Mathematisches Forschungsinstitut Oberwolfach

Report No. 57/2021

DOI: 10.4171/OWR/2021/57

Mini-Workshop: Scattering Amplitudes, Cluster Algebras, and Positive Geometries (hybrid meeting)

Organized by
Nima Arkani-Hamed, Princeton
Hugh Thomas, Montreal
Lauren Williams, Cambridge MA

5 December – 11 December 2021

Abstract. Cluster algebras were developed by Fomin and Zelevinsky about twenty years ago. While the initial motivation came from within algebra (total positivity, canonical bases), it quickly became clear that cluster algebras possess deep links to a host of other subjects in mathematics and physics. In a separate vein, starting about ten years ago, Arkani-Hamed and his collaborators began a program of reformulating the bases of quantum field theory, motivated by a desire to discover the basic rules of quantum mechanics and spacetime as arising from deeper mathematical principles. Their approach to the fundamental problem of particle scattering amplitudes entails encoding the solution in geometrical objects, "positive geometries" and "amplituhedra". Surprisingly, cluster algebras have been found to be tightly woven into the mathematics needed to describe these geometries. The purpose of this workshop is to explore the various connections between cluster algebras, scattering amplitudes, and positive geometries.

Mathematics Subject Classification (2020): 81U10, 13F60, 14M15, 51M20.

Introduction by the Organizers

The hybrid mini-workshop *Scattering Amplitudes, Cluster Algebras, and Positive Geometries* was attended by about 20 participants, split roughly equally between mathematicians and physicists, with representation from Europe, North America, and Asia. Four participants attended in person, while the rest were remote. The workshop consisted of 14 talks and two discussion sessions. While most of the talks were on the latest research in this area—often focusing on unpublished work in progress—a few of the talks served as an invitation to the central themes connecting this physics and mathematics that have emerged over the past few years.

There were talks on cutting edge topics within cluster algebras (Dani Kaufman, Marcus Spradlin), and new light cast on the mysterious appearance of cluster variables in the polylogarithmic structure of Feynman integrals (Song He). A number of talks focused on new developments in the mathematics and physics of amplituhedra (Nima Arkani-Hamed, Steven Karp, Lauren Williams, Tomasz Łukowski). A novel understanding of the "T-duality" symmetry, first encountered in connecting different physical descriptions of amplituhedra, was extended in the discussion (Pavel Galashin) of "shift maps" on the positive Grassmannian. Two talks (Thomas Lam, Claudia Fevola) were related to certain important classes of "stringy" integrals, defined on cluster varieties and beyond, with natural links to algebraic statistics and likelihood maps. New connections between polytopes and amplitudes were explored in a pair of talks, providing the first "all-loop-order" analog of the amplituhedra —"surfacehedra"— for the scattering amplitudes of a wide class of theories (Giulio Salvatori), and suggesting interesting mathematical generalizations of particle scattering (Nick Early). Finally, two complementary approaches to the construction of stringy integrals for surfacehedra were described, naturally connected to a "global" description of Teichmüller space on the one hand (Hadleigh Frost), and to representations of gentle algebras on the other (Pierre-Guy Plamondon).

Conversations after the talks ranged widely, including both the in-person and virtual participants. This brought people who had not previously met or collaborated into fruitful contact, for instance triggering discussions about the relation between Dani Kaufman's affine associahedra and the polytopes appearing in Giulio Salvatori's talk, and clarifying the positivity conditions yielding real solutions of Schubert problems in interactions between Steven Karp and Nima Arkani-Hamed, Thomas Lam, and Marcus Spradlin.

The two discussion sessions were aimed to facilitate communication between the mathematicians and physicists. The sessions were entitled "Ask the mathematicians a question" and "Ask the physicists a question" and were amongst the most enjoyable activities of the workshop. All in all, despite the difficulties associated with the hybrid nature of the event, the participants were actively engaged in the talks and discussions, and the meeting has already stimulated a number of new lines of inquiry.

Mini-Workshop (hybrid meeting): Scattering Amplitudes, Cluster Algebras, and Positive Geometries

Table of Contents

Abstracts

Kinematics, cluster algebras and Feynman integrals

Song He
(joint work with Zhenjie Li, Qinglin Yang)

For scattering amplitudes in $N = 4$ supersymmetric Yang–Mills theory (SYM) in the planar limit, cluster algebras have played an important role not only for its all-loop integrand, but also for the functions after integration. There has also been significant progress in computing and studying finite, conformal Feynman integrals contributing to amplitudes; remarkably, cluster algebras and the so-called cluster adjacency conditions seem to apply to individual Feynman integrals. To systematically study them, we identify cluster algebras for planar kinematics of conformal Feynman integrals in four dimensions, as sub-algebras of that for top-dimensional $G(4, n)$ corresponding to n-point massless kinematics. We work with general planar kinematics for Feynman integrals with "massive" corners, which only depends on a subset of the n dual points. A priori it is unclear at all if one can find any sub-algebra of the $G(4, n)$ cluster algebra which parametrizes such a kinematics. We show that it is indeed the case by going through all kinematics with $n \leq 8$. The basic idea is to find a suitable quiver of $G(4, n)$, where we can *freeze* some mutable variables such that the remaining sub-quiver with d mutable nodes becomes independent of the removed dual points.

We provide evidence that they encode information about singularities of such Feynman integrals, including all-loop ladders with symbol letters given by cluster variables and algebraic generalizations. As a highly-nontrivial example, we apply the D_3 cluster algebra to an $n = 8$ three-loop wheel integral, which contains a new square root. Based on the D_3 alphabet and three new algebraic letters essentially dictated by the cluster algebra, we bootstrap its symbol, which is strongly constrained by the cluster adjacency. By sending a point to infinity, our results have implications for non-conformal Feynman integrals, *e.g.* up to two loops the alphabet of two-mass-easy kinematics is given by a limit of this generalized D_3 alphabet. We also find that the reduction to three dimensions is achieved by *folding* and the resulting cluster algebras may encode singularities of amplitudes and Feynman integrals in ABJM theory, at least through $n = 7$ and two loops. More details and relevant references can be found in the preprint [1].

References

[1] S. He, Z. Li, Q. Yang, *Kinematics, cluster algebras and Feynman integrals*, arXiv:2112.11842 [hep-th].

Introduction to stringy integrals

Thomas Lam

In this talk, I give an introduction to "stringy integrals" [2] intended for cluster algebra experts in the audience. Stringy integrals are integral functions of the form

$$I(S) = (\alpha')^d \int_{\mathbb{R}^d_{\geq 0}} \prod_{j=1}^{r} p_j(x_1, x_2, \ldots, x_d)^{-\alpha' S_i} \frac{dx_1}{x_1} \frac{dx_2}{x_2} \cdots \frac{dx_d}{x_d} \tag{1}$$

where $p_j(x)$ are Laurent polynomials with nonnegative coefficients, α' is a fixed positive real "string length", and the integral is considered as a function of the parameters $S_1, S_2, \ldots, S_r$. The basic example of a string integral is Euler's beta function, which is in turn an example of a tree-level scattering amplitude in open string theory. The latter are certain integrals over the moduli space $M_{0,n}$ of n-pointed genus zero curves of importance in physics. Stringy integrals can also be associated to cluster algebras: the stringy integral for the type A_n-cluster algebra is the $(n+3)$-point open string tree amplitude.

I discuss in this talk the convergence of the integrals (1), and the field theory limit $\lim_{\alpha' \to 0} I(S)$, which turns out to be the canonical rational function of a polytope, as studied in [1].

References

[1] N. Arkani-Hamed, Y. Bai, and T. Lam, Positive geometries and canonical forms, JHEP November 2017, Article 39.

[2] N. Arkani-Hamed, S. He, and T. Lam, Stringy canonical forms, JHEP February 2021, Article 69.

Shift maps, poset associahedra, and totally nonnegative critical varieties

Pavel Galashin

I will discuss some recent results on totally nonnegative parts of critical varieties and the associated polytopes [4, 5, 6]. Along the way, I will highlight a surprising relation between these results and the *shift map* (also known as *T-duality* [11]) that has previously appeared in relation to scattering amplitudes.

The *shift map* is a conjectural bijection $\mathrm{Gr}_{\geqslant 0}(k,n) \dashrightarrow \mathrm{Gr}_{\geqslant 0}(k-1,n)$ between certain subsets of two adjacent totally nonnegative Grassmannians [12]. Each space $\mathrm{Gr}_{\geqslant 0}(k,n)$ is stratified into positroid cells $\Pi_f^{>0}$ indexed by permutations f, and the shift map is expected to send $\Pi_f^{>0}$ to $\Pi_{f^\downarrow}^{>0}$, where the permutation $f^\downarrow$ is obtained from f by shifting all of its values down by 1. This map appears in the *BCFW triangulation* [2] of the *amplituhedron* [1], and later it was also conjectured to give a topological equivalence between the spaces of planar Ising and electrical networks; see [9] and [7, Question 9.2].

When restricted to *critical* planar Ising and electrical networks, the shift map is straightforward to define. In [4], we introduce the critical part $\mathrm{Crit}_f^{>0}$ of an arbitrary positroid cell $\Pi_f^{>0}$, building on the results of [8, 10]. We study shift maps between different critical cells, as well as the topology of their closures $\mathrm{Crit}_f^{\geqslant 0}$. We show in [6] that in the case of the top positroid cell $\Pi_{k,n}^{>0} \subset \mathrm{Gr}_{\geqslant 0}(k,n)$, $\mathrm{Crit}_{k,n}^{\geqslant 0}$ is homeomorphic to the *second hypersimplex* $\Delta_{2,n}$. This polytope does not depend on k, in agreement with the shift map prediction. For lower-dimensional positroid cells $\Pi_f^{>0} \subset \mathrm{Gr}_{\geqslant 0}(k,n)$, studying the topology of $\mathrm{Crit}_f^{\geqslant 0}$ leads to new interesting families of polytopes which we call *poset associahedra* [5].

Poset associahedra are naturally defined in terms of *tubings* on a poset, similarly to the construction of *graph associahedra* introduced in [3]. In special cases, poset associahedra recover the Stasheff associahedron and the permutohedron. In [6], we obtain each totally nonnegative critical variety as an image of an *affine poset cyclohedron*, which is an affine analog of a poset associahedron. It remains an open problem to decide whether each $\mathrm{Crit}_f^{\geqslant 0}$ is itself a polytope.

References

[1] Nima Arkani-Hamed and Jaroslav Trnka. The amplituhedron. *Journal of High Energy Physics*, **2014**(10):30, Oct 2014.

[2] Ruth Britto, Freddy Cachazo, Bo Feng, and Edward Witten. Direct proof of the tree-level scattering amplitude recursion relation in Yang–Mills theory. *Phys. Rev. Lett.*, **94**(18):181602, 4, 2005.

[3] Michael P. Carr and Satyan L. Devadoss. Coxeter complexes and graph-associahedra. *Topology Appl.*, **153**(12):2155–2168, 2006.

[4] Pavel Galashin. Critical varieties in the Grassmannian. arXiv:2102.13339, 2021.

[5] Pavel Galashin. Poset associahedra. arXiv:2110.07257, 2021.

[6] Pavel Galashin. Totally nonnegative critical varieties. arXiv:2110.08548, 2021.

[7] Pavel Galashin and Pavlo Pylyavskyy. Ising model and the positive orthogonal Grassmannian. *Duke Math. J.*, **169**(10):1877–1942, 2020.

[8] R. Kenyon. The Laplacian and Dirac operators on critical planar graphs. *Invent. Math.*, **150**(2):409–439, 2002.

[9] Thomas Lam. Electroid varieties and a compactification of the space of electrical networks. *Adv. Math.*, **338**:549–600, 2018.

[10] Suho Oh, Alexander Postnikov, and David E. Speyer. Weak separation and plabic graphs. *Proc. Lond. Math. Soc. (3)*, **110**(3):721–754, 2015.

[11] Matteo Parisi, Melissa Sherman-Bennett, and Lauren Williams. The $m = 2$ amplituhedron and the hypersimplex: signs, clusters, triangulations, Eulerian numbers. arXiv:2104.08254, 2021.

[12] Alexander Postnikov. Total positivity, Grassmannians, and networks. Preprint, http://math.mit.edu/~apost/papers/tpgrass.pdf, 2006.

Affine quotient associahedra

Dani Kaufman

(joint work with Zachary Greenberg)

This talk was based on joint work with Z. Greenberg [1]. We define and analyse an analog of the generalized associahedra associated to finite type cluster algebras for affine type cluster algebras.

When a cluster algebra has finitely many clusters, it is called a *finite type* cluster algebra, and all of the relevant information it provides can be easily computed and studied. We are motivated by considering the properties of cluster algebras and cluster complexes as they transition from finite to infinite type in hopes of extending the foundational work of [3].

Our primary goal is to construct a quotient of the cluster complex of an affine cluster algebra by a finite index normal subgroup of its automorphism group. The dual complex to this finite quotient will be the "quotient affine associahedron", and we analyse it in a similar fashion to that of the usual generalized associahedron.

Thinking of affine cluster algebras as "singly extended", we take our analysis one step further to the "doubly extended" case.

Definition 1. An *affine type* cluster algebra is a cluster algebra which has an acyclic seed whose exchange matrix corresponds to an orientation of an affine Dynkin diagram. Such exchange matrices can be described by directed graphs with integer weight nodes called *weighted quivers.*

Our new method for analysing affine and doubly-extended cluster algebras stems from considering a new family of weighted quivers called $T_{\mathbf{n},\mathbf{w}}$ quivers that contains seeds for these algebras.

Let $\mathbf{n} = (n_1, n_2, \ldots, n_m)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_m)$ with $n_i > 1, w_i > 0$ be m tuples of positive integers. We consider a weighted quiver, $T_{\mathbf{n},\mathbf{w}}$, with $n = \sum(n_i - 1) + 2$ nodes constructed in the following way: First consider the star shaped quiver $T'_{\mathbf{n},\mathbf{w}}$ with $n-1$ nodes consisting of one central node, N_1 of weight 1 and m tails of length $n_i - 1$ of weight w_i nodes $i_2, \ldots, i_{n_i}$ connected in a source-sink pattern with N_1 as a source.

$T_{\mathbf{n},\mathbf{w}}$ is constructed from $T'_{\mathbf{n},\mathbf{w}}$ by adding an additional weight 1 node N_∞ along with a double arrow from N_∞ to N_1 and single arrows from each of the m other neighbours of N_1 to N_∞, as shown in Figure 1.

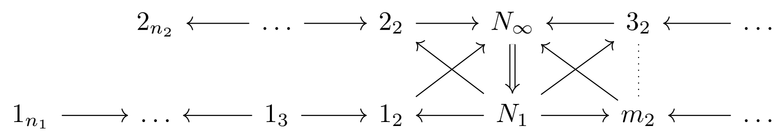


FIGURE 1. The quiver $T_{\mathbf{n},\mathbf{w}}$.

We prove the following theorem:

Theorem 2. *Let* $\mathbf{n}, \mathbf{w}$ *be* m *dimensional vectors of positive integers. Let* $\chi(T_{\mathbf{n},\mathbf{w}}) = \sum(w_i(n_i^{-1} - 1)) + 2$. *Then we have the following:*

1. *If* $\chi > 0$, *then* $T_{\mathbf{n},\mathbf{w}}$ *provides a seed of an affine cluster algebra.*
2. *If* $\chi = 0$, *then* $T_{\mathbf{n},\mathbf{w}}$ *provides a seed of a doubly extended cluster algebra.*
3. *If* $\chi < 0$, *then* $T_{\mathbf{n},\mathbf{w}}$ *provides a seed of an infinite mutation type cluster algebra.*

Moreover almost every affine and doubly extended cluster algebra has a seed with underlying quiver isomorphic to a $T_{\mathbf{n},\mathbf{w}}$ *for some* $\mathbf{n}, \mathbf{w}$.

Let us assume now that we are considering $T_{\mathbf{n},\mathbf{w}}$ quivers with $\chi > 0$. The key reason for considering these quivers as seeds of affine cluster algebras is that they encode a simple finite index normal subgroup of the cluster modular group. Let

$$\gamma = \{\mu_{N_1}, (N_1 N_\infty)\}$$

be the cluster modular group element given by mutation at node N_1 and the permutation which swaps nodes N_1 and N_∞. Then we prove that γ is equivalent to χ^{-1} source-sink mutations on a corresponding affine Dynkin seed and that the group generated by γ is a finite index normal subgroup.

Definition 3. Let $C(A)$ be the cluster complex associated to the affine cluster algebra $\mathcal{A}$. The *affine associahedron* is the dual complex to $C(A)/\langle\gamma\rangle$. The 1-skeleton of an affine associahedron is the *quotient exchange complex* of an affine cluster algebra.

We call cluster variables in an affine cluster algebra "finite" if freezing them gives a finite subalgebra, and affine otherwise. The following theorem allows us to count the facets of each codimension of this complex.

Theorem 4. *Each finite cluster variable appears in a seed whose quiver is an orientation of the associated affine Dynkin diagram. Every affine cluster variable appears on a node other than* N_1 *or* N_∞ *in a* $T_{\mathbf{n},\mathbf{w}}$ *seed.*

We prove the following theorem about the counts of codimension 1 and dimension 0 facets of the quotient affine associahedron.

Theorem 5. *The number of distinct cluster variables in an affine cluster algebra up to the action of* $\langle\gamma\rangle$ *is given by*

$$\sum_i (n_i - 1)n_i + \frac{n}{\chi}. \tag{1}$$

The number of distinct clusters in an affine cluster algebra up to the action of $\langle\gamma\rangle$ *is given by*

$$\frac{2}{\chi}\prod_i \binom{2n_i - 1}{n_i}. \tag{2}$$

These theorems are analogues to similar facet counts of the usual generalized associahedra of finite cluster algebras, see theorem 5.1 of [2].

We wish to further understand the quotient exchange complex. A possible way to accomplish this is by introducing a special framing of a $T_{\mathbf{n},\mathbf{w}}$ quiver, and consider two clusters the same if their unordered lists of c-vectors are the same with respect to this framing.

Consider a framing, $T^f_{\mathbf{n},\mathbf{w}}$, obtained from $T_{\mathbf{n},\mathbf{w}}$ by adding a frozen node for vertices $i_2, \ldots, i_{n_i}$ in each tail and one vertex associated with the double edge. For each tail node i add frozen nodes of weight w_i labeled $f_{i,2}, \ldots, f_{i,n_i}$ with a single arrow from i_j to $f_{i,j}$. Then add a frozen node f_1 of weight 1 along with single arrows N_1 to f_1 and f_1 to N_∞.

Conjecture 6. *Two clusters in the exchange graph of a* $T_{\mathbf{n},\mathbf{w}}$ *cluster algebra are in the same orbit of the action of* $\langle\gamma\rangle$ *if and only if their c-vectors with respect to this framing are the same.*

The "if" part of the statement follows since the framing is preserved by the action of γ. However, it is not clear that the only quivers which are identified are the ones which are in the same γ orbit.

We take our analysis one step further and construct quotients of the cluster complexes of doubly extended cluster algebras, i.e. the case that $\chi = 0$. For these algebras, the group $\langle\gamma\rangle$ is no longer a finite index normal subgroup. Thus we consider (in most cases) the normal closure of γ and quotient the cluster complex by this subgroup and call the dual of this complex the "doubly extended associahedron".

We conjecture the following about the topology of these complexes:

Conjecture 7.

(1) *The affine generalized associahedron of an affine cluster algebra of rank* $n+1$ *is homeomorphic to a sphere of dimension* n.
(2) *The cluster complex of a doubly extended cluster algebra of rank* $n+2$ *is homotopy equivalent to* S^{n-1}.
(3) *The doubly extended associahedron associated with a doubly extended cluster algebra is homeomorphic to* $S^{n-1} \times S^2$ *in all cases other than* $E_8^{(1,1)}$ *where it instead is homeomorphic to* $S^7 \times S^1 \times S^1$.

References

[1] Greenberg Z., Kaufman D., *Cluster Modular Groups of Affine and Doubly Extended Cluster Algebras*, arXiv:2107.10334 (2021).
[2] Fomin S., Reading N., *Root systems and generalized associahedra*, Geometric combinatorics, 63–131, IAS/Park City Math. Ser., 13, Amer. Math. Soc., Providence, RI (2007).
[3] Fomin, S., Zelevinsky, A. *Cluster algebras II: Finite type classification.* Invent. math. 154, 63–121 (2003).

Some new frontiers in amplitudes and amplituhedra

Nima Arkani-Hamed

In this informal talk, intended to stimulate discussion, I sketch a number of new directions connecting the combinatorics and geometry of the positive Grassmannian and amplituhedra not just to the "integrand" of scattering amplitudes, but more directly to the actual amplitudes themselves.

(1) The amplituhedron determines scattering amplitudes in planar $\mathcal{N} = 4$ super Yang–Mills by a single "positive geometry" in the space of kinematic and loop variables. I discuss a closely related definition of the amplituhedron for the simplest case of four-particle scattering, given as a sum over complementary "negative

geometries", which provides a natural geometric understanding of the exponentiation of infrared divergences of the amplitude, as well as a new geometric definition of an IR finite observable which is directly determined by these negative geometries. This provides a long-sought direct link between canonical forms for positive (negative) geometries, and a completely well-defined, finite physical observable, expressed in perturbation theory as a polylogarithmic function of a single kinematic variable. An especially simple class of negative geometries at all loop orders, associated with a "tree" structure can easily be determined by an interesting non-linear differential equation immediately following from the combinatorics of negative geometries, allowing the computation of these "tree" contributions non-perturbatively, for all values of the coupling constant, remarkably exhibiting the qualitative behavior expected of the transition from "gluons" at weak coupling to "strings in AdS space" at strong coupling.

(2) The appearance of cluster variables in the arguments of polylogs for amplitudes continues to be a wonderfully mysterious fact, to be more deeply understood. I suggest a geometric question naturally associated with planar diagrams, which may hold some clues as to where this phenomenon comes from. The duals of planar loop diagrams are graphs, whose nodes are associated with lines in $\mathbb{P}^3$, corresponding to the loop variables, as well as external data. Any such diagram can be interpreted as giving rise to a "leading singularity", known to play a central role in the physics, by demanding that any pairs of lines connected by an edge in the graph intersect. For the simplest case of a "box" diagram, this yields the first classic "Schubert" problem—to find a line in $\mathbb{P}^3$ intersecting a given set of four lines. Remarkably, when the external data is positive, these Schubert problems have real solutions. Furthermore each Schubert line has four points on it, and these four points are always ordered, so long as the external data are positive in the sense of the positive Grassmannian. Finally the cross-ratio of these four points give the arguments of the dilogarithm associated with performing the loop integration for the box diagram. This connection between the geometry of Schubert lines, and the polylogarithmic functions associated with planar loop diagrams, appears to hold in many other examples: the Schubert problems associated with the leading singularities have real, ordered solutions, with non-trivial cross-ratios of points on the Schubert lines having positivity properties. Furthermore these cross-ratios can be recognized as cluster variables of the $G(4, n)$ of the external kinematical data, and appear as the arguments of the polylogs. There are however also some peculiar counterexamples to this pattern, where some of the solutions of the Schubert problem are not cluster variables, and do not enjoy positivity properties. The systematics of this emerging story must clearly be better understood.

(3) Finally, the notion of positivity in the setting of the positive Grassmannian and the amplituhedron is naturally invariant under what physicists think of as "parity", which translates mathematically to the rather non-trivial property that the "twist map" preserves positivity. Only a small subset of minors are manifestly parity invariant; these are also the variables of immediate relevance in the computation of amplitudes. It is then natural to ask: what is the image of the positive

part of the Grassmannian, purely in this parity-invariant subspace of minors, and can we cut out this image by polynomial inequalities? I discuss a number of small examples showing that the very simple statement of the positivity of all minors, translates into non-trivial polynomial inequalities on the set of parity-invariant minors. Fascinatingly, these polynomials also naturally show up from completely standard physical considerations, as characterizing the locus of solutions of "Landau equations" controlling the possible branch points of amplitudes. A natural hope is that these two sets of polynomials are in fact identical, with the mathematically natural question of a parity-invariant characterization of the positive Grassmannian, being solved by producing solutions of Landau equations associated with planar graphs. In a precise sense, this would directly show how the loop diagrams encoding physics compatible with the principles of spacetime and quantum mechanics, are just a machine to produce the answer to a simple and natural mathematical question about positivity. The story of the amplituhedron already shows how this can happen for the integrand of amplitudes, but such a connection would take a significant step towards seeing how positivity can determine physics of the full amplitude.

Likelihood degenerations

Claudia Fevola

(joint work with Daniele Agostini, Taylor Brysiewicz, Lukas Kühne, Bernd Sturmfels, Simon Telen; with an appendix by Thomas Lam)

Computing all critical points of a monomial on a very affine variety is a fundamental task in algebraic statistics [6, 7, 8, 9], particle physics [3, 4] and other fields. We introduce degeneration techniques that are inspired by the soft limits in the theory developed by Cachazo–Early–Guevara–Mizera (CEGM), and we answer several questions raised in the physics literature.

More precisely, a central theme in CEGM theory in particle physics is the count of the number of critical points to the *potential function* or equivalently the number of solutions to the *scattering equations.* The analogous fundamental task in algebraic statistics is to compute all critical points of a monomial on a very affine variety. A *very affine variety* X is a closed subvariety of an algebraic torus $(\mathbb{C}^*)^n$. For any integer vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$, the Laurent monomial $\mathbf{z}^\mathbf{u} = z_1^{u_1} \cdots z_n^{u_n}$ is a regular function on $(\mathbb{C}^*)^n$, and we are interested in the set of critical points of $\mathbf{z}^\mathbf{u}$ on X. The natural approach is via the gradient of the *log-likelihood function* $\log(\mathbf{z}^\mathbf{u}) = \sum_{i=1}^n u_i \log(z_i)$. This makes sense for any complex vector $\mathbf{u} \in \mathbb{C}^n$. The coordinates of $\nabla \log(\mathbf{z}^\mathbf{u})$ are rational functions, and we seek points $\mathbf{z} \in X$ at which that gradient vector lies in the normal space. This leads to a system of rational function equations whose solutions are the critical points. Their number is independent of $\mathbf{u}$, provided $\mathbf{u}$ is generic. This is an invariant of X, denoted MLdegree(X), and known as the *maximum likelihood degree.* Whenever X is smooth, we know from [8, Theorem 1] that it coincides with the signed Euler characteristic of X. In particle physics, very affine varieties arise from *scattering* *equations* [3, 4, 5, 11]. Their solutions are the critical points of $\log(\mathbf{z}^{\mathbf{u}})$ on the moduli space $\mathcal{M}_{0,m} = \mathrm{Gr}(2,m)^{\circ}/(\mathbb{C}^*)^m$ of genus zero curves with m marked points. In the more general context of CEGM amplitudes [3], one considers the very affine varieties $X(k,m) = \mathrm{Gr}(k,m)^{\circ}/(\mathbb{C}^*)^m$ with $k \geq 2$. These are moduli spaces of m points in $\mathbb{P}^{k-1}$ in linearly general position, natural generalizations of $\mathcal{M}_{0,m}$. While the maximum likelihood degree $\mathrm{MLdegree}(X(2,m))$ is known to be $(m-3)!$, much less is known for $k \geq 3$. The connection between maximum likelihood and scattering equations was developed in [11].

The term *degeneration* in algebraic geometry represents the idea of studying the properties of a general object X_t for $t \neq 0$ by letting it degenerate to a more special object X_0, which is often easier to understand. This corresponds to finding a nice compactification of the variety $\mathcal{X}^0 = \cup_{t\neq 0} X_t$ to a variety $\mathcal{X}$, by adding the special fiber X_0. Cachazo, Umbert and Zhang [5] introduced a class of degenerations called *soft limits*. The present article arose from our desire to gain a mathematical understanding of that construction from physics. We succeeded in reaching that understanding, and we here share it from multiple perspectives: algebraic geometry, combinatorics and numerical mathematics. The soft limits in [5] are special instances of likelihood degenerations that are well adapted to the geometry of configurations. We explain how these are related to the deletion maps

$$\pi_{k,m} : X(k,m+1) \to X(k,m). \tag{1}$$

These maps are shown to be stratified fibrations. We discuss both the strata and the fibers. This sets the stage for the computation of Euler characteristics by combinatorial methods. This topological approach is applied to examine the space $X(3,m)$ of m points in general position in the projective plane $\mathbb{P}^2$. Cachazo, Umbert and Zhang [5] report that the ML degree of $X(3,m)$ equals 26 for $m=6$, 1 272 for $m=7$, and 188 112 for $m=8$. We present a topological proof of these results, and we prove the conjecture made in [5, §6]. This involves a careful study of the stratified fibration (1).

An additional approach we use to compute the number of critical points to the scattering equations comes from numerical algebraic geometry. In particular, these techniques are applied to configurations of eight points in projective 3-space. Based on our computational results, we predict that the ML degree of $X(4,8)$ is equal to 5 211 816. This is the number of solutions to the likelihood equations, found numerically by the software `HomotopyContinuation.jl` [2]. We present a detailed analysis of the tropical geometry of soft limits in this case. This confirms the combinatorial predictions made in [5, Table 2], and offers a blueprint for future research that connects tropical geometry and numerical analysis.

References

[1] D. Agostini, T. Brysiewicz, C. Fevola, L. Kühne, B. Sturmfels, and S. Telen: *Likelihood Degenerations*. arXiv preprint `arXiv:2107.10518` (2021).

[2] P. Breiding and S. Timme: *HomotopyContinuation.jl: A Package for Homotopy Continuation in Julia*, Math. Software – ICMS 2018, 458–465, Springer International Publishing (2018).

[3] F. Cachazo, N. Early, A. Guevara and S. Mizera: *Scattering equations: from projective spaces to tropical Grassmannians*, J. High Energy Phys. **39** (2019).

[4] F. Cachazo, S. He and E. Y. Yuan: *Scattering equations and Kawai–Lewellen–Tye orthogonality*, Physical Review D **90** (2014) 065001.

[5] F. Cachazo, B. Umbert and Y. Zhang: *Singular solutions in soft limits*, J. High Energy Phys. **148** (2020).

[6] F. Catanese, S. Hoşten, A. Khetan and B. Sturmfels: *The maximum likelihood degree*, American Journal of Mathematics **128** (2006) 671–697.

[7] E. Gross and J. Rodriguez: *Maximum likelihood geometry in the presence of data zeros*, ISSAC 2014 – 39th Internat. Symposium on Symbolic and Algebraic Computation, 232–239, ACM, New York (2014).

[8] J. Huh: *The maximum likelihood degree of a very affine variety*, Compos. Math. **149** (2013) 1245–1266.

[9] J. Huh and B. Sturmfels: *Likelihood geometry*, Combinatorial Algebraic Geometry, Lecture Notes in Mathematics 2108, Springer Verlag, 63–117 (2014).

[10] D. Maclagan and B. Sturmfels: *Introduction to Tropical Geometry*, Graduate Studies in Mathematics, Vol. 161, American Mathematical Society (2015).

[11] B. Sturmfels and S. Telen: *Likelihood equations and scattering amplitudes*, Algebraic Statistics, to appear, `arXiv:2012.05041`.

Aspects of positive geometry and noncrossing combinatorics on the Grassmannian, planar kinematics polytope and generalized worldsheet

Nick Early

In the study of scattering amplitudes, positive geometries [1, 2] can be used to capture the recursive nature of residues of amplitudes for certain quantum field theories; in this talk, we focus on the positive geometry of the biadjoint scalar partial amplitude $m_n(\alpha, \beta)$ for a pair of cyclic orders α, β. When $\alpha = \beta = \mathbb{I}$ is the standard cyclic order $\mathbb{I} = (1, 2, \ldots, n)$, then the relevant positive geometry is obtained by partially compactifying the torus quotient of the positive Grassmannian $G_+(2, n)$, that is to say, the configuration space $M_{0,n}$ of n distinct points on $\mathbb{CP}^1$. Then the strata in the compactification are closely related to the residues of the function $m_n(\mathbb{I}, \mathbb{I})$. This is our starting point.

In recent work [5], we introduce a deformation of the positive Grassmannian $G_+(3, n)$, and from its torus quotient a certain *generalized worldsheet associahedron*, by replacing the $\binom{n}{3}$ Plücker coordinates with Laurent polynomials, related to certain cluster variables first appearing in the $G(3, 8)$ cluster algebra. We gave an explicit extension of the construction to all $k \geq 3$.

We studied the polyhedral fan obtained by tropicalizing the deformation; and formulated a conjecture that, (in fact for deformations of $G_+(k, n)$ for any $k \geq 3$), the face poset of the polyhedral fan is isomorphic to the noncrossing complex of k-element subsets, whose maximal collections are known to be enumerated by the k-dimensional Catalan numbers. This is in analogy with the partial compactification of $M_{0,n}$, where the codimension zero strata are known to be enumerated by size $n-3$ collections of pairwise noncrossing pairs $\{i, j\}$. We ran numerous consistency checks, including comparing the combinatorial formula with the numerical value obtained using the $k = 3$ case of the formula due to Cachazo–Early–Guevara–Mizera, in terms of the generalized scattering equations. Working by analogy with the correspondence between the positive tropical Grassmannian $\text{Trop}^+G(2,n)$ and the associahedron, we were able to formulate a certain higher rank $k \geq 3$ generalization of the set of positive roots $e_i - e_j$. The main combinatorial result here was to prove that this gives rise to a complete simplicial fan consisting of $C^{(k)}_{n-k}$ maximal simplices which are in bijection with pairwise size $(k-1)(n-k-1)$ collections of pairwise noncrossing k-element subsets, again in perfect analogy with the well-known triangulation of the standard root polytope studied by Gelfand–Graev–Postnikov [6]. This proves a conjecture from [4] that the volume of the root polytope $\mathcal{R}_{k,n}$ is the k-dimensional Catalan number $C^{(k)}_n$.

In this talk, we will survey the story described above, and time permitting we will discuss relations to joint works with Cachazo [3, 4] on the Planar Kinematics polytope and the so-called mirror superpotential, as studied by Marsh, Rietsch and Williams in various works [8, 9, 10], see also [7].

References

[1] N. Arkani-Hamed, S. He, T. Lam, and H. Thomas. "Binary geometries, generalized particles and strings, and cluster algebras." arXiv preprint arXiv:1912.11764 (2019).

[2] N. Arkani-Hamed, T. Lam, and M. Spradlin. "Positive configuration space." Communications in Mathematical Physics (2021): 1–46.

[3] F. Cachazo and N. Early. *Minimal Kinematics: An All k and n Peek into* $Trop^+G(k,n)$. Symmetry, Integrability and Geometry: Methods and Applications 17, no. 0 (2021): 78-22.

[4] F. Cachazo and N. Early. *Planar Kinematics: Cyclic Fixed Points, Mirror Superpotential, k-Dimensional Catalan Numbers, and Root Polytopes.* arXiv:2010.09708 (2020).

[5] N. Early. *Planarity in Generalized Scattering Amplitudes: PK Polytope, Generalized Root Systems and Worldsheet Associahedra.* arXiv:2106.07142 (2021).

[6] I. M. Gelfand, M. Graev, and A. Postnikov. *Combinatorics of hypergeometric functions associated with positive roots.* In The Arnold–Gelfand mathematical seminars, pp. 205–221. Birkhauser Boston, 1997.

[7] S. Karp. "Moment curves and cyclic symmetry for positive Grassmannians." Bulletin of the London Mathematical Society 51, no. 5 (2019): 900–916.

[8] R. Marsh and K. Rietsch. *The B-model connection and mirror symmetry for Grassmannians.* Advances in Mathematics 366 (2020): 107027.

[9] K. Rietsch. *A mirror construction for the totally nonnegative part of the Peterson variety.* Nagoya Mathematical Journal 183 (2006): 105–142.

[10] K. Rietsch and L. Williams. *Newton–Okounkov bodies, cluster duality, and mirror symmetry for Grassmannians.* Duke Mathematical Journal 168, no. 18 (2019): 3437–3527.

Surfacehedra

Giulio Salvatori

(joint work with Nima Arkani-Hamed, Hadleigh Frost, Pierre-Guy Plamondon, Hugh Thomas)

A recurrent theme of this workshop is that Scattering Amplitudes can be thought of as canonical differential forms of Positive Geometries. This fact was first discovered in the context of $\mathcal{N} = 4$ SYM [1], but more recently the same interpretation was found to hold in ϕ^3 theory at tree level where the relevant positive geometry is a polytope known as the Associahedron [2]. In this talk we illustrate how the picture can be generalized beyond the tree level and furthermore beyond the planar limit of the theory. This has led us to the discovery of a new class of polytopes, which we called *Surfacehedra*, whose face lattice encodes the combinatorics of curves on surfaces. Roughly speaking, amplitudes of a colored version of ϕ^3 theory can be identified with approximations to the canonical differential forms of Surfacehedra.

Let us begin by defining the field theory and the physical quantities of interest. Consider a scalar field $\phi_{i,j}$ carrying two indices in a fundamental representation of U(N) or SU(N). We consider an interaction of the form $\mathcal{L}_{\text{int}} = g\ \text{Tr}[\phi^3]$. It is well known [3] that the Feynman diagrammatic expansion of scattering amplitudes takes an elegant form in terms of the t'Hooft coupling $\lambda = g^2 N$. One finds that the amplitude A_n for the scattering of n particles is given by

$$A_n = \sum_S C_S A_S,$$

where the sum runs over all orientable surfaces S, considered up to homeomorphism, with n marked points on the boundary ∂S. In other words, the sum runs over all choices of genus, number of punctures and number of boundary components with the constraint that precisely n marked points have to be chosen on the boundary. For each surface S the factor C_S absorbs color factors as well as powers of the coupling constant λ and of $\frac{1}{N}$ which depend on the topology of S; its computation is straightforward and we will not therefore focus on it. Our interest will be centered instead on the kinematic factor A_S, which depends only on the momenta of the scattered particles and is given by a sum over all cubic Feynman diagrams which can be drawn on the surface S, more precisely those that are dual to a triangulation of S with vertices at the marked points or punctures of S. Because only a subset of all the Feynman graphs with n external legs contribute to A_S these are also called color ordered amplitudes. Feynman rules state that the contribution of a diagram Γ to the amplitude A_S is the integral

$$A_\Gamma = \int \prod_{i=1}^{L} dk_i\ I_\Gamma. \tag{1}$$

The integrand I_Γ is a rational function given by

$$I_\Gamma = \prod_{e\in\Gamma} \left(P(e)^2 - m^2\right)^{-1}, \tag{2}$$

where the product runs over all internal edges e of the graph Γ and m is the mass of the particles, a fixed parameter of the theory. To each edge e is associated a momentum $P(e)$ obtained by solving momentum conservation relations at each vertex of the graph, which states $\sum_{i=1}^{3} \sigma(e_i)P(e_i) = 0$, where $\sigma(e_i) = \pm 1$ is an arbitrary orientation associated to the edge e_i. Note that such orientation drops out in I_Γ since the latter only depends on the Minkowski norm of the momentum $P(e)$. Solving momentum conservation allows one to express the momentum of any edge of the graph in terms of the momenta of the external legs, which are those of the particles being scattered, and of an arbitrary basis $\{k_i\}_{i=1,\ldots,L}$ for the first homology group of the graph. In physics the elements k_i of this basis are called loop momenta for obvious reasons.

Once again this arbitrary choice of basis is irrelevant, since it is being integrated over in (1) to produce the final contribution A_Γ associated to the graph Γ. However it matters for the definition of the integrand I_Γ, a fact which poses an immediate and yet long-standing obstacle in extending the ideas of [1, 2] beyond the planar limit approximation, i.e. beyond the case where S is a disk with n marked points on the boundary and from which a total of L among punctures and disks are then removed. In this case there is a canonical choice for the basis $\{k_i\}_{i=1,\ldots,L}$ which allows one to define a unique integrand I_S by

$$A_S = \sum_\Gamma A_\Gamma = \sum_\Gamma \int dk_i I_\Gamma = \int dk_i \sum_\Gamma I_\Gamma = \int dk_i I_S, \tag{3}$$

where $I_S := \sum_\Gamma I_\Gamma$. It is the rational function I_S which can then be understood as the canonical form of a positive geometry. Beyond the planar limit, however, there is no obvious generalization of this canonical choice of homological basis and therefore it is unclear whether the notion of a unique integrand for a color ordered amplitude makes sense. In other words, the first problem to be addressed is to find a meaningful way to compare the choice of loop momenta made across the various diagrams, so that one can define an integrand by exchanging the order of the sum over diagrams and integration over loops as done in (3). The first step toward this is to exploit the fact that all diagrams Γ contributing to A_S by definition can be embedded on S in such a way that the external legs end on the marked points on ∂S. Moreover, when Γ is embedded on S each of its internal edges e uniquely corresponds to an element of the relative homology $\mathrm{H}_1(S, \partial S)$. Therefore by making a choice of basis for this homological group one has a common choice of loop momenta for all diagrams Γ. The fine print is that the construction still depends on a choice of embedding for each of the diagrams; if we denote by $\mathcal{D}$ this set of embeddings we have then defined an integrand $I_S^{\mathcal{D}}$ which depends on both $\mathcal{D}$ and S. In the planar case, the homological class $[\ell] \in \mathrm{H}_1(S, \partial S)$ of a curve ℓ depends only on the location of the endpoints of ℓ and not on the homotopy class of ℓ, therefore the dependence of the integrand on $\mathcal{D}$ drops out and indeed one recovers the standard definition of integrand used for example in [1]. On the other hand, beyond the planar limit the form of the integrand $I_S^{\mathcal{D}}$ as a function of the external momenta $\{p_j\}_{j=1,\ldots,n}$ and loop momenta $\{k_i\}_{i=1,\ldots,L}$ depends on the choice of $\mathcal{D}$. This obviously affects the pole structure of the integrand, which is at the heart of the connection between scattering amplitudes and positive geometries.

It is at this stage that the existence of Surfacehedra become crucial in that they provide an organizational principle behind how the choice of $\mathcal{D}$ is reflected on the pole structure of $I_S^{\mathcal{D}}$. Before explaining this point, it is helpful to take an extra level of abstraction and think of the integrand $I_S^{\mathcal{D}}$ directly as a function of homotopy classes of curves on S. Concretely, we imagine associating a variable X_ℓ to each homotopy class of curves ℓ on S and we replace $(P(e)^2 - m^2)$ with X_ℓ in (2). One immediate advantage of this operation is that the newly defined integrand has at most simple poles as a function of the variables X_ℓ. This is in contrast with the original integrand which can have double and higher poles arising from the fact that different homotopy classes of curves can have the same homological class. This makes it possible for the new integrand to be connected with a canonical form of a positive geometry, which by definition has at most simple poles. A deeper reason is that, as will be explained in the talk, the face lattice of Surfacehedra captures the combinatorics of homotopy classes of curves on surfaces: to each curve ℓ on S corresponds a facet of the Surfacehedron $\mathcal{P}_S$ and two such facets are compatible if the corresponding curves do not cross. This implies that each and every embedding of a graph Γ contributing to A_S corresponds to some vertex of the Surfacehedron $\mathcal{P}_S$ and moreover that the pole structure of $I_S^{\mathcal{D}}$ can be understood in terms of the facet structure of $\mathcal{P}_S$. This is not exactly the same as saying that the integrand coincides with the canonical form of $\mathcal{P}_S$: such equality is destroyed by the choice of $\mathcal{D}$ or equivalently by a choice of vertices of the Surfacehedron. One possibility is that the integrands $I_S^{\mathcal{D}}$ should be more naturally thought of in connection with certain complexes discussed during Dani Kaufman's talk. Furthermore, there are vertices of $\mathcal{P}_S$ which do not correspond to Feynman diagrams at all. They live on facets of the Surfacehedron which are labelled by closed loops on S rather than by curves with endpoints on ∂S. A precise physical interpretation for these facets is still lacking but an intriguing idea is that they could signal the need to complete the ϕ^3 theory by introducing colorless particles, in a similar way as in string theory one cannot consider open strings alone without introducing also closed ones.

While the above ideas are interesting and worthy of further investigation, they are not addressed in this talk, which instead focuses on the definition of Surfacehedra as well as explaining their still conjectural convex realization. The starting point is asking how to parametrize the set of all curves on the surface S, which we do by borrowing the notion of geometric and shear vectors from [4] and from the Teichmüller theory as developed by Thurston and Penner [5]. The Surfacehedron is then defined by intersecting an infinite dimensional simplex with a finite dimensional subspace

$$\mathcal{P}_S = \{X_\ell \geq 0, \ell \text{ curve on } S\} \cap \{X_\ell + X_{\ell'} - \sum_i X_{\ell_i} = c_{\ell,\ell'}, \ell \cap \ell' \neq \emptyset\};$$

the subspace is given by a set of relations which we call tropical skein relations due to their analogy with the non-linear relations defining the cluster (or skein) algebra associated to the surface S [6]. Beyond the simple case where S is a disk with marked points, as considered in [2], the constants $c_{\ell,\ell'}$ have to satisfy non trivial requirements. It is still a conjecture that such constants can actually be found, but a promising idea to prove their existence is by building them out of geodesic lengths computed using an hyperbolic metric on S.

References

[1] N. Arkani-Hamed and J. Trnka, *The Amplituhedron*, JHEP **10** (2014), 030.

[2] N. Arkani-Hamed, Y. Bai, S. He and G. Yan, *Scattering Forms and the Positive Geometry of Kinematics, Color and the Worldsheet*, JHEP **05** (2018), 096.
[3] G. t'Hooft, *A planar diagram theory for strong interactions*, Nuclear Physics B **72** (3): 461, (1974).
[4] V.V. Fock and A.B. Goncharov, *Dual Teichüller and lamination spaces*. Handbook of Teichmüller theory, **1**(11), 647–684, (2007).
[5] R. Penner, *Decorated Teichmüller theory (Vol. 1)*, European Mathematical Society, (2012).
[6] S. Fomin, M. Shapiro and D. Thurston, *Cluster algebras and triangulated surfaces. Part I: Cluster complexes*, Acta Mathematica, 201(1), 83–146, (2008).

The 'u equations' for hyperbolic surfaces, and string integrals

Hadleigh Frost

(joint work with Nima Arkani-Hamed, Pierre-Guy Plamondon, Giulio Salvatori, Hugh Thomas)

The cross ratios of geodesic lengths on hyperbolic surfaces give rise to natural coordinates, 'u variables', on the Teichmüller space, which satisfy relations, the 'u equations'. We conjecture the u equations for an arbitrary surface and show how these variables give new formulas for the functions that arise in string theory.

1. Background

Let $\mathbb{H}$ be the upper half plane, with the Poincaré metric, and with the Möbius action of $PSL_2\mathbb{R}$. Fix a hyperbolic surface with boundary, Σ, with geodesic boundaries, and marked cusp points on the boundary. Recall that the *Teichmüller space* $T(\Sigma)$ is the moduli space of Fuchsian groups $\Gamma \leq SL_2\mathbb{R}$ (up to conjugation) such that $\Gamma \simeq \pi_1(\Sigma)$ and $\mathbb{H}/\Gamma$ is homeomorphic to Σ. If Σ has marked boundary points, then each point in $T(\Sigma)$ includes the added data of a 'marking': i.e. a labelling of the boundary cusps of $\mathbb{H}/\Gamma$.

Example. Let Σ be the annulus with one marked point on each boundary. A family of Fuchsian groups is generated by $z \mapsto \lambda z$, for a real number $\lambda > 1$. For some real $w > 0$, the two marked cusp points can be taken to be -1 and w, respectively (i.e. the images of -1 and w in the surface $\mathbb{H}/\Gamma$). $T(\Sigma)$ is two dimensional, with coordinates $\lambda > 1$ and $w > 0$.

2. The Hyperbolic Disk

When Σ is the disk with n boundary marked points, $T(\Sigma)$ is just the moduli space, $\mathcal{M}_{0,n}(\mathbb{R})$, of n points on the projective line. The cross-ratios of the positions of points on $\mathbb{P}^1$ have a direct interpretation in hyperbolic geometry, and can be used as coordinates on $\mathcal{M}_{0,n}(\mathbb{R})$. In particular, let $z_1 < z_2$ and $z_3 < z_4$ be four distinct marked points on the boundary of $\mathbb{H}$. Write $\gamma_{1,2}$ for the geodesic in $\mathbb{H}$ from z_1 to z_2. Write $\gamma_{1,2|3,4}$ for the shortest geodesic between $\gamma_{1,2}$ and $\gamma_{3,4}$. The cross ratio of the four points,

$$u(12,34) = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)},$$

can also be computed as the ratio of 'λ-lengths', which are the lengths of geodesics in $\mathbb{H}$. In particular, the length of $\gamma_{1,2|3,4}$ goes to zero if and only if $u(12,34) \to 0$.

The cross-ratios satisfy

$$u(12,34) + u(23,41) = 1. \tag{1}$$

Moreover, if $z_1 < z_2 < ... < z_{n-1} < z_n$, and $z_{n+1} < z_{n+2} < ... < z_m$, then

$$u(1n, n+1m) = \prod_{i=1}^{n-1} \prod_{j=n+1}^{m-1} u(ii+1, jj+1). \tag{2}$$

Now fix a hyperbolic disk with n marked cusp points. Denote by $I, J, K, ...$ an 'arc' on this disk: i.e. a geodesic between two boundary segments. Let $z_1 < z_2 < z_3 < z_4$ be four of the cusp points, cyclically ordered. These points form a quadrilateral bounded by the geodesics $\gamma_{12}, \gamma_{23}, \gamma_{34}, \gamma_{41}$. Then the 'generalized u equation' associated to this quadrilateral is:

$$\prod_{I \not\sim \gamma_{12}, \gamma_{34}} u_I + \prod_{J \not\sim \gamma_{23}, \gamma_{41}} u_J = 1, \tag{3}$$

where the first product is over all arcs I that intersect both γ_{12} and γ_{34}, and the second product is analogous. This follows directly from (1) and (2).

3. The u equations

Let Σ be a more general hyperbolic surface, with hyperbolic boundary and labelled boundary cusp points. Then Σ is homeomorphic to $\mathbb{H}/\Gamma$ for some Γ. We conjecture the following theorem.

Theorem. *For arcs I on a surface Σ, the associated cross-ratios u_I satisfy*

$$u_I + \prod_J u_J^{\langle I,J \rangle} = 1, \tag{4}$$

where $\langle I, J\rangle$ is the intersection number of the arcs I, J, and the product is over all boundary-boundary arcs J.

The idea of the proof is that (4) is a consequence of (3), by identifying Σ with $\mathbb{H}/\Gamma$. The point is that if I intersects J some k times on Σ, then a preimage of I in $\mathbb{H}$ intersects k copies of the preimage of J.

Example. Consider again the annulus with one point on each boundary, Σ. Write R_i for the arc beginning and ending at point A, that goes around the annulus $i \geq 2$ times. Note that all of these arcs are self-intersecting. Write C_i (for i an integer) for the arc beginning at point A, ending at point B, that spirals around the annulus $+i$ times in the clockwise direction with respect to the surface's orientation.

In the universal cover, $\mathbb{H}$, R_i is the arc from $[w, \lambda w]$ to $[w\lambda^i, w\lambda^{i+1}]$ (or any of its images under $z \mapsto \lambda z$). Likewise, C_i is the arc from $[-\lambda, -1]$ to $[w\lambda^i, w\lambda^{i+1}]$ (or any of its images). The associated u variables are the cross-ratios

$$u(A_i) = \frac{(\lambda^{i-1}-1)(\lambda^{i+1}-1)}{(\lambda^i - 1)^2}, \qquad u(C_i) = \frac{(w\lambda^i + 1)^2}{(w\lambda^{i-1}+1)(w\lambda^{i+1}+1)}.$$

It is verified that these satisfy

$$u(A_i) + \left(\prod_{j=2}^{i-1} u(A_j)^{2(j-1)}\right)\left(\prod_{j=i} u(A_j)\right)^{2(j-1)}\left(\prod_{j} u(C_j)\right)^{i-1} = 1,$$

and

$$u(C_i) + \left(\prod_{j=2} u(A_j)^{j-1}\right)\left(\prod_{j=2} u(B_j)^{j-1}\right)\left(\prod_{j} u(C_j)^{j-1}\right) = 1.$$

These equations agree with (4): the exponents are intersection numbers.

4. String Theory

In string theory, the tree level generalized Veneziano amplitude is written as an integral

$$A(X_{ij}) = \int_{[0,1]^{n-3}} \mathrm{d}u_{13}...\mathrm{d}u_{1,n-1} \prod_{\substack{\text{non adjacent}\\ i,j}} u_{ij}^{X_{ij}}, \tag{5}$$

where the u_{ij} are the u variables for a disk, and they satisfy the u equations, (3).

Generalizing this, let Σ be a hyperbolic surface with hyperbolic boundary and marked cusp boundary points. Suppose Σ has n boundary segments, h distinct boundary components, and genus g. Write $\partial\Sigma$ for the boundary of Σ (not including cusps). Then $\dim H_1(\Sigma, \partial\Sigma) = n + h + 2g - 4$. Fix a homomorphism

$$f : H_1(\Sigma, \partial\Sigma) \to \mathbb{R}[k_1, ..., k_n, \ell_1, ..., \ell_{h+2g-1}],$$

such that the k_i are the images of geodesic boundary segments. We take the k_i and ℓ_i to be D-dimensional vectors in some vector space. For an arc I with class $[I] \in H_1(\Sigma, \partial\Sigma)$ we define the variable X_I as the norm squared, $X_I = ||f([I])||^2$. Then we form the following integral, generalizing (5),

$$A(k_1, ..., k_n) = \int \prod_{i=1}^{L} \mathrm{d}^D\ell_i \int_{[0,1]^{n-3+2L}} \prod_a \mathrm{d}u_{I_a} \left(\prod_I u_I^{X_I}\right),$$

where the product over arcs I includes arcs that are self-intersecting.

In the case of the annulus, the above prescription agrees with the expectation from string theory, which suggests that the products of u variables will give rise to a new family of representations of string theory amplitudes.

References

[1] Arkani-Hamed, N., et al. *Binary Geometries, Generalized Particles and Strings, and Cluster Algebras.* ArXiv:1912.11764 [Hep-Th], 2020.

[2] Brown, Francis C. S. *Multiple Zeta Values and Periods of Moduli Spaces* $\overline{\mathfrak{M}}_{0,N}$. Annales Scientifiques de l'École Normale Supérieure, vol. 42, no. 3, 2009, pp. 371–489.

Solving u-equations with quiver representations

Pierre-Guy Plamondon
(joint work with Nima Arkani-Hamed, Hadleigh Frost, Giulio Salvatori, Hugh Thomas)

We have seen the u-equations appear in the talks of Hadleigh Frost and Giulio Salvatori. We will now see how we can solve them using the representation theory of quivers.

1. A primer on quiver representations

Fix a field k. A *quiver* is an oriented graph. Given a quiver Q, its vertices will be labelled $1, \dots, n$. We denote by kQ the set of k-linear combinations of paths in Q; when equipped with the concatenation of paths, it becomes a k-algebra. A *relation* is a linear combination of paths of length at least 2. A *quiver with relations* is a pair (Q, R) where Q is a quiver and R is a finite set of relations of Q.

Fix a quiver Q. A *representation* V of Q is given by

- for all vertices i of Q, a k-vector space V_i (assumed to be finite-dimensional in this talk), and
- for all arrows $\alpha : i \to j$ in Q, a linear map $V_\alpha : V_i \to V_j$.

If R is a set of relations on Q, then a representation of (Q, R) is a representation V of Q such that, for any relation in R, the corresponding linear combination of compositions of the V_α's vanishes.

Example 1. Let $Q = 1 \xrightarrow{\alpha} 2 \xrightarrow{\beta} 3$ and $R = \{\beta\alpha\}$. Then $V = k \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} k^2 \xrightarrow{\begin{bmatrix}0 & 1\end{bmatrix}} k$ is a representation of (Q, R).

Fix a representation V of a quiver with relation (Q, R).

- The *dimension vector* of V is the vector $\underline{\dim}(V) = (\dim V_1, \dots, \dim V_n)$.
- The *direct sum* $V \oplus W$ of two representations V and W is defined vertex-wise. The representation V is *indecomposable* if it is not isomorphic to the direct sum of two non-zero representations.
- A *subrepresentation* of V is a collection U of subvector spaces $U_i \subset V_i$ for $i \in Q_0$ such that, for each arrow $\alpha : i \to j$, we have that $V_\alpha(U_i) \subset U_j$. The set of subrepresentations of V of dimension vector $\mathbf{d}$ is a projective variety called the *Grassmannian of subrepresentations* and denoted by $\mathrm{Gr}_{\mathbf{d}}(V)$. It is a subvariety of the product of classical Grassmannians $\prod_{i=1}^n \mathrm{Gr}_{d_i}(V_i)$.

The main object we will need is a device to keep track of the "number" of subrepresentations of a given representation. Since this number can be infinite, we will instead be "measuring" the Grassmannians of subrepresentations by computing their Euler characteristics. From now on, $k = \mathbb{C}$.

Definition 2. The F*-polynomial* of a representation V of (Q, R) is

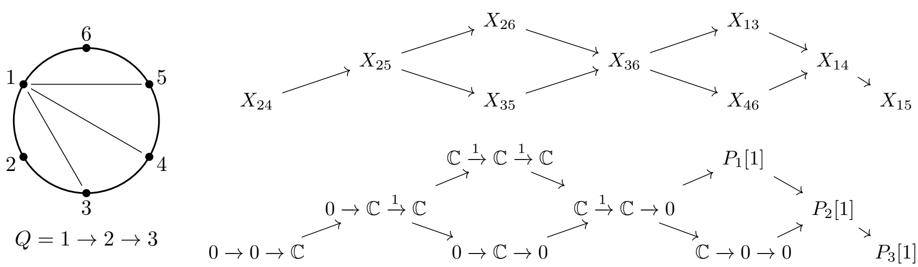
$$F_V(y_1, \ldots, y_n) = \sum_{\mathbf{d} \in \mathbb{Z}^n_{\geq 0}} \chi\big(\mathrm{Gr}_{\mathbf{d}}(V)\big) y_1^{d_1} \cdots y_n^{d_n}.$$

Although F-polynomials seem like natural objects, they have appeared relatively recently in connection with cluster algebras [3]. Note that, despite having a definition similar to those of classical Grassmannians, the Grassmannians of subrepresentations do not behave remotely as well; in fact, any complex projective variety is isomorphic to a Grassmannian of subrepresentations [6].

Example 3. If $V = \mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathbb{C}$, then $F_V = 1 + y_3 + y_2 y_3 + y_1 y_2 y_3$.

2. Relation with u-equations from polygons

The u-equations in type A_n [5, 1] can be illustrated with diagonals of an $(n+3)$-gon with vertices labelled 1 to $n+3$. To each diagonal a, associate a variable u_a; the u-equations are then $u_a + \prod_b u_b^{(a,b)} = 1$, where (a, b) is the intersection number of a and b. If one fixes a triangulation of the surface, one can associate a quiver with relations: vertices are diagonals of the triangulations, arrows are angles and relations correspond to consecutive angles in a triangle [2]. There is then a bijection between diagonals not in the triangulation and indecomposable representations of the quiver with relations; diagonals i of the triangulation are sent to symbols $P_i[1]$ called "shifted projectives". This is illustrated for a triangulation of a hexagon.



On the top right are listed all diagonals, organized in a way reminiscent of cluster variables in a cluster algebra of type A_3. On the bottom right is the *Auslander–Reiten quiver* of Q, whose vertices are isomorphism classes of indecomposable representations, and arrows are irreducible morphisms. The *Auslander–Reiten translation* τ is the action of going one step to the left. Under the bijection associating arcs to representations, the original u-equation $u_a + \prod_b u_b^{(a,b)} = 1$ becomes $u_a + \prod_b u_b^{\dim \mathrm{Hom}(a, \tau b) + \dim \mathrm{Hom}(b, \tau a)} = 1$. This example illustrates that the u-equations have a representation-theoretic interpretation.

3. Results

Our main result is a solution for systems of u-equations defined for a large class of quivers with relations.

Theorem 4. *Let (Q,R) be a quiver with relations admitting only finitely many isomorphism classes of indecomposable representations. For each indecomposable V, let u_V be a variable. Then the system of u-equations*

$$u_V + \prod_{W \text{ indec.}} u_W^{\dim \mathrm{Hom}(V,\tau W)+\dim \mathrm{Hom}(W,\tau V)} = 1, \quad V \text{ indecomposable}$$

is solved by

$$u_V = \begin{cases} \frac{F_{\mathrm{rad}\,P}}{F_P} & \text{if } V = P \text{ is projective,} \\ y_i \frac{F_E}{F_V F_{\tau V}} & \text{if } V = P_i[1], \\ \frac{F_E}{F_V F_{\tau V}} & \text{otherwise,} \end{cases}$$

where $\tau V \to E \to V$ is almost split.

The proof uses Auslander–Reiten theory and a result of [4] on F-polynomials. We can prove similar results by working in cluster categories of finite type instead of the category of representations of a quiver with relations.

4. Work in progress

If (Q,R) admits infinitely many indecomposable representations, the u-equation

$$u_V + \prod_{W \text{ indec.}} u_W^{\dim \mathrm{Hom}(V,\tau W)+\dim \mathrm{Hom}(W,\tau V)} = 1$$

still makes sense, even though the product is now infinite. We can prove that the F-polynomials still allow us to solve the system of u-equations in many nice cases. These include gentle algebras arising from triangulations of surfaces more general than a polygon.

References

[1] Nima Arkani-Hamed, Song He, Thomas Lam and Hugh Thomas, *Binary geometries, generalized particles and strings, and cluster algebras*, arXiv:1912.11764.

[2] Ibrahim Assem, Thomas Brüstle, Gabrielle Charbonneau-Jodoin and Pierre-Guy Plamondon, *Gentle algebras arising from surface triangulations* Algebra Number Theory 4 (2010), no. 2, 201–229.

[3] Philippe Caldero and Frédéric Chapoton, *Cluster algebras as Hall algebras of quiver representations*, Comment. Math. Helv. 81 (2006), no. 3, 595–616.

[4] Salomón Dominguez and Christof Geiss, *A Caldero–Chapoton formula for generalized cluster categories*, J. Algebra 399 (2014), 887–893.

[5] Zirô Koba and Holger Bech Nielsen, *Reaction amplitude for n-mesons: A generalization of the Veneziano–Bardakçi–Ruegg–Virasoro model*, Nuclear Physics B10 (1969) 633–665.

[6] Markus Reineke, *Every projective variety is a quiver Grassmannian*, Algebr. Represent. Theory 16 (2013), no. 5, 1313–1314.

Gradient flows, adjoint orbits, and the topology of totally nonnegative flag varieties

Steven N. Karp
(joint work with Anthony M. Bloch)

Given integers $1 \le k_1 < \cdots < k_l \le n-1$, let $\mathrm{Fl}_{k_1,\dots,k_l;n}$ denote the partial flag variety consisting of all chains of subspaces $(V_{k_1} \subset \cdots \subset V_{k_l})$ in $\mathbb{C}^n$, where each V_k has dimension k. One can view $\mathrm{Fl}_{k_1,\dots,k_l;n}$ as an adjoint orbit inside the Lie algebra $\mathfrak{u}_n$ of $n \times n$ skew-Hermitian matrices. Explicitly, let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n$ be a weakly decreasing sequence with descents precisely in positions $k_1, \dots, k_l$, and let $\mathcal{O}_\lambda$ be the adjoint orbit of $\mathfrak{u}_n$ consisting of all matrices with eigenvalues $\mathrm{i}\lambda_1, \dots, \mathrm{i}\lambda_n$, where $\mathrm{i} = \sqrt{-1}$. Then we have an isomorphism $\mathcal{O}_\lambda \xrightarrow{\cong} \mathrm{Fl}_{k_1,\dots,k_l;n}$, sending a matrix to its partial flag of generalized eigenvectors.

Lusztig [10, 11] introduced the totally nonnegative part $\mathrm{Fl}^{\ge 0}_{k_1,\dots,k_l;n}$ of $\mathrm{Fl}_{k_1,\dots,k_l;n}$, which defines a totally nonnegative part $\mathcal{O}^{\ge 0}_\lambda$ of $\mathcal{O}_\lambda$. We use the orbit context to study totally nonnegative flag varieties from an algebraic, geometric, and dynamical perspective. One of our motivations was to relate work of Bloch, Flashcka, and Ratiu [2] with recent work of Galashin, Karp, and Lam [8, 9].

We highlight some of our main contributions, and conclude by posing some open problems. For further details, see our paper [5].

The twist map. Let $\mathrm{Fl}_n := \mathrm{Fl}_{1,\dots,n-1;n}$ denote the *complete flag variety.* We introduce an involution ϑ on $\mathrm{Fl}_n^{\ge 0}$ called the *twist map*, defined as follows. We represent a given $V = (V_1 \subset \cdots \subset V_{n-1}) \in \mathrm{Fl}_n^{\ge 0}$ by the unique orthogonal matrix g whose left-justified minors are all nonnegative, so that each V_k is spanned by the first k columns of g. Then $\vartheta(V)$ is defined to be the element represented by the matrix $((-1)^{i+j} g_{j,i})_{1 \le i,j \le n}$, which is obtained by inverting (or transposing) g and changing the sign of every other entry. Amazingly, this operation is compatible with positivity:

Theorem 1. *The twist map ϑ defines an involution on $\mathrm{Fl}_n^{\ge 0}$.*

For example, the twist map ϑ sends

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \\ \frac{\sqrt{3}}{4} & \frac{1}{4\sqrt{2}} & -\frac{5}{4\sqrt{2}} \\ \frac{1}{4} & \frac{3\sqrt{3}}{4\sqrt{2}} & \frac{\sqrt{3}}{4\sqrt{2}} \end{bmatrix} \text{ to } \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{4\sqrt{2}} & -\frac{3\sqrt{3}}{4\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{5}{4\sqrt{2}} & \frac{\sqrt{3}}{4\sqrt{2}} \end{bmatrix} \text{ in } \mathrm{Fl}_3^{\ge 0}.$$

We call ϑ the 'twist map' since it is analogous to the twist maps introduced by Berenstein, Fomin, and Zelevinsky [3, 7], but with the key difference that our map is based on the Iwasawa (or QR-) decomposition of GL_n, rather than the Bruhat decomposition.

We also obtain a corresponding involution $\vartheta_\lambda : \mathcal{O}^{\ge 0}_\lambda \to \mathcal{O}^{\ge 0}_\lambda$ on the totally nonnegative part of any generic adjoint orbit, via the isomorphism $\mathcal{O}_\lambda \cong \mathrm{Fl}_n$. This generalizes (in type A) a map defined by Bloch, Flaschka, and Ratiu [2] on an isospectral manifold of Jacobi matrices appearing in the study of the Toda lattice. Further connections with the Toda lattice are discussed in [5, Section 8].

Gradient flows. Inspired by [8, 9], we study flows on $\mathcal{O}_\lambda$ which *strictly preserve positivity*, meaning that $\mathcal{O}_\lambda^{\geq 0}$ is sent inside its interior $\mathcal{O}_\lambda^{>0}$ after any positive time. We focus on gradient flows for height functions of the form $L \mapsto \operatorname{tr}(LN)$ (coming from the Killing form) for fixed $N \in \mathfrak{u}_n$, and work in three different Riemannian metrics: the Kähler, normal (or standard), and induced metrics. In several cases we are able to classify which flows strictly preserve positivity.

One such case is when $\mathcal{O}_\lambda$ is isomorphic to a *Grassmannian* $\mathrm{Gr}_{k,n} := \mathrm{Fl}_{k;n}$ for some k, in which case the three metrics coincide up to dilation. We then have the following classification:

Theorem 2. *Let* $2 \leq k \leq n-2$. *Then the gradient flow of* $L \mapsto \operatorname{tr}(LN)$ *on* $\mathcal{O}_\lambda \cong \mathrm{Gr}_{k,n}$ *strictly preserves positivity if and only if* $\mathrm{i}N$ *is real,* $N_{i,j} = 0$ *for* $i - j \not\equiv -1, 0, 1 \pmod{n}$,

$$\mathrm{i}N_{1,2},\, \mathrm{i}N_{2,3},\, \ldots,\, \mathrm{i}N_{n-1,n},\, (-1)^{k-1}\mathrm{i}N_{n,1} \geq 0,$$

and at least $n-1$ *of the* n *inequalities above are strict.*

When $\mathcal{O}_\lambda$ is not isomorphic to a Grassmannian, then the three metrics are different, and their gradient flows exhibit markedly different behavior with respect to positivity. In the case of the Kähler metric, the flows admit a beautiful explicit solution, which we use to obtain a complete classification, similar to Theorem 2. By contrast, in the normal metric, in the generic case (i.e. when $\mathcal{O}_\lambda \cong \mathrm{Fl}_n$) there are no flows which strictly preserve positivity; we leave the consideration of non-generic orbits to future work. For the induced metric, our preliminary investigations for $\mathcal{O}_\lambda \cong \mathrm{Fl}_3$ indicate that the existence of gradient flows which strictly preserve positivity may depend on the spacing between the entries of λ.

Topology. Galashin, Karp, and Lam [8, 9] used certain flows which strictly preserve positivity to show that the totally nonnegative part of a partial flag variety (in arbitrary Lie type) is homeomorphic to a closed ball. We rephrase their argument in the orbit language for gradient flows in the Kähler metric, and show that the height function provides a strict Lyapunov function for such a flow. This implies that certain invariant subsets of $\mathcal{O}_\lambda$ are homeomorphic to closed balls.

We apply the framework above to study the topology of *amplituhedra* $\mathcal{A}_{n,k,m}(Z)$. These are generalizations of the totally nonnegative Grassmannian $\mathrm{Gr}_{k,n}^{\geq 0}$, introduced by Arkani-Hamed and Trnka [1] in order to give a geometric basis for calculating scattering amplitudes in planar $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. The amplituhedron $\mathcal{A}_{n,k,m}(Z)$ depends on a certain auxiliary $(k+m) \times n$ matrix Z, where m is an additional parameter satisfying $k + m \leq n$. It is believed that every amplituhedron $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed ball, which is known in many special cases. We apply the methods of [8] to show that a new family of amplituhedra are homeomorphic to closed balls. These are *twisted Vandermonde amplituhedra*, for which the matrix Z arises by applying the twist map ϑ to a *Vandermonde flag*.

Theorem 3. *Every twisted Vandermonde amplituhedron (in particular, every amplituhedron $\mathcal{A}_{n,k,m}(Z)$ with $n-k-m \leq 2$) is homeomorphic to a closed ball.*

Open problems. We state several problems raised by our work.

Problem 4. *Does the Plücker-nonnegative part of* $\mathrm{Fl}_{k_1,\ldots,k_l;n}$ *(which differs from Lusztig's totally nonnegative part unless $k_1,\ldots,k_l$ are consecutive integers* [6]*) have nice properties?*

The Plücker-nonnegative part of $\mathrm{Fl}_{1,3;n}$ was studied by Bai, He, and Lam [4].

Problem 5. *Do the twisted Vandermonde amplituhedra appearing in Theorem 3 have any other distinguishing properties?*

Problem 6. *Can we classify gradient flows preserving $\mathcal{A}_{n,k,m}(Z)$? Does there exist such a flow for every Z, allowing one to show that $\mathcal{A}_{n,k,m}(Z)$ is homeomorphic to a closed ball?*

References

[1] Nima Arkani-Hamed and Jaroslav Trnka. The amplituhedron. *J. High Energy Phys.*, (10):33, 2014.
[2] A. M. Bloch, H. Flaschka, and T. Ratiu. A convexity theorem for isospectral manifolds of Jacobi matrices in a compact Lie algebra. *Duke Math. J.*, 61(1):41–65, 1990.
[3] Arkady Berenstein, Sergey Fomin, and Andrei Zelevinsky. Parametrizations of canonical bases and totally positive matrices. *Adv. Math.*, 122(1):49–149, 1996.
[4] Yuntao Bai, Song He, and Thomas Lam. The amplituhedron and the one-loop Grassmannian measure. *J. High Energy Phys.*, (1):112, front matter+41, 2016.
[5] Anthony M. Bloch and Steven N. Karp. Gradient flows, adjoint orbits, and the topology of totally nonnegative flag varieties. `arXiv:2109.04558`.
[6] Anthony M. Bloch and Steven N. Karp. On two notions of total positivity for partial flag varieties. In preparation.
[7] Sergey Fomin and Andrei Zelevinsky. Double Bruhat cells and total positivity. *J. Amer. Math. Soc.*, 12(2):335–380, 1999.
[8] Pavel Galashin, Steven N. Karp, and Thomas Lam. The totally nonnegative Grassmannian is a ball. *Adv. Math.* (to appear), `arXiv:1707.02010`.
[9] Pavel Galashin, Steven N. Karp, and Thomas Lam. The totally nonnegative part of G/P is a ball. *Adv. Math.*, 351:614–620, 2019.
[10] G. Lusztig. Total positivity in reductive groups. In *Lie theory and geometry*, volume 123 of *Progr. Math.*, pages 531–568. Birkhäuser Boston, Boston, MA, 1994.
[11] G. Lusztig. Total positivity in partial flag manifolds. *Represent. Theory*, 2:70–78, 1998.

Towards cluster super-algebras

Marcus Spradlin
(joint work with S. James Gates, Jr., S.-N. Hazel Mak, Anastasia Volovich)

This talk is based on the preprint [1]. The workshop focuses on connections between scattering amplitudes and cluster algebras. There have recently been several apparently distinct approaches to the construction of cluster superalgebras (see for example [2, 3, 4, 5, 6, 7]), and it is natural to wonder whether these make any appearance in physics. In the talk I review the definition proposed by Ovsienko and Shapiro in [5] and explain that while in general the cluster superalgebras constructed in this manner are infinite, there is a finite algebra that is based on the A_2 cluster algebra and has 15 cluster supervariables. I also mention an alternate definition of cluster superalgebras based on promoting ordinary cluster variables to superfields and comment on some possible applications of cluster superalgebras to various aspects of scattering amplitudes.

References

[1] S. J. Gates, S. N. Hazel Mak, M. Spradlin and A. Volovich, *Cluster Superalgebras and Stringy Integrals*, arXiv:2111.08186 (2021).
[2] V. Ovsienko, *Cluster superalgebras*, arXiv:1503.01894 (2015).
[3] L. Li, J. Mixco, B. Ransingh and A. K. Srivastava, *An Introduction to Supersymmetric Cluster Algebras*, arXiv:1708.03851 (2017).
[4] E. Shemyakova and T. Voronov, *On super Plücker embedding and cluster algebras*, arXiv:1906.12011 (2019).
[5] V. Ovsienko and M. Shapiro, *Cluster algebras with Grassmann variables*, Electronic Research Announcements **26** (2019), 1.
[6] G. Musiker, N. Ovenhouse and S. W. Zhang, *An Expansion Formula for Decorated Super-Teichmüller Spaces*, arXiv:2102.09143 (2021).
[7] G. Musiker, N. Ovenhouse and S. W. Zhang, *Double Dimer Covers on Snake Graphs from Super Cluster Expansions*, arXiv:2110.06497 (2021).

Enumeration techniques with applications to positroids, amplituhedra, and their friends

Lauren Williams

(joint work with Robert Moerman)

The Exponential Formula and its generalizations (which fit into the framework of Joyal's *theory of species*), give techniques to count objects which are built by choosing a set S, breaking it up into components, and placing a structure on each component.

The *momentum amplituhedron* $\mathcal{M}_{n,k}$, introduced by Damgaard–Ferro–Łukowski–Parisi in [1], has a boundary stratification which was first described in [2]. In [3], we reformulate the strata in terms of *Grassmannian forests*, which are acyclic planar graphs embedded in a disk, in which each vertex is decorated by a *helicity*.

By enumerating Grassmannian forests while keeping track of helicity and the corresponding dimension statistic in the momentum amplituhedron, in [3] we were able to come up with an explicit formula for the number of r-dimensional boundary strata in the momentum amplituhedron $\mathcal{M}_{n,k}$.

In what follows, the notation $F^{\langle -1\rangle}(x)$ denotes the compositional inverse of the power series $F(x)$. That is, $F^{\langle -1\rangle}(F(x)) = F(F^{\langle -1\rangle}(x)) = x$. *Lagrange Inversion* allows one to compute the coefficients of $F^{\langle -1\rangle}(x)$ in terms of those of $F(x)$.

Theorem 1. *The number of boundaries of the momentum amplituhedron $\mathcal{M}_{n,k}$ of dimension r is given by the coefficient $[x^n y^k q^r]\mathcal{G}_{forest}(x,y,q)$ where*

$$x\mathcal{G}_{forest}(x,y,q) = \left(\frac{x}{1+\mathcal{G}_{tree}(x,y,q)}\right)^{\langle -1\rangle},$$

and

$$\mathcal{G}_{tree}(x,y,q) = x\left(1+y+yq\,C^{\langle -1\rangle}(x,y,q)\right), \text{ with}$$
$$C(x,y,q) = \frac{x(1-x(1+y)q^2 - x^2yq^2(1+q-q^2) - x^4y^2q^5(1+q))}{(1+xq)(1+xyq)(1-xq^2)(1-xyq^2)}.$$

Note that the compositional inverse above is taken with respect to the variable x. Equivalently,

$$[x^n]\mathcal{G}_{forest}(x,y,q) = \frac{1}{n+1}[x^n]\left(1+\mathcal{G}_{tree}(x,y,q)\right)^{n+1}.$$

When we substitute $q=-1$ in the above formulas, they simplify dramatically and allow us to check that the Euler characteristic of $\mathcal{M}_{n,k}$ is always 1. See [3] for a table of data.

References

[1] D. Damgaard, L. Ferro, T. Lukowski, M. Parisi, *The momentum amplituhedron*, J. High Energy Phys. **8** (2019).

[2] L. Ferro, T. Lukowski, R. Moerman, *From momentum amplituhedron boundaries to amplitude singularities and back*, J. High Energy Phys. **7** (2020).

[3] R. Moerman, L. Williams, *Grass trees and forests: Enumeration of Grassmannian trees and forests, with applications to the momentum amplituhedron*, arXiv:2112.02061.

On the geometry of the orthogonal momentum amplituhedron

Tomasz Łukowski

(joint work with Robert Moerman, Jonah Stalknecht)

In recent years we have seen tremendous interest in positive geometries [1] that encode observables in quantum field theories, and in particular the ones that can be employed to study scattering amplitudes. A particularly fruitful theory where positive geometries can be defined is $\mathcal{N}=4$ super Yang–Mills, where the amplituhedron $\mathcal{A}_{n,k}$ [2] and the momentum amplituhedron $\mathcal{M}_{n,k}$ [3] have been introduced to encode the tree-level scattering amplitudes. Both of these geometries are defined as the image of the positive Grassmannian through a linear map. More recently, a similar construction has been proposed for ABJM theory tree-level scattering amplitudes [4, 5] using the orthogonal Grassmannian and its positive part. The resulting geometry, denoted $\mathcal{O}_k$, is $(2k-3)$-dimensional and can be thought of as a deformation of the ABHY associahedron A_{2k-3} [6] where faces corresponding to even-particle planar Mandelstam variables are squashed to lower dimensional boundaries.

In this talk I will describe the properties of $\mathcal{O}_k$, and in particular provide a complete classification of its boundaries. To this end we have used the algorithm developed in [7] where it was successfully applied to find all boundaries of the amplituhedron $\mathcal{A}^{(2)}_{n,k}$, and which has been subsequently used to find all boundaries of the momentum amplituhedron $\mathcal{M}_{n,k}$ in [8]. Applying this algorithm to the orthogonal momentum amplituhedron $\mathcal{O}_k$, we observed that all boundaries can be labelled by a particular class of graphs, which we called orthogonal Grassmannian forests, that correspond to all possible factorizations of ABJM amplitudes. This observation is analogous to the one that has been made for $\mathcal{N} = 4$ sYM in [8], where the boundaries of the momentum amplituhedron $\mathcal{M}_{n,k}$ can be labelled using Grassmannian forests, see [9]. Both form a subset of the Grassmannian graphs introduced in [10]. In this talk I will summarise our explorations of the boundaries for $\mathcal{O}_k$ for $k \leq 7$, and provide a conjecture on the boundary stratification for all k. In particular, using the methods developed in [9], we provided a generating function for the number of boundaries of a given dimension and this allows us to show that the Euler characteristic for the orthogonal momentum amplituhedron equals one. This story parallels the one developed for the momentum amplituhedron.

Moreover, it has been shown in [5] that both the interior of the ABHY associahedron A_{2k-3} and the interior of the orthogonal momentum amplituhedron $\mathcal{O}_k$ are diffeomorphic to the positive part of the moduli space of n points on the Riemann sphere and therefore are diffeomorphic to each other. This is however not true for their closures. Nevertheless, we showed that there is a simple diagrammatic way to understand how the boundaries of the associahedron naturally reduce to the boundaries of the orthogonal momentum amplituhedron.

References

[1] N. Arkani-Hamed, Y. Bai and T. Lam, JHEP **11**, 039 (2017) doi:10.1007/JHEP11(2017)039 [arXiv:1703.04541 [hep-th]].

[2] N. Arkani-Hamed and J. Trnka, JHEP **10**, 030 (2014) doi:10.1007/JHEP10(2014)030 [arXiv:1312.2007 [hep-th]].

[3] D. Damgaard, L. Ferro, T. Lukowski and M. Parisi, JHEP **08**, 042 (2019) doi:10.1007/JHEP08(2019)042 [arXiv:1905.04216 [hep-th]].

[4] Y. t. Huang, R. Kojima, C. Wen and S. Q. Zhang, [arXiv:2111.03037 [hep-th]].

[5] S. He, C. K. Kuo and Y. Q. Zhang, [arXiv:2111.02576 [hep-th]].

[6] N. Arkani-Hamed, Y. Bai, S. He and G. Yan, JHEP **05**, 096 (2018) doi:10.1007/JHEP05(2018)096 [arXiv:1711.09102 [hep-th]].

[7] T. Lukowski, [arXiv:1908.00386 [hep-th]].

[8] T. Łukowski and R. Moerman, Comput. Phys. Commun. **259**, 107653 (2021) doi:10.1016/j.cpc.2020.107653 [arXiv:2002.07146 [hep-th]].

[9] R. Moerman and L. K. Williams, [arXiv:2112.02061 [math.CO]].

[10] A. Postnikov, [arXiv:1806.05307 [math.CO]].

Reporter: Steven N. Karp

Participants

Prof. Dr. Nima Arkani-Hamed
School of Natural Sciences
Institute for Advanced Study
1 Einstein Drive
Princeton, NJ 08540
UNITED STATES

Dr. James Drummond
School of Physics and Astronomy
University of Southampton
Highfield Campus
Southampton SO17 1BJ
UNITED KINGDOM

Dr. Nick Early
Max-Planck-Institut für Physik und Astrophysik
Werner Heisenberg-Inst. f. Physik
Föhringer Ring 6
80805 München
GERMANY

Dr. Livia Ferro
Arnold Sommerfeld Center for Theoretical Physics
Mathematical Physics - String theory
Ludwig-Maximilians-Universität München
Theresienstr. 37
80333 München
GERMANY

Claudia Fevola
Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstr. 22 - 26
04103 Leipzig
GERMANY

Dr. Hadleigh Frost
Mathematical Institute
Oxford University
Andrew Wiles Building
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Pavel Galashin
Department of Mathematics
University of California, Los Angeles
UCLA
Los Angeles CA 90095-1555
UNITED STATES

Dr. Song He
CAS Key Laboratory of Theoretical Physics
Institute of Theoretical Physics
Chinese Academy of Sciences
Chinese Academy of Sciences
Beijing 100190
CHINA

Dr. Steven Karp
LaCIM
Université du Québec à Montréal
CP 8888, Succ. Centre-ville
Montréal QC H3C 3P8
CANADA

Dr. Dani Kaufman
Mathematisches Institut
Universität Heidelberg
Im Neuenheimer Feld 205
69120 Heidelberg
GERMANY

Prof. Dr. Thomas Lam
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Tomasz Łukowski
University of Hertfordshire
AL10 9AB Hatfield, Herts
UNITED KINGDOM

Dr. Bethany Marsh
School of Mathematics
University of Leeds
Leeds LS2 9JT
UNITED KINGDOM

Antonela Matijasic
Max-Planck-Institut für Physik und Astrophysik
Werner Heisenberg-Inst. f. Physik
Föhringer Ring 6
80805 München
GERMANY

Matteo Parisi
Mathematical Institute
University of Oxford
Andrew Wiles Building
Radcliffe Observatory Quarter
Woodstock Road
Oxford OX2 6GG
UNITED KINGDOM

Prof. Dr. Pierre-Guy Plamondon
Laboratoire de Mathématiques de Versailles
Université de Versailles Saint-Quentin
45, av. des Etats-Unis
78035 Versailles Cedex
FRANCE

Prof. Dr. Konstanze Rietsch
Department of Mathematics
King's College London
Strand
London WC2R 2LS
UNITED KINGDOM

Dr. Giulio Salvatori
Perimeter Institute for Theoretical Physics
31 Caroline Street N.
Waterloo ON N2L 2Y5
CANADA

Melissa Sherman-Bennett
Department of Mathematics
University of Michigan
530 Church St
Ann Arbor 48109-1043
UNITED STATES

Dr. Marcus Spradlin
Department of Physics
Brown University
Box 1917
Providence, RI 02912
UNITED STATES

Prof. Dr. Hugh R. Thomas
Département de Mathématiques
Université du Québec à Montréal
CP 8888, succursale Centre-Ville
Montréal, Québec H3C 3P8
CANADA

Dr. Anastasia Volovich
Department of Physics
Brown University
Box 1917
Providence, RI 02912
UNITED STATES

Prof. Dr. Lauren K. Williams
Department of Mathematics
Harvard University
Science Center
One Oxford Street
02138 Cambridge MA 02138-2901
UNITED STATES