Mathematisches Forschungsinstitut Oberwolfach

Report No. 50/2021

DOI: 10.4171/OWR/2021/50

Arbeitsgemeinschaft: Thin Groups and Super-approximation (hybrid meeting)

Organized by
Alireza Salehi Golsefidy, San Diego
Alex Kontorovich, Piscataway
Hee Oh, New Haven

10 October – 15 October 2021

Abstract. The aim of this workshop was to discuss the super-approximation of thin groups, its dynamical implications in terms of the mixing of geodesic flows, and applications to various problems in arithmetic, geometry, and dynamics.

Mathematics Subject Classification (2010): 22E40, 37F35, 11N45.

Introduction by the Organizers

The workshop *Thin groups and super-approximation*, was well attended with over 50 participants (in person + zoom participants) with broad geographic representation from all continents. This workshop was a nice blend of researchers with various backgrounds.

Let us recall that a thin group refers to a finitely generated Zariski dense subgroup of an S-arithmetic subgroup of a (semisimple real) algebraic group that is of infinite covolume. Given a thin group Γ of a semisimple algebraic group G, we consider a family, say $\mathcal{F}$, of its congruence subgroups. Fixing a finite symmetric generating subset Ω of Γ, the super-approximation for Γ with respect to $\mathcal{F}$ means that the family of Cayley graphs $(\Gamma/\Gamma_n, \Omega)$, $\Gamma_n \in \mathcal{F}$ forms an expander family.

The main theme of this workshop as explained in our program was divided into the following three topics:

(1) Overview the super-approximation;
(2) Dynamical implications in terms of uniform exponential mixing of geodesic flows;
(3) Applications to problems in arithmetic, geometry, and dynamics.

The lectures followed carefully the scheduled program: 6 of the lectures were devoted to the first topic, 7 devoted to the second, and 5 devoted to the third.

The first theme of this meeting was to understand what the super-approximation is, and go over proofs of some of the main results on this topic. There have been many advances in our understanding of the spectral gap properties of random walks in (locally) compact groups, and some of the progress manifested itself in super-approximation property of thin groups. In the first lecture, Pham went over the basics of random walk in a compact group and mentioned some of the main results of super-approximation due to Bourgain-Gamburd, Bourgain-Varjú, Golsefidy-Varjú, and Golsefidy. In the second lecture on this topic, Sert went over the Bourgain-Gamburd machine and showed how these techniques answer Lubotzky's 1-2-3 problem. As part of the Bourgian-Gamburd machine, one needs to understand the approximate subgroups. The first result of this type is due to Helfgott. Dona explained the main ideas of Helfgott's proof. The Bourgain-Gamburd machine works in a *single scale* setting. In a multiscale setting, one needs to focus on various scales separately. Kogler explained how Bourgain-Gamburd and Benoit-de Saxcé overcame these difficulties in the Archimedean setting: compact semisimple Lie groups. Winkel went through Varjú's thesis and explained how he dealt with the direct product of certain finite groups. Finally, Machado explained how Golsefidy used understanding of approximate submodules and a quantitative p-adic open function theorem to prove the super-approximation for powers of primes.

The second topic is to understand the implication of super-approximation in terms of the uniform exponential mixing of the geodesic flow on congruence covers of a hyperbolic manifold of the form $\Gamma\backslash\mathbb{H}^n$ where $\Gamma < \mathrm{SO}(n,1)$ is a thin geometrically finite subgroup. Thanks to the well-developed theory of Patterson-Sullivan, there has been quite a lot of progress on this. After one lecture on preliminaries on hyperbolic geometry given by Islam, the second lecture on this topic by Luethi discussed the mixing and its consequence on matrix coefficients on a fixed geometrically finite manifold. There are two approaches to prove exponential mixing: one using the representation theory, following Edwards-Oh paper, on which we had 2 lectures by Dabeler and Han and the other using the so-called Dolgopyat machine, following Oh-Winter and Sarkar, on which we had 2 lectures by Chow and Corso. As an application of mixing, the last lecture on this topic by Lee was devoted to the discussion on circle-counting problem for Apollonian circle packings, following the works of Kontorovich-Oh, and Oh-Shah.

The third topic concerns the combination of the previous two with other techniques (most notably, the affine sieve and the orbital circle method) to make progress on (or solve outright) a wide range of (initially unrelated) problems spanning arithmetic, geometry, and dynamics. In the first lecture on these topics, Sarkar gave a basic overview of the types of problems that are amenable to attack, setting up the next few lectures. Jones explained the local-global problem for integral Apollonian packings and generalizations, and sketched how the orbital circle method is used in work of Bourgain-Kontorovich and others to show a density-one analogue. Zaremba's conjecture, as well as its origins in and applications to numerical integration and pseudorandom sequences, was explained by Zhang, who also discussed some of the ideas in the proof of a density-one analogue. This lecture and the previous one exhibited how the orbital circle method, while broadly applicable as a "tool," is not a quotable "theorem," as its actual use in practice varies greatly, with rather different machinery responsible for both the major arcs and minor arcs analyses in these two settings. The last two lectures in this series concerned applications of the affine sieve. Kim's lecture stated the Einsiedler-Lindenstrauss-Michel-Venkatesh conjecture on low-lying, closed, fundamental geodesics on the modular surface, and sketched its resolution by Bourgain-Kontorovich, using methods from the "Beyond Expansion" program; the latter gives exponents of distribution greatly exceeding those coming from spectral gap methods alone. The final lecture, by Litman, discussed progress on the Pythagorean Prime Triples problems, spanning work of Kontorovich-Oh, Hong-Kontorovich, and Bourgain-Kontorovich, each reducing the number of prime factors of, say, the hypotenuse, in a thin (but not too thing) Zariski-dense orbit of Pythagorean triples.

In order to accommodate the different time zone, each day began only at 3 pm and ended at 10 pm at the MFO and the participants devoted the morning time for group discussion and to go over lectures from previous days. There were a lot of topics which were covered, and we felt we had a very good panoramic view of the recent developments of these topics.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Arbeitsgemeinschaft (hybrid meeting): Thin Groups and Super-approximation

Table of Contents

Abstracts

Random walks on compact groups and super-approximation: an overview

Lam Pham

As the first lecture of the workshop, we give an introduction to random walks on compact groups and give the relevant definitions of spectral gap and super-approximation, pointing out the connections with strong approximation and expander families. We also survey some relevant results.

1. Random Walks on Compact Groups

1.1. **Spectral Gap.** Let G be a compact group with normalized Haar measure m_G. We consider the regular representation λ_G acting on $L^2(G, m_G)$ by left translations:

$$\lambda_G(g)f(x) = f(g^{-1}x), \quad g, x \in G,\ f \in L^2(G).$$

Let

$$L_0^2(G) := \left\{ f \in L^2(G) \,\middle|\, \int_G f \mathrm{d}m_G = 0 \right\} = \mathbb{C}^{\perp},$$

the space orthogonal to the constants in $L^2(G)$. Given a symmetric probability measure μ on G (symmetric in the sense that $\mu(A) = \mu(A^{-1})$ for any measurable set A), we consider the convolution operator $A_\mu := \lambda_G(\mu)$ defined by

$$(A_\mu f)(x) = \int_G f(g^{-1}x)\mathrm{d}\mu(g), \quad f \in L_0^2(G).$$

This operator has norm $\|A_\mu\| \leq 1$, and we say that μ has a *spectral gap* if $\|A_\mu\| < 1$.

A random walk is given by a probability measure μ and the sequence of convolution powers $\{\mu^{*n}\}_{n\geq 1}$. The random walk with law μ has a spectral gap if μ has a spectral gap.

1.2. **Other representations.** More generally, given a unitary representation on a Hilbert space $\pi : G \to \mathrm{U}(\mathcal{H})$, we can consider the operator $\pi(\mu)$ acting on $\mathcal{H}$ given by

$$\pi(\mu)\xi = \int_G \pi(g)\xi \mathrm{d}\mu(g), \quad \xi \in \mathcal{H}.$$

Then, $A_\mu = \lambda_G(\mu)$, $\|\pi(\mu)\| \leq 1$ and we can study the spectral gap of this operator. Of course, in order to have $\|\pi(\mu)\| < 1$, π should not have any G-invariant vectors.

1.3. **Expander Graph.** Let $\epsilon > 0$. A finite connected k-regular graph X is said to be an *ϵ-expander* if for every subset A of vertices in X, with $|A| \leq \frac{1}{2}|X|$, one has the following *isoperimetric inequality*:

$$|\partial A| \geq \epsilon |A|,$$

where ∂A denotes the set of edges of X which connect a point in A to a point in its complement. The optimal ϵ as above is sometimes called the *discrete Cheeger constant*:

$$h(X) = \inf \left\{ \frac{|\partial A|}{|A|} \middle| A \subset X, |A| \leq \tfrac{1}{2}|X| \right\}.$$

1.4. **Spectral Gap and Expanding Constant.** Given a k-regular graph X, one can consider the operator

$$Pf(x) = \frac{1}{k} \sum_{x \sim y} f(y), \tag{1}$$

where $x \sim y$ means that x and y are connected by an edge.

This operator is self-adjoint on $\ell^2(X)$ and $\|P\| \leq 1$, so its spectrum is real and contained in $[-1, 1]$. We can write the eigenvalues of P in decreasing order as

$$\mu_0 = 1 \geq \mu_1 \geq \cdots \geq \mu_n \quad (n = |X|).$$

The constant functions are eigenvectors of P with eigenvalue 1, and if X is connected, the eigenvalue 1 has multiplicity 1 (if $Pf = f$ and f achieves its maximum at x, then f must take the same value $f(x)$ at each neighbor of x, and this value spreads to the entire graph). The *spectral gap* of the graph X is $\lambda_1(X) := 1 - \mu_1$.

The following key inequality connecting the Cheeger constant and the spectral gap is known as the *Discrete Cheeger-Buser Inequality*: if X is connected and k-regular,

$$\frac{1}{2}\lambda_1(X) \leq \frac{1}{k} h(X) \leq \sqrt{2\lambda_1(X)}.$$

1.5. **Family of expanders.** Let $k \geq 3$. Let $\epsilon > 0$. A family $(X_n)_n$ of k-regular graphs is said to be a *ϵ-expanding family* if

$$\lim_{n \to \infty} |X_n| = \infty \quad \text{and} \quad \lambda_1(X_n) \geq \epsilon.$$

1.6. If S is a finite symmetric set in a compact group G, we consider the symmetric probability measure $\mu_S = \frac{1}{|S|} \sum_{s \in S} \delta_s$.

If we have a sequence of sets S_n in finite groups Γ_n, and $\langle S_n \rangle = \Gamma_n$, the Cayley graphs of Γ_n with respect to S_n will be expanders if and only if the probability measures μ_n have a (uniform) spectral gap.

When S generates a *dense* subgroup of a compact group, we are interested in the spectral gap of μ_S.

1.7. Margulis' expanders. With $\Gamma = \mathrm{SL}(2,\mathbb{Z}) \ltimes \mathbb{Z}^2$, Margulis showed that given a finite generating set of Γ, the infinite family of Schreier graphs $(\mathbb{Z}/n\mathbb{Z})^2$ of Γ was an expanding family.

It uses the facts that $\mathrm{SL}(2,\mathbb{R}) \ltimes \mathbb{R}^2$ has the relative property (T) with respect to the normal subgroup $\mathbb{R}^2$, and that the lattice Γ inherits this property.

This observation implies, via theorems of Selberg and Kazhdan, that for $d \geq 2$, the family of Cayley graphs $\{\mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z}) \mid n \in \mathbb{N}\}$ with respect to the image of a finite generating set of $\mathrm{SL}_d(\mathbb{Z})$ is an expander family.

2. Strong and Super Approximation

2.1. Strong Approximation. Suppose we have a family of polynomials

$$f_\alpha(x_1, \ldots, x_d) \in \mathbb{Z}[x_1, \ldots, x_d], \quad \alpha \in I,$$

and we let $X \subset \mathbf{A}^d_{\mathbb{Z}}$ denote the closed affine subscheme defined by these polynomials. Thus, for any $\mathbb{Z}$-algebra R, the scheme X has the following set of R-points:

$$X(R) = \{(a_1, \ldots, a_d) \in R^d \mid f_\alpha(a_1, \ldots, a_d) = 0,\ \forall \alpha \in I\}.$$

Then for any integer $m \geq 1$, we have a natural reduction modulo m map

$$\rho_m : X(\mathbb{Z}) \to X(\mathbb{Z}/m\mathbb{Z}),$$

and the question is whether these maps are *surjective* for all m. For $m \mid n$, we have a canonical homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$, and therefore we have a natural map

$$\pi^n_m : X(\mathbb{Z}/n\mathbb{Z}) \to X(\mathbb{Z}/m\mathbb{Z}).$$

The system $\{(X(\mathbb{Z}/m\mathbb{Z}), \pi^n_m)\}$ is an inverse system, so we can form the inverse limit:

$$\varprojlim X(\mathbb{Z}/m\mathbb{Z}) = X(\hat{\mathbb{Z}}), \quad \text{where} \quad \hat{\mathbb{Z}} = \varprojlim \mathbb{Z}/m\mathbb{Z},$$

and we identify

$$X(\hat{\mathbb{Z}}) \simeq \prod_p X(\mathbb{Z}_p),$$

where $\mathbb{Z}_p$ is the ring of p-adic integers.

The following are equivalent:

(1) $\rho_m : X(\mathbb{Z}) \to X(\mathbb{Z}/m\mathbb{Z})$ is surjective for all integers $m \geq 1$;
(2) the natural embedding $\iota : X(\mathbb{Z}) \hookrightarrow X(\hat{\mathbb{Z}})$ has dense image.

We say that X has *strong approximation* if any of these is satisfied.

Instead of varieties, we can focus on algebraic groups, and more generally, on finitely generated Zariski-dense subgroups.

2.2. Subgroups of $\mathrm{GL}_n(\mathbb{Q})$. Let Γ be a finitely generated subgroup of $\mathrm{GL}_n(\mathbb{Q})$. Then there is a finite set $S = \{p_1, \ldots, p_r\}$ of primes so that if $A = S^{-1}\mathbb{Z} = \mathbb{Z}[p_1^{-1}, \ldots, p_k^{-1}]$, then $\Gamma \subset \mathrm{GL}_n(A)$. Two measures of "how large Γ is" are obtained by determining:

(a) the Zariski-closure $\mathbf{G}_A$ of Γ in $(\mathrm{GL}_n)_A$, and
(b) the closure $\bar{\Gamma}$ of Γ in the profinite group $\mathrm{GL}_n(\hat{A})$ where $\hat{A} = \prod_{p \notin S} \mathbb{Z}_p$.

If $\mathbf{G} \subset \mathrm{GL}_n$ is an algebraic $\mathbb{Q}$-group and $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is a finitely generated Zariski-dense subgroup, then $\Gamma \subset \mathbf{G}_A(A)$. Then, reduction Γ_p modulo p is well-defined if p is large enough.

Theorem 1 (Strong Approximation). *Let $\mathbf{G} \subset \mathrm{GL}_n$ be a connected, semisimple, simply connected, algebraic group defined over $\mathbb{Q}$, and let $\Gamma \subset \mathbf{G}(\mathbb{Q})$ be a finitely generated Zariski-dense subgroup. Then for all sufficiently large prime numbers p, there exists an algebraic group $\mathbf{G}^{(p)}$ defined over $\mathbb{F}_p$ such that $\Gamma_p = \mathbf{G}^{(p)}(\mathbb{F}_p)$.*

The *Super Approximation Theorem* is a strengthening of the Strong Approximation Theorem: the reduction modulo p map is not only surjective, but it defines an expander family.

Theorem 2 (Super Approximation Theorem). *Suppose $\mathbf{G}$ is a connected, simply connected, semi-simple algebraic group defined over $\mathbb{Q}$, and let $\Gamma \leq \mathbf{G}(\mathbb{Q})$ be a Zariski-dense subgroup generated by a finite set S. Then there is $\varepsilon = \varepsilon(S) > 0$ such that for all large enough prime numbers p, the reduction Γ_p of Γ is equal to $\mathbf{G}^{(p)}(\mathbb{F}_p)$ and the associated Cayley graph $\mathrm{Cay}(\mathbf{G}^{(p)}(\mathbb{F}_p), S_p)$ is an ε-expander.*

3. Lubotzky's 1-2-3 problem and the Bourgain-Gamburd "Expansion machines"

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. Consider the finite symmetric sets

$$S^{(j)} := \left\{ \begin{pmatrix} 1 & j \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -j \\ 0 & 1 \end{pmatrix} \right\} \subset \Gamma, \quad j \in \{1, 2, 3\}.$$

The set $S^{(1)}$ generates Γ, while the set $S^{(2)}$ generates a finite-index subgroup of Γ. In both cases, one readily obtains that the corresponding family of Cayley graphs

$$\{\mathrm{Cay}(\pi_p(\Gamma), \pi_p(S)) \mid p \text{ prime}\}$$

is an expanding family. However, $S^{(3)}$ generates an *infinite index* subgroup of Γ. On the other hand, the subgroup $\langle S^{(3)} \rangle$ is still Zariski-dense – Γ is called a *thin* subgroup – so strong approximation still applies, and $\pi_p(S^{(3)})$ still generates $\mathrm{SL}_2(\mathbb{F}_p)$.

Lubotzky's 1-2-3 problem asks if, in such a situation, the corresponding family of Cayley graphs is still an expanding family. The following summary of results gives a positive answer to this problem in a very strong sense.

3.1. Bourgain-Gamburd [4]**.** Let $S \subset \mathrm{SL}_2(\mathbb{Z})$ be a finite symmetric set generating a not virtually solvable subgroup. Then, the family $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{F}_p), \pi_p(S))$ is an expanding family.

3.2. Bourgain-Gamburd [2]**.** Let $S \subset \mathrm{SL}_d(\mathbb{Z})$ be finite and symmetric. Assume that S generates a subgroup $\Gamma \subset \mathrm{SL}_d(\mathbb{Z})$ which is Zariski dense in SL_d. Then $\mathrm{Cay}(\pi_q(\Gamma), \pi_q(S))$ form a family of expanders when q ranges through the integers. Moreover, there is an integer q_0 such that $\pi_q(\Gamma) = \mathrm{SL}_d(\mathbb{Z}/q\mathbb{Z})$ if q is coprime to q_0.

3.3. Salehi-Golsefidy-Varju [7]**.** Let $\Gamma \subset \mathrm{GL}_d(\mathbb{Z}[1/q_0])$ be the group generated by a symmetric set S. Then $\mathrm{Cay}(\pi_q(\Gamma), \pi_q(S))$ form a family of expanders when q ranges over square-free integers coprime to q_0 if and only if the connected component of the Zariski-closure of Γ is perfect[1].

On the other hand, an appropriate adaptation of the Bourgain-Gamburd method enables to prove a spectral gap property for finitely generated dense subgroups of compact Lie group:

3.4. Bourgain-Gamburd [3]**.** Let S be a finite symmetric set of $\mathrm{SU}(2)$. If $S \subset \mathrm{M}_2(\bar{\mathbb{Q}})$ and the subgroup $\Gamma = \langle S \rangle$ it generates is Zariski-dense[2] in SL_2 (over $\mathbb{C}$), then μ_S has a spectral gap.

3.5. Bourgain-Gamburd [2]**.** Let S be a finite symmetric set of $\mathrm{SU}(d)$. If $S \subset \mathrm{M}_d(\bar{\mathbb{Q}})$ and the subgroup $\Gamma = \langle S \rangle$ it generates is Zariski-dense in SL_d (over $\mathbb{C}$), then μ_S has a spectral gap.

The most general result was proved by Benoist and de Saxcé in the context of compact Lie groups. A probability measure μ is *adapted* if its support generates a dense subgroup.

3.6. Benoist and de Saxcé [1]**.** Let G be a connected compact simple Lie group and $\mathcal{U}$ a fixed basis for its Lie algebra. Let μ be an adapted probability measure on G and assume that for any $g \in \mathrm{supp}(\mu)$, the matrix of $\mathrm{Ad}\, g$ in the basis $\mathcal{U}$ has algebraic entries. Then μ has a spectral gap.

References

[1] Yves Benoist and Nicolas de Saxcé. A spectral gap theorem in simple Lie groups. *Invent. Math.*, 205(2):337–361, 2016.

[2] J. Bourgain and A. Gamburd. A spectral gap theorem in SU(d). *J. Eur. Math. Soc. (JEMS)*, 14(5):1455–1511, 2012.

[3] Jean Bourgain and Alex Gamburd. On the spectral gap for finitely-generated subgroups of SU(2). *Invent. Math.*, 171(1):83–121, 2008.

[4] Jean Bourgain and Alex Gamburd. Uniform expansion bounds for Cayley graphs of $\mathrm{SL}_2(\mathbb{F}_p)$. *Ann. of Math. (2)*, 167(2):625–642, 2008.

[5] Jean Bourgain, Alex Gamburd, and Peter Sarnak. Affine linear sieve, expanders, and sum-product. *Invent. Math.*, 179(3):559–644, 2010.

[6] Jean Bourgain and Péter P. Varjú. Expansion in $\mathrm{SL}_d(\mathbf{Z}/q\mathbf{Z})$, q arbitrary. *Invent. Math.*, 188(1):151–173, 2012.

[7] A. Salehi Golsefidy and Péter P. Varjú. Expansion in perfect groups. *Geom. Funct. Anal.*, 22(6):1832–1891, 2012.

[8] Weikun He and Nicolas de Saxcé. Linear random walks on the torus, 2020.

[9] Alireza Salehi Golsefidy. Super-approximation, I: $\mathfrak{p}$-adic semisimple case. *Int. Math. Res. Not. IMRN*, (23):7190–7263, 2017.

[10] Alireza Salehi Golsefidy. Super-approximation, II: the p-adic case and the case of bounded powers of square-free integers. *J. Eur. Math. Soc. (JEMS)*, 21(7):2163–2232, 2019.

[11] Péter P. Varjú. Expansion in $\mathrm{SL}_d(\mathscr{O}_K/I)$, I square-free. *J. Eur. Math. Soc. (JEMS)*, 14(1):273–305, 2012.

[1] an algebraic group $\mathbf{G}$ is *perfect* if $\mathbf{G}^\circ = [\mathbf{G}^\circ, \mathbf{G}^\circ]$.

[2] The Zariski-dense assumption is equivalent to the assumption that Γ is dense in SU(2).

Preliminaries on dynamics on geometrically finite hyperbolic manifolds

Mitul Islam

In this talk, we discuss the construction of Patterson-Sullivan density, Bowen-Margulis-Sullivan measure, and Burger-Roblin measure associated to geometrically finite subgroups of $\mathrm{Isom}_+(\mathbb{H}^d) = \mathrm{SO}$ when $d \geq 2$. We will also see some applications of these measures in studying the properties of the limit set and the dynamics of the geodesic flow.

Basics of Hyperbolic Geometry. Let $d(\cdot, \cdot)$ denote the distance on $\mathbb{H}^d$ induced by the Riemannian metric of curvature -1. Suppose Γ is a discrete subgroup of SO. The limit set of Γ is $\Lambda_\Gamma := \overline{\Gamma \cdot x} \cap \partial\,\mathbb{H}^d$ where $x \in \mathbb{H}^d$ is some (hence any) fixed basepoint. The group Γ is called:

(1) convex co-compact if Γ acts co-compactly on the convex hull $\mathcal{C}(\Lambda_\Gamma)$ of Λ_Γ in $\mathbb{H}^d$, and
(2) geometrically finite if the 1-neighbourhood of $\mathcal{C}(\Lambda_\Gamma)$ has finite volume.

If Γ is geometrically finite, then $\Lambda_\Gamma = \Lambda_\Gamma^c \sqcup \Lambda_\Gamma^p$ where Λ_Γ^c is the set of conical limit points and Λ_Γ^p is the set of bounded parabolic points. Note that $\xi \in \Lambda_\Gamma^c$ if there exist infinitely many $\gamma_n \in \Gamma$ and $R_\xi > 0$ such that $d(\gamma_n x, [x, \xi)) < R_\xi$ where $[x, \xi)$ is a geodesic ray in $\mathbb{H}^d$. On the other hand, $\xi \in \Lambda_\Gamma^p$ if $(\Lambda_\Gamma \setminus \{\xi\})/\mathrm{Stab}_\Gamma(\xi)$ is compact. Finally, Γ is convex co-compact if and only if $\Lambda_\Gamma = \Lambda_\Gamma^c$. Examples of geometrically finite groups are Schottky groups and Apollonian circle packing groups. See [6] or [3] for details.

The Poincaré series is given by ($x, y \in \mathbb{H}^d$):

$$g_s(x, y) := \sum_{\gamma \in \Gamma} e^{-s d(x, \gamma y)}.$$

The critical exponent of Γ is $\delta_\Gamma := \inf\{s > 0 : g_s(x, y) < \infty\}$. It is positive whenever Γ is non-elementary, i.e. Λ_Γ is an infinite set.

Conformal density and Patterson-Sullivan density. The main references for this section are [5, 7, 2].

Definition 1. Suppose $\Gamma \leq \mathrm{SO}$ is a discrete group. A Γ-conformal density of dimension α is a family of finite measures $\{\mu_x\}_{x \in \mathbb{H}^d}$, each supported on $\partial\,\mathbb{H}^d$, such that: for any $x, y \in \mathbb{H}^d, \gamma \in \Gamma$, and $\xi \in \partial\,\mathbb{H}^d$,

(1) $\gamma_* \mu_x = \mu_{\gamma x}$
(2) $\dfrac{d\mu_y}{d\mu_x}(\xi) = e^{\alpha \beta_\xi(x, y)}$

where $\beta_\xi : \mathbb{H}^d \times \mathbb{H}^d \to \mathbb{R}$ is the Busemann function based at ξ, i.e. $\beta_\xi(x, y) := \lim_{\xi_t \to \xi}(d(x, \xi_t) - d(y, \xi_t))$.

Theorem 2. [7, 5] *If $\Gamma \leq \mathrm{SO}$ is a non-elementary discrete group, then there exists a Γ-conformal density of dimension δ_Γ that is supported on Λ_Γ.*

More precisely, if we assume that $g_{\delta_\Gamma}(x,y)=\infty$, then the conformal density in the above theorem is constructed by taking a weak-* limit of

$$\mu_s(x) := \frac{1}{g_s(y,y)} \sum_{\gamma\in\Gamma} e^{-sd(x,\gamma y)} 1_{\gamma y}$$

as $s \to \delta_\Gamma^+$. A similar construction yields the result when $g_{\delta_\Gamma}(x,y)<\infty$, see [7, 2].

A conformal density that is constructed as above will be called a Patterson-Sullivan density. The following is an application of Patterson-Sullivan densities for understanding the limit sets of convex co-compact subgroups of SO. Recall that $\Lambda_\Gamma = \Lambda_\Gamma^c$ when Γ is convex co-compact.

Theorem 3. [7, 5] *Suppose $\Gamma \le$ SO is a discrete, non-elementary, convex co-compact subgroup, and $\{\mu_x\}$ is a Patterson-Sullivan density. Then:*

(1) μ_x and the Hausdorff δ_Γ-measure on Λ_Γ have the same measure class,
(2) the Hausdorff dimension of Λ_Γ is δ_Γ,
(3) Γ action on (Λ_Γ, μ_x) is ergodic,
(4) $\{\mu_x\}$ is the unique (up to scaling) Γ-conformal density (of any dimension) that gives positive measure to Λ_Γ, and
(5) $g_{\delta_\Gamma}(x,y)=\infty$.

For geometrically finite subgroups of SO, we have similar results [7, 2]. More precisely, when $\Gamma <$ SO is a discrete, non-elementary, geometrically finite subgroup, then we have: $\{\mu_x\}$ does not have any atoms [8, 2]; the Hausdorff dimension of Λ_Γ is δ_Γ; $\{\mu_x\}$ is the unique (up to scaling) Γ-conformal density (of any dimension) that gives positive measure to the conical limit set Λ_Γ^c; Γ-action on (Λ_Γ, μ_x) is ergodic; and $g_{\delta_\Gamma}(x,y)=\infty$.

Bowen-Margulis-Sullivan measure. The main references for this section are [7, 2, 6].

A Γ-conformal density $\{\mu_x\}$ can be used to construct a geodesic flow $\{g_t\}_{t\in\mathbb{R}}$ invariant measure on $\mathrm{T}^1\mathbb{H}^{\mathrm{d}}/\Gamma$ called the Bowen-Margulis-Sullivan measure and denoted by μ_{BMS}. In order to do this, we first identify $\mathrm{T}^1\mathbb{H}^{\mathrm{d}}$ with $(\partial\,\mathbb{H}^d \times \mathbb{H}^d \setminus \Delta)\times \mathbb{R}$ via the Hopf parametrization

$$u \mapsto (u_+, u_-, \beta_{u_+}(x, \pi(u))).$$

Here $x \in \mathbb{H}^d$ is a fixed basepoint and $\pi : \mathrm{T}^1\mathbb{H}^{\mathrm{d}} \to \mathbb{H}^d$ is the obvious projection map. We now define the $\{g_t\}$ and Γ-invariant measure on $\mathrm{T}^1\mathbb{H}^{\mathrm{d}}$ which then descends to $\mathrm{T}^1\mathbb{H}^{\mathrm{d}}/\Gamma$:

$$d\mu_{\mathrm{BMS}}(u_+,u_-,t) = e^{\delta_\Gamma \beta_{u_+}(x,\pi(u))} e^{\delta_\Gamma \beta_{u_-}(x,\pi(u))} d\mu_x(u_+) d\mu_x(u_-) dt$$

Sullivan showed that μ_{BMS} is a finite measure whenever $\Gamma \le$ SO is a geometrically finite subgroup, see [7] or [2, Theorem 9.2.2]. The support of μ_{BMS} consists of all vectors $u \in \mathrm{T}^1\mathbb{H}^{\mathrm{d}}$ such that $u_\pm \in \Lambda_\Gamma$. The following theorem is called the Hopf-Tsuji-Sullivan-Roblin dichotomy for subgroups of SO, see for instance [2] or [6].

Theorem 4. *Suppose* $\Gamma \leq \mathrm{SO}$ *is a non-elementary discrete group and* $\{\mu_x\}$ *is a* Γ*-conformal density of dimension* δ_Γ*. Then the following are equivalent:*

(1) $\mu_x(\Lambda_\Gamma^c) > 0$ *(resp.* $\mu_x(\Lambda_\Gamma^c) = 0$*),*
(2) Λ_Γ^c *is a full measure subset of* $(\partial\,\mathbb{H}^d, \mu_x)$ *(resp.* $\mu_x(\Lambda_\Gamma^c) = 0$*),*
(3) the geodesic flow on $(\mathrm{T}^1\mathbb{H}^{\mathrm{d}}/\Gamma, \mu_{\mathrm{BMS}})$ *is conservative and ergodic (resp.* $\{g_t\}$ *is completely dissipative and non-ergodic)*
(4) Γ*-action on* $(\partial\,\mathbb{H}^d \times \partial\,\mathbb{H}^d, \mu_x \otimes \mu_x)$ *is conservative and ergodic (resp.* Γ *action is completely dissipative and non-ergodic),*
(5) $g_{\delta_\Gamma}(x,y) = \infty$ *(resp.* $g_{\delta_\Gamma}(x,y) < \infty$*).*

Burger-Roblin measure. The main references for this section are [1, 6, 4, 3].

Let $\{\nu_x\}$ be the Lebesgue measure on $\partial\,\mathbb{H}^d$ obtained by identifying the unit ball in $\mathrm{T}_x\,\mathbb{H}^d$ with $\partial\,\mathbb{H}^d$. Then $\{\nu_x\}$ is a Γ-conformal density of dimension $(d-1)$. The Burger-Roblin measure [6, 4] is a Radon (i.e. locally finite) measure on $\mathrm{T}^1\mathbb{H}^{\mathrm{d}}/\Gamma$ which is defined on $\mathrm{T}^1\mathbb{H}^{\mathrm{d}}$ as:

$$d\mu_{\mathrm{BR}}(u_+, u_-, t) := e^{(d-1)\beta_{u_+}(x,\pi(u))} e^{\delta_\Gamma \beta_{u_-}(x,\pi(u))} d\nu_x(u_+) d\mu_x(u_-) dt.$$

Note that $(g_t)_*\mu_{\mathrm{BR}} = e^{(d-1-\delta_\Gamma)t}\mu_{\mathrm{BR}}$. Thus, if $\Gamma \leq \mathrm{SO}$ is geometrically finite, then μ_{BR} is an infinite measure unless $\delta_\Gamma = d-1$ (equivalently, unless Γ is a lattice). The support of μ_{BR} is

$$\mathcal{F} := \{u \in \mathrm{T}^1\mathbb{H}^{\mathrm{d}} : u_- \in \Lambda_\Gamma\}.$$

Consider $d = 2$. Then μ_{BR} is invariant under the horocycle flow given by the action of $N := \left\{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} : s \in \mathbb{R}\right\}$ on $\mathrm{T}^1\,\mathbb{H}^2/\Gamma$. In particular, Burger [1] showed that if $\mathbb{H}^2/\Gamma$ is a convex co-compact surface with $\delta_\Gamma > 1/2$, then μ_{BR} is (up to scaling) the unique N-invariant Radon measure supported on $\mathcal{F}$. This was generalized to geometrically finite subgroups of SO (more generally, CAT(-1) spaces) by Roblin [6].

References

[1] M. Burger, *Horocycle flow on geometrically finite surfaces*, Duke Math. J., 61(3):779–803, 1990
[2] P. Nicholls, *The Ergodic Theory of Discrete Groups*, London Mathematical Society Lecture Note Series, Cambridge University Press, 1989.
[3] H. Oh, *Dynamics on geometrically finite hyperbolic manifolds with applications to Apollonian circle packings and beyond*, Proceedings of the International Congress of Mathematicians. Volume III, 1308–1331, 2010.
[4] H. Oh and N. Shah, *Equidistribution and Counting for orbits of geometrically finite hyperbolic groups*, Journal of AMS, 26:511–562, 2013.
[5] S. J. Patterson, *The limit set of a Fuchsian group*, Acta Mathematica, 136:241–273, 1976.
[6] T. Roblin, *Ergodicit é et équidistribution en courbure n égative* M ém. Soc. Math. Fr. (N.S.), 95:vi+96, 2003.
[7] D. Sullivan, *The density at infinity of a discrete group of hyperbolic motions*, Publications Mathématiques de L'Institut des Hautes Scientifiques 50:171–202, 1979.
[8] D. Sullivan, *Entropy, Hausdorff measures old and new, and limit sets of geometrically finite Kleinian groups*, Acta Math., 153, 259–277, 1984.

Overview of Thin Groups and Applications

Pratyush Sarkar

We refer the reader to [5, 24, 15] for a more detailed overview of thin groups. Let us begin by discussing arithmetic lattices. A classical example is $\mathrm{SL}_2(\mathbb{Z}) < \mathrm{SL}_2(\mathbb{R})$. More generally, if $\mathbf{G}$ is a semisimple algebraic group defined over $\mathbb{Q}$, then $\mathbf{G}(\mathbb{Z}) < \mathbf{G}(\mathbb{R})$ is an arithmetic lattice by a theorem of Borel–Harish-Chandra [4]. A discrete subgroup is called a *lattice* when the associated quotient is of finite volume: $\mathrm{vol}(\mathbf{G}(\mathbb{Z})\backslash\mathbf{G}(\mathbb{R})) < \infty$. Lattices are Zariski dense in $\mathbf{G}$ by a theorem of Borel [3]. Moreover, any finite index subgroup is again a lattice.

Remark 1. One can also consider semisimple algebraic groups and arithmetic lattices over other number fields but for simplicity we will not do so in this report. In fact we will mostly consider $\mathrm{SL}_2(\mathbb{Z})$ or $\mathrm{SL}_n(\mathbb{Z})$.

A thin group, coined by Sarnak, is essentially the other cases of discrete subgroups in the arithmetic lattice $\mathbf{G}(\mathbb{Z})$. The following is the formal definition.

Definition 2. A *thin group* $\Gamma < \mathbf{G}(\mathbb{Z})$ is a finitely generated infinite index subgroup which is Zariski dense in $\mathbf{G}$.

Note that for a thin group $\Gamma < \mathbf{G}(\mathbb{Z})$, we have $\mathrm{vol}(\Gamma\backslash\mathbf{G}(\mathbb{R})) = \infty$. Such groups have been studied as long as 100–150 years ago. However, there were no tools to study them in the context of arithmetic.

1. Strong approximation and superstrong approximation

Let $q \in \mathbb{N}$. Define the reduction map

$$\begin{aligned}\pi_q : \mathrm{SL}_2(\mathbb{Z}) &\longrightarrow \mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z})\\ \begin{pmatrix} a & b \\ c & d\end{pmatrix} &\longmapsto \begin{pmatrix} a+q\mathbb{Z} & b+q\mathbb{Z} \\ c+q\mathbb{Z} & d+q\mathbb{Z}\end{pmatrix}.\end{aligned}$$

Then π_q can be shown to be surjective using the Chinese remainder theorem. Although it is harder to show, the generalization to arbitrary semisimple algebraic groups also holds. This phenomenon is called *strong approximation (SA)*. See the book [22] for more details.

Recall that given a group G and a finite symmetric generating set $S \subset G$, i.e., $s^{-1} \in S$ if $s \in S$, its *Cayley graph* $\mathrm{Cay}(G, S)$ is an undirected graph whose vertex set is G and edge set is $\{(g, sg) : g \in G, s \in S\}$ (see fig. 1). Note that such graphs are always $|S|$-regular, i.e., there are exactly $|S|$ number of edges emanating from any vertex. Now, let $S \subset \mathrm{SL}_2(\mathbb{Z})$ be any finite symmetric generating set. Then $\{\mathrm{Cay}(\pi_q(\mathrm{SL}_2(\mathbb{Z})), \pi_q(S))\}_{q\in\mathbb{N}} = \{\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \pi_q(S))\}_{q\in\mathbb{N}}$ forms an expander as defined below. This phenomenon is called *superstrong approximation (SSA)*. Lubotzky coined a related notion called property (τ).

Definition 3 (expander)**.** Let $k \in \mathbb{N}$. An infinite sequence of k-regular graphs $\{\mathcal{G}_q = (\mathcal{V}_q, \mathcal{E}_q)\}_{q\in\mathbb{N}}$ with $|\mathcal{V}_q| \xrightarrow{q\to\infty} \infty$ is called an *expander* if the corresponding graph Laplacians $\{\Delta_{\mathcal{G}_q}\}_{q\in\mathbb{N}}$ (recall $\Delta_{\mathcal{G}_q} = \mathrm{Id}_{L^2(\mathcal{G}_q)} - \frac{1}{k}A_{\mathcal{G}_q}$ where $A_{\mathcal{G}_q}$ is the

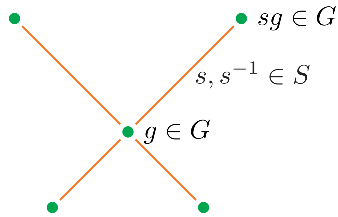
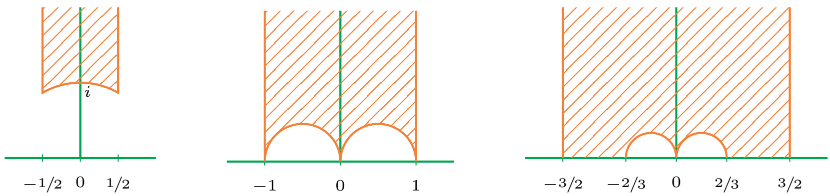


FIGURE 1. Cayley graph.



(A) Γ_1 (B) Γ_2 (C) Γ_3

FIGURE 2. Fundamental domains.

adjacency operator) have a spectral gap, i.e., there exists $\epsilon > 0$ such that denoting $0 = \lambda_0(\Delta_{\mathcal{G}_q}) \leq \lambda_1(\Delta_{\mathcal{G}_q}) \leq \cdots$ for the combinatorial spectrum, we have $\lambda_1(\Delta_{\mathcal{G}_q}) \geq \epsilon$.

For $\mathrm{SL}_2(\mathbb{Z})$, SSA is known using the following Selberg's 3/16 theorem [25] regarding an explicit spectral gap. For $\mathrm{SL}_n(\mathbb{Z})$ with $n \geq 3$ (which has property (T)), SSA is known from a general result that property (T) implies property (τ).

Theorem 4 (Selberg)**.** *Denoting* $X_q = \ker(\pi_q)\backslash\mathbb{H}^2$ *and* $0 = \lambda_0(\Delta_{X_q}) \leq \lambda_1(\Delta_{X_q}) \leq \cdots$ *for the Archemedian (L^2) spectrum, we have* $\lambda_1(\Delta_{X_q}) \geq \frac{3}{16}$ *for all* $q \in \mathbb{N}$.

For general semisimple algebraic groups, SSA is known from developments toward the more general Ramanujan conjectures [11, 12, 23].

Let us now return to thin groups $\Gamma < \mathrm{SL}_n(\mathbb{Z})$. Due to Matthews–Vaserstein–Weisfeiler [17], SA continues to hold with a small modification. We state their more general theorem for SL_n.

Theorem 5 (Matthews–Vaserstein–Weisfeiler)**.** *Let* $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ *be a finitely generated discrete subgroup which is Zariski dense in* SL_n*. There exists* $q_0 \in \mathbb{N}$ *such that* $\pi_q|_\Gamma : \Gamma \to \mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z})$ *is surjective for all* $q \in \mathbb{N}$ *with* $(q, q_0) = 1$.

There are further generalizations of the above in [30, 19, 21].

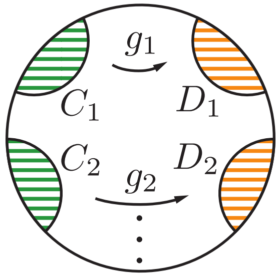


FIGURE 3. Schottky group.

Lubotzky in 1993–1994 raised his 1-2-3 problem: Of the following finitely generated subgroups of $\mathrm{SL}_2(\mathbb{Z})$,

$$\Gamma_1 = \left\langle \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix} \right\rangle,$$
$$\Gamma_2 = \left\langle \begin{pmatrix} 1 & \pm 2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 2 & 1 \end{pmatrix} \right\rangle,$$
$$\Gamma_3 = \left\langle \begin{pmatrix} 1 & \pm 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm 3 & 1 \end{pmatrix} \right\rangle,$$

the first in fact coincides with the whole group $\mathrm{SL}_2(\mathbb{Z})$ and hence satisfies SSA, the second is of finite index and also satisfies SSA, but the third is of infinite index (see fig. 2)—does it still satisfy SSA? Lubotzky suggested that SSA should continue to hold. This is now known in vast generality after the works of several people: Sarnak–Xue, Gamburd, Helfgott, Bourgain–Gamburd, Balog–Szemerédi, Gowers, Pyber–Szabó, Bourgain–Gamburd–Sarnak, Breuillard–Green–Tao, Varjú, Bourgain–Varjú, Golsefidy–Varjú, He–de Saxcé. We present here the theorem for SL_n which is due to Bourgain–Varjú [9].

Theorem 6 (Bourgain–Varjú)**.** *Let $\Gamma < \mathrm{SL}_n(\mathbb{Z})$ be a finitely generated discrete subgroup which is Zariski dense in SL_n. Then, there exists $q_0 \in \mathbb{N}$ such that $\{\mathrm{Cay}(\pi_q(\Gamma), \pi_q(S))\}_{q\in\mathbb{N},(q,q_0)=1} = \{\mathrm{Cay}(\mathrm{SL}_n(\mathbb{Z}/q\mathbb{Z}), \pi_q(S))\}_{q\in\mathbb{N},(q,q_0)=1}$ forms an expander.*

2. Examples of thin groups

2.1. **Schottky groups.** Let $\{g_1, g_2, \ldots, g_k\} \subset \mathrm{SL}_2(\mathbb{Z})$ be a finite generating set which plays ping-pong, i.e., there exist mutually disjoint open balls $C_1, C_2, \ldots, C_k$, $D_1, D_2, \ldots, D_k \subset \partial_\infty \mathbb{H}^2$ such that $g_j(\mathrm{ext}(C_j)) = \overline{D_j}$ for all $1 \le j \le k$ (see fig. 3). Then $\Gamma = \langle g_1, g_2, \ldots, g_k \rangle$ is called a *Schottky group* and is in fact a free group. This construction can be generalized for other semisimple algebraic groups which we omit (see [1]). It is not difficult for such groups to be Zariski dense (e.g., we simply require $k \ge 2$ in SL_2). Since they are always of infinite index in $\mathrm{SL}_2(\mathbb{Z})$, we obtain a large class of thin groups.

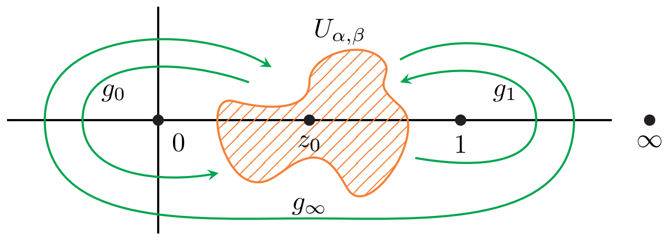


FIGURE 4. Monodromy group.

2.2. **Reflection groups.** This is the work of Vinberg [29] and Nikulin [18]. Let f be an integral quadratic form of signature $(n, 1)$. Consider the arithmetic subgroup $\mathrm{O}_f(\mathbb{Z}) \subset \mathrm{O}_f(\mathbb{R})$. A reflection subgroup $R_f < \mathrm{O}_f(\mathbb{Z})$ is one which is generated by all hyperbolic reflections which are in $\mathrm{O}_f(\mathbb{Z})$. In fact it is a normal subgroup and if it is nontrivial, then it is also Zariski dense in O_f. Moreover, there exists only finitely many f's (up to integral equivalence) for which $R_f < \mathrm{O}_f(\mathbb{Z})$ is of finite index. So for all other f's, we know $R_f < \mathrm{O}_f(\mathbb{Z})$ is thin (provided it is nontrivial).

2.3. **Monodromy groups.** Consider the differential equation

$$D_{\alpha,\beta}(u) = 0$$

on $\hat{\mathbb{C}}$ where

$$\begin{aligned} D_{\alpha,\beta} &= \left(z\frac{d}{dz} + \beta_1 - 1\right) \cdots \left(z\frac{d}{dz} + \beta_n - 1\right) - z\left(z\frac{d}{dz} + \alpha_1\right) \cdots \left(z\frac{d}{dz} + \alpha_n\right) \\ \alpha &= (\alpha_1, \alpha_2, \ldots, \alpha_n) \in [0,1)^n \\ \beta &= (\beta_1, \beta_2, \ldots, \beta_n) \in [0,1)^n. \end{aligned}$$

This equation is regular away from $\{0, 1, \infty\} \subset \hat{\mathbb{C}}$ and there are n linearly independent solutions in terms of hypergeometric functions: $z^{1-\beta_j}{}_nF_{n-1}(1 + \alpha_1 - \beta_j, \ldots, 1 + \alpha_n - \beta_j; 1 + \beta_1 - \beta_j, \ldots, \widecheck{(1 + \beta_j - \beta_j)}, \ldots, 1 + \beta_n - \beta_j | z)$ for $1 \le j \le n$ where $\check{}$ is used to signify omission and the hypergeometric functions are defined as

$${}_nF_{n-1}(\zeta_1, \ldots, \zeta_n; \eta_1, \ldots, \eta_{n-1} | z) = \sum_{j=0}^{\infty} \frac{(\zeta_1)_k \cdots (\zeta_n)_k z^k}{(\eta_1)_k \cdots (\eta_{n-1})_k k!}$$

and the Pochhammer symbol is defined as

$$(\eta)_k = \eta(\eta + 1) \cdots (\eta + k - 1) = \frac{\Gamma(\alpha + k)}{\Gamma(k)}.$$

Note that $\hat{\Gamma} := \pi_1(\hat{\mathbb{C}} \setminus \{0, 1, \infty\}) = \langle g_0, g_1, g_\infty | g_\infty g_1 g_0 = 1 \rangle$ which is simply a free group on two generators. Let $V_{\alpha,\beta}$ be a local solution space, say on some open neighborhood $U_{\alpha,\beta}$ of $z_0 = \frac{1}{2}$. See fig. 4 for a diagram of the setup.

We get a monodromy representation $M_{\alpha,\beta} : \hat{\Gamma} \to \mathrm{GL}(V_{\alpha,\beta}) \cong \mathrm{GL}_n(\mathbb{C})$ called *hypergeometric monodromy representation* by analytic continuation of solutions in

$V_{\alpha,\beta}$ along nontrivial loops based at z_0. Let $\Gamma_{\alpha,\beta} = M_{\alpha,\beta}(\hat{\Gamma})$. The following is a fundamental theorem of Beukers–Heckman [2].

Theorem 7 (Beukers–Heckman)**.** *Let $G_{\alpha,\beta}$ be the Zariski closure of $\Gamma_{\alpha,\beta}$. Then $G_{\alpha,\beta}$ is one of SL_n, SO_n, Sp_n or a finite primitive reflection group listed in the table of* [26].

We now want $\Gamma_{\alpha,\beta} < \mathrm{GL}_n(\mathbb{Z})$ to obtain thin groups. This is in fact possible with certain technical conditions. Moreover, for other technical reasons $\alpha, \beta \in \mathbb{Q}^n$. These properties actually imply that there are finitely many possible $\alpha, \beta \in \mathbb{Q}^n$ for a fixed $n \in \mathbb{N}$. See [13] for details. We are interested in the following question.

Question 8. Which $\Gamma_{\alpha,\beta}$ are arithmetic and which $\Gamma_{\alpha,\beta}$ are thin?

It is known that $\Gamma_{\alpha,\beta}$ are arithmetic for $n \in \{2, 3\}$. There have been further work for the above question [10, 28, 27] when $G_{\alpha,\beta} = \mathrm{Sp}_n$.

We now briefly describe the work of Fuchs–Meiri–Sarnak [13]. Suppose $G_{\alpha,\beta} = \mathrm{SO}_n$. Suppose further that $G_{\alpha,\beta}(\mathbb{R}) \cong \mathrm{SO}(n-1, 1)$. In this case, the authors call $\Gamma_{\alpha,\beta}$ a *hyperbolic hypergeometric monodromy group (HHMG)*. This setup allows one to exploit hyperbolic geometry. It turns out that there are 7 families and some sporadic HHMGs. They find a certificate for thinness and prove that some of these families and many of the sporadic ones are thin. They make the following conjecture.

Conjecture 9 (Fuchs–Meiri–Sarnak)**.** *All but finitely many HHMGs are thin.*

3. Applications

3.1. Generalization of Selberg's 3/16 theorem and affine sieve.

Let $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ be a finitely generated discrete subgroup which is Zariski dense in SL_2. Denote by δ_Γ the Hausdorff dimension of the limit set of Γ in $\partial_\infty \mathbb{H}^2$. As mentioned above, for lattices, SSA was initially derived from Selberg's 3/16 theorem. In the general case, once SSA is established, one can now "go backwards" and prove an analogue of Selberg's 3/16 theorem. This was initiated by Bourgain–Gamburd–Sarnak [7] as in the following theorem. Note that the original square-free hypothesis on $q \in \mathbb{N}$ is not required due to the improved SSA result of Bourgain–Varjú (see theorem 6).

Theorem 10 (Bourgain–Gamburd–Sarnak)**.** *Suppose $\delta_\Gamma > \frac{1}{2}$. Denote $X_q = \ker(\pi_q|_\Gamma)\backslash\mathbb{H}^2$ for all $q \in \mathbb{N}$. The following are equivalent:*

(1) The sequence $\{\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}/q\mathbb{Z}), \pi_q(S))\}_{q\in\mathbb{N},(q,q_0)=1}$ forms an expander.
(2) There exists $\epsilon > 0$ such that $\lambda_1(\Delta_{X_q}) \geq \lambda_0(\Delta_{X_q}) + \epsilon$ for all $q \in \mathbb{N}$ with $(q, q_0) = 1$.

An important fact is that there is *no spectral gap* for the $\delta_\Gamma \leq \frac{1}{2}$ regime. One must turn to resonances of the resolvent of the Laplacian for analogues. They obtained sufficient results for the resolvent for their affine sieve application which we recount below. A much stronger result and the right generalization of spectral gap for the $\delta_\Gamma \leq \frac{1}{2}$ regime is due to Oh–Winter [20].

Let $f \in \mathbb{Q}[(x)_{j,k}]$ be integral on the orbit $\Gamma \subset \mathrm{Mat}_{2\times 2}(\mathbb{R})$. Without loss of generality, assume that $\gcd\{f(x) : x \in \Gamma\} = 1$. Let $f = f_1 f_2 \cdots f_r$ for some $r \in \mathbb{N}$ be its factorization in the UFD $\mathbb{Q}[(x)_{j,k}]/(\det((x)_{j,k}) - 1)$ and assume the factors are irreducible in the UFD $\overline{\mathbb{Q}}[(x)_{j,k}]/(\det((x)_{j,k}) - 1)$. Let $\|\cdot\|$ denote the Frobenius norm. The following is the affine sieve application in [7].

Theorem 11 (Bourgain–Gamburd–Sarnak). *There exists $R \in \mathbb{N}$ such that*

$$|\{x \in \Gamma : \|x\| \leq T \text{ and } f_j(x) \text{ is prime for all } 1 \leq j \leq r\}| \ll \frac{T^{2\delta_\Gamma}}{(\log T)^r},$$
$$|\{x \in \Gamma : \|x\| \leq T \text{ and } f(x) \text{ has at most } R \text{ prime factors}\}| \gg \frac{T^{2\delta_\Gamma}}{(\log T)^r}.$$

The above theorem was proved using affine sieve techniques from their earlier work [6]. In that work, they prove a related counting result and obtain the following theorem.

Theorem 12 (Bourgain–Gamburd–Sarnak). *There exists $R \in \mathbb{N}$ such that $\{x \in \Gamma : f(x)$ has at most R prime factors$\}$ is Zariski dense in* SL_2.

3.2. **Zaremba's conjecture.** Let $\mathcal{A} \subset \mathbb{N}$ be a finite subset. Denote by $\mathfrak{D}_\mathcal{A}$ the set of denominators d of reduced rational numbers

$$\frac{b}{d} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_k}}}}$$

whose coefficients $a_1, a_2, \ldots, a_k$ in its continued fraction expansion are all in $\mathcal{A}$. The following was conjectured by Zaremba, motivated by applications in random numbers and numerical integration.

Conjecture 13 (Zaremba). *There exists $A \in \mathbb{N}$ such that $\mathfrak{D}_{\{1,2,\ldots,A\}} = \mathbb{N}$.*

The work of Bourgain–Kontorovich [8] was a breakthrough toward Zaremba's conjecture. It uses counting results coming from SSA among other techniques.

Theorem 14 (Bourgain–Kontorovich). *We have* $\lim_{N\to\infty} \frac{1}{N}|\mathfrak{D}_{\{1,2,\ldots,50\}} \cap [1,N]| = 1.$

The above theorem is equivalent to

$$|\mathfrak{D}_{\{1,2,\ldots,50\}} \cap [1,N]| = N + o(N) \qquad \text{as } N \to \infty.$$

In fact, they obtain a better error term of $O\big(Ne^{-c\sqrt{\log(N)}}\big)$ for some $c > 0$. We mention that the work of Huang [14] shows $A = 5$ suffices and the work of Magee–Oh–Winter [16] shows the error term can be improved to $O(N^{1-\epsilon})$ for some $\epsilon \in (0,1)$.

References

[1] Y. Benoist, *Propriétés asymptotiques des groupes linéaires*, Geom. Funct. Anal. **7** (1997), 1–47.

[2] F. Beukers and G. Heckman, *Monodromy for the hypergeometric function* ${}_nF_{n-1}$, Invent. Math. **95** (1989), 325–354.

[3] A. Borel, *Density properties for certain subgroups of semi-simple groups without compact components*, Ann. of Math. **72**(2) (1960), 179–188.

[4] A. Borel and Harish-Chandra, *Arithmetic subgroups of algebraic groups*, Ann. of Math. **75**(2) (1962), 485–535.

[5] J. Bourgain, *Some Diophantine applications of the theory of group expansion*, Math. Sci. Res. Inst. Publ. **61**, 1–22.

[6] J. Bourgain, A. Gamburd, and P. Sarnak, *Affine linear sieve, expanders, and sum-product*, Invent. Math. **179** (2010), 559–644.

[7] J. Bourgain, A. Gamburd, and P. Sarnak, *Generalization of Selberg's* $\frac{3}{16}$ *theorem and affine sieve*, Acta Math. **207** (2011), 255–290.

[8] J. Bourgain and A. Kontorovich, *On Zaremba's conjecture*, Ann. of Math. **180**(2) (2014), 137–196.

[9] J. Bourgain and P. P. Varjú, *Expansion in* $SL_d(\mathbf{Z}/q\mathbf{Z})$, *q arbitrary*, Invent. Math. **188** (2012), 151–173.

[10] C. Brav and H. Thomas, *Thin monodromy in Sp(4)*, Compos. Math. **150** (2014), 333–343.

[11] M. Burger and P. Sarnak, *Ramanujan duals. II*, Invent. Math. **106** (1991), 1–11.

[12] L. Clozel, *Démonstration de la conjecture* τ, Invent. Math. **151** (2003), 297–328.

[13] E. Fuchs, C. Meiri, and P. Sarnak, *Hyperbolic monodromy groups for the hypergeometric equation and Cartan involutions*, J. Eur. Math. Soc. **16** (2014), 1617–1671.

[14] S. Huang, *An improvement to Zaremba's conjecture*, Geom. Funct. Anal. **25** (2015), 860–914.

[15] A. Kontorovich, D. D. Long, A. Lubotzky, and A. W. Reid, *What is ... a thin group?*, Notices Amer. Math. Soc. **66**, 905–910.

[16] M. Magee, H. Oh, and D. Winter, *Uniform congruence counting for Schottky semigroups in* $\mathrm{SL}_2(\mathbf{Z})$, J. Reine Angew. Math. **753** (2019), 89–135. With an appendix by Jean Bourgain, Alex Kontorovich and Michael Magee.

[17] C. R. Matthews, L. N. Vaserstein, and B. Weisfeiler, *Congruence properties of Zariski-dense subgroups. I*, Proc. London Math. Soc. **48**(3) (1984), 514–532.

[18] V. V. Nikulin, *Discrete reflection groups in Lobachevsky spaces and algebraic surfaces*, Proceedings of the International Congress of Mathematicians (Berkeley, Calif., 1986) **1**, 654–671.

[19] M. V. Nori, *On subgroups of* $\mathrm{GL}_n(\mathbf{F}_p)$, Invent. Math. **88** (1987), 257–275.

[20] H. Oh and D. Winter, *Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$, J. Amer. Math. Soc. **29** (2016), 1069–1115.

[21] R. Pink, *Strong approximation for Zariski dense subgroups over arbitrary global fields*, Comment. Math. Helv. **75** (2000), 608–643.

[22] V. Platonov and A. Rapinchuk, *Algebraic groups and number theory*, **139** of Pure and Applied Mathematics, Academic Press, Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

[23] P. Sarnak, *Notes on the generalized Ramanujan conjectures*, Clay Math. Proc. **4**, 659–685.

[24] P. Sarnak, *Notes on thin matrix groups*, Math. Sci. Res. Inst. Publ. **61**, 343–362.

[25] A. Selberg, *On the estimation of Fourier coefficients of modular forms*, Proc. Sympos. Pure Math. **8**, 1–15.

[26] G. C. Shephard and J. A. Todd, *Finite unitary reflection groups*, Canad. J. Math. **6** (1954), 274–304.

[27] S. Singh and T. N. Venkataramana, *Arithmeticity of certain symplectic hypergeometric groups*, Duke Math. J. **163** (2014), 591–617.

[28] T. N. Venkataramana, *Monodromy of cyclic coverings of the projective line*, Invent. Math. **197** (2014), 1–45.
[29] E. B. Vinberg, *Absence of crystallographic groups of reflections in Lobachevskiĭ spaces of large dimension*, Trudy Moskov. Mat. Obshch. **47** (1984), 68–102, 246.
[30] B. Weisfeiler, *Strong approximation for Zariski-dense subgroups of semisimple algebraic groups*, Ann. of Math. **120**(2) (1984), 271–315.

Bourgain–Gamburd machine

Cagri Sert

The talk consisted of two parts. The first part included a discussion about expander families and (super-)strong approximation. In the second part, we explained the proof of the general result (the Bourgain–Gamburd machine) underlying [1, Theorem 3] of Bourgain–Gamburd. For the exposition, we mainly followed [1] and the useful notes of Tao [2].

1. Expander graphs and (super-)strong approximation

1.1. **Expander graphs.** Let $\mathcal{G}$ be a k-regular graph. One can associate to $\mathcal{G}$ an adjacency matrix A (real, symmetric). Let $\lambda_1 \geq \ldots \geq \lambda_n$ be the eigenvalues of A, where $n = |V\mathcal{G}|$, the cardinality of vertices of $\mathcal{G}$. It is not hard to see that $k = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \geq -k$.

Definition 1 (Spectral definition)**.** For $\varepsilon > 0$, a k-regular graph $\mathcal{G}$ is called an ε-expander if $\lambda_2 \leq (1-\varepsilon)k$.

As such, any connected graph is an ε-expander for certain $\varepsilon > 0$. The interesting question is to find (or determine which) families of k-regular graphs are expanders. We shall call a family of graphs ε-expander family if all graphs in the family are ε-expanders. In all discussion below, the degree k of graphs in a family are fixed.

First proof of existence of expander graphs was probabilistic (Pinsker). The first constructive proof was given by Margulis in early '70s. He showed that certain regular Cayley graphs $\mathcal{C}(G_n, S_n)_{n\in\mathbb{N}}$ constitute an expander family (more precisely $G_n = \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z})$ and S_n is the projection of a generating set of $\mathrm{SL}_d(\mathbb{Z})$ (always) not containing the identity, and $d \geq 3$). Later on, a stronger property (Ramanujan) was shown by Lubotzky–Philips–Sarnak for $d = 2$ for n's ranging over primes.

The subsequent question of knowing whether infinite index but "large" (Zariski-dense) subgroups of $\mathrm{SL}_d(\mathbb{Z})$ also share this property of projecting onto expanders was motivated by progresses in strong approximation which we now mention.

1.2. **(Super-)strong approximation.** The surjectivity in the following particular case of a result by Matthews–Vasserstein–Weisfeiler and Nori is what we will refer to as strong approximation property.

Theorem 2 (Strong approximation)**.** *Suppose that* $\Gamma < \mathrm{SL}_d(\mathbb{Z})$ *is Zariski-dense in* SL_d*. Then, there exists* $q_0 \in \mathbb{N}$ *such that for every* $q \in \mathbb{N}$ *with* $(q, q_0) = 1$*, the projection* $\Gamma \to \mathrm{SL}_d(\mathbb{Z}/q\mathbb{Z})$ *is surjective.*

In particular, when we have $\Gamma < \mathrm{SL}_2(\mathbb{Z})$ that is non-elementary (equivalently, not virtually nilpotent) and S is a generating set of Γ, then the Cayley graphs $\mathcal{C}(\mathrm{SL}_2(\mathbb{F}_p), S_p)$ when p ranges over (large enough) primes and S_p denotes the projection as before, is connected. In particular, each of them are ε_p-expanders for some $\varepsilon_p > 0$. *Super-strong approximation* refers to this expansion being uniform, i.e. that $\mathcal{C}(\mathrm{SL}_2(\mathbb{F}_p), S_p)$ constitutes an ε-expander family for some $\varepsilon > 0$.

Before the work of Bourgain–Gamburd [1], the understanding of the extent of the super-strong approximation property was very limited (except for some significant progress by Gamburd in his thesis). The back then open 1-2-3 question of Lubotzky reflected this fact. The following result of Bourgain–Gamburd provided the answer and a fundamental progress, as much by the result itself as its proof.

Theorem 3 (Bourgain–Gamburd, [1]). *Let S be a subset of $\mathrm{SL}_2(\mathbb{Z})$. Then $\mathcal{C}(\mathrm{SL}_2(\mathbb{F}_p), S_p)$ is a family of expanders*[1] *(p prime $\gg 1$) if and only if the group $\langle S\rangle$ generated by S is non-elementary.*

Underlying the proof of this result is a scheme that produces expansion out of certain phenomena. This scheme or parts of it have been repeatedly used in various extensions of the previous theorem as well as in other major results. The rest of this notes is devoted to its discussion.

2. Bourgain–Gamburd machine

Below is the statement of this scheme whose formulation we borrow from [2].

Theorem 4 (Bourgain–Gamburd expansion machine). *Suppose G is a finite group, $S \subset G$ is a symmetric set of cardinality k and that there exist constants $0 < \kappa < 1 < \Lambda$ such that*

1) *(High multiplicity or quasi-randomness, $QR(\kappa)$) Smallest dimension of a non-trivial representation of G is $|G|^\kappa$,*
2) *(Approximate subgroups or product theorem, AS) For every $\delta > 0$, there exists $\delta' = \delta'(\delta) > 0$ such that if $A \subset G$ is a $|G|^{\delta'}$-approximate subgroup of size contained between $|G|^\delta$ and $|G|^{1-\delta}$, then $\langle A\rangle$ is a proper subgroup of G.*
3) *(Non-concentration, NC)*

There exists an even number $n \leq \Lambda \log |G|$ such that $\sup_{H \leq G} \mu^{*n}(H) < |G|^{-\kappa}$. Then, (expand): $\mathcal{C}(G, S)$ is an ε-expander with $\varepsilon = \varepsilon(\kappa, \Lambda, \delta', k)$.

In fact, there is an elementary probabilistic characterization of (two-sided) expanders via equidistribution (with uniform exponential speed) of random walks and this is what the previous results establishes. One notable aspect of this statement is that it shows uniform equidistribution to prove spectral gap (usually, the latter is used to prove the former).

[1] In fact, Bourgain–Gamburd proves that this is a two-sided family of expanders, for this notion see e.g. [2]

2.1. **Quasi-randomness.** To indicate the scheme of proof of Theorem 4, for $\beta > 0$ and a probability measure on a finite group G, let us call the following property as ℓ^2-flat(β): there exists a constant $C = C(\beta)$ such that $\|\mu^{*n}\|_2 \leq |G|^{-1/2+\beta}$ for some $n \leq C|G|$, where μ^{*n} denotes the n^{th}-step distribution of the μ-random walk on G.

By a principle going back to Sarnak–Xue exploiting high multiplicity coming from quasi-randomness, one shows (see [2, Proposition 1.3.7]) that $QR(\kappa) + \ell^2$-flat(β) with $\beta < \kappa/2$ implies the expander property. Therefore, the proof of Theorem 4 (which is $QR + AS + NC \implies$ (expand)) boils down to showing $AS + NC \implies \ell^2$-flat(β) with small enough $\beta > 0$. Showing this is indeed the core of the proof of Theorem 4.

2.2. **ℓ^2-flattening and Balog-Szemerédi-Gowers lemma.** To prove this last ℓ^2-flattening statement (see below), Bourgain–Gamburd bring a combinatorial tool, namely the Balog-Szemerédi-Gowers (BSG) lemma, which was previously related to the current setting of product-set expansion in non-commutative groups by Tao. The following is a by-product of Bourgain-Gamburd's use of Tao's BSG lemma ([2, Lemma 1.4.1]). Below, all implied constants are universal.

Lemma 5 (Weighted-BSG)**.** *Let ν be a symmetric probability measure on a group G, and $K \geq 1$. Then, either*

1) *(ℓ^2-flattening) $\|\nu * \nu\|_2 \leq \frac{1}{K}\|\nu\|_2$, or*
2) *(some structure is charged) there exist a $O(K^{O(1)})$-approximate subgroup $H \subset G$ with $|H| \ll K^{O(1)}/\|\nu\|_2^2$ and $x \in G$ with $\nu(xH) \gg K^{-O(1)}$.*

Equipped with this ingredient, one proves the implication $AS + NC \implies \ell^2$-flat(β) discussed above:

Lemma 6 (ℓ^2-flattening)**.** *Suppose there exists $n \geq \frac{1}{2}\Lambda \log|G|$ such that $\|\mu^{*n}\|_2 \geq |G|^{-1/2+\kappa}$. Then, $\|\mu^{*2n}\|_2 \leq |G|^{-\varepsilon}\|\mu^{*n}\|_2$ for some $\varepsilon = \varepsilon(\kappa, \delta')$.*

The lemma is proved by arguing by contradiction: one applies Lemma 5 to get a contradiction to the non-concentration hypothesis (NC) thanks to the product theorem (i.e. hypothesis (AS)). These prove Theorem 4.

2.3. **Back to Theorem 3.** In the setting of Theorem 3, among the assumptions of Theorem 4,

1) (QR) is a result of Frobenius, which says that the group $G_p = \mathrm{SL}_2(\mathbb{F}_p)$ is $(p-1)/2$ quasi-random. Note that $(p-1)/2 \sim |G_p|^{1/3}$.
2) (AS) is a back-then-recent major result of Helfgott (later generalized by Breuillard–Green–Tao and Pyber–Szabó) – which itself uses a (recent) result of Bourgain–Katz–Tao among others,
3) (NC) is proved by Bourgain–Gamburd by a nice argument using results of Dickson (on subgroups of $\mathrm{SL}_2(\mathbb{F}_p)$), Tits (on free subgroups in linear groups) and Kesten (the rate of return to identity for random walks on free groups).

References

[1] J. Bourgain and A. Gamburd. *Uniform expansion bounds for Cayley graphs of* $\mathrm{SL}_2(\mathbb{F}_p)$, Annals of Mathematics (2008), 625–642.

[2] T. Tao *Expansion in finite simple groups of Lie type*, Vol. 164. American Mathematical Soc., 2015.

Bourgain and Kontorovich's work on Zaremba's Conjecture

Xin Zhang

The goal of this note is to give a short description of Bourgain and Kontorovich's work on Zaremba's Conjecture.

It is well known that every number $x \in (0,1)$ admits a continued fraction expansion:

$$x = [a_1, a_2, \cdots] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}}$$

The integers $a_1, a_2, \cdots$ are called partial quotients for x. If x is rational, the length of the expansion is finite.

Motivated by constructing sets with minimal discrepancy for numeric integration, Zaremba was looking for rationals b/d with d arbitrarily large and yet all partial quotients of b/d are controlled. This led him to make the following conjecture:

Conjecture 1 (Zaremba, 1972, [1]). *Every natural number is the denominator of a reduced fraction whose partial fractions are bounded. That is, there exists some* $A > 1$ *so that for each* $d \in \mathbb{N}$*, there is some* $b \in \mathbb{N}$ *with* $(b, d) = 1$*, so that* $b/d = [a_1, \cdots, a_k]$ *with* $\max\{a_i\}_{i=1}^k \le A$.

In 2014, Bourgain and Kontorovich made a major breakthrough towards Zaremba's conjecture:

Theorem 2 (Bourgain-Kontorovich, 2014, [2]). *Almost every number is the denominator of a reduced fraction whose partial quotients are bounded by 50.*

Note that a reduced fraction b/d admits an expansion $b/d = [a_1, a_2, \cdots, a_k]$ if and only if

$$\begin{pmatrix} * & b \\ * & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & a_1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & a_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & a_k \end{pmatrix}.$$

If we let Γ be the semigroup generated by $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & a \end{pmatrix} : 1 \le a \le 50 \right\}$ and $v_0 = (0,1)^{\mathrm{t}}$, then the statement of Theorem 2 can be rephrased as

$$\#\{\langle \gamma(v_0), v_0 \rangle : \gamma \in \Gamma\} \cap [1, N] = N + o(N)$$

as $N \to \infty$.

The main technique employed in the proof of Theorem 2 is the Hardy-Littlewood circle method. Let

$$\Omega_N = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \sqrt{a^2+b^2+c^2+d^2} \leq 100N \right\}$$

(here we simplified the much more complicated original definition of Ω_N in [2] in order to present some main ideas.) We consider

$$R(n) := \sum_{\gamma \in \Omega_N} \mathbf{1}\{\langle \gamma(v_0), v_0 \rangle = n\}.$$

Then $R(n) > 0$ implies that n is a denominator of a reduced fraction with all partial quotients bounded by 50. Therefore, to prove Theorem 2, it suffices to prove that $R(n) > 0$ for almost all $n \in \mathbb{N}$.

It is difficult to analyse $R(n)$ directly. On the other hand, the Fourier transform $\hat{R}(\theta) := \sum_{\gamma \in \Omega_N} e(\langle \gamma(v_0), v_0 \rangle \theta)$ is more comprehensible, where $e(x) = e^{2\pi i x}$. For instance, the total input

$$\hat{R}(0) = \sum_{\gamma \in \Omega_N} 1 \sim cN^{2\delta}$$

by some counting techniques from thermodynamics formalism for finitely generated sub-semigroups of $\mathrm{SL}_2(\mathbb{Z})$. Here δ is the Hausdorff dimension of the set

$$\{[a_1, a_2, \cdots] : a_i \in [1, 50], i \in \mathbb{N}\}.$$

More generally, for q small we also have good estimate for $\hat{R}(a/q)$ by asymptotic counting of points of $\Gamma(q)$, the congruence subsemigroup of Γ of level q.

Suppose we have obtained sufficient information from $\hat{R}$, we can retrieve $R(n)$ by the Fourier Inversion Formula:

$$R(n) = \int_0^1 \hat{R}(\theta) e(-n\theta) d\theta.$$

The Hardy-Littlewood circle method is to split

$$\int_0^1 \hat{R}(\theta) e(-n\theta) d\theta = \int_{\mathfrak{M}} \hat{R}(\theta) e(-n\theta) d\theta + \int_{\mathfrak{m}} \hat{R}(\theta) e(-n\theta) d\theta := M(n) + E(n),$$

where the "major arcs" $\mathfrak{M}$ consists of small neighbourhoods of rationals with small denominators, and the "minor arcs" $\mathfrak{m}$ is the complement of $\mathfrak{M}$ in $[0, 1)$. The reason for this split is that in general, if f is an arithmetic function, $|\hat{f}(x)|$ is usually large when x is near a rational with small denominator. We expect the "main term" $M(n)$ gives the major contribution to $R(n)$ and $E(n)$ is the error term.

Applying the aforementioned counting techniques, Bourgain–Kontorovich showed that for $n \in [N/2, N]$,

$$M(n) \gg N^{2\delta - 1 + o(1)}.$$

This is to say that an asymptotic positive-density portion of the total input is roughly equidistributed among $[N/2, N]$.

For the error term $E(n)$, If one were able to prove that

$$|E(n)| \ll N^{2\delta-1-\epsilon} \tag{1}$$

for some $\epsilon > 0$, one then proved Zaremba's conjecture up to verification of integers bounded by an effectively computable large number. However, the bound (1) seems beyond the reach of current techniques. Instead, Bourgain-Kontorovich proved the following ℓ^2-bound for $E(n)$:

$$\sum_{n\in\mathbb{Z}} |E(n)|^2 \ll N^{4\delta-1-\epsilon}, \tag{2}$$

which then implies (1) for almost all $n \in [N/2, N]$, and thus completes the proof Theorem 2. Some techniques involved in obtaining (2) are Kloosterman's refinement of the circle method, estimates of exponential sums, and Vinogradov's method for estimating bilinear forms.

References

[1] Zaremba, S. K. *La méthode des "bons treillis" pour le calcul des intègrales multiples*, Applications of number theory to numerical analysis (1972), 39–119.
[2] Bourgain, J. and Kontorovich, A. *On Zaremba's conjecture*, Annals of Mathematics (2014), 137–196.

Approximate subgroups

Daniele Dona

For a group G and a $K \geq 1$, a set $A \subseteq G$ is a K-*approximate subgroup* if there is some $X \subseteq G$ with $|X| \leq K$ and $A^2 \subseteq XA$. We focus here on the product theorem by Helfgott [5], which classifies approximate subgroups for $G = \mathrm{SL}_2(\mathbb{F}_p)$. Then we present the Balog-Szemerédi-Gowers lemma, in which approximate subgroups appear as the only obstacles to a "random" behaviour: originally from a discrete setting [3], the lemma is given here in a measure-theoretic version (see [7] and [1]).

1. A product theorem in $\mathrm{SL}_2(\mathbb{F}_p)$

We have the following theorem. All references in this section are to [5].

Theorem 1 (Kep Prop. (a))**.** *Let p be a prime and $G = \mathrm{SL}_2(\mathbb{F}_p)$. Fix $\delta > 0$.*
Then there exist constants $C, \varepsilon > 0$, depending only on δ, such that for any $A \subseteq G$ with $\langle A\rangle = G$ and $|A| < p^{3-\delta}$ we have $|A^3| \geq C|A|^{1+\varepsilon}$.

In line with other results about approximate subgroups, any slowly growing set A is either too small (and choosing a small C is enough), or too large (meaning $|A| > |G|^{1-\delta}$), or sits in a proper subgroup. We need only prove $|A^h| \geq C|A|^{1+\varepsilon}$ for any fixed $h \geq 3$ and for only A symmetric, thanks to Ruzsa [6, Thm. 4.2].

We start with a random walk whose number of elements we estimate directly.

Proposition 2 (Lemma 4.7)**.** *Let $V \subseteq \mathrm{SL}_2(K)$ be a set of simultaneously diagonalizable matrices, with common eigenvectors v_1, v_2. Let g be such that $gv_i \neq \lambda v_j$ for all λ, i, j. Then $|VgVg^{-1}V| \gg |V|^3$.*

The proof is straightforward: after conjugation by the same element, V is made of diagonal matrices and g has all nonzero entries, so that checking entries gVg^{-1} must have $\gg |V|$ elements and $V(gVg^{-1})V$ must have $\gg |V| \cdot |V|^2$ elements.

The bound in Proposition 2 would be enough, as long as we fill two holes:

(1) we want $V \subseteq A^k$ with k bounded and $|V|$ large with respect to $|A|$, and
(2) we want $g \in A^k$ with k bounded.

If such V, g exist, it would imply both $|VgVg^{-1}V| \leq |A^{5k}|$ and $|V|^3 \gg |A|^{1+\varepsilon}$.

For the second hole, *escape from subvarieties* is enough (see [4, §3]).

Proposition 3 (Lemma 4.4)**.** *Let $G \leq \mathrm{GL}_N(K)$, acting on the affine space $\mathbb{A}^N$. Let W be a closed variety in $\mathbb{A}^N$ over K, and let $x \in \mathbb{A}^N(K)$ with $G \cdot x \not\subseteq W(K)$.*

Then there are $k, \eta > 0$ depending only on $\dim(W), \deg(W)$ such that, for any $A \subseteq G$ with $\langle A \rangle = G$, there are $\max\{1, \eta|A|\}$ elements $g \in A^k$ with $g \cdot x \notin W(K)$.

$\mathrm{SL}_2(\mathbb{F}_p)$ acts on $\mathbb{A}^4(\mathbb{F}_p)$ via matrix multiplication, and the "bad" condition of having $gv = \lambda w$ for fixed v, w and some λ defines a variety $W_{v,w}$ in $\mathbb{A}^4$. Thus, we can find a "good" $g \in A^k$, using $x = e$ and $W = W_{v_1,v_1} \cup W_{v_1,v_2} \cup W_{v_2,v_1} \cup W_{v_2,v_2}$.

We move now to the first hole. Matrices are simultaneously diagonalizable if and only if they sit in the same centralizer $C(g)$, when their trace is not ± 2: we have many elements with $\mathrm{Tr} \neq \pm 2$, using Proposition 3 again and passing to A^2. Hence, our aim is a lower bound for $|C(g) \cap A^k|$. $C(g)$ is the stabilizer of conjugation, so we can use the *orbit-stabilizer theorem* in a version valid for sets: after combining it with the pigeonhole principle, we obtain that there is some $g \in A^2$ for which

$$|C(g) \cap A^4| \geq \frac{|A^2|}{\#\{hgh^{-1} | h \in A^2\}} \geq \frac{|A^2| \#\{\mathrm{Cl}(h) | \mathrm{Cl}(h) \cap A^2 \neq \emptyset\}}{|A^6|}.$$

In SL_2 and outside the $\mathrm{Tr} = \pm 2$ case, conjugacy classes are in bijection with characteristic polynomials, which in turn are in bijection with traces. Therefore

$$|C(g) \cap A^4| \geq \frac{|A^2||\mathrm{Tr}(A^2)|}{|A^6|}.$$

Our new problem becomes finding a lower bound for $\mathrm{Tr}(A^{k'})$. By an element counting process similar to the one in Proposition 2, we have the following.

Proposition 4 (Prop. 4.10)**.** *Let $G = \mathrm{SL}_2(K)$. Then there exists some k' such that, for all $A \subseteq G$ with $\langle A \rangle = G$, we have $|\mathrm{Tr}(A^{k'})| \gg |A|^{\frac{1}{3}}$.*

This is not enough to give growth. Our counting arguments have played on the principle that, for a "random enough" set $X \subseteq G$ and a variety $V \subseteq G$, we should have $|X \cap V| \approx |X|^{\frac{\dim(V)}{\dim(G)}}$: $A^{k'}$ is indeed random enough, and in Propositions 2-4 we bounced between SL_2 and the variety of diagonals (of dimensions 3 and 1).

We need one more idea: through traces, we can use a growth result in $\mathbb{F}_q$, a *sum-product theorem.* Such results go back to [2]: for any $X \subseteq \mathbb{F}_p$ with $p^\delta < |X| < p^{1-\delta}$, we have $\max\{|X + X|, |XX|\} \gg_\delta |X|^{1+\varepsilon}$ for some $\varepsilon > 0$ depending only on δ.

As a matter of fact, instead of max, any "reasonable" polynomial involving both field operations will do. We resort to this for our trace problem: we have in fact

$$\operatorname{Tr}\left[\begin{pmatrix} x & \\ & x^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} y & \\ & y^{-1}\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1}\right] = ad(xy+x^{-1}y^{-1})-bc(xy^{-1}+x^{-1}y),$$

and then we can apply the following version of the sum-product theorem.

Proposition 5 (Prop. 3.3)**.** *Let $q = p^\alpha$, and let $X \subseteq \mathbb{F}_q^*$ with $|X| < p^{1-\delta}$. Then there is some $\varepsilon > 0$ depending only on δ such that, for any $a_1, a_2 \in \mathbb{F}_q^*$,*

$$\#\{a_1(xy + x^{-1}y^{-1}) + a_2(xy^{-1} + x^{-1}y)|x, y \in X^{20}\} \gg_\delta |X|^{1+\varepsilon}.$$

Thus, $|\mathrm{Tr}(A^{k'})| \gg |V|^{1+\varepsilon}$ for some bounded k'. Putting together all results,

$$|A^{5k}| \geq |VgVg^{-1}V| \gg |V|^3 = |C(g') \cap A^{2k'}|^3 \gg \frac{|A^{k'}|^3|\mathrm{Tr}(A^{k'})|^3}{|A^{3k'}|^3}$$
$$\gg \frac{|A^{k'}|^3|V|^{3+3\varepsilon}}{|A^{3k'}|^3} \gg \frac{|A^{k'}|^3}{|A^{3k'}|^3}\left(\frac{|A^{k'}||\mathrm{Tr}(A^{k'})|}{|A^{3k'}|}\right)^{3+3\varepsilon} \gg \frac{|A^{k'}|^3}{|A^{3k'}|^3}\left(\frac{|A|^{\frac{4}{3}}}{|A^{3k'}|}\right)^{3+3\varepsilon},$$

implying that $|A^{k''}|^{7+3\varepsilon} \gg |A|^{7+4\varepsilon}$. The single ε we have gained in the process is enough to prove Theorem 1, using the already mentioned simplifications of Ruzsa.

2. The Balog-Szemerédi-Gowers lemma

Theorem 1 deals with finite groups, and the classical Balog-Szemerédi-Gowers lemma deals with the discrete group $\mathbb{Z}$. One may want to treat non-discrete groups: there, it is more useful to describe growth in terms of *measure* or *entropy.*

In such cases, approximate subgroups are so only with respect to a *scale.* Definitions are expressed quantitatively in terms of some $\delta > 0$. In a compact metric group with distance d, we use the following notation: for a measure μ and a set S,

$$\mu_\delta = \mu * \frac{1_{B(e,\delta)}}{|B(e,\delta)|}, \qquad \text{where } B(x,\delta) = \{y \in G | d(x,y) < \delta\} \ \ (\text{a } \delta\text{-}\textit{ball}),$$
$$S^{(\delta)} = \bigcup_{s\in S} B(s,\delta) = \{y \in G | d(y,S) < \delta\} \ \ (\text{the } \delta\text{-}\textit{neighbourhood} \text{ of } S),$$
$$\mathcal{N}_\delta(S) = \min\{\#\mathcal{S} | \mathcal{S} \text{ is a cover of } S \text{ made of } \delta\text{-balls}\} \ \ (\text{the } \textit{metric entropy} \text{ of } S).$$

In each of them, δ codifies "how finely" we can see. In the result below, we bargain between strength and finesse: the smaller δ is, the worse the estimates become.

Theorem 6. *Let G be a compact Lie group, let μ be a symmetric probability measure on G, and fix $\alpha > 0$. Assume that $\|\mu_\delta\|_2^2 \geq \delta^{-\alpha}$. Then there exists $\varepsilon > 0$ such that, for all $\delta > 0$ small enough, either*

(1) $\|\mu_\delta * \mu_\delta\|_2 \leq \delta^\varepsilon \|\mu_\delta\|_2$ (flattening), or

(2) there is a $\delta^{-O(\varepsilon)}$-approximate subgroup H with $\mathcal{N}_\delta(H) \leq \delta^{-(\dim(G)-\alpha)-O(\varepsilon)}$ and there is some $x \in G$ such that $\mu_\delta(xH^{(\delta)}) \geq \delta^{O(\varepsilon)}$ (concentration).

The analogies with Theorem 1 are evident. The fastness of the set growth becomes here the measure flattening by δ^ε in (1). Conversely, in the slow growth scenario of (2), there is a coset of an approximate subgroup H that greatly overlaps with μ. Furthermore, H is of the "correct dimension", meaning that if μ is large enough to have $\|\mu_\delta\|_2^2 \geq \delta^{-\alpha}$ then H has essentially the expected codimension α.

References

[1] Y. Benoist and N. de Saxcé. A spectral gap theorem in simple Lie groups. *Invent. Math.*, 205:337–361, 2016.
[2] J. Bourgain, N. Katz, and T. Tao. A sum-product estimate in finite fields, and applications. *Geom. Funct. Anal.*, 14:27–57, 2004.
[3] A. Balog and E. Szemerédi. A statistical theorem of set addition. *Combinatorica*, 14(3):263–268, 1994.
[4] A. Eskin, Sh. Mozes, and H. Oh. On uniform exponential growth for linear groups. *Invent. Math.*, 160(1):1–30, 2005.
[5] H. A. Helfgott. Growth and generation in $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. *Ann. of Math. (2)*, 167:601–623, 2008.
[6] I. Z. Ruzsa. Sums of finite sets. In D. V. Chudnovsky, G. V. Chudnovsky, and M. B. Nathanson, editors, *Number Theory: New York Seminar 1991-1995*, pages 281–293. Springer-Verlag, New York (USA), 1996.
[7] T. Tao. Product set estimates for non-commutative groups. *Combinatorica*, 28(5):547–594, 2008.

Local-global I: Apollonian packings

Edna Jones

We discussed the strong asymptotic local-global conjecture for integral Apollonian circle packings and progress towards proving this conjecture. An *Apollonian circle packing* is a circle packing that is constructed by repeatedly placing circles tangent to configurations of three mutually tangent circles already in the packing. A circle packing is *integral* if the bend (1/radius) of each circle in the packing is an integer. An integral circle packing is called *primitive* if the greatest common divisor of all of the bends in the packing is 1.

To start, we discussed how integral Apollonian circle packings exist. Using a poem written by Soddy [10], we introduce Descartes circle theorem. (The theorem was written in a letter by Descartes [3, pp. 45–50] in 1643.)

Theorem 1 (Descartes circle theorem)**.** *If b_1, b_2, b_3, b_4 are bends of four mutually tangent circles, then*

$$(b_1 + b_2 + b_3 + b_4)^2 = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2).$$

Using Descartes circle theorem, we show that the Apollonian group acts on quadruples of bends of mutually tangent circles. The *Apollonian group* Γ is

$$\Gamma = \langle M_1, M_2, M_3, M_4 \rangle,$$

where

$$M_1 = \begin{pmatrix} -1 & 2 & 2 & 2 \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} 1 & & & \\ 2 & -1 & 2 & 2 \\ & & 1 & \\ & & & 1 \end{pmatrix},$$

$$M_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ 2 & 2 & -1 & 2 \\ & & & 1 \end{pmatrix}, \qquad M_4 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ 2 & 2 & 2 & -1 \end{pmatrix}.$$

Let $\mathbf{v_0}$ be an integer quadruple of bends of mutually tangent circles in an Apollonian packing $\mathcal{P}$. The orbit $\Gamma\mathbf{v_0}$ contains all quadruples of bends of mutually tangent circles in $\mathcal{P}$, so all of the bends in $\mathcal{P}$ are integers.

Now that we know that integral Apollonian circle packings exist, we ask which integers can appear as bends in a fixed integral Apollonian circle packings. A number theorist might ask if there are any local or congruence restrictions on the bends. This motivates the following definition of admissibility:

Definition 2 (Admissible integers for circle packings)**.** Let $\mathcal{P}$ be an integral circle packing. An integer m is *admissible (or locally represented)* if for every $q \geq 1$

$$m \equiv \text{bend of some circle in } \mathcal{P} \pmod{q}.$$

Fuchs [4] characterized the congruence restrictions on the bends in a fixed primitive integral Apollonian circle packing.

Theorem 3 (Fuchs)**.** *Let $\mathcal{P}$ be a primitive integral Apollonian circle packing. Then m is admissible if and only if m is in certain congruence classes modulo 24. (The congruence classes depend on the packing.)*

The congruence conditions modulo 24 do not completely characterize the bends that can appear in a fixed primitive integral Apollonian circle packing. However, it is conjectured that if an integer is admissible and sufficiently large, then it is a bend in a fixed primitive integral Apollonian circle packing.

Conjecture 4 (Graham, Lagarias, Mallows, Wilks, and Yan [6])**.** *The bends of a fixed primitive integral Apollonian circle packing $\mathcal{P}$ satisfy a strong asymptotic local-global principle.*

That is, there is an $N_0 = N_0(\mathcal{P})$ so that, if $m > N_0$ and m is admissible, then m is the bend of a circle in the packing.

The first progress toward this conjecture was made by Graham, Lagarias, Mallows, Wilks, and Yan [6].

Observation 1 (Graham–Lagarias–Mallows–Wilks–Yan)**.** *There exists a $c_1 > 0$ such that at least $c_1 N^{1/2}$ of all integers less than N appear as bends in a fixed primitive integral Apollonian circle packing.*

They proved this observation by looking at the largest entries of $(M_1 M_2)^k \mathbf{v_0}$, where $\mathbf{v_0}$ is the root quadruple of bends and $k > 0$.

The next result towards the strong asymptotic local-global conjecture was proved by Sarnak [9].

Theorem 5 (Sarnak)**.** *There exists a $c_2 > 0$ such that at least $c_2 N/\sqrt{\log(N)}$ of all integers less than N appear as bends in a fixed primitive integral Apollonian circle packing.*

To prove this, Sarnak used the Descartes quadratic form Q (with signature (3,1)) defined by

$$Q(\mathbf{v}) = 2(b_1^2 + b_2^2 + b_3^2 + b_4^2) - (b_1 + b_2 + b_3 + b_4)^2.$$

Sarnak showed that there is spin homomorphism $\rho\colon \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_Q(\mathbb{R})$ such that

$$\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \mapsto M_4 M_3 \qquad \text{and} \qquad \pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \mapsto M_2 M_3.$$

It is well-known that $\pm\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $\pm\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ generate

$$\Lambda(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod 2 \right\}.$$

For any $x, y \in \mathbb{Z}$ and $\gcd(2x, y) = 1$, there is a matrix of the form $\begin{pmatrix} * & 2x \\ * & y \end{pmatrix} \in \Lambda(2)$. Using this, Sarnak showed that

$$4(b_1 + b_2)x^2 + 2(b_1 + b_2 - b_3 + b_4)xy + (b_1 + b_4)y^2 - b_1 \tag{1}$$

with $x, y \in \mathbb{Z}$ and $\gcd(2x, y) = 1$ is a bend in our packing. The number of integers up to N satisfying (1) is known to be of size $N/\sqrt{\log(N)}$. (See [8].)

Bourgain and Fuchs [1] proved that a positive density of integers appear as bends in a fixed primitive integral Apollonian circle packing.

Theorem 6 (Bourgain–Fuchs)**.** *There exists a $c_3 > 0$ such that at least $c_3 N$ of all integers less than N appear as bends in a fixed primitive integral Apollonian circle packing.*

This was proved by looking at multiple orbits of $\rho(\Lambda(2))$ in the packing.

The best result we have towards the strong asymptotic local-global conjecture is by Bourgain and Kontorovich [2].

Theorem 7 (Bourgain–Kontorovich)**.** *Almost every admissible number is the bend of a circle in the Apollonian circle packing $\mathcal{P}$. Quantitatively, the number of exceptions up to N is bounded by $O(N^{1-\eta})$, where $\eta > 0$ is effectively computable.*

This theorem has been extended by Fuchs, Stange, and Zhang [5] to certain other Kleinian circle packings.

We discussed a proof sketch of Theorem 7 outlined in [7].

References

[1] J. Bourgain and E. Fuchs, *A proof of the positive density conjecture for integer Apollonian circle packings*, J. Amer. Math. Soc. **24** (2011), no. 4, 945–967.

[2] J. Bourgain and A. Kontorovich, *On the local-global conjecture for integral Apollonian gaskets*, Invent. Math. **196** (2014), no. 3, 589– 650.

[3] R. Descartes, *Oeuvres de Descartes*, Correspondance IV, C. Adam and P. Tannery, eds., Leopold Cerf, Paris, 1901.

[4] E. Fuchs, *Strong approximation in the Apollonian group*, J. Number Theory **131** (2011), 2282–2302.

[5] E. Fuchs, K. Stange, and X. Zhang, *Local-global principles in circle packings*, Compos. Math. **155** (2019), no. 6, 1118–1170.

[6] R. L. Graham, J. C. Lagarias, C. L. Mallows, A. R. Wilks, and C. H. Yan, *Apollonian circle packings: number theory*, J. Number Theory **100** (2003), no. 1, 1–45.

[7] A. Kontorovich, *From Apollonius to Zaremba: local-global phenomena in thin orbits*, Bull. Amer. Math. Soc. (N.S.) **50** (2013), no. 2, 187–228.

[8] E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, Arch. der Math. u. Phys. **13** (1908), no. 3, 305–312.

[9] P. Sarnak, *Letter to J. Lagarias about integral Apollonian packings* (2007). `web.math.princeton.edu/sarnak/AppolonianPackings.pdf`.

[10] F. Soddy, *The kiss precise*, Nature **137** (1936), 1021.

Mixing and the decay of matrix coefficients

Manuel Luethi

1. Introdution

Let $Q:\mathbb{R}^{d+1}\to\mathbb{R}$ denote the isotropic quadratic form

$$Q(x_1,\dots,x_{d+1}) = x_1^2+\cdots+x_d^2-x_{d+1}^2.$$

Let $\mathrm{SO}_{d,1}(\mathbb{R}) < \mathrm{SL}_{d+1}(\mathbb{R})$ be the special isometry group of Q and let G be its connected component. Let $K=\mathrm{SO}_d(\mathbb{R})$ embedded in the top-left corner of $\mathrm{SL}_{d+1}(\mathbb{R})$. Then $K<G$ is a maximal compact subgroup and, denoting by $e_1,\dots,e_{d+1}$ the standard basis of $\mathbb{R}^{d+1}$, we have $K=\mathrm{Stab}_G(e_{d+1})$. Recall that $\mathbb{H}^d\cong G/K$, i.e., $\mathbb{H}^d$ identifies with the e_{d+1}-component in $\{Q=-1\}$. The space $\mathbb{H}^d\subseteq\mathbb{R}^{d+1}$ is an embedded Riemannian submanifold which is simply connected with constant negative curvature. We let $d_{\mathbb{H}^d}$ denote the induced.

We say that two geodesic rays $\sigma_1,\sigma_2:[0,\infty)\to\mathbb{H}^d$ are equivalent, if

$$\sup_{t\geq 0} d_{\mathbb{H}^d}\big(\sigma_1(t),\sigma_2(t)\big)<\infty$$

and we define $\partial\mathbb{H}^d$ as the set of equivalence classes of geodesic rays. Each geodesic $\sigma:\mathbb{R}\to\mathbb{H}^d$ can be identified with a tuple in $(\sigma^+,\sigma^-)\in\partial\mathbb{H}^d\times\partial\mathbb{H}^d$. Here σ^+ denotes the equivalence class of the geodesic ray $t\mapsto\sigma(t)$ and σ^- denotes the equivalence class of the geodesic ray $t\mapsto\sigma(-t)$. The equivalence classes σ^+,σ^- will be called the endpoints of σ. This identification can be shown to give a bijection between the set of geodesics parametrized with unit speed and $(\partial\mathbb{H}^d\times\partial\mathbb{H}^d)-\Delta$, where Δ denotes the diagonal, i.e.., any pair of distinct equivalence classes (v^+,v^-)

can be connected by a unique geodesic $\sigma_{(v^+,v^-)}$ of unit speed. In particular, we obtain an identification

$$T^1\mathbb{H}^d \cong (\partial\mathbb{H}^d \times \partial\mathbb{H}^d) - \Delta \times \mathbb{R}$$

given by

$$(v^+, v^-, t) \mapsto \dot{\sigma}_{(v^+,v^-)}(t).$$

We also recall the identification $T^1\mathbb{H}^d \cong \mathbb{H}^d/M$, where $M < K$ is the centralizer of the diagonal subgroup $A < G$ given by

$$A = \left\{ a_t = \begin{pmatrix} \mathrm{Id}_{d-1} & 0 & 0 \\ 0 & \cosh(t) & \sinh(t) \\ 0 & \sinh(t) & \cosh(t) \end{pmatrix} \right\}.$$

Using the above identification, the geodesic flow $\mathcal{G}_\bullet$ on $T^1\mathbb{H}^d$ is described by the action of A, namely $\mathcal{G}_t(gM) = ga_t^{-1}M$ and, similarly, the *frame flow* $\mathcal{F}_\bullet$ on G is given by $\mathcal{F}_t(g) = ga_t^{-1}$.

Let $\Gamma < G$ be discrete torsion-free and consider the manifold $M = \Gamma\backslash\mathbb{H}^d$. Given a Borel measure m on $T^1M \cong \Gamma\backslash T^1\mathbb{H}^d$, we abuse notaition and denote by $\tilde{m}$ both the Γ-invariant lift to $T^1\mathbb{H}^d$ as well as the $\Gamma \times M$-invariant lift to G. As M and $\mathbb{H}^d$ are locally isometric and as left- and right-multiplication commute, both the geodesic flow and the frame flow descend to T^1M and $\Gamma\backslash G$ respectively with analogous representation by the action of A. We denote by Leb the Lebesgue measure on Euclidean space of implicit dimension.

Definition 1. The measure $\tilde{m}$ is a quasi-product measure if there is a Radon measure μ on $\partial\mathbb{H}^d \times \partial\mathbb{H}^d$ equivalent to a product of probability measures such that $\tilde{m} = \mu \otimes \mathrm{Leb}$.

Examples of quasi-product measures include the lifts of the Bowen-Margulis-Sullivan, the Burger-Roblin, and the Haar measures on T^1M, which we abbreviate by m^{BMS}, m^{BR} and m_*^{BR}, m^{Haar} respectively; cf. [3, Def. 2.1] for the corresponding quasi-product structures. The Bowen-Margulis-Sullivan measure is an invariant measure for the geodesic flow and, in fact, if it is finite, then it is the unique measure of maximal entropy; cf. [4].

2. Mixing for the BMS-measure

The first main result is mixing for the BMS-measure with respect to the action of A.

Theorem 2 (cf. [1, Thm. 1] and [6, Thm. 1.1])**.** *Assume that $\Gamma < G$ is Zariski dense and that m^{BMS} is finite. Then the geodesic flow and the frame flow are mixing for $\tilde{m}^{\mathrm{BMS}}$.*

Remark 3. (1) The theorem by Babillot [1, Thm. 1] applies to the geodesic flow (i.e., M-invariant test-functions) and is more general than what is stated here. Instead of proving mixing for the geodesic flow for Zariski-dense Γ, the original theorem proves that the geodesic flow is mixing unless the length spectrum generates a discrete subgroup of $\mathbb{R}$.

(2) The theorem by Winter [6, Thm. 1.1] proves mixing for the frame flow for Zariski-dense Γ. The role of the length spectrum is taken by the *transitivity group* [6, Lem. 3.1] and a large part of the proof is concerned with showing that Zariski-density of Γ implies that the transitivity group is generically dense.

Description of the proof. The proof is by contradiction, i.e., we assume that the flow is not mixing. In particular, there exists a compactly supported continuous function ϕ of vanishing integral and a divergent sequence $(t_n)_{n\in\mathbb{N}}$ such that $\phi\circ a_{t_n}$ does not converge to 0 in the weak-$*$ topology. In particular, there exists a non-constant function $\psi \in L^2(\tilde{m}^{\mathrm{BMS}})$ and a sequence $(s_n)_{n\to\infty}$ such that $s_n \to \infty$ and ψ is the weak-$*$ limit both for $\phi\circ a_{s_n}$ and $\phi\circ a_{-s_n}$ [1, Lem. 1]. By the Hopf argument [2, Thm. 4.1], lifting ψ to a Γ-invariant function and smoothing along the A-direction, we obtain a non-constant function which is invariant both under the stable and the unstable foliation. Using Zariski density of Γ, one can show that this invariance implies that the function is essentially constant, which is absurd; cf. [6, Thm. 3.14, Thm. 4.2, Prop. 5. 1]. □

3. Local mixing for the Haar measure

The second result presented was local mixing for the Haar measure.

Theorem 4 ([5, Thm. 3.4] and [6, Thm. 1.4])**.** *Assume that Γ is Zariski-dense with critical exponent δ_Γ and assume that and m^{BMS} is finite. Let $\psi_1, \psi_2 \in \mathrm{C}_c(\Gamma\backslash G)$. Then*

$$\lim_{t\to\infty} e^{t(d-1-\delta_\Gamma)} \int_{\Gamma\backslash G} (\psi_1\circ a_t)\psi_2 \mathrm{d}m^{\mathrm{Haar}} = \frac{m^{\mathrm{BR}}(\psi_1)m_*^{\mathrm{BR}}(\psi_2)}{\|m^{\mathrm{BMS}}\|}.$$

Remark 5. (1) The result by Roblin [5, Thm. 3.4] proves local mixing for the geodesic flow, i.e., for M-invariant test functions.

(2) Mixing of BMS-measure for the frame flow [6, Thm. 1.1] combined with the argument of proof for [5, Thm. 3.4] extends the result from M-invariant to general test functions.

(3) The argument applied in [5] actually shows that local mixing for the Haar measure is equivalent to mixing for the BMS measure.

Ingredients to the proof. The proof uses two main ingredients. The first ingredient is a local product structure for quasi-conformal measures with respect to the conditional measures on stable and unstable horospheres; cf. [5, Lem. 1.15]. The second ingredient is mixing of the flow for the conditional measure along horospheres [5, Cor. 3.2]. □

References

[1] M. Babillot, *On the mixing property for hyperbolic systems*, Israel J. Math. **129** (2002), 61–76.

[2] Y. Coudène, *Ergodic theory and dynamical systems*, translated from the 2013 French original by Reinie Erné. Universitext. Springer-Verlag London, Ltd., London; EDP Sciences, [Les Ulis], 2016.

[3] A. Mohammadi and H. Oh, *Matrix coefficients, counting and primes for orbits of geometrically finite groups*, J. Eur. Math. Soc. (JEMS) **17** (2015), no. 4, 837–897.
[4] J.-P. Otal and M. Peigné, *Principe variationnel et groupes kleiniens*, Duke Math. J. **125** (2004), 15–44.
[5] Th. Roblin, *Ergodicité et équidistribution en courbure négative*, Mém. Soc. Math. Fr. (N.S.) No. 95 (2003).
[6] D. Winter, *Mixing of frame flow for rank one locally symmetric spaces and measure classification*, Israel J. Math. **210** (2015), 467–507.

Exponential decay of matrix coefficients: the case of the critical exponent bigger than $\frac{d}{2}$

Antje Dabeler

In this talk we studied the asymptotic behavior of matrix coefficients of representations of $G = \mathrm{SO}^0(d+1,1)$ according to the work of Edwards and Oh [1]. These results can be used to study the exponential mixing of the geodesic flow with respect to the Bowen-Margulis-Sullivan measure [1, Theorem 1.2].

The aim of this talk was to understand the following theorem [1, Theorem 4.8].

Theorem 1 (Edwards-Oh)**.** *There exists $m \in \mathbb{N}$ such that for any complementary series representation $U(v,s)$ containing a non-trivial M-invariant vector, for all $u, w \in S^m(v,s)$ and $t \geq 0$,*

$$(1)\quad \langle U(v,s)(a_t)u, w\rangle_{U(v,s)} = e^{(s-d)t}\left(\sum_{\tau_1,\tau_2\in\widehat{K}} \langle T_{\tau_1}^{\tau_2} c_+(s) P_{\tau_1} u, P_{\tau_2} w\rangle_{U(v,s)}\right) + \mathcal{O}\left(e^{(s-d-\eta_s)t}||u||_{S^m(v,s)}||w||_{S^m(v,s)}\right)$$

and the sum converges absolutely. Moreover, the implied constant is uniformly bounded over s in compact subsets of the interval $\mathcal{I}_v$.

Here, c_+ denotes the Harish-Chandra c-function, $S^m(v,s)$ the Sobolev space associated to $U(v,s)$ and P_τ the orthogonal projection onto the K-type τ. The parameter η_s is given by $\min\{2s-d,1\}$.

An important ingredient is the so-called spectral gap property for the quasi-regular representation $L^2(G/\Gamma)$, where Γ is a discrete torsion-free geometrically finite subgroup of G with critical exponent $\delta > \frac{d}{2}$. Before we give the result, we recall the construction of complementary series representations of G.

All representations of G were classified by Hirai [2]. The complementary series representations can be written as

$$(2)\qquad \mathrm{Ind}_{MAN}^{G}(v, \exp s, 1_N),$$

with $v \in \widehat{M}$ an irreducible unitary representation of M and $s \in \mathcal{I}_v \subseteq \left(\frac{d}{2}, d\right)$. Here MAN is the Langlands decomposition of a parabolic subgroup of G. The interval $\mathcal{I}_v$ is determined by the highest weight of v:

- n even: $\left(j_1, \ldots, j_{\frac{d}{2}}\right) \in \mathbb{Z}^{\frac{d}{2}}$ with $j_1 \geq j_2 \geq \cdots \geq \left|j_{\frac{d}{2}}\right|$,
- n odd: $\left(j_1, \ldots, j_{\frac{d-1}{2}}\right) \in \mathbb{Z}^{\frac{d-1}{2}}$ with $j_1 \geq j_2 \geq \cdots \geq j_{\frac{d-1}{2}} \geq 0$.

Define $l = l(v) := \min\{l' \mid j_{l'} \neq 0\}$ and set $l(1_M) = 0$. Then, $\mathcal{I}_v := \left(\frac{d}{2}, d-l\right)$.

Now, we can look at the quasi-regular representation $\rho : G \to \mathcal{B}\left(L^2(G/\Gamma)\right)$,

(3) $$(\rho(g)f)(x) = f(xg) \text{ for } f \in L^2(G/\Gamma), g \in G, x \in G/\Gamma.$$

The following theorem is due to Lax and Phillips [3],

Theorem 2 (Lax-Phillips)**.** *The intersection of the spectrum of the Laplace-Beltrami operator with the interval* $\left[0, \frac{d^2}{4}\right]$*, viewed as an unbounded operator on* $L^2(\Gamma\backslash\mathbb{H}^{d+1})$*, consists of a finite set of eigenvalues* $\{\lambda_i = s_i(d-s_i)\}_{0\leq i\leq l}$ *satisfying* $0 < \lambda_0 = \delta(d-\delta) < \lambda_1 \leq \cdots \leq \lambda_l < \frac{d^2}{4}$.

From this result, Mohammadi and Oh concluded that ρ has the strong spectral gap property [4, Proposition 3.24]:

Proposition 3. $L^2(\Gamma\backslash G)$ *does not weakly contain any complementary series representation* $U(v,s)$ *with* $v \in \widehat{M}$ *and* $s > \delta$.

The proof of Theorem 1 can be divided into three steps:

(1) Asymptotic expansion of the matrix coefficients with respect to $\langle\ ,\ \rangle_K$ (the scalar product on $L^2(K)$, where $K \subseteq G$ is a maximal compact subgroup),
(2) Intertwining operators,
(3) Asymptotic expansion of the matrix coefficients with respect to $\langle\ ,\ \rangle_{U(v,s)}$ (the scalar product making $U(v,s)$ unitary).

In the talk we focused our attention on the second part of the proof.

Edwards and Oh [1, Section 4.2] study the operators $\mathcal{A}(v,s)$ satisfying

- $\langle u, w\rangle_{U(v,s)} = \langle u, \mathcal{A}(v,s)w\rangle_K$ for all K-finite vectors $u, w \in U(v,s)$,
- $\mathcal{A}(v,s)U^s(g) = U^{d-s}(g)\mathcal{A}(v,s)$ for all $g \in G$.

By Schur's lemma, $\mathcal{A}$ acts as a scalar on the K-types of $U(v,s)$, i.e. $\mathcal{A}(v,s) = \sum_{v\subset\tau} a(v,s,\tau)P_\tau$. The relation $v \subset \tau$ is defined, for $v \in \widehat{M}$ and $\tau \in \widehat{K}$, using the interlacing property relating the highest weights of v and τ, respectively: Let $v = \left(v_1, \ldots, v_{\lfloor\frac{d}{2}\rfloor}\right)$ and $\tau = \left(\tau_1, \ldots, \tau_{\lfloor\frac{d}{2}\rfloor}\right)$. Then they satisfy the interlacing property, if,

(4) $$\text{for } d = 2m-1 : \tau_1 \geq v_1 \geq \tau_2 \geq v_2 \geq \cdots \geq v_{m-1} \geq |\tau_m|,$$

(5) $$\text{for } d = 2m : \tau_1 \geq v_1 \geq \tau_2 \geq v_2 \geq \cdots \geq \tau_m \geq |v_m|.$$

For a complementary series representation $U(v,s)$ with non-trivial M-invariant vector, we can deduce that v has the form $(v_1, 0, \ldots, 0)$ and subsequently, that every K-type of $U(v,s)$ has the form $(t_1, t_2, 0, \ldots, 0)$. With these properties, we can bound $\frac{a(v,s,\tau_2)}{a(v,s,\tau_1)}$ for any two K-types τ_1 and τ_2 of $U(v,s)$ (see [1, Proposition 4.6]).

Thus, the second step allows us to transfer the results from the first step with respect to $\langle\ ,\ \rangle_K$ to the scalar product $\langle\ ,\ \rangle_{U(v,s)}$.

References

[1] S. Edwards and H. Oh, *Spectral gap and exponential mixing on geometrically finite hyperbolic manifolds*, Duke Math. J. **170** (2021), 3417–3459.
[2] T. Hirai, *On irreducible representations of the Lorentz group of n-th order*, Proc. Japan Acad. **38** (1962), 258–262.
[3] P.D. Lax and R.S. Phillips, *The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces* , J. Funct. Anal. **46** (1982), 280–350.
[4] A. Mohammadi and H. Oh, *Matrix coefficients, counting and primes for orbits of geometrically finite groups*, J. Eur. Math. Soc. **17** (2015), 837–897.

Uniform exponential decay of matrix coefficients: the case of the critical exponent bigger than $d/2$

Zhicheng Han

The content of this talk is closely related to the previous talk in the report on exponential decay of matrix coefficients in the case of a single geometrically finite hyperbolic manifold. For the sake of completeness we reiterate the theorem here:

Theorem 1 ([2])**.** *Given Γ a discrete subgroup of $G = SO^0(d+1,1)$ with critical exponent δ. Assuming $\delta > d/2$ and set $\eta := \min\left\{\delta - \frac{d}{2} + \sqrt{(\frac{d}{2})^2 - \lambda_1}, 1\right\}$. Then there exists $m > \frac{d(d+1)}{2}$ such that for any $\epsilon > 0$ and functions ψ_1, ψ_2 on $T^1(\mathcal{M})$ with $\|\psi_1\|_{\mathcal{S}^m}, \|\psi_2\|_{\mathcal{S}^m} < \infty$, then we have as $t \to +\infty$:*

$$e^{(d-\delta)t}\int_{T^1(\mathcal{M})}\psi_1(\mathcal{G}^t(x))\psi_2(x)dx = \frac{1}{m^{\mathrm{BMS}}(T^1(\mathcal{M}))}m^{\mathrm{BR}}(\psi_1)m^{\mathrm{BR}_*}(\psi_2) + \mathrm{error}$$

with explicit error term expressed in L^2-Sobolev norm: $O_\epsilon(e^{(-\eta+\epsilon)t}\|\psi_1\|_{\mathcal{S}^m}\cdot\|\psi_2\|_{\mathcal{S}^m})$. Similarly for ψ_1, ψ_2 bounded function on $T^1(\mathcal{M})$ supported on 1-neighbourhood of $\mathrm{supp}(m^{\mathrm{BMS}})$ we have similar asymptotic behaviour as $t \to \infty$:

$$\int_{T^1(\mathcal{M})}\psi_1(\mathcal{G}^t(x))\psi_2(x)dm^{\mathrm{BMS}}(x) = \frac{1}{m^{\mathrm{BMS}}(T^1(\mathcal{M}))}m^{\mathrm{BMS}}(\psi_1)m^{\mathrm{BMS}}(\psi_2) + \mathrm{error}_1$$

with error term by C^m norm $O(e^{-\beta t}\|\psi_1\|_{C^m}\|\psi_1\|_{C^m})$ and explicitly computable term $\beta = \beta(\eta) > 0$.

Recall $\mathcal{G}^t$ the geodesic flow on unit tangent bundle of the geometrically finite hyperbolic manifold $\mathcal{M} := \Gamma\backslash G/K = \Gamma\backslash\mathbb{H}^d$, and $dx, m^{\mathrm{BMS}}, m^{\mathrm{BR}}, m^{\mathrm{BR}_*}$ denotes the Liouville, the Bowen-Margulis-Sullivan, the unstable and the stable Burger-Roblin measures on $T^1(\mathcal{M})$ respectively. We refer the readers to the talk by Antje Gabeler in this report as well as the original paper cited here for further details.

The main result of this talk is to yield an uniform exponent η and β for (almost) all congruence subcovers. Consider now Γ a Zariski-dense subgroup of an arithmetic subgroup $G(\mathbb{Z})$ of G, in particular the case when $\delta > 1/2$ satisfy this condition. Denote $\Gamma_q = \left\{\gamma \in \Gamma \middle| \gamma \equiv e \mod q\right\}$ the q-th congruence subgroup, then one can uniformly bound the decay rate among those good q's

Theorem 2 ([2, Corollary 1.3]). *The error exponents η and η in Theorem 1 can be chosen uniformly over all congruence covers $T^1(\Gamma_q\backslash\mathbb{H}^{d+1})$ for all q square-free integers with no factors in a fixed set S. We denote this set of q as $\mathcal{F}$.*

The key step to obtain such an uniform bound is to observe the spectral gap of the Laplacian on congruence subcovers, is equivalent to the property of all finite quotients forming an expander family. This is commonly known as the archimedean spectral gap is equivalent to the combinatorial spectral gap. The theorem in case of real hyperbolic manifolds can be explicitly stated as follows:

Theorem 3 (modified [1, Theorem 1.2]). *Let $G = O(d+1, 1)$ and Γ be its geometrically finite discrete subgroup with $\delta(\Gamma) > \frac{d}{2}$ generated by a finite symmetric set S. Then the Cayley graphs $\mathrm{Cay}(\Gamma/\Gamma_q, S \mod q)$ form a family fo expanders if and only if there is a uniform spectral gap among subcovers, i.e., there is a $\varepsilon(\Gamma) > 0$ such that $\inf_{q\in\mathcal{F}} \lambda_1(\Gamma/\Gamma_q) - \lambda(\Gamma/\Gamma_q) \geq \epsilon$.*

This is a slightly generalized version of [1]. Assuming this, theorem 2 can be readily derived from the following theorem of Salehi Golsefidy and Varjú, the original version of which we state here can be adapted to a much more general scene:

Theorem 4 ([4, Corollary 6]). *Let $\Gamma \subseteq GL_d(k)$ be the group generated by a symmetric set S, where k is a number field. If the Zariski-closure of Γ is semisimple, then $\mathrm{Cay}(\pi_{\mathfrak{q}}(\Gamma), \pi_{\mathfrak{q}}(S))$ form a family of expanders when $\mathfrak{q}$ ranges over square-free ideals of the ring of integers $\mathcal{O}_k$ with large prime factors.*

We now give a brief sketch of the proof of theorem 3. Note the necessary condition was proved in [3] using Fell's continuity argument, so we only sketch the sufficient condition here. Strategy of general proof is similar to that of $SL(2) = SO(2,1)$-case. We hence satisfied with sketching the original proof, aside with remarks of generalizations. A detailed version can be found in [1] and [5] in a more detailed form.

First note it suffices to restrict our attention to the case when $\Gamma/\Gamma_q = SL_2(\mathbb{F}_q)$. This (and the general case) follows from [7] for q large enough. Fix now a fundamental domain $\mathcal{F}$ of Γ in $\mathbb{H}^2$. Then any functions $f \in \Gamma_q\backslash\mathbb{H}^2$ can be lifted to a Γ_q-invariant function $\tilde{f}$ on $\mathbb{H}^2$. Consequently such f can be regarded as vector valued function F defined on $\Gamma\backslash\mathbb{H}^2$ by setting:

$$F(z) = (\tilde{f}(\gamma z))_{\gamma\in\Gamma/\Gamma_q}$$

with $z \in \mathcal{F}$ identified with $\Gamma\backslash\mathbb{H}^2$. Note this is equivalent to the right regular action of $SL_2(\mathbb{F}_q)$ on F: $R(\gamma) \cdot F(z) = F(\gamma z)$. We denote such space as H and denote H_0 its subspace orthogonal to λ_0-eigenspaces. Then the archimedean spectral gap is equivalent to the claim that for any F in H_0:

$$\frac{\int_{\mathcal{F}} \|\nabla F\|^2 \, d\mu}{\int_{\mathcal{F}} \|F\|^2 \, d\mu} \geq \lambda_0 + \varepsilon \tag{1}$$

Fix now S the generating set of Γ. Then the expander graph property amounts to say for each $z \in \mathcal{F}$ and for all $F \in H_0$, there exists a $\gamma \in S$ and $\epsilon = \epsilon(S) > 0$

$$\|F(z) - F(\gamma z)\|^2 \geq \lambda_1(G(\mathbb{F}_q)) \|F(z)\|^2$$

Observe now we can focus on the $f = \|F\|$ being the λ_0-eigenspaces, as otherwise we can orthogonally decompose f into λ_0-eigenvectors and its orthogonal complement b_0, and then computing 1 above already gives a nontrivial bound away from λ. Note here we make a critical use of [6], which in particular states the hyperbolic manifolds has 1-dimensional λ_0-eigenspaces.

In the later case, this amounts to choose area of integration carefully, and by integrating the gradient along geodesic perpendicular boundary direction of the fundamental domain, and by choosing Fermi coordinates with respect to these. Then assuming 1 contradicts with the expander property, which amounts to say teh derivative is lower bounded. Therefore the theorem is proved.

we conclude this discussion by mentioning a few results branched out from this. First a similar result of theorem 1 was derived in the case Γ a convex cocompact subgroup of $SO^0(d, 1)$ by [8] and [9]. See also the article by Emilio Corso in this report for more details.

It seems the whole story of uniform spectral gap (in the spirit of Selberg's 3/16 theorem) ends at the critical exponent greater than $\delta(\Gamma) > d/2$ for twofold reason: First the Laplacian on $\Gamma\backslash\mathbb{H}^d$ have no discrete spectrum between $[0, d^2/4)$ so the natural replacement is by studying the resonances as poles of the meromorphic continuation of the resolvent. We refer the reader to [1, Section 5 to 12] for more details.

All the theorems cited in this report have already seen many applications. To name a few, the uniform spectral gap is a crucial ingredient in the execution of affine linear sieve in the archimedean norm, whereas the uniform exponential mixing has seen its application to integral Apollonian packings. The readers is welcome to the reference in cited articles for more details.

References

[1] Bourgain, J., Gamburd, A., and Sarnak, P. Generalization of Selberg's $\frac{3}{16}$ theorem and affine sieve. *Acta Math. 207*, 2 (2011), 255–290.

[2] Edwards, S., and Oh, H. Spectral gap and exponential mixing on geometrically finite hyperbolic manifolds. *Duke Math. J. 170*, 15 (oct 2021).

[3] Gamburd, A. On the spectral gap for infinite index "congruence" subgroups of SL2(Z). *Isr. J. Math.* (2002).

[4] Golsefidy, A. S., and Varjú, P. P. Expansion in perfect groups. *Geom. Funct. Anal. 22*, 6 (2012), 1832–1891.

[5] Kim, I. Counting, mixing and equidistribution of horospheres in geometrically finite rank one locally symmetric manifolds. *J. für die reine und Angew. Math. (Crelles Journal) 2015*, 704 (jan 2015).

[6] Lax, P. D., and Phillips, R. S. The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Funct. Anal.* (1982).

[7] Matthews, C. R., Vaserstein, L. N., and Weisfeiler, B. Congruence properties of zariski-dense subgroups I. *Proc. London Math. Soc.* (1984).

[8] Oh, H., and Winter, D. Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$. *J. Am. Math. Soc. 29*, 4 (nov 2015), 1069–1115.
[9] Sarkar, P. Generalization of Selberg's 3/16 theorem for convex cocompact thin subgroups of $\mathrm{SO}(n,1)$.

Beyond Expansion Lecture I: Einsiedler-Lindenstrauss-Michel-Venkatesh Problem

Wooyeon Kim

In this talk, we describe the formulation and sieve method solution [2] to the ELMV problem [3] on low lying fundamental geodesics on the modular surface.

Let $\mathbb{H} = \{x+iy : y > 0\}$ denote the Poincaré upper half plane, let $T^1\mathbb{H}$ be its unit tangent bundle with Riemannian metric $\|\zeta\|_z := \frac{|\zeta|}{\mathrm{Im} z}$, where $z \in \mathbb{H}$ is the position and $\zeta \in T_z\mathbb{H} \cong \mathbb{C}$ with $\|\zeta\|_z = 1$ is the direction vector. The fractional linear action of the group $G = PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/\{\pm I\}$ on $\mathbb{H}$ is defined by $g(z,\zeta) := \left(\frac{az+b}{cz+d}, \frac{v}{(cz+d)^2}\right)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ and $(z,\zeta) \in T^1\mathbb{H}$. Let $\Gamma = PSL_2(\mathbb{Z})$ and X be the unit tangent bundle of the modular surface $\Gamma\backslash\mathbb{H}$. The above action of G on $T^1\mathbb{H}$ induces the identification $X \cong \Gamma\backslash G$. Under this identification, the time-t geodesic flow on X corresponds to right multiplication by $a_t := \begin{pmatrix} e^{\frac{t}{2}} & \\ & e^{-\frac{t}{2}} \end{pmatrix}$ on $\Gamma\backslash G$. We now state the main result of [2]:

Theorem 1. *There exists a compact region $Y \subset X$ such that Y contains infinitely many fundamental closed geodesics.*

It is not yet clear what "a closed geodesic is fundamental" means. To clarify this, we need a correspondence between closed geodesics in X and equivalence classes of integral binary quadratic forms. In other words, given closed geodesic, we can attach a equivalence class of binary quadratic form as follows: Pick a point $\Gamma g \in \Gamma\backslash G$ on the given closed geodesic $\gamma \in X$, then there exists $l > 0$ and $M \in \Gamma$ such that $ga_l = Mg$. The hyperbolic matrix M is determined up to Γ-conjugation and $\mathrm{tr}M = 2cosh\frac{l}{2}$. We also define the visual point α from g by

$$\alpha = \lim_{t\to\infty} ga_t \cdot i = \frac{a-d+\sqrt{\mathrm{tr}^2 M - 4}}{2c},$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $Q(x,y)$ be the primitive integral binary quadratic form such that α is a root of $Q(x,1) = 0$. More explicitly, we take $Q(x,y) = \frac{\mathrm{sgn}(\mathrm{tr}(M))}{\gcd(c,d-a,b)}(cx^2 + (d-a)xy - by^2)$. We observe that this binary quadratic form is determined up to $SL_2(\mathbb{Z})$-change of coordinates. It is said that the geodesic γ is fundamental if the discriminant $D \in \mathbb{N}$ of $Q(x,y)$ is fundamental, i.e. D is the discriminant of a real quadratic field $K_D = \mathbb{Q}(D)$.

We now give a brief sketch of the proof of Theorem 1. Theorem 1 is actually from the following stronger statement.

Theorem 2. *For any $\epsilon > 0$, there exists a compact region $Y \subset X$, $\mathcal{D} \subset \{$fund. disc.$\}$ such that*

- *For any $D \in \mathcal{D}$, $|\{\gamma \in \mathcal{C}_D : \gamma \subset Y\}| > |\mathcal{C}_D|^{1-\epsilon}$,*
- *For sufficiently large $T > 0$, $|\mathcal{D} \cap [1, T]| \gg T^{\frac{1}{2}-\epsilon}$.*

Here, $\mathcal{C}_D$ is the class group of $\mathbb{Q}(D)$ and the elements of $\mathcal{C}_D$ can be considered as closed geodesics with discriminant D.

The visual point $\alpha \in \mathbb{R}$ of the closed geodesic γ is a quadratic irrational. Let $\alpha = [\overline{a_0, a_1, \cdots, a_k}]$ be its continued fraction which is eventually periodic (by an appropriate choice of g, we may assume it is periodic). Note that the corresponding hyperbolic matrix $M \in \Gamma$ of γ is then $\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_k & 1 \\ 1 & 0 \end{pmatrix}$. It is also well-known that the excursion of γ to the cusp is controlled by the upper bound of partial quotients a_i's.

Fact. For any compact region $Y \subset X$, there exists $A_Y \in \mathbb{N}$ such that if γ is contained in Y, then $a_i \leq A_Y$ for any $1 \leq i \leq k$.

To make use of this fact, we consider a thin group in $SL_2(\mathbb{Z})$

$$\Gamma_A := \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} : 1 \leq a \leq A \right\rangle^+ \cap SL_2(\mathbb{Z}),$$

where the superscript "+" denotes generation as a semigroup. Here are useful facts about Γ_A:

- Γ_A is Zariski dense if $A \geq 2$,
- $|\Gamma_A \cap B_N| \asymp N^{2\delta_A}$, where B_N is the N-ball in $SL_2(\mathbb{Z})$ with respect to Frobenius norm and $\delta_A := \mathrm{Hdim}\{[a_0, a_1, a_2, \cdots] : 1 \leq a_j \leq A\}$ [4],
- $\delta_A = 1 - \frac{6}{\pi^2 A} + o(\frac{1}{A})$ [5].

A sufficient condition of being fundamental is that $D = \mathrm{tr}^2(M) - 4$ is square-free. Using the above **Fact**, one can deduce Theorem 2 from the following proposition.

Proposition 3. *For any $\eta > 0$, there exists $A(\eta) \in \mathbb{N}$ such that*

$$|\{M \in \Gamma_A \cap B_N : tr^2(M) - 4 \text{ is square-free}\}| > N^{2\delta_A - \eta}.$$

The main ingredients of the proof of this proposition are sieving theorems and "spectral gap" or "expander" property of Γ_A. A theorem of [1] roughly says that we have equidistribution in Γ_A mod q with power saving error rate. In our setting, we have the following estimate:

$$|\{M \in \Gamma_A \cap B_N : \mathrm{tr}^2(M) - 4 \equiv 0 (\mathrm{mod}\ q)\}| = \frac{1}{q}|\Gamma_A \cap B_N| + O(N^{2\delta_A - \theta})$$

for some $\theta > 0$. Applying the sieving theorem for this estimate, one can count the number of M's such that $\mathrm{tr}^2(M) - 4$ is almost-prime, and finally get the estimate of Proposition 3.

References

[1] Jean Bourgain, Alex Gamburd, and Peter Sarnak, *Generalization of Selberg's $\frac{3}{16}$th theorem and affine sieve*, Acta Math, 207, (2011), 255–290.
[2] Jean Bourgain and Alex Kontorovich, *Beyond expansion II: low-lying fundamental geodesics*, J. Eur. Math. Soc. (JEMS), 19, **5** (2017), 1331–1359.
[3] Manfred Einsiedler, Elon Lindenstrauss, Philippe Michel, and Akshay Venkatesh, *Distribution of periodic torus orbits on homogeneous spaces*, Duke Math. J, 148, **1** (2009), 119–174.
[4] Doug Hensley, *The distribution of badly approximable numbers and continuants with bounded digits*, In Théorie des nombres (Quebec, PQ, 1987), de Gruyter, Berlin, (1989), 371–385.
[5] Doug Hensley, *Continued fraction Cantor sets, Hausdorff dimension, and functional analysis*, J. Number Theory, 40, **3** (1992), 336–358

On Spectral Gap of Compact Simple Lie Groups

Constantin Kogler

Let $G \subset \mathrm{GL}_d(\mathbb{C})$ be a compact connected simple Lie group (e.g. $G = \mathrm{SU}(d)$) and consider a symmetric Borel probability measure μ on G. We aim to understand the distribution of μ^{*n}.

Definition 1. The symmetric measure μ is called **non-degenerate** if its support generates a dense subgroup of G.

For example consider the measure

$$\mu_{g_1,g_2} = \frac{1}{4}(\delta_{g_1} + \delta_{g_1^{-1}} + \delta_{g_2} + \delta_{g_2^{-1}}). \tag{1}$$

As a Lie group has only countably many conjugacy classes of closed subgroups and using that G is connected, it follows that μ_{g_1,g_2} is almost surely non-degenerate for g_1 and g_2 to be chosen uniformly at random from G.

If μ is non-degenerate, then we show below that μ^{*n} equidistributes to the Haar measure m_G on G, i.e. for $f \in C(G)$,

$$\lim_{n\to\infty} \int f \, d\mu^{*n} \longrightarrow \int f \, dm_G. \tag{2}$$

Denote by λ_G the left-regular representation of G and let $\widehat{G}$ be the set of irreducible representations of G. As G is compact, every irreducible representation is finite dimensional and the Peter-Weyl Theorem holds:

$$\lambda_G = \bigoplus_{\pi \in \widehat{G}} \pi^{\oplus \dim \pi}. \tag{3}$$

Furthermore we have the Fourier inversion formula for $f \in C^\infty(G)$ and all $g \in G$,

$$f(g) = \sum_{\pi \in \widehat{G}} \dim(\pi) \langle \pi(f), \pi_g \rangle_{\mathrm{HS}}, \tag{4}$$

where HS is the Hilbert-Schmidt inner product, i.e. $\langle T, S\rangle_{\mathrm{HS}} = \mathrm{tr}(TS^*)$ for T, S bounded operators on a finite dimensional complex vector space.

Lemma 2. *Let μ be a symmetric non-degenerate probability measure on a compact group G and let $\pi \in \widehat{G}$ be an irreducible non-trivial representation. Then*

$$||\pi(\mu)||_{\mathrm{op}} < 1.$$

Proof. As π is finite dimensional and μ symmetric, the operator $\pi(\mu)$ is diagonalizable. Furthermore μ is a probability measure, implying $||\pi(\mu)||_{\mathrm{op}} \leq 1$. It thus suffices to show that $\pi(\mu)$ has no eigenvalue of modulus one. Assume for a contradiction that $\pi(\mu)v = \int \pi_g v \, d\mu(g) = \lambda v$ for a non-zero element $v \in \mathscr{H}_\pi$ and $\lambda \in \mathbb{S}^1$. As π is finite-dimensional and $||\pi_g v|| = ||\lambda v|| = ||v||$, it must hold that $\pi_g v = \lambda v$ for almost all $g \in \mathrm{supp}(\mu)$. Now consider the subgroup

$$H = \{g \in G \,:\, \pi_g v = v\}$$

and fix an element $g_0 \in \mathrm{supp}(\mu)$ such that $\pi_{g_0} v = \lambda v$. Then almost surely $\mathrm{supp}(\mu) \subset g_0 H$. As μ is symmetric, $\mathrm{supp}(\mu)^2 \subset (g_0 H)^{-1}(g_0 H) = H$, showing that $G = H$ as μ is non-degenerate. However this implies that $\langle v \rangle$ is an invariant subspace, contradicting the assumption that π is irreducible and non-trivial. □

Lemma 2 together with (3) or (4) implies (2). We next want to discuss whether one can prove Lemma 2 uniformly over all non-trivial irreducible representations of G. This leads to the notion of spectral gap. Notice that the Peter-Weyl Theorem has as a consequence

$$||\lambda_G(\mu)|_{L^2_0(G)}||_{\mathrm{op}} = \sup_{\pi \in \widehat{G} \setminus \{1_G\}} ||\pi(\mu)||_{\mathrm{op}}.$$

Definition 3. The measure μ has **spectral gap** if

$$||\lambda_G(\mu)|_{L^2_0(G)}||_{\mathrm{op}} < 1.$$

Conjecture 4. *(Spectral Gap Conjecture) Let μ be a symmetric probability measure on a compact connected simple Lie group. Then μ has spectral gap if and only if μ is non-degenerate.*

As argued above, spectral gap implies equidistribution of μ^{*n} and hence that μ is non-degenerate. Equivalently, if the support of μ is trapped in a closed subgroup of G, then μ cannot have spectral gap. The Spectral Gap Conjecture claims that this is the only obstruction to μ having a spectral gap. In particular, in view of the discussion around (1), the conjecture claims that spectral gap is a generic condition.

If μ has spectral gap, using (4) one can deduce equidistribution of μ^{*n} with exponential speed. Namely there is $c > 0$ such that for $f \in C^\infty(G)$,

$$\int f \, d\mu^{*n} = \int f \, dm_G + O(\mathcal{S}(f)e^{-cn}),$$

where $\mathcal{S}(f)$ is a Sobolev norm of sufficiently high degree on G.

Furthermore, if μ has spectral gap, one has exponential mixing of the operator $\lambda_G(\mu)$: For $f_1, f_2 \in L^2(G)$,

$$|\langle \mu^{*n} * f_1, f_2 \rangle - \langle f_1, f_2 \rangle| \leq e^{-cn} ||f_1||_2 \, ||f_2||_2 \tag{5}$$

for $c = -\log ||\lambda_G(\mu)|_{L^2_0(G)}||_{\text{op}}$.

A consequence of (5) is that μ^{*n} gives only very little mass to small sets. For instance for $\varepsilon > 0$ small,

$$\mu^{*n}(B_\varepsilon(e)) \ll \varepsilon^{\dim(G)} + e^{-cn}. \tag{6}$$

If $\varepsilon = e^{-c_1 n}$, then the right hand side of (6) decays exponentially fast. A similar conclusion holds if $B_\varepsilon(e)$ is replaced by $B_\varepsilon(H) = \{x \in G : d(x,H) < \varepsilon\}$ for H a closed subgroup of G. This shows that if μ has spectral gap, then G satisfies the weak Diophantine property defined as follows:

Definition 5. The measure μ is called **weakly Diophantine** if there exist $c_1, c_2 > 0$ such that for n large enough

$$\sup_{H<G} \mu^{*n}(B_{e^{-c_1 n}}(H)) \leq e^{-c_2 n},$$

where the supremum is taken over all closed subgroups of G.

The weak Diophantine property is a non-commutative generalization of the notion of a Diophantine number on the real line (see introduction of [2]). The main result of this talk reduces spectral gap to the weak Diophantine property.

Theorem 6. *(*[3] *for* $G = \mathrm{SU}(d)$*,* [5] *for general case) Let* μ *be a symmetric probability measure on a compact connected simple Lie group. Then* μ *has spectral gap if and only if* μ *is weakly Diophantine.*

It remains to study the weak Diophantine property in more detail.

Theorem 7. *(*[3] *for* $G = \mathrm{SU}(d)$*,* [5] *for general case) Let* μ *be a finitely supported non-degenerate probability measure on* G*. Assume that* $\mathrm{supp}(\mu)$ *consists of matrices with algebraic entries. Then* μ *is weakly Diophantine. In particular,* μ *has spectral gap.*

The final consequence of Theorem 7 was first proven by Bourgain and Gamburd for $SU(2)$ in [2] for measures of the form μ_{g_1,g_2} as defined in (1) for $g_1, g_2 \in SU(2) \cap M_2(\overline{\mathbb{Q}})$ generating a free subgroup. They generalized their results to $SU(d)$ in [3]. Building on the method by Bourgain and Gamburd, Benoist and de Saxcé clarified in [5] the relationship between spectral gap and the weak Diophantine property and further generalized the results to simple Lie groups.

We briefly discuss the proofs of Theorem 6 and 7. Concerning Theorem 7, for simplicity consider the measure μ_{g_1,g_2} for g_1 and g_2 generating a dense subgroup of G. Then by Kesten's Theorem [6], since the group generated by g_1 and g_2 is non-amenable, it follows that

$$\mu^{*n}_{g_1,g_2}(e) \leq e^{-c_1 n} \tag{7}$$

for $c_1 > 0$. For the reader unfamiliar with Kesten's Theorem, consider the case when the group generated by g_1 and g_2 is free. Then $\mu^{*n}_{g_1,g_2}(e)$ is the probability that the natural symmetric random walk on the free group that starts at the identity returns to the identity after n steps, a quantity that decays exponentially by direct calculation.

Notice that the above argument does not require g_1 and g_2 to have algebraic entries, however in (7) we only show exponential decay at the identity. To pass to decay for an exponentially small neighborhood around the identity, as required by the weak Diophantine property, the current methods require the further assumption that $g_1, g_2 \in M_d(\overline{\mathbb{Q}})$. Denote by K the number field generated by the entires of g_1 and g_2. The height of the entries of $\mathrm{supp}(\mu_{g_1,g_2}^{*n})$ can be bounded by $e^{c_2 n}$ for some $c_2 > 0$. Using that there are no matrices in $B_{e^{-c_2 n}}(e) \cap M_d(K)$ with entries of height $\leq O(e^{[K:\mathbb{Q}]c_2 n})$ in $B_{e^{-c_2 n}}(e)$, it follows for $n \geq 1$ that

$$\mathrm{supp}(\mu_{g_1,g_2}^{*n}) \cap B_{e^{-c_2 n}}(e) = \{e\}, \tag{8}$$

by altering the constant c_2. Combining (7) and (8), the weak Diophantine property follows for the trivial subgroup. To further treat closed subgroups $H < G$, one requires stronger results from the theory of random matrix products to derive (7) for H and one moreover uses the effective Nullstellensatz for an analogue of (8) (see Proposition 3.3 and 3.12 in [5]).

Returning to Theorem 6, the strategy of proof is analogous to the method of Bourgain and Gamburd to show expansion of Cayley graphs of finite simple groups of Lie type that was exposed in previous talks.

Denote $P_\delta = \frac{1_{B_\delta(e)}}{m_G(1_{B_\delta(e)})}$ the uniform probability measure on $B_\delta(e)$. For a given measure ν, we consider the δ-discretization of ν given as

$$\nu_\delta = \nu * P_\delta.$$

The Bourgain-Gamburd method relies on analyzing the behavior of μ^{*n} at scale δ (or δ-discretized) and to deduce uniform results for all scales. The following proposition shows that if μ^{*n} is flat enough at all scales (for n depending on the scale) then μ has spectral gap.

Proposition 8. *(Lemma 2.9 of* [5]*) Let μ be a non-degenerate probability measure on a compact semisimple Lie group. Assume that for δ small enough,*

$$||(\mu^{*C\log(\frac{1}{\delta})})_\delta||_2 \leq \delta^{-1/4}.$$

Then μ has spectral gap.

In analogy to quasirandomness for finite groups, the proof relies on high multiplicity of the irreducible representations of G.

Proof. (Sketch for SU(2)) Recall that for SU(2) there is a unique representation π_k for each dimension $k \geq 1$. By Lemma 2 it suffices to consider representations of large dimension. Using the Plancharel formula for SU(2) (which follows from (4)),

$$\sum_{k\geq 1} k||\pi_k(\mu)^{*C\log(\frac{1}{\delta})}\pi_k(P_\delta)||_{\mathrm{HS}}^2 = ||(\mu^{*C\log(\frac{1}{\delta})})_\delta||_2^2 \leq \delta^{-1/2}$$

As $||\pi_k(P_\delta) - \mathrm{Id}_k|| \leq k\delta$, we can omit the term $\pi_k(P_\delta)$ in the above expression for $k \sim \delta^{-1}$. Thus

$$||\pi_k(\mu)||_{\mathrm{HS}}^{2C\log(\frac{1}{\delta})} \ll k^{-1}\delta^{-1/2} = \delta^{1/2},$$

implying the claim for δ small enough and hence for k large enough. ☐

Finally, it is necessary to establish an analogue of Helfgott's Theorem for arbitrary subsets of G. Having proven the latter, the proof of Theorem 6 is similar to the setting of expander graph – using a scaled version of the Balog-Szemeredi-Gowers Lemma (see Lemma 2.5 of [5]).

The proof of Helfgott's Theorem relied on the Sum-Product Theorem [4] for $\mathbb{Z}/p\mathbb{Z}$. This theorem was inspired by the Erdös-Szemeredi Theorem, stating that there is $\varepsilon > 0$ such that for any finite subset $A \subset \mathbb{R}$,

$$|A + A| + |A \cdot A| \gg |A|^{1+\varepsilon}. \tag{9}$$

As we want to consider arbitrary subsets of G, as a preliminary step one needs to generalize (9) to arbitrary subsets of $\mathbb{R}$. This was achieved by Bourgain [1].

Instead of studying the cardinality of arbitrary subsets of $\mathbb{R}$, we, as before, discretize the setting and analyze the metric entropy $N_\delta(A)$, defined as the minimal number of δ-balls necessary to cover $A \subset \mathbb{R}$.

Theorem 9. *(Discretized Sum-Product Theorem* [1]*, Simplified Version) Given $\sigma \in (0,1)$ there is $\varepsilon = \varepsilon(\sigma) > 0$ such that the following holds for $\delta > 0$ small enough.*

Assume that $A \subset [0,1]$ satisfies:

(i) (Size Assumption) $\delta^{-(\sigma-\varepsilon)} \leq N_\delta(A) \leq \delta^{-(\sigma+\varepsilon)}$.
(ii) (Non-concentration) For all $x \in \mathbb{R}$ and $\rho \geq \delta$,

$$N_\delta(A \cap B_\rho(x)) \leq \delta^{-\varepsilon}\rho^{-\sigma}.$$

Then

$$N_\delta(A + A) + N_\delta(A \cdot A) \geq \delta^{-\varepsilon} N_\delta(A).$$

Assumption (i) is necessary in order to ensure that $N_\delta(A)$ can grow at least $\delta^{-\varepsilon}$, whereas (ii) excludes the counterexamples $[x, x+\rho]$ and unions of a small number of such sets.

As discussed above, one further needs to generalize Theorem 9 to subsets of G. This was done in [3] for $\mathrm{SU}(d)$ and in [7] for an arbitrary connected simple Lie group G. We refer the reader to the references for a further discussion of these results.

References

[1] Bourgain, J. of On the Erdős-Volkmann and Katz-Tao ring conjectures, Geom. Funct. Anal. **13** (2002), 334–365.
[2] Bourgain, J. and Gamburd, A., *On the spectral gap for finitely-generated subgroups of* $\mathrm{SU}(2)$, Invent. Math. **171** (2008), 83–121.
[3] Bourgain, J. and Gamburd, A., *A spectral gap theorem in* $\mathrm{SU}(d)$, J. Eur. Math. Soc. (JEMS) **14** (2012), 1455–1511.
[4] Bourgain, J. and Katz, N. and Tao, T., *A sum-product estimate in finite fields, and applications*, Geom. Funct. Anal. **14** (2004), 27–57.
[5] Benoist, Yves and de Saxcé, Nicolas, *A spectral gap theorem in simple Lie groups*, Invent. Math. **205** (2016), 337–361.

[6] Kesten, Harry, *Symmetric random walks on groups*, Trans. Amer. Math. Soc. **92** (1959), 336–354.
[7] de Saxcé, Nicolas, *A product theorem in simple Lie groups*, Geom. Funct. Anal. **25** (2015), 915–941.

Exponential mixing: the case of convex cocompact manifolds

Michael Chow

The aim of this talk is to present the exponential mixing of the Bowen-Margulis-Sullivan measure m^{BMS} for convex cocompact hyperbolic manifolds $\Gamma\backslash\mathbb{H}^n$. To state the exponential mixing theorem precisely, let $G = \mathrm{Isom}^+(\mathbb{H}^n) \cong \mathrm{SO}^\circ(n,1)$ and $\Gamma < G$ be a Zariski dense convex cocompact subgroup. Then the frame bundle of $\Gamma\backslash\mathbb{H}^n$ identifies with $\Gamma\backslash G$ and the frame flow corresponds to the action on the right by a one-parameter subgroup of semisimple elements $\{a_t\}_{t\in\mathbb{R}}$. Stoyanov [3] proved exponential mixing for the geodesic flow (on $\Gamma\backslash G/M$) and later this was extended to the frame flow by Sarkar-Winter [2]:

Theorem 1 (Exponential mixing)**.** *There exists $C, \eta > 0$ and $r \in \mathbb{N}$ such that for every $\varphi \in C_c^r(\Gamma\backslash G)$, $\psi \in C_c^1(\Gamma\backslash G)$ and $t > 0$,*

$$\left| \int_{\Gamma\backslash G} \varphi(xa_t)\psi(x)\mathrm{d}m^{\mathrm{BMS}}(x) - m^{\mathrm{BMS}}(\varphi)m^{\mathrm{BMS}}(\psi) \right| \leq C^{-\eta t}\|\varphi\|_{C^r}\|\psi\|_{C^1}$$

One of the applications of this result is to deduce exponential decay of matrix coefficients using Roblin's transvere intersection argument:

Theorem 2 (Exponential decay of matrix coefficients)**.** *There exists $\eta > 0$ and $r \in \mathbb{N}$ such that for every $\varphi \in C_c^r(\Gamma\backslash G)$, $\psi \in C_c^1(\Gamma\backslash G)$, there exists $C > 0$ depending only on* $\mathrm{supp}(\varphi)$ *and* $\mathrm{supp}(\psi)$ *such that for all $t > 0$,*

$$\left| e^{(n-1-\delta_\Gamma)t} \int_{\Gamma\backslash G} \varphi(xa_t)\psi(x)\mathrm{d}x - m^{\mathrm{BR}}(\varphi)m^{\mathrm{BR}_*}(\psi) \right| \leq C^{-\eta t}\|\varphi\|_{C^r}\|\psi\|_{C^1}$$

On the proof

Since Γ is convex cocompact, a classical construction of Bowen and Ratner gives the existence of a Markov section for the geodesic flow, giving a symbolic coding and allowing us to apply symbolic dynamics and thermodynamic formalism. Keeping track of the holonomy, we obtain a suspension space model for the frame flow. More precisely, we study the $\mathbb{R}$-translation flow on the suspension space $\Sigma^+ \times M \times \mathbb{R}/\sim$ where (Σ^+, σ) is a one-sided shift space on a finite alphabet and $(x, m, s) \sim (\sigma(x), m\vartheta(x)^{-1}, s - \tau(x))$, where $\tau : \Sigma^+ \to \mathbb{R}$ is the first return time map and $\vartheta : \Sigma^+ \to M$ is the holonomy map.

The main object of study then becomes transfer operators since by the classical work of Pollicott, we can prove exponential mixing of the geodesic flow from spectral bounds of transfer operators. In the geodesic flow case where we do not need to consider holonomy, for each complex parameter $\xi = a + ib \in \mathbb{C}$, we have a transfer operator $\mathcal{L}_\xi : C(\Sigma^+) \to C(\Sigma^+)$ defined by

$$\mathcal{L}_\xi(H)(x) = \sum_{x' \in \sigma^{-1}(x)} e^{\xi\tau(x')} H(x')$$

In the case of small frequencies $|b| \ll 1$, the required spectral bounds are a consequence of the Ruelle-Perron-Frobenius theorem and perturbation theory. For large frequencies $|b| \gg 1$, spectral bounds were obtained by Dolgopyat [1] first and his results were generalized by Stoyanov. One of the key ingredients to Dolgopyat's method is the local non-integrability condition (LNIC) which will tell us that the first return time map τ has large oscillations.

When considering the frame flow, we need to consider transfer operators with holonomy, that is, the transfer operators from before twisted by representations in the unitary dual $\hat{M}$ of M. More precisely, given $\xi \in \mathbb{C}$ and $\rho : M \to V_\rho$ in $\hat{M}$, the transfer operator with holonomy $\mathcal{L}_{\xi,\rho} : C(\Sigma^+, V_\rho^{\oplus \dim(\rho)}) \to C(\Sigma^+, V_\rho^{\oplus \dim(\rho)})$ is defined by

$$\mathcal{L}_{\xi,\rho}(H)(x) = \sum_{x' \in \sigma^{-1}(x)} e^{\xi\tau(x')} \rho(\vartheta(x')^{-1}) H(x')$$

The LNIC required in this setting must deal with an AM-valued map Φ which incorporates both the first return time map τ and the holonomy map ϑ. The LNIC is proved by using Lie theoretic arguments and Zariski denseness of Γ (hence, it is necessary to modify the above constructions using the smooth structure on G, but we will not address this), and it will tell us Φ has large oscillations when b is large or when ρ is nontrivial.

In addition to the the LNIC, in the frame flow case, a non-concentration property (NCP) is required as well. The NCP essentially says that for Zariski dense Γ, the limit set $\Lambda_\Gamma \subset \partial\mathbb{H}^n \cong \mathbb{R}^{n-1} \cup \{\infty\}$ does not concentrate along any particular direction:

Proposition (NCP). *There exists $\delta \in (0,1)$ such that for all $x \in \Lambda_\Gamma \cap \mathbb{R}^{n-1}$, $\varepsilon \in (0,1)$ and unit vector $w \in \mathbb{R}^{n-1}$, there exists $y \in \Lambda_\Gamma \cap B_\varepsilon(x)$ such that $|\langle y - x, w\rangle| \geq \varepsilon\delta$.*

Having the LNIC and NCP, we can follow the Dolgopyat's method and construct Dolgopyat operators from which we can deduce bounds for the transfer operator. The mechanism of Dolgopyat's method is summarized in the following theorem, where for $B > 0$,

$$K_B = \{h \in C^1(\Sigma^+) \mid h > 0, \|(dh)_x\|_{\mathrm{op}} \leq Bh(x) \text{ for all } x \in \Sigma^+\}.$$

Theorem 3 (Dolgopyat's method). *There exists $m \in \mathbb{N}$, $\tilde{\eta} \in (0,1)$, $E, a_0, b_0 > 0$ and for each $(b,\rho) \in \mathbb{R} \times \hat{M}$ with $|b| > b_0$ or $\rho \neq 1$, there exists a finite indexing set $\mathcal{J}(b,p)$ and a set of Dolgopyat operators*

$$\{\mathcal{N}_{a,J}^H : C^1(\Sigma^+) \to C^1(\Sigma^+) \mid H \in C^1(\Sigma^+, V_\rho^{\oplus \dim(\rho)}), |a| < a_0, J \in \mathcal{J}(b,\rho)\}$$

satisfying the following

(1) $\mathcal{N}_{a,J}(K_{E\|(b,\rho)\|}) \subset K_{E\|(b,\rho)\|}$ *(cone preserving)*
(2) $\|\mathcal{N}_{a,J}(h)\|_2 \leq \tilde{\eta}\|h\|_2$ *for all* $h \in K_{E\|(b,\rho)\|}$ *(uniform contraction)*
(3) If $h \in K_{E\|(b,\rho)\|}$ *satisfies*
- $\|H(x)\|_2 \leq h(x)$ *for all* $x \in \Sigma^+$ *and*
- $\|(dH)_x\|_{\text{op}} \leq E\|(b,\rho)\|h(x)$ *for all* $x \in \Sigma^+$*,*

then there exists $J \in \mathcal{J}(b,\rho)$ *such that*
- $\|\mathcal{L}^m_{\xi,\rho}(H)(x)\|_2 \leq \mathcal{N}_{a,J}(h)(x)$ *for all* $x \in \Sigma^+$ *and*
- $\|(d\mathcal{L}^m_{\xi,\rho}(H))_x\|_{\text{op}} \leq E\|(b,\rho)\|\mathcal{N}_{a,J}(h)(x)$ *for all* $x \in \Sigma^+$*,*

Given h and H satisfying the hypotheses of (3), by replacing h and H with $\mathcal{N}_{a,J}(h)$ and $\mathcal{L}^m_{\xi,\rho}(H)$, we see that (3) provides a mechanism for an induction argument and this will give the required bounds for the transfer operator, from which exponential mixing can be derived.

Theorem 4 (Spectral bounds for transfer operators)**.** *There exists* $C, \eta, a_0, b_0 > 0$ *such that if* $|a| < a_0$*, and* $|b| > b_0$ *or* $\rho \neq 1$*, then for every* $k \in \mathbb{N}$ *and for all* $H \in C(\Sigma^+, V_\rho^{\oplus \dim(\rho)})$*, we have*

$$\|\mathcal{L}^k_{\xi,\rho}(H)\| \leq Ce^{-\eta k}\|H\|_{1,\|(b,\rho)\|}$$

References

[1] D. Dolgopyat, *On decay of correlations in Anosov flows.* Annals of Math, **147** (1998), no. 2, 357–390.

[2] P. Sarkar, D. Winter *Exponential mixing of frame flow for convex cocompact hyperbolic manifolds.* Preprint, ArXiv:1612.00909.

[3] L. Stoyanov, *Spectra of Ruelle transfer operators for Axiom A flows.* Nonlinearity, **24** (2011), no. 4, 1089–1120.

Expansion of $\mathbf{SL_2}(\mathcal{O}_K/I)$ with I square-free

Jeroen Winkel

In this talk we will see how we can make expanders of Cayley graphs of quotients of the group $\mathrm{SL}_2(\mathbb{O}_K)$, where K is a number field. First, we will consider a theorem shown in 2008 by Bourgain and Gamburd [2]. For any prime p consider the projection $\pi_p \colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p)$, which is always surjective.

Theorem 1. *Let* $S \subseteq \mathrm{SL}_2(\mathbb{Z})$ *be a finite symmetric set, such that* $\langle S \rangle$ *is Zariski dense in* $\mathrm{SL}_2(\mathbb{Z})$*. Then the Cayley graphs* $\mathrm{Cay}(\mathrm{SL}_2(\mathbb{Z}/p), \pi_p(S))$ *form an expander sequence, where* p *ranges over all large enough primes.*

Today we talk about a theorem by Varjú generalizing this result in two directions: we will replace $\mathbb{Z}$ by a ring of integers and p by a square-free ideal. Before stating the theorem we need some definitions.

Definition 2. A number field is a field K containing $\mathbb{Q}$ such that the dimension $[K : \mathbb{Q}]$ is finite.

Any $\alpha \in K$ is algebraic, meaning that there are $a_0, \ldots, a_{k-1}$ such that $\alpha^k + a_{k-1}\alpha^{k-1} + \ldots + a_0 = 0$. Moreover, if we insist on k to be minimal, then the $a_0, \ldots, a_{k-1}$ will be unique. We say that α is an *algebraic integer* if all $a_0, \ldots, a_{k-1}$ are integers. The set of algebraic integers in K is called $\mathcal{O}_K$. It can be shown that this is a subring of K, called the *ring of integers*.

For example, if $K = \mathbb{Q}[i]$, we get $\mathcal{O}_K = \mathbb{Z}[i]$. More generally, if $K = \mathbb{Q}[\sqrt{m}]$ for a square-free integer m, we get $\mathcal{O}_K = \mathbb{Z}[\sqrt{m}]$ if m is 2 or 3 modulo 4, and $\mathcal{O}_K = \mathbb{Z}[\frac{1+\sqrt{m}}{2}]$ if m is 1 modulo 4.

We can do number theory with the ring of integers $\mathcal{O}_K$, instead of the usual integers $\mathbb{Z}$. We say an algebraic integer $p \in \mathcal{O}_K$ is *prime* if p is not invertible, and there are no $a, b \in \mathcal{O}_K$ that are not invertible with $p = ab$. Each algebraic integer can be written as a product of primes, but unfortunately, we do not have unique prime factorisation. For example, if $K = \mathbb{Q}[\sqrt{-5}]$, we have $6 = 2 \cdot 3 = (1+\sqrt{-5}) \cdot (1-\sqrt{-5})$. The algebraic integers $2, 3, 1+\sqrt{-5}, 1-\sqrt{-5}$ are all primes in $\mathcal{O}_K$.

The situation gets better if we look at ideals in $\mathcal{O}_K$. For all (non-trivial) ideals $I \subseteq \mathcal{O}_K$ there are *unique* prime ideals $P_1, \ldots, P_k$ such that $I = P_1 \cdot \cdots \cdot P_k$. For example in $\mathbb{Q}[\sqrt{-5}]$ we have $(6) = (2, 1+\sqrt{-5}) \cdot (2, 1-\sqrt{-5}) \cdot (3, 1-\sqrt{-5}) \cdot (3, 1+\sqrt{-5})$. The unique prime factorisation of 6 goes wrong exactly because these are not principal ideals. Now we can make some definitions.

Definition 3. Let $I, J \subseteq \mathcal{O}_K$ be ideals. We say I is square-free if all prime factors occur only once. We say I and J are coprime if the prime factors of I and J are disjoint (equivalently, $I + J = \mathcal{O}_K$). Let π_I be the map $\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(\mathcal{O}_K/I)$.

For a number field K of degree $[K : \mathbb{Q}] = r$, there are r different embeddings $\sigma_1, \ldots, \sigma_r \colon K \to \mathbb{C}$. Denote the direct sum by $\hat{\sigma} \colon K \to \mathbb{C}^r$. Also denote by $\hat{\sigma}$ the obvious map $\mathrm{SL}_2(\mathcal{O}_K) \to \mathrm{SL}_2(\mathbb{C})^r$. Now we can finally state the theorem.

Theorem 4. *Let K be a number field and let $S \subseteq \mathrm{SL}_2(\mathcal{O}_K)$ be a finite symmetric set. Let $\Gamma = \langle S \rangle$. Suppose $\hat{\sigma}(\Gamma)$ is Zariski dense in $\mathrm{SL}_2(\mathbb{C})^r$. Then there is an ideal $J \subseteq \mathrm{SL}_2(\mathcal{O}_K)$ such that the Cayley graphs $\mathrm{Cay}(\mathrm{SL}_2(\mathcal{O}_K/I), \pi_I(S))$ form an expander sequence, when I ranges over the square-free ideals coprime to J.*

We will now show the whole proof but we will show some theorems that go into it. Fix the notation $G = \mathrm{SL}_2(\mathcal{O}_K)$ and $G_I = \mathrm{SL}_2(\mathcal{O}_K/I)$. Expander sequences have something to do with fast covergens of the random wal and this is the first step of the proof. Let χ_S denote the uniform probability measure on S. Let $\chi_S^{(k)} = \chi_S * \cdots * \chi_S$ denote the k-fold convolution with itself. This is a probability measure on G, corresponding to the probability distribution of a random walk after k steps. Now for any ideal I, the measure $\chi_S^{(k)}$ should converge to the uniform distribution on G_I, which has norm $|G_I|^{-1/2}$. Moreover, this distribution should happen exponentially quickly. Using some linear algebra it can be shown that the theorem is equivalent to:

for each $\epsilon > 0$ there is a constant C such that

$$\|\pi_I^{(C \log N(I))}\|_2 \leq |G_I|^{\epsilon - 1/2} \tag{1}$$

for each square-free ideal I coprime to J.

To prove 1 we need two theorems.

Theorem 5. *There is a finite symmetric $S' \subseteq \Gamma$ and constants δ, C such that for any even integer $l \geq \log N(I)$ and every subgroup $H \subseteq G_I$ we have*

$$\pi_I(\chi_{S'}^{(l)})(H) \leq C \cdot [G_I : H]^{-\delta}.$$

Note that the left-hand side is the probability that we are in the subgroup H after l steps of random walk. So the theorem shows that we leave each proper subgroup quite quickly. The proof of the theorem is based on a ping-pong argument, but we skip it completely.

To state the other theorem we need to make another definition.

Definition 6. A probability measure μ on G_I is called η-flattening if for all probability measures ν on G_I we have

$$\|\mu * \nu\|_2 \leq \|\mu\|_2^{1/2+\epsilon} \cdot \|\nu\|_2^{1/2}.$$

Theorem 7. *For every $\epsilon > 0$ there is $\eta > 0$ such that any probability measure μ that satisfies $\|\mu\|_2 \geq |G_I|^{\epsilon - 1/2}$, and $\mu(gH) \leq [G_I : H]^{-\epsilon}$ for all subgroups $H \subseteq G_I$ and $g \in G_I$, is η-flattening.*

We will not say much about the proof, just that to prove it, we have to write $G_I = G_{P_1} \times \cdots \times G_{P_k}$ where the P_i are the prime factors of I, and we prove the theorem first for the G_{P_i} and then use this to prove it for the product as well. To prove it for a prime ideal P, we need Helfgott's triple product theorem [3]: there is $\delta > 0$ such that for every generating set $A \subseteq G_P$ we have

$$|A \cdot A \cdot A| \geq |A| \cdot \min(|A|, |G_P|/|A|)^{\delta}.$$

Finally we show how Equation 1 follows from the theorems above. First of all, since being an expander is independent of generating set, we may as well replace S' by S.

Next we find the ideal J. Let $P_1, \ldots, P_k$ be some prime ideals such that for each i, the projection $\pi_{P_j}(\Gamma)$ is a strict subgroup of $\mathrm{SL}_2(G_{P_j})$. Let J be the product of them (we want J to be the product of all such prime ideals but we do not know yet that there are finitely many). By quasi-randomness, there is some positive α such that the index of $\pi_{P_j}(\Gamma)$ is at least $|G_P|^{\alpha}$. Then the index of $\pi_J(\gamma)$ in $\mathrm{SL}_2(\mathcal{O}_K/J)$ is at least $|G_J|^{\alpha}$. Applying the theorem on $H = \Gamma$ gives

$$1 = \pi_J(\chi_S^{(l)})(\Gamma) \leq C[G_J : \Gamma]^{-\delta} \leq C|G_J|^{-\alpha\delta}.$$

So $|G_J| \leq C^{1/\alpha\delta}$. This is a finite number. This shows that there can be only finitely many 'bad' prime ideals, so we can indeed let J be the product of all of them.

For all square-free ideals I coprime to J, the map $\pi_I \colon \Gamma \to G_I$ will then be a surjection. We can replace the constant C by 1 by possibly making δ smaller. Now apply the theorem on $H = \{1\}$. This gives $\pi_I(\chi_S^{(\log N(I))})(g) \leq |G_I|^{-\delta}$ for $g \in G_I$. As a result, $\|\pi_I(\chi_S^{(\log N(I))})\|_2 \leq |G_I|^{-\delta/2}$.

Now let k be an integer and suppose that $\|\pi_I(\chi_S^{2^k \log N(I)})\|_2 \geq |G_I|^{\epsilon-1/2}$. Then by the theorems, $\pi_I(\chi_S^{(2^k \log N(I))})$ will be η-flattening. So

$$\|\pi_I(\chi_S^{(2^{k+1} \log N(I))})\|_2 \leq \|\pi_I(\chi_S^{(2^k \log N(I))})\|_2^{1+\eta}.$$

By induction, we get

$$\|\pi_I(\chi_S^{(2^k \log N(I))})\|_2 \leq |G|^{-\delta/2(1+\eta)^k}$$

as long as the norm is at least $|G_I|^{\epsilon-1/2}$. So for some $k < \frac{\log(1/\delta)}{\log(1+\eta)}$ we get $\|\pi_I(\chi_S^{(2^k \log N(I))})\|_2 \leq |G|^{\epsilon-1/2}$. This shows Equation 1.

References

[1] P. Varjú, *Expansion in $SL_d(\mathcal{O}_K/I), I$ square-free*, JEMS 14(2012), no. 1, 273–305.

[2] Jean Bourgain and Alex Gamburd. *Uniform expansion bounds for Cayley graphs of* $\mathrm{SL}_2(F_p)$. Ann. of Math., 167 (2008), 625–642.

[3] Harald Helfgott. *Growth and generation in* $\mathrm{SL}_2(Z/pZ)$. Ann. of Math. (2) 167 (2008), no. 2, 601–623.

Uniform exponential mixing: the case of convex cocompact manifolds

Emilio Corso

The purpose of this talk was to present the results of Oh-Winter ([9]) and Sarkar ([12]) concerning uniform exponential rates of mixing for the geodesic flow on congruence covers of convex cocompact hyperbolic manifolds. Compared to what had been previously established in [8] and [4], the theorems do not require any restriction on the critical exponent of the convex cocompact subgroup defining the base manifold. In higher dimensions, analogous statements hold for the frame flow (cf. [13]); however, we confined our exposition to the case of the geodesic flow for the purposes of illustration.

In order to phrase the main theorem presented in the talk conveniently, it is necessary to introduce an arithmetic setup. For any integer $n \geq 2$, $\mathbb{H}^n$ denotes the n-dimensional hyperbolic space. Let $\mathbb{K}$ be a totally real number field, whose ring of integers is denoted by $\mathcal{O}_{\mathbb{K}}$. Consider a quadratic form $Q\colon \mathbb{C}^{n+1} \to \mathbb{C}$ defined over the field $\mathbb{K}$, whose restriction to $\mathbb{R}^{n+1}$ has signature $(n, 1)$, and let $\mathrm{SO}_Q = \mathrm{SO}_Q(\mathbb{C}) = \{g \in \mathrm{GL}_{n+1}(\mathbb{C}) : Q(g(x)) = Q(x) \text{ for any } x \in \mathbb{C}^{n+1}\}$ be the associated special orthogonal group. The group of real points $\mathrm{SO}_Q(\mathbb{R})$ is isomorphic to $\mathrm{SO}_{n,1}(\mathbb{R})$ as a real Lie group, so that its connected component $\mathrm{SO}_Q(\mathbb{R})^\circ$ can be identified with the group of orientation-preserving isometries of $\mathbb{H}^n$. We shall assume that, for any non-trivial embedding $\sigma\colon \mathbb{K} \hookrightarrow \mathbb{R}$, the group $(\mathrm{SO}_Q)^\sigma(\mathbb{R})$, obtained by applying σ to the coefficients of the polynomial equations defining $\mathrm{SO}_Q(\mathbb{R})$, is compact.

Remark 1. The latter condition ensures that the group of integer points $\mathrm{SO}_Q(\mathcal{O}_\mathbb{K})$ embeds as a discrete subgroup inside $G = \mathrm{SO}_Q(\mathbb{R})^\circ$ (see [7, Chap. I.3]). As an example for $n = 2$, we might consider the field $\mathbb{K} = \mathbb{Q}[\sqrt{2}]$ and the quadratic form $Q(x, y, z) = x^2 + y^2 - \sqrt{2}z^2$.

Let $\Gamma < \mathrm{SO}_Q(\mathcal{O}_\mathbb{K})$ be a Zariski dense convex cocompact subgroup. For any nontrivial ideal $\mathfrak{q}$ of $\mathcal{O}_\mathbb{K}$, denote by $N_\mathbb{K}(\mathfrak{q}) = [\mathcal{O}_\mathbb{K} : \mathfrak{q}]$ its ideal norm, and let $\Gamma_\mathfrak{q}$ be the congruence subgroup of level $\mathfrak{q}$ of Γ, that is, the intersection of Γ with the kernel of the canonical projection $\mathrm{SO}_Q(\mathcal{O}_\mathbb{K}) \to \mathrm{SO}_Q(\mathcal{O}_\mathbb{K}/\mathfrak{q})$. The unit tangent bundle of the hyperbolic manifold $\Gamma_\mathfrak{q}\backslash\mathbb{H}^n$ can be identified with double-coset space $\Gamma_\mathfrak{q}\backslash G/M$, for a compact subgroup $M < G$; similarly for $\Gamma\backslash\mathbb{H}^n$. The manifolds $\Gamma_\mathfrak{q}\backslash\mathbb{H}^n$ are locally isometric covers of the base manifold $\Gamma\backslash\mathbb{H}^n$, and the same applies to the respective unit tangent bundles, once an appropriate choice of Riemannian metrics has been made.

Let $(a_t)_{t\in\mathbb{R}}$ be the one-parameter diagonalizable subgroup of G whose action by right translations on $\Gamma\backslash G/M$ (and all its congruence covers) induces the geodesic flow; we endow the homogeneous spaces $\Gamma_\mathfrak{q}\backslash G$ with the Bowen-Margulis-Sullivan (BMS) measure $m_\mathfrak{q}^{\mathrm{BMS}}$ induced by the BMS measure on $\Gamma_\mathfrak{q}\backslash G/M$.

Assume that Γ satisfies the strong approximation property. Then the exponential mixing rates of the geodesic flow on $\Gamma_\mathfrak{q}\backslash G/M$, with respect to the BMS measures, are uniform over $\mathfrak{q}$. The precise statement reads as follows:

Theorem 2 (Oh-Winter, Sarkar)**.** *There exist real constants $C, \eta > 0$ and a nontrivial proper ideal $\mathfrak{q}_0$ of $\mathcal{O}_\mathbb{K}$ such that, for any square-free ideal $\mathfrak{q} \subset \mathcal{O}_\mathbb{K}$ coprime to $\mathfrak{q}_0$ and any pair of M-invariant functions $\phi, \psi \in C^1(\Gamma_\mathfrak{q}\backslash G)$,*

(1)
$$\left|\int_{\Gamma_\mathfrak{q}\backslash G} \phi(xa_t)\psi(x)\,\mathrm{d}m_\mathfrak{q}^{\mathrm{BMS}}(x) - \frac{1}{m_\mathfrak{q}^{\mathrm{BMS}}(\Gamma_\mathfrak{q}\backslash G)} m_\mathfrak{q}^{\mathrm{BMS}}(\phi) m_\mathfrak{q}^{\mathrm{BMS}}(\psi)\right| \leq C N_\mathbb{K}(\mathfrak{q})^C e^{-\eta t} \|\phi\|_{C^1} \|\psi\|_{C^1} .$$

A few comments about the statement of Theorem 2 in order:

- Since the BMS measures rescale as $m_\mathfrak{q}^{\mathrm{BMS}}(\Gamma_\mathfrak{q}\backslash G) = [\Gamma_\mathfrak{q} : \Gamma_{\mathfrak{q}'}]\ m_{\mathfrak{q}'}^{\mathrm{BMS}}(\Gamma_{\mathfrak{q}'}\backslash G)$ whenever $\mathfrak{q} \subset \mathfrak{q}'$, it is clear that a power of $N_\mathbb{K}(\mathfrak{q})$ should appear in any attainable uniform statement for this setting. Taking for instance the case $n = 2$ and $\mathbb{K} = \mathbb{Q}$, this observation also indicates that the optimal power rate C should correspond to the power growth rate of $[\Gamma : \Gamma_q]$ (identifying $\mathfrak{q}$ with its unique positive generator q), that is, $C = 3$.
- By the usual Roblin's transverse intersection argument, Theorem 2 admits as a corollary a uniform exponential decay rate of correlations with respect to the Haar measure on $\Gamma_\mathfrak{q}\backslash G$ for compactly supported observables ϕ, ψ, where the numerator in the limiting term is replaced by the product $m_\mathfrak{q}^{\mathrm{BR}}(\phi) m_\mathfrak{q}^{\mathrm{BR}_*}(\psi)$, and $m_\mathfrak{q}^{\mathrm{BR}}$ (resp. $m_\mathfrak{q}^{\mathrm{BR}_*}$) is the unstable (resp. stable) Burger-Roblin measure on $\Gamma_\mathfrak{q}\backslash G$ (see [12, Cor. 1.1.1]).

- It is expected that Theorem 2 holds without the assumption that $\mathfrak{q}$ is square-free; the latter comes into play owing to the use of the expander machinery of Golsefidy and Varju ([5]), which is instrumental in the proof.
- When the critical exponent δ_Γ of Γ is strictly larger than $\frac{n-1}{2}$, Theorem 2 is due to Edwards and Oh ([4]), building on previous work of Mohammadi and Oh ([8]). Also, for a single convex cocompact manifold $\Gamma\backslash\mathbb{H}^n$, exponential mixing was earlier established by Stoyanov ([14]).

We now proceed to give a brief outline of the proof of Theorem 2, referring to [9] and [12] for all the details.

The first essential ingredient, common to any modern treatment of exponential mixing of Anosov flows ([3, 6, 14]), is the symbolic coding of the geodesic flow on $\Gamma_\mathfrak{q}\backslash G/M$. To be in a position to treat all congruence covers simultaneously, it is beneficial to consider a Markov section (cf. [2, 11]) for the flow on the base manifold $\Gamma\backslash G/M$ having sufficiently small size compared to the injectivity radius, so that it can be lifted isometrically to the universal cover G/M, completed through the dynamics to a Markov section for the flow on G/M, and projected down to form compatible Markov sections for the flow on each cover $\Gamma_\mathfrak{q}\backslash G/M$. The symbolic coding amounts to identifying, measure-theoretically, the geodesic flow restricted to the (compact) support of the BMS measure on $\Gamma_\mathfrak{q}\backslash G/M$ with a suspension flow (also known as special flow in the literature) $((R^\tau_\mathfrak{q}, (\mathcal{G}^t_\mathfrak{q})_t, \nu^\tau_\mathfrak{q}))$ over a subshift of finite type; the roof function determining the suspension coincides with the first return time τ to the Markov section $R_\mathfrak{q}$, while the (symbolic) dynamics on the base is given by the Poincaré first return map σ. This description allows to leverage the Markov nature of the shift process via the theory of transfer operators, reducing the proof of exponentially-decaying bounds for correlations functions to spectral bounds on such operators.

Since useful spectral bounds on transfer operators are tipically to be expected for non-invertible systems, we consider instead the suspension flow $(U^\tau_\mathfrak{q}, (\mathcal{G}^t_\mathfrak{q})_t, \nu^\tau_{U,\mathfrak{q}})$ over the associated one-sided shift; integrating along the strong stable direction of the geodesic flow, it is possible to reduce the bound in (1) to a corresponding decay of correlations $\mathcal{C}_{\phi,\psi}(t)$ for sufficiently regular observables ϕ, ψ defined on the suspension space $U^\tau_\mathfrak{q}$. Adapting an exceedingly fruitful idea of Pollicott ([10]), it possible to express the Laplace transform

$$\hat{\mathcal{C}}_{\phi,\psi}(\xi) = \int_0^\infty \mathcal{C}_{\phi,\psi}(t)e^{-\xi t}\mathrm{d}t, \quad (\Re\xi > 0)$$

of $\mathcal{C}_{\phi,\psi}$, via the Ruelle-Perron-Frobenius (RPF) theorem (cf. [12, Thm. 2.7]), as an infinite sum of terms involving appropriately defined *congruence* transfer operators. These act on functions defined on the underlying shift space $U_\mathfrak{q} \simeq U \times G_\mathfrak{q}$, where U (resp. $U_\mathfrak{q}$) is the union of rectangles along the unstable direction forming the Markov section on the base $\Gamma\backslash G/M$ (resp. on the congruence cover $\Gamma_\mathfrak{q}\backslash G/M$) and $G_\mathfrak{q} = \Gamma/\Gamma_\mathfrak{q}$. Specifically, the congruence transfer operators are defined (up to some renormalization coming from the RPF theorem) for any non-trivial ideal

$\mathfrak{q} \subset \mathcal{O}_{\mathbb{K}}$ and any $\xi = a + ib \in \mathbb{C}$ as

$$\mathcal{M}_{\xi,\mathfrak{q}}(H)(u) = \sum_{\sigma(u')=u} e^{-(a+\delta_\Gamma - ib)\tau(u')} c_{\mathfrak{q}}(u')^{-1} H(u') \tag{2}$$

for any function $H \in C(U_{\mathfrak{q}}, \mathbb{C}) \simeq C(U, L^2(G_{\mathfrak{q}}; \mathbb{C}))$, where $c_{\mathfrak{q}} \colon U \to \Gamma/\Gamma_{\mathfrak{q}}$ is a cocycle keeping track of how the geodesic flow on $\Gamma_{\mathfrak{q}} \backslash G/M$ moves between different fundamental domains for the covering map $\Gamma_{\mathfrak{q}} \backslash G/M \to \Gamma \backslash G/M$.

The crucial uniform bounds on the operator norm of iterates of $\mathcal{M}_{\xi,\mathfrak{q}}$ is contained in the following proposition:

Proposition 3 ([12, Thm. 2.13])**.** *There exists constants $C_1, \eta_1, a_0 > 0$ and a non-trivial proper ideal $\mathfrak{q}_0' \subset \mathcal{O}_{\mathbb{K}}$ such that, for every $\xi = a + ib$ with $|a| < 2a_0$, any square-free ideal $\mathfrak{q}$ coprime to $\mathfrak{q}_0'$ and any Lipschitz function $H \colon U \to L^2(G_{\mathfrak{q}}, \mathbb{C})$ with zero-mean values, it holds*

$$\|\mathcal{M}^k_{\xi,\mathfrak{q}}(H)\|_2 \leq C_1 N_{\mathbb{K}}(\mathfrak{q})^{C_1} e^{-\eta_1 k} \left(\|H\|_\infty + \frac{1}{\max\{1, |b|\}} \mathrm{Lip}(H) \right) \quad \textit{for every } k \in \mathbb{Z}_{\geq 1}.$$

As a result of Proposition 3, the series defining $\hat{\mathcal{C}}_{\phi,\psi}$ on the half-plane $\{\Re\xi > 0\}$ converges absolutely on the strip $\{|\Re\xi| < 2a_0\}$, thereby providing a holomorphic extension for the Laplace transform to the left-side of the imaginary axis. The correlation function $\mathcal{C}_{\phi,\psi}$ might at this point be retrieved via inverse Laplace transform $\mathcal{C}_{\phi,\psi}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\mathcal{C}}_{\phi,\psi}(-a_0 + ib) e^{(-a_0+ib)t} \mathrm{d}b$, and the relevant upper bounds for $\mathcal{C}_{\phi,\psi}(t)$ are derived from the corresponding bounds for $\hat{\mathcal{C}}_{\phi,\psi}(\xi)$.

The bulk of the work in [12] lies therefore in the proof of Proposition 3. The employed methods differ substantially according to the size $|b|$ of the frequency. For large frequencies $|b| > b_0$, the resulting high-oscillatory nature of $\mathcal{M}_{\xi,\mathfrak{q}}$, in conjunction with the fundamental geometric observation that the cocycles $c_{\mathfrak{q}}$ are locally constant ([12, Lem. 2.11]), allows to adapt the construction of Dolgopyat's operators (already instrumental for the case of a single manifold in [3, 14]). The resulting bounds are entirely analogous to the single-manifold case, and do not depend on the ideal norm $N_{\mathbb{K}}(\mathfrak{q})$ (cf. [12, Thm. 4.1]).

On the other hand, uniformity for small frequencies $|b| \leq b_0$ demands an ingenious use of the expanding machinery developed by Bourgain-Gamburd-Sarnak ([1]) and subsequently extended by Golsefidy-Varju ([5]). Specifically, for a fixed integer $s \geq 1$, it is possible to estimate the value of $\mathcal{M}^s_{\xi,\mathfrak{q}}(H)(u)$ via a finite sum of convolution operators $\sum_{\alpha_s,\ldots,\alpha_1} \mu^{u,\xi,\mathfrak{q}}_{(\alpha_s,\ldots,\alpha_1)} * \phi^{\mathfrak{q}.H}_{(\alpha_s,\ldots,\alpha_1)}$, which should be interpreted as finite-scale approximations of the transfer operator (namely they depend only on finitely many coordinates $\alpha_s, \ldots, \alpha_1$ of the sequences). The L^2-norm of each addend $\mu * \phi$ of the sum (omitting subscripts and superscripts for notational simplicity) can be bounded from above via an L^2-flattening type of lemma ([12, Lem. 3.14]): there exists a constant $\bar{C} > 0$ such that

$$\|\mu * \phi\|_{L^2(G_{\mathfrak{q}})} \leq \bar{C} N_{\mathbb{K}}(\mathfrak{q})^{-\frac{1}{2}} \|\nu\|_{L^1(G_{\mathfrak{q}})} \|\phi\|_{L^2(G_{\mathfrak{q}})}\ , \tag{3}$$

where ν is a positive measure on $G_{\mathfrak{q}}$ defined analogously to the complex measure μ but removing the oscillatory terms $e^{ib\tau}$. The flattening lemma follows, albeit in a rather intricate manner, from uniform spectral gap for the family of expanders $\mathrm{Cay}(G_{\mathfrak{q}})$, where we recall that $G_{\mathfrak{q}} \simeq \mathrm{SO}_Q(\mathcal{O}_{\mathbb{K}}/\mathfrak{q})$ for all $\mathfrak{q}$ coprime to a fixed $\mathfrak{q}_1$, as Γ is assumed to satisfy strong approximation. Using (3), what remains can be majorized uniformly in a strip $\{|\Re\xi| < 2a_0\}$ by means of standard spectral bounds for transfer operators on the base manifold $\Gamma\backslash G/M$. Similar but more technical estimates can be performed for the Lipschitz constant of $\mathcal{M}^s_{\xi,\mathfrak{q}}(H)$; for these, the restriction $|b| \leq b_0$ is essential (cf. [12, Lem. 3.23]). Combined with the bounds on the supremum norm of $\mathcal{M}^s_{\xi,\mathfrak{q}}(H)$ discussed above, they allow to run an inductive argument leading to the desired estimate in Proposition 3.

References

[1] J. Bourgain, A. Gamburd and P. Sarnak, *Affine linear sieve, expanders, and sum-product*, Invent. Math. **179** (2010), 559–644.

[2] R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1971), 725–747.

[3] D. Dolgopyat, *On decay of correlations in Anosov flows*, Ann. of Math. **147** (1998), 357–390.

[4] S. Edwards, H. Oh, *Spectral gap and exponential mixing on geometrically finite hyperbolic manifolds*, arXiv:2001.03377 (2020).

[5] A. Salehi Golsefidy, P. Varju, *Expansion in perfect groups*, Geom. Funct. Anal. **22** (2012), 1832–1891.

[6] C. Liverani, *On contact Anosov flows*, Ann. of Math. **159** (2004), 1275–1312.

[7] G.A. Margulis. *Discrete Subgroups of Semisimple Lie Groups*. Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1991.

[8] A. Mohammadi, H. Oh, *Matrix coefficients, counting and primes for orbits of geometrically finite groups*, J. Eur. Math. Soc. **17** (2015), 837–897.

[9] H. Oh, D. Winter *Uniform exponential mixing and resonance free regions for convex cocompact congruence subgroups of* $\mathrm{SL}_2(\mathbb{Z})$, J. Amer. Math. Soc. **29** (2016), 1069–1115.

[10] M. Pollicott, *On the rate of mixing of Axiom A flows*, Invent. Math. **81** (1985), 413–426.

[11] M. Ratner, *Markov partitions for Anosov flows on n-dimensional manifolds*, Israel J. Math. **15** (1973), 92–114.

[12] P. Sarkar, *Uniform exponential mixing for congruence covers of convex cocompact hyperbolic manifolds*, arXiv:1903.00825 (2019).

[13] P. Sarkar, *Generalization of Selberg's* 3/16 *theorem for convex cocompact thin subgroups of* $\mathrm{SO}(n,1)$, arXiv:2006.07787 (2020).

[14] L. Stoyanov, *Spectra of Ruelle transfer operators for Axiom A flows*, Nonlinearity **24** (2011), 1089–1120.

Beyond Expansion II: Achieving or Exceeding the Ramanujan-quality Exponent of Distribution

Matthew Litman

The aim of this talk is to introduce the exponent of distribution in the affine sieve, Pythagorean triples, and how the exponent arises in the story of Pythagorean triples. Let's first take a look at our setting which makes understanding the sieve a bit easier.

1. Pythagorean Triples

A *Pythagorean triple* $\mathbf{x} = (x, y, z) \in (\mathbb{Z}_{\geq 0})^3$ is an integral point on the cone $F = 0$, where

$$F(\mathbf{x}) := x^2 + y^2 - z^2.$$

Let $G = \mathrm{SO}_F^o(\mathbb{R})$ and take $\Gamma < G(\mathbb{Z})$ to be a geometrically finite subgroup with no unipotent elements other that I. For a fixed triple, say $\mathbf{x} = (1, 0, 1)$ or $(3, 4, 5)$, one can consider the orbit of $\mathbf{x}$ under Γ,

$$\mathcal{O} := \mathbf{x} \cdot \Gamma.$$

On this orbit, it is natural to consider the following integer-valued functions (where the constants in front are to remove any extraneous factors induced by congruence conditions on the coordinates):

$$\begin{aligned} f_{\mathcal{H}}(x, y, z) &= z && \text{(hypotenuse)} \\ f_{\mathcal{A}}(x, y, z) &= \frac{1}{12}xy && \text{(normalized area)} \\ f_{\mathcal{P}}(x, y, z) &= \frac{1}{60}xyz && \text{(product of the coordinates)} \end{aligned}$$

One question that pops into mind for any of these quantities is "how many prime factors does $f(\mathbf{x})$ have?" As a first step, recall the following parametrization of such triples given by the Babylonians with $gcd(x, y, z) = 1$, x odd, and y even:

$$(x, y, z) = (c^2 - d^2, 2cd, c^2 + d^2)$$

for some $c, d \in \mathbb{Z}$. Written this way, we see $z = c^2 + d^2$ and since every prime $p \equiv 1 \pmod 4$ can be written as a sum of two squares, $f_{\mathcal{H}}(\mathbf{x})$ can have as little as one prime factor. This inspires yet another question, "does the collection of triples $\mathbf{x}$ with $f_{\mathcal{H}}(\mathbf{x})$ prime 'fill out' the cone $F = 0$?" More precisely, we are asking if this collection is Zariski dense in the closure of $\mathcal{O}$, given by $\mathrm{Zcl}(\mathcal{O}) = \{\mathbf{x} \in (\mathbb{R}_{\geq 0})^3 : F(\mathbf{x}) = 0\}$. It is conjectured that this collection is Zariski dense, but so far progress has given us the following: let

$$\begin{aligned} \mathcal{O}(f, R) &:= \{\mathbf{x} \in \mathcal{O} : f(\mathbf{x}) \text{ is } R\text{-almost prime}\} \\ R_0(\mathcal{O}, f) &:= \min\{R \leq \infty : \mathrm{Zcl}(\mathcal{O}(f, R)) = \mathrm{Zcl}(\mathcal{O})\} \end{aligned}$$

where an integer is *R-almost prime* if it has at most R prime factors.

Theorem 1 ([5]). *For Γ as above and δ_Γ (the critical exponent of Γ) sufficiently close to 1, one has*

$$R_0(\mathcal{O}, f) \leq \begin{cases} 7 & \text{if } f = f_{\mathcal{H}} \\ 25 & \text{if } f = f_{\mathcal{A}} \\ 37 & \text{if } f = f_{\mathcal{P}}. \end{cases}$$

These results rely on the existence of a spectral gap θ (which exists if $\delta_\Gamma > \frac{1}{2}$ and satisfies $\theta \in [\frac{1}{2}, \delta_\Gamma)$ [2]), and on acceptable upper bounds (given by $\theta = \frac{5}{6}$ if $\delta_\Gamma > \frac{5}{6}$ [4]). If $\theta = \frac{1}{2}$, we say Γ has a *Ramanujan-quality spectral gap*. The above

R_0 values have since been lowered by going "beyond expansion", utilizing methods of abelian harmonic analysis to overcome limitations of the spectral gap approach. Instead of expanding on this, we illuminate roughly how the value for $f_{\mathcal{H}}$ present in Theorem 1 is attained.

By the standard spin double cover $\iota : \mathrm{SL}_2(\mathbb{R}) \to \mathrm{SO}_F^o(\mathbb{R})$, taking $\mathbf{x}_0 = (1, 0, 1)$ yields

$$f(\mathbf{x}_0 \cdot \iota \begin{pmatrix} a & b \\ c & d \end{pmatrix}) = c^2 + d^2 = z.$$

Therefore we can consider Γ living in $G = \mathrm{SL}_2(\mathbb{R})$, $\mathbf{x}_0 = (0, 1)$, and our problem on R_0 for hypotenuses is reduced to studying $\mathcal{O} = \mathbf{x}_0 \cdot \Gamma$, $f(c, d) = c^2 + d^2$, and $f(\mathcal{O}) = \{c^2 + d^2 : \gamma \in \Gamma\}$. In other words, we are asking what R_0 gives us that $\{\gamma \in \Gamma : c^2 + d^2 \text{ is } R_0\text{−almost prime}\}$ is Zariski dense in $\mathrm{SL}_2(\mathbb{Z})$. To approach this, we call on the affine sieve.

2. Affine Sieve

We take this machinery as a black box but utilize what it implies. The sieve wants Γ, $\mathbf{x}_0$ (to form an orbit under Γ), f integral on the orbit, and q square-free to give us estimates for R_0. This is done by understanding the distribution of Γ on multiples of q. Let $\Gamma_x = \{\gamma \in \Gamma : \|\gamma\| < x\}$ (here $\|\gamma\|$ can be thought of as $\mathrm{tr}(\gamma^T \gamma)$) and define

$$r_q(x) := \#\{\gamma \in \Gamma_x : c^2 + d^2 \equiv 0 \pmod q\} - \frac{1}{q}\#\{\gamma \in \Gamma_x\},$$
$$N := \max\{n \geq 1 : \gamma \in \Gamma_x,\ c^2 + d^2 = n\} \asymp x^2.$$

We want to pick Q such that the "error" satisfies

$$\mathcal{E} := \sum_{q<Q} |r_q(x)| = o(\#\{\gamma \in \Gamma_x\}), \qquad (*)$$

where $\#\{\gamma \in \Gamma_x\} = x^{2\delta_\Gamma}$. Such a Q is called a *level of distribution.* Our goal is to get as large a Q as possible for $(*)$ to hold, and to express it in terms of N:

$$Q = N^\alpha,$$

where α is called an *exponent of distribution.* With this α, we obtain upper bounds R for R_0 by taking $R = \lceil \frac{1}{\alpha} + \varepsilon \rceil$.

By unpacking $\#\{\gamma \in \Gamma_x : c^2 + d^2 \equiv 0 \pmod q\}$, we get a better understanding of the remainder $|r_q(x)|$ (with a fair amount of crude estimates along the way and

$\mathbb{1}_A$ denoting the indicator function for the event A):

$$\#\{\gamma \in \Gamma_x : \ c^2 + d^2 \equiv 0 \ (q)\} = \sum_{\gamma \in \Gamma_x} \mathbb{1}_{c^2+d^2 \equiv 0 (q)} = \sum_{\gamma_0 \in \mathrm{SL}_2(q)} \sum_{\substack{\gamma \in \Gamma_x \\ \gamma \equiv \gamma_0 (q)}} \mathbb{1}_{c_0^2+d_0^2 \equiv 0 (q)}$$

$$= \sum_{\gamma_0 \in \mathrm{SL}_2(q)} \mathbb{1}_{c_0^2+d_0^2 \equiv 0 (q)} \sum_{\substack{\gamma \in \Gamma_x \\ \gamma \equiv \gamma_0 (q)}} 1 \overset{(1)}{=} \sum_{\gamma_0 \in \mathrm{SL}_2(q)} \mathbb{1}_{c_0^2+d_0^2 \equiv 0 (q)} \left[\frac{1}{|\mathrm{SL}_2(q)|} \sum_{\gamma \in \Gamma_x} 1 + O(x^{2\theta}) \right]$$

$$= \sum_{\gamma_0 \in \mathrm{SL}_2(q)} \mathbb{1}_{c_0^2+d_0^2 \equiv 0 (q)} \left[\frac{1}{|\mathrm{SL}_2(q)|} x^{2\delta} + O(x^{2\theta}) \right] \overset{(2)}{=} \frac{1}{q} x^{2\delta} + O(q^2 x^{2\theta}),$$

where in (1) we used results on effective counting in congruence towers to estimate the sum (see [5]) and in (2) used that there are roughly q^2 elements in $\mathrm{SL}_2(q)$ with $c^2 + d^2 \equiv 0 \pmod q$. If we now sum over these remainder terms, we arrive at the exponent of distribution,

$$\mathcal{E} = \sum_{q<Q} |r_q(x)| = O(Q^3 N^\theta) \quad \Rightarrow \quad Q^3 N^\theta \asymp N^\delta \quad \Rightarrow \quad Q = N^{\frac{\delta-\theta}{3}} \text{ and } \alpha = \frac{\delta - \theta}{3}.$$

By taking δ very close to 1 and $\theta = \frac{5}{6}$, we get $\alpha \approx \frac{1}{18}$ and $R = 19$. This value is far from optimal but serves as a backbone for the arguments used in [5, 6]. The achievements of these papers also lie in their reducing of the power of q present in the remainder, where in [6] they reduce the power of q by 1 to obtain $R = 13$ and in [5] they are able to remove q all together and bring the remainder down to $|r_q(x)| = O(x^{2\theta})$, resulting in $\alpha = \delta - \theta$ and $R = 7$. Using the $\frac{5}{6}$-spectral gap, this is best result that current methods based off expansion can give. If we instead assume a Ramanujan-quality spectral gap, we can get $\alpha \approx 1 - \frac{1}{2} = \frac{1}{2}$ and $R = 3$ which is better than $R = 4$ (making our title a slight misnomer), but this is conditional on a rather large assumption.

In the seminal paper of Bourgain and Kontorovich [1], they go beyond expansion by decomposing the indicator function along its primitive harmonics to unconditionally achieve the exponent of distribution $\alpha = \frac{7}{24} - \varepsilon$ to retrieve $R = 4$ for $f_{\mathcal{H}}$. Later, Ehrman expanded their approach to all three functions f considered above to raise the exponent and arrive at the best unconditional results we have:

Theorem 2. [1, 3] *If Γ has no parabolic elements and $\delta_\Gamma > \delta_0$, one has*

$$R_0(\mathcal{O}, f) \leq \begin{cases} 4 & \text{if } f = f_{\mathcal{H}}, \ \delta_0 = 0.984 \\ 18 & \text{if } f = f_{\mathcal{A}}, \ \delta_0 = 0.9955 \\ 26 & \text{if } f = f_{\mathcal{P}}, \ \delta_0 = 0.9963. \end{cases}$$

References

[1] J. Bourgain and A. Kontorovich, *The Affine Sieve Beyond Expansion I: Thin Hypotenuses*, IMRN **2015 (19)** (2015), 9175–9205.

[2] J. Bourgain, A. Gamburd, and P. Sarnak, *Affine linear sieve, expanders, and sum-product* Invent. Math., **179 (3)** (2010), 559–644.

[3] M. Ehrman, *Almost Primes in Thin Orbits of Pythagorean Triangles*, IMRN **2019 (11)** (2019), 3498–3526.

[4] A. Gamburd, *On the spectral gap for infinite index "congruence" sub-groups of* $SL_2(\mathbb{Z})$, Israel J. Math., **127** (2002), 157–200.
[5] J. Hong and A. Kontorovich, *Almost Prime Coordinates for Anisotropic and Thin Pythagorean Orbits*, Israel J. Math. **209 (1)** (2015), 397–420.
[6] A. Kontorovich and H. Ohh, *Almost prime Pythagorean triples in thin orbits*, J. Reine Angew. Math. **667** (2012), 89–131.

Multi-scale setting III: the p-adic case

Simon Machado

1. Statements of the main results

Fix q_0 and n_0 two positive integers, as well as Ω a finite subset of $GL_{n_0}(\mathbb{Z}[\frac{1}{q_0}])$ and $\Gamma := \langle\Omega\rangle$ the group it generates. For every integer m coprime to q_0 let $\pi_m : GL_{n_0}(\mathbb{Z}[\frac{1}{q_0}]) \to GL_{n_0}(\mathbb{Z}/m\mathbb{Z})$ denote the reduction of coefficients mod m. The main result we discussed in this talk can be stated elementarily in terms of Cayley graphs and expansion:

Theorem 1 ([3])**.** *The family*

$$\{Cay(\pi_{p^n}(\Gamma), \pi_{p^n}(\Omega))\}_{p,n},$$

where p *runs through all prime numbers* $p \nmid q_0$ *and* n *runs through* $\mathbb{N}$*, is a family of expanders* ***if and only if*** *the connected component* $\mathbb{G}^0$ *of the Zariski-closure* $\mathbb{G}$ *of* Γ *is perfect i.e.* $[\mathbb{G}^0, \mathbb{G}^0] = \mathbb{G}^0$.

Theorem 1 gives a complete classification of subsets for which expansion happens. If we denote now μ the uniform probability measure on Ω, then Theorem 1 essentially reduces to a bound on the top eigenvalue of the convolution operators $f \mapsto \mu * f$ defined for $f \in L^2_0(\pi_p(\Gamma))$ i.e. for all L^2 maps with mean zero. If $\lambda(\pi_{p^n}(\Gamma), \Omega)$ denotes the modulus of the top eigenvalue of this operator, then Theorem 1 asserts that $\sup_{p\nmid q_0, n\geq 0} \lambda(\pi_{p^n}(\Gamma), \Omega) < 1$ if and only if $\mathbb{G}^0$ is perfect. This observation, combined with the Peter–Weyl theorem, enables us to rephrase Theorem 1 in the framework of spectral gap over p-adic fields. For a prime number $p \nmid q_0$ we can see Γ as a subgroup of $GL_{n_0}(\mathbb{Z}_p)$ via the natural embedding $GL_{n_0}(\mathbb{Z}[\frac{1}{q_0}]) \to GL_{n_0}(\mathbb{Z}_p)$. Define Γ_p as the closure, in the Hausdorff topology, of Γ in $GL_{n_0}(\mathbb{Z}_p)$. We have:

Corollary 2 ([3])**.** *The inequality*

$$\sup_{p\nmid q_0} \lambda(\Gamma_p, \Omega) < 1$$

holds if and only if the connected component $\mathbb{G}^0$ *of the Zariski-closure* $\mathbb{G}$ *of* Γ *is perfect.*

Note, finally, that the proof of the "only if" direction is extremely robust. In particular, we have the stronger statement:

Lemma 3. *Let* $(m_k)_{k\in\mathbb{N}}$ *be a family of integers coprime to* q_0 *that goes to infinity. If the family*

$$\{Cay(\pi_{m_k}(\Gamma), \pi_{m_k}(\Omega))\}_{k\geq 0}$$

is a family of expanders, then $\mathbb{G}^0$ *is perfect.*

As a consequence:
$\lambda(\Gamma_p, \Omega) < 1$ for some $p \Rightarrow \mathbb{G}^0$ is perfect $\Rightarrow \sup_{p\nmid q_0} \lambda(\Gamma_p, \Omega) < 1$.

2. Sketch of the proof

Very much like for spectral gap in compact simple Lie groups, the proof boils down to two results: one about approximate subgroups and one about representations at various *scales*. This reduction is a result of an application of the Bourgain–Gamburd machine to our problem.

The first result is about *quasi-randomness* of the family of groups $\{\pi_{p^n}(\Gamma)\}_{p,n}$. It is easy to see that these groups are not quasi-random, as $\pi_{p^n}(\Gamma)$ always admits a small factor of the form $\pi_p(\Gamma)$. We can prove however that these factors are the only obstructions to quasi-randomness. In other words, Γ is quasi-random *at each scale* :

Proposition 4 (Quasi-randomness at every scale, Prop. 18 [3])**.** *There is a constant* $C > 0$ *such that all but finitely many irreducible representations* ϕ *of* $\pi_{p^n}(\Gamma)$ *that do not factor through* $\pi_{p^{n-1}}(\Gamma)$ *have dimension at least* p^{Cn}*.*

This is proved through essentially standard means - one can see it as a consequence of Howe's kirillov theory for compact p-adic analytic groups applied to the groups Γ_p.

The second ingredient is a product theorem - a result stating that powers of subsets must grow quickly - for subsets that are *generic* at a given scale Q. To be more precise, we fix $Q = p^n$ for $p \nmid q_0$ prime and $n \geq 0$ and denote by μ the uniform measure on Ω. We will consider subsets that are generic in the sense that they have a large weight under the measure $(\pi_Q)_*\mu^{*l}$ for some large enough l. In [3] it is proved that:

Theorem 5 (Product theorem, Thm. 20 [3])**.** *For all* $\epsilon > 0$ *there is* $\delta > 0$ *such that the following is true for* Q *large enough depending on* ϵ*:*

Let A *be a subset of* $\pi_Q(\Gamma)$ *such* $(\pi_Q)_*\mu^{*l}(A) > Q^{-\delta}$ *for some* $l \geq \frac{1}{\delta}\log Q$ *and* $|AAA| \leq |A|^{1+\delta}$*. Then* $|A| \geq |\pi_Q|^{1-\epsilon}$*.*

When $\mathbb{G}$ has its unipotent radical $\mathbb{U}$ that is abelian, the proof of Theorem 5 can be summed up as follows. First, prove this result in the case of $\mathbb{G}$ semi-simple using sum-product phenomena - as in the proof of spectral gap in compact Lie groups. (This was first proved by Bourgain–Gamburd in the case of Γ Zariski-dense in $SL_n(\mathbb{Z})$ in [1]; subsequently, Salehi Golsefidy [2] extended it to $\mathbb{G}$ any semi-simple group). The second step consists in showing that there must exist an element v in A^C - for some small constant C - that lies in $\pi_Q(\Gamma) \cap \pi_Q(\mathbb{U})$. This step is one of the main difficulties overcome in [3] and uses a key diophantine property that yields:

Proposition 6 (Key proposition, Prop. 37 [3]). *There is $\delta > 0$ depending on Ω such that if H is a subgroup of $\pi_Q(\Gamma)$, then*

$$(\pi_Q)_*\mu^{*l}(H) \leq [\pi_Q(\Gamma) : H]^\delta$$

for all $l \geq \delta^{-1}\log(Q)$.

Finally, use the action by conjugation of A on v as well as the first step to prove that $\pi_Q(A^{C'}) \cap \pi_Q(\mathbb{U})$ is large for some constant C'. This method enables us to prove Theorem 5 by induction on the dimension of $\mathbb{U}$.

References

[1] Jean Bourgain, Alex Gamburd, *Expansion and random walks in SLd(Z/pnZ): II*, J. Eur. Math. Soc. **11** (2009), 1057–1103.

[2] Alireza Salehi Golsefidy, *Super-approximation, I: p-adic semisimple case*, International Mathematics Research Notices **Volume 2017, Issue 23** (2017), 7190–7263.

[3] Alireza Salehi Golsefidy, *Super-approximation, II: the p-adic case and the case of bounded powers of square-free integers*, J. Eur. Math. Soc. **21** (2019), 2163–2232.

Application to equidistribution and counting

Minju Lee

This is a summary of the talk given by the author at MFO, October 7th of 2021, as a part of the program "Arbeitsgemeinschaft: Thin groups and Super-approximation". The goal of the lecture is to give an application of mixing to an orbital counting problem using the strategy from [2], in the infinite volume setup. The main references are survey papers [5] and [6].

Let us start with a concrete example. Let $\mathcal{P}$ be a bounded Apollonian circle packing, $\Gamma < \mathrm{PSL}(2,\mathbb{C})$ be the associated Apollonian group with the critical exponent δ_Γ. For a bounded Borel set $E \subset \mathbb{C}$, define

$$N_{\mathcal{P}}(T,E) := \{C \subset \mathcal{P} : \mathsf{curv}(C) \leq T, \text{ and } C \cap E \neq \emptyset\}, \tag{1}$$

where $\mathsf{curv}(C)$ is the inverse of the Euclidean radius of C. We have:

Theorem 1. [7] *For any bounded Borel set E with $\mathcal{H}^{\delta_\Gamma}(\partial E) = 0$,*

$$N_{\mathcal{P}}(T,E) \sim c_A \mathcal{H}^{\delta_\Gamma}(E) T^{\delta_\Gamma}$$

where $\mathcal{H}^{\delta_\Gamma}$ is the δ_Γ-dimensional Hausdorff measure on the limit set Λ_Γ and $c_A > 0$ is a constant independent of $\mathcal{P}$.

In fact, the counting problem (1) can be realized as a special case of a more general counting problem which we discuss below.

Asymptotic distribution of orbits in $H\backslash G$. Let

$$G = \mathrm{PSL}(2,\mathbb{C}), H = \mathrm{PSU}(1,1) \cup \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \mathrm{PSU}(1,1)$$

and $\Gamma < G$ be a Zariski dense discrete subgroup. Let $[e] \in H\backslash G$ and assume that $[e]\Gamma$ is discrete. For an increasing family of subsets $\mathcal{B}_T \subset H\backslash G$, we want to understand the asymptotic behavior of $\#([e]\Gamma \cap \mathcal{B}_T)$ as $T \to \infty$. When Γ is a lattice inside G, under the assumptions

(1) $\mathrm{vol}(\Gamma \cap H\backslash H) < \infty$ and
(2) $\mathcal{B}_T$ are "well-rounded" with respect to $\mathrm{vol}_{H\backslash G}$,

we have

$$\#([e]\Gamma \cap \mathcal{B}_T) \sim \frac{\mathrm{vol}(\Gamma \cap H\backslash H)}{\mathrm{vol}(\Gamma\backslash G)} \mathrm{vol}_{H\backslash G}(\mathcal{B}_T)$$

as shown in [2]. The second condition accounts for the negligibility of the boundary of $\mathcal{B}_T$ with respect to $\mathrm{vol}_{H\backslash G}$; more precisely, it means

$$\limsup_{\epsilon\to 0} \limsup_{T\to\infty} \frac{\mathrm{vol}_{H\backslash G}(\mathcal{B}_{T+\epsilon}) - \mathrm{vol}_{H\backslash G}(\mathcal{B}_{T-\epsilon})}{\mathrm{vol}_{H\backslash G}(\mathcal{B}_T)} = 0.$$

When Γ is not necessarily of finite co-volume, we will consider a measure $\mu^{\mathrm{PS}}_{\Gamma\cap H\backslash H}$ on $\Gamma \cap H\backslash H$, whose total mass will be referred to as skinning constant sk_Γ. In [4], an explicit locally finite Borel measure $\mathcal{M}_{H\backslash G}$ on $H\backslash G$ is given so that the following analogue of [2] holds:

Theorem 2 ([4],[7]). *Let Γ be a geometrically finite group. Assume that* (1) $\mathrm{sk}_\Gamma < \infty$*, and* (2) $\mathcal{B}_T$ *are well-rounded with respect to* $\mathcal{M}_{H\backslash G}$*. Then*

$$\#[e]\Gamma \cap \mathcal{B}_T \sim \mathcal{M}_{H\backslash G}(\mathcal{B}_T) \text{ as } T \to \infty.$$

In the rest of the lecture, we will

- give the definition of the skinning constant,
- explain the connection between Theorem 1 and Theorem 2,
- and give the idea of the proof of Theorem 2.

Skinning constant. We now explain the measure $\mu^{\mathrm{PS}}_{\Gamma\cap H\backslash H}$. Let ν be a Patterson-Sullivan measure of dimension δ_Γ. For $g \in G$, let $g^+ \in \partial\mathbb{H}^3$ be its image under the visual map. Define a locally finite Borel measure μ^{PS}_H on H by:

$$d\mu^{\mathrm{PS}}_H(h) := e^{\delta_\Gamma \beta_{h^+}(e,h)} d\nu(h^+).$$

By the definining relation of Patterson-Sullivan density, it is $\Gamma \cap H$-invariant, and hence descends to a measure on $\Gamma \cap H\backslash H$, denoted by $\mu^{\mathrm{PS}}_{\Gamma\cap H\backslash H}$. We define $\mathrm{sk}_\Gamma := |\mu^{\mathrm{PS}}_{\Gamma\cap H\backslash H}|$. A criterion for the finiteness of sk_Γ as well as the compactness for the support of $\mu^{\mathrm{PS}}_{\Gamma\cap H\backslash H}$ is known, and the finiteness of sk_Γ implies that $H\Gamma$ is closed in G [7].

Transition from Theorem 1 to Theorem 2. Note that $\mathcal{P}$ is a union of finitely many Γ-orbits of a circle. For simplicity we will focus on

$$\#\{C \in \Gamma C_0 : \mathsf{curv}(C) \leq T, \text{ and } C \cap E \neq \emptyset\}$$

and further assume that C_0 is the unit disc $\{z : |z| = 1\}$, so that $\mathrm{Stab}_G(C) = H$. Set

$$n_z := \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, a_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}.$$

Let $N = \{n_z : z \in \mathbb{C}\}$, $A = \{a_t : t \in \mathbb{R}\}$ and $K = \mathrm{PSU}(2)$ so that we have the Iwasawa decomposition $G = NAK$. Observe that

$$\begin{aligned} &\mathsf{curv}(\gamma C_0) \leq T \text{ and } \gamma C_0 \text{ meets } E \\ &\Leftrightarrow \mathsf{hull}(\gamma C_0) \text{ meets } N_E A_{t \geq \log T^{-1}} j \\ &\Leftrightarrow \gamma HK \cap N_E A_{t \geq \log T^{-1}} K \neq \emptyset \end{aligned}$$

where $N_E = \{n_z : z \in E\}$, $A_{t \geq \log T^{-1}} = \{a_t : t \geq \log T^{-1}\}$ and $j \in \mathbb{H}^3$ is the point stabilized by K. It follows that

$$\begin{aligned} &\#\{C \in \Gamma C_0 : \mathsf{curv}(C) \leq T, \text{ and } C \cap E \neq \emptyset\} \\ &= \#\{[\gamma] \in \Gamma/\Gamma \cap H : \gamma HK \cap N_E A_{t \geq \log T^{-1}} K \neq \emptyset\} \\ &= \#\{[\gamma] \in \Gamma \cap H \backslash \Gamma : \gamma \in HKA_{t \leq \log T} N_{-E}\} \\ &= \#([e]\Gamma \cap \mathcal{B}_T) \end{aligned}$$

where $\mathcal{B}_T := HKA_{t \leq \log T} N_{-E} \subset H\backslash G$. After this transition, one needs to verify $\mathcal{M}_{H\backslash G}(\mathcal{B}_T) \sim c_A \mathcal{H}^{\delta_\Gamma}(E) T^{\delta_\Gamma}$ as $T \to \infty$. The detail of the last step can be found in [7].

Outline of the proof of Theorem 2. The idea of empolying mixing to get a counting result first appears in Margulis [3]. Counting problem for orbital points in an affine symmetric space $H\backslash G$ was treated by Duke-Rudnick-Sarnak [1] and Eskin-McMullen [2]. Using Roblin's transversal intersection argument [8], we can adapt the strategy of [2] in the infinite volume setup to get a counting result. More precisely, we first deduce the following equidsitribution of translation of $\Gamma \cap H \backslash H a_t$ as $t \to \infty$:

Theorem 3. *For all bounded $f \in L^1(\mu^{\mathrm{PS}}_{\Gamma \cap H \backslash H})$, we have:*

$$e^{(2-\delta_\Gamma)t} \int_{\Gamma \cap H \backslash H} f([h]a_t)\, dh \to \int_{\Gamma \backslash G} f(x)\, dx \text{ as } t \to \infty.$$

Let us say few words on the proof of Theorem 3. When Γ is geometrically finite, it is known that the frame flow is mixing with respect to the Bowen-Margulis-Sullivan measure m^{BMS} [9]. Combined with Margulis' banana argument, one can first deduce an equidistribution result for the Patterson-Sullivan measure on H; this is possible due to the local product structure of m^{BMS}, which is referred to as Roblin's transversal intersection argument [8]. Finally, one gets an equidistribution result for the Haar measure on H from that of the Patterson-Sullivan measure on H (Theorem 3).

As in [2], we define the counting function $F_T : G \to \mathbb{R}$ as follows:

$$F_T(g) = \sum_{\gamma \in \Gamma \cap H \backslash \Gamma} \mathbb{1}_{\mathcal{B}_T}([e]\gamma g).$$

Note that F_T descends to $\Gamma\backslash G$ and $F_T([e]) = \#([e]\Gamma \cap \mathcal{B}_T)$. Choose a nonnegative $\rho \in C_c(\Gamma\backslash G)$ with $\int_G \rho = 1$ whose support is contained in $[e]\mathcal{O}_\epsilon$ where $\mathcal{O}_\epsilon$ is an ϵ-neighborhood of e. We will approximate $F_T([e])$ by $\langle F_T, \rho\rangle_{L^2(\Gamma\backslash G)}$. By "unfolding" the latter expression, we have

$$\begin{aligned} \langle F_T, \rho\rangle_{L^2(\Gamma\backslash G)} &= \int_{\Gamma\backslash G} \sum_{\gamma \in \Gamma \cap H \backslash \Gamma} \mathbb{1}_{\mathcal{B}_T}([e]\gamma g)\rho_\epsilon(g)\, dg \\ &= \int_{\Gamma \cap H\backslash G} \mathbb{1}_{\mathcal{B}_T}([e]g)\rho_\epsilon(g)\, dg \\ &= \int_{H\backslash G} \left(\int_{\Gamma\cap H\backslash H} \rho_\epsilon(hg)\, dh \right) \mathbb{1}_{\mathcal{B}_T}(Hg)\, d(Hg). \end{aligned} \tag{2}$$

Hence, to get the asymptotic behavior of $\langle F_T, \rho\rangle_{L^2(\Gamma\backslash G)}$, we are led to investigate the asymptotic behavior of $\int_{\Gamma\cap H\backslash H} \rho_\epsilon(hg)\, dh$ as $g \to \infty$ in $H\backslash G$. When $g \to \infty$ in A^+, this is given by Theorem 3. In general, we can further decompose the Haar measure with respect to the generalized Cartan decomposition $G = HA^+K$ in (2) to get the candidate of $\mathcal{M}_{H\backslash G}$ [4]. When $\mathcal{B}_T$ is well-rounded, $\mathcal{M}_{H\backslash G}(\mathcal{B}_T)$ gives the correct asymptotic formula for $\#([e]\Gamma \cap \mathcal{B}_T)$.

References

[1] W. Duke, Z. Rudnick and P. Sarnak, *Density of integer points on affine homogeneous varieties.* , Duke Math. J. 71 (1993), no. 1, 143–179.

[2] A. Eskin and C. McMullen, *Mixing, counting, and equidistribution in Lie groups.* , Duke Math. J. 71 (1993), no. 1, 181–209.

[3] G. Margulis, *On some aspects of the theory of Anosov systems.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2004. vi+139 pp.

[4] A. Mohammadi and H. Oh, *Matrix coefficients, counting and primes for orbits of geometrically finite groups. (English summary)* J. Eur. Math. Soc. (JEMS) 17 (2015), no. 4, 837–897.

[5] H. Oh, *Apollonian circle packings: dynamics and number theory.* , Jpn. J. Math. 9 (2014), no. 1, 69–97.

[6] H. Oh, *Harmonic analysis, ergodic theory and counting for thin groups.* , Thin groups and superstrong approximation, 179–210, Math. Sci. Res. Inst. Publ., 61, Cambridge Univ. Press, Cambridge, 2014.

[7] H. Oh and N. Shah, *Equidistribution and counting for orbits of geometrically finite hyperbolic groups.* , J. Amer. Math. Soc. 26 (2013), no. 2, 511–562.

[8] T. Roblin, *Ergodicité et équidistribution en courbure négative.* , Mém. Soc. Math. Fr. (N.S.) No. 95 (2003), vi+96 pp.

[9] D. Winter, *Mixing of frame flow for rank one locally symmetric spaces and measure classification* , Israel J. Math. 210 (2015), no. 1, 467–507.

Reporter: Nikolai Edeko

Participants

Dr. Jitendra Bajpai
Fakultät für Mathematik
Technische Universität Dresden
01062 Dresden
GERMANY

Dr. Timothée Bénard
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Christian Bernert
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstraße 3-5
37073 Göttingen
GERMANY

Aaron Calderon
Department of Mathematics
Yale University
Box 208 283
New Haven CT 06520
UNITED STATES

Fernando Camacho Cadena
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Michael Chow
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Ping Ngai Chung
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago, IL 60637-1514
UNITED STATES

Emilio Corso
Departement Mathematik
ETH Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Antje Dabeler
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Dr. Daniele Dona
Einstein Institute of Mathematics
The Hebrew University
Givat Ram
91904 01 Jerusalem
ISRAEL

Reuben Drogin
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Nikolai Edeko
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Sara Edelman-Munoz
Department of Mathematics
Rice University
P.O. Box 1892
Houston TX 77005
UNITED STATES

Prof. Dr. Livio Flaminio
Département de Mathématiques
Université de Lille
Cité Scientifique - Bâtiment M2
59655 Villeneuve d'Ascq Cédex
FRANCE

Elijah Fromm
Department of Mathematics
Yale University
10 Hillhouse Avenue
New Haven CT 06511
UNITED STATES

Prof. Dr. Alireza Salehi Golsefidy
Department of Mathematics
UCSD
9500 Gilman Drive #0112
La Jolla CA 92093-0112
UNITED STATES

Dr. Jiyoung Han
Department of Mathematics, Tata
institute of fundamental research,
Dr. Homi Bhabha Road, Navy Nagar,
Colaba
Mumbai 40001
INDIA

Zhicheng Han
Mathematisches Institut
Georg-August-Universität Göttingen
Bunsenstrasse 3-5
37073 Göttingen
GERMANY

Dr. Mitul Islam
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Edna Jones
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway 08854-8019
UNITED STATES

Dongryul Kim
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Wooyeon Kim
Departement Mathematik
ETH-Zentrum
Rämistrasse 101
8092 Zürich
SWITZERLAND

Constantin Kogler
Wolfson College
University of Cambridge
Barton Road
Cambridge CB3 9BB
UNITED KINGDOM

Prof. Dr. Alex Kontorovich
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway NJ 08854-8019
UNITED STATES

Dr. Or Landesberg
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Homin Lee
Department of Mathematics
Indiana University at Bloomington
831 East 3rd St.
Bloomington IN 47405-7106
UNITED STATES

Min Ju Lee
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Matthew C. Litman
Department of Mathematics
University of California, Davis
One Shields Avenue
Davis, CA 95616-8633
UNITED STATES

Dr. Manuel Luethi
Section des Mathématiques
EPFL
1015 Lausanne
SWITZERLAND

Simon Machado
Centre for Mathematical Sciences
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Dr. Keivan Mallahi-Karai
School of Science
Jacobs University Bremen
Campus Ring I
28725 Bremen 28759
GERMANY

Prof. Dr. Hee Oh
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Sungjin Park
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Dr. Lam Pham
Department of Mathematics
Brandeis University
Waltham, MA 02454-9110
UNITED STATES

Weite Pi
Department of Mathematics
Yale University
Box 208 283
New Haven CT 06520
UNITED STATES

Prof. Dr. Oleg Pikhurko
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UNITED KINGDOM

Pratyush Sarkar
Department of Mathematics
Yale University
10 Hillhouse Avenue
New Haven CT 06511
UNITED STATES

Dr. Cagri Sert
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
SWITZERLAND

Prof. Dr. Ralf J. Spatzier
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Srivatsa Srinivas
Department of Mathematics
University of California, San Diego
(UCSD)
United States of America (USA)
9500 Gilman Drive # 0112
La Jolla CA, 92093-0112
UNITED STATES

Stefan Stadlöder
Fakultät für Mathematik
Universität Regensburg
Universitätsstrasse 31
93053 Regensburg
GERMANY

Dr. Francois Thilmany
73 Avenue de Tervueren
1040 Bruxelles
BELGIUM

Prof. Dr. Andreas B. Thom
Institut für Geometrie
Fakultät für Mathematik
Technische Universität Dresden
01062 Dresden
GERMANY

Amy Wang
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Jeroen Winkel
Mathematisches Institut
Universität Münster
Einsteinstraße 62
48149 Münster
GERMANY

Xin Zhang
Department of Mathematics
City University of Hong Kong
Hong Kong
CHINA