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Analysis, Geometry and Topology of Positive Scalar Curvature Metrics (hybrid meeting)

Organized by
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Abstract. The investigation of Riemannian metrics with lower scalar curvature bounds has been a central topic in differential geometry for decades. It addresses foundational problems, combining ideas and methods from global analysis, geometric topology, metric geometry and general relativity. Seminal contributions by Gromov during the last years have led to a significant increase of activities in the area which have produced a number of impressive results. Our workshop reflected the state of the art of this thriving field of research.

Mathematics Subject Classification (2010): 53C20. Secondary: 53C21, 53C27, 53E20, 58J20, 58J32,35J60, 19K56.

Introduction by the Organizers

The workshop *Analysis, Geometry and Topology of Positive Scalar Curvature Metrics*, organized by Bernd Ammann (Regensburg), Bernhard Hanke (Augsburg) and Anna Sakovich (Uppsala) was the third in a series of workshops with the same title, one in 2014, one in 2017 and the current one. It was attended by 18 participants in presence, mainly from Europe, and 18 online participants from Europe, the US, and China, both numbers including some doctoral and postdoctoral researchers.

On Monday Carla Cederbaum (Tübingen) and Richard Bamler (Berkeley) delivered two extended 80 minute survey lectures on initial data sets in general relativity and the uniqueness of solutions to the Ricci flow, respectively. Important applications of the Ricci flow include the long sought-after result that the space of positive scalar curvature metrics on the 3-sphere is contractible and a definition of pointwise lower scalar curvature bounds for C^0-metrics, the latter being the topic of a separate talk later in the week.

Recently metric properties of manifolds with lower scalar curvature bounds have become accessible to the Dirac operator method by the use of appropriate potentials. A number of research talks reported on this important development, including index theoretic proofs of Gromov's band width, long neck and cube inequalities. This discussion included geometric boundary conditions such as lower mean curvature bounds, which can also be addressed systematically in the context of h-principles as pointed out in another lecture.

Minimal hypersurfaces are an important and by now classical tool in positive scalar curvature geometry. This method can be refined using volume functionals with counterforce. One of the talks used stable minimizers of such functionals, so-called μ-bubbles, to provide an alternative approach to band width inequalities which competes with the Dirac method.

Spaces of metrics with lower scalar curvature bounds were addressed from several perspectives, exploring the natural action of the diffeomorphism group, the flexibility under surgery in codimension at least three and the dominant energy condition. A related discussion concerned spaces of metrics for which the first eigenvalue of certain Laplace type operators is non-negative, a property that plays a role in minimal hypersurfaces, the Yamabe problem and the Ricci flow with surgery.

A number of talks were devoted to the positive mass theorem, which naturally arises in the investigation of initial data sets in general relativity, via the constraint equations. We heard about the positive mass theorems for ALF, ALG and asymptotically hyperbolic manifolds. This theme can also be approached from a spin geometric perspective, as pointed out in another lecture.

The positive scalar curvature problem exhibits remarkable aspects in low dimensions and on manifolds with restricted geometric properties. Besides the aforementioned results emerging from the Ricci flow, some talks of the workshop participants covered a variety of related topics such as rigidity results for (not necessarily uniform) positive scalar curvature metrics on contractible 3-manifolds, the use of spacetime harmonic functions for constructing obstructions to positive scalar curvature, the rich interplay of the Yamabe invariant and complex geometry in dimension four, and global and local scalar curvature rigidity properties of Einstein manifolds.

Most research talks had a length of 60 minutes with some additional 40 minutes talks contributed by younger participants. On Thursday evening a problem session fostered a lively exchange of ideas and perspectives. Similar to the preceding meetings in this series we observed an intense interaction of scientists with different mathematical backgrounds throughout the workshop, integrating the in-person as well as the online participants.

We could always rely on the perfect working conditions at the Oberwolfach institute and the great support by its staff. In particular we appreciated the

possibility to invite a significant number of in-person participants who enjoyed the traditional, stimulating Oberwolfach atmosphere, which was indispensible for the success of our workshop.

Workshop (hybrid meeting): Analysis, Geometry and Topology of Positive Scalar Curvature Metrics

Table of Contents

Abstracts

Overview lecture: Initial data sets in general relativity: when, why, and how should we take into account their extrinsic curvature?

Carla Cederbaum

[1] A relativistic initial data set (IDS) is a tuple (M, g, K) consisting of a smooth Riemannian manifold (M, g) and a smooth, symmetric $(0,2)$-tensor field K on M. Typically, M will have dimension $n = 3$ or $n \geq 3$; sometimes, we consider initial data sets of dimension $n = 2$ as toy models or arising from a symmetry reduction.

1. Where do initial data sets come from?

As the name indicates, initial data sets (M, g, K) arise as hypersurfaces of constant ("initial") time in "spacetimes" $(\mathfrak{L}, \mathfrak{g})$; here, a *spacetime* is a time-oriented Lorentzian manifold and M is an orientable spacelike hypersurface or "slice" in $(\mathfrak{L}, \mathfrak{g})$ with induced metric g and second fundamental form K (with respect to the future pointing normal). The following are important examples of spacetimes and initial data sets therein:

(1) The Minkowski spacetime $(\mathbb{R}^{n+1}, \eta = -dt^2 + \delta)$, δ the Euclidean metric on $\mathbb{R}^n$, with
  (a) $(\{t = 0\}, \delta, K = 0)$, the *canonical initial data set*,
  (b) or more generally the *boosted hyperplanes* $(M = \{t = \vec{a}\cdot\vec{x}+c\}, g, K = 0)$, with $\vec{a} \in \mathbb{R}^n$, $|\vec{a}|_\delta < 1$, $c \in \mathbb{R}$, with (M, g) isometric to $(\mathbb{R}^n, \delta)$,
  (c) either sheet (M_R, g_R, K_R) of the two-sheeted hyperboloid of radius $R > 0$, where (M_R, g_R) is isometric to the hyperbolic space of radius R, $(\mathbb{H}^n, b_R)$, and K_R is proportional to g_R, the *hyperboloidal initial data sets.*

(2) The de Sitter (dS, $\Lambda > 0$) and Anti de Sitter (AdS, $\Lambda < 0$) spacetimes $(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{S}^{n-1}, -\left(1 - \frac{2\Lambda r^2}{n(n-1)}\right) dt^2 + \frac{1}{1-\frac{2\Lambda r^2}{n(n-1)}} dr^2 + r^2\Omega_{\mathbb{S}^{n-1}})$, with $\Omega_{\mathbb{S}^{n-1}}$ the canonical metric on $\mathbb{S}^{n-1}$ containing the *canonical initial data set* $\{t = 0\}$ with $K = 0$. For $\Lambda < 0$, $(\{t = 0\}, g)$ is isometric to $(\mathbb{H}^n, b_{R(\Lambda)})$.

(3) The Schwarzschild spacetime of "mass" $m > 0$, $(\mathbb{R}\times ((2m)^{\frac{1}{n-2}}, \infty) \times \mathbb{S}^{n-1}, -\left(1 - \frac{2m}{r^{n-2}}\right) dt^2 + \frac{1}{1-\frac{2m}{r^{n-2}}} dr^2 + r^2\Omega_{\mathbb{S}^{n-1}})$, with its *canonical initial data set* $\{t = 0\}$ having $K = 0$, modeling the exterior of a static, spherically symmetric black hole.

[1]We refer the interested reader to [9] for more information and references.

2. Features of initial data sets

We care about the asymptotics of initial data sets (M, g, K), most notably:

- M is closed (compact without boundary): in (mathematical) cosmology, and
- (M, g, K) is *asymptotically Euclidean*, i.e.,

$$M \approx_\Phi \text{compact set } \cup \bigcup_{i=1}^{I} \left(\mathbb{R}^n \setminus \overline{B_{r_i}(0)}\right),$$

  $(\Phi_* g)_{ij} = \delta_{ij} +$ l.o.t., $(\Phi_* K)_{ij} =$ l.o.t. as $r \to \infty$ in each *end* $\mathbb{R}^n \setminus \overline{B_{r_i}(0)}$,
- (M, g, K) is *asymptotically hyperboloidal*, i.e., M as above and $(\Phi_* g)_{ij} = (b_R)_{ij}+$ l.o.t., $(\Phi_* K)_{ij} \propto (b_R)_{ij}+$ l.o.t. as $r \to \infty$ in each end (cf. Example 1c), and
- (M, g, K) is *asymptotically hyperbolic*, i.e., M as above and $(\Phi_* g)_{ij} = (b_R)_{ij} +$ l.o.t., $(\Phi_* K)_{ij} =$ l.o.t. as $r \to \infty$ in each end (cf. Example 2).

Another aspect of initial data sets we care about is whether they satisfy the *(Einstein) constraint equations* $\mathrm{R}_g + (\operatorname{tr}_g K)^2 - |K|_g^2 - 2\Lambda = \frac{16\pi G}{c^2}\mu$, $\operatorname{div}_g(K - \operatorname{tr}_g K\, g) = -\frac{8\pi G}{c^4} J$, where G, c, Λ denote the gravitational constant, the speed of light, and the cosmological constant, μ and J denote the energy and momentum density, and R_g, $|\cdot|_g$, and div_g denote the scalar curvature, induced tensor norm, and induced divergence on (M, g, K), respectively. The constraint equations arise from combining the Einstein equation in the ambient spacetime for some matter model with the Gauss–Codazzi–Mainardi equations for spacelike hypersurfaces. We typically focus on either specific matter models (resulting in specific choices of μ and J) such as *vacuum* ($\mu = J = 0$) or we consider matter models satisfying certain energy conditions (resulting in conditions on μ and J), most notably the *dominant energy condition (DEC)* which implies $\mu \geq |J|_g$.

We say that an IDS (M, g, K) is *time-symmetric* or *Riemannian* if $K = 0$, corresponding to an infinitesimal time-reflection symmetry of the ambient spacetime around said IDS.

3. Typical questions about initial data sets

Typical questions about IDSs include questions on their evolution in time (keywords: global hyperbolicity, dynamical stability, hyperbolic PDEs, gravitational waves, numerical simulation), on existence of solutions of the constraint equations displaying physically relevant features (keywords: gluing, regularity, numerical simulation), on mapping of the solution space of the constraint equations (keywords: conformal method and variants thereof), on asymptotic properties (keywords: asymptotic geometric invariants, special asymptotic coordinate charts), and on behavior in the presence of (spacetime) symmetries (keywords: Killing initial data (KIDs), staticity, stationarity).

4. Important theorems and open questions on initial data sets

Two central theorems about initial data sets are the "positive mass theorem (PMT)" and the "Riemannian Penrose inequality (RPI)". The PMT states that any geodesically complete, asymptotically Euclidean IDS of dimension $n = 3$ satisfying the DEC will have positive "ADM-mass" (an asymptotic geometric invariant), with equality if and only if the IDS sits isometrically inside the Minkowski spacetime. It was first proven with different approaches by Schoen and Yau and by Witten. Today, we know many versions of the PMT: different asymptotic conditions (combined with different asymptotic geometric invariants and different choices of cosmological constant), higher dimensions, lower regularity, etc.

The RPI states that any asymptotically Euclidean, time-symmetric IDS of dimension $n = 3$ satisfying the DEC and being geodesically complete up to a minimal surface inner boundary ∂M has ADM-mass m satisfying $m \geq \sqrt{\frac{|\partial M|}{16\pi}}$, with equality if and only if the IDS is isometric to the canonical slice of a Schwarzschild spacetime. It was first proven with different approaches by Huisken and Ilmanen and by Bray. Today, we know many versions of the RPI: different asymptotic conditions, higher dimensions, lower regularity, etc. However, we still lack a (full) generalization of the Penrose inequality for initial data sets (replacing minimal surface with "MOTS" inner boundary).

Many other results on time-symmetric IDSs have also not yet been generalized to general IDSs. Relevant results and techniques include the quasi-spherical approach to solving the constraint equations by Bartnik [1] and other methods for constructing suitable "extensions" of certain inner boundary data such as the recent construction by Mantoulidis and Schoen [11, 3]. Another branch of questions and ideas where very little is known beyond time-symmetry (but see [2]) is the question of "stability" of geometric inequalities such as the PMT and the (R)PI, meaning whether IDSs that almost saturate the inequalities will need to be geometrically "close" to the saturating ones (see e.g. [7, 10, 4]).

Methods for extending results from the Riemannian (time-symmetric) context to general IDSs include the Jang equation approach (e.g. for the PMT), small K results obtained by applying the implicit function theorem around $K = 0$, but also generalizations guided by physical/geometric intuition. To give an example of the latter, let us consider the beautiful idea of studying the "center of mass" of an asymptotically Euclidean IDS via an asymptotic foliation by constant mean curvature (CMC) surfaces first developed by Huisken and Yau [8]. As we have described with Nerz [5], this approach has subtle convergence issues (e.g. in certain graphical IDSs in the Schwarzschild spacetime). With Sakovich [6], we have developed a spacetime covariant approach to defining the center of mass of asymptotically Euclidean IDSs, using instead asymptotic foliations by constant spacetime mean curvature (STCMC), i.e., such that the Lorentzian length of the codimension 2 mean curvature vector is constant. This approach remedies in particular the convergence issues addressed in [5].

References

[1] R. Bartnik, *Quasi-spherical metrics and prescribed scalar curvature*, J. Diff. Geom. **37** (1993), 31–71.

[2] E. Bryden and M. Khuri and C. Sormani, *Stability of the Spacetime Positive Mass Theorem in Spherical Symmetry*, J. Geom. Anal. **31** (2021), 4191–4239.

[3] A.J. Cabrera Pacheco and C. Cederbaum, *A survey on extensions of Riemannian manifolds and Bartnik mass estimates*, Memorias de la reunión de Matemáticos Mexicanos en el Mundo 2018, AMS Contemporary Mathematics series (2019).

[4] A.J. Cabrera Pacheco and C. Cederbaum, *Minimizing sequences for Bartnik's quasi-local mass*, in prep.

[5] C. Cederbaum and C. Nerz, *Explicit Riemannian manifolds with unexpectedly behaving center of mass*, Ann. Henri Poincaré **16 (7)** (2015), 1609–1631.

[6] C. Cederbaum and A. Sakovich, *On center of mass and foliations by constant spacetime mean curvature surfaces for isolated systems in General Relativity*, Calc. Var. PDE (to appear).

[7] L.-H. Huang and D.A. Lee and C. Sormani, C., *Intrinsic flat stability of the positive mass theorem for graphical hypersurfaces of Euclidean space*, J. Reine Angew. Math. **727** (2017), 269–299.

[8] G. Huisken and S.-T. Yau, *Definition of Center of Mass for Isolated Physical Systems and Unique Foliations by Stable Spheres with Constant Mean Curvature*, Invent. Math. **124** (1996), 281–311.

[9] D.A. Lee, *Geometric Relativity*, AMS Graduate Studies in Mathematics **201** (2019).

[10] D.A. Lee and C. Sormani, *Near-equality of the Penrose inequality for rotationally symmetric Riemannian manifolds*, Ann. Henri Poincaré **13 (7)** (2012), 1537–1556.

[11] C. Mantoulidis and R. Schoen, *On the Bartnik mass of apparent horizons*, Class. Quantum Grav. **32** (2015), 205002–205018.

Spacetime harmonic functions and applications to relativity

Sven Hirsch

(joint work with Hubert Bray, Demetre Kazaras, Marcus Khuri, Pengzi Miao, Tin-Yau Tsang, Yiyue Zhang)

In [7] the crucial observation has been made that Bochner's identity, the Gauss equation and Gauss-Bonnet's theorem can be combined to study scalar curvature via harmonic maps. For instance, this allows for the followinsg simple proof that there is no PSC metric on T^3:

By elementary topological considerations there exist non-trivial maps from T^3 to S^1. Minimizing in this class leads to a harmonic map $u : T^3 \to S^1$. For simplicity we assume $|\nabla u| \neq 0$. Since u is harmonic, Bochner's identity yields

$$\Delta|\nabla u|^2 = 2\,\mathrm{Ric}(\nabla u, \nabla u) + 2|\nabla^2 u|^2. \tag{1}$$

Using the Gauss equations, the first term can be rewritten as

$$2\,\mathrm{Ric}(\nabla u, \nabla u) = |\nabla u|^2(R - R_\Sigma + H^2 - |A|^2). \tag{2}$$

Here R, R_Σ are the scalar curvature of T^3 and the level sets Σ of u, A is the second fundamental form of Σ, and H the mean curvature. Integrating those identities yields

$$0 = \int_{T^3} \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + R|\nabla u| - R_\Sigma |\nabla u| \right) dV. \tag{3}$$

Thus, the result follows from Gauss-Bonnet's theorem, the co-area formula and the fact that T^3 contains no non-separating spheres.

Harmonic functions can also be used to yield a new proof of the Riemannian positive mass theorem which has been the subject of [3]. However, harmonic functions are by no means the only functions where the combination of Bochner's formula, Gauss equations and Gauss-Bonnet's theorem yield new information about PSC metrics. In general, we have the formula

$$\int_M \frac{(\Delta u)^2}{|\nabla u|} dV - \int_{\partial M} 2H|\nabla u| dA = \int_M \left(\frac{|\nabla^2 u|^2}{|\nabla u|} + (R - 2K)|\nabla u| \right) dV \tag{4}$$

which holds true for any function on any 3-manifold M, which is constant on ∂M. Thus, by prescribing Δu carefully, new results about PSC metrics can be obtained:

This has led in [4] to a new proof of the spacetime positive mass theorem using the equation $\Delta u = -\operatorname{Tr}_g(k)|\nabla u|$ where k is the symmetric two tensor associated to an initial data set. In [1] we obtained a new proof of the positive mass theorem with charge using the equation $\Delta u = \langle E, \nabla u \rangle$ where E is the electric field, and in [2] to a new proof of the hyperbolic positive mass theorem using again the equation $\Delta u = -\operatorname{Tr}_g(k)|\nabla u|$. In fact, even the Hawking energy monotonicity formula which has been crucially exploited in [6] can be obtained from equation (4) by inserting the formula $\Delta u = \frac{\langle \nabla |\nabla u|, \nabla u \rangle}{|\nabla u|} + 2\frac{|\nabla u|^2}{u}$.

These level set techniques are by no means just a way to reprove old theorems, but in fact they have several advantages over minimal surfaces and spinors to obtain proofs which would otherwise be not possible. For instance, consider then convergence result for the Brown-York mass established in [5]. The proof crucially relies on the mass formula for the positive mass theorem with corners, and exploits that the formula is even valid in case the manifold does not have non-negative scalar curvature. This differs from the corresponding spinor formula which relies on the manifold having non-negative scalar curvature in order to obtain a coercivity estimate which is necessary to show existences of such harmonic spinors.

References

[1] H. Bray, S. Hirsch, D. Kazaras, M. Khuri, and Y. Zhang, *Spacetime harmonic functions and applications to mass*

[2] H. Bray, S. Hirsch, D. Kazaras, M. Khuri, and Y. Zhang, *Spacetime harmonic functions and the mass of 3-dimensional asymptotically hyperboloidal initial data for the Einstein equations*, in preparation.

[3] H. Bray, D. Kazaras, M. Khuri, and D. Stern, *Harmonic functions and the mass of 3-dimensional asymptotically flat Riemannian manifolds*, preprint, 2019. arXiv:1911.06754

[4] S. Hirsch, D. Kazaras, M. Khuri, *Spacetime Harmonic Functions and the Mass of 3-Dimensional Asymptotically Flat Initial Data for the Einstein Equations*, J. Differential Geom., to appear, arXiv:2002.01534.

[5] S. Hirsch, P. Miao, and T.-Y. Tsang. *Mass of asymptotically flat* 3*-manifolds with boundary.*, Comm. Analysis and Geometry, to appear, arXiv:2009.02959.

[6] G. Huisken, and T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom., **59** (2001), 353–437.

[7] D. Stern, *Scalar curvature and harmonic maps to* S^1, J. Differential Geom., to appear. arXiv:1908.09754

Scalar and mean curvature comparison via the Dirac operator

Rudolf Zeidler

(joint work with Simone Cecchini)

In recent years, Gromov has initiated a programme of metric inequalities in the presence of lower scalar curvature bounds which are vaguely reminiscent of classical comparison geometry. This has lead to a number of conjectures, two of which we now recall to set the stage.

Conjecture 1 ([6, 11.12, Conjecture C])**.** Let M be a closed connected manifold of dimension $n-1 \neq 4$ such that M does not admit a metric of positive scalar curvature. Let g be a Riemannian metric on $V = M \times [-1, 1]$ of scalar curvature bounded below by $n(n-1) = \mathrm{scal}_{\mathrm{S}^n}$. Then

$$\mathrm{width}(V, g) \leq \frac{2\pi}{n},$$

where $\mathrm{width}(V, g) := \mathrm{dist}_g(\partial_- V, \partial_+ V)$ is the distance between the two boundary components of V with respect to g.

Conjecture 2 ([5, p. 87, *Long neck problem*])**.** Let (M, g) be a compact connected n-dimensional Riemannian manifold with boundary such that its scalar curvature is bounded below by $n(n-1)$. Suppose that $\Phi\colon M \to S^n$ is a smooth area non-increasing map which is locally constant near the boundary. If

$$\mathrm{dist}_g(\mathrm{supp}(\mathrm{d}\Phi), \partial M) \geq \frac{\pi}{n},$$

then the mapping degree of Φ is zero.

One of Gromov's first results on these questions [6] was a proof of Conjecture 1 for the torus and related manifolds via the geometric measure theory approach to positive scalar curvature going back to the minimal hypersurface method of Schoen and Yau. Subsequently, in the recent articles [9, 2, 10], we have demonstrated that the spin Dirac operator method can also be used to approach such questions.

The goal of this talk was to present a novel point of view towards these and other conjectures, based again on the Dirac operator, which appeared in the recent joint work with S. Cecchini [1]. In our new approach, the mean curvature of the boundary becomes one of the main players. Indeed, the general idea is that in these situations one can establish a precise quantitative relationship between the scalar curvature, the mean curvature of the boundary and a relevant distance quantity. In certain situations, our new point of view also allows to formulate and prove rigidity statements. As a central example showcasing these ideas, the following theorem was presented:

Theorem 1 (cf. [1]). *Let (V, g) be a* Riemannian spin band*, that is, a compact spin manifold V together with a decomposition $\partial V = \partial_- V \sqcup \partial_+ V$ into non-empty unions of components, such that $\hat{\mathsf{A}}(\partial_- V) \neq 0$ and $\mathrm{scal}_g \geq n(n-1)$. Then the following holds:*

(1) If $\mathrm{H}_g \geq -\tan(nl/2)$ for some $0 < l < \pi/n$, then

$$\mathrm{width}(V, g) = \mathrm{dist}_g(\partial_- V, \partial_+ V) \leq 2l.$$

(2) If, in addition, equality in the above estimate is attained, then V is isometric to $M \times [-l, l]$,

$$g = \cos(nx/2)^{2/n} g_M + \mathrm{d}x^2,$$

for some spin manifold (M, g_M) that admits a parallel spinor.

(3) In particular, $\mathrm{width}(V, g) < 2\pi/n$.

Our results in [1] also yield (1) in many other situations, including bands over all even-dimensional enlargeable manifolds. Moreover, the equality (3) was established previously by both Cecchini and Zeidler in cases with non-vanishing Rosenberg index; see [9, 2, 10] for details. The latter is a sophisticated index invariant living real K-theory of group C*-algebras, which makes sense and has a rich set of non-trivial examples in all parities of the dimension.

Our main result corresponding to Conjecture 2 reads as follows:

Theorem 2 (cf. [1]). *Let (M, g) be a compact connected n-dim. Riemannian spin manifold with boundary such that $\mathrm{scal}_g \geq n(n-1)$ on M, where $n \geq 2$ is even. Let $f\colon M \to \mathrm{S}^n$ be a smooth area non-increasing map. Suppose that for some $0 < l < \pi/n$ the following estimates hold:*

- *$\mathrm{dist}_g(\partial M, \mathrm{supp}(\mathrm{d}f)) \geq l$.*
- *$\mathrm{H}_g \geq -\tan(nl/2)$ on ∂M,*

Then the mapping degree of f is zero.

One of the crucial observations here is that as $l \to \pi/n$, the lower mean curvature bound in the hypotheses of both theorems tends to $-\infty$. In other words, as l approaches this threshold and assuming that there is a non-trivial index invariant or mapping degree, the mean curvature must explode somewhere at the boundary.

Our techniques also yield rigidity result for annuli in space forms as follows:

Theorem 3 (cf. [1]). *Let $n \geq 3$ be odd, $\kappa \in \mathbb{R}$ and (M_κ, g_κ) be the n-dimensional simply connected space form of curvature κ. Consider the* annulus *around a basepoint $p_0 \in M_\kappa$*

$$\mathrm{A}_{t_-, t_+} = \{p \in M_\kappa \mid t_- \leq d_{g_\kappa}(p, p_0) \leq t_+\},$$

where $0 < t_- < t_+ < t_\infty$ with $t_\infty = \pi/\sqrt{\kappa}$ if $\kappa > 0$ and $t_\infty = +\infty$ otherwise.

Then any Riemannian metric g on A_{t_-,t_+} such that

- $g \geq g_\kappa$,
- $\mathrm{scal}_g \geq \mathrm{scal}_{g_\kappa} = \kappa n(n-1)$,
- $\mathrm{H}_g \geq \mathrm{H}_{g_\kappa} = \pm\,\mathrm{ct}_\kappa(t_\pm)$,

must satisfy $g = g_\kappa$.

On a technical level, our proofs are based on augmenting the spinor Dirac operator by a Lipschitz potential and subjecting it to Chiral boundary conditions which are tailor-made for this problem. We develop all of this inside a general framework via a new notion of so-called *relative Dirac bundles* (see [1, Sections 2-4] for details). The main principle is that we can use these tools to compare certain spin manifolds to model spaces which are suitable warped products, provided that one can produce a non-trivial solution of a boundary value problem associated to the augmented Dirac operator on the given manifold (see [1, Section 8] for details). The warped product appearing in the rigidity part of Theorem 1 is just one particular example of this. Moreover, the potential used to augment the Dirac operator has a direct geometric interpretation: Up to a constant it corresponds to the mean curvature of the cross sections in the model warped product space.

In in light of the previous paragraph, our techniques have a formal similarity to *μ-bubbles* or *generalized soap bubbles*, which have recently led to substantial advances via the geometric measure theory approach to scalar curvature, see [5, Section 5], [3, 4, 7] for examples. In this context, analogous estimates as stated in Theorem 1 for bands were also exhibited recently by Räde [8]. It may be a fruitful endeavour for the future to investigate if there are deeper connections between these two approaches beyond this formal similarity.

References

[1] S. Cecchini and R. Zeidler, *Scalar and mean curvature comparison via the Dirac operator*, arXiv:2103.06833 [math.DG]

[2] S. Cecchini, *A long neck principle for Riemannian spin manifolds with positive scalar curvature*, Geom. Funct. Anal. 30, No. 5, 1183–1223 (2020)

[3] O. Chodosh and C. Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv:2008.11888 [math.DG]

[4] M. Gromov, *No metrics with Positive Scalar Curvatures on Aspherical 5-Manifolds*, arXiv:2009.05332 [math.DG]

[5] M. Gromov, *Four Lectures on Scalar Curvature*, arXiv:1908.10612v3 [math.DG]

[6] M. Gromov, *Metric inequalities with scalar curvature*, Geom. Funct. Anal. 28, No. 3, 645–726 (2018)

[7] M. Lesourd, R. Unger, S.-T. Yau, *The Positive Mass Theorem with Arbitrary Ends*, arXiv:2103.02744 [math.DG]

[8] D. Räde, *Scalar and mean curvature comparison via μ-bubbles*, arXiv:2104.10120 [math.DG].

[9] R. Zeidler, *Band width estimates via the Dirac operator*, J. Differential Geom. (to appear)

[10] R. Zeidler, *Width, Largeness and Index Theory*, SIGMA 16 (2020), 127, 15 pages

Uniqueness of Weak Solutions to the Ricci Flow and Topological Applications

Richard Bamler
(joint work with Bruce Kleiner)

In this talk I surveyed recent work [1, 2, 3] with Kleiner in which we verify the following two topological conjectures using Ricci flow.

Theorem 1 (Generalized Smale Conjecture)**.** *Let $M = S^3/\Gamma$ be a 3-dimensional spherical space form. Then its diffeomorphism group and isometry group are homotopy equivalent:* $\mathrm{Diff}(M) \simeq \mathrm{Isom}(M)$*. More specifically, the inclusion map* $\mathrm{Isom}(M) \to \mathrm{Diff}(M)$ *is a homotopy equivalence.*

Theorem 2. *Let M be a closed, orientable 3-manifold and denote by* $\mathrm{Met}_{PSC}(M)$ *the space of Riemannian metrics on M with positive scalar curvature (equipped with the C^∞-topology). Then* $\mathrm{Met}_{PSC}(M)$ *is either empty or contractible.*

Theorem 1 gives an alternative proof of the Smale Conjecture, concerning $M = S^3$, which was originally due to Hatcher [6]. Theorem 2 can be seen as an extension of a result by Marques [8] who proved connectedness of $\mathrm{Met}_{PSC}(S^3)$ and $\mathrm{Met}_{PSC}(M)/\mathrm{Diff}(M)$.

Our proof is based on a new uniqueness theorem for singular Ricci flows. Singular Ricci flows were inspired by Perelman's proof of the Poincaré and Geometrization Conjectures [9, 10, 11], which relied on Ricci flow with surgery. A Ricci flow with surgery is a flow in which singularities are removed by a certain surgery construction. Since this surgery construction depends on various auxiliary parameters, the resulting flow is not uniquely determined by its initial data. Perelman therefore conjectured that there must be a canonical, weak Ricci flow that automatically "flows through its singularities" at an infinitesimal scale.

In [7] Kleiner and Lott showed the existence of such a flow:

Theorem 3. *Given any closed, orientable Riemannian 3-manifold (M, g) there is a singular Ricci flow $\mathcal{M}$ "through its singularities" with initial time-slice $(\mathcal{M}_0, g_0) \cong (M, g)$.*

In this flow, change of topology essentially occurs at an infinitesimal scale. Existence of this flow was obtained, via a compactness theorem, as a subsequential limit of a sequence of Ricci flows with surgery starting from (M, g) with surgery scale $\delta_i \to 0$. Due to the use of the compactness theorem, the proof of Kleiner and Lott did not imply uniqueness of $\mathcal{M}$; this was achieved by Bamler and Kleiner using other techniques [1]:

Theorem 4. *Any singular, 3-dimensional Ricci flow $\mathcal{M}$ is uniquely determined, up to isometry, by its initial time-slice $(\mathcal{M}_0, g_0)$.*

Theorems 3, 4 fully resolve Perelman's conjecture. The proof of Theorem 4 furthermore implies the following continuity statement:

Theorem 5. *Let M be a closed 3-manifold. For any Riemannian metric g on M let $\mathcal{M}^g$ be a singular Ricci flow with initial condition (M, g). Then the map $g \mapsto \mathcal{M}^g$ is continuous in a certain sense.*

Theorem 5 allowed us to evolve continuous families of metrics by singular Ricci flows and prove Theorems 1, 2. To illustrate our strategy, it is helpful to consider the 2-dimensional case first. In this case, the flow behaves much nicer:

Theorem 6 (Chow, Hamilton [4, 5])**.** *Any Ricci flow on S^2 converges, modulo rescaling, to a metric of constant curvature* 1*.*

Thus—modulo some technical details—we may view Ricci flow as a deformation retraction from $\mathrm{Met}(S^2)$ to $\mathrm{Met}_{K\equiv 1}(S^2)$, where the latter denotes the space of metrics of constant curvature 1 (those are isometric to the round metric). Since Ricci flow preserves the positive scalar curvature condition, it can also be seen as a deformation retraction from $\mathrm{Met}_{PSC}(S^2)$ to $\mathrm{Met}_{K\equiv 1}(S^2)$. This implies that

$$\mathrm{Met}_{PSC}(S^2) \simeq \mathrm{Met}_{K\equiv 1}(S^2) \simeq \mathrm{Met}(S^2) \simeq *$$

So all spaces are contractible. A standard topological argument involving the long exact sequence of the fiber bundle $\mathrm{Isom}(S^2) \to \mathrm{Diff}(S^2) \to \mathrm{Met}_{K\equiv 1}(S^2)$ eventually implies that $\mathrm{Isom}(S^2) \to \mathrm{Diff}(S^2)$ is a homotopy equivalence.

Our proofs of Theorems 1, 2 replicate this strategy and rely on Theorem 5 for continuous dependence of the flow on its initial data. However, the possible occurrence of singularities and the resulting change in topology implies that we cannot describe the singular flows $\mathcal{M}$ by continuous paths in $\mathrm{Met}(M)$. To overcome this issue we have devised methods of converting a continuous family of singular Ricci flows into the desired homotopy in $\mathrm{Met}(M)$. The first method in [2] is easily accessible, but can only be used to show Theorem 2 in the case if $M \not\approx S^3, \mathbb{R}P^3$. The second method in [3] is significantly more technical; it employs a new concept called "partial homotopy" and allows the proof of both Theorems 1, 2.

References

[1] Richard H. Bamler and Bruce Kleiner, *Uniqueness and stability of Ricci flow through singularities,* (2017), available at `http://arxiv.org/abs/1709.04122`

[2] Richard H. Bamler and Bruce Kleiner, *Ricci flow and diffeomorphism groups of 3-manifolds,* (2017), available at `http://arxiv.org/abs/1712.06197`.

[3] Richard H. Bamler and Bruce Kleiner, *Ricci flow and contractibility of spaces of metrics,* (2019), available at `https://arxiv.org/abs/1909.08710`

[4] Bennett Chow, *The Ricci flow on the 2-sphere,* J. Differential Geom. 33 (1991), no. 2, 325–334. MR1094458

[5] Richard S. Hamilton, *The Ricci flow on surfaces,* Mathematics and general relativity (Santa Cruz, CA, 1986), 1988, pp. 237-262. MR954419

[6] Allen Hatcher, *A proof of the Smale conjecture, $Diff(S^3) \simeq O(4)$,* Ann. of Math. (2) 117 (1983), no. 3, 553–607.

[7] Bruce Kleiner and John Lott, *Singular Ricci flows I,* Acta Mathematica 219 (2017), no. 1, 65–134.

[8] Fernando Codaá Marques, *Deforming three- manifolds with positive scalar curvature,* Ann. of Math. (2) 176 (2012), no. 2, 815–863. MR2950765

[9] Grisha Perelman, *The entropy formula for the Ricci flow and its geometric applications (2002),* available at `http://arxiv. org/abs/math/0211159`.
[10] Grisha Perelman, *Ricci flow with surgery on three- manifolds,* (2003), available at `http:// arxiv.org/abs/math/0303109`.
[11] Grisha Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds,* (2003), available at `http:// arxiv.org/abs/math/0307245`.

Index theory and Gromov's conjectures on positive scalar curvature

Zhizhang Xie

In the past several years, Gromov has formulated an extensive list of conjectures and open questions on scalar curvature [2, 3]. This has given rise to new perspectives on scalar curvature and inspired a wave of recent activity in this area. In this talk, I shall present my recent work on a quantitative relative index theorem (Theorem 2) that serves as a conceptual framework for solving some of these conjectures and open questions.

For example, we answer the following conjecture of Gromov for all dimensions with a suboptimal constant.

Conjecture 1 (Gromov's $\square^n$ inequality, [3, section 3.8])**.** Let g be a Riemannian metric on the cube $I^n = [0,1]^n$. If $\mathrm{Sc}(g) \geq n(n-1)$, then

$$\sum_{j=1}^{n} \frac{1}{\ell_j^2} \geq \frac{n^2}{4\pi^2},$$

where $\ell_j = \mathrm{dist}(\partial_{j_-}, \partial_{j_+})$ is the g-distance between the pair of opposite faces ∂_{j_-} and ∂_{j_+} of the cube. Consequently, we have

$$\min_{1\leq j\leq n} \mathrm{dist}(\partial_{j_-}, \partial_{j_+}) \leq \frac{2\pi}{\sqrt{n}}.$$

In fact, our method proves Gromov's $\square^{n-m}$ inequality conjecture in the spin case, which is a generalization of Conjecture 1, for all dimensions with a suboptimal constant (cf. Theorem 3 below).

One of the key ingredients for the proof of Conjecture 1 is the following quantitative relative index theorem.

Theorem 2 ([7, Theorem I])**.** *Let Z_1 and Z_2 be two closed n-dimensional Riemannian manifold and $\mathcal{S}_j$ a Euclidean $\mathrm{C}\ell_n$-bundle*[1] *over Z_j for $j = 1, 2$. Suppose D_j is a $\mathrm{C}\ell_n$-linear Dirac-type operator acting on $\mathcal{S}_j$ over Z_j. Let $\widetilde{Z}_j$ be a Galois Γ-covering space of Z_j and $\widetilde{D}_j$ the lift of D_j. Let X_j be a subset of Z_j and $\widetilde{X}_j$ the preimage of X_j under the covering map $\widetilde{Z}_j \to Z_j$. Denote by $N_r(Z_j\backslash X_j)$ the open r-neighborhood of $Z_j\backslash X_j$. Suppose there is $r > 0$ such that all geometric data on $N_r(Z_1\backslash X_1)$ and $N_r(Z_2\backslash X_2)$ coincide, i.e. there is an orientation preserving Riemannian isometry $\Phi\colon N_r(Z_1\backslash X_1) \to N_r(Z_2\backslash X_2)$ such that Φ lifts to an isometric $\mathrm{C}\ell_n$-bundle isomorphism $\Phi\colon \mathcal{S}_1|_{N_r(Z_1\backslash X_1)} \to \mathcal{S}_2|_{N_r(Z_2\backslash X_2)}$. Assume that*

[1] Here $\mathrm{C}\ell_n$ is the real Clifford algebra of $\mathbb{R}^n$. See [4, II.§7 and III. §10] for more details on $\mathrm{C}\ell_n$-vector bundles and the Clifford index of $\mathrm{C}\ell_n$-linear Dirac operators.

(1) *there exists* $\sigma > 0$ *such that*

$$\mathcal{R}_j(x) \geq \frac{(n-1)\sigma^2}{n}$$

for all $x \in X_j$*, where* $\mathcal{R}_j$ *is the curvature term appearing in* $D_j^2 = \nabla^*\nabla + \mathcal{R}_j$*,*

(2) *and* $D_1 = \Phi^{-1} D_2 \Phi$ *on* $N_r(Z_1 \backslash X_1)$*.*

Then there exists a universal constant $C > 0$ *such that if* $\sigma \cdot r > C$*, then we have*

$$\mathrm{Ind}_\Gamma(\widetilde{D}_1) - \mathrm{Ind}_\Gamma(\widetilde{D}_2) = 0$$

in $KO_n(C^*_{\max}(\Gamma; \mathbb{R}))$*, where* $\mathrm{Ind}_\Gamma(\widetilde{D}_j)$ *denotes the maximal higher index of* $\widetilde{D}_j$ *and* $C^*_{\max}(\Gamma; \mathbb{R})$ *is the maximal group* C^**-algebra of* Γ *with real coefficients.*

Our numerical estimates show that the universal constant C is ≤ 40.65. As an application of Theorem 2, we have the following theorem, which proves Gromov's $\square^{n-m}$ inequality conjecture in the spin case, a generalization of Conjecture 1, for all dimensions with a suboptimal constant.

Theorem 3 ([7, Theorem II])**.** *Let* X *be an* n*-dimensional compact connected spin manifold with boundary. Suppose* $f\colon X \to [-1,1]^m$ *is a smooth map that sends the boundary of* X *to the boundary of* $[-1,1]^m$*. Let* $\partial_{j\pm}, j = 1, \ldots, m$*, be the pullbacks of the pairs of the opposite faces of the cube* $[-1,1]^m$*. Suppose* $Y_\pitchfork$ *is an* $(n-m)$*-dimensional closed submanifold (without boundary) in* X *that satisfies the following conditions:*

(1) $\pi_1(Y_\pitchfork) \to \pi_1(X)$ *is injective;*

(2) $Y_\pitchfork$ *is the transversal intersection of* m *orientable hypersurfaces* $\{Y_j\}_{1\leq j\leq m}$ *of* X*, each of which separates* ∂_{j-} *from* ∂_{j+}*;*

(3) *the higher index* $\mathrm{Ind}_\Gamma(D_{Y_\pitchfork})$ *does not vanish in* $KO_{n-m}(C^*_{\max}(\Gamma; \mathbb{R}))$*, where* $\Gamma = \pi_1(Y_\pitchfork)$*.*

If $\mathrm{Sc}(X) \geq n(n-1)$*, then the distances* $\ell_j = \mathrm{dist}(\partial_{j-}, \partial_{j+})$ *satisfy the following inequality:*

$$\sum_{j=1}^m \frac{1}{\ell_j^2} \geq \frac{n^2}{(\frac{8}{\sqrt{3}}C + 4\pi)^2}.$$

where C *is the universal constant from Theorem 2.*

Subsequently, with Wang and Yu [6], I proved Theorem 3 with the optimal constant via a different method, hence completely solves Conjecture 1 and its generalization for spin manifolds in all dimensions. We point out that Cecchini [1] and Zeidler [8, 9] proved a special case of Theorem 3 when $m = 1$ with the optimal constant.

As another application of our quantitative relative index theorem, I proved the following λ-Lipschitz rigidity result for hemispheres. This answers (asymptotically) an open question of Gromov on the sharpness of the constant λ_n for the λ-Lipschitz rigidity of positive scalar curvature metrics on hemispheres [3, section 3.8].

Theorem 4 ([7, Theorem V])**.** *Let (X, g_0) be the standard unit hemisphere $\mathbb{S}^n_+$. If a Riemannian metric g on X satisfies that*

(1) there is a λ_n-Lipschitz homeomorphism $\varphi\colon (X, g) \to (X, g_0)$,
(2) and $\mathrm{Sc}(g) \geq n(n-1) = \mathrm{Sc}(g_0)$,

then

$$\lambda_n \geq (1 - \sin \frac{\pi}{\sqrt{n}})\sqrt{1 - \frac{8C^2}{\pi^2 n}}$$

where C is the universal constant from Theorem 2. Consequently, the lower bound for λ_n approaches 1, as $n \to \infty$.

The above theorem is asymptotically optimal in the sense that the lower bound for λ_n becomes sharp, as $n = \dim \mathbb{S}^n \to \infty$. A key geometric concept behind the proof of Theorem 4 is the following notion of wrapping property for subsets of $\mathbb{S}^n$.

Definition 5 ([7, Definition 1.3])**.** [Subsets with the wrapping property] A subset Σ of the standard unit sphere $\mathbb{S}^n$ is said to have *the wrapping property* if for all sufficiently small $\varepsilon > 0$, the complement of the ε-neighborhood $N_\varepsilon(\Sigma)$ of Σ is path-connected and furthermore there exists a smooth distance-contracting map $\Phi\colon \mathbb{S}^n \to \mathbb{S}^n$ such that the following are satisfied:

(1a) if n is even, Φ equals the identity map on $N_\varepsilon(\Sigma)$;
(1b) if n is odd, Φ equals either the identity map or the antipodal map on each of the connected components of $N_\varepsilon(\Sigma)$;
(2) and the degree $\deg(\Phi)$ of Φ is not equal to 1.

Roughly speaking, a subset $\Sigma \subset \mathbb{S}^n$ has the wrapping property if its geometric size is "relatively small". For example, if Σ is contained in a geodesic ball of radius $< \frac{\pi}{2}$, then Σ has the wrapping property [7, Lemma 5.3]. Moreover, if Σ is contained in a pair of antipodal geodesic balls of radius $< \frac{\pi}{6}$ in an odd dimensional sphere, then Σ also satisfies the wrapping property [7, Lemma 5.5]. Motivated by theorems of Llarul [5, theorem A] and Gromov [3, section 3.9], we propose the following open question.

Open Question (Rigidity for positive scalar curvature metrics on $\mathbb{S}^n \backslash \Sigma$)**.** *Let Σ be a subset with the wrapping property in the standard unit sphere $\mathbb{S}^n$. Let (X, g_0) be the standard unit sphere $\mathbb{S}^n$ minus Σ. If a (possibly incomplete) Riemannian metric g on X satisfies*

(1) $g \geq g_0$,
(2) and $\mathrm{Sc}(g) \geq n(n-1) = \mathrm{Sc}(g_0)$,

then does it imply that $g = g_0$?

References

[1] Simone Cecchini. A long neck principle for Riemannian spin manifolds with positive scalar curvature. *Geom. Funct. Anal.*, 30(5):1183–1223, 2020.

[2] Misha Gromov. Metric inequalities with scalar curvature. *Geom. Funct. Anal.*, 28(3):645–726, 2018.

[3] Misha Gromov. Four lectures on scalar curvature. arXiv:1908.10612, 2019.

[4] H. Blaine Lawson, Jr. and Marie-Louise Michelsohn. *Spin geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1989.
[5] Marcelo Llarull. Sharp estimates and the Dirac operator. *Math. Ann.*, 310(1):55–71, 1998.
[6] Jinmin Wang, Zhizhang Xie, and Guoliang Yu. A proof of Gromov's cube inequality on scalar curvature. arXiv:2105.12054, 2021.
[7] Zhizhang Xie. A quantitative relative index theorem and Gromov's conjectures on positive scalar curvature. arXiv:2103.14498, 2021.
[8] Rudolf Zeidler. Band width estimates via the Dirac operator. *Journal of Differential Geometry*, 2020.
[9] Rudolf Zeidler. Width, largeness and index theory. *SIGMA Symmetry Integrability Geom. Methods Appl.*, 16:Paper No. 127, 15, 2020.

Dominant energy condition and Dirac-Witten operators

Jonathan Glöckle

Energy conditions are a major ingredient for the famous singularity theorems of General Relativity. In this talk based on [1] we discussed the *dominant energy condition* (dec) from the perspective of initial data sets: An embedded spacelike hypersurface M of a time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ carries an induced Riemannian metric g and a second fundamental form $k \in \Gamma(T^*M \otimes T^*M)$ (with respect to the future unit normal e_0 on M). The constraint equations

$$\begin{aligned} 2\rho &= \mathrm{scal}^g + \mathrm{tr}(k)^2 - \|k\|^2 \\ j &= \mathrm{div}\, k - \mathrm{dtr}(k) \end{aligned} \tag{1}$$

allow to recover certain components of the Einstein curvature Ein of $(\overline{M}, \overline{g})$. By the Einstein field equations, $\rho = \mathrm{Ein}(e_0, e_0)$ and $j = \mathrm{Ein}(e_0, -) \in \Omega^1(M)$ can be given the interpretation of the *energy density* and the *momentum density*, respectively, as experienced by an observer moving e_0-direction. The dominant energy condition for the space-time now implies that $\rho \geq \|j\|$, and thus imposes a condition on initial data sets (g, k) via (1).

We conjecture that the converse of this is also true in the following sense: Given an initial data set (M, g, k) subject to $\rho \geq \|j\|$, then M embeds as a spacelike hypersurface into some time-oriented Lorentzian manifold $(\overline{M}, \overline{g})$ satisfying dec in such a way that (g, k) is the induced pair on M. Evidence for this is provided in two cases. Firstly, suppose there is a matter model for which the Cauchy problem is solved and that satisfies dec. When (M, g, k) extends to an initial data set for the Cauchy problem with this kind of matter, then the solution of the corresponding evolution equations yields a space-time $(\overline{M}, \overline{g})$ as desired. In particular, this applies when (g, k) is a solution of the vacuum constraints $\rho = 0$ and $j = 0$. However, in general it is unclear whether every initial data set (M, g, k) with $\rho \geq \|j\|$ comes from initial data of such a Cauchy problem with matter. Secondly, when (g, k) satisfies the strict inequality $\rho > \|j\|$ then the required space-time $(\overline{M}, \overline{g})$ also exists. In this case $\overline{g} = -\mathrm{d}t^2 + g + 2tk + t^2h$ induces (g, k) on $M \times \{0\}$ and, for a suitable choice of h, satisfies dec on $M \times \{0\} \subseteq M \times \mathbb{R}$ in a very strict sense that is taylored to be an open condition.

The dominant energy condition $\rho \geq \|j\|$ for initial data sets plays an important role in the various versions of the positive mass theorem. Here, the strategy is often to look at the time-symmetric case $k = 0$ as a first step, where it reduces to $\mathrm{scal}^g \geq 0$. Motivated by this, we ask whether also other known results for non-negative (or positive) scalar curvature have a generalization to initial data sets (g, k) statisfing (strict) dec.

More concretely, there has been recent progress in constructing non-trivial elements in the homotopy groups of the space $\mathcal{R}^>(M)$ of positive scalar curvature (psc) metrics on a compact spin manifold M, cf. e. g. [2, 3, 4]. Do these yield non-trivial elements in the homotopy groups of $\mathcal{I}^>(M)$, the space of initial data sets (g, k) subject to the strict dec $\rho > \|j\|$?

The first step is to define a suitable comparison map between $\mathcal{R}^>(M)$ and $\mathcal{I}^>(M)$ (for compact M). The main observation here is that for $\lambda \in \mathbb{R}$ the pair $(g, \lambda g)$ satisfies strict dec both in the case where $g \in \mathcal{R}^>(M)$ and $\lambda \in \mathbb{R}$ is arbitrary as well as when g is arbitrary and $|\lambda|$ is large enough (the lower bound is given in terms of the minimum of scal^g on M). As the space of all metrics is contractible, this allows to "cone off" the inclusion $\mathcal{R}^>(M) \hookrightarrow \mathcal{I}^>(M)$, $g \mapsto (g, 0)$ both in the upper region where $\lambda > 0$ and in the lower region with $\lambda < 0$. Together, this yields a map from the suspension $\Phi\colon \mathrm{Susp}(\mathcal{R}^>(M)) \to \mathcal{I}^>(M)$. It should be noted that although writing down such a map explicitly involves a number of choices, its homotopy class is uniquely defined by the requirement that the upper/lower cone is closed in the upper/lower region.

The second step consists in constructing an invariant that is able to detect non-triviality of the induced elements in $\pi_k(\mathcal{I}^>(M))$. For this, we note that in all of the above-mentioned results on $\mathcal{R}^>(M)$, Hitchin's α-invariant $\alpha\text{-diff}\colon \pi_k(\mathcal{R}^>(M)) \to \mathrm{KO}^{-n-k-1}(\{\mathrm{pt}\})$ is shown to be non-trivial. For a compact spin manifold M, we get the following:

Theorem 1. *There exists a homomorphism* $\overline{\alpha}\text{-diff}\colon \pi_k(\mathcal{I}^>(M)) \to \mathrm{KO}^{-n-k}(\{\mathrm{pt}\})$ *such that for all* $k \geq 0$ *the following diagram commutes:*

$$\begin{array}{ccccc} \pi_k(\mathcal{R}^>(M)) & \xrightarrow{\mathrm{Susp}} & \pi_{k+1}(\mathrm{Susp}(\mathcal{R}^>(M))) & \xrightarrow{\Phi_*} & \pi_{k+1}(\mathcal{I}^>(M)) \\ & {\scriptstyle \alpha\text{-diff}} \searrow & & \swarrow {\scriptstyle \overline{\alpha}\text{-diff}} & \\ & & \mathrm{KO}^{-n-k-1}(\{*\}) & & \end{array}$$

Recalling that the α-invariant is essentially the index of the family of (Cl_n-linear) Dirac operators associated to a family of metrics, the construction of $\overline{\alpha}$-diff relies on identifying an appropriate replacement for the Dirac operator in the context of initial data sets. This job is done by the Dirac-Witten operator, which was first introduced by Witten to give his spinorial proof of the positive mass theorem [5], or rather its $\mathrm{Cl}_{n,1}$-linear version. The Dirac-Witten operator $\overline{D}$ is a zero-order perturbation of the Dirac operator D on the so-called *hypersurface spinor bundle* of M:

$$\overline{D} = D - \frac{1}{2}\mathrm{tr}(k) e_0 \cdot \tag{2}$$

When M is embedded as spacelike hypersurface in $\overline{M}$, then the hypersurface spinor bundle is the restriction of the spinor bundle of $\overline{M}$ to M and $e_0\cdot$ is the involution defined by Clifford multiplication with the future unit normal on M. Together with Witten's remarkable Schrödinger-Lichnerowicz type formula

$$\overline{D}^2 = \overline{\nabla}^*\overline{\nabla} + \frac{1}{2}(\rho - j^\sharp \cdot e_0\cdot) \tag{3}$$

showing that $\overline{D}$ is invertible if $(g,k) \in \mathcal{I}^>(M)$, the comparison formula (2) provides all the (analytic) properties needed to define $\overline{\alpha}$-diff in complete analogy to α-diff.

Whereas higher homotopy groups of $\mathcal{I}^>(M)$ might mainly be of theoretical interest, identifying its path-components seems to have more direct applications to physics. For this reason, we also have a look at the "$k = -1$"-case of the above theorem.

Theorem 2. *There is a map* $\overline{\alpha}\text{-diff}\colon \pi_0(\mathcal{I}^>(M)) \to \mathrm{KO}^{-n}(\{\mathrm{pt}\})$ *sending the path component containing* $(g, -\lambda g)$ *– relative to the basepoint* $(g, \lambda g)$ *– to the* α*-index* $\alpha(M)$*. Thereby,* g *is any metric on* M *and* $\lambda > 0$ *is large enough so that* $(g, \lambda g) \in \mathcal{I}^>(M)$.

This shows in particular that if $\alpha(M) \neq 0$, then there cannot be a spacetime $(\overline{M}, \overline{g})$ with a foliation $M \times [-1,1] \to \overline{M}$ into spacelike hypersurfaces $M \times \{t\}$ such that dec is strictly satisfied on every slice and such that the induced initial data set on $M \times \{-1\}$ is in the "Big Bang" compontent containing $(g, \lambda g)$ and the one on $M \times \{1\}$ is in the "Big Crunch" component containing $(g, -\lambda g)$. There are two drawbacks here: Firstly, we would like to get rid of the strictness assumption so that one can simply assume $(\overline{M}, \overline{g})$ to satisfy dec. This passage, which is similar in spirit to the transition from positive to non-negative scalar curvature performed by Schick-Wraith [6], has been recently discussed in joint work with Bernd Ammann in [7]. Secondly, precisely in spacelike dimension 3, which would be most relevant to physics, the target KO-group is zero. This can be overcome by considering more refined index obstructions to psc that take the fundamental group into account. Here, in a paper that is about to be finished, I show that the conclusion about $\pi_0(\mathcal{I}^>(M))$ also holds if M satisfies Gromov-Lawson's enlargeability obstruction, which includes many examples such as tori or spin manifolds admitting non-positive sectional curvature.

Nevertheless, the dominant energy condition is far from being understood and there are many techniques from the study of psc metrics that still await their counterpart in the setting of initial data sets.

References

[1] J. Glöckle, *Homotopy of the space of initial values satisfying the dominant energy condition strictly*, preprint arXiv:1906.00099 (2019).

[2] B. Hanke, T. Schick and W. Steimle, *The space of metrics of positive scalar curvature*, Publ. Math. IHES **120**(1) (2014), 335–367.

[3] B. Botvinnik, J. Ebert and O. Randal-Williams, *Infinite loop spaces and positive scalar curvature* Invent. Math. **209**(3) (2017), 749–835.

[4] D. Crowley, T. Schick and W. Steimle, *Harmonic spinors and metrics of positive curvature via the Gromoll filtration and Toda brackets*, J. Topol. **11**(4) (2018), 1077–1099.
[5] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), 381–402.
[6] T. Schick and D. Wraith, *Non-negative versus positive scalar curvature*, preprint arXiv:1607.00657 (2016–2020).
[7] B. Ammann and J. Glöckle, *Dominant energy condition and spinors on Lorentzian manifolds*, preprint arXiv:2103.11032 (2021).

μ-Bubbles and the Band Width Conjecture

Daniel Räde

1. Summary

This talk is based on the preprint [10]. Let X be a connected smooth compact ($n \geq 2$)-dimensional manifold with boundary $\partial X \neq \emptyset$ and let $\sigma > 0$ be a positive real number. By Gromov's h-principle X admits a Riemannian metric g with scalar curvature $Sc(X,g) \geq \sigma$.

In case the open manifold $X \backslash \partial X$ does not admit a *complete* metric with positive scalar curvature, it is an active field of research to study the geometry of (X,g). One instance of this is the following conjecture [7, Conjecture 11.12 C] by Gromov:

Conjecture 1. Let Y be a closed smooth manifold of dimension $n-1 \neq 4$, which does not admit a metric with positive scalar curvature. If g is a Riemannian metric on $X = Y \times [-1,1]$ with $Sc(V,g) \geq \sigma > 0$, then

$$\mathrm{width}(X,g) = \mathrm{dist}_g\left(Y \times \{-1\}, Y \times \{1\}\right) \leq 2\pi\sqrt{\frac{n-1}{\sigma n}}.$$

In [7, Section 2] Gromov provided a proof for $X = T^{n-1} \times [-1,1]$ and related bands in dimension $n \leq 7$. In case Y is a spin manifold Conjecture 1 has been studied by Zeidler and Cecchini [4, 5, 12, 13]. They showed in particular that it holds true if the Rosenberg index $\alpha(Y)$ does not vanish [4, Theorem D].

Remark 2. The case $n-1 = 4$ is excluded in Conjecture 1 because of some counterexamples, which are peculiar to dimension 4. Using Seiberg-Witten theory one can show that there exists a simply connected spin 4-manifold Y such that Y does not admit a metric with positive scalar curvature but $Y \times S^1$ does [11, Counterexample 4.16]. Consequently $Y \times \mathbb{R}$ admits a complete metric with uniformly positive scalar curvature.

In his survey [8, Section 5.2] Gromov presented an alternative approach towards Conjecture 1 involving so called *μ-bubbles*, which in short are hypersurfaces minimizing a warped volume functional. His ideas were first made precise by Jintian Zhu [14] and recently this technique was used by Chodosh and Li [6, Theorem 2] and Gromov [9, Section 7] to prove that closed aspherical manifolds of dimension ≤ 5 do not admit a metrics with positive scalar curvature.

We denote $c_n := \sqrt{\frac{n-1}{n}}$ and use the μ-bubble technique to prove the following:

Theorem A. Let Y^{n-1} be a closed connected oriented manifold which does not admit a metric with positive scalar curvature and $n-1 \in \{1,2,3,5,6\}$. Let g be a metric on $X = Y \times [-1,1]$ with $Sc(X,g) \geq \sigma > 0$. If the mean curvature of the boundary satisfies

$$H(\partial_- X) \geq \sqrt{\sigma} c_n \tan\left(\sqrt{\sigma}(2c_n)^{-1}\ell_-\right) \text{ and } H(\partial_+ X) \geq -\sqrt{\sigma} c_n \tan\left(\sqrt{\sigma}(2c_n)^{-1}\ell_+\right)$$

for some $-\pi(\sqrt{\sigma})^{-1}c_n < \ell_- < \ell_+ < \pi(\sqrt{\sigma})^{-1}c_n$, then $\mathrm{width}(X,g) \leq \ell_+ - \ell_-$.

Remark 3. In dimension $n-1=4$ the result still holds if Y is a closed oriented manifold, which can not be dominated by a psc-manifold, or Y is spin with nonvanishing Rosenberg index $\alpha(Y)$. By adapting ideas of Gromov [9] in the language of Chodosh and Li [6] we show that this covers the case of closed aspherical 4-manifolds.

As an immediate Corollary to Theorem A we obtain a sharpened version (replace '$\leq$' by '$<$') of Conjecture 1 in all of the above cases. Our result is inspired by recent work of Cecchini and Zeidler [5, Theorem 7.6], who observed and quantified the contribution of the mean curvature $H(\partial_\pm X)$ in the spin setting. Even in dimension $n-1=4$ and if Y is spin our Theorem A generalizes their result as they are limited to using lower index theory by some technical issues (see [5, Section 1.4] for a discussion). On the other hand their approach, which uses Dirac operators with potentials, works in dimensions $n > 7$, while for μ-bubbles one runs into problems with singularities.

It is expected, that the band width estimate Theorem A is accompanied by the following rigidity statement (compare [5, Theorem 9.1]).

Conjecture 4. Let Y be a closed manifold of dimension $n-1 \neq 4$, which does not admit a metric with positive scalar curvature. Let g be a Riemannian metric on $X = Y \times [-1,1]$ and $\sigma > 0$. If the following conditions hold:

- $Sc(X,g) \geq \sigma$,
- $H(\partial_- X) \geq \sqrt{\sigma} c_n \tan\left(\sqrt{\sigma}(2c_n)^{-1}\ell_-\right)$ and $H(\partial_+ X) \geq -\sqrt{\sigma} c_n \tan\left(\sqrt{\sigma}(2c_n)^{-1}\ell_+\right)$ for some $-\pi(\sqrt{\sigma})^{-1}c_n < \ell_- < \ell_+ < \pi(\sqrt{\sigma})^{-1}c_n$,
- $\mathrm{width}(X,g) \geq \ell_+ - \ell_-$,

then (X,g) is isometric to a warped product $(Y \times [\ell_-,\ell_+], \varphi^2 g_Y + dt^2)$, where

$$\varphi(t) = \cos\left(\frac{\sqrt{\sigma}}{2c_n}t\right)^{\frac{2}{n}}$$

and g_Y is a Ricci flat metric on Y. In particular equality holds in all three conditions.

We present some progress towards Conjecture 4 in dimension $n-1 \leq 6$. Here we use ideas from [14, Section 3], where Zhu proves a rigidity result for certain 3-dimensional bands with positive sectional curvature.

If verified in its generality Conjecture 4 would be an instance of a *comparison* type theorem involving scalar curvature. To prove it one needs to work out some analytical subtleties regarding μ-bubbles. As they are at the center of some new and exciting developments in the field, this seems like a worthy undertaking.

References

[1] C. Bär, B. Hanke, *Boundary conditions for scalar curvature*, arXiv:2012.09127v1, December 2020.

[2] S. Brendle, F. Marques, A. Neves, *Deformations of the hemisphere that increase scalar curvature*, Invent. Math. **185** (2011), 175-197.

[3] A. Carlotto, C. Li, *Constrained deformations of positive scalar curvature metrics*, arXiv:1903.11772v2, April 2019.

[4] S. Cecchini, *A long neck principle for Riemannian spin manifolds with positive scalar curvature*, Geom. Funct. Anal. **30** (2020), 1183-1223.

[5] S. Cecchini, R. Zeidler, *Scalar and mean curvature comparison via the Dirac operator*, arXiv:2103.06833v1, March 2021.

[6] O. Chodosh, C. Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv:2008.11888v3, September 2020.

[7] M. Gromov, *Metric inequalities with scalar curvature*, Geom. Funct. Anal. **28** (2018), 645-726.

[8] M. Gromov, *Four Lectures on Scalar Curvature*, arXiv:1908.10612v4, October 2020.

[9] M. Gromov, *No metrics with positive scalar curvature on aspherical 5-manifolds*, arXiv:2009.05332v1, September 2020.

[10] D. Raede, *Scalar and mean curvature comparison via μ-bubbles*, arXiv:2104.10120v2, May 2021.

[11] J. Rosenberg, S. Stolz, *Metrics of positive scalar curvature and connections with surgery*, In: *Surveys on surgery theory, Vol. 2.* Vol. 149. Ann. of Math. Stud. Princeton Univ Press, Princeton, NJ, 2001. 353-386.

[12] R. Zeidler, *Band width estimates via the Dirac operator*, arXiv:1905.08520v3, February 2020.

[13] R. Zeidler, *Width, Largeness and Index Theory*, SIGMA **16** (2020), 127, 15 pages.

[14] Jintian Zhu, *Width estimate and doubly warped product*, Trans. Amer. Math. Soc. **374** (2021), 1497-1511.

Diffeomorphisms and positive (scalar) curvature

Georg Frenck

This talk is about my recent preprint [Fre21]. A video abstract can be found here: `https://www.youtube.com/watch?v=wuaBwTMsqsc`.

1. Introduction

Throughout this note, let M be a simply connected closed manifold of dimension d. The classification of Riemannian metrics on M satisfying a given curvature condition is a central problem in Riemannian geometry. Of course, open conditions like positive scalar curvature are preserved under small perturbations of a metric. Hence there cannot be a unique metric satisfying them. Therefore it is more reasonable to study uniqueness "up to continuous deformation", which translates into the following question:

Is the space of Riemannian metrics on M
satisfying a given curvature condition contractible?

It has been shown that the space $\mathcal{R}^+(M)$ of positive scalar curvature metrics has many nontrivial homotopy groups, see [HSS14] or [BERW17]. One way to obtain such elements is to consider the action of the diffeomorphism group $\mathrm{Diff}(M)$ of M on $\mathcal{R}^+(M)$ via pushforward. It has been first observed by Hitchin in [Hit74], that for every *Spin*-manifold of dimension $d \equiv 0, 1(8)$, there exists a diffeomorphism f of M such that f^*g and g are not isotopic for every $g \in \mathcal{R}^+(M)$. His proof works by constructing an M-bundle over the circle with nontrivial α-invariant, which obstructs positive scalar curvature on the total space. His result follows from the following, more general detection principle.

Proposition 1. *Let $[f_t] \in \pi_k(\mathrm{Diff}(M))$ and let $g \in \mathcal{R}^+(M)$. If the total space of the bundle clutched by $[f_t]$ does not admit a metric of positive scalar curvature, then $[f_t^* g] \in \pi_k(\mathcal{R}^+(M))$ is nontrivial.*

Hitchins result was later generalized by Crowley–Schick and Crowley–Schick–Steimle [CS13, CSS18]. However, those results only applies, if the total space of the bundle is of dimension $\equiv 1, 2(8)$, where existence of positive scalar curvature is obstructed by the α-invariant. The alpha-invariant is also non-zero in dimensions $\equiv 0(4)$, where it agrees with the $\hat{\mathcal{A}}$-genus. It has been abstractly shown by Hanke–Schick–Steimle, that such bundles exists, but until very recently, no explicit examples were known. One reason is, that there are obstructions to finding such bundles:

Theorem 2 ([HSS14, Proposition 1.9], [Wie19, Lemma 2.3])**.** *If all rational Pontryagin classes of M are trivial or if $k > 2d - 1$, then every M-bundle over S^{k+1} has vanishing $\hat{\mathcal{A}}$-genus.*

On the other hand, an explicit example of such a bundle was very recently found by Krannich–Kupers–Randal-Williams:

Theorem 3 ([KKR21])**.** *There exists an $\mathbb{HP}^2$-bundle $E \to S^4$ with $\hat{\mathcal{A}}(E) \neq 0$.*

It turns out, that the methods used in [KKR21] can be applied to quite general manifolds and we use them to prove the following generalization.

Theorem 4 ([Fre21, Theorem A])**.**

(1) Let M be a simply connected manifold with at least one non-vanishing rational Pontryagin class and let $k \leq \min(\frac{d-1}{3}, \frac{d-5}{2})$. Then there exists an M-bundle $E \to S^{k+1}$ with $\hat{\mathcal{A}}(E) \neq 0$.

(2) If additionally M admits a Spin-structure and a metric of positive scalar curvature, then E can be chosen to admit a Spin-structure and there is a cross-section with trivial normal bundle.

Note, that this theorem is almost optimal, as it precisely excludes the two obstructions from Theorem 2. The bound on k can be slightly improved if M is highly connected. Since the $\hat{\mathcal{A}}$-genus is an integer-valued invariant, we also deduce the following corollary:

Corollary 5. *Let M be a simply connected Spin-manifold that has a non-vanishing rational Pontryagin class and let k be as above. Let furthermore $\mathcal{R}_C(M) \subset \mathcal{R}^+(M)$ be a diffeomorphism invariant, non-empty subset. Then*

$$\pi_k(\mathcal{R}_C(M)) \otimes \mathbb{Q} \neq 0.$$

The same holds true with M replaced by $M\#N$ for any *Spin-manifold N of dimension d.*

Let us conclude by listing examples, to which this corollary applies.

Example 6. (1) The most immediate examples for the subspaces $\mathcal{R}_C(M)$ are the spaces positive Ricci and positive sectional curvature. One could also take any positive lower bounds on scalar, Ricci or sectional curvature. By a Ricci-flow argument carried out in [FR21, Proposition 3.3], the corollary also hold for the space of non-negative scalar curvature metrics.

(2) The class of manifolds as in the corollary includes the projective spaces $\mathbb{CP}^{2n+1}$, $\mathbb{HP}^n$ and $\mathbb{OP}^2$. One can also take $S^n \times K$ (for K a $K3$-surface) or $S^n \times B$ (for B the Bott manifold) if $n \geq 2$. Furthermore, this class is also closed under products and connected sums with arbitrary manifolds.

2. Outline of the argument

Before we start, let us remark again, that the underlying construction is the same as in [KKR21]. Instead of constructing an actual fiber bundle they construct a so-called block bundle. The advantage of working with block bundles is that the k-th homotopy group $\pi_k(\mathrm{hAut}(M)/\widetilde{\mathrm{Diff}}(M))$ of the classifying space for fiber homotopy trivial block bundles is isomorphic to the structure set $\mathcal{S}_\partial(D^k \times M)$ from surgery theory. The latter is accessible through the surgery exact sequence

$$L_{k+d+1}(\mathbb{Z}\pi_1 M) \longrightarrow \mathcal{S}_\partial(D^k \times M) \longrightarrow \mathcal{N}_\partial(D^k \times M) \xrightarrow{\sigma} L_{k+d}(\mathbb{Z}\pi_1 M)$$

where $\mathcal{N}_\partial$ denotes the set of normal invariants. We are interested in the case, where M is simply connected, so the L-groups are given by 0 on the left and by $\mathbb{Z}$ on the right. Hence, in order to construct an M-bundle it suffices to construct a normal invariant η with $\sigma(\eta) = 0$. It turns out, that the set of normal invariants is (rationally) isomorphic to the reduced real K-theory of $S^k \wedge M_+$ which allows one to construct a normal invariant and hence a (fiber homotopy trivial) block bundle with prescribed Pontryagin classes. Fiber homotopy trivial block bundles over S^k can (rationally) be given the structure of an actual fiber bundle if k is (roughly) smaller than $d/3$ by a classical result of Burghelea–Lashof (cf. [BL82]).

The main work in the present article lies in solving the combinatorial problem of choosing appropriate normal invariants. This way we can ensure that certain Pontryagin classes and numbers of the total space of the corresponding block bundle are zero or nonzero.

References

[BERW17] B. Botvinnik, J. Ebert, and O. Randal-Williams. Infinite loop spaces and positive scalar curvature. *Invent. Math.*, 209(3):749–835, 2017. http://doi.org/10.1007/s00222-017-0719-3.

[BL82] D. Burghelea and R. Lashof. Geometric transfer and the homotopy type of the automorphism groups of a manifold. *Trans. Amer. Math. Soc.*, 269(1):1–38, 1982. http://doi.org/10.2307/1998592.

[CS13] D. Crowley and T. Schick. The Gromoll filtration, KO-characteristic classes and metrics of positive scalar curvature. *Geom. Topol.*, 17(3):1773–1789, 2013. http://doi.org/10.2140/gt.2013.17.1773.

[CSS18] D. Crowley, T. Schick, and W. Steimle. Harmonic spinors and metrics of positive curvature via the Gromoll filtration and Toda brackets. *J. Topol.*, 11(4):1077–1099, 2018. http://doi.org/10.1112/topo.12081.

[Fre21] G. Frenck. *Diffeomorphisms and positive curvature.* https://arxiv.org/abs/2104.10595

[FR21] G. Frenck and J. Reinhold. Bundles with non-multiplicative Â-genus and spaces of metrics with lower curvature bounds. *International Mathematics Research Notices*, Jan 2021. http://doi.org/10.1093/imrn/rnaa361.

[HSS14] B. Hanke, T. Schick, and W. Steimle. The space of metrics of positive scalar curvature. *Publ. Math. Inst. Hautes Études Sci.*, 120:335–367, 2014. http://doi.org/10.1007/s10240-014-0062-9.

[Hit74] N. Hitchin. Harmonic spinors. *Advances in Math.*, 14:1–55, 1974. http://doi.org/10.1016/0001-8708(74)90021-8.

[KKR21] M. Krannich, A. Kupers, and O. Randal-Williams. An HP^2-bundle over S^4 with nontrivial $\hat{A}$-genus. *C. R., Math., Acad. Sci. Paris*, 359(2):149–154, 2021.

[Wie19] M. Wiemeler. On moduli spaces of positive scalar curvature metrics on highly connected manifolds, 2019, http://arxiv.org/abs/1610.09658.

Existence of static vacuum extensions

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(joint work with Zhongshan An)

1. Bartnik's static extension conjecture

Let $n \geq 3$ and (M, g) be an n-dimensional Riemannian manifold. We say that (M, g) is *static vacuum* if there is a scalar-valued function u on M satisfying

$$\begin{aligned} -u\mathrm{Ric}_g + \nabla_g^2 u &= 0 \\ \Delta_g u &= 0. \end{aligned}$$

Such u is called a *static potential.* Let the number $q \in (\frac{n-2}{2}, n-2)$. We say that (M, g) is *asymptotically flat* (at the rate q) if there is a compact subset $K \subset M$ such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_1$ and that the metric g has the asymptotics $g_{ij} = \delta_{ij} + O(|x|^{-q})$ with respect to the pull-back Cartesian coordinates $\{x_1, \ldots, x_n\}$ on $M \setminus K$.

Example 1. The family of (Riemannian) Schwarzschild metrics g_m is a family of asymptotically flat, static vacuum manifolds:

$$g_m = \left(1 - \tfrac{2m}{r^{n-1}}\right)^{-1} dr^2 + r^2 g_{S^{n-1}} \quad \text{defined on } \mathbb{R}^n \setminus B_{(2m)^{\frac{1}{n-1}}},$$

with the static potential $u_m = \sqrt{1 - \frac{2m}{r^{n-1}}}$, where $g_{S^{n-1}}$ is the standard metric on the unit sphere S^{n-1}. When $m = 0$, the Schwarzschild metric becomes the Euclidean metric. When $m > 0$, the Schwarzschild manifold has an outermost minimal hypersurface boundary.

The celebrated theorem on Uniqueness of Static Black Holes says that a 3-dimensional asymptotically flat, static vacuum manifold with minimal surface boundary is uniquely characterized as a Schwarzschild manifold. In contrast to the above uniqueness result, Robert Bartnik conjectured the following Plateau-type problem for static vacuum manifolds [5, Conjecture 7].

Conjecture 2 (Static Extension Conjecture)**.** Let (Ω, g_0) be a compact manifold with scalar curvature $R_{g_0} \geq 0$. Suppose the mean curvature H_{g_0} is not everywhere ≤ 0 on the boundary Σ. Then there exists a unique asymptotically flat, static vacuum manifold (M, g) with boundary $\partial M \cong \Sigma$ satisfying

$$\begin{rcases} g_0^{\intercal} = g^{\intercal} \\ H_{g_0} = H_g \end{rcases} \quad \text{on } \Sigma.$$

We shall refer to the geometric boundary data $(g_0^{\intercal}, H_{g_0})$ as the *Bartnik boundary data.* Let us also remark on the assumption that H_{g_0} is not everywhere less or equal to zero. The conjecture would fail without this assumption because if such extension, if exist, would contain a minimal hypersurface homologous to the boundary, and thus the extension must be Schwarzschild, which put further restriction on possible $g_0^{\intercal}$.

There are some progresses toward Conjecture 2 if $n = 3$ and if (g_0, H_{g_0}) is sufficiently close to the induced Bartnik boundary data from the flat metric on a round sphere; that is, if (g_0, H_{g_0}) is sufficiently close to $(g_{S^2}, 2)$. See Miao [7], Anderson-Khuri [4], and Anderson [3].

In this report, we discuss the results from [1] that confirm existence and local uniqueness of Conjecture 2 for large classes of boundary data, including those close to the induced boundary data on either a convex surface or a generic hypersurface in the Euclidean space.

2. Main results

Let Ω be a bounded open subset in $\mathbb{R}^n$ whose boundary $\Sigma = \partial\overline{\Omega}$ is an embedded hypersurface in the Euclidean space $(\mathbb{R}^n, \bar{g})$. Our first main result gives a general criterion for the existence and local uniqueness of static vacuum pairs.

Theorem 3. *Suppose the boundary Σ is static regular in $\mathbb{R}^n \setminus \Omega$. Then there exist positive constants ϵ_0, C such that for each $\epsilon < \epsilon_0$, if (τ, ϕ) satisfies $\|(\tau, \phi) - (\bar{g}^{\intercal}, H_{\bar{g}})\|_{C^{2,\alpha}(\Sigma)\times C^{1,\alpha}(\Sigma)} < \epsilon$, then there exists a unique asymptotically flat pair*

(g,u) with $\|(g,u)-(\bar{g},1)\|_{C^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)} < C\epsilon$ such that (g,u) is a static vacuum pair in $\mathbb{R}^n\setminus\Omega$ having the Bartnik boundary data $(g^{\intercal}, H_g) = (\tau,\phi)$ on Σ and satisfying both the static-harmonic gauge and the orthogonal gauge.

We shall refer the definition of the weighted Hölder spaces $C^{2,\alpha}_{-q}(\mathbb{R}^n\setminus\Omega)$ in [1]. Note that Conjecture 2 can be formulated as a boundary value problem for a system of differential equations. In a loose sense, we say that Σ is *static regular* in $\mathbb{R}^n\setminus\Omega$ if any solution (h,v) (with the suitably fall-off rate at infinity) to the *linearized* boundary value problem must satisfy a stronger boundary condition. (See [1, Definition 2] for a precise definition.) The static regular condition enables us to show that the linearized boundary value has only "trivial" solutions, i.e. those arising from diffeomorphisms.

Furthermore, we can show that large classes of hypersurfaces are static regular. We begin with convex surfaces in $\mathbb{R}^3$.

Theorem 4. *Let Ω be a bounded open subset in $\mathbb{R}^3$ whose boundary $\Sigma = \partial\overline{\Omega}$ has positive Gauss curvature. Then Σ is static regular in $\mathbb{R}^3\setminus\Omega$.*

We are also able to show that a generic hypersurfaces in $\mathbb{R}^n$ is static regular in the following concrete sense.

Theorem 5. *Let $t\in[-\delta,\delta]$ and each $\Omega_t\subset\mathbb{R}^n$ be a bounded open subset with hypersurface boundary $\Sigma_t = \partial\overline{\Omega_t}$ embedded in $\mathbb{R}^n$. Suppose the boundaries $\{\Sigma_t\}$ form a smooth generalized foliation. Then there is an open dense subset $J\subset(-\delta,\delta)$ such that Σ_t is static regular in $\mathbb{R}^n\setminus\Omega_t$ for all $t\in J$.*

By a dilation argument, the above theorem implies that a round sphere S^{n-1} is static regular in $\mathbb{R}^n\setminus B_1$.

3. Remarks and open questions

While our proofs heavily use that the background metric is the Euclidean metric, it gives a roadmap to solve Bartnik's static extension conjecture when the background metric is a general static vacuum manifold, such as the family of Schwarzschild manifolds. In forthcoming work [2], we show how to extend Theorem 3 and Theorem 5 to weakly mean-convex hypersurfaces Σ homologous to (but not intersect with) the minimal boundary of a Schwarzschild manifold. Our results should give better understanding on the structure of static vacuum manifolds. For example, as a direct consequence of our results, we can get vast examples of non-Schwarzschild, asymptotically flat, static vacuum manifolds whose boundary has constant mean curvature arbitrary close to zero.

Example 6. Given any $\epsilon > 0$, there is an asymptotically flat, static vacuum manifold $(\mathbb{R}^n\setminus B_1, g)$ such that the boundary S^{n-1} has constant mean curvature $H_g = c$ with $0 < c < \epsilon$ but g is not isometric to Schwarzschild.

In particular, those examples give an interesting contrast with uniqueness when $H_g = 0$ from Uniqueness of Static Black Holes mentioned above.

One can also consider the "dual" problem to extension question. For a given Bartnik boundary data, one can ask whether there exists a static vacuum *fill-in*, or more generally a nonnegative scalar curvature fill-in. This question has already be touched upon in the work of Bray and of Jauregui, and there are known obstructions to existence. The following problem was recently proposed by Gromov [6, Section 3.12].

Problem 7. *Let (Y, h) be a closed $(n-1)$-dimensional Riemannian manifold and ϕ a smooth function on Y. Find conditions such that, for a given number σ, there exists a complete n-dimensional Riemannian manifold (X, g) with $R_g \geq \sigma$, $\partial X = Y$, and the Bartnik boundary data of g realizes (h, ϕ); namely, $(g^\intercal, H_g) = (h, \phi)$ on Σ.*

References

[1] Zhongshan An and Lan-Hsuan Huang, *Existence of static vacuum extensions with prescribed Bartnik boundary data*, arXiv:2103.15887 [math.DG]

[2] Zhongshan An and Lan-Hsuan Huang, *Existence of static vacuum extensions with prescribed Bartnik boundary data near Schwarzschild*, in preparation

[3] Michael T. Anderson, *Local existence and uniqueness for exterior static vacuum Einstein metrics*, Proc. Amer. Math. Soc. **143** (2015), no. 7, 3091–3096. MR 3336633

[4] Michael T. Anderson and Marcus A. Khuri, *On the Bartnik extension problem for the static vacuum Einstein equations*, Classical Quantum Gravity **30** (2013), no. 12, 125005, 33. MR 3064190

[5] Robert Bartnik, *Mass and 3-metrics of non-negative scalar curvature*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 231–240. MR 1957036

[6] Misha Gromov, *Four lectures on scalar curvature*, arXiv:1908.10612 [math.DG]

[7] Pengzi Miao, *On existence of static metric extensions in general relativity*, Comm. Math. Phys. **241** (2003), no. 1, 27–46. MR 2013750

Kodaira Dimension and the Yamabe Problem, Revisited

Claude LeBrun

(joint work with Michael Albanese)

The *Yamabe invariant* $\mathscr{Y}(M)$ of a smooth compact n-manifold M is the diffeomorphism invariant defined by

$$\mathscr{Y}(M) := \sup_{\gamma} \inf_{g \in \gamma} \frac{\int_M s_g \, d\mu_g}{\left[\int_M d\mu_g\right]^{\frac{n-2}{n}}} \tag{1}$$

where γ varies over all possible conformal classes of Riemannian metrics, and where s_g and $d\mu_g$ respectively denote the scalar curvature and volume measure of an arbitrary metric g. It is not hard to to show that $\mathscr{Y}(M) > 0$ iff M admits a metric of scalar curvature $s > 0$. On the other hand, $\mathscr{Y}(M) \geq 0$ just means that M admits a unit-volume metric of scalar curvature $s > -\epsilon$ for every $\epsilon > 0$.

When $n = 4$ and M is the underlying smooth manifold of a compact complex surface (M^4, J), Donaldson [6] discovered that certain well-known complex-analytic invariants are actually diffeomorphism invariants of M, although they aren't homeomorphism invariants. Witten's introduction [18] of Seiberg-Witten invariants then seemed to hint that this phenomenon must somehow be closely tied to the behavior of the scalar curvature.

One such systematic relationship turns out to involve the *Kodaira dimension*, a basic complex-analytic invariant of (M^4, J) that is defined by

$$\mathrm{Kod}(M,J) = \limsup_{j\to+\infty} \frac{\log \dim H^0(M, \mathcal{O}(K^{\otimes j}))}{\log j}$$

where $K = \Lambda^{2,0}$ is the *canonical line bundle* of (M, J). For any compact complex surface, this invariant belongs to $\{-\infty, 0, 1, 2\}$, because it in fact coincides [3] with the largest complex dimension of an image of $M \dashrightarrow \mathbb{P}[H^0(M, \mathcal{O}(K^{\otimes j}))^*]$ among all the "pluricanonical" maps associated with the line bundles $K^{\otimes j}$, $j \in \mathbb{Z}^+$, albeit after imposing the idiosyncratic convention that $\dim \varnothing := -\infty$ in this context. For (M, J) of Kähler type, the speaker previously combined Seiberg-Witten arguments with constructions of suitable sequences of Riemannian metrics to show [13] that the Kodaira dimension is related to the Yamabe invariant in the following manner:

Theorem (L '98)**.** *Let M be the underlying smooth 4-manifold of a compact complex surface (M^4, J) of Kähler type. Then*

$$\begin{aligned} \mathscr{Y}(M) > 0 &\iff \mathrm{Kod}(M,J) = -\infty, \\ \mathscr{Y}(M) = 0 &\iff \mathrm{Kod}(M,J) = 0 \text{ or } 1, \\ \mathscr{Y}(M) < 0 &\iff \mathrm{Kod}(M,J) = 2. \end{aligned}$$

Here the Kähler-type condition is equivalent [3, 4, 17] to requiring that $b_1(M)$ be even, and this is actually the same [8] as requiring that (M^4, J) be deformation-equivalent to a smooth projective-algebraic variety. Note the that this pattern does not generalize to higher dimensions, because while the Kodaira dimension can be defined for compact complex manifolds of any complex dimension, it is not [5, 12, 15] a diffeomorphism invariant in any complex dimension ≥ 3, and so does not correlate with the Yamabe invariant in higher dimensions.

However, a curious limitation of the above result is that it only applies when the first Betti number $b_1(M)$ is even. Fortunately, our first new result is an improvement that does not depend on the parity of the first Betti number:

Theorem A. *Let M be the underlying smooth 4-manifold of any compact complex surface (M^4, J) of Kodaira dimension $\neq -\infty$. Then*

$$\begin{aligned} \mathscr{Y}(M) = 0 &\iff \mathrm{Kod}(M,J) = 0 \text{ or } 1, \\ \mathscr{Y}(M) < 0 &\iff \mathrm{Kod}(M,J) = 2. \end{aligned}$$

One cornerstone of Kodaira's classification of complex surfaces [3, 9] is the *blow-up* operation, which replaces a point of a complex surface Y with a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(-1)$; this then produces a new complex surface M that is diffeomorphic to $Y \# \overline{\mathbb{CP}}_2$, where $\overline{\mathbb{CP}}_2$ denotes the smooth oriented 4-manifold obtained by equipping $\mathbb{CP}_2$ with the non-standard orientation. Conversely, any complex surface M containing a $\mathbb{CP}_1$ of normal bundle $\mathcal{O}(-1)$ can be *blown down* to produce a new complex surface Y of which M then becomes the blow-up. While this blow-down procedure can in principle be iterated, the process necessarily terminates after finitely many steps, because each blow-down decreases b_2 by 1. When a complex surface X cannot be blown down, it is called *minimal*, and the upshot is that any complex surface M can be obtained from a minimal complex surface X by blowing up finitely many times. In this situation, one then says that X is a *minimal model* of M. Blowing up or down always leaves the Kodaira dimension unchanged. Moreover, the minimal model of a complex surface is actually *unique* whenever Kod $\neq -\infty$.

Our proof of Theorem A also yields the the following additional new result, which was previously proved in [11, 13] for $b_1(M)$ even:

Theorem B. *Let (M, J) be a compact complex surface with* Kod $\neq -\infty$*, and let (X, J') be its minimal model. Then*

$$\mathscr{Y}(M) = \mathscr{Y}(X).$$

In fact, most cases of Theorems A and B were already proved in [13], leaving only the case of Kod $= 1$ and b_1 odd to be settled. Moreover, even in this outstanding case, the results of [13] showed that that these remaining manifolds admit sequences of Riemannian metrics with $\int s^2 \, d\mu \to 0$, and therefore have $\mathscr{Y} \geq 0$. Thus, it only remained to show that properly elliptic complex surface with b_1 odd can never admit Riemannian metrics of positive scalar curvature. The first step in our proof is to use a covering argument to reduce this claim to the following narrower assertion:

Lemma. *Let Σ denote a compact Riemann surface of genus ≥ 2, and let $N \to \Sigma$ be a non-trivial circle bundle. Set $X = N \times S^1$, and let $M = X \# k\overline{\mathbb{CP}}_2$ for some integer $k \geq 0$. Then M does not admit any Riemannian metric g of positive scalar curvature.*

We then give two entirely different proofs [2, 14] of this lemma. Our first proof is based on the Schoen-Yau stable-minimal-hypersurface method [16], and is presented in a way that simultaneously implies other relevant results. Our second proof involves the Seiberg-Witten equations, and so is closer in spirt to the techniques used [11, 13] in the Kähler-type case. Despite a result of Biquard that shows that these manifolds do not carry any Seiberg-Witten basic classes, an argument due to Kronheimer [10] shows that controlled perturbations of the Seiberg-Witten equations nonetheless have solutions on arbitrarily high covers of the manifold, for specific spin^c structures, and this leads to systematic scalar-curvature estimates downstairs which we codify by introducing the new concept of a mock-monopole class. Either argument, in conjunction with the results of [13], then implies that any compact complex surface with b_1 odd and Kod $= 1$ necessarily has $\mathscr{Y} = 0$.

Theorems A and B have been formulated to exclude complex surfaces with Kod $= -\infty$ and b_1 odd. These complex surfaces, which actually all have $b_1 = 1$, were named surfaces of class VII by Kodaira [9], and inhabit a realm where algebraic geometry holds no sway. These include the familiar Hopf surfaces, and because blowing up such an examples turns out to change the Yamabe invariant [7], the exclusion of class VII in Theorem B is absolutely necessary. On the other hand, class-VII surfaces must also be excluded from Theorem A, because while Hopf surfaces and their blow-ups have $\mathscr{Y} > 0$, the so-called Inoue-Bombieri surfaces and their blow-ups also belong to class VII, but instead [1, 2] have $\mathscr{Y} = 0$.

References

[1] M. Albanese, *The Yamabe invariants of Inoue surfaces, Kodaira surfaces, and their blowups*, Ann. Global Anal. Geom., 59 (2021), pp. 179–195.

[2] M. Albanese and C. LeBrun, *Kodaira dimension and the Yamabe problem, II.* e-print arXiv:2106.14333 [math.DG], 2021.

[3] W. Barth, C. Peters, and A. Van de Ven, *Compact Complex Surfaces*, vol. 4 of Ergebnisse der Mathematik und ihrer Grenzgebiete (3), Springer-Verlag, Berlin, 1984.

[4] N. Buchdahl, *On compact Kähler surfaces*, Ann. Inst. Fourier (Grenoble), 49 (1999), pp. 287–302.

[5] F. Catanese and C. LeBrun, *On the scalar curvature of Einstein manifolds*, Math. Res. Lett., 4 (1997), pp. 843–854.

[6] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, Oxford, 1990.

[7] M. Gursky and C. LeBrun, *On Einstein manifolds of positive sectional curvature*, Ann. Glob. An. Geom.

[8] K. Kodaira, *On compact analytic surfaces. III*, Ann. of Math. (2), 78 (1963), pp. 1–40.

[9] ———, *On the structure of complex analytic surfaces. IV*, Amer. J. Math., 90 (1968), pp. 1048–1066.

[10] P. B. Kronheimer, *Minimal genus in $S^1 \times M^3$*, Invent. Math., 135 (1999), pp. 45–61.

[11] C. LeBrun, *Four-manifolds without Einstein metrics*, Math. Res. Lett., 3 (1996), pp. 133–147.

[12] ———, *Einstein metrics and the Yamabe problem*, in Trends in mathematical physics (Knoxville, TN, 1998), Amer. Math. Soc., Providence, RI, 1999, pp. 353–376.

[13] ———, *Kodaira dimension and the Yamabe problem*, Comm. Anal. Geom., 7 (1999), pp. 133–156.

[14] ———, *On the scalar curvature of 4-manifolds.* e-print arXiv:2105.10785 [math.DG], 2021.

[15] R. Răsdeaconu, *The Kodaira dimension of diffeomorphic Kähler threefolds*, Proc. Amer. Math. Soc., 134 (2006), pp. 3543–3553.

[16] R. M. Schoen and S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math., 28 (1979), pp. 159–183.

[17] Y. Siu, *Every K3 surface is Kähler*, Inv. Math., 73 (1983), pp. 139–150.

[18] E. Witten, *Monopoles and four-manifolds*, Math. Res. Lett., 1 (1994), pp. 809–822.

Spaces of metrics with positive scalar curvature on manifolds with boundary

Christian Bär

(joint work with Bernhard Hanke)

The talk is based on the results in [2]. A video abstract for this paper can be found at `https://vimeo.com/530490182`. Throughout let M be an oriented compact manifold with boundary.

1. A nonexistence result

Definition 1. We call a Hermitian vector bundle E over M with connection *admissible* if it is isomorphic to the trivial bundle with trivial connection over a neighborhood of the boundary and it has at least one nontrivial Chern number.

Definition 2. We say that an even-dimensional orientable compact connected Riemannian manifold M with boundary has *infinite K-area* if for each $\varepsilon > 0$ there exists an admissible E whose curvature satifies $\|R^E\| < \varepsilon$.

This property is independent of the Riemannian metric on M. Changing the metric changes the definition of the norm of R^E but since M is compact, the norms coming from two different metrics are equivalent.

Definition 3. We say that an orientable compact connected Riemannian manifold M with boundary has *stably infinite K-area* if $T^k \times M$ has infinite K-area for some k (and hence for all $k' = k + 2\ell$).

Note that this definition is also meaningful for odd-dimensional M. A large class of manifolds with stably infinite K-area is given by area-enlargeable manifolds. The following non-existence and rigidity result goes back to Gromov and Lawson ([6, Thm. 5.8] and [4, Sec. $5\frac{1}{4}$]) in the closed case.

Theorem 4. *Let M be a compact connected spin manifold with boundary. Assume M has stably infinite K-area. Then each Riemannian metric g on M with* $\mathrm{scal} \geq 0$ *and $H \geq 0$ is Ricci-flat and satisfies $H \equiv 0$. In particular, M does not admit a Riemannian metric with* $\mathrm{scal} > 0$ *and $H \geq 0$.*

The same holds for $N \times M$ if N is a closed connected spin manifold with nontrivial $\hat{\mathsf{A}}$-genus.

2. The main deformation result

Given a Riemannian metric g on M we denote by

- ▷ $\mathrm{scal}_g\colon M \to \mathbb{R}$ the scalar curvature of g,
- ▷ $g|_{\partial M} \in C^\infty(\partial M; (T^*M \otimes T^*M)|_{\partial M})$ the restriction of g to ∂M,
- ▷ $g_0 \in C^\infty(\partial M; T^*\partial M \otimes T^*\partial M)$ the metric induced on ∂M,
- ▷ II_g the second fundamental form of $\partial M \subset M$ with respect to the interior unit normal,
- ▷ $H_g = \frac{1}{n-1}\mathrm{tr}_g(\mathrm{II}_g)\colon \partial M \to \mathbb{R}$ the mean curvature of ∂M.

Let $\mathscr{R}(M)$ be the space of smooth Riemannian metrics on M with positive scalar curvature (psc metrics), equipped with the weak C^∞-topology.

The condition of having mean convex boundary ($H \geq 0$) can be replaced by other convexity assumptions. The following deformation result allows us to compare those different boundary conditions.

Theorem 5. *Let K be a compact Hausdorff space and let*

$$g\colon K \to \mathscr{R}(M)$$

be continuous. Let $k\colon K \to C^\infty(\partial M; T^*\partial M \otimes T^*\partial M)$ *be a continuous family of symmetric* $(2,0)$*-tensor fields satisfying* $\frac{1}{n-1}\mathrm{tr}_{g_0}(k(\xi)) \le H_{g(\xi)}$ *for all* $\xi \in K$.

Then for each neighborhood $\mathscr{U}$ *of* ∂M *there is a continuous map*

$$f\colon K \times [0,1] \to \mathscr{R}(M)$$

so that the following holds for all $\xi \in K$ *and* $s \in [0,1]$*:*

(a) $f(\xi, 0) = g(\xi)$;
(b) $f(\xi, s)|_{\partial M} = g(\xi)|_{\partial M}$, *in particular* $f(\xi, s)_0 = g(\xi)_0$;
(c) $\mathrm{II}_{f(\xi,s)} = (1-s)\mathrm{II}_{g(\xi)} + sk(\xi)$, *in particular,* $\mathrm{II}_{f(\xi,1)} = k(\xi)$;
(d) $f(\xi, s) = g(\xi)$ *on* $M \setminus \mathscr{U}$.

The proof uses the flexibility lemma ([1, Thm. 1.2 and Addendum 3.4]) and a refined deformation analysis. It even holds if we replace the condition $\mathrm{scal}_g > 0$ in the definition of $\mathscr{R}(M)$ by the lower bound $\mathrm{scal}_g > \sigma$ where $\sigma : M \to \mathbb{R}$ is any prescribed continuous function.

3. An application: comparison of different boundary conditions

We denote by $\mathscr{R}^{H\ge 0}(M)$ the subspace of $\mathscr{R}(M)$ consisting of all psc metrics for which the boundary is mean convex. Similar notation is applied to other boundary conditions. By $\mathscr{R}^{\mathsf{doubl}}(M)$ we denote the space of all psc metrics which are doubling, i.e. which yield a smooth metric after doubling the manifold along the boundary.

Theorem 5 can now be used to show:

Theorem 6. *Each of the inclusions in*

$$\begin{array}{ccccccc}
 & & & \mathscr{R}^{H=0}(M) & & & \\
 & & \nearrow & & \searrow & & \\
\mathscr{R}^{\mathsf{doubl}}(M) \hookrightarrow & \mathscr{R}^{\mathrm{II}=0}(M) & & & & \mathscr{R}^{H\ge 0}(M) & \hookleftarrow \mathscr{R}^{H>0}(M) \\
 & & \searrow & & \nearrow & & \\
 & \mathscr{R}^{\mathrm{II}>0}(M) \hookrightarrow & & \mathscr{R}^{\mathrm{II}\ge 0}(M) & & &
\end{array}$$

is a weak homotopy equivalence.

An inclusion being a weak homotopy equivalence means in particular that if the larger space in nonempty then so is the smaller space. For instance, if M admits a psc metric with $H \ge 0$ then it also admits one which is doubling. For $H > 0$ is fact has been shown by Gromov and Lawson in [5, Thm. C.3]

One popular boundary condition is missing here, namely that of having a metric product structure near the boundary. Indeed, sometimes $\mathscr{R}^{\mathsf{doubl}}(M)$ is nonempty while M does not carry any psc metric of product type near the boundary. As an example let $M = D^2 \times T^{n-2}$. Giving D^2 the metric of a round hemisphere and T^{n-2} a flat metric, the product metric on M will be a doubling psc metric.

But M cannot carry a positive scalar curvature metric with product structure near the boundary. If it did, the boundary would inherit a positive scalar curvature metric which is impossible since $\partial M = T^{n-1}$.

If one modifies the definition of $\mathscr{R}(M)$ and additionally requires that the induced metric on the boundary also has positive scalar curvature, then Theorem 6 still holds and includes the space of metrics with product structure near the boundary.

There are further applications of Theorem 5, for instance to the discussion of the counterexample by Brendle, Marques, and Neves ([3]) to the Min-Oo conjecture and to metrics with mean convex hypersurface singularities.

The homotopy equivalences in Theorem 6 would not be very interesting if the spaces turned out to always be contractible. But this is not the case.

Theorem 7. *For each $m \geq 0$ there exists a connected compact spin manifold M with nonempty boundary such that the m-th homotopy of every space in the diagram of Theorem 6 contains nontrivial classes.*

The proof combines ideas from [7] with Theorem 4.

References

[1] C. Bär and B. Hanke, *Local flexibility for open partial differential relations*, to appear in Commun. Pure Appl. Math., https://doi.org/10.1002/cpa.21982

[2] C. Bär and B. Hanke, *Boundary conditions for scalar curvature*, https://arxiv.org/abs/2012.09127

[3] S. Brendle, F. Marques, and A. Neves, *Deformations of the hemisphere that increase scalar curvature*, Invent. Math. **185** (2011), 175–197.

[4] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, in: Functional analysis on the eve of the 21st century, Vol. II, Birkhäuser, Boston, MA (1996), 1–213.

[5] M. Gromov and B. Lawson, *Spin and scalar curvature in the presence of a fundamental group. I*, Ann. of Math. (2) **111** (1980), 209–230.

[6] M. Gromov and B. Lawson, *Positive scalar curvature and the Dirac operator on complete Riemannian manifolds*, Inst. Hautes Études Sci. Publ. Math. **58** (1983), 83–196

[7] B. Hanke, T. Schick, and W. Steimle, *The space of metrics of positive scalar curvature*, Inst. Hautes Études Sci. Publ. Math. **120** (2014), 335–367.

Scalar curvature rigidity of Einstein manifolds

Mattias Dahl

(joint work with Klaus Kröncke)

We say that a Riemannian Einstein manifold is *scalar curvature rigid* if there are no compactly supported perturbations of the metric which strictly increases the scalar curvature. From the rigidity parts of the positive mass theorems we know that for example the Euclidean and hyperbolic spaces are scalar curvature rigid.

A symmetric 2-tensor with vanishing trace and divergence is called a TT-tensor. The space of compactly supported TT-tensors on M is denoted by $TT_{cs}(M)$. The *Einstein operator* $\Delta_E : C^\infty(S^2M) \to C^\infty(S^2M)$ is defined by $\Delta_E = \nabla^*\nabla - 2\mathring{R}$, where $\mathring{R}h_{ij} = h^{kl}R_{iklj}$. An open Einstein manifold (M, g) is called *linearly stable* if the number

$$\lambda_1(\Delta_E, M) = \inf \left\{ (\Delta_E h, h)_{L^2} \mid h \in TT_{cs}(M), \|h\|_{L^2}^2 = 1 \right\}$$

is non-negative and *linearly unstable* otherwise.

The goal of this talk was to present the following result from [1].

Theorem 1. *Suppose (M, g) is a Riemannian Einstein manifold which is not locally a warped product. Then (M, g) is scalar curvature rigid if and only if it is linearly stable.*

To prove one of the implications in the Theorem, we prove a stronger statement on 1-parameter deformations with prescribed scalar curvature and volume form, and a symmetric two-tensor h prescribed as the first derivative of the deformation. For an unstable manifold we can choose this deformation to increase scalar curvature.

Proposition 2. *Assume that (M, g) is Einstein and not locally a warped product. Let f_t be a 1-parameter family of smooth functions whose supports are contained in the relatively compact open set $\Omega \subset M$. Assume that $h \neq 0$ is a smooth TT-tensor with support in Ω satisfying*

$$\int_M \langle \Delta_E h, h \rangle \,\mathrm{div} = -2 \int_M f_0 \,\mathrm{div}\,.$$

Then there exists a 1-parameter family g_t of metrics with $g_0 = g$ and $\frac{d}{dt} g_t|_{t=0} = h$ such that $\mathrm{scal}^{g_t} = \mathrm{scal}^g + \frac{t^2}{2} f_t$, $\mathrm{div}^{g_t} = \mathrm{div}^g$, and $g_t = g$ outside of Ω.

The construction of the family g_t is made through an Ansatz of the form $g_t := g + t h_t + \frac{t^2}{2} k_t$ where h_t, k_t are families of compactly supported two-tensor with $h_0 = h$. The freedom to set the trace and the trace-free part of k_t indepently is used to prescribe the scalar curvature and the volume form simultanously. To get the exact solution for all small t we use an iteration scheme modelled on a proof of a second order implicit function theorem. Crucial for the construction are results by Delay [2] which allow us to solve for a compactly supported trace-free part of k_t with second divergence prescribed. To apply these results we need the assumption that the manifold is not locally a warped product.

To prove the other implication in the Theorem, we use a modified version of the lambda functional. Let $(M, \overline{g})$ be a compact Riemannian manifold with smooth boundary and let $\mathcal{M}$ be the set of smooth metrics on M such that $g - \overline{g}$ vanishes to all orders at ∂M. Let $C^\infty(M)$ be the set of smooth functions on M. For $\alpha > 0$, define

$$F_\alpha : \mathcal{M} \times C^\infty(M) \to \mathbb{R}, \qquad F_\alpha(g, f) := \int_M \left(\mathrm{scal}^g + \alpha |\nabla f|^2 \right) e^{-f} \,\mathrm{div},$$

and

$$\lambda_\alpha(g) := \inf \left\{ F_\alpha(g, f) \mid f \in C^\infty(M), \int_M e^{-f} \,\mathrm{div} = 1 \right\}.$$

For closed manifolds and with $\alpha = 1$, this is the λ-functional introduced by Perelman. Computations show that Einstein metrics are critical points of λ_α with respect to volume-preserving deformations. The second order variation of λ_α at an Einstein metric with respect to volume-preserving deformations is computed in terms of Δ_E. Using this we can prove the following.

Proposition 3. *Let (M, g) be a compact Einstein manifold with boundary and assume that the first non-zero Neumann eigenvalue of the Laplacian satisfies*

$$\mu_1^N(M, \Delta^g) > \frac{\mathrm{scal}^g}{n-1}$$

where $n = \dim M$. Choose $\alpha > 0$ so small that

$$\left(1 - \frac{n-2}{n-1}\alpha\right)\mu_1^N(M, \Delta^g) > \frac{\mathrm{scal}^{\hat{g}}}{n-1}.$$

Then, if the first Dirichlet eigenvalue of the Einstein operator on TT tensors satisfies

$$\mu_1^D(M, \Delta_E^g|_{TT}) > 0,$$

there is a neighbourhood $\mathcal{V}$ of g in $\mathcal{M}$ such that $\lambda_\alpha(\tilde{g}) \leq \lambda_\alpha(g)$ for every $\tilde{g} \in \mathcal{V}$ with $\mathrm{vol}(M, \tilde{g}) = \mathrm{vol}(M, g)$ and equality holds if and only if g is Einstein.

For closed manifolds, the inequality for the Laplacian eigenvalue is given by the Lichnerowicz-Obata eigenvalue estimate. The implication in the Theorem follows since a scalar curvature increasing deformation will increase the value of λ_α.

Examples of unstable Einstein manifolds can be found among the so called gravitational instantons. The Riemannian Schwarzschild metric and the Taub-Bolt metric are complete Ricci-flat unstable manifolds, and the Riemannian AdS Schwarzschild is a complete negative Einstein unstable manifold. They all have negative bottom of L^2-spectrum of Δ_E, which means that Δ_E has negative Dirichlet eigenvalues over sufficiently large precompact open sets Ω. The corresponding eigensection on such an Ω is automatically a TT-tensor, and we can again use the results of Delay [2] to find compactly supported TT-tensors with a negative integral as in the first Proposition. We conclude that these space are not scalar curvature rigid.

References

[1] M. Dahl, K. Kröncke, *Local and global scalar curvature rigidity of Einstein manifolds*, preprint (2021).

[2] E. Delay, *Smooth compactly supported solutions of some underdetermined elliptic PDE, with gluing applications*, Comm. Partial Differential Equations **37** (2012), no. 10, 1689–1716.

L^p-stability and positive scalar curvature rigidity of Ricci-flat ALE manifolds

Klaus Kröncke
(joint work with Oliver Lindblad Petersen)

A complete Riemannian manifold (M, g) is called *asymptotically locally Euclidean* (ALE for short) of order $\tau > 0$, if there exists a compact set $K \subset M$, a radius $R > 0$ and a diffeomorphism $\varphi : M \setminus K \to (\mathbb{R}^n \setminus B_R)/\Gamma$ such that $|\partial^k(\varphi^* g - g_{eucl})| = O(r^{-\tau-k})$ for all $k \in \mathbb{N}_0$. Here, $\Gamma \subset \mathrm{SO}(n)$ is a discrete subgroup acting freely on S^{n-1}.

If Γ is trivial, one recovers the notion of an *asymptotically Euclidean* (AE) manifold. Note that Γ has to be trivial if n is odd. If (M, g) is ALE and Ricci-flat, it is ALE of order $n-1$. If in addition, (M, g) is Kähler or if $n = 4$, it is ALE of order n. In any case, the fallof rate at infinity is so fast that ADM-mass vanishes for these manifolds. In particular, any Ricci-flat AE metric is isometric tl o Euclidean space by the positive mass theorem. Consequently, all non-flat Ricci-flat ALE manifolds examples occur in even dimensions.

Kronheimers classification of hyperkähler ALE four-manifolds [6] provides a large list of Ricci-flat ALE manifolds, including flat space and the Eguchi-Hanson manifold as its simples examples. In higher dimensions, Calabi-Yau non-hyperkähler and Spin(7) manifolds of ALE asymptotics were also discovered [3, 4, 5]. All known Ricci-flat ALE manifolds have special holonomy and thus carry a parallel spinor. It is a major open problem, whether examples of full holonomy do exist.

We are interested in the dynamical stability of Ricci-flat ALE metrics under the Ricci flow. The dynamical stability problem for the Ricci flow has been studied extensively in the case of compact Einstein manifolds over the last two decades and is well understood by now. Much less is known in the noncompact case, where the study rather focused on particular examples so far. For a more detailed discussion on this subject and references, see [8, p. 2].

Up to a gauge term, the linearization of the Ricci curvature is given by the Lichnerowicz Laplacian Δ_L. Motivated by this fact, we call a Ricci-flat manifold

- *linearly stable*, lf $\Delta_L \geq 0$ in the L^2-sense and
- *integrable*, if every $h \in \ker_{L^2}(\Delta_L)$ is tangent to a family of Ricci-flat metrics.

Our approach in proving dynamical stability consists of two main steps. At first, we consider the linearized version of the problem, i.e. the heat kernel of the Lichnerowicz Laplacian. In a second step, we construct a contraction mapping acting on time-dependent tensor fields whose unique solution is the Ricci(-de Turck) flow. Thereby, we prove longtime existence and convergence of the Ricci flow at once. Our result on the linear part of the problem is as follows:

Theorem 1 (Heat kernel estimates [7])**.** *Let (M^n, g) be a Ricci-flat ALE manifold equipped with a parallel spinor. Let $1 < p \leq q < \infty$, $k \in \mathbb{N}_0$ and $\epsilon > 0$. Then we have*

(i) $\ker_{L^2}(\Delta_L) \subset O(r^{-n})$,

(ii) $\left\|\nabla^k \circ e^{-t\Delta_L}|_{\ker^\perp}\right\|_{L^p\to L^q} \leq \begin{cases} C\cdot t^{-\alpha(p,q,k)}, & \text{if } \alpha(p,q,k) < \frac{n}{2p}, \\ C(\epsilon)\cdot t^{-\frac{n}{2p}+\epsilon}, & \text{if } \alpha(p,q,k) \geq \frac{n}{2p}, \end{cases}$

(iii) $\left\|D\mathrm{Ric}\circ \circ e^{-t\Delta_L}|_{\ker^\perp}\right\|_{L^p\to L^q} \leq C\cdot t^{-\alpha(p,q,2)}$.

Let us put these results in context: For the solution u of a Laplace-type equation on an ALE manifold with $u\to 0$ as $r\to\infty$, one has in general the expansion

$$u = A\cdot r^{2-n} + B(\varphi)\cdot r^{1-n} + C(\varphi)\cdot r^{-n} + \ldots \text{ as } r\to\infty, \tag{1}$$

where φ denotes the spherical variable. Point (i) states that we can get rid of the first two terms of this expansion in our setting. For the Euclidean Laplacian, we have the estimate $\left\|\nabla^k\circ e^{-t\Delta}\right\|_{L^p\to L^q} \leq Ct^{-\alpha(p,q,k)}$ for all $1\leq p\leq q\leq\infty$ and $k\in\mathbb{N}_0$. Our point (ii) states that this estimate may still hold for more general elliptic operators and for suitable values of p,q,k on the orthogonal complement of the kernel. However, the decay rate can not be faster than the threshold rate $\frac{n}{2p}$. Therefore, for values of p,q,k with large $\alpha(p,q,k)$, the decay rate will be slower than in the Euclidean case. In contrast, as (iii) asserts, for some specific differential operators such as the Frechét derivative of the Ricci tensor DRic, we always get the same decay rate as in the Euclidean case. The essential reason is that DRic commutes with Δ_L and hence also with its heat flow.

The assumption of having a parallel spinor appears for the following reason: Wang [9] constructed an isometric embedding of vector bundles $\Phi: T^*M\odot T^*M\to T^*M\otimes S$ via which $\Delta_L\sim D^2$. Here, S is the spinor bundle and D is the twisted Dirac operator on vector spinors. This has various consequnces: (M,g) is linearly stable and the identification Φ allows us to deduce (i) by analyzing the terms in the asymptotic expansion (1) more carefully. Furthermore, it allows us to compare each covariant derivative ∇^k with the respective elliptic operator D^k and to conclude (ii) from elliptic estimates on weighted Sobolev spaces.

Theorem 2 (Dynamical stability [8])**.** *Let $(M^n,\hat{g})$ be a Ricci-flat ALE manifold, which carries a parallel spinor and is integrable. Then for each $q\in(1,n)$ and each $L^q\cap L^\infty$-neighbourhood $\mathcal{U}\subset\mathcal{M}$ of $\hat{g}$ in the space of metrics, there exists another $L^q\cap L^\infty$-neighbourhood $\mathcal{V}\subset\mathcal{U}$ of $\hat{g}$ with the following property:*
For each metric $g_0\in\mathcal{V}$ on M, the Ricci flow $\{g_t\}_{t\geq 0}$ starting at g_0 exists for all time and there is a family of diffeomorphisms $\{\phi_t\}_{t\geq 0}$ such that $\phi_t^* g_t\in\mathcal{U}$ for all $t\geq 0$ and $\phi_t^* g_t$ converges to a Ricci-flat metric h_∞ as $t\to\infty$.
Moreover, if $g_0-\hat{g}\in L^p$ for some $p\in(1,q]$, there exists a smooth family of Ricci-flat metrics h_t, such that we have the convergence rates

$$\|h_t - h_\infty\|_{C^k} \leq C\cdot t^{1-\frac{n}{p}+\epsilon}, \tag{2}$$

$$\|\phi_t^* g_t - h_t\|_{C^k} \leq C\cdot t^{-\frac{n}{2p}+\epsilon}, \tag{3}$$

$$\|\mathrm{Ric}_{g_t}\|_{C^k} \leq C\cdot t^{-\frac{n}{2p}-1+\epsilon}, \text{ if } p < n/2, \tag{4}$$

where C depends on $k\in\mathbb{N}_0$ and $\epsilon>0$.

The proof builds up on a careful analysis of the space of Ricci-flat metrics, Theorem 1 and an iteration argument. One first proves longtime existence and convergence with convergence rates as in (2)–(4) with p replaced by q. If $g_0 - \hat{g} \in L^p$, a bootstrapping argument shows that these estimates can be improved to the faster rates (2)–(4).

For $q \geq n$, the iteration mechanism breaks down. The reason is the failure of optimal decay rates for the heat flow in Theorem 1 (ii). It is unclear whether the assertion of the theorem is true or not for $q \geq n$. Nevertheless, Theorem 2 is a substantial improvement of a result by Deruelle and the first author [2], who prove the $L^2 \cap L^\infty$-stability of linearly stable and integrable Ricci-flat ALE metrics without establishing any rate of convergence. Let us also point out that Theorem 2 applies to all known examples of Ricci-flat ALE manifolds. A consequence of our result is the following rigidity statement:

Theorem 3 (Positive scalar curvature rigidity [8])**.** *Let $(M^n, \hat{g})$ be a Ricci-flat ALE manifold which carries a parallel spinor and is integrable. Then for each $q \in (1, n)$, there exists a $L^q \cap L^\infty$-neighbourhood $\mathcal{U}$ of $\hat{g}$ in the space of metrics such that any smooth metric $g \in \mathcal{U}$ on M satisfying*

$$\mathrm{scal}_g \geq 0, \qquad \textit{and} \qquad \|g - \hat{g}\|_{L^p} < \infty$$

for some $p < n/(n-2)$ is Ricci-flat.

An analogous result was previously shown by Appleton [1] for Euclidean space with $q = \infty$. His result uses the stability of Euclidean space under Ricci flow and the proof of our result builds up on the same ideas: Under the Ricci flow, the scalar curvature satisfies the evolution inequality

$$\partial_t \mathrm{scal}_{g_t} + \Delta_{g_t} \mathrm{scal}_{g_t} = 2|\mathrm{Ric}_{g_t}|^2_{g_t}.$$

Thus if the scalar curvature is initially nonnegative everywhere, the maximum principle implies that it becomes positive everywhere for any positive time if the initial metric is not Ricci-flat. Estimates on the scalar heat kernel imply in this situation that

$$\mathrm{scal}_{g_t} = O(t^{-n/2}), \qquad \text{but} \qquad \mathrm{scal}_{g_t} \neq O(t^{-n/2+\epsilon}) \qquad \text{as } t \to \infty.$$

On the other hand, estimate (4) in Theorem 2 implies that

$$\mathrm{scal}_{g_t} = O(t^{-n/2p-1+\epsilon}) \qquad \text{for every } \epsilon > 0,$$

which yields a contradiction, if $p < n/(n-2)$ and proves the result. Note that at the end, this builds up on the crucial observation that $D\mathrm{Ric}$ commmutes with Δ_L, which enables us to prove part (iii) in Theorem 1 (from which in turn we deduce (4)).

References

[1] A. Appleton, *Scalar curvature rigidity and Ricci De Turck flow on perturbations of Euclidean space*, Calc. Var. Partial Differ. Equ. **57** (2018), no. 5, 132

[2] A. Deruelle and K. Kröncke, *Stability of ALE Ricci-flat manifolds under Ricci flow*, J. Geom. Anal. **31** (2021) no. 3, 2829–2870

[3] D. Joyce, *A new construction of compact 8-manifolds with holonomy* Spin(7), J. Differ. Geom. **53** (1999) no. 1, 89–130

[4] D. Joyce, *Compact manifolds with special holonomy*, Oxford University Press (2000)

[5] D. Joyce, *Asymptotically locally Euclidean metrics with holonomy* SU(m), Ann. Global Anal. Geom. **19** (2001) no. 1, 55–73
[6] P. B. Kronheimer, *The construction of ALE spaces as hyper-Kähler quotients*, J. Differ. Geom. **29** (1989) no. 3, 665–683
[7] K. Kröncke and O. L Petersen, *Long-time estimates for heat flows on ALE manifolds*, preprint, arXiv:2006.06662
[8] K. Kröncke and O. L Petersen, *L^p-stability and positive scalar curvature rigidity of Ricci-flat ALE manifolds*, preprint, arXiv:2009.11854
[9] M. Y. Wang, *Preserving parallel spinors under metric deformations*, Indiana Univ. Math. J. **40** (1991), no. 3, 815–844

Pointwise lower scalar curvature bounds for C^0 metrics via regularizing Ricci flow

Paula Burkhardt-Guim

In 2014, Gromov showed the following theorem on uniform limits of metrics satisfying a pointwise lower bound on the scalar curvature; see [4, p.1118] and [1, Theorem 1]:

Theorem 1. *Let M be a smooth manifold and $\kappa : M \to R$ a continuous function on M. Suppose g_i is a sequence of C^2 metrics on M that converges locally uniformly to a C^2 metric g on M. If $R(g_i) \geq \kappa$ everywhere on M for $i = 1, 2 \ldots$, then $R(g) \geq \kappa$ everywhere on M as well.*

Gromov's proof involves formulating positive scalar curvature as a C^0 quantity, by considering the mean convexity and dihedral angles of small cubes. Bamler provided an alternative proof of Theorem 1 in [1], which used the evolution of the scalar curvature under Ricci flow and some results of Koch and Lamm [6] concerning the Ricci-DeTurck flow for a class of possibly nonsmooth initial data on Euclidean space.

In light of Bamler's approach to Theorem 1, it is natural to ask whether it is possible to use Ricci flow to formulate a generalized definition of lower scalar curvature bounds for C^0 metrics. In [2] we proposed a class of such definitions, and proved some related results. Generally speaking, the lower scalar curvature bounds in [2] are determined by using the Ricci flow to "regularize" the singular metric and then observing the scalar curvature of the flow at small positive times. Our objective was to formulate the definition such that, for any constant κ, we would have

(1) **Stability under greater-than-second-order perturbation:** If g' and g'' are two C^0 metrics that agree to greater than second order around a point x_0, i.e. if, for some fixed smooth background metric, we have $|g'(x) - g''(x)| \leq cd^{2+\eta}(x, x_0)$ for some $c, \eta > 0$ and all x in a neighborhood of x_0, then g' should have scalar curvature bounded below by κ in the weak sense at x_0 if and only if g'' does. Moreover, if g' and g'' are C^0 metrics on different manifolds which merely agree to greater than second order under pullback by a locally defined diffeomorphism, the conclusion should still hold.

(2) **Preservation of global lower bounds under the Ricci flow:** If g is a C^0 metric on a closed manifold that has scalar curvature bounded below by κ in the weak sense at every point, and $\tilde{g}_t$ is a regularizing Ricci flow for g, then $\tilde{g}_t$ should have scalar curvature bounded below by κ at every point for all $t > 0$ for which the flow is defined. This is true for Ricci flows starting from smooth initial data.

(3) **Agreement with the classical notion for C^2 metrics:** If g is a C^2 metric with scalar curvature bounded below by κ at x_0 in the weak sense for C^0 metrics, then g should have scalar curvature bounded below by κ at x_0 in the classical sense. Conversely, if g has scalar curvature bounded below by κ at x_0 in the classical sense, then the same should hold in the weak sense.

We showed that, for C^0 initial data, there exists a Ricci flow in the following sense:

Theorem 2. *Let M be a closed manifold and g_0 a C^0 metric on M. Then there exists a time-dependent family of smooth metrics $(\tilde{g}_t)_{t\in(0,T]}$ and a continuous surjection $\chi : M \to M$ such that the following are true:*

(a) The family $(\tilde{g}_t)_{t\in(0,T]}$ is a Ricci flow, and

(b) There exists a smooth family of diffeomorphisms $(\chi_t)_{t\in(0,T]} : M \to M$ such that

$$\chi_t \xrightarrow[t\to 0]{C^0} \chi \quad \text{and} \quad ||(\chi_t)_*\tilde{g}_t - g_0||_{C^0(M)} \xrightarrow[t\to 0]{} 0.$$

Moreover, for any $x \in M$, $diam_{\{\chi_s(x)\,:\, s\in(0,t]\}} \le C\sqrt{t}$ for some constant $C > 0$ independent of x, where the diameter is measured with respect to a fixed smooth background metric, and any two such families are isometric, in the sense that if $\tilde{g}'_t$ is another such family with corresponding continuous surjection χ', then there exists a stationary diffeomorphism $\alpha : M \to M$ such that $\alpha^*\tilde{g}_t = \tilde{g}'_t$ and $\chi \circ \alpha = \chi'$.

The pair $((\tilde{g}_t)_{t\in(0,T]}, \chi)$ is called a *regularizing Ricci flow* for g_0.

Definition 3. Let M^n be a closed manifold and g_0 a C^0 metric on M. For $0 < \beta < 1/2$ we say that g_0 has scalar curvature bounded below by κ at x in the *β-weak sense* if there exists a regularizing Ricci flow $((\tilde{g}_t)_{t\in(0,T]}, \chi)$ for g_0 such that, for some point $y \in M$ with $\chi(y) = x$, we have

$$\inf_{C>0}\left(\liminf_{t\searrow 0}\left(\inf_{B_{\tilde{g}(t)}(y,Ct^\beta)} R^{\tilde{g}}(\cdot,t)\right)\right) \ge \kappa, \tag{1}$$

where $B_{\tilde{g}(t)}(y, Ct^\beta)$ denotes the ball of radius Ct^β about y, measured with respect to the metric $\tilde{g}(t)$, and $R^{\tilde{g}}(\cdot,t)$ denotes the scalar curvature of $\tilde{g}_t$.

Remark 4. In fact, Definition 3 is independent of choice of y, so it is equivalent to require that (1) hold at y for *all* y with $\chi(y) = x$. Moreover, the fact that any two regularizing Ricci flows for g_0 are isometric (see Theorem 2) implies that Definition 3 holds for some regularizing Ricci flow if and only if it holds for all regularizing Ricci flows for g_0.

It is straightforward to show that Definition 3 satisfies Item 3. Towards Items 1 and 2, we showed:

Theorem 5. *Suppose g' and g'' are two C^0 metrics on closed manifolds M' and M'' respectively, and that there is a locally defined diffeomorphism $\phi : U \to V$ where U is a neighborhood of x_0' in M' and V is a neighborhood of x_0'' in M'' with $\phi(x_0') = x_0''$. Suppose furthermore that g' and $\phi^* g''$ agree to greater than second order around x_0', i.e. with respect to some fixed smooth background metric, $|g'(x) - \phi^* g''(x)| \leq c d^{2+\eta}(x, x_0)$ for some $c, \eta > 0$ and all x in a neighborhood of x_0'. Then there exist regularizing Ricci flows $(\tilde{g}_t', \chi')$ and $(\tilde{g}_t'', \chi'')$ for g' and g'' respectively such that, for $1/(2+\eta) < \beta < 1/2$, $C > 0$, and t sufficiently small depending on C, β, and η, we have*

$$\sup_{B(x_0', Ct^\beta)} |R^{(\chi_t')_* \tilde{g}_t'} - \phi^* R^{(\chi_t'')_* \tilde{g}_t''}| \leq c t^\omega, \tag{2}$$

where ω is some positive exponent, c is a constant that does not depend on t or C, $R^{(\chi_t')_* \tilde{g}_t'}$ and $R^{(\chi_t'')_* \tilde{g}_t''}$ denote the scalar curvatures with respect to $(\chi_t')_* \tilde{g}_t'$ and $(\chi_t'')_* \tilde{g}_t''$ respectively, and (χ_t') and (χ_t'') are the smooth families of diffeomorphisms for $\tilde{g}_t'$ and $\tilde{g}_t''$ respectively, whose existence is given by (b) in Theorem 2.

In particular, Definition 3 holds for g' at x_0' if and only if it holds for g'' and x_0''.

Theorem 6. *Suppose that g_0 is a C^0 metric on a closed manifold M, and suppose there is some $\beta \in (0, 1/2)$ such that g_0 has scalar curvature bounded below by κ in the β-weak sense at all points in M. Suppose also that $(\tilde{g}(t))_{t \in (0,T]}$ is a Ricci flow starting from g_0 in the sense of Theorem 2. Then the scalar curvature of $\tilde{g}(t)$, $R(\tilde{g}(t))$, satisfies $R(\tilde{g}(t)) \geq \kappa$ everywhere on M, for all $t \in (0, T]$.*

We also showed the following:

Theorem 7. *Let g be a C^0 metric on a closed manifold M which admits a uniform approximation by C^2 metrics g^i that have $R(g^i) \geq \kappa_i$, where κ_i is some sequence of numbers such that $\kappa_i \xrightarrow[i\to\infty]{} \kappa$ for some number κ. Then g admits a uniform approximation by smooth metrics with scalar curvature bounded below by κ.*

Question 8. *Is Definition 3 equivalent to Gromov's polyhedral formulation* [4] *of lower scalar curvature bounds for C^0 metric, in the case of global or local lower bounds?*

Question 9. *For metrics in $C^0 \cap W^{1,n}_{loc}$, is Definition 3 equivalent to the notion of nonnegative distributional scalar curvature of* [7]*?*

Question 10. *Suppose $g = g_{ij} dx^i \otimes dx^j$ is a metric on a neighborhood of the origin in $\mathbb{R}^n$, and that we may write*

$$g_{ij}(x) = \delta_{ij} + r^2 G_{ij}(\frac{x}{r}) + O(|x|^{2+\eta}),$$

where the G_{ij} are functions on $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ satisfying $x^i x^j G_{ij}(x) = 0$. Is there an explicit characterization of metrics of this form that have nonnegative scalar curvature at the origin, in the sense of Definition 3?

References

[1] R.H. Bamler, *A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature*, Math. Res. Lett. **23** (2016), 325–337.

[2] P. Burkhardt-Guim, *Pointwise lower scalar curvature bounds for C^0 metrics via regularizing Ricci flow*, Geom. Funct. Anal. **29** (2019), 1703–1772.

[3] P. Burkhardt-Guim, *Defining pointwise lower scalar curvature bounds for C^0 metrics with regularization by Ricci flow*, SIGMA **16** (2020), 10 pages.

[4] M. Gromov, *Dirac and Plateau billiards in domains with corners*, Cent. Eur. J. Math. **12** (2014), 1109–1156.

[5] H. Koch and T. Lamm, *Geometric flows with rough initial data*, Asian J. Math **16** (2012), 209–236.

[6] H. Koch and T. Lamm, *Parabolic equations with rough data*, Math. Bohem. **14** (2015), 457–477.

[7] D.A. Lee and P.G. LeFloch, *The positive mass theorem for manifolds with distributional curvature*, Comm. Math. Phys. **339** (2015), 99–120.

Positive mass theorems of ALF and ALG manifolds

Yuguang Shi

(joint work with Peng Liu, Jintian Zhu)

The positive mass theorem (PMT) of asymptotically flat (AF) manifolds is one of basic results both in geometry of scalar curvature and the General Relativity (see [15] and references therein). It states that the mass of an AF manifolid with nonnegative scalar curvature is nonnegetive and vanishes if and only if the manifold is isometric to the Euclidean space. The philosophy behind PMT is that the flat Minkowski spacetime $\mathbb{R}^{3,1}$ is a trivial solution of Einstein fields equations, it is the solution with the least energy among all those with nonnegative energy density. On the other hand, we do have many nontrivial static solutions of Einstein fields equations. So, it is natural to ask if there are any other PMT on manifolds with more general asymptotical structure at the infinity (see [1], [11], [2] and references therein). From view point of geometry, another motivation to study PMT on manifolds with general asymptotic structure is to explore the Gromov fill-in problems (Problem A,B, p.1 in [4], and Section 3.12.1 in [5]), some interesting works on this fill-in problems on two dimensional case can be find [6], [7], [8], [9], and [13], [12] for higher dimensional case. Indeed, fill-in problems have deep relations with PMT on manifolds (see Theorem 1.3, the proof of Theorem 1.4 in [12]). To state our main result, we need the following

Definition 1. A complete noncompact Riemannian manifold (M^n, g) is ALF with asymptotic order μ if it satisfies:

- there is a compact set $K \subset M$ so that $M - K$ is diffeomorphic to $(\mathbb{R}^{n-1} - \mathbb{B}^{n-1}) \times \mathbb{S}^1$, where $\mathbb{B}^{n-1}$ is the unit ball in $\mathbb{R}^{n-1}$;
- when $n = 3$, $g = dr^2 + \beta^2 r^2 d\phi^2 + d\theta^2 + \sigma$ on $M - K$, $\beta \in \mathbb{R}^+$, (r, ϕ) is the polar coordinates on $\mathbb{R}^2$, θ is the standard coordinate on $\mathbb{S}^1$, we say g is of conical type;

- when $n \geq 4$, the metric g has the expression $g = \left(1 + \frac{m_1}{2r^{n-3}}\right)^{\frac{4}{n-3}} dx^2 + \left(1 + \frac{m_2}{2r^{n-3}}\right)^{\frac{4}{n-3}} d\theta^2 + \sigma, \quad \sigma = o_2(r^{-\mu}), \quad \mu \geq n-3$, here m_1, m_2 are two constants, we say g is of m_1-m_2 type;
- Denote $r = |x|$ with the Euclidean norm $|\cdot|$. The error term σ satisfies

$$\sigma = o_2(r^{-\mu}) \quad \text{as} \quad r \to \infty, \quad \text{with} \quad \mu > \frac{n-3}{2}.$$

Here and in the sequel, the notation $\sigma = o_s(r^{-\mu})$ with $s \in \mathbb{N}_+$ means

$$\sum_{k=0}^{s} r^k |\nabla_{g_0}^k \sigma| = o(r^{-\mu}),$$

where $g_0 = \mathrm{d}x^2 + \mathrm{d}\theta^2$ and ∇_{g_0} is the covariant derivative with respect to the metric g_0.

Remark 2.

- for $n \geq 4$, we may define the total mass of an ALF manifold by the similar integral formula as that in AF case, and get $m(M,g) = (n-2)m_1 + m_2$;
- for $n = 3$, $m(M,g) \geq 0$ if and only if $1 - \beta \geq 0$.

In this talk, we discuss the following main results that were obtained in our preprint [10].

Theorem 3. *For $n \leq 7$, let (M^n, g) be an ALF manifold with scalar curvature $R_g \geq 0$ such that the inclusion map $\pi_1(M-K) \to \pi_1(M)$ is non-trivial. The following statement holds:*

- *If $n = 3$ and (M,g) is of conical type, then $\beta \leq 1$. If we further have $\sigma = o_4(r^{-\mu})$, then equality yields that (M,g) is isometric to $\mathbb{R}^2 \times \mathbb{S}^1$.*
- *If $n \geq 4$ and (M,g) is of m_1-m_2 type with $\mu \geq n-3$, i.e, $g = \left(1 + \frac{m_1}{2r^{n-3}}\right)^{\frac{4}{n-3}} dx^2 + \left(1 + \frac{m_2}{r^{n-3}}\right)^2 d\theta^2 + \sigma$, $\sigma = o_2(r^{-\mu})$, then $m(M,g) \geq 0$. If we further have $\sigma = o_4(r^{-\mu})$, then equality yields that (M,g) is isometric to $\mathbb{R}^{n-1} \times \mathbb{S}^1$.*

Remark 4. The assumption that the inclusion map $\pi_1(M-K) \to \pi_1(M)$ is non-trivial cannot be removed, otherwise there would be a counterexample, see [1] or [11, Page 953].

As an application of Theorem 3 in fill-in problem when Σ is diffeomorphic to $\mathbb{S}^p \times \mathbb{S}^1$, we prove

Theorem 5. *Let Σ_0 be a convex hypersurface or codimension one curve in the Euclidean space $\mathbb{R}^n$ with induced metric, whose total mean or geodesic curvature is T_0. If (Ω, g) is an admissible fill-in of the product manifold $\Sigma_0 \times \mathbb{S}^1(l)$ such that the circle component is homotopically non-trivial in Ω, then for $n \leq 6$ it holds*

$$\int_{\partial\Omega} H_{\partial\Omega} \, \mathrm{d}\sigma_g \leq 2\pi l T_0, \tag{1}$$

where $H_{\partial\Omega}$ *is the mean curvature of* $\partial\Omega$ *with respect to the unit outer normal and* $\mathrm{d}\sigma_g$ *is the area element of* $\partial\Omega$*. If* (Ω, g) *is an admissible fill-in of* $\Sigma_0 \times \mathbb{S}^1(l)$ *with the equality above, then* (Ω, g) *is flat.*

Some similar results for ALG manifolds was also proved in [10].

In fact, PMT for ALF and ALG manifolds deeply related to the following problem which is far from being solved.

Problem 6. *Let* M *be a smooth manifold admits no metric with positive scalar curvature(PSC), under what kind of surgery on* M*, the resulting manifold admits no PSC metric either?*

For surgery on manifolds with PSC metrics see [14], [3].

References

[1] D. Brill, and G.T. Horowitz, *Negative energy in string theory*, Phys. Lett. B **262** (1990), 437–443.

[2] X. Dai, *A positive mass theorem for spaces with asymptotic SUSY compactification*, Comm. Math. Phys. **224** (2004), 335–345

[3] M. Gromov, H.B.Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. **111** (1980), no. 3, 423–434.

[4] M. Gromov., *Scalar curvature of manifolds with boundaries: natural questions and artificial constructions*, 2019, arXiv:1811.04311v2.

[5] M. Gromov., *Four lectures on scalar curvature*, 2021, arXiv:1908.10612v5.

[6] J. Jauregui, *Fill-ins of nonnegative scalar curvature, static metrics, and quasi-local mass*, Pacific J. Math. **261** (2013), 417–444.

[7] J. Jauregui, P. Miao , L.-F.Tam, *Extensions and fill-ins with nonnegative scalar curvature*, Class. Quantum Grav. **30** (2013), 195007.

[8] C. Mantoulidis, P. Miao, *Total mean curvature, scalar curvature, and a variational analog of Brown-York mass*, Comm. Math. Phys. **352** (2017), no. 2, 703–718.

[9] C. Mantoulidis, P. Miao, L.-F.Tam., *Capacity, quasi-local mass, and singular fill-ins*, J. Reine Angew. Math. **768** (2020), 55–92.

[10] P.Liu, Y.G. Shi and J.T., Zhu *Positive mass theorem for ALF and ALG manifolds*, arXiv math.DG 2103.11289 (2021).

[11] V. Minerbe *A mass for ALF manifolds*, Comm. Math. Phys. **289** (2009), 925–955

[12] Y. G. Shi, W. L. Wang , G.D. Wei Total mean curvature of the boundary and nonnegative scalar curvature fill-ins arXiv:2007.06756v2.

[13] Y. G. Shi, W. L. Wang , G.D. Wei, J.T.Zhu, *On the fill-in of nonnegative scalar curvature metrics*, Math. Ann. **379** (2021) 235–270.

[14] R. Schoen R, S.-T.Yau , *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159–183.

[15] R. Schoen and S.-T. Yau, *Positive Scalar Curvature and Minimal Hypersurface Singularities*, arXiv math.DG 1704.05490 (2017).

Hyperbolic positive energy theorems

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(joint work with Erwann Delay and Gregory Galloway)

It is convenient to start this report with a few definitions. We say that a Riemannian manifold (M, g) is *conformally compact* if there exists a compact manifold with boundary $\widehat{M}$ such that the following holds: First, we allow M to have a boundary, which is then necessarily compact. Next, M is the interior of $\widehat{M}$, whose boundary is the union of the boundary of M and of a number of new boundary components, at least one, which form the *conformal boundary at infinity.* Further, there exists on $\widehat{M}$ a smooth function $\Omega \geq 0$ which is positive on M, and which vanishes precisely on the new boundary components of $\widehat{M}$, with $d\Omega$ nowhere vanishing there. Finally, the tensor field $\Omega^2 g$ extends to a smooth metric on $\widehat{M}$.

We will say that a conformally compact manifold (M, g) is *asymptotically locally hyperbolic* (ALH) if all sectional curvatures approach minus one as the conformal boundary at infinity is approached. An ALH metric has an *asymptotically hyperbolic* (AH) component of its boundary at infinity, or an AH end, if the conformal metric on that component of the boundary is conformal to a round sphere.

A useful global invariant of an ALH-but-not-AH end is its mass m, while for AH ends we have an energy-momentum vector $\mathbf{m} \equiv (m_\mu)$ [12, 15, 33] (compare [1, 16]). For this one considers metrics g which asymptote, at a suitable rate, to a background metric $\mathring{g}$. It is assumed that $\mathring{g}$ admits nontrivial *static potentials* which, in dimension n, are defined as solutions of the overdetermined system of equations

$$(1) \qquad \mathring{D}_i \mathring{D}_j V = \big(\mathring{R}_{ij} - \frac{\mathring{R}}{n-1}\mathring{g}_{ij}\big)V\,.$$

Here $\mathring{D}$ is the covariant derivative of the background metric $\mathring{g}$, while $\mathring{R}_{ij}$ is its Ricci tensor and $\mathring{R}$ is the trace $\mathring{g}^{ij}\mathring{R}_{ij}$. To every static potential V and asymptotic end ∂M one associates a mass $m = m(V, \partial M)$ by the formula [22] (compare [4, Equation (IV.40)])

$$(2) \qquad m(V, \partial M) \quad = \quad -\lim_{x\to 0}\int_{\{x\}\times\partial M} D^j V (R^i{}_j - \frac{R}{n}\delta^i_j)\, d\sigma_i\,,$$

where R_{ij} is the Ricci tensor of the metric g, R its trace, and we have ignored an overall dimension-dependent positive multiplicative factor which is often used in the physics literature. Here ∂M is a component of conformal infinity, and x is a coordinate near ∂M so that ∂M is given by the equation $\{x = 0\}$.

The difference between AH ends and general ALH ends arises from the dimension of the space of static potentials. Indeed, the AH case is the only one where this dimension is larger than one. Then $\mathring{g}$ is taken to be the hyperbolic metric, which can be written as the following metric on $\mathbb{R}^n$:

$$(3) \qquad \mathring{g} = \frac{dr^2}{1+r^2} + r^2 d\Omega^2_{n-1}\,,$$

where $d\Omega^2_{n-1}$ is the unit round metric on S^{n-1}. In this coordinate system a basis of the space of static potentials is provided by the functions

$$V_0 = \sqrt{r^2+1}\,, \qquad V_i = x^i\,.$$

One defines the components m_μ of the *energy-momentum vector* $\mathbf{m}$ as

$$m_\mu := m(V_\mu)\,. \tag{4}$$

One checks that $\mathbf{m}$ transforms as a Lorentz vector under conformal transformations of S^{n-1}, so that its Lorentzian norm is a geometric invariant.

For all remaining ALH ends, the mass is directly an invariant [15].

Strictly speaking, a rescaling of V by a constant is always possible, and a preferred scale can be set as follows: In AH ends the standard normalisation is the one just described. In all remaining cases, in any chosen ALH end we can write $\mathring{g}$ as

$$\mathring{g} = x^{-2}(dx^2 + \mathring{h})\,, \quad \mathring{h}(\partial_x, \cdot) = 0\,, \tag{5}$$

with the volume of ∂M, calculated in the metric $\mathring{h}|_{x=0}$, normalised to one. One then normalises V so that $\lim_{x\to 0} xV = 1$.

There is a closely related definition of energy-momentum for *asymptotically flat* general relativistic initial data sets (M, g, K) which, perhaps somewhat unexpectedly, turns out to be relevant for the *asymptotically hyperbolic* problem at hand, and which is invoked in Theorem 1 below, we refer the reader to [3, 7] for details.

In the case of spherical conformal infinity, it has been known that $\mathbf{m}$ is timelike future pointing under a spin condition [12, 13, 21, 28, 33], or under restrictive hypotheses [2, 11]. In my talk in Oberwolfach, summarised here, I reported on results presented in [9, 10] where it is shown how to remove these hypotheses.

The starting point of the analysis in [9] is the following result:

Theorem 1. *Let (M, g) be an asymptotically Euclidean Riemannian manifold, where M is the union of a compact set and of an asymptotically flat region, of dimension $n \geq 3$. Suppose that the general relativistic initial data set (M, g, K) possesses a well defined energy-momentum vector* $\mathbf{m}$*. If the dominant energy condition holds, then* $\mathbf{m}$ *is timelike future pointing or vanishes. Furthermore, in the last case (M, g, K) arises from a hypersurface in Minkowski spacetime.*

A published proof of Theorem 1 in dimensions less than or equal to seven can be found in [17, 19, 25], building upon [29, 30, 31]. A proof covering all dimensions is available in preprint form in [27], with the borderline cases covered in [5, 14, 25]. Conjecturally, this result also follows in all dimensions basing on the preprint [32].

In [9] it is shown how Theorem 1, together with the perturbation results in [11] and the gluing constructions of [8], can be used to remove all unnatural restrictions in the proof of positivity of asymptotically hyperbolic mass:

Theorem 2. *Let (M, g) denote an n-dimensional Riemannian manifold which is the union of a compact set and an AH end. If the scalar curvature $R(g)$ satisfies $R(g) \geq -n(n-1)$, then the energy-momentum vector of (M, g) is causal future pointing, or vanishes.*

The impossibility of a *null* future pointing energy-momentum vector, under the hypotheses above, has been established in [24].

Theorem 2 has been generalised in [10] to allow manifolds with several ends, and with boundaries satisfying an optimal mean-curvature condition:

Theorem 3. *Let (M, g) be a conformally compact n-dimensional, $3 \leq n \leq 7$, asymptotically locally hyperbolic manifold with boundary. Assume that the scalar curvature of M satisfies $R(g) \geq -n(n-1)$, and that the boundary has mean curvature $H \leq n-1$ with respect to the normal pointing into M. Then, the energy-momentum vector $\mathbf{m}$ of every spherical component of the conformal boundary at infinity of (M, g) is future causal.*

In this theorem neither the boundary ∂M, nor the conformal boundary at infinity of M, need to be connected. The proof relies heavily on the results of [18], which assume $3 \leq n \leq 7$.

The above theorems concern AH ends, and one is led to wonder about properties of mass for ALH-but-not-AH ends. Here the following is known: First, positivity is known on manifolds with suitable spin structure [33], or under restrictive conditions [10, 11], but such (M, g) are scarce. Next, boundaryless conformally compact examples with negative mass and toroidal infinity are due to Horowitz and Myers [23]; nontrivial quotients of spheres at infinity with, again, negative mass have been constructed by Chen and Zhang [6]. Finally, a natural negative lower bound, together with a Penrose-type inequality (compare [16, 20]), has been established by Lee and Neves in [26] for a class of three dimensional models with higher genus conformal infinity.

References

[1] L.F. Abbott and S. Deser, *Stability of gravity with a cosmological constant*, Nucl. Phys. **B195** (1982), 76–96.

[2] L. Andersson, M. Cai, and G.J. Galloway, *Rigidity and positivity of mass for asymptotically hyperbolic manifolds*, Ann. H. Poincaré **9** (2008), 1–33, arXiv:math.dg/0703259. MR MR2389888 (2009e:53054)

[3] R. Bartnik, *The mass of an asymptotically flat manifold*, Commun. Pure Appl. Math. **39** (1986), 661–693. MR 849427 (88b:58144)

[4] H. Barzegar, P.T. Chruściel, and M. Hörzinger, *Energy in higher-dimensional spacetimes*, Phys. Rev. D **96** (2017), 124002, 25 pp., arXiv:1708.03122 [gr-qc].

[5] R. Beig and P.T. Chruściel, *Killing vectors in asymptotically flat spacetimes: I. Asymptotically translational Killing vectors and the rigid positive energy theorem*, Jour. Math. Phys. **37** (1996), 1939–1961, arXiv:gr-qc/9510015.

[6] J. Chen and X. Zhang, *Metrics of Eguchi-Hanson types with the negative constant scalar curvature*, Jour. Geom. Phys. **161** (2021), 104010, 10, arXiv:2007.15964 [math.DG]. MR 4180104

[7] P.T. Chruściel, *Boundary conditions at spatial infinity from a Hamiltonian point of view*, Topological Properties and Global Structure of Space–Time (P. Bergmann and V. de Sabbata, eds.), Plenum Press, New York, 1986, pp. 49–59, arXiv:1312.0254 [gr-qc].

[8] P.T. Chruściel and E. Delay, *Exotic hyperbolic gluings*, Jour. Diff. Geom. **108** (2018), 243–293, arXiv:1511.07858 [gr-qc].

[9] P.T. Chruściel and E. Delay, *The hyperbolic positive energy theorem*, (2019), arXiv:1901.05263v1 [math.DG].

[10] P.T. Chruściel and G.J. Galloway, *Positive mass theorems for asymptotically hyperbolic Riemannian manifolds with boundary*, (2021), arXiv:2107.05603 [gr-qc].

[11] P.T. Chruściel, G.J. Galloway, L. Nguyen, and T.-T. Paetz, *On the mass aspect function and positive energy theorems for asymptotically hyperbolic manifolds*, Class. Quantum Grav. **35** (2018), 115015, arXiv:1801.03442 [gr-qc].

[12] P.T. Chruściel and M. Herzlich, *The mass of asymptotically hyperbolic Riemannian manifolds*, Pacific Jour. Math. **212** (2003), 231–264, arXiv:math/0110035 [math.DG]. MR MR2038048 (2005d:53052)

[13] P.T. Chruściel, J. Jezierski, and S. Leski, *The Trautman-Bondi mass of hyperboloidal initial data sets*, Adv. Theor. Math. Phys. **8** (2004), 83–139, arXiv:gr-qc/0307109. MR MR2086675 (2005j:83027)

[14] P.T. Chruściel and D. Maerten, *Killing vectors in asymptotically flat spacetimes: II. Asymptotically translational Killing vectors and the rigid positive energy theorem in higher dimensions*, Jour. Math. Phys. **47** (2006), 022502, arXiv:gr-qc/0512042. MR MR2208148 (2007b:83054)

[15] P.T. Chruściel and G. Nagy, *The mass of spacelike hypersurfaces in asymptotically anti – de Sitter spacetimes*, Adv. Theor. Math. Phys. **5** (2001), 697–754, arXiv:gr-qc/0110014.

[16] P.T. Chruściel and W. Simon, *Towards the classification of static vacuum spacetimes with negative cosmological constant*, Jour. Math. Phys. **42** (2001), 1779–1817, arXiv:gr-qc/0004032.

[17] M. Eichmair, *The Jang equation reduction of the spacetime positive energy theorem in dimensions less than eight*, Commun. Math. Phys. **319** (2013), 575–593, arXiv:1206.2553 [math.dg]. MR 3040369

[18] M. Eichmair, G.J. Galloway, and A. Mendes, *Initial data rigidity results*, (2020), arXiv:2009.09527 [gr-qc].

[19] M. Eichmair, L.-H. Huang, D.A. Lee, and R. Schoen, *The spacetime positive mass theorem in dimensions less than eight*, Jour. Eur. Math. Soc. (JEMS) **18** (2016), 83–121, arXiv:1110.2087 [math.DG]. MR 3438380

[20] G.W. Gibbons, *Some comments on gravitational entropy and the inverse mean curvature flow*, Class. Quantum Grav. **16** (1999), 1677–1687, arXiv:hep-th/9809167. MR 1697098

[21] G.W. Gibbons, S.W. Hawking, G.T. Horowitz, and M.J. Perry, *Positive mass theorem for black holes*, Commun. Math. Phys. **88** (1983), 295–308.

[22] M. Herzlich, *Computing asymptotic invariants with the Ricci tensor on asymptotically flat and asymptotically hyperbolic manifolds*, Ann. Henri Poincaré **17** (2016), 3605–3617, arXiv:1503.00508 [math.DG]. MR 3568027

[23] G.T. Horowitz and R.C. Myers, *The AdS/CFT correspondence and a new positive energy conjecture for general relativity*, Phys. Rev. D **59** (1998), 026005, arXiv:hep-th/9808079.

[24] L.-H. Huang, H.C. Jang, and D. Martin, *Mass rigidity for hyperbolic manifolds*, Commun. Math. Phys. (2019), 1–21, arXiv:1904.12010 [math.DG].

[25] L.-H. Huang and D.A. Lee, *Equality in the spacetime Positive Mass Theorem*, Commun. Math. Phys. **376** (2020), 2379–2407, arXiv:1706.03732 [math.DG]. MR 4104553

[26] D.A. Lee and A. Neves, *The Penrose inequality for asymptotically locally hyperbolic spaces with nonpositive mass*, Commun. Math. Phys. **339** (2015), 327–352. MR 3370607

[27] J. Lohkamp, *The Higher Dimensional Positive Mass Theorem II*, (2016), arXiv:1612.07505 [math.DG].

[28] D. Maerten, *Positive energy-momentum theorem in asymptotically anti-de Sitter spacetimes*, Ann. H.Poincaré **7** (2006), 975–1011, arXiv:math.DG/0506061. MR MR2254757 (2007d:83016)

[29] R. Schoen and S.-T. Yau, *Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity*, Proc. Nat. AcadŚci. U.S.A. **76** (1979), 1024–1025. MR MR524327 (80k:58034)

[30] ———, *On the proof of the positive mass conjecture in general relativity*, Commun. Math. Phys. **65** (1979), 45–76. MR MR526976 (80j:83024)

[31] ______, *Proof of the positive mass theorem. II*, Commun. Math. Phys. **79** (1981), 231–260. MR 612249
[32] ______, *Positive Scalar Curvature and Minimal Hypersurface Singularities*, (2017), arXiv:1704.05490 [math.DG].
[33] X. Wang, *Mass for asymptotically hyperbolic manifolds*, Jour. Diff. Geom. **57** (2001), 273–299. MR MR1879228 (2003c:53044)

Positive Ricci Curvature and Generalized Surgery

Philipp Reiser

Compared to the case of positive scalar curvature, only little is known about which closed manifolds admit a Riemannian metric of positive Ricci curvature. A powerful tool to construct metrics of positive scalar curvature is the surgery theorem of Schoen–Yau [6] and Gromov–Lawson [3] and it is an open question whether a surgery theorem in the same generality holds for metrics of positive Ricci curvature. However, under additional assumptions on the metric and the dimensions involved, similar results for metrics of positive Ricci curvature were obtained by Perelman [4] and Burdick [1] [2] for connected sums and by Sha–Yang [7] and Wraith [9] for higher surgeries. In the following we recall these results and present a generalization of the surgery theorem of Wraith.

To construct metrics of positive Ricci curvature on connected sums, the following notion was introduced by Burdick [1] and is based on work by Perelman [4].

Definition 1. A metric g on a manifold M^n is called a *core metric* if it has positive Ricci curvature and if there exists an embedding $\varphi\colon D^n \hookrightarrow M$ such that

- The restriction of g to $\varphi(S^{n-1})$ is isometric to the round metric on S^{n-1}, and
- The second fundamental form on $\varphi(S^{n-1})$ is positive semi-definite with respect to the inward normal vector of $S^{n-1} \subseteq D^n$.

A consequence of Perelman's work now is the following result.

Theorem 2. *Let M_i^n, $1 \leq i \leq k$, be manifolds that admit core metrics. If $n \geq 4$ then $\#_{i=1}^k M_i$ admits a metric of positive Ricci curvature.*

Hence, by the classical theorem of Bonnet–Myers, a closed manifolds that admits a core metric is simply-connected. It was shown by Burdick [1], [2] that the following manifolds admit a core metric:

- S^n, if $n \geq 2$,
- $\mathbb{C}P^n$, $\mathbb{H}P^n$ and $\mathbb{O}P^2$,
- $M_1^n \# M_2^n$ if both M_1 and M_2 admit a core metric and $n \geq 4$, and
- total spaces of linear sphere bundles $S^{p-1} \hookrightarrow E \longrightarrow B^q$ if $p \geq 4$, $q \geq 3$ and B is a compact manifold that admits a core metric.

Since there are no examples known of closed simply-connected manifolds with a metric of positive Ricci curvature that do not admit a core metric, we can ask the following question:

Question 3. *Does every closed simply-connected manifold that admits a metric of positive Ricci curvature also admit a core metric?*

For higher surgeries Sha and Yang [7] obtained a surgery theorem, which was later extended and modified by Wraith [9] and subsequently generalized by the author [5]. To state it, suppose that

(1) (M^n, g_M) is a Riemannian manifold of positive Ricci curvature,
(2) $\iota\colon S^{p-1}(\rho) \times D^q_R(N) \hookrightarrow M$, $n = p + q - 1$, is an isometric embedding, where $S^{p-1}(\rho)$ denotes the round metric with radius ρ on S^{p-1} and $D^q_R(N)$ denotes the ball of radius R in $S^q(N)$,
(3) $S^{q-1} \hookrightarrow E \xrightarrow{\pi} B^p$ is a linear sphere bundle, where B is a compact manifold that admits a core metric g_B, and
(4) $p, q \geq 3$.

Theorem 4 ([5])**.** *Under the assumptions (1)–(4) there exists a constant $\kappa = \kappa(p, q, R/N, g_B) > 0$ such that if $\frac{\rho}{N} < \kappa$ then the manifold*

$$\hat{M} = M \setminus im(\iota) \cup_\partial \pi^{-1}(B \setminus \varphi(D^p))$$

admits a metric of positive Ricci curvature.

This theorem can be applied whenever M itself is a linear sphere bundle over a compact manifold with a metric of positive Ricci curvature and ι is the embedding of a normal neighborhood of a fiber sphere. By using a submersion metric with totally geodesic and round fibers, we can always achieve that the inequality $\frac{\rho}{N} < \kappa$ is satisfied by shrinking the fibers.

The metric on $\hat{M}$ in the theorem coincides outside a neighborhood of the gluing area with a submersion metric on E with totally geodesic and round fibers of radius r and with a scalar multiple of the metric g_M on M. In particular, we can again apply the theorem to the embedding of a normal neighborhood of a fiber sphere of E in $\hat{M}$, provided r is sufficiently small. By making use of this property we can prove the following.

Corollary 5 ([5])**.** *Let $S^2 \hookrightarrow E \to B^q$ be a linear S^2-bundle. If B is compact and admits a core metric then E admits a core metric, provided $q \geq 4$.*

Concrete applications can be given in dimension 6. Gromov and Lawson [3] showed that every closed simply-connected 6-manifold admits a metric of positive scalar curvature. Hence, we can ask the following question:

Question 6. *Does every closed simply-connected 6-manifold admit a metric of positive Ricci curvature?*

An answer to this question, however, seems to be out of reach at the moment, since, on the one hand, there is no obstruction known for the existence of metrics of positive Ricci curvature on closed simply-connected manifolds that does not already hold for metrics of positive scalar curvature. On the other hand, only few examples of closed simply-connected 6-manifolds that admit a metric of positive Ricci curvature are known. In particular, the only such manifolds with $b_2 > 3$ are connected sums of linear sphere bundles over S^2 or S^3 or have the structure of a linear S^2-bundle over a 4-manifold, see [5, Section 5.2] for an overview.

By performing iterated surgeries, combined with the classification results of Wall [8], we can construct new examples as follows.

Theorem 7 ([5]). *Let M be a closed simply-connected spin 6-manifold with torsion-free homology and let $(x_1, \ldots, x_k)$ be a basis of $H^2(M)$ such that*

(1) $x_i^2 x_j = x_i x_j^2$ for all i, j and $x_i^2 x_j = 0$ if $|i-j| \geq 2$,
(2) $x_i x_j x_k = 0$ for all $i \neq j \neq k \neq i$, and
(3) $p_1(M) \cup x_i = 4x_i^3$ for all i.

Then M admits a core metric.

Note that, by [8], assumption (3) always holds mod 24.
In the non-spin case we also obtain many new examples of manifolds with metrics of positive Ricci curvature. In fact, by performing generalized surgeries using linear S^2-bundles over $\mathbb{C}P^2$, we obtain infinitely many new examples, both spin and non-spin, with arbitrarily large second Betti number, that are not diffeomorphic to any connected sum of the known examples of 6-manifolds admitting a metric of positive Ricci curvature, see [5].

References

[1] Bradley Lewis Burdick. *Ricci-positive metrics on connected sums of projective spaces.* Differential Geom. Appl., 62:212–233, 2019.

[2] Bradley Lewis Burdick. *Metrics of positive Ricci curvature on the connected sums of products with arbitrarily many spheres.* Ann. Global Anal. Geom., 58(4):433–476, 2020.

[3] Mikhael Gromov and H. Blaine Lawson, Jr. *The classification of simply connected manifolds of positive scalar curvature.* Ann. of Math. (2), 111(3):423–434, 1980.

[4] G. Perelman. *Construction of manifolds of positive Ricci curvature with big volume and large Betti numbers.* In Comparison geometry (Berkeley, CA, 1993-94), volume 30 of Math. Sci. Res. Inst. Publ., pages 157–163. Cambridge Univ. Press, Cambridge, 1997.

[5] P. Reiser, *Generalized Surgery on Riemannian Manifolds of Positive Ricci Curvature*, 2021. arXiv: 2103.05517

[6] R. Schoen and S. T. Yau. *On the structure of manifolds with positive scalar curvature.* Manuscripta Math., 28(1-3):159–183, 1979.

[7] Ji-Ping Sha and DaGang Yang. *Positive Ricci curvature on the connected sums of $S^n \times S^m$.* J. Differential Geom., 33(1):127–137, 1991.

[8] C. T. C. Wall *Classification problems in differential topology. V. On certain 6-manifolds.* Invent. Math. I (1966), 335-374.

[9] David Wraith. *Surgery on Ricci positive manifolds.* J. Reine Angew. Math., 501:99–113, 1998.

A positive mass theorem for Riemannian spin manifolds with compact boundary

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(joint work with Rudolf Zeidler)

This talk is a report on joint work in progress with Rudolf Zeidler.

There are two main techniques to study metrics of positive scalar curvature, the Dirac operator method and the minimal hypersurface method. Both techniques have restrictions, the Dirac operator method requiring the manifold to be spin and the minimal hypersurface method requiring the dimension of the manifold to be at most 8, due to singularities occurring in higher dimension. This dimensional restriction has been recently removed by Schoen and Yau, at least in certain geometrically relevant situations; see [18, Theorem 2.6]. One main achievement of the study of metrics of positive scalar curvature is the positive mass theorem. For the notion of asymptotically flat metric and the definition of ADM mass, we refer to [10, § 3.1].

Theorem 1. *Let (M, g) be an asymptotically flat Riemannian manifold with $\mathrm{scal}_g \geq 0$. Then the ADM mass of (M, g) is nonnegative. Furthermore, if the mass is zero, then M is isometric to $\mathbb{R}^n$.*

Remark 2. This theorem was proved by Schoen and Yau [16, 17] in dimension ≤ 7 using the minimal hypersurface method. Witten [20] used the Dirac operator method to extend Theorem 1 to every dimension, under the assumption that the manifold is spin. Schoen and Yau [18] refined the minimal hypersurface method and removed the spin assumption. A different approach to the higher dimensional case has been proposed by Lohkamp [12, 13].

In this talk, we discuss whether the positive mass theorem can be localized to a single asymptotically flat end. More precisely, we discuss the following question.

Question 3. *Let (M, g) be a complete Riemannian manifold with $\mathrm{scal}_g \geq 0$. Suppose $\mathcal{E}$ is an asymptotically flat end of M. Can $\mathcal{E}$ have negative ADM mass?*

This question has been recently addressed by Lesourd, Unger and Yau [11] using the μ-bubble technique. This is a localization technique for the minimal hypersurface method due to Gromov [5, 6, 8]. It has recently led to substantial results in the study of metrics of positive scalar curvature [3, 4, 7, 14, 15, 23, 24].

Let us now state the results of Lesourd, Unger and Yau. For the notion of point of incompleteness and asymptotically Schwarzschild end, we refer to [11, §1, Definitions and conventions].

Theorem 4 (Lesourd-Unger-Yau, [11] Theorem 1.6)**.** *Let (M^n, g), $3 \leq n \leq 7$, be an asymptotically Schwarzschild manifold, not assumed to be complete or have nonnegative scalar curvature everywhere. Let U_1 and U_2 be neighborhoods of infinity with $\overline{U}_2 \subset U_1$ and let D be a positive constant. Moreover, suppose that*

(1) g has no point of incompleteness in the D-neighborhood of U_1,
(2) $\mathrm{scal}_g \geq 0$ in the D-neighborhood of U_1, and

(3) the scalar curvature satisfies

$$scal_g > \frac{32}{D}\left(\frac{8}{D} + \frac{1}{dist_g(U_2, \partial U_1)}\right)$$

on $\overline{U}_1 \setminus U_2$.

Then the ADM mass is nonnegative.

Remark 5. Let us now explain the relationship between Theorem 4 and Question 3. Suppose (M, g) is a complete Riemannian manifold of nonnegative scalar curvature and suppose $\mathcal{E} \subset M$ is an asymptotically Schwarzschild end with negative mass. Using [11, Lemma 5.1], the scalar curvature of g can be taken strictly positive on $\mathcal{E}$. Therefore, we can choose neighborhoods of infinity U_1 and U_2 in $\mathcal{E}$ such that scal_g is uniformly positive in $\overline{U}_1 \setminus U_2$ and, since M is complete, we can pick D large enough such that the hypotheses of Theorem 4 are satisfied. This implies the following consequence.

Theorem 6 (Lesourd-Unger-Yau, [11] Theorem 1.2)**.** *Let* (M^n, g)*, with* $3 \leq n \leq 7$*, be a complete noncompact Riemannian manifold with nonnegative scalar curvature. Suppose* $\mathcal{E} \subset M$ *is an asymptotically Schwarzschild end. Then the ADM mass of* $\mathcal{E}$ *is nonnegative.*

Our main result consists in extending Theorem 4 to the spin setting using the Dirac operator method. More precisely, we use potentials to localize the Witten proof of the positive mass theorem; see [20]. Localization techniques with Dirac operators have been successfully used to address questions recently raised by Gromov about metrics of positive scalar curvature; see [1, 2, 9, 19, 21, 22].

Let us now formulate our results. Note that in Theorem 4, outside the D-neighbor-hood of U_2, there are no assumptions of completeness or nonnegativity of the metric g. This suggests to study the following question.

Question 7. *Let* X *be an asymptotically flat Riemannian manifold with compact boundary and nonnegative scalar curvature. Is it possible to give metric conditions, in terms of a positive lower bound of the scalar curvature in a certain region and of the distance between this region and* ∂X*, in such a way that the ADM mass of* X *must be nonnegative?*

Note that we do not make any assumption on the metric at the boundary. Instead, we follow the point of view recently proposed by Gromov in the study of metrics of positive scalar curvature on manifolds with boundary and formulate metric conditions in terms of the lower bound of the scalar curvature in a certain region and of the distance between this region and the boundary. Our main result extends Theorem 4 to the spin setting, without dimensional restrictions.

Theorem 8 (C.-Zeidler)**.** *Let* (X, g) *be an asymptotically flat spin manifold with compact boundary whose scalar curvature is nonnegative. Let* $\Sigma \subset X^\circ$ *be a closed separating hypersurface with associated partition* $X_- \cup_\Sigma X_+$*, where* X_- *is an asymptotically flat manifold with boundary* $\partial X_- = \Sigma$ *and where* X_+ *is a compact* *manifold with boundary* $\partial X_+ = \Sigma \sqcup \partial X$*. Let* $\mathcal{N}_\delta(\Sigma) \subset X_-$ *be a collar neighborhood of* Σ *of width* δ*. Suppose that* $scal_g \geq \kappa_0 > 0$ *on* $\mathcal{N}_\delta(\Sigma)$*. Then there exists a constant* $\Lambda > 0$*, depending only on* δ *and* κ_0*, such that, if* $dist_g(\Sigma, \partial X) \geq \Lambda$*, then the ADM mass of each end of* X *is nonnegative.*

Remark 9. The collar neighborhood $\mathcal{N}_\delta(\Sigma)$ can be replaced by a "separating band" of width δ, that is, a compact manifold Y with boundary $\partial Y = \Sigma_1 \sqcup \Sigma$ sitting inside X_- such that $\mathrm{dist}_g(\Sigma_1, \Sigma) = \delta$.

Remark 10. One can ask what happens when the upper bound Λ is not achieved, that is, when $\mathrm{dist}_g(\Sigma, \partial X) < \Lambda$. Using the techniques developed in [2], it is possible to refine Theorem 8 and obtain metric inequalities relating Λ to κ_0, δ and the mean curvature of ∂X.

In the case of a complete spin manifold with an asymptotically Schwarzschild end, using [11, Lemma 5.1] in a similar fashion as in Remark 5, from Theorem 8 we deduce the following consequence.

Theorem 11 (C.-Zeidler)**.** *Let* (M^n, g) *be a complete noncompact Riemannian spin manifold with nonnegative scalar curvature. Suppose* $\mathcal{E} \subset M$ *is an asymptotically Schwarzschild end. Then the ADM mass of* $\mathcal{E}$ *is nonnegative.*

References

[1] S. Cecchini, *A long neck principle for Riemannian spin manifolds with positive scalar curvature*, GAFA, **30**, 5 (2020), 1183–1223.

[2] S. Cecchini and R. Zeidler, *Scalar and mean curvature comparison via the Dirac operator*, arXiv:2103.06833 (2021).

[3] O. Chodosh and C. Li, *Generalized soap bubbles and the topology of manifolds with positive scalar curvature*, arXiv:2008.11888 (2020).

[4] O. Chodosh, C. Li, and Y. Liokumovich, *Classifying sufficiently connected PSC manifolds in 4 and 5 dimensions*, arXiv:2105.07306 (2021).

[5] M. Gromov, *Positive curvature, macroscopic dimension, spectral gaps and higher signatures*, Functional analysis on the eve of the 21st century, Vol. II (New Brunswick, NJ, 1993). Progr. Math., vol. 132, Birkhäuser, Boston, MA, 1996, pp. 1–213.

[6] M. Gromov, *Metric inequalities with scalar curvature*, GAFA **28**, 6 (2018), 645–726.

[7] M. Gromov, *No metrics with positive scalar curvature on aspherical 5-manifolds*, arXiv:2009.05332 (2020).

[8] M. Gromov, *Four lectures on scalar curvature*, arXiv:1908.10612v6 (2021).

[9] H. Guo, Z. Xie, and G. Yu, *Quantitative K-theory, positive scalar curvature, and band width* In: Perspectives in Scalar Curvature, Ed. by M. Gromov and B. Lawson, To appear, World Scientific, 2021. arXiv:2010.01749.

[10] Dan A. Lee, *Geometric Relativity*, Graduate Studies in Mathematics, Vol. 201, AMS, 2019.

[11] M. Lesourd, R. Unger and S.-T. Yau, *The Positive Mass Theorem with Arbitrary Ends*, 2021. arXiv:2103.02744 (2021).

[12] J. Lohkamp, *The higher dimensional positive mass theorem I*, arXiv:math/0608795 (2016).

[13] J. Lohkamp, *The higher dimensional positive mass theorem II*, arXiv:1612.07505 (2017).

[14] D. Räde, *Macroscopic band width inequalities*, to appear in Algebr. Geom. Topol., arXiv:1911.13000 (2021).

[15] D. Räde, *Scalar and mean curvature comparison via μ-bubbles*, arXiv:2104.10120 (2021).

[16] R. Schoen and S.-T. Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76.

[17] R. Schoen and S.-T. Yau, *Complete manifolds with nonnegative scalar curvature and the positive action conjecture in general relativity*, Proc. Nat. Acad. Sci. U.S.A. **76** (1979), no. 3, 1024–1025.
[18] R. Schoen and S.-T. Yau, *Positive scalar curvature and minimal hypersurface singularities*, arXiv:1704.05490 (2017).
[19] J. Wang, Z. Xie, and G. Yu, *A proof of Gromov's cube inequality on scalar curvature*, arXiv:arXiv:2105.12054 (2021).
[20] E. Witten, *A new proof of the positive energy theorem*, Comm. Math. Phys. **80** (1981), no. 3, 381–402.
[21] Z. Xie, *A quantitative relative index theorem and Gromov's conjectures on positive scalar curvature*, arXiv:2103.14498 (2021).
[22] R. Zeidler, *Band width estimates via the Dirac operator*, To appear in J. Differential Geom, arXiv:1905.08520 (2019).
[23] J. Zhu, *Rigidity results for complete manifolds with nonnegative scalar curvature*, arXiv:2008.07028 (2020).
[24] J. Zhu, *Width estimate and doubly warped product*, Trans. Amer. Math. Soc. **374**, 2 (2021), 1497–1511.

Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the Riemannian Penrose Inequality

Christos Mantoulidis
(joint work with Chao Li)

In all that follows, M denotes a closed n-dimensional manifold and $\mathrm{Met}(M)$ denotes the space of smooth Riemannian metrics on M. For $k \in (0, \infty)$, we define

$$\mathcal{M}_k^{\geq 0}(M) := \{g \in \mathrm{Met}(M) : \lambda_1(-\Delta_g + kR_g) \geq 0\},$$

where $\lambda_1(-\Delta_g + kR_g)$ is the first eigenvalue of the operator $-\Delta_g + kR_g$ on M, and R_g is the scalar curvature of g. We also define

$$\mathcal{M}_\infty^{\geq 0}(M) := \{g \in \mathrm{Met}(M) : R_g \geq 0\}.$$

Finally, we define $\mathcal{M}_k^{>0}(M)$, $k \in (0, \infty]$, as above with all "$\geq$" replaced by "$>$." Note that, for $0 < k < k' \leq \infty$,

$$\begin{array}{ccc} \mathcal{M}_{k'}^{>0}(M) & \subset & \mathcal{M}_{k'}^{\geq 0}(M) \\ \cap & & \cap \\ \mathcal{M}_k^{>0}(M) & \subset & \mathcal{M}_k^{\geq 0}(M). \end{array}$$

These spaces are not generally encountered in the literature in this level of generality, so some remarks are in order about their actual geometric significance. This is discussed extensively in our paper [10]. For the purpose of this brief report, we simply highlight that for $k = \frac{1}{2}$ these spaces encode apparent horizons in time-symmetric initial data sets to Einstein's equations with the dominant energy condition, and that $\mathcal{M}_k^{>0}(M) \neq \emptyset$ for $k = \frac{n-1}{4(n-2)}$ if $n \geq 3$ and M is topologically PSC (i.e., its Yamabe constant is positive).

Our starting point was a generalization of a theorem of Codá Marques [12], who proved that the ultimate space in the filtration has a connected *moduli space*, i.e.,

$$\mathcal{M}^{>0}_{\infty}(M)/\operatorname{Diff}_+(M) \text{ is path-connected,}$$

when M is a closed orientable 3-manifold. We proved:

Theorem 1. *Let M be a closed orientable topologically PSC 3-manifold. Then, $\mathcal{M}^{>0}_{k}(M)/\operatorname{Diff}_+(M)$, $\mathcal{M}^{\geq 0}_{k}(M)/\operatorname{Diff}_+(M)$ are path-connected for all $k \in [\frac{1}{4}, \infty]$.*

To prove Theorem 1 we needed a suitable generalization of the Gromov–Lawson surgery process [8] (cf. Schoen–Yau's [13]) from $\mathcal{M}^{>0}_{\infty}(M)$ to $\mathcal{M}^{>0}_{k}(M)$. Such a surgery was first carried out by Bär–Dahl in [3, Theorem 3.1], and we give a full independent proof of it with some added details in an appendix to our paper.

The recent breakthrough of Bamler–Kleiner [4] on the path-connectedness of $\mathcal{M}^{>0}_{\infty}(M)$ implies the following two companion results when used in conjunction with Theorem 1 and, separately, the conformal method:

Theorem 2. *Let M be a closed orientable topologically PSC 3-manifold. Then, $\mathcal{M}^{>0}_{k}(M)$ and $\mathcal{M}^{\geq 0}_{k}(M)$ are path-connected for all $k \in [\frac{1}{4}, \infty]$.*

Theorem 3. *Let M be a closed orientable topologically PSC 3-manifold. Then, $\mathcal{M}^{>0}_{1/8}(M)$ is contractible and $\mathcal{M}^{\geq 0}_{1/8}(M)$ is weakly contractible.*

Our main application of these results is to the computation of the Bartnik mass of apparent horizons, and its generalization due to Bray. For n-dimensional closed orientable (M^n, g), the apparent horizon Bartnik mass is defined as

$$\mathfrak{m}_B(M, g, H = 0) = \inf\{\mathfrak{m}_{ADM}(\mathbf{M}, \mathbf{g}) : (\mathbf{M}, \mathbf{g}) \in \mathcal{E}_B(M, g, H = 0)\},$$

where $\mathcal{E}_B(M, g, H = 0)$ is the set of complete, connected, asymptotically flat $(\mathbf{M}, \mathbf{g})$ with nonnegative scalar curvature, no closed interior minimal hypersurfaces, and minimal $(H = 0)$ boundary isometric to (M, g). Such $(\mathbf{M}, \mathbf{g})$ are initial data sets for solutions of Einstein's equations with the dominant energy condition, and $\mathfrak{m}_{ADM}(\mathbf{M}, \mathbf{g})$ is the ADM mass of the initial data set [2, 1]. Using a rearrangement trick of Schoen–Yau and a delicate splitting theorem of Galloway, it follows that:

$$\mathcal{E}_B(M, g, H = 0) \neq \emptyset \implies M \text{ is topologically PSC, } g \in \mathcal{M}^{\geq 0}_{1/2}(M).$$

Thus, we are precisely in the context studied by Theorems 1, 2, 3.

There exists a nontrivial lower bound for $\mathfrak{m}_B(M, g, H = 0)$ by Bray [6] and Bray–Lee's [5] Riemannian Penrose Inequality, which says:

$$(\mathbf{M}, \mathbf{g}) \in \mathcal{E}_B(M, g, H = 0) \implies \mathfrak{m}_{ADM}(\mathbf{M}, \mathbf{g}) \geq \tfrac{1}{2}(\sigma_n^{-1} \operatorname{vol}_g(M))^{(n-1)/n},$$

when $2 \leq n \leq 6$ and σ_n is the volume of the standard round $\mathbf{S}^n$; see also Huisken–Ilmanen [9] in case $n = 2$ and M is connected. Thus of course

$$\mathfrak{m}_B(M, g, H = 0) \geq \tfrac{1}{2}(\sigma_n^{-1} \operatorname{vol}_g(M))^{(n-1)/n},$$

We computed the left hand side to be a topological invariant when $n = 3$ and M is connected. (When $n = 2$, this is due to M.–Schoen [11], Chau–Martens [7].)

Theorem 5. *For a closed connected topologically PSC 3-manifold M, either:*

- $\mathcal{E}_B(M, g, H = 0) = \emptyset$
- $\mathcal{E}_B(M, g, H = 0) \neq \emptyset$ *and* $\mathfrak{m}_B(M, g, H = 0) = \mathfrak{c}_B(M) \operatorname{vol}_g(M)^{2/3}$,

for all $g \in \mathcal{M}^{\geq 0}_{1/2}(M)$. *Here,* $\mathfrak{c}_B(M)$ *is a topological constant and* $\mathfrak{c}_B(\mathbf{S}^3) = \frac{1}{2}\sigma_3^{-2/3}$.

Unfortunately, the precise value of the apparent horizon Bartnik mass remains unknown for:

- disconnected 2- or 3-dimensional M;
- 3-dimensional M with nontrivial topology;
- all higher dimensional M, except for certain special metrics on $M = SS^n$.

While we do not have satisfactory answers for the Bartnik mass for these bullet points at this time, we know how to compute a relaxation of Bartnik's mass due to Bray [6] in near-complete generality. In this relaxation, the set $\mathcal{E}_{BB}(M, g, H = 0)$ of extensions considered is such that the boundary (M, g) is outer-minimizing minimal, rather than outermost minimal. The Bartnik–Bray mass $\mathfrak{m}_{BB}(M, g, H = 0)$ is then defined analogously. We showed:

Theorem 6. *Let M be a closed orientable topologically PSC n-manifold with $2 \leq n \leq 6$. Consider the subset of $\mathcal{M}^{\geq 0}_{1/2}(M)$ given by:*

$$\begin{aligned}\operatorname{LinClos}[\mathcal{M}^{>0}_{1/2}(M)] := \{g \in \mathcal{M}^{\geq 0}_{1/2}(M) : \ & \textit{there exists a } C^1 \textit{ path} \\ & [0,1) \ni t \mapsto g(t) \textit{ with } g(0) = g \textit{ and} \\ & \left[\tfrac{d}{dt}\lambda_1(-\Delta_{g(t)} + \tfrac{1}{2}R_{g(t)})\right]_{t=0} > 0\}.\end{aligned}$$

If $g \in \operatorname{LinClos}[\mathcal{M}^{>0}_{1/2}(M)]$ *and* $\mathcal{E}_{BB}(M, g, H = 0) \neq \emptyset$, *then*

$$\mathfrak{m}_{BB}(M, g, H = 0) = \tfrac{1}{2}(\sigma_n^{-1} \operatorname{vol}_g(M))^{(n-1)/n}.$$

We emphasize that M need not be connected and that our computation is valid as long as a single Bartnik–Bray extension exists. Note that it is known that

$$\mathcal{M}^{\geq 0}_{1/2}(M) \setminus \operatorname{LinClos}[\mathcal{M}^{>0}_{1/2}(M)] \subset \{g \in \operatorname{Met}(M) : \operatorname{Ric}_g \equiv 0\},$$

which is empty when $n = 2$, 3, and small for larger n.

References

[1] R. Arnowitt, S. Deser, and C. W. Misner, *Dynamical structure and definition of energy in general relativity*, Phys. Rev. (2) **116** (1959), 1322–1330. MR 113667

[2] ———, *Canonical variables for general relativity*, Phys. Rev. (2) **117** (1960), 1595–1602. MR 113668

[3] C. Bär and M. Dahl, *Small eigenvalues of the conformal Laplacian*, Geom. Funct. Anal. **13** (2003), no. 3, 483–508. MR 1995796

[4] Richard H. Bamler and Bruce Kleiner, *Ricci flow and contractibility of spaces of metrics*, 2019.

[5] Hubert L. Bray and Dan A. Lee, *On the Riemannian Penrose inequality in dimensions less than eight*, Duke Math. J. **148** (2009), no. 1, 81–106. MR 2515101

[6] Hubert L. Bray, *Proof of the Riemannian Penrose inequality using the positive mass theorem*, J. Differential Geom. **59** (2001), no. 2, 177–267. MR 1908823

[7] Albert Chau and Adam Martens, *Exterior schwarzschild initial data for degenerate apparent horizons*, 2020.
[8] Mikhael Gromov and H. Blaine Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **111** (1980), no. 3, 423–434. MR 577131
[9] Gerhard Huisken and Tom Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59** (2001), no. 3, 353–437. MR 1916951
[10] Chao Li, Christos Mantoulidis, *Metrics with $\lambda_1(-\Delta + kR) \geq 0$ and flexibility in the Riemannian Penrose Inequality*, 2021.
[11] Christos Mantoulidis and Richard Schoen, *On the Bartnik mass of apparent horizons*, Classical Quantum Gravity **32** (2015), no. 20, 205002, 16. MR 3406373
[12] Fernando Codá Marques, *Deforming three-manifolds with positive scalar curvature*, Ann. of Math. (2) **176** (2012), no. 2, 815–863. MR 2950765
[13] R. Schoen and S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), no. 1-3, 159–183. MR 535700
[14] Richard Schoen and Shing Tung Yau, *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. **65** (1979), no. 1, 45–76. MR 526976

Topological rigidity and positive scalar curvature

Jian Wang

Scalar curvature is a weak invariant of the local geometry but delicately linked with the global topology of a smooth manifold. Classically, there are several topological obstructions for a smooth manifold to have a complete metric with positive scalar curvature (see [4, 5] and [11, 12, 13, 14]).

A long-standing question was addressed by S.T. Yau in his problem section [15]:

Question 1. [15] *Is it possible to classify* 3*-manifold admitting complete positive scalar curvature metrics up to diffeomorphisms?*

It was done by G. Perelman for the compact case. From his proof for the celebrated Pioncaré conjecture ([8, 9, 10]), a closed and orientable 3-manifold admits a metric with positive scalar curvature if and only if it is a connected sum of spherical 3-manifolds (i.e. a quotient of $\mathbf{S}^3$ by a finite subgroup of $\mathbf{O}(4)$) and some copies of $\mathbf{S}^1 \times \mathbf{S}^2$.

However, the topological structure of non-compact 3-manifolds is much more complicated. The simplest case is that of contractible 3-manifolds, for example, the Whitehead manifold which is a contractible 3-manifold but not homeomorphic to $\mathbf{R}^3$ (see [19]).

Among the contractible 3-manifolds, $\mathbf{R}^3$ is the only one known to carry a complete metric with positive scalar curvature, for example g_0, defined by

$$g_0 = \sum_{i=1}^{3} dx_i^2 + (\sum_{i=1}^{3} x_i dx_i)^2.$$

This suggests the following topological rigidity question:

Question 2. *Is any complete contractible* 3*-manifold with positive scalar curvature homeomorphic to* $\mathbf{R}^3$*?*

Question 2 is deeply linked with the classifying question for (non-compact) 3-manifolds with positive scalar curvature. It had been intensively studied over recent decades, leading a series of interesting discoveries (see [5], [11, 12, 13, 14], [3] and [16, 17, 18]).

In a series of works [16, 17], we established a list of results concerning the non-existence of complete metrics with positive scalar curvature. One of results give a definite answer for the Whitehead manifold.

Theorem 3. [16] *The Whitehead manifold does not admit a complete metric with positive scalar curvature.*

In [18], we give a full and positive answer to Question 2

Theorem 4. [18]*A complete contractible* 3*-manifold with positive scalar curvature is homeomorphic to* $\mathbf{R^3}$.

Theorem 4 was also proved by Gromov-Lawson [5] and Chang-Weinberger-Yu [3] when the scalar curvature is uniformly positive (i.e. bounded below by a strictly positive constant). To be precise, M. Gromov and B. Lawson showed that it is *simply-connected at infinity* and then one needs the Poincaré conjecture (see [1, 7]). It also follows with the work [2] of L. Bessières, G. Besson and S. Maillot when the geometry is bounded.

The proof of Theorem 4 totally depends on the positivity of scalar curvature. We use a metric deformation due to J. Kazdan [6] to generalize it to the non-negative scalar curvature case.

Corollary 5. [18] *A complete contractible* 3*-manifold with non-negative scalar curvature is homeomorphic to* $\mathbf{R}^3$.

References

[1] L. Bessières, G. Besson, M. Boileau, S. Maillot, J. Porti, *Geometrisation of 3-manifolds*, European Mathematical Society, **Volume 13** (2010).

[2] L. Bessières, G. Besson, and S. Maillot, *Ricci flow on open 3-manifolds and positive scalar curvature*, Geom. Topol. , **15(2)** (2011), 927–975.

[3] S. Chang, S. Weinberger, and G. Yu, *Taming 3-manifolds using scalar curvature*, Geometriae Dedicata, **148(1)**(2010), 3–14.

[4] M. Gromov, B. Lawson, *The classification of simply connected manifolds of positive scalar curvature*, Annals of Mathematics, **111(3)** (1980), 423–434.

[5] M. Gromov, B. Lawson, *Positive scalar curvature and the dirac operator on complete riemannian manifolds*, Publications Mathématiques de l'Institut des Hautes Études Scientifiques, **58(1)** (1983), 83–196.

[6] J. Kazdan, *Deformation to positive scalar curvature on complete manifolds*, Mathematische Annalen, **261(2)**(1982), 227–234.

[7] J. Morgan, G. Tian, *Ricci flow and the Poincaré conjecture*, American Mathematical Soc., **Volume 3** (2007).

[8] G. Perelman, *The entropy formula for ricci flow and its geometric applications*, arXiv preprint math/0211159, 2002.

[9] G. Perelman, *Ricci flow with surgery on three-manifolds*, arXiv preprint math/0303109, 2002.

[10] G. Perelman, *Finite extinction time for the solutions to the ricci flow on certain three-manifolds*, arXiv preprint math/0307245, 2003.

[11] R. Schoen, S.T. Yau, *Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature*, Annals of Mathematics, **110(1)** (1979).
[12] R. Schoen, S.T. Yau, *On the proof of the positive mass conjecture in general relativity.* Communications in Mathematical Physics, **65(1)** (1979),45–76.
[13] R. Schoen, S.T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta mathematica, **28**(1979)159?183.
[14] R. Schoen, S.T. Yau, *Complete three-dimensional manifolds with positive ricci curvature and scalar curvature*, In Seminar on Differential Geometry, **Volume 102**(1982), 209–228, Princeton Univ. Press.
[15] S.T. Yau, *Problem section*, In Seminar on Differential Geometry, volume 102 of Ann. of Math. Stud. (1982), 669–706. Princeton Univ. Press, Princeton, N.J.
[16] J. Wang, *Contractible 3-manifolds and positive scalar curvatures (I)*, arXiv:1901.04605, 2019.
[17] J. Wang, *Contractible 3-manifolds and positive scalar curvatures (II)*, arXiv:1906.04128, 2019.
[18] J. Wang, *Topological rigidity and positive scalar curvature*, arXiv:2105.07095, 2021.
[19] J.H.C. Whitehead, *A certain open manifold whose group is unity*, The Quarterly Journal of Mathematics, **6(1)**(1935), 268–279.

On the homotopy type of the space of positive scalar curvature metrics

Michael Wiemeler
(joint work with Johannes Ebert)

When studying Riemannian metrics of positive scalar curvature, there are two basic questions:

(1) Given a closed manifold M^d , is there a metric of positive scalar curvature on M^d?
(2) If the answer to the first question is "yes", what can be said about the topology of the space $\mathcal{R}^+(M^d)$ of positive scalar curvature metrics on M^d?

For simply connected manifolds of dimension $d \geq 5$ the first question has the following answer: If M^d is non-spin then it always admits a metric of positive scalar curvature by results of Gromov and Lawson [2] and Schoen and Yau [7]. For spin manifolds M^d, there is an index-theoretic necessary condition for M d to admit a psc metric. Namely, the â-invariant $\hat{\mathfrak{a}}(M) \in ko_d$ has to be trivial ($\hat{\mathfrak{a}}(M)$ only depends on the spin cobordism class $[M]$ of M). A celebrated result of Stolz [8] states that this condition is also sufficient, when M^d is simply connected and $d \geq 5$. In this talk we discussed the second question. Our first main result is as follows.

Theorem 1. *Let M be a simply connected closed spin manifold of dimension $d \geq 5$. Then if M admits a psc metric, there is a homotopy equivalence*

$$\mathcal{R}^+(M) \simeq \mathcal{R}^+(S^d).$$

Here S^d denotes the d-dimensional sphere with standard smooth structure.

Previously it has been known by results of Chernysh [1] and Walsh [9] that the homotopy type of $\mathcal{R}^+(M)$ only depends on the spin bordism type of the simply connected spin manifold M. The main technical step in the proof of this result of Chernysh and Walsh, is to show that the space of positive scalar curvature metrics on M is homotopy equivalent to the space of such metrics which have some standard form near a submanifold of codimension at least 3 with trivial normal bundle.

Recently Kordaß [5] showed that $\mathcal{R}^+(\mathbb{H}P^2) \simeq \mathcal{R}^+(S^8)$. The proof of this result of Kordaß and of Theorem 1 uses an extension of the above mentioned technical result of Chernysh to submanifolds with non-trivial normal bundle due to Kordaß. Using this extension one can show that $\mathcal{R}^+(M)$ is homotopy equivalent to $\mathcal{R}^+(S^d)$ whenever M is spin bordant to a manifold which decomposes as a union of two disc bundles with fiber dimensions at least 3 and suitable metrics on their common boundary.

This reduces the proof to finding manifolds which decompose as unions of disc bundles in those spin bordism classes which contain manifolds with positive scalar curvature metrics. By a result of Stolz [8] and Kreck–Stolz [6] these bordism classes can be represented by total spaces of $\mathbb{H}P^2$-bundles with structure group $PSp(3)$, the isometry group of $\mathbb{H}P^2$ with the Fubini-Study metric. When the structure group of such a bundle can be reduced to $P(Sp(2) \times Sp(1))$, then the total space decomposes as a union of two disc bundles of fiber dimension 8 and 4, respectively. Therefore it suffices to prove that such a reduction of structure group is always possible up to bordism and we prove this.

This strategy of proof in principle also works for simply connected non-spin manifolds. For these we have

Theorem 2. *Let M be a simply connected closed manifold of dimension $d \geq 5$ which does not admit a spin structure. Then if $d \neq 8$, there is a homotopy equivalence*

$$\mathcal{R}^+(M) \simeq \mathcal{R}^+(W^d).$$

If $d = 8$, there is either a homotopy equivalence $\mathcal{R}^+(M) \simeq \mathcal{R}^+(W^8)$ or $\mathcal{R}^+(M) \simeq \mathcal{R}^+(\mathbb{C}P^2 \times \mathbb{C}P^2)$.

Here W^d is the nontrivial S^{d-2}-bundle over S^2 with structure group $SO(d-1)$.

For certain groups π we also get the uniqueness of the homotopy type of the space of positive scalar curvature metrics on spin manifolds with fundamental group π in a given dimension at least five. The set of groups for which this holds includes finitely generated free abelian groups.

For finite π there are only finitely many homotopy types of the space of positive scalar curvature metrics on spin manifolds with fundamental group π in a given dimension at least five.

The above results naturally lead to the following conjecture.

Conjecture 3. Let M and N be two closed d-manifolds, $d \geq 5$, with the same normal 2-type. If both, M and N, admit psc metrics, then $\mathcal{R}^+(M) \simeq \mathcal{R}^+(N)$.

Here two manifolds M_i, $i = 1, 2$, are said to have the same normal 2-type, if there is a fibration $\xi : B \to BO$ such that the classifying maps $\nu_i : M_i \to BO$ of the stable normal bundles lift to maps $\bar{\nu}_i : M_i \to B$ and both $\bar{\nu}_i$ are 2-connected.

It has been observed in [3] that the concordance–implies–isotopy conjecture for psc metrics implies our above conjecture.

The proofs of the above two theorems appeared in the preprint [4].

References

[1] V. Chernysh, *On the homotopy type of the space $\mathcal{R}^+(M)$*, Preprint (2004), arXiv:math/0405235

[2] M. Gromov, M. and H. B. Lawson, Jr., *The classification of simply connected manifolds of positive scalar curvature*, Ann. of Math. (2), **111** (1980), 423–434

[3] J. Ebert and O. Randal-Williams, *The positive scalar curvature cobordism category*, Preprint (2019), arXiv:1904.12951

[4] J. Ebert and M. Wiemeler, *On the homotopy type of the space of positive scalar curvature metrics*, Preprint (2020), arXiv:2012.00432

[5] J.-B. Kordaß *On the space of riemannian metrics satisfying surgery stable curvature conditions*, Preprint (2018), arXiv:1808.00581

[6] M. Kreck and S. Stolz $\mathbf{H}P^2$*-bundles and elliptic homology*, Acta Math., **171** (1993), 231–261

[7] R. Schoen and S. T. Yau, *On the structure of manifolds with positive scalar curvature*, Manuscripta Math. **28** (1979), 159–183

[8] S. Stolz, *Simply connected manifolds of positive scalar curvature*, Ann. of Math. (2) **136** (1992), 511–540

[9] M. Walsh, *Cobordism invariance of the homotopy type of the space of positive scalar curvature metrics*, Proc. Amer. Math. Soc. **141** (2023), 2475–2484

How to solve the constraint equations

Romain Gicquaud

Given a manifold M of dimension n, the constraint equations are equations for pairs $(\widehat{g}, \widehat{K})$, where $\widehat{g}$ is a Riemannian metric and $\widehat{K}$ is a symmetric 2-tensor field on M, that appear in the study of the Cauchy problem in general relativity.

Indeed, general relativity describes the spacetime as a $(n+1)$-dimensional manifold $\mathcal{M}$ endowed with a Lorentzian metric h (i.e. h has signature $(-+\cdots+)$). In the absence of any other field (matter fields, electromagnetic field...), the Einstein equation imposes that the metric h is Ricci flat. It can be more or less thought as a second order hyperbolic equation for h. If M is a spacelike hypersurface lying in $\mathcal{M}$ (meaning that the metric induced by h on M is Riemannian) then one can think of the induced metric $\widehat{g}$ as the space configuration at some initial time and of the second fundamental form $\widehat{K}$ as its time derivative. As such, $\widehat{g}$ and $\widehat{K}$ are natural candidates for the Cauchy data. However, it follows from the Gauss-Codazzi equations that $\widehat{g}$ and $\widehat{K}$ are related by the following two equations:

$$\begin{cases} 0 = \mathrm{Scal}(\widehat{g}) + (\mathrm{tr}_{\widehat{g}}\widehat{K})^2 - |\widehat{K}|^2_{\widehat{g}} \\ 0 = \mathrm{div}_{\widehat{g}}\widehat{K} - d(\mathrm{tr}_{\widehat{g}}\widehat{K}). \end{cases}$$

The first equation is called the Hamiltonian constraint and is a scalar equation. The second one is the momentum constraint and is a vector equation. They form together the constraint equations. It is then a celebrated result by Y. Choquet-Bruhat that, given a triple $(M, \widehat{g}, \widehat{K})$, there exists a unique spacetime $(\mathcal{M}, h)$ solving the Einstein equation with $(M, \widehat{g}, \widehat{K})$ as an embedded hypersurface.

As a consequence, to produce solutions to the Einstein equation, we have to construct triples $(M, \widehat{g}, \widehat{K})$ solving the constraint equations.

As a simple dimension counting argument shows, these equations are underdetermined and several methods exist to construct solutions.

There exist mostly two approaches. The first one is gluing two or more solutions. This method was introduced by J. Corvino and R. Schoen. The second one (historically the first one) is to freeze part of $(\widehat{g}, \widehat{K})$ to get an elliptic system of PDEs. Several such methods exist. However, due to technical difficulties, studies have been concentrated on the so called conformal method.

As its name indicates, the method consists in looking at metrics $\widehat{g}$ conformal to a given g: $\widehat{g} = \phi^{N-2} g$, where $N - 2 = 4/(n-2)$. $\widehat{K}$ then gets decomposed as follows:

$$\widehat{K} = \frac{\tau}{n}\widehat{g} + \phi^{-2}(\sigma + \mathbb{L}W)$$

where τ is a (given) function that corresponds to the mean curvature of the embedding into the spacetime (trace of $\widehat{K}$ with respect to $\widehat{g}$), σ is a TT-tensor for g (i.e. σ is trace-free and divergence-free), W is a vector field and $\mathbb{L}W$ is the trace-free part of the Lie derivative of g in the direction of W:

$$(\mathbb{L}W)_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n}\nabla^k W_k g_{ij},$$

∇ being the Levi-Civita connection associated to g. As a consequence, the splitting is as follows:

- Given data: g, τ, σ.
- Unknown data: ϕ, W.

The constraint equations then become the following system called the conformal constraint equations:

$$\text{(CCE)} \qquad \begin{cases} 0 = -\dfrac{4(n-1)}{n-2}\Delta_g \phi + \mathrm{Scal}(g)\phi + \dfrac{n-1}{n}\tau^2\phi^{N-1} - \dfrac{|\sigma + \mathbb{L}W|_g^2}{\phi^{N+1}} \\ 0 = \mathrm{div}_g \mathbb{L}W - \dfrac{n-1}{n}\phi^N d\tau. \end{cases}$$

The first equation is called the Lichnerowicz equation and is a generalization of the prescribed scalar curvature equation. Existence and uniqueness of the solution ϕ to this equation is completely known (at least on a compact manifold M) and depends only on the zero set of τ and on whether $|\sigma + \mathbb{L}W|_g^2 \equiv 0$ or not. The second equation is called the vector equation. As long as the metric g has no non zero conformal Killing vector field, it can be solved for a unique W.

Difficulties arise when studying the system (CCE). If τ is a constant (the CMC case), the vector equation reduces to $\mathrm{div}_g \mathbb{L}W = 0$ so $W \equiv 0$ and one is left with solving the Lichnerowicz equation. Perturbation arguments can be used to address the case where $d\tau$ is small (near CMC-case). However, for arbitrary τ, little is known. We discuss the two major approaches:

- The Holst-Nagy-Tsogtgerel–Maxwell method that guarantee the existence of a solution to (CCE) when g has positive Yamabe invariant and σ is small enough.
- The Dahl-G.-Humbert method that shows that if a certain equation (called the limit equation) admits no non-trivial solution, then the set of solutions to the system (CCE) is non-empty and compact.

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