Mathematisches
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OWP 2021 - 10

MARIA STELLA ADAMO, KARL-HERMANN NEEB AND
JONAS SCHOBER

Reflection Positivity and Hankel Operators- the Multiplicity Free Case

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel   +49 7834 979 50
Fax   +49 7834 979 55
Email   admin@mfo.de
URL   www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

DOI   10.14760/OWP-2021-10

Reflection positivity and Hankel operators— the multiplicity free case

Maria Stella Adamo, Karl-Hermann Neeb, Jonas Schober

January 11, 2022

Abstract

We analyze reflection positive representations in terms of positive Hankel operators. This is motivated by the fact that positive Hankel operators are described in terms of their Carleson measures, whereas the compatibility condition between representations and reflection positive Hilbert spaces is quite intricate. This leads us to the concept of a Hankel positive representation of triples (G, S, τ), where G is a group, τ an involutive automorphism of G and $S \subseteq G$ a subsemigroup with $\tau(S) = S^{-1}$. For the triples $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$, corresponding to reflection positive operators, and $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$, corresponding to reflection positive one-parameter groups, we show that every Hankel positive representation can be made reflection positive by a slight change of the scalar product. A key method consists in using the measure μ_H on $\mathbb{R}_+$ defined by a positive Hankel operator H on $H^2(\mathbb{C}_+)$ to define a Pick function whose imaginary part, restricted to the imaginary axis, provides an operator symbol for H.
Keywords: Hankel operator, reflection positive representation, Hardy space, Widom Theorem, Carleson measure,
MSC 2020: Primary 47B35; Secondary 47B32, 47B91.

Contents

Introduction

This paper contributes to the operator theoretic background of *reflection positivity*, a basic concept in constructive quantum field theory ([GJ81, JOl98, JOl00, Ja08]) that recently required some interest from the perspective of the representation theory of Lie groups (see [NÓ14, NÓ15] and the survey booklet [NÓ18] which contains further references).

The main novelty of this paper is that we analyze reflection positive representations in terms of positive Hankel operators. This is motivated by the fact that positive Hankel operators can be described nicely in terms of their Carleson measures, whereas the compatibility condition between representations and reflection positive Hilbert spaces is quite intricate. This leads us to the concept of a Hankel positive representation of a triple (G, S, τ), where G is a group, τ an involutive automorphism of G and $S \subseteq G$ a subsemigroup with $\tau(S) = S^{-1}$. For the triples $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$, corresponding to reflection positive operators, and $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$, corresponding to reflection positive one-parameter groups, we show that every Hankel positive representation can be made reflection positive by a slight change of the scalar product.

To introduce our abstract conceptual background, we define a *symmetric semigroup* as a triple (G, S, τ), where G is a group and $S \subseteq G$ is a subsemigroup satisfying $\tau(S)^{-1} = S$, so that $s^\sharp := \tau(s)^{-1}$ defines an involution on S. A *representation of the pair* (G, S) is a triple $(\mathcal{E}, \mathcal{E}_+, U)$, where $U\colon G \to \mathrm{U}(\mathcal{E})$ is a unitary representation and $\mathcal{E}_+ \subseteq \mathcal{E}$ is a closed subspace satisfying $U(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. It is said to be *regular* if $\mathcal{E}_+$ contains no non-zero $U(G)$-invariant subspace and the smallest $U(G)$-invariant subspace containing $\mathcal{E}_+$ is $\mathcal{E}$.

Additional positivity is introduced by the concept of a *reflection positive Hilbert space*, which is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, consisting of a Hilbert space $\mathcal{E}$ with a unitary involution θ and a closed subspace $\mathcal{E}_+$ satisfying

$$\langle \xi, \xi \rangle_\theta := \langle \xi, \theta\xi \rangle \geq 0 \quad \text{for} \quad \xi \in \mathcal{E}_+. \tag{1}$$

A *reflection positive representation* of (G, S, τ) is a quadruple $(\mathcal{E}, \mathcal{E}_+, \theta, U)$, where $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a reflection positive Hilbert space and $(\mathcal{E}, \mathcal{E}_+, U)$ is a representation of the pair (G, S) where U and θ satisfy the following compatibility condition

$$\theta U(g)\theta = U(\tau(g)) \quad \text{for} \quad g \in G. \tag{2}$$

Any reflection positive representation specifies three representations:

(L1) the unitary representation U of the group G on $\mathcal{E}$,

(L2) the representation U_+ of the semigroup S on $\mathcal{E}_+$ by isometries,

(L3) a $*$-representation $(\widehat{\mathcal{E}}, \widehat{U})$ of the involutive semigroup $(S, \sharp)$, induced by U_+ on the Hilbert space $\widehat{\mathcal{E}}$ obtained from the positive semidefinite form $\langle \cdot, \cdot \rangle_\theta$ on $\mathcal{E}_+$.

The difficulty in classifying reflection positive representations lies in the complicated compatibility conditions between $\mathcal{E}_+$, θ and U. For the groups $G = \mathbb{Z}$ and $\mathbb{R}$ that we study in this paper, it is rather easy, resp., classical, to understand the regular representation $(\mathcal{E}, \mathcal{E}_+, U)$ of the pair (G, S). For $(G, S) = (\mathbb{Z}, \mathbb{N})$, this amounts to describe for a unitary operator U all invariant subspaces $\mathcal{E}_+$, and for $(G, S) = (\mathbb{R}, \mathbb{R}_+)$, one has to describe for a unitary one-parameter group $(U_t)_{t\in\mathbb{R}}$ all subspaces $\mathcal{E}_+$ invariant under $(U_t)_{t>0}$. Beurling's Theorems for the disc and the upper half plane solve this problem in terms of inner functions (cf. [Pa88, Thm. 6.4], [Sh64]). Adding to such triples $(\mathcal{E}, \mathcal{E}_+, U)$ a unitary involution θ such that $(\mathcal{E}, \mathcal{E}_+, \theta, U)$ is reflection positive is tricky because the θ-positivity of $\mathcal{E}_+$ is hard to control.

Similarly, the description of all triples $(\mathcal{E}, \theta, U)$ satisfying (2) is the unitary representation theory of the semidirect product $G \rtimes \{\mathrm{id}_G, \tau\}$, which is well-known for $\mathbb{Z}$ and $\mathbb{R}$. To fit in subspaces $\mathcal{E}_+$ becomes complicated by the two requirements of θ-positivity and $U(S)$-invariance of $\mathcal{E}_+$.

The new strategy that we follow in this paper is to focus on the intermediate level (L2) of the representation U_+ of the involutive semigroup $(S, \sharp)$ by isometries on $\mathcal{E}_+$. On this level, we introduce the concept of a *U_+-Hankel operator*. These are the operators $H \in B(\mathcal{E}_+)$ satisfying

$$HU_+(s) = U_+(s^\sharp)^* H \quad \text{for} \quad s \in S. \tag{3}$$

Although it plays no role for the representations of the pair (G, S), the involution $\sharp$ on S is a crucial ingredient of the concept of a Hankel operator.

To illustrate these structures, let us take a closer look at the triple $(\mathbb{Z}, \mathbb{N}_0, -\mathrm{id}_{\mathbb{Z}})$, i.e., we study *reflection positive unitary operators* $U \in \mathrm{U}(\mathcal{E})$ on a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$, which means that

$$U\mathcal{E}_+ \subseteq \mathcal{E}_+ \quad \text{and} \quad \theta U \theta = U^*. \tag{4}$$

Classical normal form results for the isometry $S := U_+(1)$ on $\mathcal{E}_+$ (assuming regularity) imply that the triple $(\mathcal{E}, \mathcal{E}_+, U)$ is equivalent to $(L^2(\mathbb{T}, \mathcal{K}), H^2(\mathbb{D}, \mathcal{K}), U)$, where $\mathcal{K}$ is a multiplicity space,

$$\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$$

is the open unit disc, $H^2(\mathbb{D}, \mathcal{K})$ is the $\mathcal{K}$-valued Hardy space on $\mathbb{D}$, and $(U(1)f)(z) = zf(z)$, $z \in \mathbb{T}$, is the multiplication operator corresponding to the bilateral shift on $L^2(\mathbb{T}, \mathcal{K})$. Our assumption of multiplicity freeness means that $\mathcal{K} = \mathbb{C}$. In this case U_+-Hankel operators are precisely classical Hankel operators, realized as operators on $H^2(\mathbb{D})$. The difficult part in the classification of reflection positive operators consists in a description of all unitary involutions θ turning $(\mathcal{E}, \mathcal{E}_+, U) = (L^2(\mathbb{T}), H^2(\mathbb{D}), U)$ into a reflection positive representation. The compatibility with U is easy to accommodate. It means that θ is of the form

$$(\theta_h f)(z) = h(z) f(\overline{z}) \quad \text{with} \quad h \colon \mathbb{T} \to \mathbb{T}, \;\; h(\overline{z}) = \overline{h(z)} \quad \text{for} \quad z \in \mathbb{T}.$$

The hardest part is to control the positivity of the form $\langle \cdot, \cdot \rangle_\theta$ on $\mathcal{E}_+ = H^2(\mathbb{D})$. Here the key observation is that, if P_+ is the orthogonal projection $L^2(\mathbb{T}) \to H^2(\mathbb{D})$, then $H_h := P_+ \theta_h P_+^*$ is a Hankel operator whose positivity is equivalent to $(L^2(\mathbb{T}), H^2(\mathbb{D}), \theta_h)$ being reflection positive. Positive Hankel operators H on $H^2(\mathbb{D})$ are most nicely classified in terms of their Carleson measures μ_H on the interval $(-1, 1)$ via the relation

$$\langle \xi, H\eta \rangle_{H^2(\mathbb{D})} = \int_{-1}^{1} \overline{\xi(x)} \eta(x) \, d\mu_H(x) \quad \text{for} \quad \xi, \eta \in H^2(\mathbb{D}).$$

Widom's Theorem (see [Wi66] and Theorem A.1 in the appendix) characterizes these measures in very explicit terms. Our main result on positive Hankel operators on the disc asserts that all these measures actually arise from reflection positive operators on weighted L^2-spaces $L^2(\mathbb{T}, \delta\, dz)$, where δ is a bounded positive weight for which δ^{-1} is also bounded (Theorem 4.8). As a consequence, the corresponding weighted Hardy space $H^2(\mathbb{D}, \delta)$ coincides with $H^2(\mathbb{D})$, endowed with a slightly modified scalar product.

The results for reflection positive one-parameter groups concerning $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$ are similar. Here the Lax–Phillips Representation Theorem shows that a regular multiplicity free representation $(\mathcal{E}, \mathcal{E}_+, U)$ of the pair $(\mathbb{R}, \mathbb{R}_+)$ is equivalent to $(L^2(\mathbb{R}), H^2(\mathbb{C}_+), U)$, where $\mathbb{C}_+ = \{z \in \mathbb{C} \colon \operatorname{Im} z > 0\}$ is the upper half-plane, $H^2(\mathbb{C}_+)$ is the Hardy space on $\mathbb{C}_+$, and $(U(t)f)(x) =$

$e^{itx}f(x)$, $x \in \mathbb{R}$, is the multiplication representation. Again, U_+-Hankel operators are the classical Hankel operators on $H^2(\mathbb{C}_+)$ (cf. [Pa88, p. 44]). The unitary involutions compatible with U in the sense of (2) are of the form

$$(\theta_h f)(x) = h(x)f(-x) \quad \text{with} \quad h\colon \mathbb{R} \to \mathbb{T}, \;\; h(-x) = \overline{h(x)} \quad \text{for} \quad x \in \mathbb{R}.$$

Now $H_h := P_+\theta_h P_+^*$ is a Hankel operator on $H^2(\mathbb{C}_+)$ and $(L^2(\mathbb{R}), H^2(\mathbb{C}_+), \theta_h)$ is reflection positive if and only if H_h is positive. Instead of trying to determine all functions h for which this is the case, we focus on positive Hankel operators H on $H^2(\mathbb{C}_+)$ because they completely determine the $*$-representation in (L3). We prove a suitable version of Widom's Theorem for the upper half plane (Theorem 3.7) that characterizes the Carleson measures μ_H on $\mathbb{R}_+$ which are determined by

$$\langle f, Hg\rangle_{H^2(\mathbb{C}_+)} = \int_{\mathbb{R}_+} \overline{f(i\lambda)}g(i\lambda)\, d\mu_H(\lambda) \quad \text{for} \quad f, g \in H^2(\mathbb{C}_+).$$

For $\mathbb{C}_+$ we show that all these measures actually arise from reflection positive one-parameter groups on weighted L^2-spaces $L^2(\mathbb{R}, w\,dx)$, where w is a bounded positive weight for which w^{-1} is also bounded. As a consequence, the corresponding weighted Hardy space $H^2(\mathbb{C}_+, w\,dz)$ coincides with $H^2(\mathbb{C}_+)$ but is endowed with modified scalar product.

Our key method of proof is to observe that the measure μ_H defines a holomorphic function

$$\kappa\colon \mathbb{C} \setminus (-\infty, 0] \to \mathbb{C}, \quad \kappa(z) := \int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{1}{z+\lambda}\, d\mu_H(\lambda)$$

whose imaginary part defines a bounded function

$$h_H(p) := \frac{i}{\pi} \cdot \mathrm{Im}(\kappa(ip))$$

which is an operator symbol of H (Theorem 4.1). As $h(\mathbb{R}) \subseteq i\mathbb{R}$, adding real constants, we obtain operator symbols for H which are invertible in $L^\infty(\mathbb{R})$, and this is used in Subsection 4.2 to show that, for $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_\mathbb{Z})$ and $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$ all multiplicity free regular Hankel positive representations can be made reflection positive by modifying the scalar product with an invertible intertwining operator. On the level of representations, this means to pass from the Hardy space $H^2(\mathbb{D})$, resp., $H^2(\mathbb{C}_+)$ to the Hardy space corresponding to a boundary measure with a positive bounded density whose inverse is also bounded.

Since the Banach algebras $H^\infty(\mathbb{D}) \cong H^\infty(\mathbb{C}_+)$ play a central role in our arguments, we decided to discuss some of their key features in an appendix. In view of the Riemann Mapping Theorem, this can be done for an arbitrary proper simply connected domain $\Omega \subsetneq \mathbb{C}$, endowed with an antiholomorphic involution σ that is used to define on $H^\infty(\Omega)$ the structure of a Banach $*$-algebra by $f^\sharp(z) := \overline{f(\sigma(z))}$. By Ando's Theorem, this algebra has a unique predual, so that it carries a canonical weak topology, which for $H^\infty(\mathbb{D})$ and $H^\infty(\mathbb{C}_+)$ is defined by integrating boundary values against L^1-functions on $\mathbb{T}$ and $\mathbb{R}$, respectively. In the literature, what will be called weak topology on $H^\infty(\Omega)$ with respect to the canonical pairing $(H^\infty(\Omega)_*, H^\infty(\Omega))$ is also known as the weak*-topology. For this algebra we determine in particular all weakly continuous positive functionals and all weakly continuous characters.

Structure of this paper: In the short Section 1 we introduce the concepts on an abstract level. In particular, we define reflection and Hankel positive representations of symmetric semigroups (G, S, τ). In particular, we show that reflection positive representations are in particular Hankel positive and that every Hankel positive representation defines a $*$-representation of $(S, \sharp)$ by bounded operators on the Hilbert space $\widehat{\mathcal{E}}$ defined by the H-twisted scalar product on $\mathcal{E}_+$. We thus obtain the same three levels (L1-3) as for reflection positive representations.

In Section 2 we connect our abstract setup with classical Hankel operators on $H^2(\mathbb{D})$. We study reflection positive representations of the symmetric group $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_\mathbb{Z})$, i.e., reflection positive operators, and relate the problem of their classification to positive Hankel operators on the Hardy space $H^2(\mathbb{D})$. As these operators are classified by their Carleson measures on the interval $(-1,1)$, we recall in Appendix A Widom's classical theorem characterizing the Carleson measures on positive Hankel operators. In Section 3 we proceed to reflection positive one-parameter groups. In this context, the upper half plane $\mathbb{C}_+$ plays the same role as the unit disc does for the discrete context and any regular multiplicity free representation of the pair $(\mathbb{R}, \mathbb{R}_+)$ is equivalent to the multiplication representation on $(L^2(\mathbb{R}), H^2(\mathbb{C}_+))$. We show that in this context the Hankel operators coincide with the classical Hankel operators on $H^2(\mathbb{C}_+)$ and translate Widom's Theorem to an analogous result on the upper half plane, where we realize the Carleson measures on the positive half-line $\mathbb{R}_+$ (Theorem 3.7). The key result of Subsection 4.1 is Theorem 4.1 asserting that h_H is an operator symbol of H. The applications to reflection positivity are discussed in Subsection 4.2, where we prove that Hankel positive representations $(\mathcal{E}, \mathcal{E}_+, U, H)$ of $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_\mathbb{Z})$ and $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$ respectively, are reflection positive if we change the inner product on $\mathcal{E}_+$ obtained through a symbol for H. Appendix B is devoted to the Banach $*$-algebras $(H^\infty(\Omega), \sharp)$ and in the short Appendix C we collect some formulas concerning Poisson and Szegö kernels.

Notation:

- $\mathbb{R}_+ = (0,\infty)$, $\mathbb{D} = \{z \in \mathbb{C} \colon |z| < 1\}$, $\mathbb{C}_+ = \mathbb{R} + i\mathbb{R}_+$ (upper half plane), $\mathbb{C}_r = \mathbb{R}_+ + i\mathbb{R}$ (right half plane), $\mathbb{S}_\beta = \{z \in \mathbb{C} \colon 0 < \mathrm{Im}\, z < \beta\}$ (horizontal strip).
- For a holomorphic function f on $\mathbb{C}_+$, we write f^* for its non-tangential limit function on $\mathbb{R}$; likewise for functions on $\mathbb{D}$ and $\mathbb{S}_\beta$.
- We write $\omega \colon \mathbb{D} \to \mathbb{C}_+, \omega(z) := i\frac{1+z}{1-z}$ for the Cayley transform with $\omega^{-1}(w) = \frac{w-i}{w+i}$.
- On the circle $\mathbb{T} = \{e^{i\theta} \in \mathbb{C} \colon \theta \in \mathbb{R}\}$, we use the length measure of total volume 2π.
- For $w \in \mathbb{C}$ we write $e_w(z) := e^{zw}$ for the corresponding exponential function on $\mathbb{C}$.
- For a function $f \colon G \to \mathbb{C}$, we put $f^\vee(g) := f(g^{-1})$.
- We write E^* for the dual of a Banach space E.

Acknowledgment: We are most grateful to Daniel Beltiţă for pointing out the references [Pa88] and [Pe98] on Hankel operators and for suggesting the connection of our work on reflection positivity with Hankel operators. We thank Christian Berg for a nice short argument for the implication (b) $\Rightarrow$ (c) in Widom's Theorem (Theorem A.1).

MSA wishes to thank the Department of Mathematics, University of Erlangen, and the Mathematisches Forschungsinstitut Oberwolfach (MFO) for their hospitality. This work is part of two Oberwolfach Leibniz Fellowships with projects entitled "Beurling–Lax type theorems and their connection with standard subspaces in Algebraic QFT" and "Reflection positive representations and standard subspaces in algebraic QFT". MSA is part of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro applicationi (GNAMPA) of INdAM. MSA acknowledges the University of Rome "Tor Vergata" funding scheme "Beyond Borders" CUP E84I19002200005 and the support by the Deutsche Forschungsgemeinschaft (DFG) within the Emmy Noether grant CA1850/1-1. MSA wishes to thank Yoh Tanimoto for his insightful comments and suggestions. KHN acknowledges support by DFG-grant NE 413/10-1.

1 Hankel operators for reflection positive representations

In this section we first recall the concept of a reflection positive representations of symmetric semigroups in the sense of [NÓ18]. In this abstract context we introduce the notion of a Hankel operator (Definition 1.3). Below it will play a key role in our analysis of the concrete symmetric semigroups $(\mathbb{Z}, \mathbb{N}_0, -\mathrm{id}_{\mathbb{Z}})$ and $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$, where it specializes to the classical concept of a Hankel operator on $H^2(\mathbb{D})$ and $H^2(\mathbb{C}_+)$, respectively.

A *reflection positive Hilbert space* is a triple $(\mathcal{E}, \mathcal{E}_+, \theta)$, consisting of a Hilbert space $\mathcal{E}$ with a unitary involution θ and a closed subspace $\mathcal{E}_+$ satisfying

$$\langle \xi, \xi \rangle_\theta := \langle \xi, \theta \xi \rangle \geq 0 \quad \text{for} \quad \xi \in \mathcal{E}_+.$$

This structure immediately leads to a new Hilbert space $\widehat{\mathcal{E}}$ that we obtain from the positive semidefinite form $\langle \cdot, \cdot \rangle_\theta$ on $\mathcal{E}_+$ by completing the quotient of $\mathcal{E}_+$ by the subspace of null vectors. We write $q \colon \mathcal{E}_+ \to \widehat{\mathcal{E}}, \xi \mapsto \widehat{\xi}$ for the natural map.

Definition 1.1. A *symmetric semigroup* is a triple (G, S, τ), where G is a group, τ is an involutive automorphism of G, and $S \subseteq G$ is a subsemigroup invariant under $s \mapsto s^\sharp := \tau(s)^{-1}$.

In the present paper we shall only be concerned with the two examples $(\mathbb{Z}, \mathbb{N}_0, -\mathrm{id}_{\mathbb{Z}})$ and $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$. As it creates no additional difficulties, we formulate the concepts in this short section on the abstract level.

Definition 1.2. Let (G, S, τ) be a symmetric semigroup.
(a) A *representation of the pair* (G, S) is a triple $(\mathcal{E}, \mathcal{E}_+, U)$, where $U \colon G \to \mathrm{U}(\mathcal{E})$ is a unitary representation, $\mathcal{E}_+ \subseteq \mathcal{E}$ is a closed subspace and $U(S)\mathcal{E}_+ \subseteq \mathcal{E}_+$. We call $(\mathcal{E}, \mathcal{E}_+, U)$ *regular* if

$$\overline{\mathrm{span}(U(G)\mathcal{E}_+)} = \mathcal{E} \quad \text{and} \quad \bigcap_{g \in G} U(g)\mathcal{E}_+ = \{0\}. \tag{5}$$

This means that $\mathcal{E}_+$ contains no non-zero $U(G)$-invariant subspace and that $\mathcal{E}$ is the only closed $U(G)$-invariant subspace containing $\mathcal{E}_+$.
(b) A *reflection positive representation of* (G, S, τ) is a quadruple $(\mathcal{E}, \mathcal{E}_+, \theta, U)$, where $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a reflection positive Hilbert space, $(\mathcal{E}, \mathcal{E}_+, U)$ is a representation of the pair (G, S) and, in addition,

$$\theta U(g) \theta = U(\tau(g)) \quad \text{for} \quad g \in G \tag{6}$$

([NÓ18, Def. 3.3.1].

Definition 1.3. If $(S, \sharp)$ is an involutive semigroup and $U_+ \colon S \to B(\mathcal{F})$ a representation of S by bounded operators on the Hilbert space $\mathcal{F}$, then we call $A \in B(\mathcal{F})$ a U_+*-Hankel operator* if

$$AU_+(s) = U_+(s^\sharp)^* A \quad \text{for} \quad s \in S. \tag{7}$$

We write $\mathrm{Han}_{U_+}(\mathcal{F}) \subseteq B(\mathcal{F})$ for the subspace of U_+-Hankel operators.

If $U_+^\vee(s) := U_+(s^\sharp)^*$ denotes the *dual representation* of S on $\mathcal{F}$, then (7) means that Hankel operators are the intertwining operators $(\mathcal{F}, U_+) \to (\mathcal{F}, U_+^\vee)$.

Lemma 1.4. *Let (G, S, τ) be a symmetric semigroup, $(\mathcal{E}, \mathcal{E}_+, U)$ be a representation of the pair (G, S), and $P_+ \colon \mathcal{E} \to \mathcal{E}_+$ be the orthogonal projection. If $A \in B(\mathcal{E})$ satisfies*

$$AU(g) = U(\tau(g))A = U(g^\sharp)^* A \quad \textit{for} \quad g \in G, \tag{8}$$

then

$$H_A := P_+ A P_+^* \in B(\mathcal{E}_+)$$

is a U_+-Hankel operator for the representation of S in $\mathcal{E}_+$ by $U_+(s) := U(s)|_{\mathcal{E}_+}$.

If, in addition, R is unitary in $\mathcal{E}$ satisfying $RU(g)R^{-1} = U(\tau(g))$ for $g \in G$, then $A \in B(\mathcal{H})$ satisfies (8) *if and only if $A = BR$ for some $B \in U(G)'$.*

Proof. For the first assertion, we observe that, for $s \in S$ and $\xi, \eta \in \mathcal{E}_+$, we have

$$\begin{aligned} \langle \xi, H_A U_+(s)\eta \rangle &= \langle \xi, AU(s)\eta \rangle \overset{(8)}{=} \langle \xi, U(\tau(s))A\eta \rangle = \langle U(s^\sharp)\xi, A\eta \rangle \\ &= \langle U_+(s^\sharp)\xi, H_A\eta \rangle = \langle \xi, U_+(s^\sharp)^* H_A \eta \rangle. \end{aligned}$$

The second assertion follows from the fact that $B := AR^{-1}$ commutes with $U(G)$. □

Lemma 1.5. *Hankel operators have the following elementary properties:*

(a) *If $H \in \mathrm{Han}_{U_+}(\mathcal{F})$, then $H^* \in \mathrm{Han}_{U_+}(\mathcal{F})$.*

(b) *If $H \in \mathrm{Han}_{U_+}(\mathcal{F})$ and B commutes with $U_+(S)$, then HB and B^*H are Hankel operators.*

Proof. (a) If $H \in \mathrm{Han}_{U_+}(\mathcal{F})$ and $s \in S$, then

$$H^*U_+(s) = (U_+(s)^*H)^* = (U_+^\vee(s^\sharp)H)^* = (HU_+(s^\sharp))^* = U_+(s^\sharp)^*H^*.$$

(b) Let $H \in \mathrm{Han}_{U_+}(\mathcal{F})$ and suppose that B commutes with $U_+(S)$. Then

$$HBU_+(s) = HU_+(s)B = U_+^\vee(s)HB \quad \text{for} \quad s \in S$$

implies that $HB \in \mathrm{Han}_{U_+}(\mathcal{F})$. Taking adjoints, we obtain $B^*H = (H^*B)^* \in \mathrm{Han}_{U_+}(\mathcal{F})$ with (a). □

Definition 1.6. (Hankel positive representations) Let (G, S, τ) be a symmetric semigroup. Then a *Hankel positive representation* is a quadruple $(U, \mathcal{E}, \mathcal{E}_+, H)$, where $(\mathcal{E}, \mathcal{E}_+, U)$ is a representation of the pair (G, S), and $H \in \mathrm{Han}_{U_+}(\mathcal{E}_+)$ is a positive Hankel operator for the representation $U_+(s) := U(s)|_{\mathcal{E}_+}$ of S by isometries on $\mathcal{E}_+$.

Example 1.7. (a) Let $(\mathcal{E}, \mathcal{E}_+, U)$ be a representation of the pair (G, S) and $\theta\colon \mathcal{E} \to \mathcal{E}$ a unitary involution satisfying $\theta U(g)\theta = U(\tau(g))$ for $g \in G$ (see (2)). Then Lemma 1.4 implies that

$$H_\theta := P_+\theta P_+^* \in B(\mathcal{E}_+)$$

is a U_+-Hankel operator. It is positive if and only if $(\mathcal{E}, \mathcal{E}_+, \theta)$ is reflection positive.
(b) The identity $\mathbf{1} \in B(\mathcal{F})$ is a U_+-Hankel operator if and only if the two representations U and $U^\vee$ coincide, i.e., if U is a $*$-representation of the involutive semigroup $(S, \sharp)$. If $U_+(S)$ consists of isometries, this is only possible if all operators $U_+(s)$ are unitary and $U_+(s^\sharp) = U_+(s)^{-1}$. In the context of (a), this leads to the case where $U_+(s)\mathcal{E}_+ = \mathcal{E}_+$ for $s \in S$.

The following proposition shows that a positive Hankel operator H immediately leads to a $*$-representation of S on the Hilbert space defined by H via the scalar product $\langle \xi, \eta \rangle_H := \langle \xi, H\eta \rangle$.

In the context of reflection positive representations (Example 1.7), the passage from the representation $(\mathcal{E}_+, U_+)$ of S by isometries to the $*$-representation on $(\widehat{\mathcal{E}}, \widehat{U})$ by contractions is called the *Osterwalder–Schrader transform*, see [NÓ18] for details. In this sense, the following Proposition 1.8 generalizes the Osterwalder–Schrader transform.

Proposition 1.8. *Let $U_+\colon S \to B(\mathcal{F})$ be a representation of the involutive semigroup $(S, \sharp)$ by bounded operators on $\mathcal{F}$ and $H \geq 0$ be a positive U_+-Hankel operator on $\mathcal{F}$. Then*

$$\langle \xi, \eta \rangle_H := \langle \xi, H\eta \rangle_{\mathcal{F}}$$

defines a positive semidefinite hermitian form on $\mathcal{F}$. We write $\widehat{\mathcal{F}}$ for the associated Hilbert space and $q\colon \mathcal{F} \to \widehat{\mathcal{F}}$ for the canonical map. Then there exists a uniquely determined $*$-representation

$$\widehat{U}\colon (S, \sharp) \to B(\widehat{\mathcal{F}}) \quad \textit{satisfying} \quad \widehat{U}(s) \circ q = q \circ U_+(s) \quad \textit{for} \quad s \in S. \tag{9}$$

Proof. For every $s \in S$ and $\xi, \eta \in \mathcal{F}$, we have

$$\langle \xi, U_+(s)\eta \rangle_H = \langle \xi, HU_+(s)\eta \rangle = \langle \xi, U_+(s^\sharp)^*H\eta \rangle = \langle U_+(s^\sharp)\xi, H\eta \rangle = \langle U_+(s^\sharp)\xi, \eta \rangle_H. \tag{10}$$

If $q(\eta) = 0$, i.e., $\langle \eta, H\eta \rangle = 0$, then this relation implies that $q(U_+(s)\eta) = 0$. Therefore $\widehat{U}(s)q(\eta) := q(U_+(s)\eta)$ defines a linear operator on the dense subspace $\mathcal{D} := q(\mathcal{F}) \subseteq \widehat{\mathcal{F}}$. It also follows from (10) that $(\widehat{U}, \mathcal{D})$ is a $*$-representation of the involutive semigroup $(S, \sharp)$.

To see that the operators $\widehat{U}(s)$ are bounded, we observe that, for every $n \in \mathbb{N}_0$ and $\eta \in \mathcal{F}$, we have

$$\|\widehat{U}(s^\sharp s)^n q(\eta)\|_{\widehat{\mathcal{F}}}^2 = \langle U_+(s^\sharp s)^n \eta, HU_+(s^\sharp s)^n \eta\rangle \leq \|H\| \|U_+(s^\sharp s)\|^{2n} \|\eta\|^2.$$

Now [Ne99, Lemma II.3.8(ii)] implies that

$$\|\widehat{U}(s)\| \leq \sqrt{\|U_+(s^\sharp s)\|} \leq \max(\|U_+(s)\|, \|U_+(s^\sharp)\|) \quad \text{for} \quad s \in S.$$

We conclude that the operators $\widehat{U}(s)$ are contractions, hence extend to operators on $\widehat{\mathcal{F}}$. We thus obtain a $*$-representation of $(S, \sharp)$. Clearly, this representation is uniquely determined by the equivariance requirement (9). □

The construction in Proposition 1.8 shows that every Hankel positive representation $(U, \mathcal{E}, \mathcal{E}_+, H)$ of (G, S, τ) defines a $*$-representation of S by bounded operators on the Hilbert space $\widehat{\mathcal{E}}$ defined by the H-twisted scalar product on $\mathcal{E}_+$. So we obtain the same three levels (L1-3) as for reflection positive representations.

Remark 1.9. Let $\mathcal{E}$ be a Hilbert space, $\mathcal{E}_+ \subseteq \mathcal{E}$ a closed subspace, and $R \in \mathrm{U}(\mathcal{E})$ a unitary involution with $R(\mathcal{E}_+) = \mathcal{E}_+^\perp$. Then

$$A^\sharp := R^{-1} A^* R$$

defines an antilinear involution on $B(\mathcal{E})$ leaving the subalgebra

$$\mathcal{M} := \{A \in B(\mathcal{E}) \colon A\mathcal{E}_+ \subseteq \mathcal{E}_+\}$$

invariant. In fact, $A \in \mathcal{M}$ implies that $A^*\mathcal{E}_+^\perp \subseteq \mathcal{E}_+^\perp$, so that

$$A^\sharp \mathcal{E}_+ = R^{-1}A^*R\mathcal{E}_+ = R^{-1}A^*\mathcal{E}_+^\perp \subseteq R^{-1}\mathcal{E}_+^\perp = \mathcal{E}_+.$$

Examples 1.10. (a) For $\mathcal{E} = L^2(\mathbb{T}) \supseteq \mathcal{E}_+ = H^2(\mathbb{D})$ and $(Rf)(z) = \overline{z}f(\overline{z})$, we have $\mathcal{M} = H^\infty(\mathbb{D})$ ([Ni19, §1.8.3]) and the corresponding involution is given by

$$f^\sharp(z) := \overline{f(\overline{z})} \quad \text{for} \quad z \in \mathcal{D}. \tag{11}$$

For this example Hankel operators will be discussed in Theorem 2.2.
(b) For $\mathcal{E} = L^2(\mathbb{R}) \supseteq \mathcal{E}_+ = H^2(\mathbb{C}_+)$, we have $(Rf)(x) = f(-x)$ with $\mathcal{M} = H^\infty(\mathbb{C}_+)$, endowed with the involution

$$f^\sharp(z) := \overline{f(-\overline{z})} \quad \text{for} \quad z \in \mathbb{C}_+. \tag{12}$$

Let $H = P_+hRP_+^*$, $h \in L^\infty(\mathbb{R})$ be a Hankel operator on $H^2(\mathbb{C}_+)$ (cf. Theorem 3.5) and $g \in H^\infty(\mathbb{C}_+)$. Then the corresponding multiplication operator m_g on $H^2(\mathbb{C}_+)$ satisfies

$$Hm_g = P_+hRP_+^*m_g = P_+hRm_gP_+^* = P_+h(g^*)^\vee RP_+^*,$$

where $(g^*)^\vee(x) = g^*(-x)$ for $x \in \mathbb{R}$. This also is a Hankel operator, where h has been modified by $(g^*)^\vee$. We shall use this procedure in Theorem 4.5 to pass from Hankel positive representations to reflection positive ones.

2 Reflection positivity and Hankel operators

In this section we connect the abstract context from the previous section with classical Hankel operators on $H^2(\mathbb{D})$. We study reflection positive operators as reflection positive representations of the symmetric group $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_\mathbb{Z})$ and relate the problem of their classification to positive Hankel operators on the Hardy space $H^2(\mathbb{D})$ on the open unit disc $\mathbb{D} \subseteq \mathbb{C}$.

Definition 2.1. A *reflection positive operator* on a reflection positive Hilbert space $(\mathcal{E}, \mathcal{E}_+, \theta)$ is a unitary operator $U \in \mathrm{U}(\mathcal{E})$ such that

$$U\mathcal{E}_+ \subseteq \mathcal{E}_+ \quad \text{and} \quad \theta U \theta = U^*. \tag{13}$$

It is easy to see that reflection positive operators are in one-to-one correspondence with reflection positive representations of $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$: If $(\mathcal{E}, \mathcal{E}_+, \theta, U)$ is a reflection positive representation of the symmetric semigroup $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$, then $U(1)$ is a reflection positive operator. If, conversely, U is a reflection positive operator, then $U(n) := U^n$ defines a reflection positive representation of $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$. Accordingly, we say that a reflection positive operator U is *regular* if

$$\bigcap_{n\in\mathbb{Z}} U^n\mathcal{E}_+ = \bigcap_{n>0} U^n\mathcal{E}_+ \overset{!}{=} \{0\} \quad \text{and} \quad \overline{\bigcup_{n\in\mathbb{Z}} U^n\mathcal{E}_+} = \overline{\bigcup_{n<0} U^n\mathcal{E}_+} \overset{!}{=} \mathcal{E}$$

(cf. Definition 1.2). If this is the case, we obtain for $\mathcal{K} := \mathcal{E}_+ \cap (U\mathcal{E}_+)^\perp$ a unitary equivalence from $(\mathcal{E}, \mathcal{E}_+, U)$ to

$$(\ell^2(\mathbb{Z}, \mathcal{K}), \ell^2(\mathbb{N}_0, \mathcal{K}), S),$$

where S is the right shift (Wold decomposition, [SzNBK10, Thm. I.1.1]).

We would like to classify quadruples $(\mathcal{E}, \mathcal{E}_+, \theta, U)$, where U is a regular reflection positive operator, up to unitary equivalence. In the present paper we restrict ourselves to the *multiplicity free case*, where $\mathcal{K} = \mathbb{C}$, so that the triple $(\mathcal{E}, \mathcal{E}_+, U)$ is equivalent to $(\ell^2(\mathbb{Z}), \ell^2(\mathbb{N}_0), S)$, where S is the right shift. For our purposes, it is most convenient to identify $\ell^2(\mathbb{Z})$ with $L^2(\mathbb{T})$ and $\ell^2(\mathbb{N}_0)$ with the Hardy space $H^2(\mathbb{D})$ of the unit disc $\mathbb{D}$, so that the shift operator acts by $(Sf)(z) = zf(z)$ for $z \in \mathbb{T}$.

We now want to understand the possibilities for adding a unitary involution θ for which $H^2(\mathbb{D})$ is θ-positive. For $f\colon \mathbb{T} \to \mathbb{C}$ we define

$$f^\sharp(z) := \overline{f(\overline{z})} \quad \text{for} \quad z \in \mathbb{T}$$

(cf. (11) and (37) in Appendix B). Then any involution θ satisfying $\theta S \theta = S^{-1}$ has the form

$$\theta_h(f)(z) = h(z)f(\overline{z}) \quad \text{for} \quad z \in \mathbb{T},$$

where $h \in L^\infty(\mathbb{T})$ satisfies $h^\sharp = h$ and $h(\mathbb{T}) \subseteq \mathbb{T}$ (cf. Lemma 1.4). As any $h \in L^\infty(\mathbb{T})$ defines a Hankel operator

$$H_h := P_+\theta_h P_+^* \in B(H^2(\mathbb{D})), \tag{14}$$

this leads us naturally to Hankel operators on $H^2(\mathbb{D})$. If h is unimodular with $h^\sharp = h$, so that θ_h is a unitary involution, then H_h is positive if and only if $\mathcal{E}_+$ is θ_h-positive.

The following theorem characterizes Hankel operators from several perspectives. Condition (a) provides the consistency with the abstract concept of a U-Hankel operator from Definition 1.3. The equivalence of (a) and (c) is well known ([Ni02, p. 180]).

Theorem 2.2. (Characterization of Hankel Operators on the disc) *Consider a bounded operator D on $H^2(\mathbb{D})$, the shift operator $(SF)(z) = zF(z)$, and the multiplication operators m_g defined by $g \in H^\infty(\mathbb{D})$ on $H^2(\mathbb{D})$. Then the following are equivalent:*

(a) *The Rosenblum relation* [1] *$DS = S^*D$ holds for the shift operator $(Sf)(z) = zf(z)$, i.e., D is a Hankel operator for the representation of $(\mathbb{N}, +)$ on $H^2(\mathbb{D})$ defined by $U_+(n) := S^n$.*

(b) *$Dm_g = m_{g^\sharp}^* D$ for all $g \in H^\infty(\mathbb{D})$, i.e., D is a Hankel operator for the representation of the involutive algebra $(H^\infty(\mathbb{D}), \sharp)$ on $H^2(\mathbb{D})$ by multiplication operators.*

(c) *There exists $h \in L^\infty(\mathbb{T})$ with $D = P_+ m_h R P_+^*$ for $R(F)(z) := \overline{z}F(\overline{z})$, $z \in \mathbb{D}$, i.e., D is a bounded Hankel operator on $H^2(\mathbb{D})$ in the classical sense.*

[1] See [Ro66], [Ni02, p. 205].

Proof. (b) $\Rightarrow$ (a) is trivial.
(a) $\Rightarrow$ (b): We recall from Example B.2(b) that the weak topology on the Banach algebra $H^\infty(\mathbb{D})$ is defined by the linear functionals

$$\eta_f(h) = \int_{\mathbb{T}} f(z) h^*(z)\, dz \quad \text{for} \quad f \in L^1(\mathbb{T}), h \in H^\infty(\mathbb{D}). \tag{15}$$

For $f_1, f_2 \in H^2(\mathbb{D})$, we observe that

$$\langle f_1, Dgf_2\rangle = \langle D^* f_1, gf_2\rangle = \int_{\mathbb{T}} \overline{(D^*f_1)^*(z)} g^*(z) f_2^*(z)\, dz = \eta_{\overline{(D^*f_1)^*} f_2^*}(g)$$

(cf. Example B.2), and

$$\begin{aligned}\langle g^\sharp f_1, Df_2\rangle &= \int_{\mathbb{T}} \overline{g^\sharp f_1}^*(z)(Df_2)^*(z)\, dz = \int_{\mathbb{T}} \overline{f_1^*(z)} g^*(\overline{z})(Df_2)^*(z)\, dz \\ &= \int_{\mathbb{T}} (f_1^\sharp)^*(z) g^*(z)(Df_2)^*(\overline{z})\, dz = \eta_{(f_1^\sharp)^*(Df_2)^{*,\vee}}(g),\end{aligned}$$

where we use the notation $h^\vee(z) = h(z^{-1})$, $z \in \mathbb{T}$. Both define weakly continuous linear functionals on $H^\infty(\mathbb{D})$ because $L^2(\mathbb{T})L^2(\mathbb{T}) = L^1(\mathbb{T})$, which by (a) coincide on polynomials. As these span a weakly dense subspace (Lemma B.6(a)), we obtain equality for every $g \in H^\infty(\mathbb{D})$, which is (b).
(a) $\Leftrightarrow$ (c): It is well known that (a) characterizes bounded Hankel operators on $H^2(\mathbb{D})$ (cf. [Pe98, Thm. 2.6]). This relation immediately implies

$$\langle z^j, Dz^k\rangle = \langle z^j, DS^k 1\rangle = \langle z^j, (S^*)^k D1\rangle = \langle S^k z^j, D1\rangle = \langle z^{j+k}, D1\rangle, \tag{16}$$

so that the matrix of D is a Hankel matrix [Ni02, Def. 6.1.1]. The converse requires Nehari's Theorem. We refer to [Ni02, Part B, 1.4.1] for a nice short functional analytic proof. □

Remark 2.3. In the proof above we have used Nehari's Theorem ([Pe98, Thm. 2.1], [Ni02, Part B, 1.4.1], [Ni19, Thm. 4.7.1]) which actually contains the finer information that every bounded Hankel operator on $H^2(\mathbb{D})$ is of the form H_h (see (14)), where $h \in L^\infty(\mathbb{T})$ can even be chosen in such a way that

$$\|H_h\| = \|h\|_\infty.$$

As $H_h = 0$ if and only if $h \in H^\infty(\mathbb{D}_-)$ for $\mathbb{D}_- = \{z \in \mathbb{C} \colon |z| > 1\}$, bounded Hankel operators on $H^2(\mathbb{D})$ are parametrized by the quotient space $L^\infty(\mathbb{T})/H^\infty(\mathbb{D}_-)$ (cf. [Pa88, Cor. 3.4]). As

$$\|H_h\| = \mathrm{dist}_{L^\infty(\mathbb{T})}(h, H^\infty(\mathbb{D}_-)), \tag{17}$$

the embedding $L^\infty(\mathbb{T})/H^\infty(\mathbb{D}_-) \hookrightarrow B(H^2(\mathbb{D}))$ is isometric.

We now recall how positive Hankel operators can be classified by using Hamburger's Theorem on moment sequences.

Definition 2.4. (The Carleson measure μ_H) Suppose that H is a positive Hankel operator. Then (16) shows that the sequence $(a_n)_{n\in\mathbb{N}_0}$ defined by

$$a_n := \langle S^n 1, H1\rangle \tag{18}$$

satisfies

$$a_{n+m} = \langle S^{n+m} 1, H1\rangle = \langle S^n 1, HS^m 1\rangle,$$

so that the positivity of H implies that the kernel $(a_{n+m})_{n,m\in\mathbb{N}_0}$ is positive definite, i.e., $(a_n)_{n\in\mathbb{N}}$ defines a bounded positive definite function on the involutive semigroup $(\mathbb{N}_0, +, \mathrm{id})$

whose bounded spectrum is $[-1,1]$. By Hamburger's Theorem ([BCR84, Thm. 6.2.2], [Ni02, Chap. 6]), there exists a unique positive Borel measure μ_H on $[-1,1]$ with

$$\int_{-1}^{1} x^n \, d\mu_H(x) = a_n \quad \text{for} \quad n \in \mathbb{N}_0.$$

Widom's Theorem (see Theorem A.1 in Appendix A) implies that

$$\langle f, Hg \rangle_{H^2(\mathbb{D})} = \int_{-1}^{1} \overline{f(x)} g(x) \, d\mu_H(x) \quad \text{for} \quad f, g \in H^2(\mathbb{D})$$

and it characterizes the measures on $[-1,1]$ which arise in this context. In particular, all these measures are finite and satisfy $\mu_H(\{1,-1\}) = 0$. We call μ_H the *Carleson measure of* H.

We shall return to positive Hankel operators on the disc $\mathbb{D}$ in Theorem 4.8.

3 Reflection positive one-parameter groups

In this section we proceed from the discrete to the continuous by studying reflection positive one-parameter groups instead of single reflection positive operators. In this context, the upper half plane $\mathbb{C}_+$ plays the same role as the unit disc does for the discrete context.

Definition 3.1. A *reflection positive one-parameter group* is a quadruple $(\mathcal{E}, \mathcal{E}_+, \theta, U)$ defining a reflection positive strongly continuous representation of the symmetric semigroup $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_{\mathbb{R}})$. This means that $(U_t)_{t \in \mathbb{R}}$ is a unitary one-parameter group on $\mathcal{E}$ such that

$$U_t \mathcal{E}_+ \subseteq \mathcal{E}_+ \quad \text{for} \quad t > 0 \quad \text{and} \quad \theta U_t \theta = U_{-t} \quad \text{for} \quad t \in \mathbb{R}. \tag{19}$$

As in Definition 1.2, we call a reflection positive one-parameter group *regular* if

$$\bigcap_{t \in \mathbb{R}} U_t \mathcal{E}_+ = \bigcap_{t > 0} U_t \mathcal{E}_+ \overset{!}{=} \{0\} \quad \text{and} \quad \overline{\bigcup_{t \in \mathbb{R}} U_t \mathcal{E}_+} = \overline{\bigcup_{t < 0} U_t \mathcal{E}_+} \overset{!}{=} \mathcal{E}.$$

If this is the case, then the representation theorem of Lax–Philipps provides a unitary equivalence from $(\mathcal{E}, \mathcal{E}_+, U)$ to

$$(L^2(\mathbb{R}, \mathcal{K}), L^2(\mathbb{R}_+, \mathcal{K}), S),$$

where $(S_t)_{t \in \mathbb{R}}$ are the unitary shift operators on $L^2(\mathbb{R}, \mathcal{K})$ and $\mathcal{K}$ is a Hilbert space (the multiplicity space) ([NÓ18, Thm. 4.4.1], [LP64, LP67, LP81]).

To classify reflection positive one-parameter groups, we consider in this paper the *multiplicity free case*, where $\mathcal{K} = \mathbb{C}$. Again, it is more convenient to work in the spectral representation, i.e., to use the Fourier transform and to consider on $\mathcal{E} = L^2(\mathbb{R})$ the unitary multiplication operators

$$(S_t f)(x) = e^{itx} f(x) \quad \text{for} \quad x \in \mathbb{R}$$

and the Hardy space $\mathcal{E}_+ := H^2(\mathbb{C}_+)$ which is invariant under the semigroup $(S_t)_{t>0}$.

Remark 3.2. (Representations of $(\mathbb{R}, \mathbb{R}_+)$) The closed invariant subspaces $\mathcal{E}_+ \subseteq H^2(\mathbb{C}_+) \subseteq L^2(\mathbb{R})$ under the semigroup $(S_t)_{t>0}$ are of the form $hH^2(\mathbb{C}_+)$ for an inner function h. This is Beurling's Theorem for the upper half plane. It follows from Beurling's Theorem for the disc ([Pa88, Thm. 6.4]) and Lemma B.6 by translation with Γ_2 (see Theorem 3.5).

The involutions θ satisfying $\theta U_t \theta = U_{-t}$ for $t \in \mathbb{R}$ are of the form $\theta_h = hR$, where $(Rf)(x) = f(-x)$ and h is a measurable unimodular function on $\mathbb{R}$ satisfying $h^\sharp = h$, where $h^\sharp(x) = \overline{h(-x)}$ as in (12).

3.1 Hankel operators on $H^2(\mathbb{C}_+)$

Definition 3.3. For $h \in L^\infty(\mathbb{R})$, we define on $H^2(\mathbb{C}_+)$ the *Hankel operator*

$$H_h := P_+ h R P_+^*, \quad \text{where} \quad (Rf)(x) := f(-x), x \in \mathbb{R},$$

$P_+ \colon L^2(\mathbb{R}) \to H^2(\mathbb{C}_+)$ is the orthogonal projection, and hR stands for the composition of R with multiplication by h (cf. [Pa88, p. 44]).

Let

$$j_\pm : H^\infty(\mathbb{C}_\pm) \to L^\infty(\mathbb{R}, \mathbb{C}), \quad f \mapsto f^*$$

denote the isometric embedding defined by the non-tangential boundary values. Accordingly, we identify $H^\infty(\mathbb{C}_\pm)$ with its image under this map in $L^\infty(\mathbb{R}, \mathbb{C})$.

Lemma 3.4. *For $h \in L^\infty(\mathbb{R})$, the following assertions hold:*

(a) $H_h^* = H_{h^\sharp}$. *In particular H_h is hermitian if $h^\sharp = h$.*

(b) $H_h = 0$ *if and only if $h \in H^\infty(\mathbb{C}_-)$.*

(c) $\|H_h\| \leq \|h\|$.

Proof. (a) follows from the following relation for $f, g \in H^2(\mathbb{C}_+)$:

$$\begin{aligned}\langle f, H_h g\rangle &= \int_{\mathbb{R}} \overline{f^*(x)} h(x) g^*(-x)\, dx = \int_{\mathbb{R}} h(-x)\overline{f^*(-x)} g^*(x)\, dx \\ &= \int_{\mathbb{R}} \overline{h^\sharp(x) f^*(-x)} g^*(x)\, dx = \langle H_{h^\sharp} f, g\rangle.\end{aligned}$$

(b) (cf. [Pa88, Cor. 4.8]) The operator H_h vanishes if and only if

$$hH^2(\mathbb{C}_-) = \theta_h H^2(\mathbb{C}_+) \subseteq H^2(\mathbb{C}_+)^\perp = H^2(\mathbb{C}_-),$$

which is equivalent to $h \in H^\infty(\mathbb{C}_-)$.
(c) follows from $\|P_+\| = \|R\| = 1$. □

The preceding lemma shows that we have a continuous linear map

$$L^\infty(\mathbb{R})/H^\infty(\mathbb{C}_-) \to B(H^2(\mathbb{C}_+)), \quad [h] \mapsto H_h$$

which is compatible with the involution $\sharp$ on the left and $*$ on the right. By Nehari's Theorem ([Pa88, Cor. 4.7]), this map is isometric. As $H^\infty(\mathbb{C}_+) \cap H^\infty(\mathbb{C}_-) = \mathbb{C}\mathbf{1}$, we obtain in particular an embedding

$$H^\infty(\mathbb{C}_+)/\mathbb{C}\mathbf{1} \hookrightarrow L^\infty(\mathbb{R})/H^\infty(\mathbb{C}_-) \to B(H^2(\mathbb{C}_+)), \quad [h] \mapsto H_h.$$

In Proposition 1.8, we have used a positive Hankel operator H to define a new scalar product that led us to a $*$-representation of $(S, \sharp)$. Here the key ingredient was the Hankel relation, an abstract form of the Rosenblum relation in Theorem 2.2(b). As the following theorem shows, this relation actually characterizes Hankel operators on $H^2(\mathbb{C}_+)$, so that the classical definition (Definition 3.3) and Definition 1.3 are consistent.

Theorem 3.5. (Characterization of Hankel Operators on the upper half plane) *Consider a bounded operator C, the isometries $S_t f = e_{it} f$, $t \geq 0$, and the multiplication operators m_g, $g \in H^\infty(\mathbb{C}_+)$ on $H^2(\mathbb{C}_+)$. We also consider the unitary isomorphism*

$$\Gamma_2 \colon L^2(\mathbb{T}) \to L^2(\mathbb{R}), \quad (\Gamma_2 f)(x) := \frac{\sqrt{2}}{x+i} f\Big(\frac{x-i}{x+i}\Big) \tag{20}$$

from [Ni19, p. 200] *which maps $H^2(\mathbb{D})$ to $H^2(\mathbb{C}_+)$ and the operator*

$$D := \Gamma_2^{-1} C \Gamma_2 \colon H^2(\mathbb{D}) \to H^2(\mathbb{D}).$$

Then the following are equivalent:

(a) *There exists* $h \in L^\infty(\mathbb{R})$ *with* $C = H_h$*, i.e.,* C *is a Hankel operator in the sense of* Definition 3.3.

(b) $Cm_g = m_{g^\sharp}^* C$ *for all* $g \in H^\infty(\mathbb{C}_+)$*, where* $g^\sharp(z) = \overline{g(-\overline{z})}$*, i.e.,* C *is a* U_+*-Hankel operator for the representation of the involutive algebra* $(H^\infty(\mathbb{C}_+), \sharp)$ *on* $H^2(\mathbb{C}_+)$ *by multiplication operators* $U_+(g) = m_g$*.*

(c) $CS_t = S_t^* C$ *for all* $t > 0$*, i.e.,* C *is a* U_+*-Hankel operator for the representation of* $\mathbb{R}_+$ *on* $H^2(\mathbb{C}_+)$ *defined by* $U_+(t)f := e_{it} f$ *for* $t \geq 0$*.*

(d) D *is a Hankel operator on* $H^2(\mathbb{D})$*.*

Proof. (a) $\Rightarrow$ (b): Suppose that $C = H_h$ for some $h \in L^\infty(\mathbb{R})$. For $f_1, f_2 \in H^2(\mathbb{C}_+)$ we then have

$$\langle f_1, H_h g f_2\rangle = \int_{\mathbb{R}} \overline{f_1^*(x)} h(x) g^*(-x) f_2^*(-x)\, dx = \int_{\mathbb{R}} \overline{g^{*,\sharp}(x) f_1^*(x)} h(x) f_2^*(-x)\, dx = \langle g^\sharp f_1, H_h f_2\rangle,$$

which is (b).
(b) $\Rightarrow$ (c) follows from $e_{it}^\sharp = e_{it}$ for $t > 0$.
(c) $\Rightarrow$ (b): For $f_1, f_2 \in H^2(\mathbb{C}_+)$ and $g \in H^\infty(\mathbb{C}_+)$, we observe that

$$\langle f_1, Cgf_2\rangle = \langle C^* f_1, g f_2\rangle = \int_{\mathbb{R}} \overline{C^* f_1}^*(x) g^*(x) f_2^*(x)\, dx = \eta_{\overline{C^* f_1} f_2}(g)$$

(see Example B.2 for the functionals η_f) and

$$\begin{aligned}\langle g^\sharp f_1, Cf_2\rangle &= \int_{\mathbb{R}} \overline{(g^\sharp f_1)^*}(x) (Cf_2)^*(x)\, dx = \int_{\mathbb{R}} \overline{f_1^*(x)} g^*(-x) (Cf_2)^*(x)\, dx \\ &= \int_{\mathbb{R}} (f_1^\sharp)^*(x) g^*(x) (Cf_2)^*(-x)\, dx = \eta_{f_1^\sharp (Cf_2)^\vee}(g),\end{aligned}$$

where we use the notation $h^\vee(x) := h(-x)$ for $x \in \mathbb{R}$. Both define weakly continuous linear functionals on $H^\infty(\mathbb{C}_+)$, which by (c) coincide on the functions e_{it}, $t > 0$. As these span a weakly dense subspace (Lemma B.6(b)), we obtain equality for every $g \in H^\infty(\mathbb{C}_+)$, which is (b).
(b) $\Leftrightarrow$ (d): The Cayley transform $\omega\colon \mathbb{D} \to \mathbb{C}_+, \omega(z) := i\frac{1+z}{1-z}$ defines an isometric isomorphism $L^\infty(\mathbb{T}) \to L^\infty(\mathbb{R}), g \mapsto g \circ \omega^{-1}$ which restricts to an isomorphism $H^\infty(\mathbb{D}) \to H^\infty(\mathbb{C}_+)$ and satisfies

$$\Gamma_2 \circ m_g = m_{g\circ\omega^{-1}} \circ \Gamma_2.$$

Therefore (b) is equivalent to

$$Dm_{g\circ\omega} = m_{g^\sharp\circ\omega}^* D \quad \text{for} \quad g \in H^\infty(\mathbb{C}_+),$$

which is (d) by Theorem 2.2.
(d) $\Rightarrow$ (a): Suppose that $D = D_k$ as in Theorem 2.2. For $f \in H^2(\mathbb{D})$, we then have for $x \in \mathbb{R}$

$$\begin{aligned}(C\Gamma_2(f))(x) &= \Gamma_2(Df)(x) = \frac{\sqrt{2}}{x+i}(Df)^*(\omega^{-1}(x)) = \frac{\sqrt{2}k(\omega^{-1}(x))}{x+i} f^*(\overline{\omega^{-1}(x)}) \\ &= k(\omega^{-1}(x)) \frac{i-x}{(i+x)} \frac{\sqrt{2}}{(-x+i)} f^*(\omega^{-1}(-x)) = k(\omega^{-1}(x)) \frac{i-x}{i+x} \Gamma_2(f)^*(-x).\end{aligned}$$

The assertion now follows with

$$h(x) := k(\omega^{-1}(x)) \frac{i-x}{i+x} = -k(\omega^{-1}(x))\omega^{-1}(x) \tag{21}$$

(cf. [Pa88, Thm. 4.6]). $\square$

3.2 Widom's Theorem for the upper half-plane

In this subsection we translate Widom's Theorem (Theorem A.1) characterizing the Carleson measures of positive Hankel operators on the disc to a corresponding result on the upper half plane. This is easily achieved by using Theorem 3.5 for the translation process.

Let H be a positive Hankel operator on $H^2(\mathbb{C}_+)$. For $t \geq 0$, the exponential functions $e_{it}(z) = e^{itz}$ in $H^\infty(\mathbb{C}_+)$ satisfy $e_{it}^\sharp = e_{it}$. Therefore the function

$$\varphi_H : \mathbb{R}_+ \to \mathbb{R}, \quad \varphi_H(t) := \langle e_{it/2}, He_{it/2} \rangle_{H^2(\mathbb{C}_+)} \tag{22}$$

satisfies

$$\varphi_H(t+s) = \langle e_{i(t+s)/2}, He_{i(t+s)/2} \rangle_{H^2(\mathbb{C}_+)} = \langle e_{it}, He_{is} \rangle_{H^2(\mathbb{C}_+)} \quad \text{for} \quad s, t > 0,$$

so that the kernel $(\varphi_H(t+s))_{t,s>0}$ is positive definite. This means that φ_H is a positive definite function on the involutive semigroup $(\mathbb{R}_+, +, \mathrm{id})$ bounded on $[1, \infty)$. By the Hausdorff–Bernstein–Widder Theorem ([BCR84, Thm. 6.5.12], [Ne99, Thm. VI.2.10]), there exists a unique positive Borel measure μ_H on $[0, \infty)$ with

$$\varphi_H(t) = \int_0^\infty e^{-\lambda t} \, d\mu_H(\lambda) \quad \text{for} \quad t > 0. \tag{23}$$

Widom's Theorem for $\mathbb{C}_+$ (Theorem 3.7 below) now implies that

$$\langle f, Hg \rangle_{H^2(\mathbb{C}_+)} = \int_0^\infty \overline{f(i\lambda)} g(i\lambda) \, d\mu_H(\lambda) \quad \text{for} \quad f, g \in H^2(\mathbb{C}_+)$$

and it characterizes the measures μ_H on $[0, \infty)$ which correspond to positive bounded Hankel operators. In particular, all these measures satisfy $\mu_H(\{0\}) = 0$.

Definition 3.6. The measure μ_H on $\mathbb{R}_+$ is called the Carleson measure of H.

Theorem 3.7. (Widom's Theorem for the upper half-plane) *For a positive Borel measure μ on $\mathbb{R}_+$, we consider the measure ρ on $\mathbb{R}_+$ defined by*

$$d\rho(\lambda) := \frac{d\mu(\lambda)}{1 + \lambda^2}.$$

Then the following are equivalent:

(a) *There exists an $\alpha \in \mathbb{R}$ with*

$$\int_{\mathbb{R}_+} |f(i\lambda)|^2 \, d\mu(\lambda) \leq \alpha \|f\|^2 \quad \textit{for} \quad f \in H^2(\mathbb{C}_+), \tag{24}$$

i.e., μ is the Carleson measure of a positive Hankel operator on $H^2(\mathbb{C}_+)$.

(b) $\rho((0, x)) = O(x)$ *and* $\rho((x^{-1}, \infty)) = O(x)$ *for* $x \to 0+$.

If these conditions are satisfied, then $\rho(\mathbb{R}_+) < \infty$ and there exist $\beta, \gamma > 0$ such that

$$\rho((0, \varepsilon]) \leq \beta \varepsilon \quad \textit{and} \quad \rho([t, \infty)) \leq \frac{\gamma}{t} \quad \textit{for every} \quad \varepsilon, t \in \mathbb{R}_+.$$

Proof. Condition (a) is equivalent to the existence of a positive Hankel operator C on $H^2(\mathbb{C}_+)$ with $\mu = \mu_C$. Let D be the corresponding Hankel operator on $H^2(\mathbb{D})$ (Theorem 3.5) and consider the diffeomorphism

$$\gamma : \mathbb{R}_+ \to (-1, 1), \quad \gamma(\lambda) = \frac{\lambda - 1}{\lambda + 1} = \omega^{-1}(i\lambda).$$

For $f \in H^2(\mathbb{D})$, we then have

$$\begin{aligned}
&\int_{-1}^{1} |f(t)|^2 \, d\mu_D(t) = \langle f, Df \rangle_{H^2(\mathbb{D})} = \langle \Gamma_2(f), C\Gamma_2(f) \rangle_{H^2(\mathbb{C}_+)} \\
&= \int_{\mathbb{R}_+} |\Gamma_2(f)(i\lambda)|^2 \, d\mu_C(\lambda) = 2 \int_{\mathbb{R}_+} \frac{|f(\omega^{-1}(i\lambda))|^2}{(1+\lambda)^2} \, d\mu_C(\lambda) \\
&= 2 \int_{\mathbb{R}_+} \frac{|f(\gamma(\lambda))|^2}{(1+\lambda)^2} \, d\mu_C(\lambda) = 2 \int_{-1}^{1} \frac{|f(t)|^2}{(1+\gamma^{-1}(t))^2} \, d(\gamma_* \mu_C)(t).
\end{aligned}$$

As $\gamma^{-1}(t) = -i\omega(t) = \frac{1+t}{1-t}$ and $1 + \frac{(1+t)}{(1-t)} = \frac{2}{(1-t)}$, it follows that

$$d\mu_D(t) = \frac{(1-t)^2}{2} d(\gamma_* \mu_C)(t).$$

We conclude that

$$\begin{aligned}
\mu_D((1-x, 1)) &= \int_{1-x}^{1} \frac{(1-t)^2}{2} d(\gamma_* \mu_C)(t) = \int_{\gamma^{-1}(1-x)}^{\infty} \frac{(1-\gamma(\lambda))^2}{2} d\mu_C(\lambda) \\
&= \int_{\frac{2}{x}-1}^{\infty} \frac{2}{(\lambda+1)^2} d\mu_C(\lambda) = 2 \int_{\frac{2}{x}-1}^{\infty} \frac{1+\lambda^2}{(\lambda+1)^2} d\rho(\lambda).
\end{aligned}$$

Therefore $\mu_D((1-x, 1))$ has for $x \to 0^+$ the same asymptotics as $\rho((x^{-1}, \infty))$. Likewise

$$\begin{aligned}
\mu_D((-1, -1+x)) &= \int_{-1}^{-1+x} \frac{(1-t)^2}{2} d(\gamma_* \mu_C)(t) = \int_{0}^{\gamma^{-1}(x-1)} \frac{(1-\gamma(\lambda))^2}{2} d\mu_C(\lambda) \\
&= \int_{0}^{\frac{x}{2-x}} \frac{2}{(\lambda+1)^2} d\mu_C(\lambda) = 2 \int_{0}^{\frac{x}{2-x}} \frac{1+\lambda^2}{(\lambda+1)^2} d\rho(\lambda).
\end{aligned}$$

This shows that $\mu_D((-1, -1+x))$ has for $x \to 0+$ the same asymptotics as $\rho((0, x))$. Therefore the assertion follows from Widom's Theorem for the disc (Theorem A.1).

Now we assume that ρ satisfies (b). Then there exist $\beta', \gamma' > 0$ and $\varepsilon_0, t_0 \in \mathbb{R}_+$ such that

$$\frac{\rho((0, \varepsilon])}{\varepsilon} \leq \beta' \quad \text{and} \quad \rho([t, \infty))\, t \leq \gamma' \quad \text{for every} \quad \varepsilon \leq \varepsilon_0, \ t \geq t_0.$$

Then

$$\rho(\mathbb{R}_+) = \rho((0, \varepsilon_0)) + \rho([\varepsilon_0, t_0]) + \rho((t_0, \infty)) \leq \beta' + \rho([\varepsilon_0, t_0]) + \gamma' < \infty.$$

For $\varepsilon > \varepsilon_0$ and $t < t_0$, we now find

$$\frac{\rho((0, \varepsilon])}{\varepsilon} \leq \frac{\rho(\mathbb{R}_+)}{\varepsilon_0} \quad \text{and} \quad \rho([t, \infty))\, t \leq \rho(\mathbb{R}_+)\, t_0.$$

This completes the proof. □

3.3 The symbol kernel of a positive Hankel operator

Definition 3.8. Let H be a Hankel operator on $H^2(\mathbb{C}_+)$ and

$$Q(z, w) = Q_w(z) = \frac{1}{2\pi} \frac{i}{z - \overline{w}}$$

be the Szegö kernel of $\mathbb{C}_+$ (cf. Appendix C). Then we associate to H its *symbol kernel*, i.e., the kernel

$$Q_H(z, w) := \langle Q_z, HQ_w \rangle = (HQ_w)(z) = \overline{(H^* Q_z)(w)}. \tag{25}$$

Clearly, Q_H is holomorphic in the first argument and antiholomorphic in the second argument.

By [Ne99, Lemma I.2.4], the Hankel operator H is positive if and only if its symbol kernel Q_H is positive definite. Suppose that this is the case and let μ_H be the corresponding Carleson measure on $\mathbb{R}_+$. Then

$$\begin{aligned} Q_H(z,w) &= \int_0^\infty \overline{Q_z(i\lambda)} Q_w(i\lambda)\, d\mu_H(\lambda) = \frac{1}{4\pi^2} \int_0^\infty \frac{d\mu_H(\lambda)}{(-i\lambda - z)(i\lambda - \overline{w})} \\ &= \frac{1}{4\pi^2} \int_0^\infty \frac{d\mu_H(\lambda)}{(\lambda - iz)(\lambda + i\overline{w})}. \end{aligned} \tag{26}$$

Definition 3.9. From Widom's Theorem for the upper half plane (Theorem 3.7), we know that the measure $\frac{d\mu(\lambda)}{1+\lambda^2}$ is finite, so that,

$$\kappa(z) := \int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{1}{z+\lambda}\, d\mu_H(\lambda) \tag{27}$$

defines a holomorphic function on $\mathbb{C} \setminus (-\infty, 0]$ ([Do74, Ch. II, Thm. 1]).

For $z, w \in \mathbb{C}_r$, we then have

$$\kappa(z) - \kappa(w) = \int_{\mathbb{R}_+} \frac{1}{w+\lambda} - \frac{1}{z+\lambda}\, d\mu_H(\lambda) = \int_{\mathbb{R}_+} \frac{z-w}{(w+\lambda)(z+\lambda)}\, d\mu_H(\lambda),$$

so that

$$\frac{\kappa(z) - \kappa(w)}{z-w} = \int_{\mathbb{R}_+} \frac{d\mu_H(\lambda)}{(w+\lambda)(z+\lambda)} = 4\pi^2 Q_H(iz, i\overline{w}). \tag{28}$$

4 Schober's representation theorem

In this section we explain how to find for every positive Hankel operator H on $H^2(\mathbb{C}_+)$ an explicit bounded function $h_H \in L^\infty(\mathbb{R})$ with values in $i\mathbb{R}$ such that $h_H^\sharp = h_H$ and H is the corresponding Hankel operator, i.e., $H_{h_H} = H$. This supplements Nehari's classical theorem by a constructive component. Adding non-zero real constants then leads to functions f in the unit group of $L^\infty(\mathbb{R})$ with $H_f = H$, and we shall use this to shows that all Hankel positive one-parameter groups are actually reflection positive for a slightly modified scalar product.

4.1 An operator symbol for H

Theorem 4.1. *Let H be a positive Hankel operator on $H^2(\mathbb{C}_+)$ with Carleson measure μ_H and define*

$$h_H \colon \mathbb{R} \to i\mathbb{R}, \quad h_H(p) := \frac{i}{\pi} \cdot \int_{\mathbb{R}_+} \frac{p}{\lambda^2 + p^2}\, d\mu_H(\lambda).$$

Then $h_H \in L^\infty(\mathbb{R}, \mathbb{C})$ and the associated Hankel operator H_{h_H} equals H.

Proof. **Part 1:** We first show that h_H is bounded. Let $d\rho(\lambda) = \frac{d\mu_H(\lambda)}{1+\lambda^2}$ be the finite measure on $\mathbb{R}_+$ from Theorem 3.7. Then we have

$$\int_{\mathbb{R}_+} \frac{p}{\lambda^2 + p^2}\, d\mu_H(\lambda) = \int_{\mathbb{R}_+} \frac{p\left(1+\lambda^2\right)}{\lambda^2 + p^2}\, d\rho(\lambda).$$

For the integrand

$$f_p(\lambda) := \frac{p\left(1+\lambda^2\right)}{\lambda^2 + p^2} \quad \text{we have} \quad f_p'(\lambda) = \frac{2p\left(p^2 - 1\right)\lambda}{\left(\lambda^2 + p^2\right)^2}.$$

Hence the function f_p is increasing for $p \geq 1$, and therefore

$$\int_{(0,1]} f_p(\lambda)\, d\rho(\lambda) \leq f_p(1) \int_{(0,1]} d\rho(\lambda) = \frac{2p}{1+p^2} \cdot \rho((0,1]) \leq \rho((0,1]).$$

Now, let γ be the constant from Theorem 3.7. Then integration by parts (cf. Lemma A.5) leads for $p \geq 1$ to

$$\begin{aligned}
\int_{(1,\infty)} f_p(\lambda)\, d\rho(\lambda) &= \rho((1,\infty))\, f_p(1) + \int_{(1,\infty)} \rho((t,\infty))\, f_p'(t)\; dt \\
&\leq \rho((1,\infty))\, \frac{2p}{1+p^2} + \int_{(1,\infty)} \frac{\gamma}{t} \cdot \frac{2p\left(p^2-1\right)t}{\left(t^2+p^2\right)^2}\, dt \\
&\leq \rho((1,\infty)) \cdot 1 + \gamma\left(p^2-1\right)\left[\frac{\frac{tp}{t^2+p^2} + \arctan\left(\frac{t}{p}\right)}{p^2}\right]_1^\infty \\
&= \rho((1,\infty)) + \gamma\, \frac{p^2-1}{p^2}\left[\frac{\pi}{2} - \frac{p}{1+p^2} - \arctan\left(\frac{1}{p}\right)\right] \leq \rho((1,\infty)) + \frac{\gamma\pi}{2}.
\end{aligned}$$

So, for every $p \geq 1$, we have

$$\begin{aligned}
\int_{\mathbb{R}_+} \frac{p}{\lambda^2+p^2}\, d\mu_H(\lambda) &= \int_{\mathbb{R}_+} f_p(\lambda)\; d\rho(\lambda) = \int_{(0,1]} f_p(\lambda)\, d\rho(\lambda) + \int_{(1,\infty)} f_p(\lambda)\, d\rho(\lambda) \\
&\leq \rho((0,1]) + \rho((1,\infty)) + \frac{\gamma\pi}{2} = \rho(\mathbb{R}_+) + \frac{\gamma\pi}{2}.
\end{aligned}$$

For $p \in (0,1)$, the function f_p is decreasing and therefore

$$\int_{(1,\infty)} f_p(\lambda)\, d\rho(\lambda) \leq f_p(1) \int_{(1,\infty)} d\rho(\lambda) = \frac{2p}{1+p^2} \cdot \rho((1,\infty)) \leq \rho((1,\infty)).$$

Now, let β be the constant from Theorem 3.7. Then, for $p < 1$, we have

$$\begin{aligned}
\int_{(0,1]} f_p(\lambda)\, d\rho(\lambda) &= \rho((0,1])\, f_p(1) - \int_{(0,1]} \rho((0,t])\, f_p'(t)\; dt \\
&\leq \rho((0,1])\, \frac{2p}{1+p^2} - \int_{(0,1]} \beta t \cdot \frac{2p\left(p^2-1\right)t}{\left(t^2+p^2\right)^2}\, dt \\
&\leq \rho((0,1]) \cdot 1 + \beta\left(1-p^2\right)\left[\arctan\left(\frac{t}{p}\right) - \frac{tp}{t^2+p^2}\right]_0^1 \\
&= \rho((0,1]) + \beta\left(1-p^2\right)\left[\arctan\left(\frac{1}{p}\right) - \frac{p}{1+p^2}\right] \leq \rho((0,1]) + \frac{\beta\pi}{2}.
\end{aligned}$$

So, for every $p \in (0,1)$, we have

$$\begin{aligned}
\int_{\mathbb{R}_+} \frac{p}{\lambda^2+p^2}\, d\mu_H(\lambda) &= \int_{\mathbb{R}_+} f_p(\lambda)\; d\rho(\lambda) = \int_{(0,1]} f_p(\lambda)\, d\rho(\lambda) + \int_{(1,\infty)} f_p(\lambda)\, d\rho(\lambda) \\
&\leq \rho((0,1]) + \frac{\beta\pi}{2} + \rho((1,\infty)) = \rho(\mathbb{R}_+) + \frac{\beta\pi}{2}.
\end{aligned}$$

Therefore, for every $p \in \mathbb{R}_+$, we have

$$|h_H(p)| = \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{p}{\lambda^2+p^2}\, d\mu_H(\lambda) \leq \frac{1}{\pi}\rho(\mathbb{R}_+) + \frac{1}{2}\max\{\beta,\gamma\}.$$

Since $h_H(-p) = -h_H(p)$, this yields

$$\|h_H\|_\infty \le \frac{1}{\pi}\rho(\mathbb{R}_+) + \frac{1}{2}\max\{\beta,\gamma\}$$

and therefore $h_H \in L^\infty(\mathbb{R},\mathbb{C})^\sharp$, where $h_H^\sharp = h_H$ follows by $h_H(-p) = -h_H(p) = \overline{h_H(p)}$.
Part 2: For the second statement, we recall the function

$$\kappa : \mathbb{C}\setminus(-\infty,0] \to \mathbb{C}, \quad \kappa(z) = \int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{1}{\lambda+z}\,d\mu_H(\lambda)$$

from (27). Then, for $p \in \mathbb{R}^\times$, we have

$$\begin{aligned}\operatorname{Im}(\kappa(ip)) &= \operatorname{Im}\left(\int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{1}{\lambda+ip}\,d\mu_H(\lambda)\right)\\ &= \operatorname{Im}\left(\int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{\lambda-ip}{\lambda^2+p^2}\,d\mu_H(\lambda)\right) = \int_{\mathbb{R}_+} \frac{p}{\lambda^2+p^2}\,d\mu_H(\lambda),\end{aligned}$$

so

$$h_H(p) = \frac{i}{\pi}\cdot\operatorname{Im}(\kappa(ip)). \tag{29}$$

For the real part, we get

$$\begin{aligned}\operatorname{Re}(\kappa(ip)) &= \operatorname{Re}\left(\int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{1}{\lambda+ip}\,d\mu_H(\lambda)\right) = \operatorname{Re}\left(\int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{\lambda-ip}{\lambda^2+p^2}\,d\mu_H(\lambda)\right)\\ &= \int_{\mathbb{R}_+} \frac{\lambda}{1+\lambda^2} - \frac{\lambda}{\lambda^2+p^2}\,d\mu_H(\lambda) = \left(p^2-1\right)\int_{\mathbb{R}_+} \frac{\lambda}{(1+\lambda^2)(\lambda^2+p^2)}\,d\mu_H(\lambda)\end{aligned}$$

and therefore

$$\begin{aligned}|\operatorname{Re}(\kappa(ip))| &= \left|p^2-1\right|\int_{\mathbb{R}_+} \frac{\lambda}{(1+\lambda^2)(\lambda^2+p^2)}\,d\mu_H(\lambda)\\ &\le \left|p^2-1\right|\int_{\mathbb{R}_+} \frac{4\lambda}{(1+\lambda)^2(|p|+\lambda)^2}\,d\mu_H(\lambda).\end{aligned}$$

For $p \in \mathbb{R}^\times$, we now define the function

$$n_p : \mathbb{C}_+ \to \mathbb{C}, \quad n_p(z) = \frac{2\sqrt{z}}{(1-iz)(|p|-iz)},$$

where by $\sqrt{\cdot}$ we denote the inverse of the function $\mathbb{C}_r \cap \mathbb{C}_+ \to \mathbb{C}_+, z \mapsto z^2$. Then n_p is holomorphic on $\mathbb{C}_+$ and for $y > 0$, we have

$$|n_p(x+iy)|^2 = \frac{4\sqrt{x^2+y^2}}{((1+y)^2+x^2)\,((|p|+y)^2+x^2)} \le \frac{4\sqrt{x^2+y^2}}{(1+y^2+x^2)(p^2+x^2)} \le \frac{2}{p^2+x^2},$$

so

$$\sup_{y>0}\int_{\mathbb{R}} |n_p(x+iy)|^2\,dx \le \int_{\mathbb{R}} \frac{2}{p^2+x^2}\,dx = \frac{2\pi}{|p|} < \infty$$

and therefore $n_p \in H^2(\mathbb{C}_+)$. Since μ_H is a Carleson measure, by Theorem 3.7(a), there is a constant $\alpha \ge 0$ such that

$$\int_{\mathbb{R}_+} \overline{f(i\lambda)}g(i\lambda)\,d\mu_H(\lambda) \le \alpha\,\|f\|_2\,\|g\|_2 \quad \text{for every} \quad f,g \in H^2(\mathbb{C}_+).$$

Then

$$\begin{aligned}
\left|\mathrm{Re}\left(\kappa\left(ip\right)\right)\right| &\leq \left|p^2-1\right|\int_{\mathbb{R}_+}\frac{4\lambda}{\left(1+\lambda\right)^2\left(\left|p\right|+\lambda\right)^2}\,d\mu_H\left(\lambda\right) = \left|p^2-1\right|\int_{\mathbb{R}_+}\left|n_p\left(i\lambda\right)\right|^2\,d\mu_H\left(\lambda\right)\\
&\leq \left|p^2-1\right|\alpha\left\|n_p\right\|_2^2 = \alpha\left|p^2-1\right|\int_{\mathbb{R}}\frac{4\left|x\right|}{\left(1+x^2\right)\left(p^2+x^2\right)}\,dx\\
&= 4\alpha\left|p^2-1\right|\int_0^\infty\frac{2x}{\left(1+x^2\right)\left(p^2+x^2\right)}\,dx = 4\alpha\left|\int_0^\infty\frac{2x}{1+x^2}-\frac{2x}{p^2+x^2}\,dx\right|\\
&= 4\alpha\left|\left[\log\left(1+x^2\right)-\log\left(p^2+x^2\right)\right]_0^\infty\right| = 4\alpha\left|\left[\log\left(\frac{1+x^2}{p^2+x^2}\right)\right]_0^\infty\right| = 8\alpha\left|\log\left(\left|p\right|\right)\right|
\end{aligned}$$

for every $p\in\mathbb{R}^\times$. This estimate together with $\left\|h_H\right\|_\infty<\infty$ shows that, for $z,w\in\mathbb{C}_+$, the integrals

$$\int_{\mathbb{R}}\frac{\kappa\left(ip\right)}{\left(p-z\right)\left(p-w\right)}\,dp \quad \text{and} \quad \int_{\mathbb{R}}\frac{\overline{\kappa\left(ip\right)}}{\left(p-z\right)\left(p-w\right)}\,dp$$

exist. We have

$$\int_{\mathbb{R}}\frac{\kappa\left(ip\right)}{\left(p-z\right)\left(p-w\right)}\,dp = \int_{\mathbb{R}}\frac{\kappa\left(-ip\right)}{\left(p+z\right)\left(p+w\right)}\,dp = 0 \tag{30}$$

because the function $p\to\frac{\kappa(-ip)}{(p+z)(p+w)}$ is holomorphic on $\mathbb{C}_+$.

By the Residue Theorem, for $z,w\in\mathbb{C}_+$ with $z\neq w$ and $\kappa\left(-iz\right)\neq 0\neq\kappa\left(-iw\right)$, we get

$$\begin{aligned}
\int_{\mathbb{R}}\frac{\overline{\kappa\left(ip\right)}}{\left(p-z\right)\left(p-w\right)}\,dp &= \int_{\mathbb{R}}\frac{\kappa\left(-ip\right)}{\left(p-z\right)\left(p-w\right)}\,dp = 2\pi i\left(\frac{\kappa\left(-iz\right)}{z-w}+\frac{\kappa\left(-iw\right)}{w-z}\right)\\
&= 2\pi i\,\frac{\kappa\left(-iz\right)-\kappa\left(-iw\right)}{z-w} \overset{(28)}{=} \left(2\pi\right)^3 Q_H(z,-\overline{w}).
\end{aligned}$$

By continuity of both sides in z and w, we get

$$\int_{\mathbb{R}}\frac{\overline{\kappa\left(ip\right)}}{\left(p-z\right)\left(p-w\right)}\,dp = \left(2\pi\right)^3 Q_H(z,-\overline{w}) \quad \text{for every} \quad z,w\in\mathbb{C}_+. \tag{31}$$

For $z,w\in\mathbb{C}_+$, we finally obtain

$$\begin{aligned}
4\pi^2 Q_{H_{h_H}}\left(z,w\right) &= 4\pi^2\langle Q_z, h_H R Q_w\rangle = \int_{\mathbb{R}}\frac{h_H\left(p\right)}{\left(p-z\right)\left(-p-\overline{w}\right)}\,dp = \int_{\mathbb{R}}\frac{-h_H\left(p\right)}{\left(p-z\right)\left(p+\overline{w}\right)}\,dp\\
&= \int_{\mathbb{R}}\frac{-\frac{i}{\pi}\cdot\mathrm{Im}\left(\kappa\left(ip\right)\right)}{\left(p-z\right)\left(p+\overline{w}\right)}\,dp = \frac{1}{2\pi}\int_{\mathbb{R}}\frac{\overline{\kappa\left(ip\right)}-\kappa\left(ip\right)}{\left(p-z\right)\left(p+\overline{w}\right)}\,dp\\
&= \frac{1}{2\pi}\left(\int_{\mathbb{R}}\frac{\overline{\kappa\left(ip\right)}}{\left(p-z\right)\left(p+\overline{w}\right)}\,dp-\int_{\mathbb{R}}\frac{\kappa\left(ip\right)}{\left(p-z\right)\left(p+\overline{w}\right)}\,dp\right)\\
&\overset{(30)}{=} \frac{1}{2\pi}\int_{\mathbb{R}}\frac{\overline{\kappa\left(ip\right)}}{\left(p-z\right)\left(p+\overline{w}\right)}\,dp \overset{(31)}{=} 4\pi^2 Q_H(z,w)
\end{aligned}$$

This means that the operators H and H_{h_H} have the same symbol kernel, hence are equal by [Ne99, Lemma I.2.4]. □

Lemma 4.2. *Let $H\neq 0$ be a positive Hankel operator on $H^2(\mathbb{C}_+)$. Then there exist $c,a\in\mathbb{R}_+$ such that*

$$\left|h_H\left(p\right)\right|\geq c\cdot\frac{\left|p\right|}{a^2+p^2} \quad \textit{for every} \quad p\in\mathbb{R}^\times.$$

Proof. Since $H \neq 0$, we have $\mu_H \neq 0$, hence $\mu_H((0,a]) > 0$ for some $a > 0$. Then setting $c := \frac{\mu_H((0,a])}{\pi}$, for $p \in \mathbb{R}^\times$, we have

$$|h_H(p)| = \frac{1}{\pi} \int_{\mathbb{R}_+} \frac{|p|}{\lambda^2 + p^2} \, d\mu_H(\lambda) \geq \frac{1}{\pi} \int_{(0,a]} \frac{|p|}{\lambda^2 + p^2} \, d\mu_H(\lambda)$$
$$\geq \frac{1}{\pi} \int_{(0,a]} \frac{|p|}{a^2 + p^2} \, d\mu_H(\lambda) = c \cdot \frac{|p|}{a^2 + p^2}. \qquad \square$$

Choosing the measure $\mu = \delta_a$ for an $a \in \mathbb{R}_+$ shows that the estimate in this lemma is optimal.

Definition 4.3. (cf. [RR94, Thm. 5.13]) A holomorphic function on $\mathbb{C}_+$ is called an *outer function* if it is of the form

$$\mathrm{Out}(k,C)(z) = C \exp\left(\frac{1}{\pi i} \int_{\mathbb{R}} \left[\frac{1}{p-z} - \frac{p}{1+p^2}\right] \log(k(p)) \, dp\right),$$

where $C \in \mathbb{T}$ and $k \colon \mathbb{R} \to \mathbb{R}_+$ satisfies $\int_{\mathbb{R}} \frac{|\log(k(p))|}{1+p^2} \, dp < \infty$. Then $k = |\,\mathrm{Out}(k,C)^*|$. We write $\mathrm{Out}(k) := \mathrm{Out}(k,1)$. If k_1 and k_2 are two such functions, then so is their product, and

$$\mathrm{Out}(k_1 k_2) = \mathrm{Out}(k_1)\,\mathrm{Out}(k_2). \tag{32}$$

We also note that the function $k^\vee(p) = k(-p)$ satisfies

$$\mathrm{Out}(k^\vee) = \mathrm{Out}(k)^\sharp. \tag{33}$$

Theorem 4.4. *Let H be a positive Hankel operator on $H^2(\mathbb{C}_+)$. Then, for every $c \in \mathbb{R}^\times$, we have*

$$\delta := h_H + c\boldsymbol{1} \in L^\infty(\mathbb{R},\mathbb{C}) \quad \text{and} \quad \frac{1}{\delta} \in L^\infty(\mathbb{R},\mathbb{C}).$$

Further $H_\delta = H$ and there exists an outer function $g \in H^\infty(\mathbb{C}_+)^\times$ (the unit group of this Banach algebra) such that $|g^*|^2 = |\delta|$.

Proof. Since $h_H(\mathbb{R}) \subseteq i\mathbb{R}$ we have

$$\|\delta\|_\infty = \sqrt{\|h_H\|_\infty^2 + c^2} < \infty \quad \text{and} \quad \left\|\frac{1}{\delta}\right\|_\infty \leq \frac{1}{|c|},$$

which shows the first statement. For the second statement, we notice that $c\mathbf{1} \in H^\infty(\mathbb{C}_-)$ implies $H_{c\mathbf{1}} = 0$ by Lemma 3.4, so that $H_\delta = H_{h_H} + H_{c\mathbf{1}} = H + 0 = H$ by Lemma 3.4 and Theorem 4.1.

Finally, we have

$$\int_{\mathbb{R}} \frac{|\log|\delta(p)||}{1+p^2} \, dp \leq \int_{\mathbb{R}} \frac{\max\left\{\left|\log\|\delta\|_\infty\right|, \left|\log\left\|\frac{1}{\delta}\right\|_\infty\right|\right\}}{1+p^2} \, dp < \infty$$

and

$$\int_{\mathbb{R}} \frac{\left|\log\left|\left(\frac{1}{\delta}\right)(p)\right|\right|}{1+p^2} \, dp \leq \int_{\mathbb{R}} \frac{\max\left\{\left|\log\|\delta\|_\infty\right|, \left|\log\left\|\frac{1}{\delta}\right\|_\infty\right|\right\}}{1+p^2} \, dp < \infty,$$

so we obtain bounded outer functions $\mathrm{Out}(|\delta|^{1/2})$ and $\mathrm{Out}(|\delta|^{-1/2})$ whose product is $\mathrm{Out}(1) = 1$ ([RR94, §5.12]). In particular, $g := \mathrm{Out}(|\delta|^{1/2})$ is invertible in $H^\infty(\mathbb{C}_+)$ and $|g^*|^2 = |\delta|$. $\square$

4.2 From Hankel positivity to reflection positivity

For a positive Hankel operator H on $H^2(\mathbb{C}_+)$ and the corresponding function δ from Theorem 4.4, let ν be the measure on $\mathbb{R}$ with

$$d\nu(x) = |\delta(x)|\, dx.$$

As $\delta(-x) = c + h_H(-x) = c - h_H(x) = \overline{\delta(x)}$, we have $\delta^\sharp = \delta$, and in particular the function $|\delta|$ is symmetric. We consider the weighted L^2-space $L^2(\mathbb{R}, \mathbb{C}, \nu)$ with the corresponding scalar product $\langle \cdot, \cdot \rangle_\nu$. For the function

$$g := \mathrm{Out}(|\delta|^{1/2}) \in H^\infty(\mathbb{C}_+)^\times$$

we then have

$$|g^*|^2 = |\delta| \quad \text{and} \quad g^\sharp = g. \tag{34}$$

Furthermore, $gH^2(\mathbb{C}_+) = H^2(\mathbb{C}_+)$, and

$$m_{g^*} : L^2(\mathbb{R}, \nu) \to L^2(\mathbb{R}), \quad f \mapsto g^* \cdot f$$

is an isometric isomorphism of Hilbert spaces. We write

$$H^2(\mathbb{C}_+, \nu) := (H^2(\mathbb{C}_+), \|\cdot\|_\nu)$$

for $H^2(\mathbb{C}_+)$, endowed with the scalar product from $L^2(\mathbb{R}, \mathbb{C}, \nu)$, so that we obtain a unitary operator

$$m_g : H^2(\mathbb{C}_+, \nu) \to H^2(\mathbb{C}_+).$$

For the unimodular function $u := \frac{\delta}{|\delta|}$, we get with Theorem 4.4 for $a, b \in H^2(\mathbb{C}_+)$:

$$\begin{aligned} \langle a, Hb \rangle_{H^2(\mathbb{C}_+)} &= \langle a^*, \delta R b^* \rangle_{L^2(\mathbb{R})} = \langle \sqrt{|\delta|} a^*, \sqrt{|\delta|} u R b^* \rangle_{L^2(\mathbb{R})} \\ &= \langle a^*, u R b^* \rangle_{L^2(\mathbb{R}, \nu)} = \langle a, H_u b \rangle_{H^2(\mathbb{C}_+, \nu)}. \end{aligned} \tag{35}$$

As ν is symmetric and $u^\sharp = \frac{\delta^\sharp}{|\delta|^\sharp} = \frac{\delta}{|\delta|} = u$,

$$\theta_u(f)(x) := u(x) f(-x)$$

defines a unitary involution on $L^2(\mathbb{R}, \nu)$ (and on $L^2(\mathbb{R})$) for which the subspace $H^2(\mathbb{C}_+, \nu)$ is θ_u-positive by (35) (cf. Example 1.7). Therefore

$$(L^2(\mathbb{R}, \nu), H^2(\mathbb{C}_+, \nu), \theta_u, U) \quad \text{with} \quad (U_t f)(x) = e^{itx} f(x)$$

defines a reflection positive one-parameter group.

These are the essential ingredients in the proof of the following theorem:

Theorem 4.5. (Hankel positive representations are reflection positive) *Let $(\mathcal{E}, \mathcal{E}_+, U, H)$ be a regular multiplicity free Hankel positive representation of $(\mathbb{R}, \mathbb{R}_+, -\mathrm{id}_\mathbb{R})$. Then there exists an invertible bounded operator $g \in \mathrm{GL}(\mathcal{E})$ with $g\mathcal{E}_+ = \mathcal{E}_+$ commuting with $(U_t)_{t \in \mathbb{R}}$ and a unitary involution $\theta \in \mathrm{GL}(\mathcal{E})$ such that:*

(a) *$\theta U_t \theta = U_{-t}$ for $t \in \mathbb{R}$.*

(b) *θ is unitary for the scalar product $\langle \xi, \eta \rangle_g := \langle g\xi, g\eta \rangle$.*

(c) *With respect to $\langle \cdot, \cdot \rangle_g$, the quadruple $(\mathcal{E}, \mathcal{E}_+, \theta, U)$ is a reflection positive representation.*

(d) *$\langle \xi, H\eta \rangle = \langle \xi, \theta\eta \rangle_g = \langle g\xi, g\theta\eta \rangle$ for $\xi, \eta \in \mathcal{E}_+$.*

Proof. As we have seen in the introduction to Section 3, the Lax–Phillips Representation Theorem implies that, up to unitary equivalence, $\mathcal{E} = L^2(\mathbb{R})$ and $\mathcal{E}_+ = H^2(\mathbb{C}_+)$ with $(U_t f)(x) = e^{itx} f(x)$, so that H corresponds to a positive Hankel operator on $H^2(\mathbb{C}_+)$. We use the notation from the preceding discussion and Theorem 4.4. Then m_{g^*} defines an invertible operator on $L^2(\mathbb{R})$ commuting with U, and $\theta := uR$ satisfies (a) and (b). Further, (c) and (d) follow from (35). $\square$

Remark 4.6. For $H = H_\delta = P_+ \delta R P_+^*$, we see with Example 1.10(b) that $H m_g^{-2}$ also is a Hankel operator H_h with the operator symbol

$$h(x) = \frac{\delta(x)}{g^*(-x)^2} = \frac{\delta(x)}{\overline{g^*(x)}^2}.$$

As $|g^*|^2 = |\delta|$, the function h is unimodular. Further $g^\sharp = g$ and $\delta^\sharp = \delta$ imply $h^\sharp = h$, so that $\theta_h = hR$ is a unitary involution. We think of the factorization

$$H = H_h m_g^2$$

as a "polar decomposition" of H.

Remark 4.7. The weighted Hardy space $H^2(\mathbb{C}_+, \nu)$ has the reproducing kernel

$$Q^\nu(z, w) = \frac{Q(z, w)}{g(z)\overline{g(w)}}.$$

In fact, for $f \in H^2(\mathbb{C}_+, \nu)$ we have

$$\begin{aligned} \langle Q_w^\nu, f\rangle_{H^2(\mathbb{C}_+,\nu)} &= f(w) = g(w)^{-1}(fg)(w) = g(w)^{-1}\langle Q_w, fg\rangle_{H^2(\mathbb{C}_+)} \\ &= g(w)^{-1}\langle g^{-1} Q_w, f\rangle_{H^2(\mathbb{C}_+,\nu)}. \end{aligned}$$

We have a similar result for the symmetric semigroup $(\mathbb{Z}, \mathbb{N}, -\mathrm{id}_{\mathbb{Z}})$, which corresponds to single unitary operators.

Theorem 4.8. (Hankel positive operators are reflection positive) *Let $(\mathcal{E}, \mathcal{E}_+, U, H)$ be a regular multiplicity free Hankel positive operator. Then there exists an invertible bounded operator $g \in \mathrm{GL}(\mathcal{E})$ with $g\mathcal{E}_+ = \mathcal{E}_+$ commuting with U and a unitary involution $\theta \in \mathrm{GL}(\mathcal{E})$ such that:*

(a) $\theta U \theta = U^*$.

(b) *θ is unitary for the scalar product $\langle \xi, \eta\rangle_g := \langle g\xi, g\eta\rangle$.*

(c) *With respect to $\langle \cdot, \cdot\rangle_g$, the quadruple $(\mathcal{E}, \mathcal{E}_+, \theta, U)$ is a reflection positive operator.*

(d) *$\langle \xi, H\eta\rangle = \langle \xi, \theta\eta\rangle_g$ for $\xi, \eta \in \mathcal{E}_+$.*

Proof. Up to unitary equivalence, we may assume that

$$\mathcal{E} = L^2(\mathbb{T}), \quad \mathcal{E}_+ = H^2(\mathbb{D}) \quad \text{with} \quad (Uf)(z) = zf(z),$$

the shift operator (Wold decomposition), so that H corresponds to a positive Hankel operator on $H^2(\mathbb{D})$.

Let $C := \Gamma_2 H \Gamma_2^{-1}$ be the corresponding positive Hankel operator on $H^2(\mathbb{C}_+)$ (Theorem 3.5) which we write as $C = H_\delta$ as above in Theorem 4.4. Then (21) in the proof of Theorem 3.5 shows that $H = H_k$ for the function $k\colon \mathbb{T} \to \mathbb{C}$ defined by

$$k\colon \mathbb{T} \to \mathbb{C}, \quad k(z) := -\delta(\omega(z))\overline{z} \quad \text{for} \quad z \in \mathbb{T}.$$

Then $|k(z)| = |\delta(\omega(z))|$ is bounded with a bounded inverse.

We thus find an outer function $g \in H^\infty(\mathbb{D})^\times$ with $|g^*|^2 = |k|$ and consider the measure $d\nu(z) = |k(z)|\, dz$ on $\mathbb{T}$ ([Ru86, Thm. 17.16]; see also Lemma B.13). Then

$$m_g \colon H^2(\mathbb{D}, \nu) \to H^2(\mathbb{D})$$

is unitary and the unimodular function $u := \frac{k}{|k|}$ on $\mathbb{T}$ satisfies, for $a, b \in H^2(\mathbb{D})$:

$$\begin{aligned}\langle a, Hb\rangle_{H^2(\mathbb{D})} &= \langle a^*, kRb^*\rangle_{L^2(\mathbb{T})} = \langle \sqrt{|k|}a^*, \sqrt{|k|}uRb^*\rangle_{L^2(\mathbb{T})} \\ &= \langle a^*, uRb^*\rangle_{L^2(\mathbb{T},\nu)} = \langle a, H_u b\rangle_{H^2(\mathbb{D},\nu)}. \end{aligned} \tag{36}$$

Clearly, m_{g^*} defines an invertible operator on $L^2(\mathbb{T})$ commuting with U and $\theta := uR$ satisfies (a) and (b). As in the proof of Theorem 4.5, (c) and (d) follow from (36). □

A Widom's Theorem on Hankel operators on the disc

In this appendix we recall Widom's classical theorem on positive Hankel operators on the Hardy space of the unit disc. The arguments mostly follow Widom's original proof in [Wi66], including some simplifications. In particular the proof for the implication (b) ⇒ (c) was communicated to us by Christian Berg. For more information concerning Widom's Theorem, we refer to [Ni02, Thm. B.6.2.1], which contains in particular the equivalence of (a) and the inclusion $H^2(\mathbb{D}) \subseteq L^2([-1,1],\mu)$.

Theorem A.1. (Widom's Theorem; [Wi66]) *For a finite positive Borel measure μ on $[-1,1]$ with moment sequence*

$$c_j := \int_{-1}^1 x^j \, d\mu(x),$$

the following are equivalent:

(a) *The corresponding Hankel operator H on $H^2(\mathbb{D})$ is bounded, i.e., there exists an $\alpha \geq 0$ with*

$$\langle f, Hf\rangle_{H^2(\mathbb{D})} = \int_{-1}^1 |f(x)|^2 \, d\mu(x) \leq \alpha \|f\|^2 \quad \textit{for} \quad f \in H^2(\mathbb{D}).$$

(b) $c_j = O(j^{-1})$ *for* $j \to \infty$.

(c) $\mu([x,1]) = O(1-x)$ *as* $x \to 1$ *and* $\mu([-1,x]) = O(1+x)$ *as* $x \to -1$.

Proof. Since we may decompose $\mu = \mu_1 + \mu_2$ with $\mu_1([-1,0)) = 0$ and $\mu_2([0,1]) = 0$, we can reduce the discussion to measures on $[-1,0]$ and $[0,1]$. In fact, (a) holds for μ if and only if it holds for μ_1 and μ_2. The same is true for (c), where the first condition refers to μ_1 and the second one on μ_2. For (b), we write $c_j = c_j^1 + c_j^2$, according to the decomposition of μ. If (b) holds for μ_1 and μ_2, then it clearly holds for μ. If, conversely, (b) holds for μ, then the positive sequence $c_{2j} = c_{2j}^1 + c_{2j}^2$ is $O(j^{-1})$, and since both summands are positive, we get $c_{2j}^1 = O(j^{-1})$ and $c_{2j}^2 = O(j^{-1})$. As the sequences c_j^1 and $|c_j^2| = (-1)^j c_j^2$ are decreasing, it follows that c_j^1 and c_j^2 are $O(j^{-1})$.

After this discussion, it suffices to consider the case where $\mu = \mu_1$ is a measure on $[0,1]$.
(b) ⇒ (c) By (b), there exists $\beta > 0$ such that

$$\beta/n \geq c_n = \int_0^1 x^n d\mu(x) \geq \int_{1-\frac{1}{n}}^1 x^n d\mu(x) \geq \Big(1 - \frac{1}{n}\Big)^n \mu\Big(\Big[1 - \frac{1}{n}, 1\Big]\Big)$$

Using that $\left(1 - \frac{1}{n}\right)^n \to e^{-1}$ for $n \to \infty$, we find a $\gamma > 0$ with

$$\mu([1 - 1/n, 1]) \leq \gamma/n \quad \text{for all} \quad n \in \mathbb{N}.$$

Finally, since $x \mapsto \mu([1-x,1])$ is increasing we get $\mu([1-x,1]) \leq 2\gamma x$ by choosing n so that $\frac{1}{n+1} < x \leq \frac{1}{n}$.
(c) $\Rightarrow$ (b): Suppose that $\mu([1-x,1]) \leq \gamma x$ for $x > 0$ sufficiently small. Enlarging γ if necessary, we may assume that this relation holds for all $x \in [0,1]$. Integration by parts as in Lemma A.5 leads for $j > 0$ to

$$c_j = \int_0^1 x^j\, d\mu(x) = \int_0^1 jx^{j-1}\mu([x,1])\, dx \leq j\gamma\Big(\frac{1}{j} - \frac{1}{j+1}\Big) \leq \frac{\gamma}{j+1}.$$

(b) $\Rightarrow$ (a): Let $\beta > 0$ be such that $c_n \leq \beta/(n+1)$ for $n \in \mathbb{N}$. For $(a_n)_{n\in\mathbb{N}} \in \ell^2$, we then have

$$\Big|\sum_{n,m\geq 0} c_{n+m}\overline{a_n}a_m\Big| \leq \sum_{n,m\geq 0} c_{n+m}|a_n||a_m| \leq \beta \sum_{n,m\geq 0} \frac{|a_n||a_m|}{1+n+m} \leq \beta\pi\|a\|^2$$

by Hilbert's Theorem ([Ni02, Part B, 1.6.7]).
(a) $\Rightarrow$ (c): For $0 < r < 1$, we consider the function

$$f(z) := \sum_{j=0}^\infty r^j z^j = \frac{1}{1-rz}$$

in $H^2(\mathbb{D})$. Then

$$H(f,f) = \int_{-1}^1 |f(x)|^2\, d\mu(x) \leq \|H\|\|f\|^2 = \|H\|\frac{1}{1-r^2}.$$

This leads to the estimate

$$\frac{\mu([r,1]}{(1-r^2)^2} \leq \int_r^1 \frac{1}{(1-rx)^2}\, d\mu(x) \leq \int_r^1 |f(x)|^2\, d\mu(x) \leq \|H\|\|f\|^2 = \frac{\|H\|}{1-r^2}$$

and further to

$$\mu([r,1]) \leq 2\|H\|(1-r),$$

which implies (c). ☐

Definition A.2. A measure μ on $\mathbb{D}$ for which all functions in $H^2(\mathbb{D})$ are square-integrable is called a *Carleson measure* (cf. [Ca62], [Ni02, p. 327]). The implication (c) $\Rightarrow$ (a) in Widom's Theorem also follows from the much more general Theorem 1 in [Ca62] concerning measures on the disc.

Example A.3. (a) For the Lebesgue measure $d\mu(x) = dx$ on $[0,1]$, we obtain the moment sequence $c_j = \frac{1}{j+1}$, and by Hilbert's Theorem ([Ni02, Part B, 1.6.7]), the corresponding Hankel operator is bounded. In particular, there exists a constant C with

$$\int_{[0,1]} |f(x)|^2\, dx \leq C\|f\|^2 \quad \text{for} \quad f \in H^2(\mathbb{D}).$$

Note that $\int_{-1}^1 \frac{dx}{1-x^2} = \infty$.
(b) If $s > -\frac{1}{2}$, then the measure $d\mu(x) = x^s\, dx$ on $(0,1)$ is finite with moment sequence

$$c_j = \int_0^1 x^{j+s}\, dx = \frac{1}{j+1+s}.$$

Example A.4. Suppose that μ is a measure on $(-1,1)$ with $\int_{-1}^1 \frac{d\mu(x)}{1-x^2} < \infty$. On $H^2(\mathbb{D})$, the Szegö kernel is given by

$$Q_w(z) = Q(z,w) = \frac{1}{2\pi}\frac{1}{1-z\overline{w}}$$

We therefore have

$$|f(w)|^2 \leq \|f\|^2 \|Q_w\|^2 = \|f\|^2 Q(w,w) = \frac{1}{2\pi} \frac{\|f\|^2}{1-|w|^2},$$

and this shows that

$$\int_{-1}^{1} |f(x)|^2 \, d\mu(x) \leq \|f\|^2 \int_{-1}^{1} \frac{d\mu(x)}{1-x^2} < \infty.$$

For the sake of easier reference, we include the following version of integration by parts in this appendix.

Lemma A.5. (Integration by parts) *Let $a < b$ be real numbers and $f \in C^1([a,b])$. For a finite positive Borel measure μ on $[a,b]$ we then have*

$$\int_a^b f(x)\, d\mu(x) = \mu([a,b]) f(a) + \int_a^b \mu([t,b]) f'(t)\, dt$$

and

$$\int_a^b f(x)\, d\mu(x) = \mu([a,b]) f(b) - \int_a^b \mu([a,t]) f'(t)\, dt$$

Proof. With Fubini's Theorem, we obtain

$$\begin{aligned}\int_a^b f(x)\, d\mu(x) &= \int_a^b \Big(f(a) + \int_a^x f'(t)\, dt\Big)\, d\mu(x) = f(a)\mu([a,b]) + \int\int_{a\leq t\leq x\leq b} d\mu(x)\, f'(t)\, dt \\ &= f(a)\mu([a,b]) + \int_a^b \mu([t,b])\, f'(t)\, dt.\end{aligned}$$

We likewise get the second assertion. □

Remark A.6. There exists a refinement of Widom's Theorem characterizing those Hankel operators for which the measure μ lives on $[0,1)$, i.e., $\mu((-1,0)) = 0$ ([GP15]). This condition means that, not only the moment sequence $(c_n)_{n\in\mathbb{N}_0}$ of μ is positive definite on $\mathbb{N}_0$, but also the shifted sequence $(c_{n+1})_{n\in\mathbb{N}_0}$. As the shifted sequence satisfies

$$c_{n+1} = \int_{-1}^{1} t^n \cdot t\, d\mu(t) \quad \text{for} \quad n \in \mathbb{N}_0,$$

its positive definiteness is equivalent to the positivity of the measure $t\, d\mu(t)$, which is equivalent to $\mu((-1,0)) = 0$.

B The Banach $*$-algebra $(H^\infty(\Omega), \sharp)$

Let $\Omega \subseteq \mathbb{C}$ be a proper simply connected domain. By the Riemann Mapping Theorem, there exists a biholomorphic map $\varphi\colon \mathbb{D} \to \Omega$, so that $\sigma(z) := \varphi(\overline{\varphi^{-1}(z)})$ defines an antiholomorphic involution on Ω. We thus obtain on the Banach algebra $H^\infty(\Omega)$ of bounded holomorphic functions on Ω the isometric antilinear involution

$$f^\sharp(z) := \overline{f(\sigma(z))}, \tag{37}$$

turning into a Banach $*$-algebra. As this algebra and some of its subsemigroups play a key role in many of our arguments, we take in this appendix a closer look at some of its features. Its natural weak topology is of utmost importance because the weakly continuous positive functionals turn out to be closely related to Hankel operators resp., to measures on the fixed point set Ω^σ of σ on Ω (cf. Proposition B.8).

B.1 The weak topology

Lemma B.1. *Two antiholomorphic involutions on* Ω *are conjugate under the group* $\mathrm{Aut}(\Omega)$ *of biholomorphic automorphisms.*

Proof. By the Riemann Mapping Theorem, we may assume that $\Omega = \mathbb{D}$ is the unit disc. Let $\sigma\colon \mathbb{D} \to \mathbb{D}$ be an antiholomorphic involution. Then σ is an isometry for the hyperbolic metric. Therefore the midpoint of 0 and $\sigma(0)$ is fixed by σ. Conjugating by a suitable automorphism of $\mathbb{D}$, we may therefore assume that $\sigma(0) = 0$. Then $\psi(z) := \sigma(\overline{z})$ is a holomorphic automorphism fixing 0, hence of the form $\psi(z) = e^{i\theta}z$ for some $\theta \in \mathbb{R}$, so that $\sigma(z) = e^{i\theta}\overline{z} = \gamma(\overline{\gamma^{-1}(z)})$ for $\gamma(z) = e^{i\theta/2}z$. □

As all these involutions are conjugate under the group $\mathrm{Aut}(\Omega)$ by Lemma B.1, all Banach $*$-algebras $(H^\infty(\Omega), \sharp)$ are isomorphic.

According to Ando's Theorem ([An78]), the Banach space $H^\infty(\Omega) \cong H^\infty(\mathbb{D})$ has a unique predual space $H^\infty(\Omega)_* \subseteq H^\infty(\Omega)^*$, hence carries a natural *weak topology*, which is the initial (locally convex) topology defined by the elements of the predual. Note that the predual is norm-closed in $H^\infty(\Omega)^*$ because its embedding is isometric.

Example B.2. (a) (The upper half plane $\mathbb{C}_+$) We consider on $H^\infty(\mathbb{C}_+)$ the continuous linear functionals

$$\eta_f(g) := \int_{\mathbb{R}} g^*(x)f(x)\,dx, \qquad f \in L^1(\mathbb{R}), g \in H^\infty(\mathbb{C}_+).$$

Recall that $L^\infty(\mathbb{R}) \cong L^1(\mathbb{R})^*$. By [RR94, Ex. 12, p. 115], the closed subspace

$$H^\infty(\mathbb{C}_+) \cong \{g \in L^\infty(\mathbb{R})\colon gH^2(\mathbb{C}_+) \subseteq H^2(\mathbb{C}_+)\}$$

of $L^\infty(\mathbb{R})$ coincides with the annihilator of the subspace $H^1(\mathbb{C}_+)$ of $L^1(\mathbb{R})$. Therefore

$$(L^1(\mathbb{R})/H^1(\mathbb{C}_+))^* \cong H^1(\mathbb{C}_+)^\perp \cap L^\infty(\mathbb{R}) = H^\infty(\mathbb{C}_+).$$

By Ando's Theorem,

$$H^\infty(\mathbb{C}_+)_* \cong L^1(\mathbb{R})/H^1(\mathbb{C}_+) \tag{38}$$

is the unique predual of $H^\infty(\mathbb{C}_+)$. In particular, the weak topology is the initial topology with respect to the functionals η_f, $f \in L^1(\mathbb{R})$.

As the predual $H^\infty(\mathbb{C}_+)_*$ is a norm-closed subspace of $H^\infty(\mathbb{C})^*$, the image of the map

$$L^1(\mathbb{R}) \to H^\infty(\mathbb{C}_+)^*, \quad f \mapsto \eta_f$$

is closed. For $f \in L^1(\mathbb{R})$, we have $\eta_f^\sharp = \eta_{f^\sharp}$, so that η_f is symmetric if $f = f^\sharp$.

(b) (The unit disc $\mathbb{D}$) For the disc, we define for $f \in L^1(\mathbb{T})$ the functional

$$\eta_f(g) = \int_{\mathbb{T}} f(e^{it})g^*(e^{it})\,dt$$

on $H^\infty(\mathbb{D})$. Then the unique predual of $H^\infty(\mathbb{D})$ is the quotient of $L^1(\mathbb{T})$ by the subspace $\{f \in L^1(\mathbb{T})\colon \eta_f = 0\}$, which by [Ru86, Ch. 17, Ex. 2.9] is contained in $H^1(\mathbb{D})$ (see also the proof of Lemma B.6). For $g \in H^\infty(\mathbb{D})$ and $f \in H^1(\mathbb{D})$, we have

$$\frac{\eta_{f^*}(g)}{2\pi} = \int_{\mathbb{T}} f^*(e^{it})g^*(e^{it})\,\frac{dt}{2\pi} = (fg)(0) = f(0)g(0),$$

so that $\eta_{f^*} = 0$ is equivalent to $0 = f(0) = \frac{\eta_{f^*}(1)}{2\pi}$. With $H^1_0(\mathbb{D}) := \{f \in H^1(\mathbb{D})\colon f(0) = 0\}$, we thus obtain

$$H^\infty(\mathbb{D})_* \cong L^1(\mathbb{T})/H^1_0(\mathbb{D}). \tag{39}$$

Lemma B.3. *On the closed unit ball $B \subseteq H^\infty(\Omega)$, the following topologies coincide and turn B into a compact space:*

(a) *The topology τ_c of uniform convergence on compact subsets of Ω.*

(b) *The topology τ_p of pointwise convergence.*

(c) *The weak topology τ_w.*

Proof. By Montel's Theorem, (B, τ_c) is a compact space. Since τ_p is Hausdorff and $(B, \tau_c) \to (B, \tau_p)$ is continuous, the compactness of (B, τ_c) implies that $\tau_c = \tau_p$.

To show that $\tau_w = \tau_p$, we may w.l.o.g. assume that $\Omega = \mathbb{C}_+$. For each $z \in \mathbb{C}_+$ and $g \in H^\infty(\mathbb{C}_+)$, we have

$$g(z) = \int_{\mathbb{R}} P_z(x) g^*(x)\, dx,$$

where $P_z(x) = P(z, x)$ is the Poisson kernel of $\mathbb{C}_+$. As the functions P_z are L^1, it follows that point evaluations are weakly continuous. Therefore the map $(B, \tau_w) \to (B, \tau_p)$ is continuous. As (B, τ_w) is compact by the Banach–Alaoglu Theorem, this map is a homeomorphism, and thus $\tau_w = \tau_p$. □

Remark B.4. (a) For a σ-finite measure space $(X, \mathfrak{S}, \mu)$, the unique predual of $L^\infty(X, \mathfrak{S}, \mu)$ is the space $L^1(X, \mathfrak{S}, \mu)$ (Grothendieck, [Gr55]). However, the space $L^\infty(X, \mathfrak{S}, \mu)$ only depends on the measure class $[\mu]$. From this perspective, one should think of its predual as the space $\{f\mu \colon f \in L^1(X, \mathfrak{S}, \mu)\}$ of all finite measures on $(X, \mathfrak{S})$ which are absolutely continuous with respect to μ.

With this observation, it is clear how to identify the predual of $H^\infty(\mathbb{C}_+)$ in terms for weighted Hardy spaces. In particular, $H^\infty(\mathbb{C}_+)_* \cong L^1(\mathbb{R}, w\,dx)/H^1(\mathbb{R}, w\,dx)$ for any positive measurable function $w \colon \mathbb{R} \to \mathbb{R}_+$.

(b) (Saks spaces) The theory of Saks spaces, i.e., Banach spaces E with an additional locally convex topology γ satisfying certain compatibility conditions is a natural context to deal with similar structures. We refer to Cooper's monograph [Co87] for a detailed exposition of this theory. We shall not need it here. An interesting result one finds in [Co87, Prop. V.3.2] is that the space of continuous homomorphisms $(H^\infty(\mathbb{D}), \beta) \to \mathbb{C}$, where β is the topology on $H^\infty(\mathbb{D})$ defined by the Saks space structure, is homeomorphic to $\mathbb{D}$ (the point evaluations).

Proposition B.5. *The multiplication on $H^\infty(\Omega)$ is separately continuous with respect to the weak topology, i.e., the multipication maps $m_g(f) = gf$ are weakly continuous. Moreover, the involution $\sharp$ is weakly continuous.*

Proof. It suffices to verify this for $\Omega = \mathbb{C}_+$. In this case it follows from

$$\eta_f(g^\sharp) = \overline{\eta_{f^\sharp}(g)} \quad \text{and} \quad \eta_f(gh) = \eta_{fg^*}(h) \quad \text{for} \quad f \in L^1(\mathbb{R}), g, h \in H^\infty(\mathbb{C}_+). \qquad \square$$

B.2 Subsemigroups spanning weakly dense subalgebras

For $\Omega = \mathbb{D}$ we have $\sigma(z) = \overline{z}$, so that

$$f^\sharp(z) = \overline{f(\overline{z})} \quad \text{for} \quad f \in H^\infty(\mathbb{D}). \tag{40}$$

The elements $(z^n)_{n \geq 0}$ define a cyclic subsemigroup of $H^\infty(\mathbb{D})$ consisting of $\sharp$-symmetric elements and, for $f \in L^1(\mathbb{T})$, we have

$$\eta_f(z^n) = \int_{\mathbb{T}} e^{int} f(e^{it})\, dt = \widehat{f}(-n) \quad \text{for} \quad n \in \mathbb{N}_0. \tag{41}$$

For $\Omega = \mathbb{C}_+$ we have $\sigma(z) = -\overline{z}$, so that

$$f^\sharp(z) = \overline{f(-\overline{z})} \quad \text{for} \quad f \in H^\infty(\mathbb{C}_+). \tag{42}$$

The elements $(e_{it})_{t>0}$ define a one-parameter semigroup of $H^\infty(\mathbb{C}_+)$ consisting of $\sharp$-symmetric elements and, for $f \in L^1(\mathbb{R})$, we have

$$\eta_f(e_{it}) = \int_{\mathbb{R}} e^{itx} f(x)\,dx = \widehat{f}(-t) \quad \text{for} \quad t \geq 0. \tag{43}$$

Lemma B.6. (The Density Lemma)

(a) *The polynomials* $\mathbb{C}[z] \subseteq H^\infty(\mathbb{D})$ *are dense with respect to the weak topology.*

(b) *The one-parameter semigroup* $(e_{it})_{t>0}$ *spans a weakly dense subspace of* $H^\infty(\mathbb{C}_+)$.

(c) *For the strip* $\mathbb{S}_\beta = \{z \in \mathbb{C} \colon 0 < \operatorname{Im} z < \beta\}$, *the functions* $(e_{it})_{t\in\mathbb{R}}$ *span a weakly dense subspace of* $H^\infty(\mathbb{S}_\beta)$.

Proof. (a) ([Co87, Prop. V.2.2]) If $f \in L^1(\mathbb{T})$ is such that η_f vanishes on all polynomials, then all negative Fourier coefficients of f vanish:

$$\widehat{f}(-n) = \int_0^{2\pi} f(e^{it}) e^{int}\,dt = 0 \quad \text{for} \quad n > 0,$$

and [Ru86, Ch. 17, Ex. 2.9] implies that $f \in H^1(\mathbb{D})$. Now the vanishing of η_f follows from

$$\eta_f(h) = \int_{\mathbb{T}} f(e^{it}) h(e^{it})\,dt = 2\pi (fh)(0) = 2\pi f(0) h(0) = 0,$$

because $2\pi f(0) = \eta_f(1) = 0$.

(b) We have to show that, if a functional η_f, $f \in L^1(\mathbb{R})$, vanishes on each $e_{it}, t > 0$, then $\eta_f = 0$. So suppose that $\eta_f(e_{it}) = \widehat{f}(-t) = 0$ for $t > 0$. We claim that this implies that

$$\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt = 0 \quad \text{for} \quad \operatorname{Im} z < 0. \tag{44}$$

In fact, for $\operatorname{Im} z < 0$, we have

$$\frac{1}{t-z} = -i \int_0^\infty e^{itx} e^{-ixz}\,dx.$$

We thus obtain

$$\int_{\mathbb{R}} \frac{f(t)}{t-z}\,dt = -i \int_0^\infty \int_{\mathbb{R}} e^{itx} e^{-ixz} f(t)\,dt\,dx = -i \int_0^\infty \widehat{f}(-x) e^{-ixz}\,dx = 0.$$

In view of [RR94, Thm. 5.19(ii)], (44) implies that $f \in H^1(\mathbb{C}_+)$ in the sense that f is the boundary value of an H^1-function on $\mathbb{C}_+$.

We now show that this implies $\eta_f = 0$. In fact, for $h \in H^\infty(\mathbb{C}_+)$, we obtain

$$\eta_f(h) = \int_{\mathbb{R}} f(x) h(x)\,dx = 0$$

because the function $fh \in H^1(\mathbb{C}_+)$ has a continuous Fourier transform vanishing on $\mathbb{R}_-$, hence also in 0. [2]

(c) Let $f = (f_0, f_1) \in L^1(\mathbb{R}) \oplus L^1(\mathbb{R})$ be such that $\eta_f(e_{it}) = 0$ for every $t \in \mathbb{R}$. These numbers evaluate to

$$\begin{aligned}\eta_f(e_{it}) &= \int_{\mathbb{R}} e^{itx} f_0(x)\,dx + \int_{\mathbb{R}} e^{it(x+i\beta)} f_1(x)\,dx = \int_{\mathbb{R}} e^{itx} f_0(x)\,dx + e^{-t\beta} \int_{\mathbb{R}} e^{itx} f_1(x)\,dx \\ &= \widehat{f_0}(-t) + e^{-t\beta} \widehat{f_1}(-t).\end{aligned}$$

[2] In [RR94, Ex. 12, p. 115] one finds the interesting characterization that, for $1 \leq p, q \leq \infty$ and $p^{-1} + q^{-1} = 1$, a function $f \in L^p(\mathbb{R})$ is contained in $H^p(\mathbb{C}_+)$ if and only if η_f vanishes on $H^q(\mathbb{C}_+)$.

We thus arrive at the relation

$$\widehat{f_1}(t) = -e^{-t\beta}\widehat{f_0}(t) \quad \text{for} \quad t \in \mathbb{R}, \quad \text{resp.} \quad \widehat{f_1} = -e_{-\beta}\widehat{f_0}. \tag{45}$$

Let $\mathcal{E} \subseteq L^1(\mathbb{R}) \times L^1(\mathbb{R})$ be the closed linear subspace of all pairs (g_0, g_1) satisfying $\widehat{g_1} = -e_{-\beta}\widehat{g_0}$. This is a closed subspace invariant under the translation action $\alpha_s(g) = g(\cdot + s)$. In the Banach algebra $L^1(\mathbb{R})$, we consider the approximate identity

$$\delta_n(x) := \frac{n}{\sqrt{2\pi}} e^{-\frac{n^2x^2}{2}}.$$

The pairs $(\alpha_s(\delta_n * g_0), \alpha_s(\delta_n * g_1))$ for $n \in \mathbb{N}$ and $(g_0, g_1) \in \mathcal{E}$ extend to pairs of holomorphic maps $\mathbb{C} \to L^1(\mathbb{R})$, given concretely by the functions

$$\alpha_z(\delta_n * g_i)(x) = \delta_n * g_i(x + z) := \frac{n}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-\frac{n(x+z-t)^2}{2}} g_i(t)dt \quad \text{for} \quad z \in \mathbb{C},\ i = 0, 1.$$

For any such pair, the Fourier transform of $\alpha_z(\delta_n * g_0) = \delta_n * g_0(\cdot + z)$ is $e_{iz}\widehat{\delta_n}\widehat{g_0}$. For $z = \beta i$, this function coincides with $-\widehat{\delta_n}\widehat{g_1}$, so that the injectivity of the Fourier transform leads to

$$\delta_n * g_1 = -\alpha_{\beta i}(\delta_n * g_0).$$

As $\lim_{n\to\infty} \delta_n * h = h$ for any $h \in L^1(\mathbb{R})$, it follows that the set of pairs $(\delta_n * g_0, \delta_n * g_1)$ is dense in $\mathcal{E}$.

Let $h \in H^\infty(\mathbb{S}_\beta)$. Then $h_z(x) := h(z + x)$ defines a bounded weakly holomorphic family in $L^\infty(\mathbb{R})$, i.e., the function $\mathbb{S}_\beta \ni z \mapsto \eta_f(h_z) \in \mathbb{C}$ is holomorphic for every $f \in L^1(\mathbb{R})$. For $\delta_n * g = (\delta_n * g_0, \delta_n * g_1) \in \mathcal{E}$ as above, the function

$$\gamma\colon \mathbb{S}_\beta \to \mathbb{C}, \quad \gamma(z) := \int_{\mathbb{R}} \alpha_z(\delta_n * g_0)(x)h(z + x)\, dx$$

is holomorphic and bounded on $\mathbb{S}_\beta$ and extends continuously to the closed strip. Moreover, its lower boundary values are constant because of the translation invariance of Lebesgue measure. Therefore γ is constant and we obtain in particular

$$\int_{\mathbb{R}} \delta_n * g_0(x)h^*(x)\, dx = \gamma(0) = \gamma(\beta i) = \int_{\mathbb{R}} \alpha_{\beta i}(\delta_n * g_0)(x)h^*(\beta i + x)\, dx$$
$$= -\int_{\mathbb{R}} \delta_n * g_1(x)h^*(\beta i + x)\, dx,$$

which means that $\eta_{\delta_n * g}(h) = 0$. With the density argument from above, this entails that $\eta_f(h) = 0$ for each $h \in H^\infty(\mathbb{S}_\beta)$, and $f \in \mathcal{E}$. This proves that the functions $(e_{it})_{t\in\mathbb{R}}$ span a weakly dense subspace of $H^\infty(\mathbb{S}_\beta)$. □

B.3 Weakly continuous positive functionals

The compact space of characters of the commutative Banach algebra $H^\infty(\Omega)$ is a complicated space in which the evaluation functionals $\delta_z(f) = f(z)$, $z \in \Omega$, are dense by Carleson's Corona Theorem ([Ca62]). These characters satisfy $\delta_z^\sharp(f) := \overline{f^\sharp(z)} = \delta_{\sigma(z)}(f)$, so that $\delta_z^\sharp = \delta_z$ is equivalent to $z \in \Omega^\sigma$. The following proposition shows that this construction exhausts the set of weakly continuous $*$-characters.

Proposition B.7. *The weakly continuous $*$-homomorphisms $(H^\infty(\Omega), \sharp) \to \mathbb{C}$ are the maps*

$$\delta_\lambda(f) := f(\lambda) \quad \textit{for} \quad \lambda \in \Omega^\sigma = \{z \in \Omega\colon \sigma(z) = z\}.$$

Proof. In view of the Riemann Mapping Theorem, we may w.l.o.g. assume that $\Omega = \mathbb{C}_+$ with $\sigma(z) = -\overline{z}$ (Lemma B.1). Clearly, each δ_λ defines a weakly continuous $*$-homomorphism. Suppose, conversely, that $\chi\colon (H^\infty(\mathbb{C}_+), \sharp) \to \mathbb{C}$ is a weakly continuous unital $*$-homomorphism. As $(e_{it})_{t>0}$ is an involutive subsemigroup spanning a weakly dense subspace, χ is uniquely determined by its values on this semigroup. This defines a continuous non-zero homomorphism

$$\mathbb{R}_+ \to ([0,1], \cdot), \quad t \mapsto \chi(e_{it}),$$

hence is of the form $t \mapsto e^{-t\lambda} = e_{it}(i\lambda)$ for some $\lambda \geq 0$. Writing $\chi = \eta_f$ for some $f \in L^1(\mathbb{R})$, we see that $\eta_f(e_{it}) = \widehat{f}(-t)$ tends to 0 for $t \to \infty$ (Riemann–Lebesgue Lemma), so that we must have $\lambda > 0$ and thus $\chi = \delta_{i\lambda}$. ☐

Proposition B.8. *The weakly continuous positive functionals $(H^\infty(\Omega), \sharp) \to \mathbb{C}$ are the maps*

$$\eta_\mu(f) := \int_{\Omega^\sigma} f(\lambda)\, d\mu(\lambda),$$

where μ is a finite positive Borel measure on Ω^σ.

Proof. Again, we may w.l.o.g. assume that $\Omega = \mathbb{C}_+$ with $\sigma(z) = -\overline{z}$. For the elements e_{it}, $t > 0$, of $H^\infty(\mathbb{C}_+)$, the reproducing property of the Poisson kernel

$$P(z,x) = P_z(x) = \frac{1}{\pi} \frac{\operatorname{Im} z}{|z-x|^2}$$

(see (57) below) leads to

$$e^{-t\lambda} = e_{it}(i\lambda) = \frac{1}{\pi} \int_{\mathbb{R}} e^{itx} \frac{\lambda}{\lambda^2 + x^2}\, dx. \tag{46}$$

If μ is a finite positive measure on $\mathbb{R}_+ = (0, \infty)$, then (46) shows that the function

$$\psi_\mu(x) := \frac{1}{\pi} \int_0^\infty \frac{\lambda}{\lambda^2 + x^2}\, d\mu(\lambda) = \int_0^\infty P(i\lambda, x)\, d\mu(\lambda) \tag{47}$$

is L^1 on $\mathbb{R}$ with total integral $\mu((0,\infty))$. For $f \in H^\infty(\mathbb{C}_+)$, we obtain

$$\eta_{\psi_\mu}(f) = \int_{\mathbb{R}} \psi_\mu(x) f^*(x)\, dx = \int_0^\infty \Big(\int_{\mathbb{R}} P_{i\lambda}(x) f^*(x)\, dx \Big)\, d\mu(\lambda) = \int_0^\infty f(i\lambda)\, d\mu(\lambda). \tag{48}$$

Therefore all functionals η_μ are weakly continuous. That they are positive follows from

$$\eta_{\psi_\mu}(f^\sharp f) = \int_0^\infty \overline{f(i\lambda)} f(i\lambda)\, d\mu(\lambda). = \int_0^\infty |f(i\lambda)|^2\, d\mu(\lambda).$$

Suppose, conversely, that $\eta_f\colon H^\infty(\mathbb{C}_+) \to \mathbb{R}$ is a weakly continuous positive functional. As the semigroup $(e_{it})_{t\geq 0}$ spans a weakly dense subspace, η_f is determined uniquely by its restriction to this semigroup, on which it defines a continuous bounded positive definite function. All these functions are Laplace transforms $\mathcal{L}(\mu)$ of a finite positive Borel measure on $[0,\infty)$ by the Hausdorff–Bernstein–Widder Theorem ([BCR84, Thm. 6.5.12]), so that it remains to show that $\mu(\{0\}) = 0$. This follows from

$$0 = \lim_{t\to\infty} \widehat{f}(-t) = \lim_{t\to\infty} \eta_f(e_{it}) = \lim_{t\to\infty} \mathcal{L}(\mu)(t) = \mu(\{0\}) + \lim_{t\to\infty} \int_0^\infty e^{-t\lambda}\, d\mu(\lambda) = \mu(\{0\}).$$ ☐

B.4 Weakly continuous representations

Proposition B.9. *For a $*$-representation $(\pi, \mathcal{H})$ of the Banach $*$-algebra $(H^\infty(\Omega), \sharp)$, the following are equivalent:*

(a) *For every trace class operator $A \in B_1(\mathcal{H})$, the functional $\pi^A(f) := \mathrm{tr}(A\pi(f))$ is weakly continuous.*

(b) *For every $\xi \in \mathcal{H}$, the matrix coefficient $\pi^\xi(f) := \langle \xi, \pi(f)\xi \rangle$ is weakly continuous.*

(c) *There exists a dense subspace $\mathcal{D} \subseteq \mathcal{H}$ such that, for every $\xi \in \mathcal{D}$, the matrix coefficient $\pi^\xi(f) := \langle \xi, \pi(f)\xi \rangle$ is weakly continuous.*

Proof. Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (c). It remains to shows that (c) implies (a). To this end, assume (c). Then each π^ξ, $\xi \in \mathcal{D}$, is a weakly continuous positive functional, hence satisfies

$$|\pi^\xi(f)| \leq \pi^\xi(\mathbf{1})\|f\| = \|\xi\|^2\|f\|$$

by [Dix64, Prop. 2.1.4]. This implies in particular that

$$\|\pi(f)\xi\|^2 = |\pi^\xi(f^\sharp f)| \leq \|\xi\|^2\|f^\sharp f\| \leq \|\xi\|^2\|f\|^2,$$

so that $\|\pi\| \leq 1$. We now consider the map

$$\pi^* \colon B_1(\mathcal{H}) \to H^\infty(\Omega)^*, \quad \pi^*(A)(f) := \mathrm{tr}(A\pi(f))$$

which is a linear contraction. As the map

$$\mathcal{H} \times \mathcal{H} \to B_1(\mathcal{H}), \quad (\xi, \eta) \mapsto P_{\xi,\eta}, \quad P_{\xi,\eta}(v) := \langle \xi, v \rangle \eta$$

is sesquilinear and continuous, the Polarization Identity implies that $\pi^*(\{P_{\xi,\eta} \colon \xi, \eta \in \mathcal{D}\})$ consists of weakly continuous functionals. Now (a) follows from the norm-closedness of the predual $H^\infty(\Omega)_*$ in $H^\infty(\Omega)^*$ (cf. Example B.2(a)), the norm-continuity of π^*, and the density of the span of $P_{\xi,\eta}$, $\xi, \eta \in \mathcal{D}$, in $B_1(\mathcal{H})$. □

Definition B.10. Representations of the $*$-algebra $(H^\infty(\Omega), \sharp)$ satisfying the equivalent conditions in Proposition B.9 are called *weakly continuous*.

From weakly continuous positive functionals, we obtain weakly continuous cyclic $*$-representations of $(H^\infty(\Omega), \sharp)$ by Proposition B.9(c). Another source of such representations are positive Hankel operators for the multiplication representation of $(H^\infty(\Omega), \sharp)$ on $H^2(\Omega)$ (Proposition 1.8). For the sake of easier reference, we formulate the corresponding result for $\mathbb{C}_+$ explicitly:

Proposition B.11. *Suppose that $H^2(\mathbb{C}_+)$ is θ_h-positive for the operator $\theta_h(f)(x) = h(x)f(-x)$ and $h \in L^\infty(\mathbb{R})$ satisfying $h^\sharp = h$. Let $\mathcal{H}$ denote the Hilbert space defined by the positive semidefinite form $\langle f, g \rangle_h := \langle f, \theta_h g \rangle$ on $H^2(\mathbb{C}_+)$ and write $q \colon H^2(\mathbb{C}_+) \to \mathcal{H}$ for the natural map with dense range. Then there exists a $*$-representation $(\pi, \mathcal{H})$ of the Banach $*$-algebra $(H^\infty(\mathbb{C}_+), \sharp)$ which is uniquely determined by the relation*

$$q(fg) = \pi(f)q(g) \quad \text{for} \quad f \in H^\infty(\mathbb{C}_+), g \in H^2(\mathbb{C}_+). \tag{49}$$

Proof. In view of Theorem 3.5(b), the operator $H_h := P_+\theta_h P_+^*$ is a positive Hankel operator for the representation of the $*$-algebra $(H^\infty(\mathbb{C}_+), \sharp)$ on $H^2(\mathbb{C}_+)$. Hence the assertion follows from Proposition 1.8. □

Remark B.12. In the context of Proposition B.11, we have for $g \in H^2(\mathbb{C}_+)$

$$\begin{aligned}\langle q(g), \pi(f)q(g)\rangle_{\widehat{\mathcal{E}}} = \langle g, \theta_h(fg)\rangle &= \int_{\mathbb{R}} \overline{g(x)} g(-x)h(x)f(-x)\,dx \\ &= \int_{\mathbb{R}} g^\sharp(x)g(x)h(-x)f(x)\,dx = \eta_{h^\vee g^\sharp g}(f),\end{aligned} \tag{50}$$

where we use the notation $h^\vee(x) := h(-x)$ for $x \in \mathbb{R}$. These positive functionals are weakly continuous on $H^\infty(\mathbb{C}_+)$, so that the representation π is weakly continuous.

If $g \in H^2(\mathbb{C}_+)$ is an outer function, then $H^\infty(\mathbb{C}_+)g$ is dense in $H^2(\mathbb{C}_+)$, so that $H^2(\mathbb{C}_+)$ is θ_h-positive if and only if the functional $\eta_{h^\vee g^\sharp g}$ is positive. This in turn is equivalent to

$$h^\vee g^\sharp g \in \psi_\mu + H^1(\mathbb{C}_+)$$

for a finite positive Borel measure μ on $\mathbb{R}_+$ (cf. Example B.2). In this case the representation $(\pi, \mathcal{H})$ is equivalent to the GNS representation associated to the positive functional $\eta_{h^\vee g^\sharp g}$ on $H^\infty(\mathbb{C}_+)$.

B.5 The unit group of H^∞

A non-negative function $w\colon \mathbb{T} \to \mathbb{R}_+$ arises as $|f^*|$ for $f \in H^\infty(\mathbb{D})$ if and only if $\log w \in L^1(\mathbb{T})$. This means that $w = e^h$ for $h \in L^1(\mathbb{T}, \mathbb{R})$ bounded from above. If, in addition, $h \in L^\infty(\mathbb{T}, \mathbb{R})$, then $e^{\pm h}$ are bounded. This leads to the following description of the unit group of $H^\infty(\mathbb{D})$. As all proper simply connected domains $\Omega \subsetneq \mathbb{C}$ are isomorphic, it also provides a description of the unit group $H^\infty(\Omega)^\times$ in general (see in particular Definition 4.3).

Lemma B.13. *We have a surjective group homomorphism*

$$\mathrm{Out}\colon (L^\infty(\mathbb{T}, \mathbb{R}), +) \times \mathbb{T} \to (H^\infty(\mathbb{D})^\times, \cdot), \quad \mathrm{Out}(w, \zeta) = \zeta e^{q_w}, \quad q_w(z) := \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z}\, w(e^{it})\,dt.$$

In particular, all invertible elements of $H^\infty(\mathbb{D})$ are outer.

Proof. It is clear that Q is a group homomorphism whose range consists of outer functions which are invertible in $H^\infty(\mathbb{D})$. If, conversely, $f \in H^\infty(\mathbb{D})$ is invertible, then the subspace $fH^2(\mathbb{D})$ coincides with $H^2(\mathbb{D})$. As $f \in H^2(\mathbb{D})$, it follows that $fH^\infty(\mathbb{D})$ is dense in $H^2(\mathbb{D})$, and hence that f is outer, i.e., of the form ζe^{q_w} for some $w \in L^1(\mathbb{T}, \mathbb{R})$. As f and f^{-1} are bounded, w is bounded as well, hence contained in $L^\infty(\mathbb{T}, \mathbb{R})$. Therefore Out is surjective. ☐

Finer results that imply the preceding lemma can be found in [Ni02, Part A, §4.2]. For results on the operator-valued case, see [Ni02, Part A, §3.3].

B.6 The representation on $H^2(\Omega)$

If $\varphi\colon \Omega_1 \to \Omega_2$ is a biholomorphic map between simply connected proper domains in $\mathbb{C}$, then the map

$$\Gamma_\varphi\colon H^2(\Omega_2) \to H^2(\Omega_1), \quad \Gamma_\varphi(f) := \sqrt{\varphi'} \cdot f \circ \varphi \tag{51}$$

is unitary up to a positive factor, depending on the normalization of the scalar product. Here $\sqrt{\varphi'}$ denotes one of the two holomorphic square roots of $\varphi'\colon \Omega_1 \to \mathbb{C}^\times$. Its existence follows from the simple connectedness of Ω_1. Actually, this is how one can define $H^2(\Omega)$ for domains with a complicated boundary in a natural way.

Clearly, Γ_φ intertwines the multiplication action of $H^\infty(\Omega_2)$ on $H^2(\Omega_2)$ with the action of $H^\infty(\Omega_1)$ on $H^2(\Omega_1)$ in the sense that

$$\Gamma_\varphi(fg) = (f \circ \varphi) \cdot \Gamma_\varphi(g) \quad \text{for} \quad f \in H^\infty(\Omega_2), g \in H^2(\Omega_2). \tag{52}$$

To see how the Szegö kernels on Ω_1 and Ω_2 are related, we observe that, for $f \in H^2(\Omega_2)$ and $z \in \Omega_1$ the relation

$$\sqrt{\varphi'(z)}\langle Q^{\Omega_2}_{\varphi(z)}, f\rangle = \sqrt{\varphi'(z)}f\big(\varphi(z)\big) = \Gamma_\varphi(f)(z) = \langle Q^{\Omega_1}_z, \Gamma_\varphi(f)\rangle = \langle \Gamma_\varphi^{-1} Q^{\Omega_1}_z, f\rangle$$

implies that

$$Q^{\Omega_1}_z = \overline{\sqrt{\varphi'(z)}}\Gamma_\varphi(Q^{\Omega_2}_{\varphi(z)}),$$

which leads to the following transformation formula for the kernels

$$Q^{\Omega_1}(z,w) = \sqrt{\varphi'(z)}Q^{\Omega_2}(\varphi(z),\varphi(w))\overline{\sqrt{\varphi'(w)}}.$$

Example B.14. For $\Omega_1 = \mathbb{D}$ and $\Omega_2 = \mathbb{C}_+$ we have

$$Q^{\Omega_1}(z,w) = \frac{1}{2\pi}\frac{1}{1-z\overline{w}} \quad \text{and} \quad Q^{\Omega_2}(z,w) = \frac{1}{2\pi}\frac{i}{z-\overline{w}}.$$

The Cayley transform $\omega\colon \mathbb{D} \to \mathbb{C}_+, \omega(z) := i\frac{1+z}{1-z}$ with $\omega'(z) = \frac{2i}{(1-z)^2}$ satisfies

$$\sqrt{\omega'(z)}Q^{\mathbb{C}_+}(\omega(z),\omega(w))\overline{\sqrt{\omega'(w}} = \frac{\sqrt{2}}{(1-z)}\frac{1}{2\pi}\frac{1}{\frac{1+z}{1-z}+\frac{1+\overline{w}}{1-\overline{w}}}\frac{\sqrt{2}}{(1-\overline{w})} = \frac{1}{2\pi}\frac{1}{1-z\overline{w}} = Q^{\mathbb{D}}(z,w).$$

B.7 The Carleson measure

The Carleson measure μ_H of a positive Hankel operator H on $H^2(\Omega)$ with respect to the multiplication representation of the Banach $*$-algebra $(H^\infty(\Omega), \sharp)$ lives on the subset Ω^σ of σ-fixed points (cf. Proposition B.8). The abstract correspondence is

$$\langle f, Hg\rangle_{H^2(\Omega)} = \int_{\Omega^\sigma} \overline{f(\lambda)}g(\lambda)\,d\mu(\lambda) \quad \text{for} \quad f,g \in H^2(\Omega). \tag{53}$$

However, it is more convenient to parametrize this subset by real intervals. Formula (53) shows in particular that Hankel operators on $H^2(\Omega)$ are superpositions of rank-one Hankel operators H_λ corresponding to point measures δ_λ, $\lambda \in \Omega^\sigma$. If Q is the reproducing kernel of $H^2(\Omega)$, then the relation

$$\langle f, H_\lambda g\rangle_{H^2(\Omega)} = \overline{f(\lambda)}g(\lambda) = \langle f, Q_\lambda\rangle\langle Q_\lambda, g\rangle$$

shows that

$$H_\lambda = |Q_\lambda\rangle\langle Q_\lambda|$$

in Dirac's bra-ket notation. This means that

$$H_\lambda(f) = f(\lambda)Q_\lambda \quad \text{with} \quad \|H_\lambda\| = \|Q_\lambda\|^2 = Q(\lambda,\lambda). \tag{54}$$

Formula (53) can now be written as a weak integral

$$H = \int_{\Omega^\sigma} H_\lambda\,d\mu(\lambda),$$

which exists pointwise in the space of sesquilinear forms on $H^2(\Omega)$.[3]

The symbol kernel of the Hankel operator H with respect to the Szegö kernel is the kernel

$$Q_H(z,w) = \langle Q_z, HQ_w\rangle = \int_{\Omega^\sigma} \overline{Q_z(\lambda)}Q_w(\lambda)\,d\mu(\lambda) = \int_{\Omega^\sigma} Q(z,\lambda)Q(\lambda,w)\,d\mu(\lambda)$$

and

$$Q_{H_\lambda}(z,w) = Q(z,\lambda)Q(\lambda,w).$$

[3] See [Ni02, §6.3.1] for the case of the disc and for more general Carleson measures.

Remark B.15. The norm of H can be determined in terms of the kernel Q_H by

$$\|H\| = \inf\{c > 0 \colon cQ - Q_H \ \text{ positive def.}\}$$

([Ne99]). In some situations this number can be determined by restricting to finite subsets.

Remark B.16. If H is a positive Hankel operator on $H^2(\Omega)$ and μ_H the corresponding Carleson measure on Ω^σ, then we obtain for every $g \in H^2(\Omega)$ a finite positive measure $d\mu_g = |g(\lambda)|^2 \, d\mu_H(\lambda)$ representing a positive weakly continuous functional on the Banach $*$-algebra $(H^\infty(\Omega), \sharp)$:

$$\varphi_{\mu_g}(f) := \langle g, Hfg \rangle = \int_{\Omega^\sigma} f(\lambda)|g(\lambda)|^2 \, d\mu_H(\lambda) = \int_{\Omega^\sigma} f(\lambda) \, d\,\mu_g(\lambda)$$

(cf. Proposition B.8).

C Cauchy and Poisson kernels

For a proper simply connected domain $\Omega \subseteq \mathbb{C}$, the Hardy space $H^2(\Omega)$ (cf. Subsection B.6) is a reproducing kernel Hilbert space, i.e., the point evaluations

$$\mathrm{ev}_z \colon H^2(\Omega) \to \mathbb{C}, \quad f \mapsto f(z)$$

are continuous linear functionals, hence can be written as

$$f(z) = \langle Q_z, f \rangle \quad \text{for some} \quad Q_z \in H^2(\Omega).$$

The kernel

$$Q \colon \Omega \times \Omega \to \mathbb{C}, \quad Q(z,w) := Q_w(z) = \langle Q_z, Q_w \rangle$$

is called the *Szegö kernel of* Ω.

If Ω has smooth boundary, so that we have an isometric boundary value map

$$H^2(\Omega) \to L^2(\partial\Omega), \quad f \mapsto f^*,$$

then we obtain the *Poisson kernel* of Ω by the *Hua formula* (cf. [Hu63, pp. 8,98], [Ko65])

$$P \colon \Omega \times \partial\Omega \to \mathbb{R}, \quad P(z,x) = \frac{|Q(z,x)|^2}{Q(z,z)} \quad \text{for} \quad z \in \Omega, x \in \partial\Omega, \tag{55}$$

i.e.,

$$P(z,\cdot) = \frac{|Q_z^*|^2}{Q(z,z)} \in L^1(\partial\Omega).$$

Example C.1. The Szegö kernel of the disc is

$$Q(z,w) = \frac{1}{2\pi} \frac{1}{1 - z\overline{w}}.$$

For $f \in H^2(\mathbb{D})$, we have

$$f(z) = \langle Q_z, f \rangle = \int_0^{2\pi} \overline{Q_z}(e^{i\theta}) f^*(e^{i\theta}) \, d\theta = \frac{1}{2\pi} \int_{\partial\mathbb{D}} \frac{f^*(\zeta)}{1 - \overline{\zeta} z} \frac{d\zeta}{i\zeta} = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \frac{f^*(\zeta)}{\zeta - z} \, d\zeta.$$

We thus obtain from (55) the Poisson kernel

$$P(re^{i\theta}, e^{it}) = \frac{1}{2\pi} \frac{1 - r^2}{|1 - re^{i(\theta - t)}|^2} = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2} \quad \text{for} \quad z = re^{i\theta} \in \mathbb{D}, t \in [0, 2\pi].$$

Example C.2. The Szegö kernel on the upper half-plane is given by

$$Q(z,w) = \frac{1}{2\pi}\frac{i}{z-\overline{w}} \quad \text{for} \quad z,w \in \mathbb{C}_+. \tag{56}$$

This is an easy consequence of the Residue Theorem. We have

$$f(z) = \langle Q_z, f\rangle = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f^*(x)}{x-z}\,dx \quad \text{for} \quad f \in H^2(\mathbb{C}_+), z \in \mathbb{C}_+.$$

For the Poisson kernel we obtain with Hua's formula (55)

$$P(z,x) = P_z(x) = \frac{1}{\pi}\frac{\operatorname{Im} z}{|z-x|^2}. \tag{57}$$

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Department for Basic and Applied Sciences for Engineering, University of Rome "La Sapienza", Via Antonio Scarpa 16, 00161 Rome, Italy; mariastella.adamo@uniroma1.it

Department of Mathematics, Friedrich-Alexander-University of Erlangen-Nürnberg, Cauerstrasse 11, 91058 Erlangen, Germany; neeb@math.fau.de

Department of Mathematics, Friedrich-Alexander-University of Erlangen-Nürnberg, Cauerstrasse 11, 91058 Erlangen, Germany; jonas.schober@fau.de