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Mathematical Aspects of General Relativity (hybrid meeting)

Organized by
Carla Cederbaum, Tübingen
Mihalis Dafermos, Princeton
Jim Isenberg, Eugene
Hans Ringström, Stockholm

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Abstract. General relativity is an area that naturally combines differential geometry, partial differential equations, global analysis and dynamical systems with astrophysics, cosmology, high energy physics, and numerical analysis. It is rapidly expanding and has witnessed remarkable developments in recent years.

Mathematics Subject Classification (2010): 83-XX, 35-XX, 53-XX.

Introduction by the Organizers

The workshop Mathematical Aspects of General Relativity was organised by Carla Cederbaum (Tübingen), Mihalis Dafermos (Cambridge/Princeton), Jim Isenberg (Eugene) and Hans Ringström (KTH Stockholm). There were 24 on-site and 30 online participants. There were three overview talks of 80 minutes, four evening talks of 60 minutes, 19 talks of 45 minutes and seven talks of 20 minutes.

Continuing one of the major themes of this series, the problem of stability of classical black hole solutions in general relativity remains the focus of much attention in the field. Recent progress on the topic was surveyed by Holzegel in an 80 minute overview talk, where he discussed in particular his recent complete proof, in collaboration with Dafermos, Rodnianski and Taylor, of the full nonlinear stability of the Schwarzschild family, without symmetry assumptions. This result, originally announced at the 2018 Oberwolfach meeting (in a talk of Taylor), was completed in April of the past year, and builds on the previous linear stability of

Schwarzschild (discussed at the 2015 meeting of this series in a talk of Holzegel). Extension of this to the Kerr case remains ongoing work of various groups.

In her talk, Teixeira da Costa discussed her recent work, in collaboration with Shlapentokh-Rothman, on boundedness and decay for the Teukolsky equation on Kerr spacetimes in the full subextremal range of parameters $|a| < M$, the key step for showing the linear stability of these spacetimes, and in addition, her proof of mode stability for the Teukolsky equation in the extremal case $|a| = M$. This extends previous work of Dafermos, Holzegel and Rodnianski and of Ma concerning the very slowly rotating case $|a| \ll M$. On the other hand, Gajic, concerning the extremal case $|a| = M$, announced in his talk some fascinating new results showing a novel azimuthal instability on the event horizon, i.e. an instability for the fixed azimuthal wave equation with non-zero azimuthal number $m \neq 0$, which is stronger than the Aretakis instability already present for the $m = 0$ case. This result is related to various recent heuristics from the physics literature. This gives hope that a full mathematical understanding of the scalar wave equation, and even the Teukolsky equation, on extremal Kerr will be obtained by the next meeting in this series!

The above results all concern stability of vacuum solutions. Turning to electrovacuum, Giorgi discussed in her talk her recent work on the linear stability of Reissner–Nordström and her ongoing work on the linear stability of the Kerr–Newman spacetime, in the slowly rotating and small charge case.

Beyond stability per se, it is fundamental to understand finer properties of gravitational radiation from black holes. Three such fundamental issues are those of late time tails, quasinormal modes and gravitational memory. Kehrberger presented surprising work which rigorously shows that the failure of peeling at scri, arising naturally from an argument due to Christodoulou, leads to logarithmic decay terms in the asymptotic expansion of the decay of radiation along null infinity (the "tails"). This could have observational consequences. Concerning quasinormal modes, Petersen presented joint work with Vasy concerning the analyticity properties of such modes at the event horizon of Kerr and Kerr-de Sitter spacetime. Bieri presented some new developments in the understanding of Christodoulou memory and other novel gravitational memory effects.

Another topic of increasing mathematical activity is the analysis of asymptotically anti-de Sitter (AdS) spacetimes. Smulevici and Chatzikaleas in their talks discussed the problem of time periodic solutions of the Einstein–scalar field system with a negative cosmological constant, and recent results they have jointly obtained for the toy problem of a nonlinear wave equation on a fixed AdS background. This is related to the instability of pure AdS spacetime itself, continuing a theme from the 2018 meeting (where such instability results were presented by Moschidis).

An intriguing phenomenon recently uncovered by Kehle concerning Kerr–AdS black holes is the relevance of the Diophantine properties of the black hole parameters to the question of the stability or instability of the black hole *interior*. Kehle presented his remarkable recent results on the wave equation on Kerr–AdS, showing that blow up or boundedness at the Cauchy horizon is determined by these Diophantine properties. The unexpected appearance of number theory here has no analogue in the $\Lambda = 0$ or $\Lambda \geq 0$ cases. This work has implications for the problem of strong cosmic censorship.

Continuing on the topic of black hole interiors, van de Moortel presented his recent result concerning the breakdown of null singularities in black hole interiors. He shows that for the Einstein-charged scalar field model, in the one-ended case, under the assumption of spherical symmetry, a null singularity eventually necessarily gives way to a spacelike singularity (or a (locally) naked singularity emanating from the centre, but these are presumably non-generic—see below!). Thus, while *part* of the singular boundary is generically null, the entirety of it cannot be. This is in sharp distinction with the two-ended case where the entirety has been shown to be null for an open set in the moduli space of initial data. It is an interesting—but quite difficult—problem to understand whether a similar statement can be made in the gravitational collapse setting for the vacuum case without symmetry assumptions.

The importance of black holes in the theory rests to a large extent on the expectation that generically, singularities not "hidden" inside black holes, so-called "naked singularities", are in fact *unstable*, if they exist. Whereas for various Einstein–matter models under spherical symmetry naked singularities have been shown by Christodoulou both to exist and to be unstable, for the vacuum equations (where one cannot impose such symmetry), even existence has up to now been open. In his talk, Shlapentokh-Rothman presented his joint work with Rodnianski, part of which has appeared and part of which is upcoming, proving the existence of naked singularities for the Einstein vacuum equations. Their remarkable construction gives an unexpected "twist" to the scalar field examples of Christodoulou.

The overview talk by Chruściel on Mathematical General Relativity covered areas such as Lorentzian geometry, mass, evolution questions and interferometers. In particular, Chruściel discussed the recent remarkable result that global hyperbolicity of non-compact n-dimensional spacetimes with $n \geq 3$ is characterised by compactness of causal diamonds. Graf continued on the Lorentz geometric theme by describing singularity theorems in C^1-regularity.

Huisken gave an overview talk on geometric flows and the mass. In particular, he discussed recent results on parabolic geometric evolutions equations such as Ricci flow, mean curvature flow and inverse mean curvature flows. He also gave applications to asymptotically flat initial data sets and concepts of mass. In a related talk, Wolff discussed the existence of weak solutions to inverse spacetime mean curvature flow.

The positive mass theorem is a topic of central importance in the subject, and it was discussed in several talks. Galloway described a recent result (obtained with Chruściel) on a positive mass theorem for asymptotically hyperbolic manifolds that allows for the presence of internal (e.g. black hole) boundaries. Sakovich discussed how the method of Jang equation reduction, originally devised by Schoen and Yau to prove the positive mass conjecture for asymptotically Euclidean initial data sets, can be adapted to the asymptotically hyperbolic setting, yielding a non-spinor proof of the positive mass conjecture in certain cases. In a related talk, Jauregui proposed a definition of total mass for an asymptotically flat 3-manifold, based on the capacity-volume inequality. He also related this mass to the ADM mass. Stern discussed a series of results obtained in the last two years concerning the influence of scalar curvature on the geometry of level sets of harmonic functions and solutions of other PDE's on 3-manifolds. These results have applications in the study of the ADM mass of initial data sets in GR.

Huang (in joint work with Lee) introduced the concept of improvability of the dominant energy scalar. Moreover, she described how non-improvable initial data sets without local symmetries must sit inside a null perfect fluid spacetime carrying a global Killing vector field. She also provided a characterisation of Bartnik mass minimising initial data sets.

Wang (in joint work with several collaborators) gave a talk showing how the theory of quasilocal angular momentum and optimal isometric embeddings identifies a correction term to the classical definition of angular momentum, and leads to a new definition that is free from supertranslation ambiguity. Reiris described a classification theorem for the topology and for the orbital type of the null generators of compact non-degenerate Cauchy horizons of time orientable smooth vacuum $3+1$-spacetimes.

In the last year, an important breakthrough in the study of big bang formation was achieved by Fournodavlos, Rodnianski and Speck. In the physics literature, there are conjectures identifying a maximal regime for stable big bang formation in the Einstein vacuum and Einstein-scalar field settings. Speck presented a result demonstrating that big bang formation is indeed stable in this regime.

A related important result, by Luk and Fournodavlos, admits the specification of smooth data on the singularity for the vacuum equations in $3+1$-dimensions. The corresponding Kasner asymptotics are not consistent with the expected generic behaviour, but can be achieved by restricting the degrees of freedom.

Turning to the expanding direction of cosmological spacetimes, Wyatt presented remarkable recent results obtained with collaborators. One of the results presented yields future global non-linear stability of solutions to the Einstein-Dust system. Another result concerns future global existence of solutions to the irrotational relativistic Euler equations (with a linear equation of state strictly between dust and a radiation fluid) on an FLRW background expanding linearly. In Minkowski space, shocks typically form in finite time. However, on cosmological backgrounds with accelerated expansion, the expansion was, due to previous results, known to suppress the shock formation (for small initial data). The case considered by Wyatt is a borderline case: there is expansion, but it is not accelerated, only linear.

Related results were presented by Oliynyk. In particular, he described results for the relativistic Euler equations on expanding cosmological backgrounds. However, he focused on an equation of state of the form $p = K\rho$, where K is a constant satisfying $1/3 < K < 1/2$. Remarkably, he obtains stability results in this regime, in spite of previous conjectures that this regime should be unstable.

Luk (in joint work with Huneau) presented a result demonstrating that a class of small data $U(1)$ symmetric solutions to the Einstein-massless Vlasov system can be achieved as weak limits of vacuum spacetimes. This constitutes an important step towards the classification of high-frequency limits of vacuum spacetimes. Previous results were restricted to the Einstein-null dust setting.

A different perspective on cosmological spacetimes is obtained by making a priori assumptions and then deducing conclusions concerning the behaviour of solutions. Recent results of this nature were presented by Lott. Garfinkle gave a talk on numerical studies of big bang singularities. In particular, he presented results using scale invariant variables that remain regular in the direction of the singularity. He also discussed the small scale features called spikes.

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Workshop (hybrid meeting): Mathematical Aspects of General Relativity

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Abstracts

Quo Vadis, Mathematical GR?

Piotr T. Chruściel

The last five years proved that Mathematical General Relativity is more lively than ever. It got itself a Nobel Prize, with help from Roger Penrose. Some long standing major research problems, namely stability of slowly rotating Kerr black holes and positivity of mass in all dimensions, have been announced as being solved [24, 17, 26]. In my talk in Oberwolfach I reported on some progress in the field, as biased by my research interests, ignoring topics which I expected to be covered by other speakers.

The topics I discussed will be split into thematic sections, no order of importance implied.

1. Lorentzian geometry

One of the fundamental notions in Lorentzian geometry is that of *global hyperbolicity.* The standard definition proceeds as follows: A spacetime (M, g) is said to be globally hyperbolic if it is strongly causal and if for any pair of points $p, q \in M$ the *causal diamond* $J^+(p) \cap J^-(q)$ is compact, if not empty. Recall that *strong causality* is defined as the requirement that for every point p and for every neighborhood $\mathscr{O}$ of p there exists a neighborhood $\mathscr{U} \subset \mathscr{O}$ of p such that every inextendible causal curve in M intersects $\mathscr{U}$ in a connected interval.

Lorentzian geometers have been using these notions for more than 60 years by now without realising, as pointed-out by Hounnonpke and Minguzzi [14], that for non-compact n-dimensional spacetimes with $n \geq 3$, global hyperbolicity is the same as requiring *compactness of causal diamonds.* This is a dramatic simplification of the notion both at a conceptual level, and in applications.

Further noteworthy developments include proofs of incompleteness theorems with weaker hypotheses on the metric, see Melanie Graf's contribution to this volume.

A milestone in the understanding of the global structure of Lorentzian manifolds was the paper by Sbierski [22], who showed that the Kruskal-Szekeres manifold is inextendible in the class of C^0 metrics. The result was nicely complemented by the joint paper of Galloway, Ling and Sbierski [12], who show that globally hyperbolic timelike geodesically complete spacetimes are C^0 inextendible. One is then left to wonder about C^0-extendibility of several physically significant spacetimes: Friedman-Lemaitre-Robertson-Walker FLRW metrics? Kerr metric? Partial results have been obtained, but the arguments used for Schwarzschild do not adapt in any obvious way to these metrics. As a step towards an answer, Sbierski [23] introduced a new technique based on holonomy to show that FLRW metrics are $C^{0,1}$–indextendible. While the current FLRW-result is not as elegant as the C^0-indextendibility of Schwarzschild, it should be kept in mind that being Lipschitz is the borderline regularity condition under which many things go wrong with the geometry of Lorentzian metrics, or with associated wave equations, and therefore reference [23] provides highly relevant information.

The interesting questions of C^0-inextendibility of the FLRW solutions, or of Kerr, or of negative-mass Schwarzschild, remain open.

2. Mass

Positive-energy theorems are widely recognised as one of the most remarkable achievements of mathematical general relativity. They have found applications in the analysis of the Yamabe problem, or in proofs of uniqueness of asymptotically flat black holes. Several new positivity proofs for asymptotically flat initial data sets have been discovered or rediscovered in recent years. Here I would only like to mention the elegant Green function analysis by Agostiniani, Mazzieri and Oronzio [1]; further approaches are described in other contributions to this volume.

Some progress has also been made on the understanding of mass for spacetimes with negative cosmological constant. In this context it is usual to consider spacetimes which have a conformal boundary at spacelike infinity *à la Penrose*. Quite generally, the mass of a metric is defined relatively to a background metric with a Killing vector which is timelike near the conformal boundary at infinity. Within the conformally compactifiable category, the simplest case arises for metrics which asymptote to metrics of the form

$$g = V^{-2}dr^2 + r^2 h_k \,, \qquad V^2 = r^2 + k \,, \qquad k \in \{0, \pm 1\} \,, \tag{1}$$

where h_k is an Einstein metric on an $(n-1)$ dimensional manifold N with scalar curvature

$$R(h_k) = k(n-1)(n-2) \,.$$

I will refer to such metrics as *asymptotically Birmingham-Kottler (BK) metrics.* There is a whole zoo of such metrics, differing by existence of boundaries or lack thereof, and by the topology of the conformal boundary at infinity (N, h_k). Our current knowledge of the sign of the mass for such metrics [5, 7, 8, 27, 4, 6, 20, 16] is summarised in Table 1. The table makes it clear that a considerable amount of work remains to obtain a complete picture.

3. Evolution questions

As already hinted-to, authors of several key papers in this area will present their work during this meeting, and there is no point for me to duplicate this. So I will only mention some results here which will certainly not be discussed in their talks.

3.1. **Local existence.** The ADM equations have always been at the heart of most numerical calculations in general relativity, as well as of many theoretical discussions. Their mathematical status was, and widely remains, rather unsatisfactory. There exists a round-about way of solving these equations: first solve the harmonically-reduced Einstein equations, then make a transformation which brings the metric into a desired ADM form. A dramatic change in our understanding of this was brought by the work of Fournodavlos and Luk [10], who show that there exists a directly well-posed formulation of the ADM equations in a Gaussian

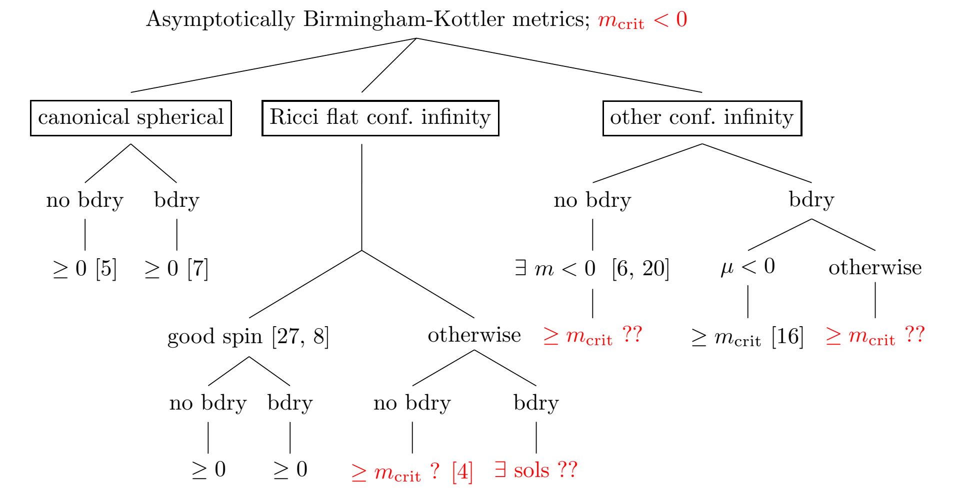

TABLE 1. Mass inequalities for asymptotically Birmingham-Kottler metrics. A double question mark indicates that no results are available; a single one indicates existence of partial results. The shorthand "bdry" refers to a black-hole boundary. "Good spin" denotes a topology where the manifold is spin *and* the spin structure admits asymptotic Killing spinors; "$\exists$ sols ??" indicates that no such vacuum solutions are known other than the Birmingham-Kottler metrics equipped with the "wrong" spin structure. The case "other conformal infinity" includes higher genus topologies when the boundary is two-dimensional, but also e.g. quotients of spheres in higher dimensions. Finally, μ is the mass aspect function. The critical value of the mass m_{crit}, assuming it exists, is expected to depend upon the conformal structure of the boundary at infinity.

slicing (zero shift and lapse equal to one). The key is a trick, how to handle the trace of the extrinsic curvature of the metric.

3.2. **Stability of de Sitter spacetime, higher dimensional $\mathscr{I}$.** One such result concerns the question of stability of even-dimensional de Sitter spacetime under small vacuum perturbations. The $3+1$ dimensional case has been settled by Friedrich in a landmark paper [11]. This has been followed by an inspired observation by Anderson [2], that the *Fefferman-Graham obstruction tensor* can be used as a tool to prove stability of higher-dimensional de Sitter spacetime with odd space-dimensions. (An alternative approach can be found in [21].) In a joint paper with Anderson [3] we used the same idea to construct even-dimensional spacetimes with vanishing cosmological constant and a smooth conformal completion at null infinity. An embarassing mistake in the proof of well-posedness of the associated evolution equations in [3], relevant both to the stability of de Sitter and to existence of null infinity, has been pointed-out and corrected in [15].

3.3. **Electrically charged Robinson-Trautman solutions.** An approach to the Einstein equations which is very familiar to an Oberwolfach audience is by solving a Cauchy problem. It is somewhat surprising that the charged Robinson-Trautman metrics cannot be handled in this way: after introducing an ansatz for the metric, and solving most of the Einstein-Maxwell equations, one ends up with a set of coupled equations, one of which is parabolic to the future, the other to the past [18]. A well posed problem for the *linearised* equations is obtained by introducing a spectral projection operator, with solutions determined by the projection of a set of free data at a characteristic surface $u = u_-$, with the complementing projection of the free data provided at a characteristic surface $u = u_+ > u_-$ [9]. The question then arises whether one can prove something similar for the nonlinear problem.

For definiteness we note that the metric takes the form

$$ds^2 = -\left(-2r\frac{P_{,u}}{P} + K - 2\frac{M}{r} + \frac{Q_0^2}{r^2}\right)du^2 - 2du dr + \frac{2r^2 d\zeta d\overline{\zeta}}{P^2}, \tag{2}$$

(quoted from [18], see [25] for a derivation). Here $P(u,\zeta,\overline{\zeta})$ is the dynamical variable determining the metric of the 2-surface $\mathcal{S}$ coordinatised by a complex coordinate ζ, $K = \Delta \ln P$ is the Gauss curvature of the metric

$$g := 2P^{-2} d\zeta d\overline{\zeta}\,,$$

Δ is the Laplace operator of g, Q_0 is a nonzero real number and $M(u,\zeta,\overline{\zeta})$ is a dynamical variable which also enters in the electromagnetic potential A:

$$A = (M - \frac{Q_0}{r})du\,.$$

The Einstein-Maxwell equations reduce to a pair of equations which involve derivatives up to order four:

$$(\ln P)_{,u} = -\frac{1}{4Q_0^2}\Delta M\,, \tag{3}$$

$$M_{,u} = -\frac{3M}{4Q_0^2}\Delta M + \frac{1}{4}\Delta^2 \ln P - \frac{1}{4Q_0^2}|dM|_g^2\,, \tag{4}$$

where $|\cdot|_g$ denotes the norm with respect to the metric g.

It turns out that the nonlinear equations with small part-initial-part-final data can be solved provided a certain functional inequality holds. In order to present the problem, for definiteness we consider a square torus

$$\mathcal{S} = \mathbb{T}^2 := [0,2\pi] \times [0,2\pi]$$

with the flat metric inherited from $\mathbb{R}^2$, and we note that a similar approach applies to other topologies. An orthonormal basis of L^2 consisting of eigenfunctions of the Laplacian is given by the collection of functions

$$f_{\vec{\ell}}(\vec{x}) = \frac{1}{2\pi}e^{i\vec{\ell}\vec{x}}\,, \quad \vec{\ell} \in \mathbb{N}^2\,. \tag{5}$$

Let $G:\mathbb{R}\to\mathbb{R}$ be a smooth function with $G(0)=0$. We denote by $(G\circ\phi)_{\vec\ell}$ the Fourier coefficients of the composition $G\circ\phi$,

$$G\circ\phi(\vec x)=\sum_{\vec\ell\in\mathbb{N}^2}(G\circ\phi)_{\vec\ell}\,f_{\vec\ell}(\vec x)\,, \tag{6}$$

and by $\phi_{\vec\ell}$ the Fourier coefficients of ϕ:

$$\phi(\vec x)=\sum_{\vec\ell\in\mathbb{N}^2}\phi_{\vec\ell}\,f_{\vec\ell}(\vec x)\,. \tag{7}$$

Consider the well known inequality: for $s>1$,

$$\|G\circ\phi\|_{H^s(\mathcal{S})}\le C(\|\phi\|_{L^\infty(\mathcal{S})})\|\phi\|_{H^s(\mathcal{S})}\,, \tag{8}$$

where the strictly increasing function C depends upon G and s. After multiplying the function C by a constant if necesary, (8) is equivalent to

$$\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}(G\circ\phi)_{\vec\ell}^{\,2}\le C(\|\phi\|_{L^\infty(\mathcal{S})})\Big(\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}\phi_{\vec\ell}^2\Big)\,. \tag{9}$$

It turns out the full Einstein-Maxwell equations for Robinson-Trautman metrics can be solved, at least in the small-data regime, if a variation of this inequality holds for functions ϕ which depend upon a time parameter t. We formulate the desired inequality as a question:

Is it true that, given a smooth function G with a first order zero at the origin and a real number $s>1$, there exists a constant $C_2=C_2(G,s)$ such that for all $\phi(t)$ satisfying

$$\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}\sup_{t\in[t_-,t_+]}\phi_{\vec\ell}^2(t)\le 1\,. \tag{10}$$

we have

$$\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}\sup_{t\in[t_-,t_+]}(G\circ\phi)_{\vec\ell}^{\,2}(t)\le C_2\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}\sup_{t\in[t_-,t_+]}\phi_{\vec\ell}^2(t)\ ? \tag{11}$$

Note that a direct application of (8) gives a different inequality:

$$\begin{aligned}\sup_{t\in[t_-,t_+]}&\Big(\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}(G\circ\phi)_{\vec\ell}^{\,2}(t)\Big)\\ &\le C(\sup_{t\in[t_-,t_+]}\|\phi(t)\|_{L^\infty(\mathcal{S})})\sup_{t\in[t_-,t_+]}\Big(\sum_{\vec\ell\in\mathbb{N}^2}(1+|\vec\ell|)^{2s}\phi_{\vec\ell}^2(t)\Big)\,.\end{aligned} \tag{12}$$

Assuming that (11) holds, we have the conditional result [9]:

Theorem 1. *Let $s,u_\pm\in\mathbb{R}$ with $s>1$ and $u_-<u_+$. If the inequality (11) holds, there exists a unique smooth solution of the Einstein-Maxwell Robinson-Trautman equations defined on $[u_,u_+]\times\mathbb{T}^2$ with part of sufficiently small spectral data of (M,P) prescribed at u_-, and the remaining part on u_+.*

The question then arises, whether (11) holds true.

4. Interferometers

Yet another recent Nobel prize for general relativity was awarded for the first direct observation of gravitational waves. The detection involved a Michelson interferometer, which works as follows: a laser sends light to a beam splitter, the two resulting beams bounce back and forth a few times between mirrors and interfere at the output port. The freshman calculation is to determine the affine parameter needed for the roundtrip along the geodesics connecting the (freely falling) beam splitter and mirrors. A somewhat more sophisticated approach is to calculate the leading perturbation in an eikonal expansion for the Maxwell equation. With some hand waving one recovers the geodesic approximation just described, but one quickly realises that the solutions so obtained are ambiguous and possibly coordinate dependent. So the question arises, is there a way of describing this problem which guarantees existence of a unique solution, with an unambiguous answer for the interference pattern. Note that geometric uniqueness will also guarantee coordinate independence of the final result.

In [19] we have observed that one can solve uniquely the Maxwell equations by describing the laser, and the mirrors, as a boundary-value problem for the Maxwell equations with analytic boundary data given on an infinite *timelike* hyperplane in a space-time with an analytic metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \epsilon A_{\mu\nu} \cos(\vec{k}\vec{x} - \omega_g t)\,, \tag{13}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $A_{\mu\nu}$ is an η-traceless tensor with constant entries satisfying $A_{0\mu} = 0 = A_{ij}k^j$. Existence of solutions, and analyticity in ϵ is guaranteed by the Cauchy-Kovalevskaya theorem. Uniqueness within the class of smooth solutions is guaranteed by a theorem of Holmgren. One can then calculate explicitly the first term in a (convergent) expansion of the solution in terms of powers of ϵ, and find that is has a Laurent expansion in terms of ω_g/ω, where ω_g is the frequency of the gravitational wave and ω is the frequency of the Maxwell wave emitted by the laser. It turns out that this is precisely what is needed to validate the application of eikonal expansions for the problem at hand. One also shows that those ambiguities which remain do not affect the interference pattern at the level of accuracy available experimentally.

The above relies heavily on the analyticity of the metric (13), which is a poor man's approximation to the physical metric. This last metric has no reason to be analytic. One is thus led to the question, whether there exists a way to justify rigorously the results of LIGO-type interferometric experiments without assuming analyticity?

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Mode stability for the entire Kerr black hole family

Rita Teixeira da Costa
(joint work with Marc Casals)

A major open problem in General Relativity, going back to the work of Regge and Wheeler [7], is that of black hole stability:

Conjecture 1 (Black hole stability)**.** *The Kerr family of black holes, which is parametrized by $M > 0$ and $|a| \leq M$, is stable, as a family of solutions to the vacuum Einstein equations,* $\mathrm{Ric}(g) = 0$*, to perturbations in the initial data.*

Conjecture 1 has recently been shown inside the $a = 0$ subfamily[1] [4]. For at least the subextremal $|a| < M$ Kerr subfamily, the roadmap set by this monumental work is clear. A resolution to Conjecture 1 must build on a very precise understanding of the linear stability problem and, in particular, of the so-called Teukolsky equation,

$$\begin{aligned}\Big\{\Box_g + \frac{2s}{\rho^2}(r-M)\partial_r + \frac{2s}{\rho^2}\left(\frac{M(r^2-a^2)}{\Delta} - r - ia\cos\theta\right)\partial_t \\ + \frac{2s}{\rho^2}\left[\frac{a(r-M)}{\Delta} + i\frac{\cos\theta}{\sin^2\theta}\right]\partial_\phi + \frac{s}{\rho^2}\left(1 - s\cot^2\theta\right)\Big\}\alpha^{[s]} = 0\end{aligned} \tag{1}$$

which, for $s = \pm 2$, governs some gauge-invariant curvature quantities at the linear level. Following earlier works in the $a = 0$ [3] and $s = 0$ [2] cases, such an analysis of (1) was carried out in collaboration with Shlapentokh-Rothman [9]:

Theorem 2 (Stability for the Teukolsky equation)**.** *Solutions for* (1) *arise from regular, compactly supported initial data remain bounded in time and, in fact, decay in time at a suitably fast inverse polynomial rate.*

The goal of the talk is to expand on what is arguably the most mysterious point of the proof of Theorem 2: (the proof of) mode stability.

[1] It is expected that a generic perturbation of an $a = 0$ black hole would asymptote towards a member of the Kerr family with $a \neq 0$. The authors consider initial data on a codimension-3 "submanifold" of the moduli space which is teleologically chosen so that the resulting spacetimes asymptotes to a member of the $a = 0$ subfamily.

Theorem 3 (Mode stability)**.** *Take $|a| < M$. Fix $s \in \frac{1}{2}\mathbb{Z}$, $\omega \neq 0$ with $\Im\omega \geq 0$, and $m - s \in \mathbb{Z}$. Consider a mode solution to* (1)*, i.e.*

$$(2) \qquad \alpha^{[s]}(t, r, \theta, \phi) = e^{-i\omega t} \cdot e^{im\phi} \cdot S^{[s],\, a\omega}_{m\Lambda}(\theta) \cdot (r^2 + a^2)^{-1/2}\Delta^{-|s|/2}u^{[s]}(r)\,,$$

where Λ is the separation constant between the r and θ variables, and assume $u^{[s]}(r)$ has outgoing boundary conditions as $r \to \infty$ and ingoing boundary conditions as $r \to r_+$. Then, $u^{[s]} \equiv 0$.

The conclusion holds for $|a| = M$ if $\omega = \frac{am}{2M^2}$ is excluded from the statement.

Theorem 3 rules out the existence of separable solutions to (1) which are exponentially growing ($\Im\omega > 0$) or bounded in time but non-decaying ($\omega \in \mathbb{R}\backslash\{0\}$). Its quantitative strengthening [8, 10] provides the only known method of obtaining a Morawetz estimate for a bounded range of frequencies, excluding $\omega \neq 0$, in the full subextremal black hole range $|a| < M$.

Theorem 3 would follow easily if one could find a conserved, coercive quantity for (1). However, on rotating $a \neq 0$ Kerr, the conserved energy associated to the stationary Killing field is not coercive if the frequencies are superradiant, i.e. if $\omega(\omega - m\upomega_+) < 0$. The redshift effect associated to subextremal black holes is not sufficient overcome to this lack of coercivity in the full $|a| < M$ range.

Theorem 3 was first shown for $|a| < M$ by Whiting [11] for $\Im\omega > 0$. His proof, as well as those of the following extensions to $\omega \in \mathbb{R}\backslash\{0\}$ [8] and $|a| = M$ [10], are based on the same method: one can define an injective integral transformation which maps $u^{[s]}(r)$ to a function $\tilde{u}(x)$ solving a new equation which itself admits a conserved and coercive quantity, i.e. which is modally stable. These classical proofs mostly rely on lucky guesswork. Such transformations are not guaranteed to exist nor can the ansatz required be very obviously constrained by the goal they are meant to achieve. Furthermore, a different spacetime means one must begin anew: for instance, Whiting's ansatz breaks down completely in the limit $|a| \to M$ so a very different strategy for the transformation was required to address extremal Kerr in [10].

The main new result introduced in this talk is a different approach to Theorem 3. In a seminal paper, Leaver [6] obtains a condition for a frequency triple (ω, m, Λ) to correspond to a nontrivial mode solution: a continued fraction equation where each term is *explicit* in the Kerr black hole parameters, frequency parameters and s. Our novel contribution is

Theorem 4 (Hidden symmetries)**.** *Consider the setup of Theorem 3 with $|a| < M$. Define "masses"*

$$(3) \qquad m_1 = s + 2iM\omega\,, \qquad m_2 = \frac{2iM^2\omega - am}{\sqrt{M^2 - a^2}}\,, \qquad m_3 = -s + 2iM\omega\,.$$

Then, Leaver's condition on the mode solution spectrum of (1) *is* ***invariant*** *under any permutations of (m_1, m_2, m_3).*

Theorem 4 was previously conjectured to hold in [1, 5]. While the invariance in exchanging m_1 and m_2 is present in the radial ODE for $u^{[s]}$, the symmetries $m_1 \leftrightarrow m_3$ and $m_2 \leftrightarrow m_3$ are not symmetries of the ODE; they are symmetries of the its point spectrum. Hence, such interchanges lead to new, isospectral, ODEs. For instance, swapping m_1 and m_3 yields (1) with s replaced by $-s$; thus Theorem 4 recovers the Teukolsky–Starobinsky identities. More interestingly, swapping m_2 and m_3 produces a completely different equation to (1), which is nevertheless familiar: Whiting's equation! The upshot is that we are able to reprove Theorem 3 bypassing the classical integral transformation method completely.

Our method is also more flexible to changes in the spacetime. As an example, we show that a similar statement to Theorem 4, with four "masses", holds for Kerr-de Sitter. This allows us to rule out existence of nontrivial mode solutions for some superradiant frequencies, uniformly in the subextremal black hole parameters.

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Singularity theorems for C^1-Lorentzian metrics

Melanie Graf

The classical singularity theorems of S. Hawking and R. Penrose show that a Lorentzian manifold with a smooth (or at least C^2) metric satisfying certain curvature and causality conditions cannot be causal geodesically complete. Since many physical spacetimes have metrics whose regularity lies significantly below C^2 it is natural to ask whether these theorems continue to hold for lower-regularity metrics – a question that was already raised in [4, Sec. 8.4] and emphasized in the review [9] but has proven to be very difficult and only recently begun being rigorously explored even for metrics that are still $C^{1,1}$, cf. [5, 6].

I presented some recent results from [3] concerning proofs of both the Hawking and the Penrose singularity theorem for C^1-Lorentzian metrics – a decisive step down in regularity from the previous results for $C^{1,1}$-metrics because (a) the curvature is now merely distributional and no longer locally bounded and (b) geodesics, defined as solutions of the geodesic equation, will generally be non-unique (we do, however, still have classical existence of solutions). The proofs required significant sharpening of earlier estimates for the curvature of approximating smooth metrics, developing stability properties of long existence times for causal geodesics and, in case of the Penrose theorem, finding an appropriate formulation of a distributional version of the null energy condition.

Concretely, the two main results of [3] are:

Theorem 1 (C^1 Hawking theorem, [3])**.** *A spacetime with $\mathcal{C}^1$-metric is future timelike geodesically incomplete (i.e., there exists an inextendible future directed timelike geodesic not defined on all of* $[0,\infty)$*) if*

(i) it satisfies the distributional strong energy condition, i.e., $\mathrm{Ric}(\mathcal{X},\mathcal{X})$ *is a non-negative distribution for all smooth timelike vector fields* $\mathcal{X}$

(ii) there exists a compact smooth spacelike hypersurface Σ *in* M *and*

(iii) the future convergence $k := g(H,\nu)$ *(where* ν *is the future unit normal to* Σ *and* H *is the mean curvature vector of* Σ*) is positive*

Theorem 2 (C^1 Penrose theorem, [3])**.** *A spacetime with $\mathcal{C}^1$-metric is future null geodesically incomplete (i.e., there exists an inextendible future directed null geodesic not defined on all of* $[0,\infty)$*) if*

(i) it satisfies the distributional null energy condition (a generalization of the classical null energy condition to this lower regularity; to be defined below),

(ii) there exists a non-compact Cauchy hypersurface in M*,*

(iii) and there exists a trapped surface S *(i.e., a compact smooth achronal spacelike co-dimension* 2 *submanifold with past-pointing timelike mean curvature vector field* H*)*

Taking the approximation approach of previous works on low-regularity singularity theorems a step further, the guiding idea for the proof was the following: If one could prove that assumptions (i)-(iii) and timelike (respectively null) geodesic completeness are stable (in a suitable topology), then we could aim to approximate g by suitable smooth metrics g_ϵ (e.g. via convolution) and for ϵ small enough, the smooth g_ϵ would contradict the classical Hawking or Penrose theorem. In trying to implement this strategy one notices readily that (ii) and (iii) are stable[1]. Looking at geodesic completeness, it can be shown that both causal and null geodesic

[1]Conditions (iii) (and (ii) in Hawking's theorem) are stable under C^1_{loc}-convergence because they (at most) involve first derivatives on a compact subset. Stability of global hyperbolicity/Cauchy hypersurfaces is well-known and holds even for merely continuous metrics and does not require convergence of derivatives (but it does require fast-enough convergence at infinity, provided by, e.g., C^0_{fine}-convergence, or alternatively, that one restricts oneself to approximate only with metrics with narrower lightcones such that any Cauchy hypersurface for the original metric is also a Cauchy hypersurface for the approximating metrics irrespective of their closeness).

completeness together with global hyperbolicity are stable under C^1_{fine}-convergence (cf. [3, Thm. 2.17]). However the proofs of the C^1 singularity theorems only need the following weaker "almost stability" result [3, Prop. 2.11]: Let $g, g_\epsilon \in C^1$, $g_\epsilon \to g$ in C^1_{loc}, let $K \subseteq TM$ be compact and assume that all g-geodesics starting in K are defined on $[0, \infty)$. Then for any $N \in \mathbb{N}$ there exists $\epsilon_0(N, K)$ such that for all $\epsilon \leq \epsilon_0(N, K)$ all g_ϵ-geodesics starting in K are defined on $[0, N]$ and the subset of TM spanned by all their tangent vectors is relatively compact.

To discuss the formulation and "almost stability" of the energy conditions, let us focus on the null energy condition: Note first that since Ric is a tensor distribution of first order, one still could formulate a distributional null energy condition by demanding non-negativity for the Ricci of C^1-null vector fields. The problem is that being null isn't stable under approximations (being null is almost stable, but since Ric is no longer locally bounded that isn't enough). Therefore, we need a definition which, roughly speaking, demands that Ric is close to non-negative for vector fields close to being null, leading to the following

Definition 3 (Distributional null energy condition, [3])**.** *For any compact $K \subseteq M$ and any $\delta > 0$ there exists $\epsilon(\delta, K)$ s.t.*

$$(\mathrm{Ric}(\mathcal{X}, \mathcal{X}))|_U > -\delta \qquad \text{(as a distribution on U)}$$

for any (local) smooth vector field $\mathcal{X} \in \mathfrak{X}(U)$ ($U \subseteq K$, open) with $||\mathcal{X}||_h = 1$ (where h is a fixed Riemannian background metric) and $|g(\mathcal{X}, \mathcal{X})| < \epsilon(\delta, K)$ on U.

This still reduces to the usual definition if g is C^2 (and also to the "a.e. non-negative for locally Lipschitz null vector fields"-null energy condition for $C^{1,1}$-metrics from [6]). The condition $||\mathcal{X}||_h = 1$ is added to really only pose a restriction on the "almost null directions": Else *all* distributions Ric_{ii} would be locally bounded from below irrespective of the causal character of ∂_i.

Using well-constructed approximations $\check{g}_\epsilon \prec g$ (defined via a standard smoothing-via-convolution of g with the addition of a small correction term to achieve that $\check{g}_\epsilon \prec g$, i.e., that $\check{g}_\epsilon \leq 0 \implies g(X, X) < 0$) and precise estimates based on Friedrich's lemma type arguments, I obtained the following almost stability statement [3, Lem. 5.5]: If a C^1-Lorentzian metric g satisfies the distributional null energy condition, then for any compact $K \subseteq M$, $c_1, c_2 > 0$ and $\delta > 0$, there exists $\epsilon_0 > 0$ (depending on K, δ and c_1, c_2) such that $\forall \epsilon < \epsilon_0$:

$$\forall X \in TM|_K \text{ with } 0 < c_1 \leq ||X||_h \leq c_2 \text{ with } \check{g}_\epsilon(X,X) = 0 :$$
$$\mathrm{Ric}[\check{g}_\epsilon](X,X) > -\delta$$

Note that, by standard index form or Raychaudhuri equation methods, for δ small enough this will imply that $\check{g}_\epsilon$-geodesics, satisfying appropriate initial conditions (e.g., null geodesics starting orthogonally to the trapped surface) and whose tangent vectors remain within the fixed compact set K (the existence of a suitable such set follows from the "almost stability" of timelike/null geodesic completeness) must encounter a focal point at a parameter less than some explicit constant depending on the initial convergence and δ. From here on out the contradiction follows essentially analogously to the classical theorems.

A next significant step would be to obtain singularity theorems based on these distributional energy conditions also for Lorentzian metrics which are only Lipschitz continuous. The importance of this regularity is twofold: On the one hand it substantially increases the amount of admissible physical examples. On the other hand, the bulk of causality theory remains valid and geodesics in the sense of Filippov are now reasonably well understood, see [7], making it a little more tractable mathematically. However, the estimates used to obtain the almost energy conditions for the approximating metrics are very sharp, so it remains to be seen whether there is room for improvement.

Further, there are already Hawking-type singularity theorems available in the new framework of synthetic Lorentzian geometry, cf. [1, 2], and it would certainly be worthwhile investigating a) how exactly these theorems relate to the C^1-Hawking theorem and b) what tools/new developments one would need for a synthetic version of a Penrose-type theorem.

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The characteristic gluing problem of general relativity

Stefan Czimek
(joint work with Stefanos Aretakis and Igor Rodnianski)

In collaboration with Aretakis and Rodnianski, the author initiated in [1] the study of the *characteristic gluing problem of general relativity*:

Given two spacelike 2-spheres S_1 and S_2 in spacetimes $(\mathcal{M}_1, \mathbf{g}_1)$ and $(\mathcal{M}_2, \mathbf{g}_2)$, respectively, is it possible to glue $(\mathcal{M}_1, \mathbf{g}_1)$ and $(\mathcal{M}_2, \mathbf{g}_2)$ along an outgoing null hypersurface $\mathcal{H}$ leading from S_1 to S_2 (as solution to the Einstein equations)?

In general, there are obstacles to characteristic gluing. First, the Raychauduri equation implies monoticity properties along null hypersurfaces which act as obstruction to gluing general spacetimes. Second, more fundamentally, in [1, 2] an ∞-dimensional space of obstructions to characteristic gluing near Minkowski at C^2-regularity is identified. In [1, 2] it is shown that this obstruction space stems from *conservation laws* of the linearized Einstein equations at Minkowski along null hypersurfaces. Importantly, the obstruction space splits into two components:

(1) An ∞-dimensional space of *gauge-dependent conserved charges* which can be adjusted by applying gauge perturbations to the gluing data.
(2) A 10-dimensional space of *gauge-invariant conserved charges* which can in general not be adjusted by gauge perturbations.

It is moreover shown that one can glue transversally to the obstruction space by using the freedom of prescribing the *characteristic seed* in this gluing problem (see [1, 2]). In other words, the conservation laws are the only obstacles in this problem near Minkowski. Concluding the above, the following is a rough summary of the so-called "codimension-10 characteristic gluing" of [1, 2].

Theorem (Codimension-10 gluing [1, 2]) *Assume that the spacetime geometries around the spheres $S_1 \subset (\mathcal{M}_1, \mathbf{g}_1)$ and $S_2 \subset (\mathcal{M}_2, \mathbf{g}_2)$ are close to the flat geometries around the round spheres of radius* 1 *and* 2 *in Minkowski spacetime, respectively. Then the following holds.*

(1) *There exists a perturbation of the sphere S_2 to a nearby sphere $S_2' \subset (\mathcal{M}_2, \mathbf{g}_2)$ such that one can characteristically glue with C^2-regularity from S_1 to S_2' along a null hypersurface $\mathcal{H}'_{[1,2]}$ up to the* 10 *gauge-invariant charges.*
(2) *Using characteristic gluing, it is possible to construct C^{m+2}-gluings for any integer $m \geq 0$ along <u>spacelike</u> hypersurfaces – up to the* 10*-dimensional space of gauge-invariant charges.*

In the characteristic gluing along a null hypersurface $\mathcal{H}$, one can glue higher-order $\mathcal{H}$-tangential derivatives *without further obstruction*: The additional obstacles to C^{m+2}-characteristic gluing are only due to *conservation laws for higher $\mathcal{H}$-transversal derivatives.* Moreover, to deduce spacelike gluing result from characteristic gluing, the local existence result [8] for the characteristic Cauchy problem is used.

In [1, 3] a geometric characterization of the 10 gauge-invariant charges in terms of the ADM integrals for *energy*, *linear momentum*, *angular momentum* and *center-of-mass* is given. Subsequently, the above characteristic gluing near Minkowski can be applied to glue asymptotically flat spacetimes to Kerr black hole spacetimes.

Theorem (Gluing to Kerr [1, 3]) *Let $(\mathcal{M}, \mathbf{g})$ be an asymptotically flat spacetime. Then the following holds.*

(1) *There exists a Kerr spacetime $(\mathcal{M}^{\mathrm{Kerr}}, \mathbf{g}^{\mathrm{Kerr}})$ such that far out in $(\mathcal{M}, \mathbf{g})$ one can characteristically glue $(\mathcal{M}, \mathbf{g})$ to $(\mathcal{M}^{\mathrm{Kerr}}, \mathbf{g}^{\mathrm{Kerr}})$ with C^2-regularity. The Kerr spacetime $(\mathcal{M}^{\mathrm{Kerr}}, \mathbf{g}^{\mathrm{Kerr}})$ and $(\mathcal{M}, \mathbf{g})$ have almost identical ADM energy, linear momentum, angular momentum and center-of-mass.*

(2) *Using characteristic gluing, it is possible to construct C^{m+2}-gluings to Kerr for any integer $m \geq 0$ along spacelike hypersurfaces.*

In particular, (2) above provides an alternative proof of the ground-breaking Corvino-Schoen gluing [6, 7] for spacelike initial data of the Einstein equations.

Another important aspect of characteristic gluing is that due to the transport nature of the null constraint equations, one can prove *localization results*. The following is a simplified version of characteristic localization, we refer to [1] for a precise statement.

Theorem (Characteristic localization [1, 2]) *Let $(\mathcal{M}, \mathbf{g})$ be a spacetime, and let $\mathcal{H} \subset \mathcal{M}$ be a null hypersurface connecting two spheres S_1 and S_2 in $\mathcal{M}$. Let $K' \subset\subset K \subset\subset \mathbb{S}^2$ be two angular regions. If the induced characteristic data on $\mathcal{H}$ is sufficiently close to the characteristic data on the null hypersurface connecting the round spheres of radius 1 and 2 in Minkowski, then there exists a localization, that is, characteristic initial data on $\mathcal{H}$ which agrees with the given characteristic initial data on S_1 and in the angular region K' along $\mathcal{H}$, and agrees with the trivial data of the round sphere of radius 2 in Minkowski in the complement of the angular region K on S_2.*

It is shown in [1] that the above characteristic localization yields an alternative proof of the Carlotto-Schoen localization [5] of asymptotically flat spacelike initial data *without loss of decay. This resolves an open problem in this direction, see Open Problem 3.18 in* [4].

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Numerical Simulations of Spacetime Singularities

David Garfinkle
(joint work with Frans Pretorius, Paul Steinhardt, Woei-Chet Lim)

This talks reports on the results of two different projects involving computer simulations of the approach to spacetime singularities. One[1] involves the so-called "ekpyrotic" cosmological scenario. The other[2] involves the appearence of small scale features called "spikes" during the formation of singularities described by the vacuum Einstein equation.

Starting with the work of Penrose[3] it has been known that spacetime singularities can form in physically realistic situations: notably gravitational collapse to form a black hole, and the Big Bang at the start of the universe. However, the Penrose theorem gives very little information about the nature of singularities, stating only that at least one causal geodesic is incomplete.

To address this issue, Belinskii, Khalatnikov, and Lifschitz[4] (usually known as BKL) conjectured that (in an appropriate foliation) spacetime singularities have the property that time derivative terms in the field equations are much larger than space derivative terms. Thus the dynamics of each spatial point effectively decouples from those at all other points, and each point is effictively its own homogeneous spacetime (though a different homogeneous spacetime at each point). To test the BKL conjecture, Berger and Moncrief[5] began a research program of performing computer simulations of the approach to spacetime singularities to see whether the singularities are accurately described by the BKL picture. For reasons of numerical convenience, most simulations of spacetime singularities are done for the case where the Cauchy surface is T^3 (thus compact Cauchy surfaces). Furthermore singularities of the BKL type are all spacelike. Thus this approach does not address the null singularities that form in asymptotically flat gravitational collapse.

The simulations reported in this talk use methods introduced by Uggla et al.[6] Here one uses a tetrad, and the variables evolved are the connection coefficients of the tetrad divided by the mean curvature. These variables are scale invariant and remain finite as the singularity is approached. The simulations use a constant mean curvature foliation and use the logarithm of the mean curvature as a time coordinate. Thus the singularity is approached in the limit of infinite coordinate time (though at finite proper time).

One striking feature of the Big Bang is that it is a very special type of singularity: homogeneous, isotropic, with vanishing Weyl tensor. How did our universe start in this special way? One possible answer (which is quite popular among cosmologists) is the inflationary scenario:[7] here the universe began with a much more general type of singularity, but a subsequent period of exponential expansion drove the universe to a homogeneous, isotropic state. An alternative is the ekpyrotic scenario:[8] here there is a contracting universe, and the mechanism that drives the universe to a homogeneous, isotropic state is a scalar field with a potential of the form $V(\phi) = -V_0 e^{c\phi}$ where V_0 and c are positive constants. To complete this scenario, one would also need a different mechanism to cause the universe to "bounce" from a contracting phase to an expanding phase.

For convenience, most treatments of either of these cosmological scenarios use universes that start off nearly homogeneous and isotropic to begin with. But this is certainly not adequate to test the contention that this is a robust smoothing mechanism that works on generic initial conditions. In contrast, we perform numerical simulations[1] using the methods described above of conditions far from homogeneity and isotropy. We find that with the potential given above these generic conditions do indeed evolve to approach homogeneity and isotropy as they approach the singularity. Essentially all the energy contained in the shear and curvature variables is transferred to the scalar field variable.

From the mathematical point of view, we note that there are proofs of the stability of Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetimes with a free scalar field.[9] We suspect that one could obtain stronger versions of such results (and that the proof would be easier) in the case where the scalar field has an ekpyrotic potential. The reason for these expectations is that in the case treated in [9] the shear variables do not become negligible: they simply do not grow any faster than the mean curvature. In contrast, our numerical simulations of the ekpyrotic scenario indicate that as the singularity is approached the shear becomes negligible compared to the mean curvature.

The simulations of [5], treating spacetimes with two spatial symmetries, found that at most spatial points the dynamics were as anticipated by BKL. However, there were isolated points at which narrow features formed, and that these features became ever narrower as the singularity was approached. These features later came to be known as "spikes." In [2] we use a combination of analytic approximation methods and exact (i.e. without any approximation) numerical simulations of the vacuum Einstein field equations to elucidate the nature of these spikes. We find that rather than being an exception to BKL dynamics, spikes are actually a consequence of them. The BKL dynamics at each spatial point is the following: there are comparatively long "Kasner epochs" in which the behavior is that of a Kasner spacetime. These epochs are punctuated by comparitively short "transitions" from one Kasner epoch to another. We find that the transitions are driven by a particular term in the Einstein field equations, and that this term vanishes on surfaces of co-dimension one. At these surfaces, spikes form. That is, transitions to the new Kasner epoch occur away from the surface, while the surface itself is stuck in the old Kasner epoch. The surfaces of co-dimension one are effectively isolated points in the case of two spatial symmetries (as in the simulations of [5]), curves in the case of one spatial symmetry, and two-surfaces in the case of no symmetry. Our analytic approximation using BKL dynamics yields formulas for the shape of the spikes and for the dependence of their width as a function of time. Since spikes are essentially co-dimension one phenomena (i.e. everything about them depends only on the spatial varible that is perpendicular distance from the spike surface) the behavior of the spike is the same regardless of the group of spatial symmetries. Our numerical simulations show that these analytic appoximations provide a good description of spike behavior. Furthermore our simulations show that eventually the spike becomes sufficiently steep that spatial derivative terms in the field equations, rather than remaining negligible, eventually become sufficiently strong to disrupt the spike and cause it to disappear. Thus our simulations show that ultimately the spikes are ephemeral.

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Bartnik mass minimizing initial data sets and improvability of the dominant energy scalar

Lan-Hsuan Huang
(joint work with Dan A. Lee)

In this talk, we introduce the concept of improvability of the dominant energy scalar, and we derive strong consequences of non-improvability. In particular, we give an initial data set characterization of a null perfect fluid whose velocity is parallel to a Killing vector field. Using those main results, we characterize ADM mass minimizing initial data sets which makes substantial progress toward Bartnik's stationary conjecture. Along the way we observe that in dimensions greater than eight there exist pp-wave counterexamples (without the optimal decay rate for asymptotically flatness) to the equality case of the spacetime positive mass theorem. As a consequence, we find counterexamples to Bartnik's stationary and strict positivity conjectures in those dimensions.

Before we describe those main results, we recall some basic definitions. An *initial data set* is a triple (U, g, k), where (U, g) is an n-dimensional Riemannian manifold and k is a symmetric $(0, 2)$-tensor on M. The *Einstein constraint map* Φ is defined by

$$\Phi(g, k) = \big(R_g - |k|_g^2 + (\mathrm{tr}_g k)^2, \mathrm{div}_g k - d(\mathrm{tr}_g k)\big) =: (2\mu, J).$$

We say that (g, k) is *vacuum* if $\mu = 0, J = 0$ everywhere in U. More generally, we define the *dominant energy scalar* $\sigma(g, k) := \mu - |J|_g$, and the *dominant energy condition* says $\sigma(g, k) \geq 0$.

1. An initial data set characterization of a null perfect fluid

We start with a fundamental observation of V. Moncrief that gives an initial data set characterization of a vacuum spacetime admitting a Killing vector field.

Theorem 1 (Moncrief [5], Cf. [3]). *Let (U, g, k) be a vacuum initial data set. Suppose that there exists a nontrivial lapse-shift pair (f, X) on U solving*

$$D\Phi|^*_{(g,k)}(f, X) = 0. \tag{1}$$

Then (U, g, k) sits inside a vacuum spacetime admitting a unique global Killing vector field $\mathbf{Y}$ such that $\mathbf{Y} = 2f\mathbf{n} + X$ along U, where $\mathbf{n}$ is the future unit normal to U. A converse statement also holds.

A spacetime $(\mathbf{N}, \mathbf{g})$ is a *null perfect fluid* with *velocity* $\mathbf{v}$, either future null or zero, and *pressure* p if its Einstein tensor $G(\mathbf{g}) := \mathrm{Ric}(\mathbf{g}) - \frac{1}{2}R(\mathbf{g})\mathbf{g}$ can be expressed as $G(\mathbf{g}) = p\mathbf{g} + \mathbf{v} \otimes \mathbf{v}$. Note that this class of spacetimes includes (1) the vacuum spacetimes with a cosmological constant (by letting $\mathbf{v} = 0$ and $p =$ constant) and (2) the null dust (by letting p identically zero).

Our first result is a generalization of Theorem 1. In particular, it gives an initial data set characterization of a null perfect fluid whose velocity $\mathbf{v}$ is parallel to a Killing vector $\mathbf{Y}$. Equation (1) for a general non-vacuum setting is replaced by

$$\begin{aligned} D\overline{\Phi}^*_{(g,k)}(f, X) &= 0 \\ 2fJ + |J|_g X &= 0. \end{aligned} \tag{$\star$}$$

The second equation in $(\star)$ is referred to as the *J-null-vector equation* for (f, X), as the corresponding spacetime vector $\mathbf{Y} = 2f\mathbf{n} + X$ is null. The operator $\overline{\Phi}_{(g,k)}$ in the first equation of $(\star)$ is the *modified* Einstein constraint operator defined by the author and J. Corvino in [2], in which they study initial data set gluing satisfying the dominant energy condition. Fix an initial data (g, k), and for an arbitrary initial data set (γ, τ) we define

$$\overline{\Phi}_{(g,k)}(\gamma, \tau) := \Phi(\gamma, \tau) + (0, \tfrac{1}{2}\gamma \cdot J)$$

where J is the one defined from (g, k).

Theorem 2 (Huang-Lee [4, Theorem 6]). *Let (U, g, k) be an initial data set. Assume there exists a nontrivial lapse-shift pair (f, X) on U solving the system $(\star)$, and assume that f is nonvanishing*[1] *in U. Then the following holds:*

[1] The condition can be relaxed in the special case that $\sigma(g, k) = 0$. See [4, Section 6.2].

(1) *The dominant energy scalar* $\sigma(g,k)$ *is constant on* U.
(2) (U,g,k) *sits inside a spacetime* $(\mathbf{N},\mathbf{g})$ *that admits a global Killing vector field* $\mathbf{Y}$ *equal to* $2f\mathbf{n}+X$ *along* U, *where* $\mathbf{n}$ *is the future unit normal to* U, *and* $(\mathbf{N},\mathbf{g})$ *is a null perfect fluid with velocity* $\mathbf{v}=\frac{\sqrt{|J|}}{2f}\mathbf{Y}$ *and pressure* $p=-\frac{1}{2}\sigma(g,k)$.
(3) *If* (g,k) *satisfies the dominant energy condition, then* $\mathbf{g}$ *satisfies the spacetime dominant energy condition.*

Conversely, let $(\mathbf{N},\mathbf{g})$ *be a null perfect fluid spacetime, with velocity* $\mathbf{v}$ *and pressure* p, *admitting a global Killing vector field* $\mathbf{Y}$. *Assume that* $\mathbf{v}=\eta\mathbf{Y}$ *for some scalar function* η. *Then* p *is constant, and for any spacelike hypersurface* U *with induced initial data* (g,k) *and future unit normal* $\mathbf{n}$, *if we decompose* $\mathbf{Y}=2f\mathbf{n}+X$ *along* U, *then the lapse-shift pair* (f,X) *satisfies the system* $(\star)$.

2. Improvability of the dominant energy scalar

The system $(\star)$ also arises from a seemly different point of view, relating to the ADM mass minimizer in the positive mass theorem and Bartnik's quasi-local mass.

In loose terms, for an initial data set (g,k), we say the dominant energy scalar $\sigma(g,k)$ is *improvable* if for small, compactly supported function u, there exists compactly supported (h,w) such that the deformed initial data set $(g+h,k+w)$ satisfies

$$\sigma(g+h,k+w)\geq\sigma(g,k)+u.$$

See [4, Definition 2] for a precise definition. The system $(\star)$ is a significant consequence of non-improvability.

Theorem 3 (Huang-Lee [4, Theorem 5])**.** *Let* (Ω,g,k) *be an initial data set (with possibly nonempty boundary). Then either the dominant energy scalar is improvable in* $\operatorname{Int}\Omega$, *or else there exists a nontrivial* (f,X) *on* $\operatorname{Int}\Omega$ *satisfying* $(\star)$.

Theorem 3 is the key result. To prove it, we introduce a new infinite-dimensional family of deformations from the modified constraint operator and show that generically, the adjoint linearizations of those modified operators are either injective, or else kernel elements satisfy a null-vector equation.

3. Mass minimizing initial data sets and Bartnik's conjectures

The most useful situation for improvability of $\sigma(g,k)$ is used to deform the dominant energy condition to the *strict* dominant energy condition in a compact subset. In [4, Section 7.1], we show that an ADM mass minimizing (asymptotically flat) initial data set (g,k) (among a suitable class of competitive initial data sets) must have $\sigma(g,k)\equiv 0$ and that $\sigma(g,k)$ is not improvable. There are several outstanding conjectures on characterizing an ADM mass minimizing initial data set, including the following one proposed by R. Bartnik in 1989.

Conjecture 4 (Bartnik's stationary conjecture)**.** *Let* (Ω_0,g_0,k_0) *be a compact initial data set with smooth boundary, satisfying the dominant energy condition, and suppose that* (M,g,k) *is a Bartnik mass minimizer for* (Ω_0,g_0,k_0). *Then*

$(\mathrm{Int}M, g, k)$ *sits inside a vacuum spacetime admitting a global timelike Killing vector field. Or in other words, it sits inside a vacuum (strongly) stationary spacetime.*

The above results imply that a Bartnik mass minimizing initial data set is a spacelike slice in a null dust spacetime having a Killing vector, which is null where the spacetime is not vacuum. On the other hand, by finding asymptotically flat spacelike hypersurfaces in a certain class of null dust spacetimes (the pp-waves) having nonzero, null ADM energy-momentum vector, we get counter-examples to Bartnik's stationary conjecture in dimensions greater than eight. Those examples are also the counter-examples to other closely-related conjectures: the equality case of the spacetime positive mass theorem without the optimal fall-off rate and Bartnik's strict positivity conjecture.

We end this report with another long-standing conjecture of Bartnik. Some recent progress has been made for the static case (i.e. $k \equiv 0$) (see [1] and the references therein), but the general case is still widely open. The *Bartnik boundary data* $B(g,k)$ on the boundary Σ of an initial data set (g,k) is defined as

$$B(g,k) = \big(g|_\Sigma, H_g, (k - (\mathrm{tr}k)g)(\nu, \cdot)\big)$$

where H_g is the mean curvature and ν is the unit normal vector on Σ.

Conjecture 5 (Bartnik's stationary vacuum extension conjecture)**.** *Let* (B^3, g_0, k_0) *be an initial data set satisfying the dominant energy condition. Suppose the expansion* $\theta := H_{g_0} + \mathrm{tr}_{S^2} k_0$ *on the boundary* S^2 *is not everywhere less or equal to zero. Then there exists a unique asymptotically flat, stationary vacuum initial data set* $(\mathbb{R}^3 \setminus B^3, g, k)$ *such that* $B(g,k) = B(g_0, k_0)$ *on* S^2.

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The non-linear stability of the Schwarzschild family of black holes

Gustav Holzegel

(joint work with Mihalis Dafermos, Igor Rodnianski, Martin Taylor)

The first part of the talk reviewed recent developments in the study of stability of black hole solutions in general relativity focussing on the Einstein vacuum equations with cosmological constant Λ,

$$Ric[g] = \Lambda g\,. \tag{1}$$

We refer here to [14], [11], [12], [18], [16, 19] and references therein.

The second and main part of the talk restricted to $\Lambda = 0$ in (1) and discussed the dynamics of (1) near the Schwarzschild family of solutions to (1),

$$g_M = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\,. \tag{2}$$

We recall that the nonlinear stability problem is naturally formulated in the context of the *Cauchy problem* [4] for (1), which associates a unique *maximal Cauchy development*, solving (1), to every *vacuum initial data set*.

Theorem 1 ([12])**.** *For vacuum initial data sets sufficiently close to appropriate Schwarzschild initial data, the resulting maximal Cauchy development*

- *(i) possesses a complete future null infinity $\mathcal{I}^+$ whose past $J^-(\mathcal{I}^+)$ is bounded to the future by a regular future complete event horizon $\mathcal{H}^+$,*
- *(ii) remains globally close to Schwarzschild* (2) *in its exterior and*
- *(iii) asymptotes back to a member of the Schwarzschild family as a suitable notion of time goes to infinity,*

provided that the initial data set itself lies on a codimension-3 "submanifold" of the moduli space of vacuum initial data.

We note that the restriction on data is a necessary condition for the asymptotic stability statement (iii) above. For as is well known, the Schwarzschild family (2) is contained as the $a = 0$ subcase of the larger Kerr family $g_{M,a}$. Outside our codimension-3 submanifold, one expects solutions to necessarily asymptote to a Kerr solution with $a \neq 0$, since the dimension of *linearised Kerr solutions fixing the mass* is equal to 3 in our parametrisation.[1]

Central to the proof of Theorem 1 is the use of a double null gauge to break the general covariance of (1). We emphasise that both the asymptotically stable codimension-3 submanifold itself and the double null gauges that we employ must be constructed "teleologically", i.e. on the basis of properties of their future evolution. The same applies to the final mass parameter M of the Schwarzschild solution g_M to which our metric g asymptotically settles down. (Indeed, in general, the only way to identify data for which (iii) holds, and to moreover determine M, is to evolve them towards the future under (1)!) Finally, the event horizon $\mathcal{H}^+$ will

[1] Theorem 1 can be compared with previous work of Klainerman–Szeftel [18], which restricted to the class of polarised axisymmetric initial data near Schwarzschild. As proven in [12], the class of [18] is contained as a set of infinite codimension in the codimension-3 set of Theorem 1.

represent an asymptotic hypersurface of our double null gauge and in particular is also only determined teleologically.

It is worth remarking that the global closeness of statement (ii) can be expressed at the top order energy level with respect to the same quantity that measures a suitable "initial" energy quantity, i.e. without loss of derivatives. In this sense, we have obtained a true orbital stability statement. As is well known, however, the supercriticality of the non-linearities of (1) means that the only path to proving the statements (i) and (ii) is through a full asymptotic stability statement (iii), and thus one does not expect to be able to obtain a statement involving (i)–(ii) alone.

The necessary starting point to proving (iii) is a robust approach to *linear stability* around the expected asymptotic state. This was provided by our (purely physical space based) method and framework introduced in [11], where we proved the linear stability of the Kerr family around Schwarzschild, in double null gauge. In particular, a corollary of [11] is that the Schwarzschild subfamily is itself linearly *asymptotically* stable precisely for data lying on a codimension-3 subspace, which, unlike the nonlinear setting, can be identified explicitly, as can the final linearised mass parameter and the location of the event horizon. We note, however, that the necessity of teleological normalisation of the double null gauge is already a key difficulty of linear theory. There it can be fully understood by the notion of (residual) *pure gauge solutions*, which are infinite dimensional families of explicit solutions to the linearised system of equations corresponding to infinitesimal diffeomorphisms preserving the double null form of the metric. Adding and subtracting such pure gauge solutions to a reference solution one can teleologically normalise the solution with respect to asymptotic hypersurfaces like the horizon $\mathcal{H}^+$ and future null infinity $\mathcal{I}^+$. A further crucial ingredient of the linear theory in [11] is to establish decay of two linearised null curvature components, which satisfy a decoupled wave equation in linear theory (the well-known Teukolsky equation [21]) and are moreover invariant under adding pure gauge solutions. This was achieved by combining transformations originally discovered by Chandrasekhar [3] in the context of mode solutions with analytical techniques developed in the last fifteen years (see [1, 8, 2, 9, 15, 10]) for understanding dispersion of waves outside of black holes. We note that decay for the Teukolsky equation has recently been obtained for the entire subextremal range $|a| < M$ of the Kerr family [20].

In addition to the linear theory of [11], the proof of Theorem 1 depends on many insights developed over the years, starting from the monumental proof of the stability of Minkowski space [7], for understanding the *nonlinearities* of (1), in particular, in the most difficult radiation zone towards null infinity $\mathcal{I}^+$, where null structure (cf. [17]) is paramount. This full understanding of the null structure elucidated by the geometric gauge used here allows one to understand in our black hole context subtle non-linear effects associated to radiation, for instance, Christodoulou memory [5].

We finally comment briefly on the modulation theory required to produce the codimension-3 family in Theorem 1. We decompose the space of initial data (which are prescribed on two characteristic hypersurfaces intersecting in a 2-sphere, see [6]) into disjoint 3-parameter families $D = D_0 + \sum_{m=-1}^{1} \lambda_m D_m^{Kerr}$, where D_0 varies over a suitable space. Here D_m^{Kerr} essentially prescribes the three $\ell = 1$ modes of the torsion on the sphere of intersection and the vector $(\lambda_{-1}, \lambda_0, \lambda_1)$ is a measure of the size of the angular momentum of the data. We prove that given any $d \in D_0$ we can find a $(\lambda_{-1}, \lambda_0, \lambda_1)$ such that the corresponding data set D converges to Schwarzschild. This is realised by bootstrapping a collection of estimates for the above 3-parameter family of initial data. The "final" angular momentum of the solution is measured at the latest asymptotic sphere of the bootstrap region and the family of data then appropriately restricted in the bootstrap to confine the future angular momentum vector to a suitably small sphere, whose smallness is inversely proportional to the Bondi-time along null infinity (consistent with how all the other modes of the solution decay). In practice it is the topological degree of the map from the space of λ's to the space of angular momentum in the future that is bootstrapped and implies that at every stage of the bootstrap the set of allowed λ's contains a tuple $(\lambda_{-1}, \lambda_0, \lambda_1)$ that gets mapped to zero angular momentum.

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On Strong Cosmic Censorship for $\Lambda < 0$

Christoph Kehle

The celebrated *Strong Cosmic Censorship conjecture* due to Penrose [1] states that for generic initial data for the Einstein equations

$$R_{\mu\nu}[g] = \Lambda g_{\mu\nu}, \tag{1}$$

or more general the Einstein–Maxwell system (or other reasonable matter models)

$$R_{\mu\nu}[g] = \Lambda g_{\mu\nu} + 2\left(F_\mu^\lambda F_{\lambda\nu} - \frac{1}{4} g_{\mu\nu} F_{\lambda\kappa} F^{\lambda\kappa}\right), \nabla^\mu F_{\mu\nu} = 0, \nabla_{[\mu} F_{\nu\lambda]}, \tag{2}$$

the corresponding unique maximal solution is inextendible as a suitably regular Lorentzian manifold. The word 'generic' is clearly necessary in the conjecture as the explicit Kerr–(A/dS) black hole solutions $(\mathcal{M}, g_{\Lambda,M,a})$ to (1) or the Reissner–Nordström–(A/dS) black hole solutions $(\mathcal{M}, g_{\Lambda,M,Q})$ to (2) admit a future boundary (a so-called *Cauchy horizon*) in the interior of the black hole across which spacetime is smoothly but non-uniquely extendible (assuming $a \neq 0$, $Q \neq 0$, respectively). Thus, a natural starting point to address the issue of regular Cauchy horizons and the validity of the Strong Cosmic Censorship is to study the problem around Kerr–(A/dS) or Reissner–Nordström–(A/dS).

In addition to specifying a suitable notion of genericity for the Strong Cosmic Censorship conjecture, one also has to make precise in which regularity class the solution is inextendible. The strongest, most definitive resolution of the conjecture would be a proof of the so-called C^0*-formulation*, where generically, spacetime is inextendible as a Lorentzian manifold with continuous metric.

In the remarkable work [2] Dafermos–Luk prove that the C^0-formulation of the Strong Cosmic Censorship conjecture is *false* for $\Lambda = 0$ and $\Lambda > 0$. They show that perturbations settling down to Kerr (or Kerr–dS) admit a Cauchy horizon across which the metric is continuously extendible. Nevertheless, it is still expected that the weaker H^1-formulation of Strong Cosmic Censorship remains true (at least for $\Lambda = 0$), for which generically the metric is conjectured to be inexendible with square integrable Christoffel symbols.

Turning now to the case of negative cosmological constant $\Lambda < 0$, a first step in addressing the validity of the C^0-formulation of Strong Cosmic Censorship for $\Lambda < 0$ is to consider its linear scalar analog for the conformal wave equation

$$\Box_g \psi - \frac{2}{3}\Lambda\psi = 0 \tag{3}$$

on Kerr–AdS and Reissner–Nordström–AdS. The linear scalar analog of the C^0-formulation of Strong Cosmic Censorship on the level of (3) states that solutions to (3) arising from generic initial data on a spacelike hypersurface with reflecting (Dirichlet or Neumann) boundary conditions at infinity blow up in amplitude at the Kerr–AdS or Reissner–Nordström–AdS Cauchy horizon. At this point, we note that the result [2] by Dafermos–Luk on the falsification for the fully nonlinear C^0-formulation of Strong Cosmic Censorship for $\Lambda \geq 0$ was preceded by the corresponding linear analog on the level of (3) in [3, 4]. A key ingredient to the proofs in [3, 4] and in [2] is an inverse polynomial (Kerr) or exponential (Kerr–dS) decay rate of solutions to (3) on the exterior of Kerr or Kerr–dS. For the present $\Lambda < 0$ case, the dynamics on the exterior are radically different from the dynamics for $\Lambda = 0$ and $\Lambda > 0$: More precisely, solutions ψ to (3) only decay at a sharp inverse logarithmic rate [5, 6] for which the methods developed by Dafermos–Luk manifestly do not apply. This could mean that the linear scalar analog of the C^0-formulation holds true after all for $\Lambda < 0$. It however turns out that despite the slow inverse logarithmic decay on the Reissner–Nordström–AdS exterior, the linear scalar analog of the C^0-formulation is *false* for Reissner–Nordström–AdS.

Theorem 1 ([7])**.** *Linear perturbations ψ solving* (3) *on Reissner–Nordström–AdS and arising from regular Cauchy data posed on a spacelike hypersurface with Dirichlet boundary conditions at infinity remain uniformly bounded*

$$|\psi| \leq C \tag{4}$$

in the black hole interior and extend continuously across the Cauchy horizon.

A key insight in the proof of Theorem 1 is that the following two sources of instability—which may lead to blow-up in amplitude at the Cauchy horizon—are decoupled. On the one hand, the reason for the slow decay and the unstable behavior on the exterior is a stable trapping phenomenon. This trapping is associated with the high frequency part ($|\omega|, \ell$ large) of the solution ψ, while the low frequency part decays superpolynomially. In the black hole interior, however, it was observed in [8] that low frequencies ($|\omega|$ small) measured with respect to the null generator $T = \partial_t$ of the Cauchy horizon may lead to blow-up at the Cauchy horizon due to an interior scattering pole at $\omega = 0$. High frequencies can be controlled in the interior by suitable energy fluxes on the event horizon. In that sense, for Reissner–Nordström–AdS, the two sources of instability decouple in Fourier space which finally led to a falsification of the linear scalar analog of the C^0-formulation of Strong Cosmic Censorship for Reissner–Nordström–AdS. One may still wonder whether at least the linear scalar analog of the weaker, H^1-formulation of Strong

Cosmic Censorship holds true for Reissner–Nordström–AdS. This question was answered in the affirmative.

Theorem 2 ([9]). *Linear perturbations ψ solving* (3) *on Reissner–Nordström–AdS and arising from generic regular Cauchy data posed on a spacelike hypersurface with Dirichlet boundary conditions at infinity have infinite local energy along hypersurfaces intersecting transversally the Cauchy horizon $\mathcal{CH}$, i.e. ψ fails to be in H^1_{loc} around any point on $\mathcal{CH}$.*

Together with Theorem 1, this settles the linear analog of the Strong Cosmic Censorship conjecture for Reissner–Nordström–AdS.

For Kerr–AdS there is yet another layer of complexity. Due to the angular rotation of the black hole, a frequency mixing phenomenon occurs and high frequencies on the exterior can at the same time be low frequency if frequency is measured with respect to the Killing generator of the Kerr–AdS Cauchy horizon $K_- = T + \omega_- \Phi$. More precisely, as in the case of Reissner–Nordström–AdS, the slowly decaying high frequency part of the solution is dominated by a quasi-discrete set of high frequencies, so-called quasimode frequencies, which can be indexed by $\omega \in (\omega_{m,n,\ell})_{n,n,\ell}$. On the other hand, the zero frequency interior scattering poles, which may lead to blow-up in amplitude of ψ at the Cauchy horizon, are located at the discrete set of frequencies $\omega = \omega_- m$, where m is the azimuthal number and ω_- is a constant only depending on the black hole parameters mass, angular momentum and cosmological constant. One may now see, that the frequency decoupling phenomenon of Reissner–Nordström–AdS does not extend to Kerr–AdS, and it is possible that high stably trapped quasimode frequencies $\omega_{n,m,\ell}$ can at the same time be low frequency measured with respect to the Killing generator of the Cauchy horizon in the sense that $\omega_{n,m,\ell} - \omega_- m$ is small. It turns out that generic solutions to (3) blow up in amplitude at the Cauchy horizon if the above two sets are sufficiently resonant. This naturally gives rise to a small divisors problem and a generalized non-Diophantine condition. In particular, this non-Diophantine condition was shown to hold true for a Baire-generic but Lebesgue-exceptional subset of dimensionless black hole parameters $\mathfrak{m} := M\sqrt{-\Lambda}, \mathfrak{a} := a\sqrt{-\Lambda}$ respecting the Hawking–Reall bound. In order to state the following theorem, we denote with $\mathcal{P} \subset \mathbb{R}^2$ the set of all dimensionless mass and angular momentum parameters $\mathfrak{m}$, $\mathfrak{a}$ satisfying the Hawking–Recall bound.

Theorem 3 ([10]). *There exists a subset $\mathcal{P}_{\text{Blow-up}} \subset \mathcal{P}$ such that the following is true. Consider a Kerr–AdS black hole with parameters $\mathfrak{m}, \mathfrak{a} \in \mathcal{P}_{\text{Blow-up}}$. Then, linear perturbations ψ solving* (3) *and arising from generic regular Cauchy data posed on a spacelike hypersurface with Dirichlet boundary conditions at infinity blow up in amplitude at the Cauchy horizon*

$$\|\psi(x)\|_{L^\infty(\mathbb{S}^2)} \to \infty \text{ as } x \to \mathcal{CH}. \tag{5}$$

Moreover, the set $\mathcal{P}_{\text{Blow-up}}$ *as a subset of* $\mathcal{P}$ *satisfies*

- $\mathcal{P}_{\text{Blow-up}}$ *is dense,*
- $\mathcal{P}_{\text{Blow-up}}$ *is Baire-generic (of second category),*
- $\mathcal{P}_{\text{Blow-up}}$ *is a Lebesgue exceptional set (of Lebesgue measure zero).*

We shall complement the above theorem with the following conjecture, which is motivated by the proof in [10].

Conjecture 4 ([10])**.** *There exists a subset* $\mathcal{P}_{\text{Bounded}} \subset \mathcal{P}$ *such that the following is true. Consider a Kerr–AdS black hole with parameters* $\mathfrak{m}, \mathfrak{a} \in \mathcal{P}_{\text{Bounded}}$*. Then, linear perturbations* ψ *solving* (3) *and arising from regular Cauchy data posed on a spacelike hypersurface with Dirichlet boundary conditions at infinity remain uniformly bounded at the Cauchy horizon*

$$|\psi| \leq C. \tag{6}$$

Moreover, the set $\mathcal{P}_{\text{Bounded}}$ *as a subset of* $\mathcal{P}$ *satisfies*

- $\mathcal{P}_{\text{Bounded}}$ *is dense,*
- $\mathcal{P}_{\text{Bounded}}$ *is Lebesgue-generic (of full Lebesgue measure),*
- $\mathcal{P}_{\text{Bounded}}$ *is a Baire-exceptional (of first category).*

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Periodic non-linear waves on an AdS background

Jacques Smulevici
(joint work with Athanasios Chatzikaleas)

The simplest solutions to the Einstein equations $Ric(g) = \Lambda g$ are the Minkowski space ($\Lambda = 0$), the de-Sitter space ($\Lambda > 0$) and the Anti-de-Sitter (AdS) space ($\Lambda < 0$). While the small data dynamics for $\Lambda \geq 0$ are now very well understood, little is known rigorously in the Anti-de-Sitter case. One of the first difficulties is the lack of global hyperbolicity of the AdS space, which forces every dynamical property of asymptotically AdS spaces to be dependent on a choice of boundary conditions imposed at infinity. In this setting, two conjectures have been formulated in the two opposite regimes of reflective and dissipative boundary conditions.

Conjecture 1 (Dafermos-Holzegel, Anderson)**.** *The AdS space is unstable for reflective boundary conditions.*

Conjecture 2 (Holzegel-Luk-Smulevici-Warnick)**.** *The AdS space is stable for dissipative boundary conditions.*

In the talk, I focused on the case of reflective boundary conditions. The instability conjecture was first seriously investigated in the work of Bizón and Rostworoski [1] for the spherically-symmetric Einstein-Klein-Gordon (EKG) system. They indeed verified numerically the presence of the instability and suggested a Fourier space mechanism as the source of the instability. This pioneering work led to many important developments, see the review [2]. While there is no rigrorous proof of the instability in this setting, based on a physical space mechanism, Moschidis has recently established the instability of the AdS space for various spherically-symmetric Einstein-matter models [3]. Another important work in this context is the one of Maliborski and Rostworowski [4] who constructed numerically small data time-periodic solutions for the EKG system. They also explained how to construct a formal serie representing their solutions via a Fourier decomposition and an expansion in terms of the smallness parameter. This shows in particular that not all data in the neighborhood of AdS are expected to exhibit the instability mechanism. Despite several subsequent works, see for instance [5], there are no proof of the existence of such time periodic solutions yet. See however [6] that rigorously prove the existence of a formal serie in the syle of Maliborski and Rostworowski and who also derived sharp asymptotics for some of the Fourier coefficients of the EKG system.

The aim of the talk was to present recent results [12] obtained in collaboration with Athanasios Chatzikaleas where we prove the existence of such time-periodic solutions for several semi-linear toy models on either a fixed AdS space or the Einstein cylinder $\mathbb{R}_t \times \mathbb{S}^3$. The Einstein cylinder appears naturally in this contect since the AdS is conformal to one-half of the Einstein cylinder, so that conformally covariant equations on an AdS space can be mapped to the Einstein cylinder, modulo appropriate boundary conditions. The equations we considered are the conformal cubic wave equation and the spherical symmetric Yang-Mill equations.

These equations have been proposed as good toy models for dynamical problems related to AdS and have been the subject of several previous works, mostly related to the behaviour of the associated resonance systems [8, 9]. In [10], an approach in the style of [4] was developped using formal pertubative series. In the same work, it is observed that small initial data which are proportional to the lowest mode of the linearized operator lead to time-periodic solutions of the spherically symetric conformal cubic wave equation. Unfortunately, this simple behaviour does not generalize to the higher modes, since data initially proportional to any of those will typically excite all the other modes.

In the talk, I focused on the conformal cubic wave equation on the Einstein cylinder within spherical symmetry, which reduces to the equation

$$-\partial_t^2 u + \frac{1}{\sin^2(x)}\partial_x\left(\sin^2(x)\partial_x u\right) - u = u^3, \tag{1}$$

where $u := u(t,x)$, $x \in (0,\pi)$. Solutions which are odd with respect to $x = \pi/2$ correspond, after a conformal transformation, to solutions to the conformal cubic wave equation on AdS with Dirichlet boundary condition.

The construction of time-periodic solutions in the presence of confined dynamics has a long history and is still an active subject of research related to the so-called KAM theory for PDEs. The starting point is to consider the linear operator

$$Lu := \frac{1}{\sin^2(x)}\partial_x\left(\sin^2(x)\partial_x u\right) - u,$$

acting on its natural domain. This is a self-adjoint operator, with compact resolvent, and thus it possess a complete set of orthonormal eigenvectors, denoted e_n and corresponding eigenvalues ω_n^2. In fact, one has $e_n(x) := \frac{\sin \omega_n x}{\sin x}$, and $\omega_n = n+1$.

Consider now a formal solution $\psi(x)$ of size $\epsilon << 1$ centered around a one mode-solution corresponding to the eigenvector e_γ and admitting a representation of the form

$$\psi(t,x) = \epsilon \cos \tau e_\gamma(x) + \sum_{\lambda=1}^{\infty}\left(\sum_{m=0} f^{(m)}_{2\lambda+1}(\tau) e_m(x)\right) e^{2\lambda+1},$$

where $\tau = \Omega t$ and $\Omega = \omega_\gamma + 0(\epsilon^2)$ is a pertubation of the original frequency associated to e_γ. After plugging the ansatz into the equations, one obtain an infinite system of harmonic oscillator equations of the form

$$\frac{d^2}{d\tau^2} f^{(m)}_{2\lambda+1} + \left(\frac{\omega_m}{\omega_\gamma}\right)^2 f^{(m)}_{2\lambda+1} = S^{(m)}_{2\lambda+1},$$

where the source terms $S^{(m)}_{2\lambda+1}$ arise from the non-linearity and involves products of the f coefficients. Depending on the oscillations of the source terms, we may then obtain periodic solutions to the above odes or solutions growing in time, the so-called *secular terms.*

The work [4] consists for instance in fine tuning the initial data to construct numerical solutions which contains no secular terms. On the other hand, it seems difficult to then prove a posteriori that the resulting formal serie actually converges to a regular solution. Thus, we do not rely on such arguments and instead we use a theorem of Bambusi and Paleari [11]. Based on this approach, we are able to prove the following theorem.

Theorem 3. *Let e_γ be an eigenvector of the linearized operator L. Let $s > 3/2$. Then, there exists an uncountable set E containing* 0 *as an accumulation point, such that any initial data*

$$u(t=0) = \epsilon e_n, \quad \partial_t u(t=0) = 0,$$

with $\epsilon \in E$, gives rise to a time–periodic solution u_ϵ to (1) *with period $\frac{2\pi}{\delta_\epsilon}$, such that*

(1) $u_\epsilon \in H^1([0, 2\pi/\delta_\epsilon]\,; H^s)$, and $\delta_\epsilon \simeq 1$,
(2) u_ϵ stays close to the solution to the linearized equation with the same initial data as above and zero initial velocity.

Remark 4. *The theorem of Bambusi and Paleari* [11] *on which the proof of Theorem* (3) *requires the existence of a non-degenerate zero for a non-linear operator obtained by averaging the equation in time along the linear flow and a projection on the space of initial data. In the simple setting of the spherically symmetric conformal cubic wave equation, the operator can be computed explictly and one can easily verifies that, after a suitable rescaling, any of the eigenvectors e_n are zeros. On the other hand, the non-degeneracy condition amount to inverting an infinite dimensional system of linear equations.*

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Some Rigorous Results about the Past and Future Behavior of Expanding Vacuum Spacetimes

John Lott

Suppose that we have a cosmological spacetime M, i.e. it is globally hyperbolic with a compact Cauchy hypersurface. I'm interested in expanding vacuum spacetimes. By Hawking's singularity theorem, there is a past singularity. Two basic questions:
1. What is the behavior as one approaches the past singularity?
2. What is the future behavior?

We will describe some rigorous results from [1, 2, 3]. For concreteness, we assume here a 3+1 dimensional spacetime, with compact spatial slices. (Some of the results work in any dimension, and for noncompact spatial slices.) We also assume a constant mean curvature (CMC) foliation by compact spatial hypersurfaces, whose mean curvatures H approach $-\infty$ in the past. Define the Hubble time by $t = -\frac{3}{H}$.

Using the foliation, one can write the metric as

$$g = -L^2 dt^2 + h(t),$$

where $L = L(t)$ is a function on the 3-manifold X and $h(t)$ is a Riemannian metric on X. Let $K(t)$ denote the second fundamental form. Let us define a rescaling as follows. Given $s > 0$, put

$$g_s = - L^2(su)\, du^2 + s^{-2} h(su).$$

It is isometric to $s^{-2}g$, and so is also a vacuum solution, with Hubble time u. Hence we put

$$\begin{aligned} L_s(u) =& L(su), \\ h_s(u) =& s^{-2} h(su), \\ K_s(u) =& s^{-1} K(su). \end{aligned}$$

We now define a scale-invariant curvature condition.

Definition 1. A CMC vacuum solution is type-I if $|\operatorname{Rm}| \leq C t^{-2}$ as $t \to 0$.

The first main result characterizes when the $t \to 0$ asymptotics are Milne-like. Let $\mathcal{M}$ be the set of flat Milne spacetimes $(0, \infty) \times H^3/\Gamma$.

Theorem 2. [3] *If* $\operatorname{dvol}_{h(t)}(x) = O(t^3)$ *as* $t \to 0$ *then the blowup rescalings* g_s *approach* $\mathcal{M}$ *as* $s \to 0$.

That is, the original spacetime becomes increasing Milne-like, as measured around the point x, as one approaches the initial singularity.

Next, we characterize Kasner-like asymptotics.

Definition 3. The CMC vacuum solution has *asymptotically nonpositive spatial scalar curvature* if $\limsup_{t \to 0} \sup_{x \in X} t^2 R(t, x) \leq 0$.

Definition 4. (Kasner-like solutions) $\mathcal{K}$ is the set of type-I CMC vacuum solutions with $R = 0$, $L = \frac{1}{3}$ and $|K|^2 = H^2$.

Theorem 5. [3] *Suppose that we have a type-I CMC vacuum solution. Suppose that it has asymptotically nonpositive spatial scalar curvature. If $t^{-1}\,\mathrm{dvol}_{h(t)}(x)$ is (positively) bounded below as $t \to 0$ then the blowup rescalings g_s approach $\mathcal{K}$ as $s \to 0$.*

Mixmaster solutions do not quite have the asymptotics of the preceding theorem. Instead, we have the following result.

Theorem 6. [3] *Suppose that we have a type-I CMC vacuum solution. Suppose that it has asymptotically nonpositive spatial scalar curvature. Suppose that for each $\beta > 0$, $\mathrm{dvol}_{h(t)}(x)$ fails to be $O\left(t^{1+\beta}\right)$ as $t \to 0$. Put $\tau = \log(1/t)$, so $\tau \to \infty$ corresponds to approaching the singularity. Then as $\tau \to \infty$, the proportion of τ-time that the solution spends near $\mathcal{K}$ goes to one.*

There are related results about the future behavior of an expanding CMC vacuum spacetime, where one takes blowdown limits rather than blowup limits [1].

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The stability of charged black holes

Elena Giorgi

One of the fundamental problems in General Relativity is to understand the final state of evolution of initial data for the Einstein equation. Through gravitational collapse and dispersion of gravitational waves, the geometry to which solutions to the Einstein equation are expected to relax outside the event horizon of a black hole is the one given by the known stationary and axisymmetric explicit solutions: the Kerr and the Kerr-Newman black hole.

The interaction between gravitational and electromagnetic fields in a spacetime is governed by the Einstein-Maxwell equation. The Kerr-Newman metric is the most general known explicit black hole solution to the Einstein-Maxwell equation, and it is a 3-parameter family which describes the gravitational field around an isolated rotating charged black hole of mass M, angular momentum Ma and electric charge Q, within the subextremal range $\sqrt{a^2 + Q^2} < M$. The Kerr-Newman metric generalizes the Reissner-Nordström solution (for $a = 0$), and also the Kerr (for $Q = 0$) and Schwarzschild metric (for $Q = a = 0$), which are solutions to the Einstein vacuum equation. As such, the Kerr-Newman spacetime plays a fundamental role in describing the final state of evolution in General Relativity.

As part of the resolution of the description of the final state, we focus on the issue of stability of the Kerr-Newman black hole, which consists in showing that solutions to the Einstein equation which are given as small perturbations of the initial data of such a black hole asymptotically converge in time to a member of the Kerr-Newman family. The stability of the Kerr-Newman family can be analyzed at different levels:

(1) the *linear stability* consists in the analysis of the linearized Einstein-Maxwell equation around the background metric $g_{M,a,Q}$. It can be further divided into (a) mode stability, and (b) full linear stability.
(2) the *non-linear stability* consists in the analysis of the full Einstein-Maxwell equation for a perturbation of a member of the Kerr-Newman family.

The mode analysis (a) of the Einstein equation consists in analyzing only special solutions, the so-called mode solutions, which are of the separated form

$$\psi(r,t,\theta,\phi) = e^{-i\omega t}e^{im\phi}R(r)S(\theta) \tag{1}$$

where $\omega \in \mathbb{C}$ is the time frequency, and m is the azimuthal mode. Because of the integrability of the geodesic flow in the Kerr-Newman metric, functions of the form (1) are solutions as long as $R(r)$ satisfies a radial ODE and $S(\theta)$ satisfies an angular ODE (which defines spheroidal harmonics $S_{\omega m\ell}$). The mode stability consists in proving that solutions of the form (1) with finite initial energy do not exponentially grow in time. The mode stability of Schwarzschild, Reissner-Nordström and Kerr black hole was obtained as a combination of many results in black hole perturbation theory by the physics community in the 70s and 80s, see [10], [13], [1], [2], [11], [12].

Extensive progress has been obtained in the last fifteen years which allowed to go beyond the mode analysis in Kerr spacetime, tackling the full linear stability (b) for the linear wave equation. A robust geometric interpretation of the redshift effect [3], a physical space analysis of the trapping region and the superradiance [5], a hierarchy of r-weighted decay [4] all contributed to a complete understanding of the boundedness of solutions to the linear wave equation.

Quite strikingly, the Kerr-Newman solution stands up as genuinely different from the similar cases of Kerr or Reissner-Nordström, even in the simplest possible form of stability, i.e. the mode stability as studied by the black hole perturbation theory community. As stated by Chandrasekhar in Section 111 of [2], "the methods that have proved to be so successful in treating the gravitational perturbations of the Kerr spacetime do not seem to be applicable (nor susceptible to easy generalizations) for treating the coupled electromagnetic-gravitational perturbations of the Kerr-Newman spacetime." The techniques applied in those early works, which relied on decomposition in frequency modes of perturbations of the solutions, failed to be extended to the case of Kerr-Newman spacetime.

The Einstein-Maxwell equation governs the interaction between the gravitational radiation, encoded in the left hand side of the equation (i.e. the curvature), and the electromagnetic radiation, encoded in the right hand side (i.e. the electromagnetic tensor). From the study of perturbations of Kerr, we know that the gravitational and the electromagnetic radiation are transported by a spin-2 field $\psi^{[2]}$ and a spin-1 $\psi^{[1]}$ respectively. When taken independently, the gravitational and electromagnetic perturbations of Kerr satisfy the Teukolsky equation [11] for spin $s = \pm 2$ or $s = \pm 1$ respectively. On the other hand, when considering coupled electromagnetic-gravitational perturbations of Kerr-Newman, one should expect a system of coupled Teukolsky equations, as in the case of Reissner-Nordström [7] [8]. The issue in the analysis of a coupled system comes from the decomposition in modes. The mode decomposition of the Teukolsky variables involves the *spin s-weighted spheroidal harmonics* $S^{[s]}_{m\ell}(a\omega, \cos\theta)$ which are eigenfunctions of the spin s-weighted laplacian. For $a = 0$, they reduce to the spherical harmonics $S^{[s]}_{m\ell}(0, \cos\theta) = Y^{[s]}_{m\ell}(\cos\theta)$. *Spin-weighted spherical harmonics* of different spins are simply related through the angular operators appearing on the right hand side of the coupled equations, and have the same eigenvalues. On the other hand, in the general axisymmetric case, as in Kerr or Kerr-Newman, the spin-weighted spheroidal harmonics of different spins are not simply related through those angular operators.

This in fact explains the "apparent indissolubility of the coupling between the spin-1 and spin-2 fields" [2] for electromagnetic-gravitational perturbations of Kerr-Newman, in contrast with Reissner-Nordström or Kerr. In treating the coupled electromagnetic-gravitational perturbations of Kerr-Newman spacetime, the decomposition in modes of the equations, which had the objective of simplifying the analysis of the perturbations, actually makes them unsolvable.

Our approach to solve this issue is to abandon the decomposition in modes, and perform a physical space analysis of the equations. Following the road map that mathematicians have taken in the last few years in interpreting in physical space the mode analysis done by the physics community, the Kerr–Newman solution may be the case where a physical space approach could succeed where the mode analysis in physics failed. Observe that our proof of boundedness of a general solution through a physical space analysis will in particular imply the absence of exponentially growing modes, therefore proving mode stability.

We derived [9] the equations governing the linear stability of Kerr-Newman spacetime to coupled electromagnetic-gravitational perturbations. The equations generalize the celebrated Teukolsky equation for curvature perturbations of Kerr, and the Regge-Wheeler equation for metric perturbations of Reissner-Nordström. Using a tensorial approach that was applied to Kerr [6], we produce a set of generalized Regge-Wheeler equations for perturbations of Kerr-Newman, which are suitable for the study of linearized stability by physical space methods. The physical space analysis overcomes the issue of coupling of spin-1 and spin-2 fields and represents the first step towards an analytical proof of the stability of the Kerr-Newman black hole.

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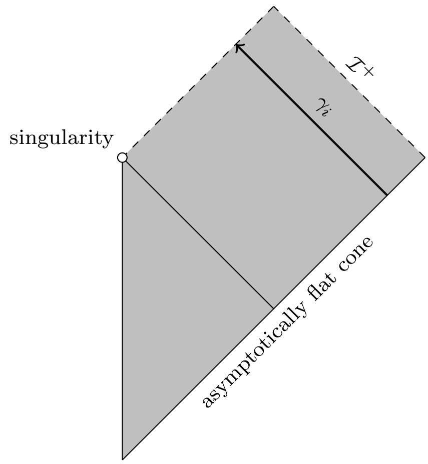
Naked Singularities for the Einstein Vacuum Equations

Yakov Shlapentokh-Rothman
(joint work with Igor Rodnianski)

The presumed validity of Penrose's Weak Cosmic Censorship Conjecture [11] (WCCC) fundamentally undergirds the modern understanding of classical General Relativity in $3+1$ dimensions:

Conjecture 1 (Weak Cosmic Censorship for the Einstein Vacuum Equations)**.** *For generic Cauchy data which is complete, regular, and asymptotically flat, singularities in the corresponding maximal development are hidden inside a black hole.*

A singular spacetime which does not contain a black hole is referred to as a "naked singularity." A representative Penrose diagram is as follows:



Here the curves $\{\gamma_i\}$ represent a family of ingoing null geodesics which terminate within a fixed affine time $A > 0$ and so that the initial velocity vectors $\{\dot{\gamma}_i(0)\}$ are all parallel transports of a fixed null vector along the asymptotically flat initial cone.

We have stated the conjecture informally here. As with other fundamental questions in General Relativity (cf. the role of the regularity of Cauchy horizons in the Strong Cosmic Censorship conjecture [4, 7, 8]), the precise meanings of "regular" and "generic" are very important and, in principle, may affect the validity of the conjecture.

Our main source of intuition regarding the WCCC is provided by a sequence of works by Christodoulou [1, 2, 3] which both established the validity of the analogue of the WCCC for the spherically symmetric Einstein-scalar field system and showed that the assumption of genericity was necessary. (In fact, it is primarily due to these works that is commonly understood that a genericity condition is needed in the WCCC.)

Theorem 2. *There exist naked singularities for spherically symmetric Einstein-scalar field system! However, generically, naked singularities do not occur.*

The naked singularities of Christodoulou are not smooth; the scalar field has a derivative which is only Hölder continuous. However, the solutions are *more* regular than the so-called "solutions of bounded variation" for which Christodoulou established a well-posedness result in [1], and thus we may consider them as legitimate examples of a naked singularity. (It is nevertheless interesting to find examples of naked singularities corresponding to smooth initial data.)

Given the existence of Christodoulou's naked singularities, one is lead to immediately ask about the existence of analogous solutions for the Einstein vacuum equations. In joint and ongoing work with Igor Rodnianski we show that such solutions do indeed exist.

Theorem 3. *There exist naked singularities for the Einstein vacuum equations. Moreover, these solutions may be considered analogous to the solutions of Christodoulou.*

As with Christodoulou's solutions, these are not smooth spacetimes; a particular null derivative of the metric is only Hölder continuous. However, this singular behavior is compensated by arbitrary regularity in all other directions and one expects a well-posedness result for the Einstein vacuum equations which encompasses such data (cf. [9, 10]).

Both Christodoulou's solutions and our solutions are *self-similar*. That is, there exists a vector field K so that $\mathcal{L}_K g = 2g$. An important role is played by our previous work [12] which showed the existence of self-similar solutions which corresponded to the formal expansions of Fefferman–Graham [5, 6]. In suitable double-null coordinate system the vectorfield K for these solutions satisfies

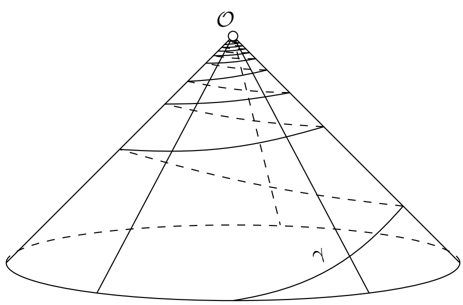
$$K = u\partial_u + v\partial_v, \tag{1}$$

and the singularity in the spacetime corresponds, formally, to $(u, v) = (0, 0)$.

However, due to an underlying rigidity for Fefferman–Grahams expansions in $3+1$ dimensional spacetimes, we cannot directly use these spacetimes in the proof of our theorem. Instead, we find a fundamentally new type of self-similarity for the Einstein vacuum equations where, along the past light cone of the singularity, the self-similar vectorfield twists relative to the null generators. (In contrast, Fefferman–Graham spacetimes always have that the self-similar vectorfield is *null* along the past light cone of the singularity.) For these new solutions, if we work in regular double-null coordinates $(u, \hat{v})$ along the past light cone of the singularity, the formula (1) must be changed to

$$K = u\partial_u + (1 - 2\kappa)\,\hat{v}\partial_{\hat{v}}, \qquad |\kappa| \ll 1. \tag{2}$$

The following schematic diagram indicates the twisting of K:



Here γ represents an orbit of the self-similar vectorfield, and the straight lines correspond to the null generators of the cone.

This "twisted" self-similarity requires us to allow for certain derivatives of the metric along this cone to be very large. A priori, one may be worried that this largeness will result, in the region exterior to the cone, in the formation of a trapped surface and hence a black hole. However, we identify a mechanism by which certain of these twisted self-similar solutions quickly dampen this largeness.

This damping mechanism allows us to globalize the construction and eventually show that suitable choices of these twisted self-similar cones may be embedded in a spacetime representing a naked singularity.

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From vacuum to dust to Vlasov

Jonathan Luk

(joint work with Cécile Huneau)

1. Main result

We present a work in progress concerning the high-frequency limit of vacuum spacetimes. Specifically, we construct examples of sequences of vacuum spacetimes, whose limit is a solution to the Einstein–massless Vlasov system (with a non-trivial Vlasov field). See Section 2 for further background on this problem.

Our setup imposes $\mathbb{U}(1)$ symmetry. More precisely, we consider metrics on $\mathcal{M} \doteq M \times \mathbb{S}^1$, with $M \doteq [0,1] \times \mathbb{R}^2$, which are translational invariant along the $\mathbb{S}^1$ direction. As is well-known, under this symmetry, the Einstein vacuum equations (resp., the Einstein–massless Vlasov system) on $\mathcal{M}$ reduce to the Eisntein–wave map system (resp., the Einstein–wave map–massless Vlasov system) on M.

In what follows, we consider the following form of the Einstein–wave map–massless Vlasov system on M:

$$(1) \qquad \begin{cases} R_{\mu\nu}(g) = 2\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{-4\phi}\partial_\mu\varpi\partial_\nu\varpi + \int_{\mathbb{S}^1} f_\omega^2\partial_\mu u_\omega\partial_\nu u_\omega \, dm(\omega), \\ \Box_g\phi + \frac{1}{2}e^{-4\phi}g^{-1}(d\varpi, d\varpi) = 0, \\ \Box_g\varpi - 4g^{-1}(d\varpi, d\phi) = 0, \\ 2g^{-1}(du_\omega, df_\omega) + (\Box_g u_\omega)f_\omega = 0, \quad \forall\omega \in \mathbb{S}^1, \\ g^{-1}(du_\omega, du_\omega) = 0, \ u_\omega \restriction_{\{t=0\}} = x\cdot\omega, \ \partial_t u_\omega \restriction_{\{t=0\}} > 0, \ \forall\omega \in \mathbb{S}^1. \end{cases}$$

where $\omega \in \mathbb{S}^1$ parametrizes the direction of the Vlasov field (f_ω, u_ω), and $m(\omega)$ is a probability measure on $\mathbb{S}^1$. (This parametrization of the Vlasov field will be convenient for the proof; see the parametrix in Section 3 below.)

We will, in addition, impose an elliptic gauge stipulating that g takes the form

$$g = -n^2dt^2 + \sum_{i,j=1}^{2} \delta_{ij}e^{2\gamma}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$$

with constant-t hypersurfaces being maximal (where δ_{ij} denotes the Kronecker delta).

The following is our main theorem:

Theorem 1 (Huneau–Luk, to appear)**.** *Given a generic, small, localized, smooth solution $(g_0, \phi_0, \varpi_0, f_0, u_0, m_0)$ to (1) on M in the elliptic gauge described above, there exists a sequence of smooth solutions $\{(g_n, \phi_n, \varpi_n)\}_{n=1}^\infty$ to the Einstein–wave map system (i.e., solutions to (1) with $f \equiv 0$) on M in the same elliptic gauge such that*

$$\|g_n - g_0\|_{L^\infty}, \ \|\phi_n - \phi_0\|_{L^\infty}, \ \|\varpi_n - \varpi_0\|_{L^\infty} \le \lambda_n,$$
$$\|\partial(g_n - g_0)\|_{L^4}, \ \|\partial(\phi_n - \phi_0)\|_{L^4}, \ \|\partial(\varpi_n - \varpi_0)\|_{L^4} \lesssim |\log\lambda_n|.$$

where λ_n is a decreasing sequence of positive numbers such that $\lambda_n \to 0$.

2. Background

- Our result can be viewed in the context of conjectures of Burnett. In [1], Burnett considered a sequence of vacuum spacetimes $\{g_n\}_{n=1}^\infty$ such that in local coordinates
$$\|\partial^i(g_n - g_0)\|_{L^\infty} \le \lambda_n^{1-i}, \quad i = 0, 1.$$
He conjectured in the forward direction that the limit g_0 must be isometric to a solution to the Einstein–massless Vlasov system, and that, in the reverse direction, any solution to the Einstein–massless Vlasov system must arise as such a limit of vacuum solutions.
- The only general result was obtained by Green–Wald [2], who showed that the stress–energy–momentum tensor corresponding to the limit spacetime must be traceless and satisfy the weak energy condition.
- In the case where the high-frequency oscillations are restricted to two null directions adapted to a system of double null coordinates, a complete

classification of the limit was achieved in [7] using the local well-posedness theory for (what the authors called) angularly regular spacetimes in [6].

- Under the $\mathbb{U}(1)$ symmetry assumption and the elliptic gauge condition as in Section 1, the forward direction of a version of Burnett's conjecture was established in our previous work [5] (see also [3]).
- As for the reverse direction of Burnett's conjecture under the same symmetry and gauge conditions, a special case of Theorem 1 has been proven in our earlier work [4]:

Theorem 2. *Theorem 1 holds under the following additional assumptions on $(g_0, \phi_0, \varpi_0, f_0, u_0, m_0)$:*

(1) The solution is, in addition, polarized, i.e., $\varpi_0 \equiv 0$.

(2) The measure $m(\omega) = \sum_{i=1}^{N} \delta_{\omega_i}(\omega)$ for some $\omega_i \in \mathbb{S}^1$, where $\delta_{\omega_i}(\omega)$ is the delta measure at ω_i.

3. Ideas of the proof

- The strategy of the proof is as the title of the talk: "from vacuum to dust to Vlasov", i.e., we make use of the construction in Theorem 2 for which the effective matter of the limiting spacetime consists of null dust, and then take the limit as the number of family of dust $\to \infty$.
- Our construction is based on a parametrix as follows. For every $\omega \in \mathbb{S}^1$, define f^ϕ, f^ϖ so that $f^2 = (f^\phi)^2 + \frac{1}{4}e^{-4\phi}(f^\varpi)^2$ and

$$\begin{cases} 2g^{-1}(du_\omega, df^\phi_\omega) + (\Box_g u_\omega) f^\phi_\omega + e^{-4\phi} g^{-1}(d\varpi, du_\omega) f^\varpi_\omega = 0, \\ 2g^{-1}(du_\omega, df^\varpi_\omega) + (\Box_g u_\omega) f^\varpi_\omega - 4g^{-1}(d\varpi, du_\omega) f^\phi_\omega - 4g^{-1}(d\phi, du_\omega) f^\varpi_\omega = 0. \end{cases}$$

Pick $\{\omega_{\mathbf{A}}\}_{\mathbf{A}=1}^N$ equally-spaced points on $\mathbb{S}^1$ so that

$$\sum_{\mathbf{A}=1}^{N} \sigma^{(N)}_{\mathbf{A}} \delta_{\omega_{\mathbf{A}}}(\omega) \to m(\omega)$$

in the weak-* topology (for constants $\sigma^{(N)}_{\mathbf{A}} \geq 0$, $\sum_{\mathbf{A}=1}^N \sigma^{(N)}_{\mathbf{A}} = 1$). Let

$$F^\phi_{\mathbf{A}} = \sqrt{\sigma^{(N)}_{\mathbf{A}}} f^\phi_{\mathbf{A}}, \quad F^\varpi_{\mathbf{A}} = \sqrt{\sigma^{(N)}_{\mathbf{A}}} f^\varpi_{\mathbf{A}}.$$

We then construct (ϕ, ϖ) by imposing

$$\phi = \phi_0 + \sum_{\mathbf{A}=1}^{N} a_{\mathbf{A}}^{-1} \lambda F^\phi_{\mathbf{A}} \cos\left(\frac{a_{\mathbf{A}} u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}^\phi,$$

$$\varpi = \varpi_0 + \sum_{\mathbf{A}=1}^{N} a_{\mathbf{A}}^{-1} \lambda F^\varpi_{\mathbf{A}} \cos\left(\frac{a_{\mathbf{A}} u_{\mathbf{A}}}{\lambda}\right) + \mathcal{E}^\varpi,$$

for well-chosen constants $a_{\mathbf{A}}$. Here, $\mathcal{E}^\phi$ and $\mathcal{E}^\varpi$ are error terms.

- The main goal is then to show that as $\lambda \to 0$, $N \to \infty$ (under a relation $\lambda^{\frac{1}{10}} e^{4N^2} \leq 1$), the solution exists on the uniform time interval $[0, 1]$. The estimates in the existence proof then imply the limiting properties.

- Compared with [4], the main difficulty is that we only have $\sum_{\mathbf{A}}[(F^{\phi}_{\mathbf{A}})^2 + (F^{\varpi}_{\mathbf{A}})^2]$ small, while the ℓ^1 norm, $\sum_{\mathbf{A}}(F^{\phi}_{\mathbf{A}} + F^{\varpi}_{\mathbf{A}}) \to \infty$ as $N \to \infty$.
- In fact, the lack of a uniformly small ℓ^1 bound as indicated above creates difficulties for local existence even for very short (λ-dependent) time. Such a local existence problem was solved by Touati in [8].
- One important new ingredient is to construct our parametrix with the exact eikonal functions corresponding to the vacuum spacetime we are studying, instead of the eikonal functions of the limit. This allows us to consider fewer terms in the parametrix compared to our previous work, though we must now also estimate the eikonal functions.
- An additional challenge, particularly for the wave energy estimates, is the lack of N-independent L^∞ estimates for $\partial\phi$, $\partial\varpi$ and ∂g.

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Analyticity of quasinormal modes in the Kerr and Kerr-de Sitter spacetimes

Oliver L. Petersen

(joint work with András Vasy)

In a recent project with András Vasy, we studied the regularity of quasinormal modes in the Kerr and Kerr-de Sitter spacetimes [2]. The main result is proven in a general setting: Let M be a real analytic spacetime and let $\mathcal{H} \subset M$ be a real analytic non-degenerate Killing horizon, i.e. there is a Killing vector field W such $W|_{\mathcal{H}}$ is lightlike (called the horizon Killing vector field) and the surface gravity is non-zero. We also assume that

$$\mathrm{Ric}(W|_{\mathcal{H}}, X) = 0$$

for all $X \in T\mathcal{H}$. Analyzing wave equations near the Killing horizon $\mathcal{H}$, one may first decompose into modes in the symmetry direction, i.e. we assume that

$$W(u) = -\sigma i u, \tag{1}$$

for some fixed $\sigma \in \mathbb{C}$, where $W(u)$ denotes the derivative of u in direction W. One should think of this as first Fourier transforming in the Killing variable and then consider the induced σ-dependent equation for each fixed mode. Together with András Vasy, we have proven:

Theorem 1 (See [2]). *Let*

$$P = \Box + \text{ lower order terms}$$

be a linear wave operator with real analytic coefficients such that

$$[P, W] = 0,$$

i.e. the coefficients commute with W. If smooth function u satisfies (1) *near $\mathcal{H}$ and Pu is real analytic near $\mathcal{H}$, then u is real analytic near $\mathcal{H}$.*

The proof is based on studying the principal symbol of P, reduced in the W-direction. We prove that the resulting bicharacteristic flow forms a stable radial point source/sink structure and we may therefore apply [1].

Quasinormal modes in the Kerr and Kerr-de Sitter spacetime are defined with respect to the *stationary* Killing vector field, which in general is not lightlike at the horizon. The *horizon* Killing vector field is given in standard Eddington-Finkelstein coordinates as

$$W := \partial_t + \frac{a}{r_0^2 + a^2}\partial_\phi \tag{2}$$

where ∂_t is the stationary Killing vector field and ∂_ϕ is the rotational Killing vector field.

Corollary 1 (See [2]). *Let M be the subextremal Kerr(-de Sitter) spacetime, extended real analytically over the future horizon(s) and let*

$$P = \Box + \text{ lower order terms}$$

be a linear wave operator with real analytic coefficients, such that

$$[P, \partial_t] = [P, \partial_\phi] = 0,$$

i.e. the coefficients commute with ∂_t and ∂_ϕ. If u is a smooth function satisfying

$$\partial_t u = -i\sigma u,$$
$$\partial_\phi u = -imu,$$

for $\sigma \in \mathbb{C}$ and $m \in \mathbb{Z}$ and Pu is real analytic, then u is real analytic.

The idea is to note that u is also a mode with respect to the horizon Killing vector field (2):

$$W(u) = \left(\partial_t + \frac{a}{r_0^2 + a^2}\partial_\phi\right) u = -i \underbrace{\left(\sigma + \frac{am}{r_0^2 + a^2}\right)}_{\in\mathbb{C}} u.$$

The analyticity near the horizon(s) thus follows by Theorem 1. The analyticity away from the horizons then follows by standard results.

Corollary 2 (See [2]). *Let M be a slowly rotating Kerr-de Sitter spacetime, with $\Lambda > 0$, extended real analytically over the future horizons and let*

$$P = \Box + \text{ lower order terms}$$

be a linear wave operator with real analytic coefficients, such that

$$[P, \partial_t] = [P, \partial_\phi] = 0,$$

i.e. the coefficients commute with ∂_t and ∂_ϕ. If u is a smooth function satisfying

$$\partial_t u = -i\sigma u,$$

for $\sigma \in \mathbb{C}$ and $Pu = 0$, then u is real analytic.

To see this, let us first write

$$u = \sum_{m \in \mathbb{Z}} e^{-i(\sigma t + m\phi)} v_{\sigma,m}(r, \theta).$$

Each summand satisfies the assumptions in Corollary 1 and is therefore real analytic. We thus have infinitely many linearly independent solutions to the homogeneous σ-dependent equation. Now, though the σ-dependent operator is non-elliptic, Vasy has proven in [3] that it is actually a Fredholm operator, which implies that the sum is finite. The assertion follows.

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Stabilizing relativistic fluids on slowly expanding spacetimes

Zoe Wyatt

(joint work with David Fajman, Maximilian Ofner)

In this talk, we present recent progress in our understanding of cosmological spacetimes filled with relativistic fluids. We consider solutions to the Einstein-relativistic Euler (ErE) system:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = T_{\mu\nu} := (\rho + p) u_\mu u_\nu + p g_{\mu\nu} \qquad \nabla^\mu T_{\mu\nu} = 0, \tag{1}$$

which describes the dynamical evolution of a spacetime containing a perfect fluid with pressure p, energy density ρ and 4-velocity vector u^μ. Equations (1) are supplemented by an equation of state. In this talk, we choose the class of linear, barotropic equations of state, $p = K\rho$ for $0 \leq K \leq 1$. This contains well-known fluid models: $K = 0$, reduces to the *Einstein-Dust* system and the case $K = 1/3$

is the *Einstein-radiation fluid* system. We also restrict to four-dimensional FLRW spacetimes of the form

$$\big((0,\infty)\times M, -\mathrm{d}t^2 + a(t)^2\cdot\gamma\big)\,, \tag{2}$$

where M is taken to be one of $\mathbb{R}^3, \mathbb{S}^3, \mathbb{H}^3$ or quotients thereof. We distinguish three classes of scale factors: $\ddot{a}(t) > 0$ are referred to as *accelerated expansion*, $\ddot{a}(t) < 0$ as *deccelerated expansion* and $\ddot{a}(t) = 0$ as *linear expansion.*

1. Background and Previous Results

Relavistic and non-relativistic fluids are well-known to form shocks in finite time. This was first observed in the general relativistic context by Oppenheimer and Snyder when they investigated the collapse of spherically symmetric clouds of dust. More recently, Christodoulou's monograph [5] demonstrated that under a very general equation of state, the constant solutions to the relativistic Euler equations on a fixed background Minkowski space are unstable, i.e. fluids form shocks from arbitrarily small initial inhomogeneities in finite time. This suggests that Minkowski spacetime is unstable as a solution to (1) for a large class of equations of state.

A powerful dispersive mechanism is clearly required to regularise fluids and to prevent finite-time shock formation. The prime example comes from cosmological models exhibiting exponential expansion. For example, a cosmological constant $\Lambda > 0$ generates expansion of the form $a(t) \sim e^{Ht}$ where $H = \sqrt{\Lambda/3}$. This creates damping terms in the equations of motion for the fluid, which dilutes the fluid and causes fluid lines to 'stretch apart', thus preventing shock formation. By contrast, a part of Minkowski spacetime can be considered as a cosmological spacetime with non-compact slices and no expansion $a(t) = 1$ in (2).

This stabilising effect was first observed by Brauer, Rendall and Reula [4] who found, for a slightly simpler Newtonian cosmological model, that the regularising effect from exponential expansion was strong enough to prevent shock formation for small inhomogeneities of initially uniformly quiet fluid states. Moving to the fully coupled Einstein-relativistic Euler system, there has been much research concerning spacetimes undergoing *exponential* expansion. For example, future stability under (1) (with Λ strictly positive!) is known to hold for uniformly quiet fluids on FLRW-spacetimes with underlying spatial manifold $M = \mathbb{T}^3$ for the parameter range $0 \le K \le 1/3$ by the works [8, 9, 10]. We also refer here to the talk by Oliynyk for his new results concerning the cases $1/3 < K < 1/2$.

1.1. Critical Expansion Rates.

Interpolating between Minkowski space and exponentially expanding spacetimes, it is clear that there must be a transition between shock formation and stability. To investigate the expansion rate for which this transition occurs, and how it depends on the equation of state, it is useful to study the stabilisation of fluids on fixed Lorentzian geometries obeying power-law inflation, where $a(t) = t^p$ for $p > 0$ and $(M,\gamma) = (\mathbb{T}^3,\delta)$. In this talk, the author presented the following table which summarises some of the state-of-the art results concerning linear equation of states $p = K\rho$:

Type	Power-law rate	Range of K	Behaviour	Reference
Accelerated	$p > 1$	$0 < K < 1/3$	Stable	[11]
Linear	$p = 1$	$K = 1/3$	Shocks	[11]
Linear	$p = 1$	$0 < K < 1/3$	Stable (irrotational)	[6]
Deccelerated	$p > \frac{1}{2}$	$K = 0$	Stable	[11]

These results suggest that slower speeds of sound reduce the tendency of shock formation, and for the case of dust, shocks are avoided even in deccelerating spacetimes.

2. Main result

The main result presented in this talk is the following:

Theorem 1 (Rough statement, [7])**.** *All four-dimensional FLRW spacetime models with compact spatial slices and negative spatial Einstein geometry are future stable solutions of the Einstein-Dust system.*

To date, all known future stability results establishing the global existence, regularity and completeness of solutions for the coupled system (1) concern the regime of accelerated expansion, and so Theorem 1 initiates the study of the ErE in the regime of non-accelerated expansion. Such a regime is also relevant in cosmology. The epoch in the early universe, shortly after a hypothetical inflationary phase, is expected to not initially have exhibited accelerated expansion.

Comments on the proof. The background geometry studied in Theorem 1 is that of the Milne model, which is known to be a stable solution solution to the Einstein vacuum equations [2, 3], with several later extensions including matter models, e.g. [1]. The standard approach in the literature must be modified, however, since there are crucial difficulties caused by a regularity problem inherent to the Einstein-Dust equations. In particular, there appears to be an inconsistency between the regularity needed to control the fluid velocity, energy density and second fundamental form. An approach to circumvent this issue, first introduced by Hadžić and Speck in [8], is to use a fluid derivative $\partial_{\mathbf{u}} \sim u^\alpha \nabla_\alpha$ as a differential operator in the highest-order energies for the perturbations of the metric and second fundamental form. Several problems still remain, and in this talk we presented the resolution of one of these issues. In particular, an additional estimate for the shift is obtained by using a *remarkable combination* of elliptic equation for the shift, commutator estimates, the Bianchi identity and the Einstein equations in the CMCSH gauge.

3. Future work

There are several avenues for future research, such as extending Theorem 1 to massive fluid models with $0 < K < 1/3$. There is also a need to better understand the relationship between linear and deccelerating spacetime expansion and fluid shock formation under the full ErE system.

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Initial data rigidity and the hyperbolic PMT with boundary

Gregory J. Galloway

(joint work with Piotr T. Chrusciel, Michael Eichmair and Abraão Mendes)

Our results involve aspects of marginally outer trapped surfaces. We begin with some basic definitions. An initial data set (M, g, K) consists of a connected Riemannian manifold (M, g) and a symmetric $(0, 2)$-tensor field K. The *local energy density* μ and the *local momentum density* J of (M, g, K) are given by

$$\mu = \frac{1}{2}\left(R - |K|^2 + (\operatorname{tr} K)^2\right) \qquad \text{and} \qquad J = \operatorname{div}\left(K - (\operatorname{tr} K)\, g\right),$$

where R is the scalar curvature of (M, g). (M, g, K) is said to satisfy the *dominant energy condition* (DEC) provided

$$\mu \geq |\, J\,|.$$

Let Σ be a closed two-sided hypersurface in M with unit normal ν; by convention we refer to ν as outward pointing. The *null 2nd fundamental forms* $\chi^\pm$ of Σ are defined as,

$$\chi^\pm = K|_{T\Sigma} \pm A$$

where A is the second fundamental form of Σ within M. The *null expansion scalars* θ^+ and θ^- of Σ are obtained by tracing the null 2nd fundamental forms,

$$\theta^\pm = \mathrm{tr}_\Sigma\, \chi^\pm = \mathrm{tr}_\Sigma(K) \pm H$$

where H is the mean curvature of Σ within M. By our sign conventions, $H = \mathrm{div}_\Sigma\, \nu$. These quantities take on natural physical/geometric meanings when the initial data set (M, g, K) is embedded in a spacetime $(\bar{M}, \bar{g})$, by which is meant that M is a spacelike hypersurface in $\bar{M}$, with induced metric g and second fundamental form K.

We say Σ is *outer trapped* if $\theta^+ < 0$. If $\theta^+ \equiv 0$ on Σ, we say that Σ is a *marginally outer trapped surface*, or MOTS for short. Finally, if Σ is a separating MOTS in (M, g, K), we say that Σ is *weakly outermost* if there are no outer trapped surfaces outside of, and homologous, to Σ.

In the paper [8], with A. Mendes and M. Eichmair, we obtained some 'initial data rigidity results' motivated by the spacetime positive mass theorem. The following version of this result was obtained by Eichmair, L.-H. Huang, D. A. Lee, R. Schoen in [9].

Theorem 1. *Let (M, g, K) be an n-dimensional asymptotically flat initial data set with ADM energy-momentum vector (E, P). Assume that $3 \le n \le 7$. If the dominant energy condition $\mu \ge |J|$ is satisfied, then $E \ge |P|$.*

In very broad terms, the proof generalizes to the spacetime setting the proof of the Riemannian positive mass theorem of Schoen and Yau [17], where now MOTS play a role analogous to minimal surfaces. (See [13] for the equality case.)

In the arxiv preprint [16], Lohkamp has presented a different proof of this result (in dimensions ≥ 3). By his approach, the proof reduces to the following result (see [16, theorem 2]).

Nonexistence of $\mu - |J| > 0$ - islands: Let (M, g, K) be an initial data set that is isometric to Euclidean space, with $K = 0$, outside some bounded open set U. Then one cannot have $\mu > |J|$ on U.

In [8, Theorem 1.2] we obtain a 'rigid version' of this result. By putting U in a large box and identifying all but one pair of opposite sides we obtain an initial data set, still calling it (M, g, K), where M is a compact manifold with boundary consisting of two flat tori Σ_0 and S, which are totally geodesic in (M, g). Moreover, since K vanishes outside of U, they are MOTS with respect to either choice of normal; in fact $\chi^\pm = 0$ along each torus. This setting applies in particular to the following more general setup; see [8, Theorem 1.2].

Theorem 2. *Let (M, g, K) be an n-dimensional, $3 \le n \le 7$, compact-with-boundary initial data set satisfying the DEC. Suppose ∂M can be expressed as a disjoint union $\partial M = \Sigma_0 \sqcup S$, such that:*

(i) $\theta^+ \le 0$ *along* Σ_0 *with respect to the normal pointing into* M*, and* $\theta^+ \ge 0$ *along* S *with respect to the normal pointing out of* M*,*

(ii) Σ_0 *satisfies the cohomology condition and* M *satisfies the homotopy condition with respect to* Σ_0*.*

Then $M \cong [0,\ell] \times \Sigma_0$ *(in particular* S *is connected), and each leaf* $\Sigma_t \cong \{t\} \times \Sigma_0$*,* $t \in [0,\ell]$*, satisfies: (a)* Σ_t *is a MOTS, in fact* $\chi^+_{\Sigma_t} = 0$*, (b)* Σ_t *is a flat torus, and (c)* $\mu + J(\nu_t) = 0$*. In particular, this implies* $\mu = |J|$*.*

The *cohomology condition* is a specific topological condition which ensures that Σ_0 does not admit a metric of positive scalar curvature. It has been used by Schoen and Yau in [18, Theorem 5.2]; see also [15, Theorem 2.28] for a nice discussion. The *homotopy condition* is a condition slightly more general than assuming the existence of a retract of M onto Σ_0. It implies that Σ_0 is connected. A priori, S is allowed to have multiple components. The proof of Theorem 2 consists of three elements:

(i) Showing that Σ_0 is a weakly outermost MOTS in M. The proof makes essential use of the cohomology condition and homotopy condition. It also makes use of the basic existence result for MOTS in dimensions $3 \le n \le 7$ due to Eichmair [6, 7] (applied to $(M, g, -K)$). This is where the dimension restriction comes in.

(ii) Using the local rigidity result proved in [10, 11] for weakly outermost MOTS to obtain a neighborhood $U \cong [0,\delta) \times \Sigma_0$ on which the conclusions (a), (b), and (c) of the theorem hold.

(iii) Extending U to all of M.

We now briefly describe work with Piotr Chruściel [3] in which Theorem 2 plays a role in establishing a positive mass theorem for asymptotically hyperbolic manifolds (AH) with boundary. Here, by an AH manifold we mean, roughly, that (N, h) admits a conformal compactification, with spherical conformal boundary, such that h approaches the hyperbolic metric at a suitable rate on the conformally compactified end. The PMT result in [3] allows for multiple ends, but, for simplicity, here we restrict to the case of a single end.

Under appropriate decay conditions, (N, h) admits an invariant, the *mass vector* $\mathbf{m} = (m_i)$, the components of which may be expressed as integrals involving certain *static potentials* in hyperbolic space; see [5] for details.

By a very elegant argument, Chruściel and Delay [2] have obtained a general version of the PMT for AH manifolds, which does not require a spin assumption, or other restrictive assumptions.

Theorem 3. *Let* (N, h) *be an AH manifold of dimension* n*. Assume that the scalar curvature of* (N, h) *satisfies,* $R(h) \ge -n(n-1)$*. Then, the mass vector* $\mathbf{m}$ *is future causal or vanishes.*

Combining with results of Huang, Jang and Martin [12], in fact $\mathbf{m}$ can't be future null, and if $\mathbf{m} = 0$ then (N, h) is isometric to hyperbolic space. Interestingly, Chruściel and Delay reduce the proof to an application of the spacetime asymptotically flat positive mass theorem, including the equality case.

Using arguments from [2] together with Theorem 2 we have obtained, with Chruściel, the following extension of Theorem 3 to AH manifolds with boundary.

Theorem 4. *Let (N, h) be an n-dimensional, $3 \leq n \leq 7$, AH manifold with (compact) boundary. Assume that the scalar curvature of N satisfies $R \geq -n(n-1)$, and that the boundary has mean curvature $H \leq n-1$ with respect to the normal pointing into M. Then, the mass vector $\mathbf{m}$ is future causal or vanishes.*

Examples show that the mean curvature assumption is sharp. A proof in the spin case in dimensions $n \geq 3$ was given by Chruściel and Herzlich [5]. Very recent work of Huang and Jang [14] shows, too, in this boundary case, that $\mathbf{m}$ can't be future null, and if $\mathbf{m} = 0$ then (N, h) is isometric to hyperbolic space.

The proof of Theorem 4 is by contradiction. Using Chruściel and Delay's localized boundary-connected sum gluing procedure, together with results in [1, 4], one ultimately obtains an initial data set (M, g, K) satisfying all the assumptions of Theorem 2. However, owing to the presence of a boundary, S has multiple components, contradicting the conclusion.

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Dynamically Stable Cosmological Singularities: The Sub-Critical Regime

Jared Speck

(joint work with Grigorios Fournodavlos, Igor Rodnianski)

1. Introduction and statement of the equations

In this extended abstract, we summarize the results of our recent work [20]. We proved stable Big Bang formation for solutions arising from open sets of initial data for the Einstein-scalar field system, where the initial data are prescribed on the D-dimensional torus $\mathbb{T}^D := [-\pi,\pi]^D$ (with the endpoints identified). The significance of our paper is that the results are sharp, that is, they rigorously confirm the dynamic stability of the singularity formation in the entire regime where heuristics in the literature have suggested it might occur.

1.1. **The Cauchy problem.** The Einstein-scalar field equations are

$$\mathbf{Ric}_{\mu\nu} = \partial_\mu\psi\partial_\nu\psi, \tag{1a}$$

$$\square_{\mathbf{g}}\psi = 0. \tag{1b}$$

The fundamental work [9] showed that sufficiently regular initial data for (1a)–(1b) give rise to a unique (up to diffeomorphism) maximal (classical) globally hyperbolic development (MGHD for short). While beautiful and conceptually foundational, [9] does not yield much information about the nature of the MGHD.

1.2. **Generalized Kasner solutions.** Our main results concern perturbations of generalized Kasner solutions on $(0,\infty)\times\mathbb{T}^D$, which satisfy the Einstein-scalar field equations (1a)-(1b) and can be expressed as follows, where $(x^1,\cdots,x^D)$ are standard coordinates on $\mathbb{T}^D$:

$$\widetilde{\mathbf{g}} = -dt\otimes dt + \widetilde{g}, \qquad \widetilde{g} := \sum_{I=1,\cdots,D} t^{2\widetilde{q}_I} dx^I \otimes dx^I, \qquad \widetilde{\psi} = \widetilde{B}\log t. \tag{2}$$

The *Kasner exponents* $\{\widetilde{q}_I\}_{I=1,\cdots,D}$ and $\widetilde{B}$ are constants constrained by

$$\sum_{I=1}^{D} \widetilde{q}_I = 1 \qquad \sum_{I=1}^{D} \widetilde{q}_I^2 = 1 - \widetilde{B}^2. \tag{3}$$

The first equation in (3) represents a gauge choice corresponding to a foliation by CMC slices Σ_t such that the trace of the second fundamental form of Σ_t is $-t^{-1}$, while the second is a consequence of the Hamiltonian constraint.

1.3. **Big Bang singularities in Kasner solutions plus some context.** Aside from the exceptional case in which one $\widetilde{q}_I$ is unity and the rest vanish, all generalized Kasner solutions exhibit curvature-blowup at $t = 0$, that is, their Kretschmann scalars satisfy $\mathbf{Riem}^{\alpha\mu\beta\nu}\mathbf{Riem}_{\alpha\mu\beta\nu} = Ct^{-4}$ for some constant C depending on the $\{\widetilde{q}_I\}_{I=1,\cdots,D}$. Our main theorem in [20], which we state in a rough form in the next section, shows that under appropriate assumptions, perturbed solutions exhibit similar curvature-blowup along an entire spacelike hypersurface.

Let us compare with the "singularity theorems" [21, 22, 28] of Hawking and Penrose. Their results show that there exist large sets of regular initial data for various Einstein-matter systems such that the corresponding solutions are causally geodesically incomplete. In particular, perturbations of the generalized Kasner solutions are guaranteed to be geodesically incomplete to the past. However, these theorems are "soft" in that they do not reveal the nature of the geodesic incompleteness. In principle, there are many conceivable reasons why a spacetime could be geodesically incomplete. For example, the breakthrough work of Dafermos–Luk [15] showed that Kerr Cauchy horizons, which are null, are stable under perturbations of the initial data for the Einstein-vacuum equations. In contrast, for the solutions covered by our main theorem in [20], we are able to show that the geodesic incompleteness is tied to curvature-blowup along an entire spacelike hypersurface.

2. Statement of the main theorem

We now provide a rough statement of the main theorem in [20].

Theorem 1 (Stable Big Bang formation (Rough statement))**.** *In $1 + D$ spacetime dimensions, consider a "background" generalized Kasner solution* (2) *whose exponents satisfy* (3) *and the "sub-criticality condition"*

$$\max_{\substack{I,J,B=1,\cdots,D\\ I<J}} \{\widetilde{q}_I + \widetilde{q}_J - \widetilde{q}_B\} < 1. \tag{4}$$

It can be shown that there exist generalized Kasner solutions satisfying (3) *and* (4) *in the presence of a scalar field when $D \geq 3$ and in vacuum (i.e., with $\widetilde{B} = 0$ in* (2)*) when $D \geq 10$. Then such generalized Kasner solutions are dynamically stable under perturbations – without symmetry – of their initial data near their Big Bang singularities, as solutions to the Einstein-scalar field equations in the case $D \geq 3$, and, when $\widetilde{B} = 0$, as solutions to the Einstein-vacuum equations in the case $D \geq 10$. Moreover, in $1 + 3$ spacetime dimensions,* ***all*** *Kasner solutions with $\widetilde{B} = 0$ are dynamically stable solutions to the Einstein-vacuum equations under perturbations – with polarized $U(1)$ symmetry – near their Big Bang singularities, even though they all violate the condition* (4)*.*

More precisely, under the above assumptions, sufficiently regular perturbations of the Kasner initial data on $\Sigma_1 := \{1\} \times \mathbb{T}^D$ give rise to maximal developments *that terminate in a Big Bang singularity to the past; the spacetime solutions in the past of Σ_1 are foliated by spacelike hypersurfaces Σ_t that are equal to the level sets of a time function t verifying the CMC condition $\mathrm{tr}k = -t^{-1}$, and $\mathbf{Riem}^{\alpha\mu\beta\nu}\mathbf{Riem}_{\alpha\mu\beta\nu}$ blows up **monotonically** like Ct^{-4} as $t \downarrow 0$.*

*In addition, the perturbed solutions exhibit **Asymptotically Velocity Term Dominated** (AVTD) behavior near the singularity. Roughly, this means that in our gauge, the spatial derivative terms in Einstein's equations become negligible compared to the time derivative terms as $t \downarrow 0$. Finally, as a consequence of the AVTD behavior, various t-rescaled solution variables have regular limits as $t \downarrow 0$.*

It is an outstanding open problem to understand how general small perturbations (e.g., without symmetry) of generalized Kasner solutions behave when the sub-criticality condition (4) *fails*. As of present, there are no rigorous results. The heuristic arguments given in the influential physics paper [8] suggest that when (4) fails to hold, solutions without symmetry might exhibit wild oscillatory behavior that is qualitatively very different than the monotonic AVTD behavior exhibited by the solutions in Theorem 1.

3. Some prior results

We now discuss some prior results. There are many contributions to the subject of singularity formation in general relativity, and we do not attempt to give an exhaustive list. The significance of the sub-criticality condition (4) was understood at a heuristic level in the influential physics papers [6, 7, 17, 26]. See also [16]. There are many prior works in which authors constructed – without proving stability – interesting families of solutions that exhibit Kasner-like singularities. See, for example, [2, 3, 5, 10, 16, 18, 19, 23, 24, 25, 27, 29, 39]. There are prior works that have shown stable spacelike singularity formation with AVTD behavior for cosmological (i.e., with compact spatial slices) solutions under symmetry assumptions such that the problem reduces to the study of ODEs or $1+1$-dimensional PDEs; see, for example, [4, 13, 14, 30, 31, 32, 33, 34]. See also [1, 11, 12] for related spacelike singularity formation results inside black holes with symmetry.

We gave the first proofs of stable curvature-blowup for cosmological solutions without symmetry assumptions in [35, 36, 37, 38]. Those results treated the case in which the spatial manifold is $\mathbb{T}^3$ or $\mathbb{S}^3$ and there is approximate spatial isotropy, which in the $\mathbb{T}^3$ case means that $\widetilde{q}_1, \widetilde{q}_2, \widetilde{q}_3 \approx 1/3$. These works rely on CMC foliations and transported spatial coordinates. The new ingredient in [20] is an orthonormal frame $\{e_I\}_{I=1,\cdots,D}$ constructed via Fermi–Walker transport. The key point is that the structure coefficients $S_{IJB} := \mathbf{g}([e_I, e_J], e_B)$ solve an approximately diagonal system $\partial_t S_{IJB} = -\frac{(\widetilde{q}_I+\widetilde{q}_J-\widetilde{q}_B)}{t} S_{IJB} + \cdots$, where $\cdots$ denotes "PDE error terms" that we control with energy and elliptic estimates. The sub-criticality condition (4) allows one to integrate this equation and conclude that the blowup-rate of $\max_{I,J,B=1,\cdots,D\, I<J} |S_{IJB}|$ as $t \downarrow 0$ is no larger than[1] t^{-q}, where

[1]In reality, to close our bootstrap argument, we have to redefine q so that it is slightly larger than LHS (4) (though still less than 1).

the constant $q := \max_{I,J,B=1,\cdots,D_{I<J}}\{\widetilde{q}_I + \widetilde{q}_J - \widetilde{q}_B\}$ is the quantity on LHS (4) – *and thus* $q < 1$. This is the main new ingredient that allows us to prove stable curvature-blowup and AVTD behavior for solutions that exhibit substantial spatial anisotropy – at the limit of what the known heuristics suggest to be tractable.

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Geometric flows and the concept of mass in General Relativity - an overview

Gerhard Huisken

Several nonlinear, parabolic geometric evolution equations have had an impact in studying concepts such as mass and center of mass in General Relativity. The lecture attempts to review recent developments and new directions of research.

The *Ricci flow* of Riemannian metrics

$$\frac{d}{dt}g_{ij} = -2Ric(g)_{ij}$$

on an asymptotically flat Riemannian 3-manifold (M^3, g) is known to preserve the ADM-mass of the 3-manifold in view of the asymptotic decay conditions on the metric and the parabolic regularity properties of the flow. Since the scalar cirvature $R(g)$ satisfies the parabolic equation

$$\frac{d}{dt}R = \Delta_g R + 2|Ric|^2,$$

it also provides an easy proof of the rigidtiy case of the positive mass theorem (PMT), namely that either the initial (M^3, g) was isometric to Euclidean space or $R > 0$ for all$t > 0$. The recent proof of the PMT by Li [8] exploits an interesting relation to Perelman's W-functional. The Hamilton-Perelman surgery construction was carried out by Johne [7] to an extended Ricci flow system that includes static metrics such as Schwarzschild amongst it's fixed points. It would be interesting to study this system with respect to boundary conditions that prescribe induced metric and mean curvature, as they appear in the static metric conjecture due to Bartnik, compare e.g. [9]. Another interesting challenge might be to find extended parabolic systems for full initial data sets $(M^3, g, K$ including the 2nd fundamental form K w.r.t. a Lorentzian ambient metric.

The evolution of hypersurfaces by *mean curvature flow* $\frac{d}{dt}F = -H\nu$ can be used to sweep out the exterior region of an asymptotically flat manifold. In [3] Brendle-Huisken showed that starting e.g. from large coordinate spheres solutions of mean curvature flow exist that are interrupted by finitely many surgeries and converge smoothly to the outermost horizon, which is known to be a weakly stable minimal surface. In the limit of small surgery parameters this solution with surgeries converges to the level-set solution of MCF which is known to be smooth for a.e. time in this dimension due to the work of White [12]. Using MCF and inverse mean curvature flow it can be shown (see e.g. [6] and [5]) that the ADM-mass can be characterised as an asymptotic deficit m_{iso} of volume in comparison with Euclidean space in the isoperimetric problem:

$$m_{iso} = \limsup_{|\partial\Omega|\to\infty} \frac{2}{|\partial\Omega|}\Big(Vol(\Omega) - \frac{1}{6\pi^{1/2}}|\partial\Omega|\Big)$$

The assumption $R(g) \geq 0$ of the PMT is also related to the isoperimetric problem in the sense that at a smooth point $p \in (M^n, g)$ we see by Taylor expansion that

$$Vol(B_r(p) - c_{iso}(n)|\partial B_r(p)|^{\frac{n}{n-1}} = c(n)R(p)r^{n+2} + O(r^{n+3}).$$

Setting

$$\theta(p) := \liminf_{r\to 0} \frac{\Big(Vol(B_r(p)) - c_{iso}(n)|\partial B_r(p)|^{\frac{n}{n-1}}\Big)}{r^{(n+2)}},$$

we may conjecture that there is a PMT for C^0-metrics that takes the form:

$$\text{If } \theta(p) \geq 0 \;\forall\, p \in (M^n, g), \text{ then } m_{iso}(M^n, g) \geq 0,$$

where $c_{iso}(n)$ is the Euclidean isoperimetric constant and m_{iso} is defined in analogue to the 3d-case. Such a strong version of PMT seems possible also in view of recent insights by Gromov [4] and Bamler [1] on C^0 convergence properties of metrics with non-negative scalar curvature, see also the overview article of Basilio-Sormani [2] and the result of Simon [11] on short time existence of Ricci-DeTurck flow for C^0 initial metrics.

Finally the lecture discusses anisotropic extensions inverse mean curvature flow $\frac{d}{dt}F = \frac{1}{H}\nu$, where weak solutions on a manifold (M^{n+1}, g) can be defined by using a variational principle in the product space $M^n \times R$, see the construction of *null inverse mean curvature flow* by Moore in [10]. Joint work with M. Wolff on a flow along *inverse space-time mean curvature* was discussed in a separate lecture, see the corresponding abstract in this workshop report.

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On the evolution of hypersurfaces along their inverse spacetime mean curvature

Markus Wolff
(joint work with Gerhard Huisken)

We consider a new inverse curvature flow of hypersurfaces bounding a region of an asymptotically flat Riemannian manifold (M, g, K), additionally equipped with a symmetric, bilinear $(0, 2)$-tensor K. Such manifolds arise naturally as Initial Data to the Einstein Equations in Lorentzian manifolds (L, h) modelling isolated gravitating systems such as black holes, stars or galaxies. In time-symmetry, $K \equiv 0$, Huisken–Ilmanen [2] were able to give a proof of the Riemannian version of the Penrose Inequality using a weak notion of inverse mean curvature flow

$$\frac{\mathrm{d}}{\mathrm{d}\,t} F = \frac{1}{H}\nu,$$

where H denotes the mean curvature, and ν the unit normal. Further regularity properties were established in [3].

For general Initial Data, Cederbaum–Sakovich [1] proposed a foliation by surfaces of constant spacetime mean curvature (STCMC), $\mathcal{H} := \sqrt{H^2 - P^2} = \text{const.}$, as a defintion of center of mass in asymptotically flat Initial Data sets (M, g, K) with nonvanishing ADM-energy $E_{ADM} \neq 0$, where $P := \operatorname{tr}_\Sigma K$ is the trace of K along a hypersurface Σ. If (M, g, K) indeed arises as a spacelike hypersurface in a Lorentzian manifold (L, h), then $\mathcal{H}$ corresponds to the Lorentzian length of the mean curvature vector $\vec{\mathcal{H}}$ of Σ as a codimension-2 surface in (L, h). In this sense, $\mathcal{H}$ is in fact a Lorentz invariant quantity.

Motivated by this, we propose a generalization of inverse mean curvature flow adapted to gneral Inital Data sets (M, g, K), where we deform hypersurfaces proportional to their inverse spacetime mean curvature. In this context, we also want to mention inverse null mean curvature flow proposed by Moore [4]. Similar to inverse mean curvature flow, smooth solutions of the parabolic equation

$$\frac{\mathrm{d}}{\mathrm{d}\,t} F = \frac{1}{\sqrt{H^2 - P^2}}\nu, \tag{1}$$

develop singularities in a natural way, so we consider a notion of weak solutions, that allows us to describe the phenomenon of jumps. To develop this notion, we first transform the above parabolic equation into a weakly elliptic problem for the time of arrival funcion $u \colon M \to \mathbb{R}$, where we now understand a smooth solution to inverse spacetime mean curvature flow (STIMCF) as a function u with $\nabla u \neq 0$ satisfying

$$\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) - \sqrt{|\nabla u|^2 + \left(\left(g^{ij} - \frac{\nabla^i u \nabla^j u}{|\nabla u|^2}\right) K_{ij}\right)^2} = 0. \tag{2}$$

Note that, due to the fixed sign on the additional anisotropic term, inverse mean curvature flow acts as a natural upper barrier to (STIMCF). To allow for jumps, i.e. open regions where $|\nabla u| = 0$ everywhere, we treat (2) as an Euler–Lagrange

Equation by freezing the lower order terms as a bulk term energy. Roughly speaking, we define weak solutions on a domain Ω as a pair (u,ν) of a Lipschitz function u and a unit vector field ν, such that u minimizes the anisotropic comparrison principle

$$\mathcal{J}_{u,\nu}(v) := \int_A |\nabla v| + v\sqrt{|\nabla u|^2 + |P_\nu|^2} \tag{3}$$

against any competitor v with $\{v \neq u\} \subset\subset A \subset\subset \Omega$, where $P_\nu := \operatorname{tr}_M K - K(\nu,\nu)$.

Definition 1. *Let (M,g,K) be an asymptotically flat Initial Data set, E_0 precompact with C^2 boundary. We call a translation invariant pair (U,ν) of a locally Lipschitz function U and measurable, unit vector field ν on $(M\setminus E_0)\times \mathrm{R}$ a weak solution to (STIMCF), if U minimizes $\mathcal{J}_{U,\nu}$ on $(M\setminus E_0)\times \mathrm{R}$, and*

- *$U \geq 0$, $U \to \infty$ as $|x| \to \infty$*
- *$\nu = \frac{\nabla U}{|\nabla U|}$ a.e. outside of jump regions*
- *in jump regions, ν is normal to a foliation of generalized apparent horizons, i.e. of hypersurfaces satifying $\mathcal{H} = 0$*

The main result presented in the talk is the existence of weak solutions on maximal Initial Data sets, i.e. where we additionally impose that $\operatorname{tr}_M K \equiv 0$ on M.

Theorem 2 (Main theorem)**.** *Let (M,g,K) be an asymptotically flat, maximal initial data set without boundary and $n < 7$. Then for any nonempty, precompact, open set $E_0 \subset M^{n+1}$ with C^2 boundary, there exists a weak solution of inverse spacetime mean curvature flow with initial condition E_0.*

Using the concept of unit normal developped in the notion of weak solutions, we can furthermore understand the development of jump regions in a precise geometric way. To this end, we introduce the following outward optimization principle, where we say a set of finite perimeter E is outward optimizing in Ω (with respect to ν), if

$$|\partial^* E| \leq |\partial^* F| - \int_{F\setminus E} |P_\nu| \tag{4}$$

for any $E \subseteq F$ with $F \setminus E \subset\subset A \subset\subset \Omega$. We call E strictly outward optimizing in Ω, if equality in (4) implies $F = E$ up to a set of measure zero, and define the strictly outward optimizing hull E' as the intersection of all strictly outward optimizing sets containing E. Note that a (strictly) outward optimizing set is (strictly) outward minimizing with respect to area. Then, if (u,ν_M) denotes the projection of a weak solution (U,ν) constructed in Theorem 2 with $\nu_M := \pi_{TM}\nu$, and we define $E_t := \{u < t\}$, $E_t^+ := \{u \leq t\}$, we find that

- E_t is outward optimizing,
- E_t^+ is locally the strictly outward optimizing hull of E_t.

Unlike inverse mean curvature flow, where the hull property of E_t^+ holds in fact globally, the phenomenon of cost-free surfaces prevents that $E_t^+ = (E_t)'$ in general.

However, we find that $E_t^+ \subseteq (E_t)'$ and $(E_t)' \setminus E_t^+ = \bigcup_i F_i$ with

$$|\partial^* F_i| = \int_{F_i} |P_\nu|$$
$$F_i \subset \{u = t_i\} \text{ for jump times } t_i > t.$$

Sufficiently far out, weak solutions become asymptotically round in the sense that a rescaling of u converges locally uniformly to the expanding sphere solution of inverse mean curvature flow in (R^n, δ). In view of the center of mass definition given by Cederbaum–Sakovich [1] via a foliation of STCMC surfaces, we anticipate that the level sets of the flow asymptotically align with the foliation and enjoy the same analytic properties.

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On the topology and geometry of Cauchy horizons in vacuum spacetimes.

Martín Reiris Ithurralde

(joint work with Ignacio Bustamante)

The occurrence of compact Cauchy horizons in some cosmological spacetimes is an enigmatic (and conceptually troublesome) phenomenon, but it manifests only when the spacetimes possess a particular type of symmetry and global structure. In [1] Isenberg and Moncrief conjectured that non-degenerate[1] compact Cauchy horizons (CH) should always be Killing horizon and began a seminal program to prove it for analytic spacetimes. Among the many results they showed that, if the spacetime and the CH ($\mathcal{C}$ below) are assumed analytic, then a sufficient condition for the CH to be Killing is the existence of an analytic, null, and nowhere-zero vector field X on $\mathcal{C}$, satisfying,

$$\nabla_X X = \kappa X, \tag{1}$$

with κ a non-zero constant (the surface gravity). In their framework, the Einstein equations and the spacetime analyticity permit extending such an X to a Killing field on the globally hyperbolic region of the spacetime, proving thus the mentioned sufficiency (see for instance [6]). Showing the existence of such analytic X on $\mathcal{C}$

[1] A non-degenerate Cauchy horizon is one having at least one incomplete null generator

is a separate endeavor. More recently and taking a different avenue, Petersen [2] and Petersen-Rácz [3] have shown that if the spacetime is smooth (non-necessarily analytic), then a smooth X on $\mathcal{C}$ with the same mentioned property (1) can always be extended to a Killing field on both sides of C, removing thus the necessity of analyticity. Finally, and closing the circle, the existence of such smooth vector field X on smooth CHs was shown by the Reiris and Bustamante in [5] (see also the preprint by Minguzzi and Gurriaran [7] extending the work [5] to include matter).

As it turns out, it is the presence of the vector field X that makes it possible not only to show that CHs of smooth spacetimes are Killing but also to provide a more complete description of their global geometric structure. Presenting that is the goal of this report.

The following theorem about the topology and the orbital type of the null generators of compact non-degenerate Cauchy horizons of time orientable smooth vacuum 3 + 1-spacetimes was proved in [4]:

Theorem 1 (Bustamante-Reiris, '21)**.** *Let $\mathcal{C}$ be a smooth compact and non-degenerate Cauchy horizon on a vacuum time-orientable space time. Then, either,*

(i) all generators are closed, or,
(ii) only two generators are closed and any other densely fills a two-torus, or,
(iii) every generator densely fills a two-torus, or,
(iv) every generator densely fills the horizon.

Furthermore, respectively to (i)-(iv)*, the horizon's manifold is either:*

(i') a Seifert manifold,
(ii') a lens space,
(iii) a two-torus bundle over a circle, or,
(iv') a three-torus.

Examples are known for each category. In the last case (iv), the spacetime can be shown to be flat Kasner, settling thus a problem posed by Isenberg and Moncrief for the so called ergodic horizons [6].

The proof of Theorem 1 relies on an observation by Oliver Linblad Petersen, that a smooth X satisfying (1) is a Killing field for the Riemannian metric on $\mathcal{C}$,

$$g = h + \omega \otimes \omega, \tag{2}$$

where h is the degenerate metric induced on C from the spacetime and ω is the 1-form, defined by X as,

$$\nabla_Y X = \omega(Y) X. \tag{3}$$

The classification of the orbital type of the null generators in Theorem 1 reduces then to the problem of understanding the orbits of a Killing field on a Riemannian manifold. In particular, it is a direct consequence of the following general theorem that can be proved by standard methods of isometric actions on Riemannian manifolds:

Theorem 2. *Let $(\mathcal{C}, g)$ be a smooth, 3-dimensional, compact and connected Riemannian manifold. Suppose that X is a nowhere vanishing Killing vector field. Then, either,*

(i) *every orbit is closed, or,*
(ii) *there are only two closed orbits, and every other orbit densely fills an embedded two-torus, or,*
(iii) *every orbit densely fills an embedded two-torus, or,*
(iv) *every orbit is dense in $\mathcal{C}$.*

The topological classification in Theorem 1 follows without much effort from the classification of the orbital types of the null generators.

We believe that Theorem 1 opens the door to provide a full parameterization of the space of manifolds and data (suitable data) on them, representing all the possible vacuum, compact, and non-degenerate Cauchy horizons.

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Asymptotically Kasner-like singularities

Grigorios Fournodavlos
(joint work with Jonathan Luk)

100 years ago, E. Kasner discovered [10] the first exact cosmological solutions to Einstein's equations in vacuum $Ric(g) = 0$,

$$g = -dt^2 + \sum_{i=1}^{3} t^{2p_i}(dx^i)^2, \qquad \sum_{i=1}^{3} p_i = \sum_{i=1}^{3} p_i^2 = 1, \qquad (t,x) \in (0,+\infty)\times\mathbb{T}^3,$$

predicting the existence of a striking new phenomenon, namely, the Big Bang singularity ($t = 0$). Since then, it has been the object of study in a great deal of research on general relativity. Nevertheless, the nature of the 'generic' Big Bang singularity is still not fully understood.

In an influential paper [13], Lifshitz–Khalatnikov considered singular solutions whose Big Bang singularity is synchronized at $t = 0$, having the asymptotic profile:

$$(1)\qquad \begin{aligned} &g \sim -dt^2 + \sum_{i=1}^{3} t^{2p_i(x)}\omega^i \otimes \omega^i, \qquad (t,x) \in (0,T] \times \mathbb{T}^3,\ T > 0,\\ &\sum_{i=1}^{3} p_i(x) = \sum_{i=1}^{3} p_i^2(x) = 1,\ p_i(x) < 1 \qquad \omega^i = \sum_{j=1}^{3} c_{ij}(x)dx^j. \end{aligned}$$

Such singularities are also called (asymptotically) Kasner-like, since along each past integral curve of ∂_t, the metric resembles a specific member of the Kasner family, as $t \to 0^+$, albeit the Kasner exponents $p_i(x)$ and principal 1-forms ω^i vary in x. The main observation in [13] is that the ansatz (1) is inconsistent with the Einstein vacuum equations, unless the following condition holds:[1]

$$(2)\qquad \omega^1 \wedge \mathrm{d}\omega^1 = 0, \qquad p_1(x) < 0.$$

However, the latter imposes a constraint on the coefficients $c_{ij}(x)$, hence, eliminating one functional degree of freedom.[2] They concluded that the class of Kasner-like singularities (1) must be non-generic (or we could say unstable).

It turns out that general Big Bang singularities in 1+3 vacuum are expected to be much more complicated than (1). A conjectural picture was set forth by Belinski–Khalatnikov–Lifshitz [3] for the 'generic' situation, where they heuristically described a mechanism that produces infinitely many oscillations as the singularity is approached. During this process, the solution between two successive oscillations is modeled by a (different each time) Kasner-like metric (1). To date, the only rigorous results that capture an oscillatory behavior of this type are restricted to spatially homogeneous solutions [15]. To make matters worse, there exist inhomogeneous Big Bang singularities with so called spikes [14], not predicted by [3], across which the asymptotic behavior of the metric is discontinuous, but otherwise Kasner-like, cf. also [8] for a study of "spike oscillations" and [4] for numerical support. This goes to show that we are still far from a complete classification of Big Bang singularities.

Nonetheless, the above investigations reveal the importance of Kasner-like singularities for the understanding of more complicated Big Bangs, since they are the building blocks of all other scenarios. We should mention that there are regimes where oscillations/spikes are silenced and Kasner-like singularities are generic, and moreover stable (see Jared Speck's abstract), but they do not include 1+3 vacuum.

Our main result provides a general method for constructing smooth solutions admitting an asymptotically Kasner-like singularity, as $t \to 0^+$, which enjoy the largest number of functional degrees of freedom, while at the same time being

[1] Due to the algebraic relations satisfied by the Kasner exponents, one of them must be negative at every spatial point x, say $p_1(x) < 0$.

[2] In 1+3 vacuum, there are in general four degrees of freedom. For solutions of the form (1), the asymptotic degrees of freedom correspond to the nine functions $p_i(x), c_{ij}(x)$. These are restricted by the two Kasner relations and the asymptotic momentum constraint (three differential equations), see Theorem 1 (3) and Remark 2, leaving four free functions.

consistent with heuristic expectations. Specifically, in 1+3 vacuum this amounts to three free functions, see (2), footnote 2, and Remark 3. Moreover, we give a simple description of the relevant metrics in a Gaussian time gauge. There is an abundance of previous results [1, 2, 6, 9, 11], constructing Kasner-like singularities (also called AVTD), but they are all restricted to symmetries or analyticity.

Theorem 1 (F.-Luk [7]). *Let $c_{ij}, p_i : \mathbb{T}^3 \to \mathbb{R}$ be smooth functions satisfying:*

(1) $\sum_{i=1}^3 p_i(x) = \sum_{i=1}^3 p_i^2(x) = 1$, $p_1(x) < p_2(x) < p_3(x) < 1$,
(2) $c_{ij}(x) = c_{ji}(x)$, $c_{11}(x), c_{22}(x), c_{33}(x) > 0$,
(3) The differential constraints[3]

$$\sum_{\ell=1}^{3} \frac{1}{2}\frac{\partial_i c_{\ell\ell}}{c_{\ell\ell}}(p_\ell - p_i) + \sum_{\ell>i} \frac{\partial_\ell(\sqrt{c_{11}c_{22}c_{33}}\kappa_i{}^\ell)}{\sqrt{c_{11}c_{22}c_{33}}} = \partial_i p_i.$$

Then there exists a smooth solution to the Einstein vacuum equations of the form:

$$g = -dt^2 + \sum_{i,j=1}^{3} a_{ij}(t,x) t^{2\max\{p_i(x),p_j(x)\}} dx_i dx_j, \qquad (t,x) \in (0,T] \times \mathbb{T}^3, \tag{3}$$

for some $T > 0$ sufficiently small, where $\lim_{t\to 0^+} a_{ij}(t,x) = c_{ij}(x)$.

Remark 2 (Asymptotic constraints). *Condition (2) ensures that (3) is a Lorentzian metric for sufficiently small $T > 0$. For technical reasons, we have incorporated in (1) the condition of distinct Kasner exponents. The differential constraints (3) on the coefficients $c_{ij}(x)$ are in fact induced by the momentum constraint on the level sets of t, asymptotically as $t \to 0^+$.*

Remark 3 (Degrees of freedom). *It is easy to see from the form of the metric (3) that the heuristic condition (2) of Lifshitz–Khalatnikov is satisfied. Indeed, comparing with (1), we find that $\omega^1 = \sqrt{c_{11}(x)}\mathrm{d}x^1$, which obviously satisfies $\sqrt{c_{11}(x)}\mathrm{d}x^1 \wedge \mathrm{d}(\sqrt{c_{11}(x)}dx^1) = 0$. The rest of the degrees of freedom are retained. Conditions (1)-(3) leave four free functions among $p_i(x), c_{ij}(x)$. However, metrics of the form (3) have an extra residual gauge freedom, namely, a change of coordinates via the rule $\widetilde{x}^1 = x^1, \widetilde{x}^2 = x^2, \widetilde{x}^3 = f(x^1,x^2,x^3)$, which can be used for example to set $c_{33}(\widetilde{x}) = 1$. Thus, there are in reality three free functions in the asymptotic data, consistent with the heuristics.*

The proof of Theorem 1 has two main steps:

• First, we compute the asymptotic expansion of the metric (3) to all orders, using the ADM equations

$$\partial_t \overline{g}_{ij} = -2\overline{g}_{aj} k_i{}^a, \qquad \partial_t k_i{}^j - \mathrm{tr}k k_i{}^j = Ric(\overline{g})_i{}^j, \tag{4}$$

where $\overline{g}_{ij}, k_{ij} = \overline{g}_{aj}k_i{}^a$ are the induced first and second fundamental forms of the level sets of t.

[3]for $\kappa_1{}^2 = (p_1 - p_2)\frac{c_{12}}{c_{22}}$, $\kappa_2{}^3 = (p_2 - p_3)\frac{c_{23}}{c_{33}}$, $\kappa_1{}^3 = -\kappa_1{}^2\frac{c_{23}}{c_{33}} + (p_1 - p_3)\frac{c_{13}}{c_{33}}$

• Then, we derive weighted energy estimates for the remainder of a truncated series, using mainly the second order system of equations satisfied by ${k_i}^j$:

$$\partial_t^2 {k_i}^j - \Delta_{\overline{g}} {k_i}^j = - \nabla_i \nabla^j {k_\ell}^\ell + \mathcal{N}(k, \partial_t k)_i{}^j. \tag{5}$$

It is well-known that (4) is not symmetric hyperbolic in a Gaussian time gauge, hence, this necessitates the use of a modified system for ${k_i}^j$. Choquet-Bruhat–Ruggeri [5] derived similar second order systems to (5), in a harmonic time gauge, demonstrating their hyperbolicity.

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Matter particles, naked singularities, and a weak second Bianchi identity

Annegret Burtscher

(joint work with Michael K.-H. Kiessling and A. Shadi Tahvildar-Zadeh)

How to best model the motion of particles (massive and charged) in general relativity? Einstein, Infeld, and Hoffmann [4, 5] determined equations of motion approximately describing the dynamics of a system of matter particles due to their gravitational interactions by interpreting them as timelike singularities in spacetimes. This approach was soon later also extended to charged particles by

Infeld's student Wallace [9]. One of the main ingredients to obtain equations of motion via this approach is that the conservation of the energy-momentum is implied by the twice-contracted second Bianchi identity in concert with the Einstein equations. A rigorous assessment, however, reveals that for spacetimes with timelike singularities this idea is not automatically true, and the issue remains how to handle the singularities correctly (see, e.g., [7] for a discussion of some problems in the original derivation and [3, 7, 8] for references on alternative approaches). The key problem is to obtain conditions for the metric tensor of a spacetime with timelike singularities that guarantee that the second Bianchi identity holds weakly.

In our joint paper [3], presented at the Oberwolfach meeting, the author together with Michael Kiessling and A. Shadi Tahvildar-Zadeh

(i) identified a class of naked singularities that are suitable from a geometric perspective,
(ii) obtained a distributional version of the second Bianchi identity for static, spherically symmetric spacetimes with such a central singularity, and
(iii) investigated for which matter fields this identity holds.

For (i) it turned out to be important to allow only particular kinds of timelike singularities that can be viewed as interior timelike boundaries of "zero area" with "regular" boundary data. More precisely, such admissible timelike singularities are those that restrict to regular zero area singularities with a negative ZAS mass on spacelike slices in the framework of Bray and Jauregui [1, 2].

Regarding (ii), it is shown that the weak second Bianchi identity, that is,

$$\int_M (R^\mu{}_\nu - \frac{1}{2} R g^\mu{}_\nu) \nabla_\mu \psi^\nu \, d\,\mathrm{vol}_g = 0, \tag{1}$$

where ψ is a vector field of compact support, holds for a large class of static, spherically symmetric spacetimes with a single timelike singularity in the center. The main technical ingredient here is the use of what we call spatial conformally flat coordinates that allow for the blow-up of the one-dimensional timelike singularity to a co-dimension-one regular timelike boundary.

Surprisingly, the well-known Reissner–Weyl–Nordström spacetime of a single point charge does not belong to the class of spacetimes with admissible timelike singularities, and indeed it can be shown directly (see [3, Sec. 3.3]) that the weak second Bianchi identity (1) does not hold (infinite self energies lead to non-integrable curvature quantities). This was our starting point in (iii) to investigate alternative electromagnetic theories which admit suitable singularities. Here we connect the ZAS mass with the (negative) bare mass of central singularity in charged spacetimes. It turns out that, for instance, Hoffmann's solution [6] with negative bare mass in the framework of the Born–Infeld electromagnetic theory is of this kind. Other theories, such as the Bopp–Landé–Thomas–Podolsky theory are also being investigated.

While a necessary condition for when the weak second Bianchi identity (1) holds was not derived, the result obtained in [3] is close to optimal in the restricted framework of static spherically symmetric spacetimes with timelike singularities.

We conjecture that the identity holds, in fact, for all spacetimes with admissible timelike singularities of the kind described in (i).

Our future plans involve the study spacetimes with less symmetries, more complicated singularities and, in particular, two or more point charges. A long-term goal is to rigorously and consistently formulate a joint initial value problem for the motion of massive charged particles and the evolution of the electromagnetic and gravitational fields they generate.

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Level set methods in the study of scalar curvature and initial data sets in GR

Daniel Stern

(joint work with Hugh Bray, Demetre Kazaras, Marcus Khuri)

The problem of understanding how lower bounds on the scalar curvature R_g of a Riemannian manifold (M^n, g) interact with the global geometry and topology is a classical one in Riemannian geometry. In dimension $n = 2$ (where scalar, Ricci, and sectional curvature all coincide) the answer has been fairly well-understood for at least a century. In higher dimension $n \geq 3$, progress began in the 1960s and accelerated in the 1970s and '80s, beginning with Lichnerowicz's discovery [11] of a Bochner-type identity for harmonic spinors involving the scalar curvature (an observation later extended dramatically and used to great effect by Gromov-Lawson and others [5, 6, 7]) and taking off with Schoen and Yau's observations about the influence of scalar curvature bounds on the geometry and topology of two-sided stable minimal hypersurfaces [12, 13, 14].

Together with Huisken and Ilmanen's inverse mean curvature flow [9] and some input from Ricci flow in dimension three, the spinorial and minimal surface methods introduced in the '60s and '70s remain the primary tools for probing the effects of scalar curvature bounds. Nonetheless, it is natural to ask how the scalar curvature interacts with other natural geometric pdes, in the hopes of getting some new information. In a recent series of papers, an interesting relationship has been observed between scalar curvature and ($\mathbb{R}$- or S^1-valued) harmonic functions and solutions of other natural elliptic equations in dimension three–through the geometry and topology of the level sets–with some intriguing consequences.

In [15], it was shown that the level sets $\Sigma_t = u^{-1}\{t\}$ of the harmonic S^1-valued map $u : M \to S^1$ representing any homotopy class $[M : S^1]$ in a closed, oriented 3-manifold (M, g) satisfy the identity

$$\int_{S^1} 2\pi\chi(\Sigma_t) \geq \frac{1}{2}\int_M (|du|^{-1}|Hess(u)|^2 + R_g|du|),$$

giving a lower bound on the average Euler characteristic in terms of the scalar curvature R_g of M. This identity–which can be derived fairly simply via the Bochner identity, the Gauss equation, the coarea formula, and a little bit of analysis near the critical points of u–yields a simple new Hodge-theoretic proof that T^3 admits no metric of positive scalar curvature, and, in a straightforward way, allows one to recover and extend a sharp estimate relating scalar curvature and the Thurston norm originally obtained by Kronheimer and Mrowka via Seiberg-Witten techniques [10].

These techniques were extended in the papers [4] and [2] to the settings of compact 3-manifolds with boundary and asymptotically flat 3-manifolds, respectively. In [2], we consider in particular an end Y of a complete, asymptotically flat three-manifold, which we require (without loss of generality) to satisfy $H_2(Y, \partial Y; \mathbb{Z}) = 0$, with boundary ∂Y consisting of minimal spheres. Letting $F : Y \to \mathbb{R}^3$ be the harmonic map asymptotic to the linear coordinate functions with homogeneous Neumann condition $\frac{\partial F}{\partial \nu}|_{\partial Y} = 0$ (whose existence follows from the techniques of [1]), it is shown that for any $v \in S^2$, the harmonic coordinate $u = \langle v, F\rangle$ satisfies

$$m_{ADM}(Y) \geq \frac{1}{16\pi}\int_Y \left(\frac{|Hess(u)|^2}{|du|} + R_g|du|\right) dvol_g.$$

In particular, the harmonic coordinates provide an S^2 of lower bounds for the ADM mass of Y, yielding a simple new proof of the Riemannian positive mass theorem in dimension three.

These techniques have since been extended in several interesting directions by various authors. From the perspective of general relativity, some of the most interesting developments (at the time of this talk) have been the level set proof of the spacetime positive mass theorem by Hirsch, Kazaras, and Khuri [8], and the new proof of the hyperbolic positive mass theorem announced by Bray, Hirsch, Kazaras, Khuri, and Zhang [3]; though many other fun results have also been obtained. One expects that still more can be proved by examining the relationship between scalar curvature and the level sets of solutions to various other natural elliptic equations.

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Supertranslation invariant angular momentum in general relativity

Mu-Tao Wang

(joint work with Po-Ning Chen, Jordan Keller, Ye-Kai Wang, Shing-Tung Yau)

The definitions of conserved quantities such as mass and angular momentum have been among the most difficult problems since the genesis of general relativity. According to Einstein's equivalence principle, there is no density for gravitation and no canonical coordinate system for spacetime. The issue is further complicated by the nonlinear nature of Einstein's eponymous equation. One of the most important problems is the definition of angular momentum for a distant observer, or angular momentum at null infinity. There have been several promising candidates [2, 11] for the definition of angular momentum at null infinity. Unfortunately, none of them have been shown to be free of coordinate ambiguities. Such ambiguities at null infinity are called supertranslations and they belong to the larger BMS symmetry group [1, 12]. In particular, the angular momentums recorded by two distant observers of the same gravitating system may not be the same. According to Penrose [10], the very concept of angular momentum gets shifted by these supertranslations and "it is hard to see in these circumstances how one can rigorously discuss such questions as the angular momentum carried away by gravitational radiation" (page 654 of [10]). In this talk, the speaker presents a new definition of angular momentum at null infinity that is free of any supertranslation ambiguities [6, 8]. The definition is derived as the limit of quasilocal angular momentum that was proposed in [4, 5] and evaluated at null infinity in [9]. Comparing with existing definitions, the new definition contains an important correction term (that has never appeared in any previous definitions), which comes from solving the optimal isometric embedding equation proposed in [13, 14, 3]. The theory also produces a definition of center of mass at null infinity which is shown to be supertranslation invariant as well. The results can be extended to the charges of all classical and extended BMS fields to obtain supertranslation invariant charges for them [7].

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Stability of relativistic fluids on expanding cosmological spacetimes

Todd A. Oliynyk

Relativistic perfect fluids on a spacetime $(M, \tilde{g})$ are governed by the relativistic Euler equations given by[1]

$$\tilde{\nabla}_i \tilde{T}^{ij} = 0$$

where

$$\tilde{T}^{ij} = (\rho + p)\tilde{v}^i\tilde{v}^j + p\tilde{g}^{ij}$$

is the stress energy tensor, ρ is the fluid proper energy density, p is the fluid pressure, and $\tilde{v}^i$ is the fluid four-velocity normalized by $\tilde{g}_{ij}\tilde{v}^i\tilde{v}^j = -1$. Here, we will focus our attention on Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes $(M, \tilde{g})$ where

$$M = (0, 1] \times \mathbb{T}^3 \quad \text{and} \quad \tilde{g} = \frac{1}{t^2} g$$

with

$$g = \begin{cases} -dt \otimes dt + \delta_{IJ} dx^I \otimes dx^J & \text{(exponential expansion)} \\ -\dfrac{1}{q^2 t^{\frac{2}{q}}} dt \otimes dt + \delta_{IJ} dx^I \otimes dx^J & \text{(power-law expansion, } q > 0) \end{cases}.$$

It is important to note that, due to our conventions, the future is located in the direction of *decreasing* t and future timelike infinity is located at $t = 0$.

The above FLRW spacetimes having exponential and power-law, with $q > 1$, expansion fall within the class of cosmological spacetimes whose expansions is accelerated. The majority of stability results in the cosmological setting that involve either matter fields on prescribed spacetimes or matter fields coupled to the Einstein equations have been established under the assumption of accelerated expansion either explicitly for prescribed spacetimes or implicitly via a homogenous background solution of the Einstein-matter field equations. For relativistic fluids, the most comprehensive results have been established on exponentially expanding FLRW spacetimes with linear equations of state

$$p = K\rho \tag{1}$$

where the equation of state parameter K lies in the range

$$0 \le K \le \frac{1}{3}. \tag{2}$$

The first such stability result in this setting was, building on the earlier stability results for the Einstein-scalar field system [14], obtained for the parameter values $0 < K < 1/3$ in the articles [16, 17] where the non-linear stability of FLRW solutions to the Einstein-Euler equations with a positive cosmological constant was established. Stability results for the end points $K = 1/3$ and $K = 0$ were established later in [10] and [5], respectively. See also [3, 7, 8, 11] for different

[1] Our indexing conventions are as follows: lower case Latin letters, e.g. i, j, k, will index spacetime coordinate indices that run from 0 to 3 while upper case Latin letters, e.g. I, J, K, will index spatial coordinate indices that run from 1 to 3.

proofs and perspectives and the articles [6, 9] for related stability results for fluids with nonlinear equations of state. One of the important aspects of all of these works is they demonstrate that spacetime expansion can suppress shock formation in fluids, which was first discovered in the Newtonian cosmological setting [19]. This should be compared to the work of [1] where it is established that arbitrary small perturbations of a class of homogeneous solutions to the relativistic Euler equations, for relatively general equations of state, on Minkowski spacetime, which is a FLRW spacetime with spatial manifold $\mathbb{R}^3$ and no expansion, form shocks in finite time.

The next most well-understood case for relativistic fluids is that of accelerated (i.e. $q > 1$) power-law expansion, again for linear equations of state where K lies in the range (2). On prescribed spacetimes, stability results in this setting have been established in the articles [18, 20]. We note also the earlier related work [15] where the non-linear stability of homogenous solutions to the Einstein-scalar field equations with accelerated power-law expansion was established. For the borderline case $q = 1$ that separates accelerated ($q > 1$) from decelerated expansion ($q < 1$), a recent stability result has been obtained in this article [2] under the assumption that the fluid is irrotational and that $0 < K < 1/3$. This result is of particular interest because stability is known to fail for $K = 1/3$. Indeed in [18], it is shown that shocks form in finite time for arbitrary small choices of the initial data. It is also worthwhile noting that the article [2] provides the first, and so-far, only nonlinear stability result for the relativistic Euler equations (non-dust, i.e. $K > 0$) on spacetimes without accelerated expansion.

For linear equations of states, the parameter K determines the square of the sound speed, and consequently, it is natural to assume that K satisfies

$$0 \leq K \leq 1 \tag{3}$$

so that the propagation speed for the fluid is less than or equal to the speed of light. When the sound speed is equal to the speed of light, that is $K = 1$, it is well known that the irrotational relativistic Euler equations coincide, under a change of variables, with the linear wave equation. In this case, the future global existence of solutions on exponentially expanding FLRW spacetimes can be inferred from standard existence results for linear wave equations. However, as shown in the recent work [4], when coupling to the Einstein equations is considered, the fluid interpretation of these type of solutions becomes problematic due to the gradient of the solution to the wave equation becoming spacelike, which ruins the perfect fluid interpretation of these solutions.

Restricting our attention to linear equations of state, the most interesting questions that remain regarding the relativistic Euler equations on exponentially expanding spacetimes involve equation of state parameters that lie in the range

$$\frac{1}{3} < K < 1. \tag{4}$$

We recall that the asymptotic behaviour of relativistic fluids on exponentially expanding FLRW spacetimes with a linear equation of state for K satisfying (3) was investigated in the article [13] by Rendall using formal expansions. In that article, Rendall observed that the formal expansions can become inconsistent for K in the range (4) if the leading order term in the expansion of the fluid 3-velocity vanishes somewhere. In that case, he speculated that inconsistent behaviour in the expansions could be due to inhomogeneous features developing in the fluid density that would lead to the density contrast blowing up. This possibility for instability in solutions to the relativistic Euler equations for the parameter range (4) was also commented on by Speck in [18, §1.2.3]. There, Speck presents a heuristic analysis that suggest uninhibited growth should set in for solutions of the relativistic Euler equations for the parameter values (4).

The arguments presented in the articles [13, 18] certainly have cast doubt on the possibility of the existence of future global solutions to the relativistic Euler equations for equation of state parameters in the range (4). However, as we established in the article [12], there does, in fact, exist an open set of inhomogeneous future global solutions to the relativistic Euler equations for K satisfying

$$\frac{1}{3} < K < \frac{1}{2}.$$

The existence of these solutions is established under a suitable smallness assumption on the initial data and their asymptotic behaviour is given by

$$\begin{aligned}
\rho(t,x) &= \frac{\rho_c t^{\frac{2(1+K)}{1-K}} e^{(1+K)\zeta(t,x)}}{\left(t^{2\mu} + e^{2(u(t)+w_1(t,x))}\right)^{\frac{1+K}{2}}},\\
\tilde{v}^0(t,x) &= -t^{1-\mu}\sqrt{e^{2(u(t)+w_1(t,x))}+t^{2\mu}},\\
\tilde{v}^1 &= t^{1-\mu}\left(\frac{e^{u(t)+w_1(t,x)}}{\sqrt{(t^\mu w_2(t,x)-t^\mu w_3(t,x))^2+(t^\mu w_2(t,x)+t^\mu w_3(t,x))^2+1}}\right),\\
\tilde{v}^2(t,x) &= t^{1-\mu}\left(\frac{(t^\mu w_2(t,x)+t^\mu w_3(t,x))e^{u(t)+w_1(t,x)}}{\sqrt{(t^\mu w_2(t,x)-t^\mu w_3(t,x))^2+(t^\mu w_2(t,x)+t^\mu w_3(t,x))^2+1}}\right)
\end{aligned}$$

and

$$\tilde{v}^3(t,x) = t^{1-\mu}\left(\frac{(t^\mu w_2(t,x)-t^\mu w_3(t,x))e^{u(t)+w_1(t,x)}}{\sqrt{(t^\mu w_2(t,x)-t^\mu w_3(t,x))^2+(t^\mu w_2(t,x)+t^\mu w_3(t,x))^2+1}}\right),$$

where

$$\mu = (3K-1)/(1-K)$$

and there exists time-independent functions ζ_*, w_I^* on $\mathbb{T}^3$ and a constant $u_* \in \mathbb{R}^3$ such that

$$u(t) = u_* + \mathrm{O}(t^{2\mu}), \quad \zeta(t,x) = \zeta_*(x) + \mathrm{O}(t^{\mu-\sigma}), \quad w_1(t,x) = w_1^*(x) + \mathrm{O}(t^{\mu-\sigma}),$$
$$t^\mu w_2(t,x) = w_2^*(x) + \mathrm{O}(t^{\mu-\sigma}) \quad \text{and} \quad t^\mu w_3(t,x) = w_3^*(x) + \mathrm{O}(t^{\mu-\sigma})$$

for $\sigma > 0$ that can be chosen as small as we like. We have further established in the article [12] that under a $\mathbb{T}^2$-symmetry assumption, future global existence results can be obtained for the full range $1/3 < K < 1$. It is unclear at the moment if one should expect that this result should still hold for K satisfying $1/2 \leq K < 1$ if the $\mathbb{T}^2$-symmetry assumption is removed. As a consequence, it is an open problem to understand what happens for K satisfying $1/2 < K < 1$ in the case of general initial data.

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On The Asymptotic Behaviour of Gravitational Radiation

Leonhard M. A. Kehrberger

1. Introduction

The goal of this talk is to motivate and discuss some research with the aim of understanding the asymptotic behaviour of gravitational radiation. Throughout this talk, we will restrict our analysis to the linear scalar wave equation,

$$\Box_g \phi = 0, \tag{1}$$

on a fixed Schwarzschild background $(\mathcal{M}_{\mathrm{Schw}}, g)$ with mass M, however, the general ideas exposed here essentially also apply to the Teukolsky equations $\mathcal{T}_g \alpha^{[\pm 2]} = 0$ on Schwarzschild, and, thus, to linearised gravity around Schwarzschild.

2. Motivation: The study of late-time asymptotics

Let's first talk about a problem that has already received a lot of attention in the literature—the problem of late-time asymptotics: Given data for (1) on some asymptotically hyperboloidal hypersurface Σ, one asks what the asymptotics of the resulting solution ϕ in a neighbourhood of future timelike infinity i^+, and, in particular, along the event horizon $\mathcal{H}^+$ and along future null infinity $\mathcal{I}^+$ are. See the upper half of Figure 1.

There are several reasons why this is an important problem to study, but we here only name one: In the idealisation of an isolated system, where detection of gravitational waves takes place at $\mathcal{I}^+$, one can hope that we will eventually be able to measure these asymptotics along $\mathcal{I}^+$. Hence, it is important to have a mathematical prediction on them. Now, the asymptotics along $\mathcal{I}^+$ will depend on the choice of data one makes on Σ—however, a priori, we do not what kind of data one should assume on Σ! This begs the question: **How *can* we make *physically meaningful* predictions on the asymptotics along $\mathcal{I}^+$?**

Before we tackle this question, let us first discuss a few different choices of initial data on Σ and how the late-time asymptotics near i^+ depend on them:

Case 1: Compactly supported data. Historically, it has typically been assumed that $\phi|_\Sigma \in C_c^\infty(\Sigma)$. Heuristic analyses dating back to Price [12] have then motivated that, generically, one gets the so-called "Price's law tails" for the ℓ-th spherically harmonic mode ϕ_ℓ (we will always suppress the m-index), namely

$$\phi_\ell|_{\mathcal{H}^+} = C'_\ell v^{-2\ell-3} + \dots, \quad r\phi_\ell|_{\mathcal{I}^+} = C_\ell u^{-2-\ell} + \dots. \tag{2}$$

In fact, a precise (and more general) version of the above statement has recently been proven in the works [1, 2, 3] and [7]. The constants C'_ℓ, C_ℓ are generically non-zero and can be computed explicitly as integrals over initial data. *However, within the context of the model of an isolated system, the assumption of compact support becomes untenable since any such system will generically have radiated for all times and, therefore, will not have vanishing asymptotics near $\mathcal{I}^+$ along any asymptotically hyperboloidal hypersurface Σ!*

Case 2: Conformally regular/"peeling" data. A choice of initial data that does not suffer from the above problem, i.e. that does not have vanishing asymptotics near $\mathcal{I}^+$, is that of *conformally regular/peeling data*, i.e. data that admit an expansion in powers of the conformal variable $x = 1/r$. Indeed, it has been shown in [2] that, roughly speaking, if $\phi|_\Sigma$ admits an asymptotic expansion $\phi|_\Sigma = \frac{A_0}{r} + \frac{A_1}{r^2} + \frac{A_2}{r^3} + \dots$, where the A_i are functions on $\mathbb{S}^2$, then the resulting late-time asymptotics will be one power worse than in eq. (2), namely

$$\phi_\ell|_{\mathcal{H}^+} = C'_\ell v^{-2\ell-2} + \dots, \quad r\phi_\ell|_{\mathcal{I}^+} = C_\ell u^{-1-\ell} + \dots, \tag{3}$$

with the C_ℓ, C'_ℓ this time being a linear combination of the ℓ-th projections of $A_1, \dots, A_{\ell+1}$. The fact that higher ℓ-modes decay faster can be traced back to certain *conservation laws*, which, in Minkowski, i.e. if $M = 0$, read

$$\partial_u(r^{-2\ell-2}(r^2\partial_v)^{\ell+1}(r\phi_\ell)) = 0. \tag{4}$$

The weighted vector field $r^2\partial_v$ provides a measure of the conformal regularity of ϕ. Since it introduces an extra weight near $\mathcal{I}^+$, one can schematically think that *if one has sufficient conformal regularity*, then each commutation with $r^2\partial_v$ in (4) can be converted into one more power of u-decay along $\mathcal{I}^+$.

However, there is again no clear physical motivation for conformally regular data; the motivation is mainly of historical or of formal nature, going back to Penrose's smooth conformal compactification of spacetime [11] and the so-called "peeling behaviour" of massless radiation [13].

Case 3: Conformally irregular data. Let us now instead assume that the data are conformally irregular, say, $\phi|_\Sigma = \frac{A_0}{r} + \frac{A^* \log r}{r^2} + \dots$. In this case, it was shown by the author in [9] that

$$\phi_0|_{\mathcal{H}^+} = C'_0 v^{-2}\log v + \dots, \quad r\phi_0|_{\mathcal{I}^+} = C_0 u^{-1}\log u + \dots. \tag{5}$$

Similarly, a preliminary analysis suggests that (cf. §1.3 of [10]), for $\ell > 0$, $r\phi_\ell|_{\mathcal{I}^+} = C_\ell u^{-1} + \dots$ etc. (this is work in progress). We want to highlight the following two points: **a)** Since one now has *finite* conformal regularity, the general structure that higher ℓ-modes decay faster is broken. **b)** The constants C'_ℓ, C_ℓ will depend entirely on the constant A^*. *In other words, A^*, which measures the breakdown of conformal regularity, determines the leading-order late-time asymptotics!*

3. Obtaining a prediction on the asymptotics along Σ

We now sketch a scattering construction which aims to answer the question what the physically relevant behaviour on Σ (and thus, along $\mathcal{I}^+$) is. This construction is meant to provide a simple model of a system of N infalling masses following asymptotically Keplerian hyperbolic orbits near i^-, with no radiation coming in from $\mathcal{I}^-$, see Figure 1: On $\mathcal{I}^-$, we impose vanishing data to capture the no incoming radiation condition: $\partial_v(r\phi)|_{\mathcal{I}^-} \equiv 0$. The timelike boundary Γ should be thought of as enclosing the infalling masses (and therefore $r|_\Gamma \sim |u|$ as $u \to -\infty$), and we try to "capture the physics" of these masses by imposing boundary data on Γ that are in accordance with the quadrupole approximation: $r\phi|_\Gamma = C_{\mathrm{in}}|u|^{-p} + \dots$ for some power p as $u \to -\infty$. We will come back to this power below. Taking this

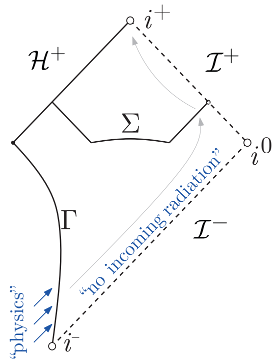

FIGURE 1. The upper half of the figure depicts the problem of late-time asymptotics discussed in section 2, i.e. how asymptotics along Σ translate into asymptotics near i^+.
The entire figure depicts the scattering construction of the present section, which aims to get a physical prediction of what the asymptotics along Σ (and thus, along $\mathcal{I}^+$) are.

as a model for N infalling masses from i^-, we now sketch how to obtain the relevant behaviour of ϕ along Σ. For brevity, we restrict to $\ell = 0$. The generalisation of (4) for $M \neq 0$ then reads (with $\partial_v r = -\partial_u r = 1 - 2M/r =: D$)

$$\partial_u \partial_v (r\phi_0) = -2MD \cdot r^{-3} \cdot r\phi_0. \tag{6}$$

Basic scattering theory ensures the unique existence of a solution to the problem outlined above, and gives us the preliminary rate $|\phi_0| \lesssim r^{-1/2}$. Inserting this bound into (6), and integrating from $\mathcal{I}^-$, where $\partial_v(r\phi_0) = 0$, we then get that $|\partial_v(r\phi_0)| \lesssim r^{-3/2}$. In turn, integrating this from Γ, we obtain that $|r\phi_0(u,v) - r\phi_0|_\Gamma| \lesssim |u|^{-1/2}$, an improvement over the initial bound (that can be traced back to the good r^{-3}-weight in (6))! One can then inductively repeat the procedure above to obtain the asymptotic estimate that $|r\phi_0(u,v) - r\phi_0|_\Gamma| \lesssim |u|^{-p-1}$. Finally, we insert this back into (6) and integrate once more from $\mathcal{I}^-$ to obtain an asymptotic estimate for $\partial_v(r\phi_0)$ (and thus ϕ_0) near $\mathcal{I}^+$. If e.g. $p = 1$, then, since $r \sim v - u$,

$$\partial_v(r\phi_0)(u,v) = \int_{-\infty}^{u} -2M \frac{C_{\mathrm{in}}}{|u'|r^3}\,\mathrm{d}u' + \dots = -\frac{-2MC_{\mathrm{in}}\log r}{r^3} + \dots. \tag{7}$$

Let's now return to the exponent p: The quadrupole approximation for N infalling masses following asymptotically Keplerian hyperbolic orbits predicts that the energy loss of gravitational radiation goes like $|u|^{-4}$ (see e.g. [4]). In the context of the scalar field, this energy loss is measured by $\int_{\mathbb{S}^2}(r\partial_u\phi)^2$. This thus predicts that $|r\partial_u\phi_0| \sim |u|^{-2}$, and thus, that $|r\phi_0| \sim |u|^{-1}$. We summarise our findings in:

Theorem 1. [8, 9] *The solutions arising from the data setup described above fail to be conformally regular near $\mathcal{I}^+$ (cf. eq. (7)). This failure of conformal regularity leaves its imprints on the late-time asymptotics near i^+: Smoothly extending the boundary data along Γ to $\mathcal{H}^+$, one obtains the late-time tails*

$$\phi_0|_{\mathcal{H}^+} = C_0' v^{-3}\log v + \dots, \qquad r\phi_0|_{\mathcal{I}^+} = C_0 u^{-2}\log u + \dots \tag{8}$$

for some constants C_0', C_0 that only depend on C_{in} and M.

A similar statement can be shown for higher ℓ-modes [10], and for solutions of the Teukolsky equation (upcoming work), where one finds that $\alpha^{[+2]} = O(r^{-4})$ near $\mathcal{I}^+$ (as opposed to the peeling rate $\alpha^{[+2]} = \mathcal{O}(r^{-5})$), see also the arguments [4, 6]. Interestingly, in the latter case, the resulting spacetimes seem to have decay rates weaker than those of [5]. It will be exciting to investigate this further.

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Time–periodic solutions to toy problems capturing the AdS (in-)stability

Athanasios Chatzikaleas

(joint work with Jacques Smulevici)

1. Abstract

Motivated by the study of small amplitudes non-linear waves in the Anti-de-Sitter spacetime and in particular the conjectured existence of periodic in time solutions, we construct families of arbitrary small time-periodic solutions to the conformal cubic wave equation and the spherically-symmetric Yang–Mills equations on the Einstein cylinder $\mathbb{R}\times\mathbb{S}^3$. For the conformal cubic wave equation, we consider both spherically-symmetric solutions and aspherical solutions with an ansatz relying on the Hopf fibration of the 3-sphere. In all three cases, the equations reduce to 1+1 semi-linear wave equations.

Our proof relies on a theorem of Bambusi–Paleari [2] for which the main assumption is the existence of a non-degenerate zero for a non-linear operator associated with the resonant system. Provided that the Fourier coefficients appearing in the resonant type system can be computed, the non-degeneracy condition amounts to solving a infinite dimensional linear system. In the simplest setting of the spherically symmetric conformal wave equation, the Fourier constants can be computed explicitly, and we prove that any mode solution of the linearized operator is a non-degenerate zero. Outside from spherical symmetry, we consider solutions depending on two frequency parameters which are associated with the torii of the Hopf fibration. We then prove the non-degeneracy condition of the linearized modes in certain high frequency regimes. The proof relies on the asymptotics of the eigenfunctions and the Fourier constants which hold in these high frequency regimes. In the Yang–Mills case, our proof relies in particular on a novel addition–like formula for the eigenfunctions of the associated linearized system.

2. The models

Specifically, we are interested in the existence of time–periodic solutions to toy models "capturing" the *Anti–de–sitter* (AdS) (in-)stability including the conformal cubic wave equation on the Einstein cylinder,

$$-\partial_t^2\phi(t,\omega)+\Delta_{\mathbb{S}^3}\phi(t,\omega)-\phi(t,\omega)=|\phi(t,\omega)|^2\,\phi(t,\omega), \tag{1}$$

for a scalar field $\phi:\mathbb{R}\times\mathbb{S}^3\longrightarrow\mathbb{C}$ with $\phi=\phi(t,\omega)$, and the equivariant Yang–Mills equation for the $SU(2)$ connection,

$$-\partial_t^2\phi(t,x)+\partial_{xx}^2\phi(t,x)+\frac{\phi(t,x)}{\sin^2(x)}=\frac{\phi^3(t,x)}{\sin^2(x)}, \tag{2}$$

for a scalar field $\phi:\mathbb{R}\times(0,\pi)\longrightarrow\mathbb{R}$ with $\phi=\phi(t,x)$, around the static solutions $\phi_0=0$ and $\phi_0=1$ respectively. We consider (1) with and without spherical symmetry. When the spherical symmetry assumption is removed, we find it convenient to introduce *Hopf coordinates* (η,ξ_1,ξ_2) instead of the spherical coordinates (ψ,ϑ,φ). These make use of the embedding of $\mathbb{S}^3\hookrightarrow\mathbb{R}^4$ and are given in terms of the standard Cartesian coordinates $(x_0,x_1,x_2,x_3)\in\mathbb{R}^4$ as follows

$$\begin{cases} x_0=\cos\xi_1\sin\eta,\\ x_1=\sin\xi_1\sin\eta,\\ x_2=\cos\xi_2\cos\eta,\\ x_3=\sin\xi_2\cos\eta,\end{cases}$$

where the new coordinates (η,ξ_1,ξ_2) vary within the bounded region

$$(\eta,\xi_1,\xi_2)\in\left[0,\frac{\pi}{2}\right]\times[0,2\pi)\times[0,2\pi)\,.$$

Note that, for any fixed $\eta\in\left(0,\frac{\pi}{2}\right)$, the coordinates (ξ_1,ξ_2) parametrize the 2−dimensional torus $\mathbb{S}^1\times\mathbb{S}^1$ where rings of constant ξ_1 and ξ_2 form orthogonal grids whereas, for fixed $\eta\in\left\{0,\frac{\pi}{2}\right\}$, the coordinates (ξ_1,ξ_2) parametrize the unit circle $\mathbb{S}^1$ embedded in $\mathbb{S}^3$. Furthermore, in these coordinates, the standard round metric on $\mathbb{S}^3$ is given by

$$h(\eta,\xi_1,\xi_2) = d\eta^2 + \sin^2\eta \; d\xi_1^2 + \cos^2\eta \; d\xi_2^2$$

and the Laplace–Beltrami operator on S^3 reads

$$\Delta^{\mathbb{S}^3}_{(\eta,\xi_1,\xi_2)}\chi = \partial_\eta^2\chi + \left(\frac{\cos\eta}{\sin\eta} - \frac{\sin\eta}{\cos\eta}\right)\partial_\eta\chi + \frac{1}{\sin^2\eta}\partial_{\xi_1}^2\chi + \frac{1}{\cos^2\eta}\partial_{\xi_2}^2\chi.$$

Then, the conformal cubic wave equation (1) can be written in Hopf coordinates as follows

(3)

$$\partial_t^2\chi - \partial_\eta^2\chi - \left(\frac{\cos\eta}{\sin\eta} - \frac{\sin\eta}{\cos\eta}\right)\partial_\eta\chi - \frac{1}{\sin^2\eta}\partial_{\xi_1}^2\chi - \frac{1}{\cos^2\eta}\partial_{\xi_2}^2\chi + \chi = -\left|\chi\right|^2\chi,$$

for $\chi(t,\eta,\xi_1,\xi_2) := u(t,\psi,\vartheta,\varphi)$. In principle, the Fourier expansion with respect to ξ_1 and ξ_2 of a generic solution $\chi(t,\eta,\xi_1,\xi_2)$ to (1) may include all possible admissible frequencies. In the following, we pick a fixed pair of frequencies (μ_1,μ_2) and force the Fourier expansion to excite only this particular pair by implementing the ansatz

$$\chi(t,\eta,\xi_1,\xi_2) = u(t,\eta)e^{i\mu_1\xi_1}e^{i\mu_2\xi_2}. \tag{4}$$

This ansatz is proposed by Oleg Evnin [3, 4] and resembles the sectors of constant μ in Figure 1 in [4]. In general, the pair (μ_1,μ_2) belongs in $\mathbb{Z}^2$. However, notice that if some of the μ_1 or μ_2 is strictly negative, then we can change coordinates $\xi_1 \longmapsto -\xi_1$ and $\xi_2 \longmapsto -\xi_2$ respectively and assume that both μ_1 and μ_2 strictly positive. Consequently, without loss of generality, we consider pairs of frequencies with

$$(\mu_1,\mu_2) \in \left(\mathbb{N}\cup\{0\}\right)^2.$$

Now, we substitute the ansatz (4) into (3) and obtain the differential equation

$$\left(\partial_t^2 + \mathsf{L}^{(\mu_1,\mu_2)}\right)u = \mathsf{f}(u), \tag{5}$$

for a scalar field $u : \mathbb{R}\times(0,\pi/2) \longrightarrow \mathbb{R}$ with $u = u(t,\eta)$ and

(6)

$$\mathsf{L}^{(\mu_1,\mu_2)}u := -\partial_\eta^2 u - \left(\frac{\cos\eta}{\sin\eta} - \frac{\sin\eta}{\cos\eta}\right)\partial_\eta u + \left(\frac{\mu_1^2}{\sin^2\eta} + \frac{\mu_2^2}{\cos^2\eta} + 1\right)u, \;\; \mathsf{f}(u) = -u^3.$$

We use the abbreviations:

- WS, for the conformal cubic wave equation in spherical symmetry (1),
- WH, for the conformal cubic wave equation out of spherical symmetry in Hopf coordinates (5)
- YM, for the Yang–Mills equation in spherical symmetry (2)

and study the evolution of the perturbations

$$u : \mathbb{R}\times I \longrightarrow \mathbb{R}, \;\; u = u(t,x),$$

where

$$I = \begin{cases} (0,\pi), & \text{for WS} \\ (0,\pi/2)\,, & \text{for WH} \\ (0,\pi), & \text{for YM} \end{cases}$$

around the static solutions

$$\phi_0 = \begin{cases} 0, & \text{for WS} \\ 0, & \text{for WH} \\ 1, & \text{for YM} \end{cases}$$

under the partial differential equations

$$\left(\partial_t^2 + \mathbf{L}\right) u = \mathbf{f}(x,u), \ \ (t,x) \in \mathbb{R}\times I \tag{7}$$

subject to the initial data with zero initial velocity,

$$\begin{cases} u_0(x) = u(0,x), \ \ x \in I, \\ u_1(x) = \partial_t u(0,x) = 0, \ \ x \in I. \end{cases}$$

Here, the linearlized operators and the non–linearities are given respectively by

$$\mathbf{L}u = \begin{cases} -\dfrac{1}{\sin^2(x)}\partial_x\left(\sin^2(x)\partial_x u\right) + u, & \text{for WS} \\ -\partial_x^2 u - \left(\dfrac{\cos x}{\sin x} - \dfrac{\sin x}{\cos x}\right)\partial_x u + \left(\dfrac{\mu_1^2}{\sin^2 x} + \dfrac{\mu_2^2}{\cos^2 x} + 1\right) u, & \text{for WH} \\ -\partial_x^2 u + \dfrac{2}{\sin^2(x)} u, & \text{for YM} \end{cases} \tag{8}$$

$$\mathbf{f}(x,u) = \begin{cases} -u^3, & \text{for WS} \\ -u^3, & \text{for WH} \\ -\dfrac{3u^2+u^3}{\sin^2(x)}, & \text{for YM} \end{cases}$$

3. Previous result

We initiated our study in [1] where we considered the spherically symmetric conformal cubic wave equation (1) and proved the existence of time–periodic solutions bifurcating from the first eigenmode to the linearized operator. Firstly, we recall our previous result.

Theorem 1 ([1]). *Let $\{e_n(\psi) : n \geq 0\}$ with*

$$e_n(\psi) := \frac{1}{\pi\sqrt{2}} U_n(\cos(\psi))$$

be the set of of all eigenfunctions to the linearized operator for (1) *in spherical symmetry where U_n stand for the Chebyshev polynomials of the second kind and* *degree n. Furthermore, pick any parameter $\epsilon > 0$. Then, all small initial data proportional to the first 1–mode*

$$u(0,\psi) = \epsilon \cdot e_0(\psi)$$

evolve into time–periodic solutions to (1) *with zero initial velocity* $\partial_t u(0,\cdot) = 0$.

4. Main result

We aim towards extending [1] and establish the following theorem.

Theorem 2 (Chatzikaleas and Smulevici). *Let $\{e_n(x) : n \geq 0\}$ with*

$$e_n(x) := \begin{cases} N_n U_n(\cos(x)), & \textit{for WS} \\ \mathsf{N}_n (1-\cos(2x))^{\frac{\mu_1}{2}} (1+\cos(2x))^{\frac{\mu_2}{2}} P_n^{(\mu_1,\mu_2)}(\cos(2x)), & \textit{for WH} \\ \mathfrak{N}_n \sin^2(x) P_n^{\left(\frac{3}{2},\frac{3}{2}\right)}(\cos(x)) & \textit{for YM} \end{cases}$$

be the set of of all eigenfunctions to the linearized operators (8) *where $P_n^{(\mu_1,\mu_2)}$ and U_n stand for the Jacobi polynomials of degree n and parameters (μ_1,μ_2) and the Chebyshev polynomials of the second kind of degree n respectively whereas N_n, N_n and $\mathfrak{N}_n$ are normalization constants. Furthermore, pick any real number $s > 3/2$, any fixed integer γ such that*

$$\gamma \in \begin{cases} \{0,1,\dots\}, & \textit{for WS} \\ \{0,1,\dots,10\}, & \textit{for WH} \\ \{0,1,\dots,10\}, & \textit{for YM} \end{cases}$$

and a pair of integers (μ_1,μ_2) with $\mu_1 = \mu_2$ and both sufficiently large. Then, given any small initial data proportional to the 1–mode $e_\gamma(x)$, there exists a family $\{u_\epsilon(t,\cdot) : \epsilon \in \mathcal{E}\}$ of time–periodic solutions to (7) *where $\mathcal{E}$ is an uncountable set that has zero as an accumulation point. In addition,*

(1) the period of $u_\epsilon(t,\cdot)$ is $2\pi/\omega_\epsilon$ where $\epsilon \longmapsto \omega_\epsilon$ that is monotone, one-to-one map that stays close to one,

$$|1-\omega_\epsilon| \lesssim \epsilon.$$

(2) $u_\epsilon \in H^1([0,2\pi/\omega_\epsilon]\,; H^s)$

(3) u_ϵ stays close to the solution to the linearized equation with the same initial data as above and zero initial velocity,

$$\sup_{t\in\mathbb{R}} \left\| u_\epsilon(t,\cdot) - \Phi^{t\omega_\epsilon}\left(\epsilon K_\gamma e_\gamma\right) \right\|_{H^s} \lesssim \epsilon^2,$$

where K_γ is a constant.

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Azimuthal instabilities of extremal Kerr black holes

Dejan Gajic

Background. The Kerr spacetimes $(\mathcal{M}_{M,a}, g_{M,a})$ constitute a 2-parameter family of stationary and axisymmetric solutions to the vacuum Einstein equations

$$\mathrm{Ric}[g] = 0, \tag{1}$$

and they are moreover expected to exhaust the *full* set of stationary black hole solutions to (1). These spacetimes describe a single isolated, asymptotically flat black hole, where $M > 0$ can be interpreted as the mass or energy of the black hole and $a \in [-M, M]$ its specific angular momentum.

In the last decade, there has been significant progress towards understanding the dynamical stability properties of *subextremal* Kerr black holes, that is to say, Kerr spacetimes satisfying the strict inequality $|a| < M$. From a full understanding of uniform boundedness and decay in the context of the toy model of the linear wave equation on a subextremal Kerr background [10] (see (2)) to recent nonlinear stability results in the non- or slowly-rotating setting [9, 15, 16] and many other foundational results. This talk discusses instead the dynamical properties of *extremal Kerr black holes*, which satisfy the equality $|a| = M$ and can be interpreted as black holes that rotate at the maximally allowed angular velocity.

Even though *exactly* extremal black holes are special in the set of all Kerr solutions, understanding their dynamics is important to be able to control uniformly (the open set of) *near-extremal* black holes and to identify possible observational signatures in gravitational radiation of (near)-extremality. They moreover play a special role in the understanding of the *weak cosmic censorship conjecture*, as they sit on the threshold between black hole solutions and naked singularities (Kerr spacetimes with $|a| > M$), as well as the *strong cosmic censorship conjecture*, as the Cauchy horizon in their black hole interiors is expected to be more stable than in the subextremal case [11, 12, 13].

Main result. I will introduce upcoming work proving the existence of new dynamical *instabilities* of extremal black holes. In particular, I will present the following main theorem, which can roughly be stated as follows:

Theorem 1. *There exists an asymptotic instability for generic solutions to the linear wave equation on extremal Kerr spacetime backgrounds that originates from non-axisymmetric azimuthal mode solutions: A) the energy density of waves blows up in time along the event horizon, and B) the total energy is non-decaying in time* *and moreover concentrates at the event horizon at late times when restricting to fixed azimuthal modes.*

Dynamics of spacetime solutions to (1) are studied in the setting of the initial value problem in general relativity. In particular, one can take initial data that are small, suitably localized perturbations of Kerr initial data and study the corresponding dynamical spacetimes. Due to the wave-like nature of key dynamical quantities for (1), a useful first step towards understanding the dynamics of Kerr perturbations is to study, using robust methods, the boundedness, decay and late-time asymptotic properties of the linear wave equation on a Kerr background:

$$\Box_{g_{M,a}}\psi = 0, \tag{2}$$

where $\Box_{g_{M,a}}$ denotes the Laplace–Beltrami operator with respect to the metric $g_{M,a}$. The instability results in Theorem 1 are obtained in the setting of (2), by considering moreover fixed azimuthal modes solutions ψ_m to (2), where $\partial_\varphi\psi_m = im\psi_m$, for ∂_φ the vector field generating axisymmetry of the background spacetimes.

Boundedness and decay of solutions to (2) in extremal Kerr was first investigated by Aretakis in [6], who restricted to axisymmetric solutions (azimuthal modes with $m = 0$). Aretakis discovered the existence of a *conservation law for* ψ_0 along the event horizon and showed that it generically results in an asymptotic horizon instability [7]: $|\partial^2\psi_0|$ blows up in time along the event horizon, but decays in time outside the horizon.

When $m \neq 0$, however, this conservation law mechanism for deriving instabilities breaks down. I will show that the stronger instabilities from Theorem 1 follow instead from precise knowledge of the late-time behaviour of ψ_m along the event horizon, i.e. the precise nature of *late-time inverse polynomial tails* in the dynamics of ψ_m. Inverse polynomial late-time tails have recently been derived in the context of subextremal Kerr spacetimes [2, 14, 5] and spherically symmetric extremal Reissner–Nordström spacetimes [3] and were originally proposed by Price [17], which is why the corresponding decay rates are also known as "Price's law" in the literature.

In this talk, I show how late-time tails for ψ_m with $m \neq 0$ take a rather different form in extremal Kerr, compared to both subextremal Kerr and extremal Reissner–Nordström, and their existence follows from two main steps:

(1) *Weighted integrated energy estimates* ("Morawetz estimates"). Controlling integrals in time of appropriately weighted energies involves quantifying the geometric phenomena of trapped null geodesics, superradiance ("energy extraction from the black hole") and their interaction. When m is fixed these are decoupled in frequency space. Controlling superradiance involves in particular quantitative mode stability near the threshold of superradiance, which was established recently in [18].

(2) *Weighted integrated energy estimates $\to$ late-time asymptotics.* This step involves the introduction of a new mechanism to convert irregularity at the event horizon/slow fall-off in r of initial data to inverse polynomial tails in the leading-order behaviour in time of ψ_m and uses the observation that, even if the original initial data is smooth and compactly supported, the initial data corresponding to a *time integral*, which can be thought of as $\widetilde{\psi} = -\int_{\tau}^{\infty} \psi \, d\tilde{\tau}$, will generically be irregular at the event horizon.

We confirm in particular decay rates in late-time tails that were first suggested by heuristics in the physics literature: away from the horizon in [1] and along the horizon in [8] for a restricted class of initial data.

Outlook for the future. In order to understand the complete instability properties of linear waves on extremal Kerr, one still has to *sum* over all the azimuthal numbers m, which requires a better understanding of the coupling between trapped null geodesics and superradiance in frequency space. Subsequently, it would be interesting to study the effects of extremal Kerr instabilities on the evolution of nonlinear wave equations. Using the methods developed in [3, 4], one can moreover initiate the study of the nonlinear dynamics of non- or slowly-rotating *charged* extremal black holes in the setting of (1) coupled to electromagnetic fields, which feature a milder instability, akin to the axisymmetric instability on extremal Kerr.

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The breakdown of weak null singularities inside black holes

Maxime Van de Moortel

What singularities lie inside a black hole formed in gravitational collapse? In a local region near time-like infinity, it is known for various models that a generic black hole has a weakly singular Cauchy horizon. The global structure of the black hole interior, however, has largely remained unexplored. I will present my recent proof that, in the spherical collapse of a charged scalar field, the weakly singular Cauchy horizon breaks down and gives way to a stronger singularity connected to the center of the collapsed star.

The breakdown result and its context

The recent result of Dafermos–Luk [4] spectacularly established the presence of a null boundary $\mathcal{CH}_{i^+}$ – the Cauchy horizon – in the interior of dynamical black holes. They also show that $\mathcal{CH}_{i^+}$ cannot be a strong singularity (since the metric is C^0-extendible); in fact, it is not known whether $\mathcal{CH}_{i^+}$ is singular at all[1]!

In models of charged spherical collapse, more is known about $\mathcal{CH}_{i^+}$, despite novel phenomena linked to the presence of matter [5, 6]. Specifically, for black holes converging to stationarity at the expected rate, the author proved that $\mathcal{CH}_{i^+}$ is non-empty and *weakly singular* [9, 11], i.e. C^2-inextendible; a scenario often dubbed in the literature as mass inflation/blue-shift instability. Previous related results include the seminal work of Luk–Oh [8]; their streamlined model does not however allow to study the global aspects of gravitational collapse, due to a topological restriction imposing time-slices to be *two-ended* (asymptotically flat).

[1] Although multiple heuristic arguments, and analogies with rigorous results on spherically symmetric models strongly suggest that, indeed, $\mathcal{CH}_{i^+}$ is weakly singular, see [4] for a discussion.

Returning to the case of charged spherical gravitational collapse, we now study the following Einstein–Maxwell–Klein–Gordon equations, with $q_0 \neq 0$ and $m^2 \geq 0$

$$\begin{aligned} Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} &= \underbrace{g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}}_{\mathbb{T}^{EM}_{\mu\nu}} \\ &\quad + \underbrace{\Re(D_\mu\phi\overline{D_\nu\phi}) - \frac{1}{2}(g^{\alpha\beta}D_\alpha\phi\overline{D_\beta\phi} + m^2|\phi|^2)g_{\mu\nu}}_{\mathbb{T}^{KG}_{\mu\nu}}, \\ \nabla^\mu F_{\mu\nu} &= \frac{q_0}{2}i(\phi\overline{D_\nu\phi} - \overline{\phi}D_\nu\phi),\ F = dA,\ D_\mu = \nabla_\mu + iq_0A_\mu, \\ g^{\mu\nu}D_\mu D_\nu\phi &= m^2\phi \end{aligned}$$

and consider spherically symmetric, one-ended initial data, as a model of astrophysical collapse more realistic [3, 7] than the two-ended case, since spacetime now includes the center Γ of the collapsed star (see Figure 1). This one-endedness assumption, which can only be considered for charged matter in spherical symmetry, is inconsequential to the local dynamics near i^+, but crucial to the global dynamics in a spectacular way (see [2], and Section 1.7 in [10] for discussions).

In this setting, the interior dynamics *away from time-like infinity* i^+ offers a fundamental problem formulated by Dafermos [3]: can the Cauchy horizon $\mathcal{CH}_{i^+}$ go all the way to the center Γ as in Figure 1, and thus "close-off" the spacetime? This problem was completely solved by the author for the above model in [10]:

Theorem ([10]). *The weak null singularity $\mathcal{CH}_{i^+}$ necessarily breaks down in gravitational collapse, in the sense that the Penrose diagram of Figure 1 is impossible. Moreover, the breakdown gives rise to the formation of a stronger singularity; namely the singular center-endpoint b_Γ is a Terminal Indecomposable (TIP).*

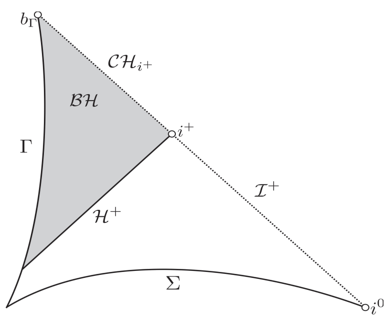

FIGURE 1. The impossible Penrose diagram if $\mathcal{CH}_{i^+}$ is weakly singular.

Thus, the answer brought in [10] to Dafermos' question is, perhaps unexpectedly, emphatically no! The underlying mechanism is a novel phenomenon discovered in [10] entitled *the breakdown of the weak null singularity.* The problem

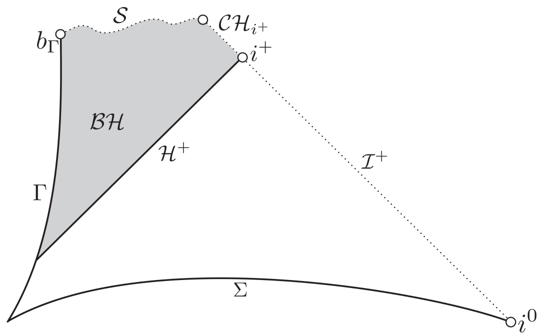

FIGURE 2. Generic black hole, resulting from the breakdown theorem of [10] and a resolution of the local Weak Cosmic Censorship.

is non-perturbative, and its resolution unexpectedly proceeds by a contradiction argument. The proof is global and exploits the presence of the center Γ (characterizing the one-ended collapse) in a crucial way. The analysis is moreover nonlinear in nature, since it relies on the Einstein null structures (Raychaudhuri equations).

The consequences of breakdown and further problems

An important further problem following our theorem, is to characterize entirely the singularity at the terminal boundary and, in particular, validate a scenario in which a strong, space-like singularity $\mathcal{S}$ bifurcates from the weaker one $\mathcal{CH}_{i^+}$ and connects to the center Γ as depicted in Figure 2. As it turns out, the theorem from [10] also provides an (almost) definite answer to the above fundamental problem.

Corollary ([10]). *Under the conditions of the above theorem, either:*

- *the terminal boundary either consists of a bifurcation between a strong $\mathcal{S} = \{r = 0\}$ and a weakly singular Cauchy horizon $\mathcal{CH}_{i^+}$ as in Figure 2.*
- *Or the center endpoint b_Γ is a locally naked singularity.*

Non-genericity of locally naked singularities and Weak Cosmic Censorship. Locally naked singularities are known to exist and cannot be excluded in the analysis of [10] in the above charged model; however, they were shown to be non-generic in Christodoulou's celebrated resolution of Weak Cosmic Censorship [1] in uncharged collapse. To generalize [1] to the charged case does not present any immediate conceptual issues, contrary to the dynamics of the Cauchy horizon; Thus, the following open problem appears within reach, and would completely characterize the interior of dynamical black holes in charged spherical collapse.

Open problem. *[Local Weak Cosmic Censorship Conjecture] Show that for generic spacetimes in charged spherical collapse, b_Γ is not a locally naked singularity.*

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An approach to mass based on capacity

Jeffrey Jauregui

The ADM mass provides a well-defined notion of the total mass of an asymptotically flat 3-manifold (M, g). However, because its formula depends on derivatives of g, the ADM mass is challenging to understand in non-smooth spaces and under non-smooth convergence. Interest in mass defined in low regularity springs from the almost-equality case of the positive mass theorem and from Bartnik's mass-minimization problem [1], for example.

In 2006 Huisken proposed a remarkable new paradigm for understanding total mass based on the isoperimetric behavior of large regions, i.e., by examining the minimal area configurations for a given volume far out in an asymptotically flat end [3], [4]. His *isoperimetric mass* is defined by:

$$m_{iso}(M, g) = \sup_{\{K_i\}} \limsup_{i\to\infty} \frac{2}{|\partial K_i|_g}\left(|K_i| - \frac{1}{6\sqrt{\pi}}|\partial K_i|^{3/2}\right),$$

where the supremum is taken over all exhaustions of M by sequences of compact sets $\{K_i\}$. Here, $|K_i|$ and $|\partial K_i|$ denote the volume and perimeter of K_i. Using the isoperimetric inequality, it is straightforward to check that $m_{iso} = 0$ in Euclidean space. With more work, it can be shown that m_{iso} agrees with the parameter m in the Schwarzschild manifold of mass $m > 0$. More generally, as proposed by Huisken, it has been established that m_{iso} agrees with the ADM mass for asymptotically flat manifolds of nonnegative scalar curvature (whose boundary, if nonempty, consists of minimal surfaces). This was carried out by Jauregui and Lee [7], building on ideas of Huisken, and separately by Chodosh, Eichmair, Shi, and Yu [2]. Consequently, since m_{iso} is defined using only areas and volumes (no derivatives of g), it is reasonable to interpret m_{iso} as a stand-in for the ADM mass for lower regularity spaces (including, for example, C^0 Riemannian manifolds).

Since m_{iso} is motivated by the classical isoperimetric inequality in $\mathbb{R}^3$, it is natural to ask if other isoperimetric-type inequalities give any insight into the ADM mass. We focus on the well-known *isocapacitary inequality* (also known as the Poincaré–Faber–Szegö inequality) for a compact region $K \subset \mathbb{R}^3$, which states:

$$\mathrm{cap}(K) \geq \left(\frac{3|K|}{4\pi}\right)^{1/3}, \tag{1}$$

where the capacity $\mathrm{cap}(K)$ is defined by minimizing the integral $\int_{\mathbb{R}^3} |\nabla\varphi|^2 dV$ over all Lipschitz functions that vanish on K and approach 1 at infinity. With some smoothness on ∂K assumed, equality is only achieved when K is a round ball. The capacity is defined in a completely analogous way in an asymptotically flat manifold, although inequality (1) will not generally hold. However, far out in the asymptotically flat end, the inequality should approximately hold, and it is natural to ask if the "deficit" carries any information about the total mass. This motivates our goals, which are to 1) carry out an analog of Huisken's program, but based alternatively on the isocapacitary inequality, and 2) identify if such an approach may have some advantages in low regularity problems.

By considering round balls in Schwarzschild space, one is naturally led to the following formula to define the "capacity–volume mass" [6]:

$$m_{CV}(M,g) = \sup_{\{K_j\}} \limsup_{j\to\infty} \left[\left(\frac{3|K_j|}{4\pi}\right)^{1/3} - \mathrm{cap}(K_j)\right].$$

This formula is of course modeled on Huisken's definition of m_{iso}, but uses only volumes and capacities in its definition. Like m_{iso} it is immediately clear that m_{CV} is a geometric invariant. The easiest example to consider is Euclidean space. By the isocapacitary inequality, $m_{CV} \leq 0$ for $\mathbb{R}^3$. But by considering an exhaustion by balls, it becomes clear that $m_{CV} = 0$. It also turns out that m_{CV} produces the value m in the Schwarzschild manifold of mass $m > 0$, as will be addressed later.

To call to m_{CV} as a "mass," it should be possible to show that it agrees with the ADM mass. We have the following results from [6]. First, a lower bound:

Theorem 1. *Let (M^3, g) be asymptotically flat with nonnegative scalar curvature outside of a compact set. Then*

$$m_{CV}(M,g) \geq m_{ADM}(M,g).$$

Next, we have an upper bound if we restrict to an exhaustion by balls:

Theorem 2. *Let (M^3, g) be asymptotically flat with nonnegative scalar curvature, with ∂M empty or union of minimal surfaces. Then*

$$\limsup_{r\to\infty}\left[\left(\frac{3|B_r|}{4\pi}\right)^{1/3} - \mathrm{cap}(B_r)\right] \le m_{ADM}(M,g).$$

Finally, we have an upper bound on m_{CV} for metrics that are harmonically flat at infinity (HF). This means that outside of a compact set, $g_{ij} = U^4\delta_{ij}$ at infinity, where U is a harmonic function approaching 1 at infinity.

Theorem 3. *Let (M, g) be a Riemannian 3-manifold that is HF with nonnegative ADM mass. Then*

$$m_{CV}(M,g) \le m_{ADM}(M,g).$$

As a corollary of Theorems 1 and 3 and the positive mass theorem, we have $m_{CV}(M,g) = m_{ADM}(M,g)$ when (M,g) is HF with nonnegative scalar curvature and empty or minimal boundary. In particular, in the Schwarzschild manifold of mass $m > 0$, we have $m_{CV}(M,g) = m$. (The latter can alternatively be shown using the fact established in [11] that round balls in positive mass Schwarzschild minimize capacity for their volume.)

We conjecture that $m_{CV}(M,g) = m_{ADM}(M,g)$ holds for general asymptotically flat metrics. In light of the above results, a reasonable approach to this would be to show that minimizers of capacity for a given large volume V approach round spheres in the AF end as $V \to \infty$. Many such results are known for isoperimetric regions (we refer to [2] and the references therein).

Now, why might it be advantageous to consider m_{CV} over m_{iso}? Both are particularly well-suited to studying the total mass of AF metrics that are not necessarily smooth (they are possibly the only known candidates), which could arise as limiting metrics in the problems mentioned in the first paragraph. Little is known about mass in low regularity, but there have been some results that show how the mass (particularly m_{iso}) behaves under low regularity convergence. We mention specifically [7] and [8] where it was shown that m_{iso} is lower semicontinuous under local C^0 convergence and, more generally, under a pointed, volume-preserving version of Sormani–Wenger intrinsic flat convergence [12], which we will call $\mathcal{VF}$ convergence. $\mathcal{VF}$ convergence seems to be well-suited for limiting problems involving nonnegative scalar curvature; see [10] for example.

In this regard, there are some technical reasons why m_{CV} might be more favorable to work with in lower regularity. First, under $\mathcal{VF}$ convergence, boundary area is lower semicontinuous (i.e., can jump down), which cause the expression defining m_{iso} to "go the wrong way" in the limit. One of the biggest challenges in [8] was overcoming this difficulty. By contrast, capacity is upper semicontinuous, which will be shown in [9], so that the expression defining m_{CV} naturally behaves in a nice way. Second, the capacity of a set is much more stable under boundary perturbations. This is significant, since the difficulty of controlling boundary area under perturbations necessitated an additional hypothesis on the limit space in [8]. While general solutions to problems such as the almost-equality case of the positive mass theorem and Bartnik's mass minimization problem remain out of reach, we anticipate m_{CV} may be a useful tool in studying them.

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New Structures in Gravitational Radiation

Lydia Bieri

Gravitational radiation has been studied in General Relativity (GR) via various mathematical methods. Studies of gravitational waves have been devoted mostly to sources such as binary black hole mergers or neutron star mergers, or generally sources that are stationary outside of a compact set. These systems are described by asymptotically-flat manifolds solving the Einstein equations with sufficiently fast decay of the gravitational field towards Minkowski spacetime far away from the source. Waves from such sources have been recorded by the LIGO/VIRGO collaboration since 2015, see [1, 2, 3]. I present new results on gravitational radiation for sources that are not stationary outside of a compact set, but whose gravitational fields decay more slowly towards infinity [5, 6]. A panorama of new gravitational effects opens up when delving deeper into these more general spacetimes. In particular, whereas the former sources produce memory effects that are finite and of purely electric parity, the latter in addition generate memory of magnetic type, and both types grow. These new effects emerge naturally from the

Einstein equations both in the Einstein vacuum case and for neutrino radiation. The latter results are important for sources with extended neutrino halos.

Gravitational waves are predicted to change the spacetime permanently, which would show in a change of the arrangement of test masses after the wave train passed. This is the so-called memory effect. This effect was found in a linear theory in 1974 by Ya. Zel'dovich and B. Polnarev [10]. Then in 1991 D. Christodoulou [8] derived within the full nonlinear theory such a memory that was much larger than expected. Together with D. Garfinkle we showed [7] that these are two different effects sourced by different events. The former is called ordinary memory the latter null memory. There has been a vast literature on memory effects. See [6] for more information and further references.

We consider the Einstein vacuum equations

$$R_{\mu\nu} = 0 \tag{1}$$

($\mu, \nu = 0, 1, 2, 3$) as well as the Einstein-null-fluid equations describing neutrino radiation in GR

$$R_{\mu\nu} \; = \; 8\pi \, T_{\mu\nu} \; . \tag{2}$$

As the $T_{\mu\nu}$ for the null fluid is traceless, the Einstein equations for a null fluid reduce to (2).

We consider various classes of asymptotically-flat spacetimes (M, g) (in 4 spacetime dimensions), which are solutions of the Einstein equations (1) respectively (2) for the following corresponding classes of initial data.

Definition 1. *((B), [4]) We define an asymptotically flat initial data set to be a (B) initial data set, if it is an asymptotically flat initial data set $(H_0, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and for which there exists a coordinate system (x^1, x^2, x^3) in a neighborhood of infinity such that with $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, it is:*

$$\bar{g}_{ij} \;=\; \delta_{ij} \;+\; o_3 \; (r^{-\frac{1}{2}}) \tag{3}$$

$$k_{ij} \;=\; o_2 \; (r^{-\frac{3}{2}}) \; . \tag{4}$$

In [4], weighted Sobolev norms of appropriate energies are controlled, yielding the most general class of spacetimes for which nonlinear stability has been proven.

Christodoulou-Klainerman (CK) studied data of the following type:

Definition 2. *((CK), [9]) We define a strongly asymptotically flat initial data set in the sense of [9] and in the following denoted by (CK) initial data set, to be an initial data set $(H, \bar{g}, k)$, where $\bar{g}$ and k are sufficiently smooth and there exists a coordinate system (x^1, x^2, x^3) defined in a neighborhood of infinity such that, as $r = (\sum_{i=1}^{3}(x^i)^2)^{\frac{1}{2}} \to \infty$, $\bar{g}_{ij}$ and k_{ij} are:*

$$\bar{g}_{ij} \;=\; (1 \;+\; \frac{2M}{r}) \; \delta_{ij} \;+\; o_4 \; (r^{-\frac{3}{2}}) \tag{5}$$

$$k_{ij} \;=\; o_3 \; (r^{-\frac{5}{2}}) \; , \tag{6}$$

where M denotes the mass.

In [9] and [4] proofs of nonlinear stability are established under appropriate smallness conditions on the initial data in weighted Sobolev norms. This ensured the existence of these spacetimes. However, the behavior along null hypersurfaces towards future null infinity is largely independent from the smallness.

More precisely, consider data of type (B). The proof [4] yielded the most general class of data for which a nonlinear stability result has been proven. The spacetimes constructed in [4] exhibit interesting new features that are important also for large data. It follows easily by a corollary that there exists a complete domain of dependence of the complement of a sufficiently large compact subset of the initial hypersurface. Thus, we have a solution spacetime with a portion of future null infinity corresponding to all values of the retarded time u not greater than a fixed constant. This provides the solid foundation to investigate the asymptotic behavior at future null infinity for large data for (B) spacetimes (that is solutions of (1) with behavior as in definition (B) but with large data), and to prove theorems on the nature of gravitational radiation. Naturally, our investigations extend to these spacetimes coupled to neutrinos via a null fluid.

We recall the decomposition of the Weyl tensor W into its electric and magnetic parts. Contracting W twice with a vectorfield X as follows gives

$$\begin{aligned} \mathcal{E}_X(W)_{\alpha\beta} &= W_{\mu\alpha\nu\beta}X^\mu X^\nu \\ \mathcal{M}_X({}^*W)_{\alpha\beta} &= {}^*W_{\mu\alpha\nu\beta}X^\mu X^\nu \quad \text{with} \quad {}^*W_{\alpha\beta\gamma\delta} = \frac{1}{2}\epsilon_{\alpha\beta\mu\nu}W^{\mu\nu}{}_{\gamma\delta} \end{aligned}$$

$\mathcal{E}_X(W)$ and $\mathcal{M}_X({}^*W)$ are symmetric, traceless tensors that are orthogonal to X. Here, we are interested in ρ (electric part) and σ (magnetic part) of the Weyl curvature. In particular, in a convenient null frame with L an incoming and $\underline{L}$ an outgoing null vectorfield these components are given by $R_{\underline{L}L\underline{L}L} = 4\rho$ respectively ${}^*R_{\underline{L}L\underline{L}L} = 4\sigma$.

It has been known, that gravitational wave memory from "traditional" sources (see above) are finite and of electric parity only. Magnetic memory does not occur for these spacetimes. In [5, 6] I showed that for spacetimes of slow decay (including data of type (B)) there exists magnetic-parity memory, and both electric and magnetic memories diverge at rate $\sqrt{|u|}$. Moreover, a multitude of new structures contribute to these effects. These spacetimes solve (1) respectively (2) for corresponding classes of initial data. A simplified version of the main theorems can be summarized and stated as follows:

Theorem 3. *((B) 2020) For classes of solutions of (1) respectively (2) (including data of type (B)) there exists magnetic memory and electric memory both diverging at rate* $\sqrt{|u|}$. *New structures contribute to these effects.*

The proof uses the Bianchi equations for ρ and σ. We investigate the behavior of the main curvature and other geometric quantities at future null infinity. In particular, we derive dynamical structures that are hidden behind the non-dynamical components. The former impact gravitational radiation and memory whereas the latter do not.

The new results describe gravitational radiation and memory for sources that are not stationary outside of a compact set, but whose gravitational fields decay more slowly towards infinity [5, 6]. In particular, these include sources with extended neutrino halos. Various applications open up.

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Positive mass theorems for asymptotically hyperbolic initial data sets

Anna Sakovich

In this talk, we presented a new proof of the positive mass theorem for asymptotically hyperbolic "hyperboloidal" initial data sets. The prototype for such initial data sets is the unit hyperboloid in Minkowski spacetime. In other words, it is assumed that both the induced metric and the second fundamental form approach the hyperbolic metric at infinity.

The underlying idea of the proof builds upon the Jang equation reduction method that was introduced by Schoen and Yau in [3]. The key observation made in [3] is that the proof of the positive mass theorem for asymptotically Euclidean initial data sets can be reduced to applying the positive mass theorem for asymptotically Euclidean manifolds that was established in [2]. More specifically, it was shown that given an asymptotically Euclidean initial data set (M, g, K) satisfying the dominant energy condition, there is a hypersurface $\Sigma \subset (M \times \mathbb{R}, g + dt^2)$ that can be equipped with an asymptotically Euclidean metric of nonnegative scalar curvature whose ADM mass does not exceed the ADM mass of (M, g, K). The hypersurface Σ arises as a solution of the so-called Jang equation, a quasilinear elliptic PDE of the form $H^\Sigma = \mathrm{tr}^\Sigma K$, where H^Σ is the mean curvature of Σ, and K is assumed to be extended to $M \times \mathbb{R}$ by setting $K(\cdot, \partial_t) = 0$. In the case when the initial data set is asymptotically Euclidean, the existence theory for this equation was established in [3].

As discussed in the talk, it turns out that the Jang equation reduction method can be implemented in the setting of asymptotically hyperbolic initial data sets, yielding a proof of the positive mass theorem. In particular, the Jang equation can be used to deform an asymptotically hyperbolic initial data set satisfying the dominant energy condition to an asymptotically Euclidean manifold with "almost nonnegative" scalar curvature, again reducing the proof to the application of the positive mass theorem for asymptotically Euclidean manifolds. However, the analysis of the Jang equation in the asymptotically hyperbolic setting poses some challenges that are not present in the asymptotically Euclidean setting of [3]. Some of these challenges are related to the fact that the expected asymptotic behaviour of the solution near infinity is more complicated. To deal with this problem, a new method for constructing the so-called barrier functions had to be designed, ensuring the desired asymptotics of the solution. Another difficulty is that the rescaling technique, which is a commonly used method for proving estimates for solutions of geometric PDEs in the asymptotically Euclidean setting, does not work on asymptotically hyperbolic manifolds. This complication was dealt with by revisiting the Jang equation and rewriting it in terms of the asymptotically Euclidean data induced on the graphs of the barrier functions, after which the rescaling argument could be applied.

The presented work is carried out in dimension 3. Please see [1] for further details.

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Reporter: Zoe Wyatt

Participants

Dr. Paul Tyler Allen
Lewis and Clark College
Department of Mathematical Sciences,
MSC 110
615 S. Palatine Hill Road
Portland OR 97219
UNITED STATES

Prof. Dr. Stefanos Aretakis
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Dr. Florian Beyer
Department of Mathematics and
Statistics
University of Otago
Science III, Room 218
P.O. Box 56
Dunedin 9054
NEW ZEALAND

Prof. Dr. Lydia Bieri
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Pieter Blue
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
King's Buildings, Mayfield Road
Edinburgh EH9 3JZ
UNITED KINGDOM

Dr. Annegret Burtscher
Department of Mathematics
Radboud University Nijmegen
P.O. Box 9010
6500 GL Nijmegen
NETHERLANDS

Prof. Dr. Carla Cederbaum
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Athanasios Chatzikaleas
Laboratoire Jacques-Louis Lions
Sorbonne Université
4 place Jussieu
75252 Paris Cedex 05
FRANCE

Prof. Dr. Piotr T. Chruściel
Fakultät für Physik/Gravitationsphysik
Universität Wien
Boltzmanngasse 5
1090 Wien
AUSTRIA

Dr. Stefan Czimek
Institute for Computational and
Experimental Research in Mathematics
(ICERM)
Brown University
121 South Main Street, 11th Floor
P.O. Box Box E
Providence RI 02912
UNITED STATES

Prof. Dr. Mihalis Dafermos
Department of Mathematics
Princeton University
Fine Hall 406
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Grigorios Fournodavlos
Laboratoire Jacques-Louis Lions
Sorbonne Université
4, place Jussieu
75005 Paris Cedex
FRANCE

Prof. Dr. Helmut Friedrich
MPI für Gravitationsphysik
Albert-Einstein-Institut
Am Mühlenberg 1
14476 Golm
GERMANY

Dr. Dejan Gajic
Department of Mathematics
Radboud Universiteit Nijmegen
P.O. Box 9010
6500 GL Nijmegen
NETHERLANDS

Prof. Dr. Gregory Galloway
Department of Mathematics
University of Miami
P.O. Box 249085
Coral Gables FL 33124
UNITED STATES

Prof. Dr. David Garfinkle
Physics Department
Oakland University
Rochester, MI 48309
UNITED STATES

Prof. Dr. Elena Giorgi
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Melanie Graf
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. David Hilditch
CENTRA, IST
University of Lisbon
1049 Lisboa
PORTUGAL

Dr. Gustav Holzegel
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Dr. Lan-Hsuan Huang
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009
UNITED STATES

Prof. Dr. Gerhard Huisken
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Prof. Dr. James A. Isenberg
Department of Mathematics
University of Oregon
Eugene, OR 97403-5203
UNITED STATES

Dr. Jeffrey Jauregui
Department of Mathematics
Union College
807 Union St.
Schenectady, NY 12308
UNITED STATES

Dr. Christoph Kehle
Institute for Theoretical Studies
ETH Zürich
Building CLV
Clausiusstrasse 47
8092 Zürich
SWITZERLAND

Leonhard Kehrberger
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WA
UNITED KINGDOM

Prof. Dr. Dan A. Lee
Department of Mathematics
Queens College, CUNY
65-30 Kissena Boulevard
New York NY 11367-1597
UNITED STATES

Dr. Oliver Lindblad Petersen
Department of Mathematics
Stanford University
Building 380
450 Jane Stanford Way
Stanford, CA 94305
UNITED STATES

Prof. Dr. John Lott
Department of Mathematics
University of California, Berkeley
970 Evans Hall
Berkeley, CA 94720-3840
UNITED STATES

Prof. Dr. Jonathan Luk
Department of Mathematics
Stanford University
Stanford CA 94305-2125
UNITED STATES

Prof. Dr. Marc Mars
Facultad de Ciencias
Universidad de Salamanca
Plaza de la Merced s/n
37008 Salamanca
SPAIN

Prof. Dr. Lorenzo Mazzieri
Dipartimento di Matematica
Università di Trento
Via Sommarive, 14
38123 Trento
ITALY

Prof. Dr. Jan Metzger
Institut für Mathematik
Universität Potsdam
Karl-Liebknecht-Straße 24/25
14476 Potsdam
GERMANY

Georgios Moschidis
Department of Mathematics
University of California, Berkeley
754 Evans Hall
Berkeley CA 94720-3860
UNITED STATES

Prof. Dr. Todd Oliynyk
MATRIX OFFICE
The University of Melbourne
Peter Hall Building
Parkville VIC 3010
AUSTRALIA

Dr. Martin Reiris-Ithurralde
Centro de Matemática
Universidad de la República
Int. 114, Oficina 3, Piso 15
CP 11400
Iguá 4225
11400 Montevideo 11400
URUGUAY

Prof. Dr. Hans Ringström
Department of Mathematics
Royal Institute of Technology
Lindstedtsvägen 25
100 44 Stockholm
SWEDEN

Prof. Dr. Anna Sakovich
Department of Mathematics
Uppsala Universitet
Box 480
751 06 Uppsala
SWEDEN

Anna Sancassani
Mathematisches Institut
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Jan J. Sbierski
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Dr. Yakov Shlapentokh-Rothman
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Jacques Smulevici
Laboratoire Jacques-Louis Lions
Sorbonne Université
4, place Jussieu
75005 Paris Cedex
FRANCE

Prof. Dr. Jared Speck
Department of Mathematics
Vanderbilt University
1326 Stevenson Center
Nashville TN 37240-0001
UNITED STATES

Prof. Dr. Iva Stavrov
Department of Mathematical Sciences
Lewis & Clark College
MSC 110
0615 S.W. Palatine Hill Road
Portland OR 97219
UNITED STATES

Dr. Daniel Stern
Department of Mathematics
University of Chicago
5734 S. University Avenue
60637 Chicago
UNITED STATES

Dr. Martin Taylor
Department of Mathematics
Imperial College London
South Kensington Campus
London SW7 2AZ
UNITED KINGDOM

Dr. Rita Teixeira da Costa
Department of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Mr. Ryan Unger
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dr. Caterina Vâlcu
CMLS - École Polytechnique
Route de Saclay
91128 Palaiseau Cedex
FRANCE

Dr. Maxime C. Van de Moortel
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Robert Wald
Enrico Fermi Institute and
Department of Physics
University of Chicago
5720 S. Ellis Avenue
Chicago, IL 60637
UNITED STATES

Prof. Dr. Mu-Tao Wang
Department of Mathematics
Columbia University, MC 4406
Math. Bldg. 509
2990 Broadway
New York NY 10027
UNITED STATES

Markus Wolff
Fachbereich Mathematik
Universität Tübingen
Auf der Morgenstelle 10
72076 Tübingen
GERMANY

Dr. Zoe Wyatt
Department of Pure Mathematics and
Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM