)$ and $\mathbf{ER}(2^\omega)$ are elementarily inequivalent.*

Proof. Inside $\mathbf{ER}(\omega)$, there are two degrees that are comparable with any other degree: $\mathbf{Id}_1$ and $\mathbf{Id}_2$. We show that this property, which is clearly expressible by first-order logic, fails in $\mathbf{ER}(2^\omega)$. Specifically, we prove the following,

$$(\dagger) \qquad E \neq \mathrm{Id}_1(2^\omega) \Rightarrow \text{ there is } F \text{ incomparable with } E.$$

So, let $E \neq \mathrm{Id}_1(2^\omega)$ and take $X_0 \not\mathrel{E} X_1$. Let $(Y_e)_{e\in\omega}$ be a list of all sets Turing equivalent to $(X_0 \oplus X_1)'$. Define the following mass problems:

(1) $\mathcal{A}_0 := \{A : A \leqslant_T X_0 \oplus X_1\}$;
(2) $\mathcal{A}_1 := \{Y_{2e} : \Phi_e(Y_{2e}) \mathrel{E} \Phi_e(Y_{2e+1})\} \cup \{Y_{2e}, Y_{2e+1} : \Phi_e(Y_{2e}) \not\mathrel{E} \Phi_e(Y_{2e+1})\}$.

Let F be $E(\mathcal{A}_0 \cup \mathcal{A}_1)$. We claim that F incomparable with E. On one hand, Lemma 1.1 ensures that $E \not\leqslant_{\mathbf{0}} F$: this is because there are two E-classes but only one F-class containing $(X_0 \oplus X_1)$-computable sets. On the other hand, for all e,

$$Y_{2e} \mathrel{F} Y_{2e+1} \Leftrightarrow \Phi_e(Y_{2e}) \not\mathrel{E} \Phi_e(Y_{2e+1}).$$

Hence, $F \not\leqslant_{\mathbf{0}} E$. So, $(\dagger)$ is proved, and therefore $\mathbf{ER}(\omega)$ and $\mathbf{ER}(2^\omega)$ are not elementarily equivalent. □

2.5. **Breaking Glimm-Effros dichotomy.** We conclude this section by focusing on an interesting local structure of $\mathbf{ER}(2^\omega)$: the interval $[\mathbf{Id}(2^\omega), \mathbf{E_0}]$. In the Borel case, the celebrated Glimm-Effros dichotomy [16] states that such an interval has precisely two elements. In fact, for any Borel equivalence relation R, either R is Borel reducible to $\mathrm{Id}(2^\omega)$, or E_0 Borel reduces to R. Miller [27] proved that the analogue fails for computable reducibility, by constructing an equivalence relation which is above $\mathrm{Id}(2^\omega)$ but is incomparable with E_0, up to computable reductions. The next theorem pushes this result forward, by proving that in fact the interval $[\mathbf{Id}(2^\omega), \mathbf{E_0}]$ (with respect to computable reducibility) has a very rich structure: e.g., any countable partial order embeds into it. Denote by $\mathbf{\Delta^0_2}$**ers** the poset of computable reducibility degrees of Δ^0_2 equivalence relations on natural numbers.

Theorem 2.16. $\mathbf{\Delta^0_2}$**ers** *embeds in in the interval* $[\mathbf{Id}(2^\omega), \mathbf{E_0}]$.

Proof. By reasoning as in the proof of Theorem 2.7, it is not hard to see that the map $R \mapsto \iota(R) \oplus \mathrm{Id}(2^\omega)$ is an embedding from $\mathbf{ER}(\omega)$ to $\mathbf{ER}(2^\omega)$ whose image is contained in the upper cone of $\mathbf{Id}(2^\omega)$. Indeed, if $f : R \leqslant_{\mathbf{0}} S$, then one can build a Turing functional Φ which reduces $\iota(R) \oplus \mathrm{Id}(2^\omega)$ to $\iota(S) \oplus \mathrm{Id}(2^\omega)$ by mapping any real in $[1]$ to itself and use the same reduction offered in Claim 2.8 for the reals in $[0]$. On the other hand, if $\Phi : \iota(R) \oplus \mathrm{Id}(2^\omega) \leqslant_{\mathbf{0}} \iota(S) \oplus \mathrm{Id}(2^\omega)$, then by continuity, Φ must map a closed equivalence class $[1^\frown X]_{\iota(R)\oplus\mathrm{Id}(2^\omega)}$ either to the class $[0^\infty]_{\iota(S)\oplus\mathrm{Id}(2^\omega)}$ or to the class $[1^\frown Y]_{\iota(S)\oplus\mathrm{Id}(2^\omega)}$ for some Y. So, that one can retrieve a reduction $f : R \leqslant_{\mathbf{0}} S$ by focusing on the reals in $[0] \smallsetminus \{0^\infty\}$, as in Claim 2.9.

So, it remains to be proved that, if R is Δ^0_2, then there is a Turing functional Φ which reduces $R \oplus \mathrm{Id}(2^\omega)$ to E_0. For sake of exposition, we begin by describing the behaviour of Φ on the transversal $(X_e)_{e\in\omega}$ of $R \oplus \mathrm{Id}(2^\omega)$, where X_e denotes the real $0^\frown\tau_{\langle e,0\rangle}{}^\frown 0^\infty$.

At any given stage s of the construction, if we say that $\Phi(X_e)$ is *hooked* to Y, for some real Y, we mean that, on further stages $t > s$, unless something different is prescribed, we set $\Phi(X_e)\restriction_t := Y\restriction_t$. Similarly, if we say that $\Phi(X_i)$ is hooked to $\Phi(X_j)$, we recursively mean that $\Phi(X_i)$ is hooked to real to which $\Phi(X_j)$ is hooked.

Fix an E_0-transversal $(Y_e)_{e\in\omega}$ such that $Y_e \not\mathrel{E_0} 0^\infty$ and for every e and i, we have $Y_e(2i) = 0$.

Fix also a Δ^0_2 approximation $\bigcup_s R\restriction_s = R$. Without loss of generality, we also assume that $R\restriction_0$ is $\mathrm{Id}(\omega)$ and, for all s, $R\restriction_s$ is closed by symmetry and transitivity.

Stage 0. For all e, let $\Phi(X_e)$ be hooked to Y_e.

Stage $s + 1 = \langle i, n\rangle$. Let k be the least number in $[i]_{R\restriction s+1}$. Let $\Phi(X_i)$ be hooked to $\Phi(X_k)$.

Observe that $i \mathrel{R} j$ if and only $\Phi(X_i) \mathrel{E_0} \Phi(X_j)$. Indeed, if $i \mathrel{R} j$ holds with $i < j$, then there is a stage s such that for all $s' \geqslant s$, we have $i \mathrel{R\restriction_{s'}} j$ and for some $k \leqslant i$, $\min[i]_{R\restriction_{s'}} = \min[i]_R = k$. By construction, at all stages $\langle i, n\rangle$ and $\langle j, n\rangle$ bigger than s, we have that $\Phi(X_i)$ and $\Phi(X_j)$ are hooked to $\Phi(X_k)$. Thus, $\Phi(X_i) \mathrel{E_0} \Phi(X_j)$. On the other hand, if $i \not\mathrel{R} j$, let $k_0 = \min[i]_R$ and $k_1 = \min[j]_R$. By construction, $\Phi(X_i)$ will be eventually hooked to $\Phi(X_{k_0})$ and $\Phi(X_j)$ will be eventually hooked to $\Phi(X_{k_1})$. Note that, for $a \in \{0, 1\}$, $\Phi(X_{k_a})$ is E_0-equivalent to Y_{k_a}. Therefore, since $(Y_e)_{e\in\omega}$ is an E_0-transversal, we conclude that $\Phi(X_i) \not\mathrel{E_0} \Phi(X_j)$.

Finally, the partial Turing functional Φ so defined can be readily extended to all Cantor space as follows:

(a) a real X from [1] is mapped to the real Y such that $Y(2i) = 1$ and $Y(2\langle i, k\rangle + 1) = X(i)$, for $i, k \in \omega$;
(b) the real 0^∞ is mapped to itself,
(c) for all e and n such that $0^\frown\tau_{\langle e,n\rangle} \subseteq A$, put $\Phi(A) := \Phi(X_e)$.

Theorem 2.16 is proved. □

3. Measuring the complexity of computability-theoretic equivalence relations

In this section, we move from the structural analysis of $\mathbf{ER}(2^\omega)$ to the application of computable reducibility to calibrate the strength of the equivalence relations induced by familiar reducibilities on reals. As a case-study, we concentrate on computability-theoretic reducibilities: 1-reducibility $\leqslant_1$, m-reducibility $\leqslant_m$, truth-table reducibility $\leqslant_{tt}$, and Turing reducibility $\leqslant_T$.

3.1. First observations.

By applying Lemma 1.1 for $X = \varnothing$, one obtains the following:

Proposition 3.1. *Each of the following relations R is neither computably reducible to $\equiv_{tt}$ nor computably reducible to $\equiv_T$:*

(1) $\mathrm{Id}(2^\omega)$,
(2) E_0,
(3) $\equiv_1$,
(4) $\equiv_m$.

We can now recover the whole picture for four equivalences: $\equiv_1$, $\equiv_m$, $\equiv_{tt}$, and $\equiv_T$.

Proposition 3.2. $\equiv_{tt} <_{\mathbf{0}} \equiv_m <_{\mathbf{0}} \equiv_1$.

Proof. The *cylindrification* of a set $X \subseteq \omega$ is the set $X \times \omega$. It is well-known (see, e.g., §7.6 of [30]) that for all sets $X, Y \subseteq \omega$, we have

$$X \leqslant_m Y \iff X \times \omega \leqslant_1 Y \times \omega.$$

Sets X so that $X \equiv_1 X \times \omega$ are called *cylinders*. So, the operator $\Phi\colon X \mapsto X \times \omega$ induces a computable reduction from $\equiv_m$ to $\equiv_1$.

Next, the *tt-cylindrification* of a set X is the set

$$X^{tt} := \{i : X \models \alpha_i\},$$

where $(\alpha_i)_{i\in\omega}$ is an effective enumeration of all propositional formulas build from the atomic ones $(n \in Y)_{n\in\omega}$, and $X \models \alpha_i$ means that α_i is true when Y is interpreted as X. It is known that

$$X \leqslant_{tt} Y \Leftrightarrow X^{tt} \leqslant_1 Y^{tt}.$$

Sets X so that $X \equiv_1 X^{tt}$ are called *tt-cylinders*. So, the operator $\Psi : X \mapsto X^{tt}$ induces a computable reduction from $\equiv_{tt}$ to $\equiv_1$. Moreover, since any *tt*-cylinder is a cylinder (see §8.4 of [30]) and the 1-degree of a cylinder coincides with its m-degree, we have that Ψ induces also a computable reduction from $\equiv_{tt}$ to $\equiv_m$.

On the other hand, just note that $\equiv_1$ has infinitely many classes with computable reals, $\equiv_m$ has three such classes, and $\equiv_{tt}$ has only one such class. By Lemma 1.1, we have $\equiv_1 \not\leqslant_{\mathbf{0}} \equiv_m$ and $\equiv_m \not\leqslant_{\mathbf{0}} \equiv_{tt}$. □

Proposition 3.3. *Turing equivalence $\equiv_T$ is $\leqslant_{\mathbf{0}}$-incomparable with $\equiv_1$, $\equiv_m$, and $\equiv_{tt}$*

Proof. Thomas (see Corollary 1.2 of [34]) proved that $\equiv_T$ is not continuously reducible to $\equiv_1$. By Proposition 3.2, this implies that $\equiv_T$ is not computably reducible to any of the other three equivalence relations. Then Proposition 3.1 ensures that $\equiv_T$ is incomparable with both $\equiv_m$ and $\equiv_1$.

It remains to be proven that $\equiv_{tt}$ is not computably reducible to $\equiv_T$. To see this, we rely on Jockusch's result [19] that each Turing degree which is not hyperimmune-free contains infinitely many tt-degrees. Let $\mathbf{d}$ be a minimal Turing degree below $\mathbf{0}'$. It is well-known that $\mathbf{d}$ is not hyperimmune-free. So, let $X, Y, Z \in \mathbf{d}$ be from three different tt-degrees. Suppose that there is a Turing functional Φ which induces a reduction from $\equiv_{tt}$ to $\equiv_T$. By Lemma 1.1 and the minimality of $\mathbf{d}$, we have that the set $\{\Phi(X), \Phi(Y), \Phi(Z)\}$ must be contained in $\mathbf{0} \cup \mathbf{d}$. In particular, since Φ is a reduction, without loss of generality, we can assume that both $\Phi(X)$ and $\Phi(Y)$ are in $\mathbf{d}$. But then, $\Phi(X) \equiv_T \Phi(Y)$ while $X \not\equiv_{tt} Y$, a contradiction. □

3.2. **Versus** $\mathrm{Id}(2^\omega)$ **and** E_0**.** Here we collect further results, which clarify the location of our computability-theoretic equivalences relative to the degrees of $\mathrm{Id}(2^\omega)$ and E_0. Recall that $\mathrm{Id}(2^\omega) <_{\mathbf{0}} E_0$.

Proposition 3.4. $\equiv_T \not\leqslant_{\mathbf{0}} E_0$*. Consequently,* $\equiv_T \not\leqslant_{\mathbf{0}} \mathrm{Id}(2^\omega)$*.*

Proof. Towards a contradiction, assume that a total Turing operator Φ provides a reduction from $\equiv_T$ to E_0.

Consider the standard enumeration $(W_e)_{e\in\omega}$ of all c.e. sets. Then we have

$$W_i \equiv_T W_j \iff \exists x_0 (\forall x \geqslant x_0)[\Phi^{W_i}(x) = \Phi^{W_j}(x)].$$

This implies that the equivalence relation

$$E := \{(i,j) \in \omega \times \omega : W_i \equiv_T W_j\}$$

is Σ^0_3. On the other hand, Theorem 5.1 of [17] proves that the relation E is Σ^0_4-complete among equivalence relations on ω. Thus, we obtain a contradiction. We deduce that $\equiv_T \not\leqslant_{\mathbf{0}} E_0$. □

Proposition 3.5. $\equiv_{tt} \not\leqslant_{\mathbf{0}} E_0$*. Consequently, by Proposition 3.2,* $\equiv_m \not\leqslant_{\mathbf{0}} E_0$ *and* $\equiv_1 \not\leqslant_{\mathbf{0}} E_0$*.*

Proof. Assume that a total Turing operator Φ_e gives a reduction from $\equiv_{tt}$ to E_0.

For a string $\sigma \in 2^{<\omega}$, let $\ell(\sigma)$ be maximal $x \leqslant |\sigma|$ such that for every $y < x$, the value $\Phi^\sigma_{e,|\sigma|}(y)$ is defined. We put $\mathrm{out}(\sigma) := \Phi^\sigma_e \upharpoonright \ell(\sigma)$.

Since Φ_e reduces $\equiv_{tt}$ to E_0, it is not hard to show the following: for a given string σ, one can effectively find two strings $\tau_0 \neq \tau_1$ such that $|\tau_0| = |\tau_1|$, and the outputs $\mathrm{out}(\sigma^\frown\tau_0)$ and $\mathrm{out}(\sigma^\frown\tau_1)$ differ in the k-th bit for some $k < \min(\ell(\sigma^\frown\tau_0), \ell(\sigma^\frown\tau_1))$.

We construct two computable reals X and Y. At stage 0, we put $X^0 = Y^0 = \epsilon$.

Stage $s+1$. Find a number $m \geqslant 1$ such that there are strings σ, τ_0, τ_1 of length m with the following property:

> the strings $\mathrm{out}(Y^{s\frown}\tau_0)$ and $\mathrm{out}(Y^{s\frown}\tau_1)$ differ in the k-th bit for some $k < \min(\ell(X^{s\frown}\sigma), \ell(Y^{s\frown}\tau_0), \ell(Y^{s\frown}\tau_1))$.

Fix these strings σ, τ_0, τ_1 and the index k. We put $X^{s+1} := X^{s\frown}\sigma$. If the strings $\mathrm{out}(X^{s+1})$ and $\mathrm{out}(Y^{s\frown}\tau_0)$ differ in the k-th bit, then set $Y^{s+1} := Y^{s\frown}\tau_0$. Otherwise, set $Y^{s+1} := Y^{s\frown}\tau_1$.

We define $X = \bigcup_s X^s$ and $Y = \bigcup_s Y^s$. The reals X and Y satisfy the following properties:

(a) Both of them are computable, thus, $X \equiv_{tt} Y$.
(b) There are infinitely many $k \in \omega$ such that $\Phi_e^X(k) \neq \Phi_e^Y(k)$.

These properties contradict our assumption that Φ_e is a reduction from $\equiv_{tt}$ to E_0. Proposition 3.5 is proved. □

Proposition 3.6. $\mathrm{Id}(2^\omega) \nleqslant_{\mathbf{0}} \equiv_1$. *Consequently,* $E_0 \nleqslant_{\mathbf{0}} \equiv_1$ *and, by Proposition 3.2, neither* $\mathrm{Id}(2^\omega)$ *nor* E_0 *is computably reducible to* $\equiv_m$ *or* $\equiv_{tt}$*.*

Proof. Assume that a total Turing operator Φ_e provides a reduction from $\mathrm{Id}(2^\omega)$ to $\equiv_1$. We use the same notations $\ell(\sigma)$ and $\mathrm{out}(\sigma)$ as in Proposition 3.5.

We construct two computable reals X_0 and X_1. At stage 0, put $X_0^0 = 0$ and $X_1^0 = 1$.

Stage $s+1$. For each $i \in \{0, 1\}$, we proceed as follows. If s is an even number, then find a non-empty string σ such that:

> the number of zeros in $\mathrm{out}(X_i^{s\frown}\sigma)$ is strictly greater than the number of zeros in $\mathrm{out}(X_i^s)$.

Notice that such a string σ exists. Indeed, assume otherwise. Then for any computable real $Y \supset X_i^s$, the number of zeros inside Φ_e^Y is the same as the number of zeros in $\mathrm{out}(X_i^s)$. This shows that Φ_e maps all these Y into the same $\equiv_1$-class, which contradicts our assumptions.

We define $X_i^{s+1} := X_i^{s\frown}\sigma$.

If s is odd, then find a non-empty string τ such that:

> the number of ones in $\mathrm{out}(X_i^{s\frown}\tau)$ is strictly greater than the number of ones in $\mathrm{out}(X_i^s)$.

Put $X_i^{s+1} := X_i^{s\frown}\tau$.

We define $X_i = \bigcup_s X_i^s$. It is clear that the real $\Phi_e^{X_i}$, $i \in \{0, 1\}$, encodes a computable, infinite and coinfinite set. Therefore, we have $X_0 \neq X_1$ and $\Phi_e^{X_0} \equiv_1 \Phi_e^{X_1}$, which gives a contradiction. Proposition 3.6 is proved. □

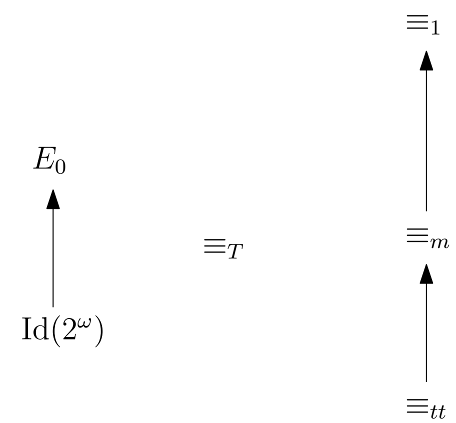

FIGURE 1. Relative to computable reducibility

Figure 1 summarizes the results obtained so far.

3.3. **Turing equivalence is not minimal.** Here we prove the following result:

Theorem 3.7. *There is an equivalence relation E such that $E <_{\mathbf{0}} \equiv_T$ and E has continuum many equivalence classes.*

Proof. Before proceeding to the construction, we discuss some conventions. As usual, the relation E is defined on 2^ω, *but* the desired computable reduction from E to $\equiv_T$ will be treated as an operator $\Psi\colon 2^\omega \to 3^\omega$. To set the meaning of Turing equivalence on the space 3^ω, we use the following convention. Consider an operator $\Theta\colon 3^\omega \to 2^\omega$, which acts as follows. Given an infinite ternary string α, it replaces every symbol of α according to the following rules:

$$0 \mapsto 00, \quad 1 \mapsto 01, \quad 2 \mapsto 10.$$

Then for $\alpha, \beta \in 3^\omega$, we say that $\alpha \leqslant_T \beta$ if and only if $\Theta(\alpha) \leqslant_T \Theta(\beta)$.

It is not hard to show that a computable reduction from E to $\equiv_T$, defined on 3^ω, induces a computable reduction from E to the standard Turing equivalence.

We define our equivalence relation E. Fix a non-computable c.e. set W and its effective approximation $(W^s)_{s\in\omega}$. Consider a set of reals

$$M = \{X : X = 0^e 1^\frown X_1, \text{ where } e \in \omega, \text{ such that } W = \Phi_e^{X_1}\}.$$

We say that $(X \mathrel{E} Y)$ if and only if one of the following holds:

(1) $X, Y \in M$ and $X \equiv_T Y$, or
(2) $X, Y \notin M$.

First, we show that E has continuum many classes. Let Y and Z be reals such that $W \leqslant_T Y$, $W \leqslant_T Z$, and $Y \not\equiv_T Z$. Fix indices i and j such that

$W = \Phi_i^Y$ and $W = \Phi_j^Z$. Then it is clear that the reals $0^i 1^\frown Y$ and $0^j 1^\frown Z$ are not E-equivalent. This argument implies that E possesses $2^{\aleph_0}$ classes.

We build a computable reduction Ψ from E to $\equiv_T$ (on the space 3^ω).

Let X be a real. At a stage s, we define the *agreement length* ℓ^s as follows.

(i) If the string $X \upharpoonright s$ does not contain ones, then set $\ell^s := -1$.
(ii) Otherwise, we already know that $X = 0^e 1^\frown X_1$ for some $e \in \omega$ and $X_1 \in 2^\omega$. Then ℓ^s is defined as the greatest number $\ell \leqslant s$ such that

$$\Phi_{e,s}^{X_1 \upharpoonright (s-e-1)}(x)\downarrow = \begin{cases} 1, & \text{if } x \in W^s, \\ 0, & \text{if } x \notin W^s, \end{cases} \tag{2}$$

for all $x < \ell$.

The ternary sequence $\Psi(X)$ is constructed as follows. Put $\Psi(X)(0) := 2$. At the beginning, we say that every bit of X is *not marked*.

Stage $s+1$. If $\ell^{s+1} \leqslant \max\{\ell^t : t \leqslant s\}$, then set $\Psi(X)(s+1) := 2$.

Suppose that $\ell^{s+1} > \max\{\ell^t : t \leqslant s\}$. Then we already know that $X = 0^e 1^\frown X_1$. In this case, we copy a fresh portion of the bits of X_1 into the output. More formally, we proceed as follows. If there exists the least k such that $e+1 \leqslant k \leqslant s$ and the k-th bit of X is not marked, then we *mark* the k-th bit and set $\Psi(X)(s+1) := X(k)$. Otherwise, put $\Psi(X)(s+1) := 2$.

This concludes the description of the operator Ψ. It is not hard to show that Ψ is computable. We prove that Ψ provides a reduction from E to $\equiv_T$. For $X \in 2^\omega$, consider the following three cases.

Case 1. If $X = 0^\infty$, then $\ell^s = -1$ for all s, and $\Psi(X) = 2^\infty$. Note that the element $\Psi(X)$ is computable.

Case 2. Suppose that $X \notin M \cup \{0^\infty\}$. Then $X = 0^e 1^\frown X_1$ and $W \neq \Phi_e^{X_1}$.

This implies that either $\Phi_e^{X_1}(x_0)$ is undefined for some $x_0 \in \omega$, or the function $\Phi_e^{X_1}$ is total and there is $y_0 \in \omega$ with

$$(y_0 \in W \text{ and } \Phi_e^{X_1}(y_0) \neq 1) \text{ or } (y_0 \notin W \text{ and } \Phi_e^{X_1}(y_0) \neq 0).$$

Thus, by (2), there is $\ell^* \in \omega$ such that $\ell^s \leqslant \ell^*$ for all s. This implies that $(\Psi(X)\ E_0\ 2^\infty)$, — again, $\Psi(X)$ is computable.

Case 3. Suppose that $X \in M$. Then $X = 0^e 1^\frown X_1$ and $W = \Phi_e^{X_1}$.

By (2), we have $\lim_s \ell^s = \infty$. Our construction ensures that there is a sequence $(\sigma_i)_{i\in\omega}$ of finite non-empty strings from $2^{<\omega}$ such that:

$$X = \sigma_0{}^\frown \sigma_1{}^\frown \sigma_2{}^\frown \dots, \text{ and}$$
$$\Psi(X) = 2^{k_0}{}^\frown \sigma_0{}^\frown 2^{k_1}{}^\frown \sigma_1{}^\frown 2^{k_2}{}^\frown \sigma_2{}^\frown \dots, \text{ where } k_i \in \omega.$$

It is clear that $X \leqslant_T \Psi(X)$. On the other hand, a not difficult analysis of the construction shows that $\Psi(X) \leqslant_T X$. Thus, we have $\Psi(X) \equiv_T X$.

These three cases show that Ψ is a reduction from E to $\equiv_T$.

Choose a non-computable c.e. set $V <_T W$. Then $\equiv_T$ has infinitely many classes containing V-computable reals, and E has only one class with a V-computable real (the class $[0^\infty]_E$). Thus, by Lemma 1.1, $\equiv_T$ is not computably reducible to E. Theorem 3.7 is proved. $\square$

3.4. **Zero-jump reductions.** In this subsection, we investigate the behavior of our computability-theoretic equivalences with respect to a weaker reducibility, which allows to use $\mathbf{0}'$ as an oracle. Comparing with the case of computable reducibility, the picture changes quite significantly.

Proposition 3.8. *There is a* $\mathbf{0}'$*-computable reduction from* $\mathrm{Id}(2^\omega)$ *to* $\equiv_T$*.*

Proof. Recall that a Π^0_1 class $\mathcal{C}$ is *computably bounded* if there is a computable, finite branching tree $T \subseteq \omega^{<\omega}$ such that $\mathcal{C} = [T]$ (i.e. $\mathcal{C}$ is the set of all infinite paths through T), and the branching function $b_T \colon T \to \omega$, i.e.

$$b_T(\sigma) = \mathrm{card}(\{n \in \omega : \sigma^\frown n \in T\}),$$

is partial computable.

Jockusch and Soare (Theorem 4.7 of [21]) constructed an infinite, computably bounded Π^0_1 class $\mathcal{C}$ such all its members are pairwise Turing incomparable. Since $\mathcal{C}$ does not contain computable members, it is known that the class $\mathcal{C}$ is perfect (see, e.g., p. 649 in [7]). Fix a computable tree T, which witnesses the computable boundedness of $\mathcal{C}$. Without loss of generality, one may assume that $T \subseteq 2^{<\omega}$.

Recall that a node $\sigma \in T$ is *extendible* if there is $X \in [T]$ such that $\sigma \subset X$. Consider a $\mathbf{0}'$-computable tree T^{ext}, which consists of all extendible nodes from T.

We define a $\mathbf{0}'$-computable sequence $(\alpha_\sigma)_{\sigma \in 2^{<\omega}}$ of binary strings with the following properties:

- every α_σ is an extendible node from T,
- $\alpha_\sigma \subseteq \alpha_\tau$ if and only if $\sigma \subseteq \tau$,
- if $|\sigma| = |\tau|$, then $|\alpha_\sigma| = |\alpha_\tau|$.

Set $\alpha_\epsilon := \epsilon$. Suppose that for all σ with $|\sigma| = s$, the nodes α_σ are already defined.

Since the class $\mathcal{C}$ is perfect, one can find nodes $\alpha_{\sigma^\frown 0}, \alpha_{\sigma^\frown 1} \in T^{\mathrm{ext}}$, for $|\sigma| = s$, such that:

- $\alpha_{\sigma^\frown i}$ extends α_σ,
- $\alpha_{\sigma^\frown 0}$ and $\alpha_{\sigma^\frown 1}$ are incomparable, and
- the lengths of all $\alpha_{\tau^\frown j}$, where $|\tau| = s$ and $j \in \{0, 1\}$, are equal.

This concludes the description of the sequence $(\alpha_\sigma)_{\sigma \in 2^{<\omega}}$.

We define a $\mathbf{0}'$-computable operator Ψ. Suppose that $X \in 2^\omega$. The output $\Psi(X)$ is defined as

$$\Psi(X) = \bigcup_{s \in \omega} \alpha_{X \upharpoonright s}.$$

Notice that $\Psi(X)$ is a member of the Π^0_1 class $\mathcal{C}$.

Consider two reals X and Y. It is clear that $X = Y$ if and only if $\Psi(X) = \Psi(Y)$. If $X \neq Y$, then by the choice of the class $\mathcal{C}$, we have

$\Psi(X) \not\equiv_T \Psi(Y)$. We deduce that Ψ induces a $\mathbf{0}'$-computable reduction from $\mathrm{Id}(2^\omega)$ to $\equiv_T$. Proposition 3.8 is proved. □

Proposition 3.9. *There is a $\mathbf{0}'$-computable reduction from E_0 to $\equiv_T$.*

Proof. The idea of the proof is similar to that of Proposition 3.8. Jockusch and Soare (Theorem 4.6 of [21]) constructed disjoint c.e. sets A and B such that:

- $A \cup B$ is coinfinite,
- if each of two sets $X, Y \subset \omega$ separates the pair (A, B) and the symmetric difference $X \triangle Y$ is infinite, then X and Y are Turing incomparable.

Suppose that the complement $\overline{A \cup B}$ equals $\{m_0 < m_1 < m_2 < \dots\}$.

We define a $\mathbf{0}'$-computable operator $\Psi \colon 2^\omega \to 2^\omega$. Given a real X and a natural number k, we set:

$$\Psi(X)(k) := \begin{cases} 0, & \text{if } k \in A, \\ 1, & \text{if } k \in B, \\ X(i), & \text{if } k = m_i \text{ for some } i \in \omega. \end{cases}$$

The choice of the sets A and B implies that for all $X, Y \in 2^\omega$, we have

$$(X \; E_0 \; Y) \;\Leftrightarrow\; (\Psi(X) \; E_0 \; \Psi(Y)) \;\Leftrightarrow\; (\Psi(X) \equiv_T \Psi(Y)).$$

Hence, we deduce that Ψ is a reduction from E_0 to $\equiv_T$. Proposition 3.9 is proved. □

Observe that the last proof can be readily adapted to show that E_0 is $\mathbf{0}'$-computably reducible to $\equiv_{tt}$. Indeed, the operator Ψ defined in the proof maps E_0-equivalent reals to reals that differ only finitely (and thus tt-equivalent), and E_0-inequivalent reals to reals that are T-inequivalent (and thus tt-inequivalent).

Figure 2 summarizes the results about the relative complexity of computability-theoretic equivalences, with respect to $\mathbf{0}'$-computable reductions.

Note that we leave open the following problem:

Question 2. *(1) Let $E \in \{\equiv_{tt}, \equiv_m, \equiv_1\}$. Is there a $\mathbf{0}'$-computable reduction from E to $\equiv_T$?*
(2) Are the arrows for $\{\equiv_{tt}, \equiv_m, \equiv_1\}$ in Figure 2 non-reversible? For example, is there a $\mathbf{0}'$-computable reduction from $\equiv_1$ to $\equiv_m$?

3.5. **Degree spectra of reductions.** Previously we proved that, even though there is no computable reduction from $\mathrm{Id}(2^\omega)$ to $\equiv_T$, such a reduction is computable in $\mathbf{0}'$. Then, it is natural to ask whether there are oracles $\mathbf{d} \not\geqslant \mathbf{0}'$ so that $\mathrm{Id}(2^\omega) \leqslant_{\mathbf{d}} \equiv_T$. More generally, given equivalence relations E and F on reals, a nice measure of the complexity of computing reductions from E to F is provided by the following spectra of degrees, which lift to 2^ω the definition given in [14] for countable equivalence relations.

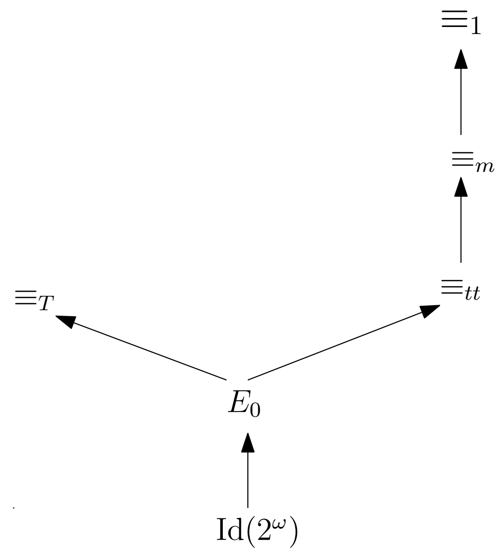

FIGURE 2. Relative to $\mathbf{0}'$-jump reducibility

Definition 3.10. The *reducibility spectrum* of equivalence relations $E, F \subseteq 2^\omega \times 2^\omega$ is the collection of Turing degrees $\{\mathbf{d} : E \leqslant_{\mathbf{d}} F\}$.

3.5.1. *PA degrees in the spectrum of* $\{\mathbf{d} : \mathrm{Id}(2^\omega) \leqslant_{\mathbf{d}} \equiv_T\}$.

Theorem 3.11. *Any PA degree computes a reduction from* $\mathrm{Id}(2^\omega)$ *to* $\equiv_T$, *that is, a perfect tree whose any pair of paths are Turing incomparable.*

Lemma 3.12 (Kučera [23])**.** *Let* $\mathcal{P}$ *be a closed set and let* σ *be a string such that* $\lambda(\mathcal{P} \mid [\sigma]) > 2^{-c}$. *Then there exists two distinct strings* $\tau_0, \tau_1 \succeq \sigma$ *such that* $|\tau_0| = |\tau_1| = |\sigma| + c + 1$ *and such that* $\lambda(\mathcal{P} \mid [\tau_i]) > 2^{-c-1}$.

Definition 3.13. *Let* $f : \omega \to \omega$ *be an order function. An* f*-tree is a set* $T \subseteq 2^{<\omega}$ *such that every node of* T *is of length* $f(n)$ *for some* n *and such that for every* σ *of* T *of length* $f(n)$ *there are exactly two extensions of* σ *in* T *of length* $f(n+1)$.

Lemma 3.14. *Let* $\mathcal{P}$ *be a closed set of measure greater than* 2^{-c}. *There exists a computable order function* $f : \omega \to \omega$, *and an* f*-tree* T *(not necessarily computable) such that* $[T] \subseteq \mathcal{P}$. *The computation of* f *is uniform in* c *and does not depend on* $\mathcal{P}$.

Proof. By a direct application of Kučera, the function f being given by $f(0) = 0$ and $f(n+1) = f(n) + c + n + 1$. □

Lemma 3.15. *Let* T_1 *be an* f_1*-tree and Let* T_2 *be an* f_2*-tree for computable functions* f_1, f_2. *Let* Φ_e *be a functional. One can compute a function* $g_1 \geqslant f_1$ *such that for any* $\sigma \in T_2$ *there is a* g_1*-tree* $Q_1 \subseteq T_1$ *and some immediate extension* τ *of* σ *in* T_2 *for which* $\Phi_e(X) \not\succeq \tau$. *The computation of* g_1 *is uniform in* f_1.

Proof. Let $T_1^* : 2^{<\omega} \to 2^{<\omega}$ be the bijection associated with T_1, that is, for each $\sigma \in 2^{<\omega}$, $T_1^*(\sigma 0)$ and $T_1^*(\sigma 1)$ are respectively the leftmost and rightmost extensions of $T_1^*(\sigma)$ in T_1. For $X \in 2^\omega$ we write $T_1^*(X)$ for the corresponding path of T_1.

Let τ_0, τ_1 be any two incomparable extensions of σ. For $i \in \{0,1\}$, let $\mathcal{U}_i$ be the open set consisting in the union of cylinders $[\rho]$ for strings ρ such that $\Phi_e(T_1^*(\rho)) \succeq \tau_i$.

If the measure of $\mathcal{U}_0$ is smaller than 1/2 we let $\mathcal{P}$ be its complement and τ be τ_0. Otherwise we let $\mathcal{P}$ be the complement of $\mathcal{U}_1$ and τ be τ_1. Note that as $\mathcal{U}_0 \cap \mathcal{U}_1 = \varnothing$ we must have $\lambda(\mathcal{P}) \geqslant 1/2$. Also for any $X \in \mathcal{P}$ we have $\Phi_e(T_1^*(X)) \nsucceq \tau$. By Lemma 3.14 we have a computable function g and a g-tree P such that $[P] \subseteq \mathcal{P}$. We can then pull the tree P inside T_1 by letting Q_1 be the set of strings $T_1^*(\sigma)$ for $\sigma \in P$. From f_1 and g one easily compute the function f_2 such that Q_1 is and f_2-tree. ☐

Lemma 3.16. *There exists an f-tree T for some computable function f such that for any distinct paths $X, Y \in [T]$ and any e we have $\Phi_e(X) \nsucc Y \upharpoonright f(e)$.*

Proof. We build T and f step by step. At each step s we have our function f specified on $0, \dots, s$ and a finite f-tree T_s of level s. We also have a computable function g_s such that for each leaf σ of T_s we have a g-tree Q_σ whose first branching node is σ. We will in particular have $\sigma \preceq \tau$ implies $Q_\tau \subseteq Q_\sigma$. Note that in this process, each g_s will be computable uniformly in s, but Q_σ may not be computable, and T_s will not be computable uniformly in s.

We start with T_0 to be ϵ, the empty string and Q_ϵ to be the whole space and thus f_0 to be the identity function. Suppose a finite complete tree T_s of level s is defined at step s, together with all its corresponding elements mentioned in the above paragraph. Let $\{\sigma_i\}_{i \in 2^{s+1}}$ be a list of the first left and right extension of each leaf σ of T_s, within their respective subtree T_σ. For each $i < 2^{s+1}$, letting $\sigma \prec \sigma_i$ be the leaf of T_s that σ_i extends, we let $Q_i \subseteq \mathcal{Q}_\sigma$ be the $(n \mapsto g(n+1))$-tree below σ_i, obtained by restricting the g-tree Q_σ to extensions of σ_i.

We will extend each string σ_i in finitely many substeps, restricting in the same time their associated trees. The extension of σ_i at step t will be denoted by $\sigma_{i,t}$, so that $\sigma_i = \sigma_{i,0} \preceq \sigma_{i,1} \preceq \dots$. We will also build theirs corresponding g_t-tree $Q_{i,t}$ whose first branching node is $\sigma_{i,t}$. Each g_t will also be a computable function, uniformly in t.

At step 0 we let $\sigma_{i,0} = \sigma_i$, $g_0 = n \mapsto g(n+1)$ and $Q_{i,0} = Q_i$. The number of substep t corresponds to the number of ordered pair of distinct integers smaller than 2^{s+1}. Suppose all the mentioned elements are defined at step t. Let (i,j) be the pair corresponding at the step $t+1$. We apply Lemma 3.15 on $Q_{i,t}$ and $Q_{j,t}$, to find the leftmost or rightmost extension $\sigma'_{j,t+1}$ of $\sigma_{j,t}$ in $Q_{j,t}$, and a g_{t+1}-tree $Q'_{i,t+1} \subseteq Q_{i,t}$ (where $g_{t+1} \geqslant g_t$ is computable) such that $\Phi_s(X) \nsucceq \sigma'_{j,t+1}$ for any $X \in [Q'_{i,t+1}]$. Then for any $k \neq i$ we defined $Q'_{k,t+1} \subseteq Q_{k,t}$ so that $Q'_{k,t+1}$ is a g_{t+1}-tree, making sure that for $k = j$ the tree contains extensions of $\sigma'_{j,t+1}$. Then we define $\sigma_{j,t+1} \succeq \sigma'_{j,t+1}$ as the first branching extension of $\sigma'_{j,t+1}$ in $Q'_{j,t+1}$. For any $k \neq j$ we define an extension $\sigma_{k,t+1} \succeq \sigma_{k,t}$ as the leftmost node of $Q'_{k,t+1}$ of length $|\sigma_{j,t+1}|$. Finally, we define for any k the tree $Q_{k,t+1} \subseteq Q'_{k,t+1}$ as the restriction of $Q'_{k,t+1}$ to nodes extending $\sigma_{k,t+1}$. The function g_{t+1} is then computed accordingy.

Let m be the last of these substeps. Once we have dealt with each pair of leaves, we put each $\sigma_{i,m}$ in T_{s+1} for $i \leqslant 2^{s+1}$, we define the computable function $g_{s+1} = g_m$ and we define the g_{s+1}-trees $\mathcal{Q}_{\sigma_i} = \mathcal{Q}_{i,m}$. The value $f(s+1)$ is defined as the length of each $\sigma_{i,m}$, which can be computably obtained from the whole process. □

Proof of Theorem 3.11. Using Lemma 3.16, there is a computable order function $f : \omega \to \omega$ such that the Π^0_1 class

$$\left\{ T \subseteq 2^{<\omega} \ : \ \begin{array}{l} T \text{ is an } f\text{-tree and for all } e \in \omega \\ \text{for all } \sigma \in T \text{ with } |\sigma| = f(e) \\ \text{for all } \tau \in T \text{ incomparable with } \sigma, \\ \text{for all } t \in \omega,\ \Phi_e(\tau)[t] \not\succeq \sigma \end{array} \right\}$$

is non-empty. □

Question 3. *If X is of PA degree, can it compute a reduction from E_0 to $\equiv_T$.*

3.5.2. *ANR degrees in the spectrum of $\{\mathbf{d} : E_0 \leqslant_{\mathbf{d}} \equiv_T\}$.* We now show that any sufficiently hyperimmune degree computes a reduction from E_0 to the Turing degrees.

Definition 3.17. *A set X is array non-computable, if for every function $f \leqslant_{wtt} \emptyset'$, there exists an X-computable function $g \leqslant_T X$ such that $g(n) > f(n)$ for infinitely many n.*

Definition 3.18. *Let $g : \omega \to \omega$ be a computable order function. A function $f : 2^{<\omega} \to 2^{<\omega}$ is g-approximable if there is a computable function $h : 2^{<\omega} \times \omega \to 2^{<\omega}$ such that $f(\sigma) = \lim_{s\to\infty} h(\sigma, s)$ and such that $|\{s \in \omega : h(\sigma, s) \neq h_{\sigma,s+1}\}| \leqslant g(|\sigma|)$.*

Definition 3.19. *Let $g : \omega \to \omega$ be a computable order function. A g-approximable dense set of strings is the image of a g-approximable function $f : 2^{<\omega} \to 2^{<\omega}$ with $\sigma \preceq f(\sigma)$.*

Theorem 3.20 (Downey, Jockusch, and Stob [12]). *Suppose X is array non-computable. Then for every computable order function $g : \omega \to \omega$, X computes a set Y which meets every g-approximable dense set of strings.*

Theorem 3.21. *If X is array non-computable, it computes a reduction from E_0 to $\equiv_T$:*

Proof. We work in 3^ω and we consider elements $X \in 3^\omega$ as encoding for strongly uniform trees, where the digit 2 in X represent a splitting in the corresponding tree. For instance 012110201 corresponds to the finite tree with 01 as first branching node, then 010110 and 011110 as leftmost and rightmost extensions of 01, with then 010110001, 010110101 and 011110001, 011110101 as the nodes of next levels in the tree.

Equivalently the tree consists of all the possible completions of X where the digit 2 is seen as empty with the possibility to fill it with 0 or 1. We show that if $G \in 3^\omega$ is 2^n-approximable generic then its corresponding tree T_G has the property that any path which differ on infinitely many bits are on distinct Turing degrees.

Given $e \in \omega$, let $f_e : 3^{<\omega} \to 3^{<\omega}$ be the 2^n-approximable function defined as follow: given a string $\sigma \in 3^{<\omega}$, the function starts by outputting as a first version of its answer the string σ itself. Then it computes the finite tree T_σ coded by σ. Note that T_σ has at most $2^{|\sigma|}$ leaves. Then for any leaf $\tau \in T_\sigma$, the function f searches for a string ρ such that $\Phi_e(\tau\rho)$ is incomparable with $\tau'\rho$ for some leaf $\tau' \in T_\sigma$. Whenever f finds such a string ρ it outputs an extension of σ corresponding to the tree T_σ where every leaf is extended by ρ, then f restart the search for the new leaves of the tree, except for the extension of the ones on which the search was already conclusive. Note that $f_e(\sigma)$ may change at most 2^n times, because there are at most 2^n leaves.

Now if G is a 2^n-approximable generic set, for every e there exists σ such that $f_e(\sigma) \prec G$. In particular if we let $T_{f,\sigma}$ be the encoded by $f_e(\sigma)$, by assumption, for any leaves $\tau_1, \tau_2 \in T_{f,\sigma}$, either $\Phi_e(\tau_1)$ is incomparable with τ_2, or for any possible extension $\rho \succeq \tau_1$, we will have $\Phi_e(\tau_1\rho)$ comparable with $\Phi_e(\tau_2\rho)$. In this case for any $X \succeq \tau_1$, if $\Phi(X) = Y$ then X and Y differ only by a finite prefix. In particular for any X, Y extending leaves of $T_{f,\sigma}$ and such that $(X, Y) \notin E_0$, then $\Phi_e(X) \neq Y$.

As it is the case for every e, if G is 2^n-approximable generic, any two path of the tree coded by G which differ on infinitely many bits are Turing incomparable. □

Corollary 3.22. *The class of reals computing a reduction from* $\mathrm{Id}(2^\omega)$ *to* $\equiv_T$ *is co-meager.*

We now see classes of reals which cannot compute a reduction from $\mathrm{Id}(2^\omega)$ to $\equiv_T$:

Theorem 3.23. *The following class of degrees do not compute a reduction from* $\mathrm{Id}(2^\omega)$ *to* $\equiv_T$*:*

(1) The 2*-random degrees*
(2) The computably dominated and not DNC degrees

Proof. Barmpalias and Lewis [5], and Lewis [24] proved that for any set X in this class and any perfect X-computable tree T, there exists a perfect X-computable subtree $Q \subseteq T$ such that any path of Q computes X. It follows that T contains many paths in the degree of X. □

Corollary 3.24. *The class of reals computing a reduction from* $\mathrm{Id}(2^\omega)$ *to* $\equiv_T$ *is of measure* 0*.*

3.5.3. *Computable Vitali.* We define a computable version of the Vitali relation. Recall that Δ is defined on sets of integers as $(X\Delta Y)(n)$ equals 1 if $X(n) \neq Y(n)$ and $(X\Delta Y)(n)$ equals 0 otherwise .

Definition 3.25. *Let V_c be the equivalence relation defined by XV_cY iff $X\Delta Y$ is computable.*

We shall now show that V_c is continuously reducible to E_0 : in fact Z can compute a reduction from V_c to E_0 iff Z is of high degree, that is, equivalently :

- $Z' \geqslant_T \emptyset''$
- Z can compute a function eventually above any computable function
- Z can uniformly list all the computable sets (possibly with repetitions).

Theorem 3.26. *A set Z is of high degree iff there is a Z-computable reduction from V_c to E_0.*

We split the theorem in two lemmas.

Lemma 3.27. *Suppose Z is of high degree. Then there is a Z-computable reduction from V_c to E_0.*

Proof. By Lemma 4.21 (see below) it is enough to give a Z-computable hyperfinite presentation of V_c. Using Z we can list the computable elements of 2^ω : $C_0, C_1, C_2, \ldots$.

Given n let A_n be the set of all possible combinations of the form

$$C_{i_0}\Delta C_{i_1}\ldots\Delta C_{i_k}$$

for $1 \leqslant k \leqslant n$ and $i_0, i_1, \ldots, i_k \leqslant n$. Note that each A_n is the smallest set containing $C_0, \ldots, C_n$ and closed by symmetric difference of its elements. Let

$$F_n = \{(X, Y) \ : \ X\Delta Y = C_{n,i} \text{ for some } C_{n,i} \in A_n\}$$

Suppose $X\Delta Y = C_{n,i}$ and $Y\Delta Z = C_{n,j}$. Then $X\Delta Z = X\Delta 0\Delta Z = X\Delta Y\Delta Y\Delta Z = C_{n,i}\Delta C_{n,j}$. Thus $X\Delta Z \in F_n$.

For each X and each C there exists exactly one set Y such that $X\Delta Y = C$. It follows that each F_n is finite as each A_n is finite.

Also it is clear that $F_n \subseteq F_{n+1}$. Furthermore each F_n can be described by a computable pruned tree, uniformly in Z : For every X, Y, C we have $X\Delta Y = C$ iff for every n we have $X \upharpoonright n\Delta Y \upharpoonright n = C \upharpoonright n$. Also whenever $\sigma\Delta\tau = C \upharpoonright n$, there exist extensions $\sigma' \succ \sigma, \tau' \succ \tau$ of length $n+1$ such that $\sigma'\Delta\tau' = C \upharpoonright n+1$. □

Lemma 3.28. *Let $\bigcup_n \mathcal{F}_n$ be a $\Sigma^0_2(Z)$ class containing exactly the computable points. Then Z is of high degree.*

Proof. We elaborate on a trick used by Jockusch [20] to show that any listing of all the computable sets is of high degree. Consider any Π^0_2 set $A = \{e \ : \ \forall n\ \exists m\ \Phi(n, m, e)\}$ where Φ is a computable predicate. Let us show that A is $\Sigma^0_2(Z)$. Let us define a total computable function f with the following properties :

(1) If $e \in A$ then $n \mapsto g(e,n)$ is a total computable function
(2) If $e \notin A$ then $n \mapsto g(e,n)$ is a partial computable function, which has no computable completion.

Let e be fixed. We describe a process uniform in e : At stage t, for every value n smaller than t such that g has not halted yet, we do the following : If $\Phi_n(n)[t] \downarrow\neq 0$ we set $g(n) = 0$. Otherwise if $\Phi_n(n)[t] \downarrow\neq 1$ we set $g(n) = 1$. Otherwise if for every $k \leqslant n$ there exists $m_k \leqslant t$ such that $\Phi(k, m_k, e)$ holds, then we set $g(n) = 0$.

The process is clearly computable. Let us show (1). Suppose $e \in A$. Then for any n there exists a smallest t such that for every $k \leqslant n$ there exists $m_k \leqslant t$ for which $\Phi(k, m_k, e)$ holds. When this happens then $g(n)$ takes a value at stage t if it did not take one before. It follows that g is total. Let us show (2). Suppose $e \notin A$. Let n be the largest such that for every $k \leqslant n$ there exists m_k for which $\Phi(k, m_k, e)$ holds. For any $m > n$ we have that $g(m)$ halts iff $\Phi_m(m)$ halts, in which case $g(m) \neq \Phi_m(m)$.

Suppose for contradiction that g has a computable completion. Then it has a computable completion with code $a > n$. In this case $g(a) = \Phi_a(a)$ which contradicts the definition of g. Therefore we have (2).

Let us now give a $\Sigma^0_2(Z)$ definition of A. For each n let T_n be a Z-computable tree whose infinite paths are the elements of F_n. Recall that g is the computable function such that for each e the function $n \mapsto g(e,n)$ has property (1) and (2) above. In what follows let us denote the function $n \mapsto g(e,n)$ by $g(e,.)$.

Let $P(e)$ be the $\Sigma^0_2(Z)$ statement:

$$\exists n\ \forall m\ \forall t\ \exists \sigma \in T_n \text{ with } |\sigma| = m \text{ s.t. } \sigma \text{ is a completion of } g(e,.) \upharpoonright m \text{ at stage } t.$$

We claim that $e \in A$ iff $P(e)$. Let us suppose $e \in A$. Then by (1) $g(e,.)$ is a total computable $\{0,1\}$-valued function. It follows that it belongs to some F_n and therefore $P(e)$ is true. Let us suppose $P(e)$. Note that if σ is a completion of $g(e,.) \upharpoonright m$ at stage t then for $k \leqslant m$ we also have that $\sigma \upharpoonright k$ is a completion of $g(e,.) \upharpoonright k$ at stage t. It follows that by compactness, there is an infinite path X in some $[T_n]$ such that for every m and every t we have that $X \upharpoonright m$ is a completion of $g(e,.) \upharpoonright m$ at stage t. It follows that X is a completion of $g(e,.)$. As X is computable, it must be that $g(e,.)$ is computable and thus it must be that $e \in A$. □

We now turn to the proof of Theorem 3.26.

Proof of Theorem 3.26. If Z is of high degree then using Lemma 3.27 there is a Z-computable reduction from V_c to E_0. Suppose now that there is a Z-computable function f such that $(X,Y) \in V_c$ iff $(f(X), f(Y)) \in E_0$.

Let $f_0 = f(0^\infty), f_1, f_2, \ldots$ be a Z computable list of all finite modifications of $f(0^\infty)$. Note that each set $f^{-1}(f_n)$ is a $\Pi^0_1(Z)$ class uniformly in n, containing only computable points, and that $\bigcup_{n\in\omega} f^{-1}(f_n)$ is a $\Sigma^0_2(Z)$ class containing exactly the computable points. We conclude using Lemma 3.28. □

4. How effective is Dougherty-Jackson-Kechris' theorem

In this section, we will focus on checking the computable complexity of some classical results related to hyperfiniteness and similar notions.

Recall that an equivalence relation is finite if all of its equivalence clases are finite. An equivalence relation is hyperfinte if it is the increasing union of finite equivalence relations. By a theorem of Dougherty-Jackson-Kechris [10], a Borel equivalence relation is hyperfinite if and only if it is reduced by E_0. To explore how effective is this notion, we need to re-prove some well known results first.

4.1. **Luzin-Novikov.** The purpose of this subsection is to re-prove the Luzin-Novikov theorem, which is essential in the study of countable Borel equivalence relation. A different proof can be found in [28].

Theorem 4.1 (Recursion theoretic Luzin-Novikov)**.** *Let $\mathbb{R} \subseteq 2^\omega \times 2^\omega$ be a Δ^1_1 relation such that for all X the class $\{Y \in 2^{\mathbb{N}} : (X,Y) \in \mathbb{R}\}$ is countable (possibly empty). Then there exists a computable ordinal α such that for every X, if $(X,Y) \in \mathbb{R}$ then $Y \leqslant_T X^{(\alpha)}$.*

In what way has 4.1 anything to do with Luzin-Novikov? Well, we can easily obtain a Borel uniformization of $\mathbb{R}$ by considering the function $f_0 : 2^\omega \to 2^\omega$ which to X associates $\Phi_e(X^{(\alpha)})$ for the smallest e such that $\Phi_e(X^{(\alpha)})$ is defined and such that $(X, \Phi_e(X^{(\alpha)})) \in \mathbb{R}$ (if no such e exists the value $f_0(X)$ is undefined). We can then inductively define $f_{n+1} : 2^\omega \to 2^\omega$ which to X associates $\Phi_e(X^{(\alpha)})$ for the smallest $e > f_n(X)$ such that $\Phi_e(X^{(\alpha)})$ is defined and such that $(X, \Phi_e(X^{(\alpha)})) \in \mathbb{R}$ (if no such e exists or if $f_n(X)$ is undefined then the value $f_{n+1}(X)$ is undefined). In the end we have $\mathbb{R} = \bigcup_n \mathcal{P}_n$ where $\mathcal{P}_n$ is the graph of f_n.

The proof of 4.1 is a consequence of the following technical lemma:

Lemma 4.2. *Let $\mathcal{A} \subseteq \omega \times 2^\omega$ be a Π^1_1 set. Suppose*

$$\forall X\ \exists a \in O^X\ (a, X) \in \mathcal{A}$$

Then there exists a computable ordinal β such that:

$$\forall X\ \exists a \in O^X_{<\beta}\ (a, X) \in \mathcal{A}$$

Proof. Suppose $\forall X\ \exists a \in O^X\ (a, X) \in \mathcal{A}$. Suppose also for contradiction the lemma is false, that is, $\forall \beta < \omega_1^{ck}\ \exists X\ \forall a \in O^X_{<\beta}\ (a, X) \notin \mathcal{A}$. Then we can give a Σ^1_1 description of Kleene's $\mathcal{O}$, which is a contradiction. We pretend that $b \in \mathcal{O}$ iff

$$(*) \qquad \exists X\ \forall a \in \omega\ (a \notin \mathcal{O}^X \vee (a, X) \notin \mathcal{A} \vee b \in \mathcal{O}_{<|a|^X})$$

Suppose $b \in \mathcal{O}$. Then by hypothesis there exists X such that $\forall a \in O^X_{<|b|+1}$ $(a, X) \notin \mathcal{A}$. Thus for all $a \in \omega$ either $a \notin \mathcal{O}^X$ or $a \in \mathcal{O}^X$ in which case either $|b| < |a|^X$ and then $b \in \mathcal{O}_{<|a|^X}$ or we have $|a|^X \leqslant |b|$ and we have $a \in \mathcal{O}^X_{<|b|+1}$ in which case $(a, X) \notin \mathcal{A}$. Thus (*) is true.

Suppose $b \notin \mathcal{O}$. Let $X \in 2^\omega$. By assumption there exists $a \in \mathcal{O}^X$ such that $(a, X) \in \mathcal{A}$. In particular we have $(a \in \mathcal{O}^X \wedge (a, X) \in \mathcal{A} \wedge b \notin \mathcal{O}_{<|a|^X})$. Thus (*) is false.

Then $b \in \mathcal{O}$ iff (*) is true, which gives a Σ^1_1 description of $\mathcal{O}$, which is a contradiction. □

We are now ready to prove our recursion theoretic version of Luzin-Novikov.

Proof of 4.1. For any X the class $\{Y \in 2^\omega : (X, Y) \in \mathbb{R}\}$ is a countable $\Delta^1_1(X)$ set which then contains only elements which are $\Delta^1_1(X)$. In particular for any Y such that $(X, Y) \in \mathbb{R}$ there is an ordinal $\alpha < \omega_1^X$ such that $Y \leqslant_T X^{(\alpha)}$. We thus have

$$\forall X\ \forall Y\ (X, Y) \notin \mathbb{R} \vee \exists \alpha < \omega_1^X\ X^{(\alpha)} \geqslant_T Y$$

and thus

$$\forall X\ \forall Y\ \exists \alpha < \omega_1^X\ (X, Y) \notin \mathbb{R} \vee X^{(\alpha)} \geqslant_T Y$$

By 4.2 we have a computable ordinal β such that :

$$\forall X\ \forall Y\ \exists \alpha < \beta\ (X, Y) \notin \mathbb{R} \vee X^{(\alpha)} \geqslant_T Y$$

□

Corollary 4.3. *Let $E \subseteq 2^\omega \times 2^\omega$ be a countable Δ^1_1 equivalence relation. Then there exists a computable ordinal α such that the equivalence classes of E are a refinement of the α-degrees, that is, $(X, Y) \in E$ implies $X \leqslant_T Y^{(\alpha)}$ and $Y \leqslant_T X^{(\alpha)}$.*

Proof. A direct application of 4.1 □

4.2. Π^0_1 **Equivalence relations.** We show here that Π^0_1 equivalence relations are as simple as they can be: always smooth. This is a known result (see for example [15]). However, we obtain a computable refinement. We make a strong use of compactness here, and this result would not be true for any Polish space.

Lemma 4.4. *Let $\mathcal{F}$ be a Π^0_1 equivalence relation on 2^ω. Then there is a code e of a functional Φ_e such that :*

1. *For every $X \in \mathrm{dom}\mathcal{F}$, $\Phi_e(X)$ codes for tree whose infinite paths are the elements in relation with X.*
2. *Whenever $(X, Y) \in \mathcal{F}$ then $\Phi_e(X)$ and $\Phi_e(Y)$ code for the exact same tree.*

Proof. Let T be the computable tree of $2^{<\omega} \times 2^{<\omega}$ describing $\mathcal{F}$. We can assume without loss of generality that a For every $\sigma \in 2^{<\omega}$ we have $(\sigma, \sigma) \in T$

b For any $\sigma, \tau \in 2^{<\omega}$, we have $(\sigma, \tau) \in T$ iff $(\tau, \sigma) \in T$.

Let $\Phi(\sigma_1, \sigma_2, m)$ be the statement :

$$\exists(\tau_1, \tau_2) \succeq (\sigma_1, \sigma_2) \text{ with } (\tau_1, \tau_2) \in T \text{ and } |\tau_1| = |\tau_2| = m$$

Given n let $m_n = 0$ and m_{n+1} be the smallest integer strictly bigger than $\max(m_n, n)$ such that for all $\sigma_1, \sigma_2, \sigma_3$ of length n we have $\Phi(\sigma_1, \sigma_2, m_{n+1})$ and $\Phi(\sigma_2, \sigma_3, m_{n+1})$ implies $\Phi(\sigma_1, \sigma_3, m_{n+1})$.

Let us argue that $n \mapsto m_n$ is well-defined : suppose there exists n such that for infinitely many m we have three strings $\sigma_1, \sigma_2, \sigma_3$ of length n with $\Phi(\sigma_1, \sigma_2, m)$ and $\Phi(\sigma_2, \sigma_3, m)$ but not $\Phi(\sigma_1, \sigma_3, m)$. As the set of strings of length n is finite, then by the pigeonhole principle, we have three strings $\sigma_1, \sigma_2, \sigma_3$ of length n such that for infinitely many m we have $\Phi(\sigma_1, \sigma_2, m)$ and $\Phi(\sigma_2, \sigma_3, m)$ but not $\Phi(\sigma_1, \sigma_3, m)$. Let m_* be the smallest such m. By König's lemma it follows that we have three elements X_1, X_2, X_3 such that $(X_1, X_2), (X_2, X_3) \in T$, but such that we don't have $\Phi(X_1 \upharpoonright n, X_3 \upharpoonright n, m_*)$. It follows that for no extension $\tau_1 \succeq X_1 \upharpoonright n$ and no extension $\tau_3 \succeq X_3 \upharpoonright n$ with $|\tau_1| = |\tau_3| = m_*$ we have $(\tau_1, \tau_3) \in T$. It follows that $(X_1, X_3) \notin T$. Therefore $\mathcal{F}$ is not an equivalence relation. Thus, for every n for almost every m for all strings $\sigma_1, \sigma_2, \sigma_3$ of length n, we have $\Phi(\sigma_1, \sigma_2, m)$ and $\Phi(\sigma_2, \sigma_3, m)$ implies $\Phi(\sigma_1, \sigma_3, m)$. Thus, $n \mapsto m_n$ is well-defined.

We define the relation A_n on elements of 2^n (the set of strings of length n) by $A_n = \{(\sigma_1, \sigma_2) : \Phi(\sigma_1, \sigma_2, m_n)\}$. We claim that A_n is an equivalence relation. By (a) and (b) above we have reflexivity and symmetry. By the choice of m_n we have transitivity.

A set $X \in \text{dom}\mathcal{F}$ now computes the tree whose nodes of length n are the nodes τ such that $(X \upharpoonright n, \tau) \in A_n$ (and such that every prefix of τ is already in the tree). In order to show (1) and (2), let us argue that

$$(X, Y) \in \mathcal{F} \text{ iff } \forall n \ (X \upharpoonright n, Y \upharpoonright n) \in A_n$$

Suppose $(X, Y) \in \mathcal{F}$. Then for any n and any $m \geqslant n$ we have $\Phi(X \upharpoonright n, X \upharpoonright n, m_n)$ and then for any n we have $(X \upharpoonright n, Y \upharpoonright n) \in A_n$. Suppose now $(X, Y) \notin \mathcal{F}$. Thus there must be some n such that $(X \upharpoonright n, Y \upharpoonright n) \notin T$ and therefore such that $\neg\Phi(X \upharpoonright n, X \upharpoonright n, m_n)$.

Let us now show (2). Suppose $(X, Y) \in \mathcal{F}$. Then for any n we have $(X \upharpoonright n, Y \upharpoonright n) \in A_n$ and therefore by the fact that A_n is an equivalence relation, X and Y will always output the same nodes of levels n. Thus we have (2). Let us now show (1). If $(X, Y) \in \mathcal{F}$ then $\forall n \ (X \upharpoonright n, Y \upharpoonright n) \in A_n$ and thus Y will be in the tree computed by X. If $(X, Y) \notin \mathcal{F}$ then $\exists n \ (X \upharpoonright n, Y \upharpoonright n) \notin A_n$ and thus Y will not be in the tree computed by X. Thus we have (1). □

Corollary 4.5. *Every Π^0_1 equivalence relation on 2^ω is smooth via some computable function.*

Proof. We use the code e of the previous lemma so that every X computes a tree whose infinite paths are the elements in relation with X. Note that if X is not in relation with Y the trees must be different as one contains all prefixes of X and not the other one. □

Corollary 4.6. *Every Π^0_1 equivalence relation on 2^ω has a Δ^0_2 selector.*

Proof. We use the code e of the previous lemma so that every X computes a tree whose infinite paths are the elements in relation with X. Using X' we then output the leftmost path of every such tree. □

The previous corollary is optimal : Π^0_1 equivalence relation does not always have a continuous selector:

Proposition 4.7. *There is a Π^0_1 equivalence relation on 2^ω with no continuous selector.*

Proof. In the following, given a string $\sigma \in 2^{<\omega}$ we write $\#\sigma$ to denote the number of 1's in σ. Let $C \subseteq 2^\omega \times 2^\omega$ be defined by

$$\{(X, Y) \ : \ \forall n \ \#X \upharpoonright n > n/2 \wedge \#Y \upharpoonright n > n/2\}$$

Let $R \subseteq 2^\omega \times 2^\omega$ be the Π^0_1 equivalence relation on 2^ω defined by $C \cup \mathrm{Id}(2^\omega)$: there is one equivalence class containing all elements of C, and outside of this equivalence class we have equality.

Suppose we have a continuous selector f. Let X be the element selected in C. Let $Y \in C$ with $Y \neq X$. As f is continuous it must be that some prefix σ of Y is sent to a prefix of X sufficiently long to be incomparable with σ. It follows that $\sigma 0^\infty$ is not sent to $\sigma 0^\infty$ and therefore f is not a selector. □

4.3. **Reducible to E_0 implies hyperfinite.** By Dougherty–Jackson–Kechris a relation E is hyperfinite iff E is Borel reducible to E_0. The purpose of the following subsections is to study how effective is this equivalence. We restrict for now to Σ^0_2 equivalence relations and lower, as the question is already interesting at this level. In this section we focus on one of the directions, while in the next subsection we will focus on the other one.

Let us introduce for the purpose of this study the following definition:

Definition 4.8. *An hyperfinite Σ^0_2 presentation $\bigcup_n \mathcal{F}_n$ of some equivalence relation is a uniform increasing countable union of finite Π^0_1 equivalence relations.*

Definition 4.9. *An hyperfinite computable presentation $\bigcup_n \mathcal{F}_n$ of some equivalence relation is a uniform increasing countable union of finite Π^0_1 equivalence relations such that each of them is uniformly presented as the infinite paths of a computable pruned tree (a tree with no dead end).*

Note that Π^0_1 classes of the Cantor space can always be represented as the infinite paths of computable trees, but it is in general not the case that these trees can be computably pruned.

We use here Luzin-Novikov to show that a countable equivalence relation which is Δ^1_1 reducible to E_0 is hyperfinite as a uniform union of Δ^1_1 sets.

Theorem 4.10 (Dougherty–Jackson–Kechris)**.** *Let E be a countable equivalence relation on 2^ω such that $(X,Y) \in E$ iff $(f(X), f(Y)) \in E_0$ for some Δ^1_1 function f. Then E is hyperfinite as a uniform union of Δ^1_1 sets.*

Proof. Given an element X we write $\sigma X \upharpoonright_{|\sigma|}$ for the sequence X with the $|\sigma|$ first bits replaced by σ.

Consider the Δ^1_1 relation $R \subseteq 2^\omega \times 2^\omega$ defined by $(X,Y) \in R$ iff $Y \in f^{-1}(\bigcup_\sigma \sigma X \upharpoonright_{|\sigma|})$. Note that given any X the class $\{Y : (X,Y) \in R\}$ is at most countable. Thus from 4.1 there is a computable ordinal α such that for every X, if $(X,Y) \in R$ then $Y \leqslant_T X^{(\alpha)}$. It follows that whenever $(X,Y) \in E$ then both $X, Y \leqslant_T f(X)^{(\alpha)}$ and $X, Y \leqslant_T f(Y)^{(\alpha)}$. Let $\bigcup_n D_n$ be the canonical hyperfinite presentation of E_0 and let F_n be the set:

$$\left\{ \begin{array}{lll} (X,Y) \ : \ X = Y & \text{or} & (f(X), f(Y)) \in D_n \text{ and } \exists Z \in [f(X)]_{D_n} \\ & & \text{and } \exists a, b \leqslant n \text{ s.t.} \\ & & \Phi_a(Z^{(\alpha)}) = X \text{ and } \Phi_b(Z^{(\alpha)}) = Y \end{array} \right\}$$

Note that the set $[f(X)]_{D_n}$ is the equivalence class of $f(X)$ in D_n and thus that the quantification $\exists Z \in [f(X)]_{D_n}$ is merely $\Delta^0_1(f(X))$.

Let us show that each F_n is an equivalence class. For every X we have by definition $(X,X) \in F_n$. It is also clear that $(X,Y) \in F_n$ iff $(Y,X) \in F_n$. Suppose $(X_1, X_2) \in F_n$ and $(X_2, X_3) \in F_n$. Then by transitivity of D_n we have $(f(X_1), f(X_3)) \in D_n$. Furthermore, we also have $\Phi_a(Z^{(\alpha)}) = X_1 \wedge \Phi_b(Z^{(\alpha)}) = X_3$ for some $a, b \leqslant n$. Thus F_n is an equivalence relation. It is clear that each F_n is finite and that $F_n \subseteq F_{n+1}$. ☐

One can ask how effective is 4.10. The proof itself uses quite powerful tools. Also, even if the reduction from E to E_0 is computable, the set $f^{-1}(f(X))$, as a countable $\Pi^0_1(X)$ closed set, may contain points of arbitrarily high complexity in the X-hyperarithmetic Turing degrees. The intuition is then that the argument used in the proof of 4.10 cannot be simplified, and that we need the power of $X^{(\alpha)}$ for arbitrarily large α.

But let us start simple. Suppose E is Σ^0_2 and suppose the reduction $f : E \to E_0$ is computable (resp. continuous). In this case does E necessarily has a Σ^0_2 hyperfinite presentation? We shall now see that this is not the case. We start with the following:

Proposition 4.11. *Let $\mathcal{F} = \bigcup_n \mathcal{F}_n$ be a hyperfinite $\Sigma^0_2(Z)$ presentation of the equivalence relation $\mathcal{F}$. Then the equivalence classes of $\mathcal{F}$ are a refinement of the Turing degrees relative to Z, that is, $(X,Y) \in \mathcal{F}$ implies $X \oplus Z \equiv_T Y \oplus Z$.*

Proof. Suppose $(X,Y) \in \mathcal{F}$. Let n be the smallest such that $(X,Y) \in \mathcal{F}_n$. Then the class $\{W \in 2^\omega : (X,W) \in \mathcal{F}_n\}$ is a finite $\Pi^0_1(X \oplus Z)$ class containing Y and the class $\{W : (Y,W) \in \mathcal{F}_n\}$ is a finite $\Pi^0_1(Y \oplus Z)$ class containing X. It follows that Y is computable from $X \oplus Z$ and that X is computable from $Y \oplus Z$. ☐

We now use the knowledge of countable Π^0_1 classes of the Cantor space. These have been extensively studied by various authors [18, 11, 8, 9]. Among what is known, it is worth mentioning that elements of countable Π^0_1 classes can be arbitrarily high in the hyperarithmetic Turing degrees. However, being a member of a countable Π^0_1 class is not a degree invariant notion and not every hyperarithmetic Turing degree has a member in a countable Π^0_1 class.

We reprove here a very small part of the results we just mentioned, the one needed for our argument:

Lemma 4.12. *There is a code e such that for any X, $\Phi_e(X)$ codes for a tree $T \subseteq 2^{<\omega}$ with countably many infinite paths and such that one path is Turing equivalent to X'.*

Proof. Let us first show that there is a code e such that for any $X \in 2^\omega$, $\Phi_e(X)$ codes for a tree $T \subseteq \omega^{<\omega}$ (in the Baire space, bit the Cantor space) which contains exactly one path, and this path is Turing equivalent to X'.

We define T by $\sigma \in T$ iff for all $n < |\sigma|$ we have:

$$(\sigma(n) = 0 \text{ and } \Phi_n(X, n)[|\sigma|] \uparrow) \text{ or } (\Phi_n(X, n)[\sigma(n)] \downarrow \\ \text{ and } \forall t < \sigma(n)\ \Phi_n(X, n)[t] \uparrow)$$

The tree T is clearly computable. It also contains one unique infinite path f such that $f(n) = 0$ iff $\Phi_n(X, n) \uparrow$ and $f(n)$ is the smallest t such that $\Phi_n(X, n)[t] \downarrow$ otherwise. We clearly have $f \equiv_T X'$.

From a tree T of the Baire space we compute a tree S of the Cantor space whose paths encode the paths in T : $\sigma \in S$ for $\sigma = 0^{n_0}10^{n_1}1 \dots 10^{n_k}$ iff there is a string $\sigma \in T$ of length k such that $\sigma(i) = n_i$ for each $i < k$: it is clear that for $f \in [T]$ we have $0^{f(0)}10^{f(1)}1 \dots$ in $[S]$. Due to the compactness of 2^ω we also of course add unwanted infinite paths in S, but we claim that there are all finite sets : suppose $X \in S$ contains infinitely many 1's. Thus for any prefix $\sigma_k = 0^{n_0}10^{n_1}1 \dots 10^{n_k}$ with $\sigma_k \prec X$, there exists $\tau \succeq \sigma_k$ which has been added in S as an encoding of some string in T. Thus it means that there is a string $\sigma_k \in T$ with $\sigma_k(i) = n_i$ for every $i < k$. As we have $\sigma_0 \prec \sigma_1 \prec \dots$ we then have an infinite path in T corresponding to X.

It follows that S is countable: it contains finite sets together with a path encoding for X'. □

We are now ready to show that Σ^0_2 sets computably below E_0 need not to have a hyperfinite Σ^0_2 presentation. In fact not even a $\boldsymbol{\Sigma^0_2}$ one.

Theorem 4.13. *There a countable Π^0_1 equivalence relations which does not have a hyperfinite $\boldsymbol{\Sigma^0_2}$ presentation.*

Proof. Let e be a code such that for any X, $\Phi_e(X)$ codes for a computable tree $T^X \subseteq 2^{<\omega}$ with countably many infinite paths and such that one path is Turing equivalent to X'.

We define the Π^0_1 equivalence relation E as

$$(X_0 \oplus X_1, Y_0 \oplus Y_1) \in E \text{ iff } X_0 = Y_0 \text{ and } (X_1 = Y_1 \text{ or } X_1, Y_1 \in T^{X_0})$$

Suppose now E has an hyperfinite $\Sigma^0_2(Z_0)$ presentation for some Z_0. Consider the infinite path Z_1 in T^{Z_0} Turing equivalent to Z_0'. Consider also some finite path $Y \in T^{Z_0}$. Then we have $(Z_0 \oplus Z_1, Z_0 \oplus Y) \in E$. But we cannot have $Z_0 \oplus Z_1 \equiv_T Z_0 \oplus Y$ as $Z_0 \oplus Y \equiv_T Z_0$ and as $Z_0 \oplus Z_1 \equiv Z_0'$. We then have a contradiction with 4.11. □

Corollary 4.14. *There is a (Π^0_1) hyperfinite equivalence relations which is computably reducible to E_0 but which does not have a hyperfinite $\mathbf{\Sigma^0_2}$ presentation.*

Proof. From 4.5 every Π^0_1 equivalence relation is smooth via a computable function and thus computably reducible to E_0. □

Note that 4.13 has in fact little to do with hyperfiniteness in itself : in particular, it has no importance if our presentation is *increasing* or not. Using Luzin–Novikov (4.1) it is possible to show that every countable Δ^1_1 equivalence relation can be described as a Δ^1_1 set of the form $\bigcup_n A_n$ where the equivalence classes of each A_n contains exactly two elements. The union is of course not increasing. Also, what 4.13 really shows is that some Π^0_1 equivalence relation cannot be described as a union of closed set, every equivalence class of which is finite (even if we don't care about them being increasing).

Can this be iterated? How about a relation which is computably reducible to E_0 but which does not have a $\mathbf{\Sigma^0_3}$ presentation? Here again this is possible to do, but the argument needs to be different: 4.11 works because Π^0_1 singletons of the Cantor space are computable. But already Π^0_2 singletons can be arbitrarily high in the hyperarithmetic Turing degrees. Let us start with an iteration of 4.11 through the computable ordinals. By singleton we means subset of 2^ω containing exactly one element.

Proposition 4.15. *Let $\mathcal{F} = \bigcup_n \mathcal{F}_n$ be a hyperfinite $\Sigma^0_{\alpha+1}(Z)$ presentation of the equivalence relation $\mathcal{F}$. If $(X, Y) \in \mathcal{F}$ then X is a $\Pi^0_\alpha(Y \oplus Z)$ singleton and Y is a $\Pi^0_\alpha(X \oplus Z)$ singleton.*

Proof. Suppose $(X, Y) \in \mathcal{F}$. Let n be the smallest such that $(X, Y) \in \mathcal{F}_n$. Then the class $\{W \in 2^\omega : (X, W) \in \mathcal{F}_n\}$ is a finite $\Pi^0_\alpha(X \oplus Z)$ class containing Y and the class $\{W \in 2^\omega : (Y, W) \in \mathcal{F}_n\}$ is a finite $\Pi^0_\alpha(Y \oplus Z)$ containing X. Also there must be prefixes $\sigma_Y \prec Y$ and $\sigma_X \prec X$ such that Y and X are the only elements extending their respective prefixes in each class. Thus, by restricting each class to element extending their respective prefixes we have that Y is a $\Pi^0_\alpha(X \oplus Z)$ singleton and that X is $\Pi^0_\alpha(Y \oplus Z)$ singleton. □

In order to repeat the previous argument with $\mathbf{\Sigma^0_3}$ in place of $\mathbf{\Sigma^0_2}$, we need to uniformly have countable $\Pi^0_1(X)$ classes containing elements which cannot be $\Pi^0_2(X)$ singletons. It turns out that this is possible to build, using a result of Downey [11]. We first need the following lemma:

Lemma 4.16. *Suppose X is a $\Pi^0_2(Z)$ singleton which is not Z computable. Then $X \oplus Z$ is not computably dominated relative to Z, that is, $X \oplus Z$ computes a function bounded by no Z-computable function.*

Proof. The function is simply given by the time at which X enters the $\Pi^0_2(Z)$ class $\bigcap_n \mathcal{U}_n$: $f(n) = \min t$ s.t. $X \in \mathcal{U}_n[t]$. Suppose now that there exists a Z-computable function g bounding f. Thus $\bigcap_n \mathcal{U}_n[g(n)]$ is a $\Pi^0_1(Z)$ singleton containing X. Thus X is Z-computable. □

Theorem 4.17 (Downey)**.** *There is a code e such that for any Z, $\Phi_e(Z)$ codes for a Z-computable tree $T \subseteq 2^{<\omega}$ with countably many infinite paths, such that one path X of $[T]$ is not Z-computable and such that $Z \oplus X$ is computably dominated relative to Z.*

Downey's result is a very complex and intricate infinite injury priority argument. We use it in the form of the following corollary:

Corollary 4.18. *There is a code e such that for any Z, $\Phi_e(Z)$ codes for an Z-computable tree $T \subseteq 2^{<\omega}$ with countably many infinite paths and such that one path X is not a $\Pi^0_2(Z)$ singleton.*

Proof. By combining it with 4.16. □

Theorem 4.19. *There a countable Π^0_1 equivalence relations which does not have a hyperfinite $\mathbf{\Sigma^0_3}$ presentation.*

Proof. It is merely a repetition of the proof of 4.13, but using the Π^0_1 classes $T^X \subseteq 2^{<\omega}$ with countably many infinite paths and such that one path is not a $\Pi^0_2(Z)$ singleton. □

Corollary 4.20. *There is a (Π^0_1) hyperfinite equivalence relations which is computably reducible to E_0 but which does not have a hyperfinite $\mathbf{\Sigma^0_3}$ presentation.*

Is it possible to iterate the argument? All we would need to do at step α is to uniformly find a countable $\Pi^0_1(X)$ class containing a point which is not a Π^0_α singleton. However given the complexity of building such a class for Π^0_2 singleton, the argument might be very hard to iterate. We however conjecture that this is true:

Conjecture 4.1. *For any computable ordinal α, uniformly in X there a countable $\Pi^0_1(X)$ class containing a non-$\Pi^0_\alpha(X)$ singleton. Therefore for any computable α there is a equivalence relation computably reducible to E_0 but which does not have a hyperfinite $\mathbf{\Sigma^0_\alpha}$ presentation.*

4.4. **Hyperfinite implies reducible to E_0.** The proof of this direction is also highly non-trivial. We proceed in two steps: first we show the result for hyperfinite computable presentations. Then we show how the general result follows from the one for computable presentation. Recall that a computable presentation of a hyperfinite equivalence relation is given by a Σ^0_2 set $\bigcup_n F_n$ such that each F_n is finite, such that $F_n \subseteq F_{n+1}$ and such that each F_n can be uniformly described as the paths of a computable pruned tree.

Lemma 4.21. *Let $\bigcup_n F_n$ be a computable hyperfinite presentation of some equivalence relation E on 2^ω. Then there is a computable reduction from E to E_0.*

Proof. Each F_n is given as a computable pruned tree T_n. For any X let T_n^X be the tree given by $\{\sigma \in 2^{<\omega} : (X \upharpoonright |\sigma|, \sigma) \in T_n\}$. Note that each T_n^X is also pruned and that $(X, Y) \in F_n$ implies $T_n^X = T_n^Y$. Furthermore each $[T_n^X]$ is finite and $T_n^X \subseteq T_{n+1}^X$ for every X and every n.

For every X and every n we define a computable set A_{n+1}^X as follows: For every $m \leqslant n+1$, for every string $\sigma \in T_n^X$ of length m we put $(\sigma, 0) \in A_{n+1}^X$. For every $m \in \omega$, and for every string $\sigma j \in T_{n+1}^X$ of length m for $j \in \{0, 1\}$ if $\sigma \in T_n^X$ and $\sigma j \notin T_n^X$, we put $(\sigma j, 1) \in A_{n+1}^X$. The reduction from E to E_0 is given by $g(X) = \oplus_n A_n^X$, the disjoint union of each A_n^X.

Let us first show that each A_{n+1}^X is finite. We have that $(\sigma j, 1) \in A_{n+1}^X$ for a string σj of length bigger than $n+1$ iff $\sigma \in T_n^X$ and σ is a splitting point in T_{n+1}^X but not in T_n^X. As T_{n+1}^X is pruned and contains finitely many elements, there is a longest splitting point in T_{n+1}^X and thus A_{n+1}^X is finite.

Let us now show $(X, Y) \in E$ iff $(g(X), g(Y)) \in E_0$. Suppose $(X, Y) \in E$. Then there exists n such that for every $m \geqslant n$ we have $(X, Y) \in F_m$ and thus $T_m^X = T_m^Y$. It follows that for every $m \geqslant n$ we have $A_{m+1}^X = A_{m+1}^Y$. As each A_i^X and A_i^Y is finite for $i \leqslant n$ we have $(g(X), g(Y)) \in E_0$.

Suppose now $(X, Y) \notin E$. For every n we have $X \in [T_n^X]$ and $X \in [T_{n+1}^X]$. It follows that for every n and every prefix σ of X we have $\sigma \in T_n^X$ and $\sigma \in T_{n+1}^X$. Therefore for every n and every prefix σ of X we have $(\sigma, 1) \notin A_{n+1}^X$. Furthermore, for every prefix σ of X of length smaller than $n+1$ we have $(\sigma, 0) \in A_{n+1}^X$.

Suppose now that for infinitely many prefixes σ of X we have $(\sigma, 1) \in A_{n+1}^Y$ for some n. Then $(g(X), g(Y)) \notin E_0$. Suppose now that for only finitely many prefixes σ of X we have $(\sigma, 1) \in A_{n+1}^Y$ for some n. Let us show that there exists m such that for every $n \geqslant m$ and every prefix σ of X we have $(\sigma, i) \in A_{n+1}^Y$ implies $i = 0$. Fix k such that for every prefix σ of X of length bigger than k we have $(\sigma, i) \in A_{n+1}^Y$ for some n implies $i = 0$. As $T_n^Y \subseteq T_{n+1}^Y$ there is a smallest m such that for every $n_1, n_2 \geqslant m$ the trees $T_{n_1}^Y$ and $T_{n_2}^Y$ have the same strings of length smaller than or equal to k. It follows that for $n \geqslant m$, whenever $(\sigma, i) \in A_{n+1}^Y$, either $|\sigma| \leqslant k$ and thus $\sigma \in T_n^Y$ and thus $i = 0$, or $|\sigma| > k$ and thus $i = 0$ by the choice of k.

Let us then fix m such that for every $n \geqslant m$ and every prefix σ of X we have $(\sigma, i) \in A_{n+1}^Y$ implies $i = 0$. Let us show that for every $n \geqslant m$ if some prefix of X belongs to T_{n+1}^Y it also belongs to T_n^Y. Suppose some prefix σ of X belongs to T_{n+1}^Y but not to T_n^Y. Then there is a largest $\tau j \preceq \sigma$ (where

$j \in \{0,1\}$) such that $\tau \in T_n^Y$ and $\tau j \notin T_n^Y$. It follows that $(\tau j, 1) \in A_{n+1}^X$ for $n \geqslant m$, whereas $\tau j \prec X$, which contradicts the choice of m. By induction, it follows that for every $n > m$ if some prefix of X belongs to T_n^Y, then it belongs to T_m^Y.

Suppose now for contradiction that for infinitely many prefixes σ of X we have $(\sigma, 0) \in A_{n+1}^Y$ for some n. As only strings of length smaller than $n+1$ are such that $(\sigma, 0) \in A_{n+1}^Y$, it follow that for infinitely many prefixes σ of X we have $(\sigma, 0) \in A_{n+1}^Y$ for some $n > m$. It follows that infinitely many prefixes σ of X belongs to T_n^Y for some $n > m$. It follows that infinitely many prefixes σ of X belongs to T_m^Y. It follows that $X \in [T_m^Y]$ and thus that $(X, Y) \in E$ which is a contradiction. Thus for only finitely many prefixes σ of X we have $(\sigma, 0) \in A_{n+1}^Y$ for some n. Thus $\oplus_n A_n^X \neq \oplus_n A_n^Y$ and thus $(g(X), g(Y)) \notin E_0$. ☐

We now iterate for any hyperfinite relation. We first make a special case for Σ_2^0 hyperfinite presentation, as the situation seems a little different than for the general case.

Lemma 4.22. *Let $\bigcup_n F_n$ be a Σ_2^0 hyperfinite presentation of some equivalence relation E. Then there is a Δ_2^0 reduction from E to E_0.*

Proof. Each F_n is a finite Π_1^0 equivalence relation. From 4.4 there are computable functions f_n such that for every X the set $f_n(X)$ codes for a tree whose infinite path are the elements in relation with X and such that whenever $(X, Y) \in F_n$ then $f_n(X)$ and $f_n(Y)$ codes for the exact same tree.

Using X' we can compute $T_n(X)$, a pruned version of $f_n(X)$. We can then repeat the argument of 4.21 to get the reduction. ☐

It seems that the jump is needed for Σ_2^0 hyperfinite presentation. We however could not find counterexemple, and therefore address the following question:

Question 4. *Does there exists a Σ_2^0 hyperfinite presentation of some equivalence relation E for which there is no continuous (computable?) reduction from E to E_0?*

We now turn to the proof in the general case, where due to the fact that Π_2^0 finite sets can already contains elements of arbitrarily large hyperarithmetic Turing degrees, there is certainly no hope to find a bound on the complexity of the reduction, given the complexity of the hyperfinite presentation.

Theorem 4.23. *Let $\bigcup_n F_n$ be a Δ_1^1 hyperfinite presentation of some equivalence relation E. Then there is a Δ_1^1 reduction from E to E_0.*

Proof. From Lusin-Novikov (4.1) there exists a computable ordinal α such that whenever $(X, Y) \in E$ we have $X^{(\alpha)} \geqslant Y$ and $Y^{(\alpha)} \geqslant X$. Given X one can compute with $X^{(\alpha+2)}$ a pruned tree T_n^X whose infinite path are exactly the elements in relation with X in F_n: $X^{(\alpha+2)}$ puts τ in the tree if there exists e such that $\Phi_e(X^{(\alpha)})$ is total, if $(X, \Phi_e(X^{(\alpha)})) \in F_n$ and if $\tau \prec \Phi_e(X^{(\alpha)})$.

We can then repeat the argument of 4.21 to get the reduction. $\square$

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