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Applications of Nijenhuis geometry III: Frobenius pencils and compatible non-homogeneous Poisson structures

Alexey V. Bolsinov[*] & Andrey Yu. Konyaev[†] & Vladimir S. Matveev[‡]

Abstract

We consider multicomponent local Poisson structures of the form $\mathcal{P}_3 + \mathcal{P}_1$, under the assumption that the third order term $\mathcal{P}_3$ is Darboux-Poisson and non-degenerate, and study the Poisson compatibility of two such structures. We give an algebraic interpretation of this problem in terms of Frobenius algebras and reduce it to classification of Frobenius pencils, i.e. of linear families of Frobenius algebras. Then, we completely describe and classify Frobenius pencils under minor genericity conditions. In particular we show that each such Frobenuis pencil is a subpencil of a certain *maximal* pencil. These maximal pencils are uniquely determined by some combinatorial object, a directed rooted in-forest with vertices labeled by natural numbers whose sum is the dimension of the manifold. These pencils are naturally related to certain (polynomial, in the most nondegenerate case) pencils of Nijenhuis operators. We show that common Frobenius coordinate systems admit an elegant invariant description in terms of the Nijenhuis pencil.

MSC classes: 37K05, 37K06, 37K10, 37K25, 37K50, 53B10, 53A20, 53B20, 53B30, 53B50, 53B99, 53D17

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1 Introduction

1.1 Foreword

Nijenhuis operator is a (1,1)-tensor field $L = L^i_j$ on a manifold M of dimension n such that its Nijenhuis torsion vanishes. Nijenhuis geometry as initiated in [6] (where also all necessary definitions can be found) and further developped in [7, 8, 23] studies Nijenhuis operators and their applications. There are many topics in mathematics and mathematical physics where Nijenhuis operators appear naturally; this paper is devoted to the study of ∞-dimensional compatible Poisson brackets of type $\mathcal{P}_3 + \mathcal{P}_1$, where the lower index i indicates the order of the homogeneous bracket $\mathcal{P}_i$ (the necessary definitions will be given in Section 1.2). Nijenhuis geometry allows us to reformulate the initial problem, originated from mathematical physics, first into the language of algebra and then into the language of differential geometry and finally solve it using the machinery of differential geometry in combination with that of algebra. Translating back the results gives a full description of (nongenerate) compatible Poisson brackes of type $\mathcal{P}_3 + \mathcal{P}_1$ such that the 3-component is Darboux-Poisson.

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1.2 Mathematical setup

The construction below is a special case of the general approach suggested in [21]. For $n = 1$, the construction can be found in [20], see also [10, 30, 36].

We work in an open disc $U \subset \mathbb{R}^n$ with coordinates $u^1, ..., u^n$. Our constructions are invariant with respect to coordinate changes so one may equally think of $(u^1, ..., u^n)$ as a coordinate chart on a smooth manifold M.

Consider the jet bundles (of curves) over U. Recall that for a point $\mathsf{p} \in U$, the k^{th} jet space $J^k_{\mathsf{p}}U$ at this point is an equivalence class of smooth curves $c : (-\varepsilon, \varepsilon) \to U$ such that $c(0) = \mathsf{p}$. The parameter of the curves c will always be denoted by x. The equivalence relation is as follows: two curves are equivalent if they coincide at $c(0)$ up to terms of order $k + 1$.

For example, for $k = 0$ the space $J^0_{\mathsf{p}}U$ contains only one element and the definition of $J^1_{\mathsf{p}}U$ coincides with one of the standard definitions of the tangent space $T_{\mathsf{p}}U$.

It is known that $J^k_{\mathsf{p}}U$ is naturally equipped with the structure of a vector space of dimension $n \times k$ with coordinates denoted by

$$(u^1_x, ..., u^n_x, u^1_{x^2}, ..., u^n_{x^2}, ..., u^1_{x^k}, ..., u^n_{x^k}). \tag{1}$$

Namely, a curve $c(x) = (u^1(x), ..., u^n(x))$ with $c(0) = \mathsf{p}$ viewed as an element of $J^k_{\mathsf{p}}U$ has coordinates

$$\begin{aligned} & \left(u^1_x, ..., u^n_x, u^1_{x^2}, ..., u^n_{x^2}, ..., u^1_{x^k}, ..., u^n_{x^k}\right) \\ = & \left(\tfrac{d}{dx}(u^1), ..., \tfrac{d}{dx}(u^n), \tfrac{d^2}{dx^2}(u^1), ..., \tfrac{d^2}{dx^2}(u^n), ..., \tfrac{d^k}{dx^k}(u^1), ..., \tfrac{d^k}{dx^k}(u^n)\right)_{|x=0}. \end{aligned} \tag{2}$$

We denote by J^kU the union $\bigcup_{\mathsf{p}\in U} J^k_{\mathsf{p}}U$. It has a natural structure of a $k\times n$-dimensional vector bundle over U. The coordinates $(u^1, ..., u^n)$ on U and (1) on $J^k_{\mathsf{p}}U$ generate a coordinate system

$$(u^1, ..., u^n, u^1_x, ..., u^n_x, u^1_{x^2}, ..., u^n_{x^2}, ..., u^1_{x^k}, ..., u^n_{x^k})$$

on J^kU adapted to the bundle structure. Any C^∞ curve $c : [a, b] \to U,\ x \mapsto (u^1(x), ..., u^n(x))$ naturally lifts to a curve $\hat{c}$ on J^kU by

$$\hat{c} : [a, b] \to J^kU \ , \quad x_0 \mapsto \left(u^1, ..., u^n, \tfrac{d}{dx}(u^1), ..., \tfrac{d}{dx}(u^n), ..., \tfrac{d^k}{dx^k}(u^1), ..., \tfrac{d^k}{dx^k}(u^n)\right)_{|x=x_0}. \tag{3}$$

Next, for every $\mathsf{p} \in U$ denote by $\Pi[J^k_{\mathsf{p}}U]$ the algebra of polynomials in variables (1) on $J^k_{\mathsf{p}}U$. It has a natural structure of an infinite-dimensional vector bundle over U. Let $\mathfrak{A}_k$ denote the algebra of C^∞-smooth sections of the bundle $\Pi[J^k_{\mathsf{p}}U]$. Notice that we have natural inclusion $\mathfrak{A}_k \subset \mathfrak{A}_{k+1}$ and set $\mathfrak{A} = \bigcup_{k=0}^{\infty} \mathfrak{A}_k$. In simple terms, the elements of $\mathfrak{A}$ are finite sums of finite products of coordinates

$$(u_x^1, ..., u_x^n, u_{x^2}^1, ..., u_{x^2}^n, ..., u_{x^k}^1, ..., u_{x^k}^n, ...) \tag{4}$$

with coefficients being C^∞-functions on U. The summands in this sum, i.e., terms of the form $a_{i_1...i_n}^{j_1...j_n}(u)(u_{x^{i_1}}^1)^{j_1}...(u_{x^{i_n}}^n)^{j_n}$ with $a_{i_1...i_n}^{j_1...j_n}(u) \not\equiv 0$ will be called *differential monomials*. The *differential degree* of such a differential monomial is the number $i_1 j_1 + i_2 j_2 + ... + i_n j_n$. For example, $f(u) u_{x^2}^1 (u_x^2)^2$ has differential degree $2 + 2 \times 1 = 4$. Differential degree of an element of $\mathfrak{A}$ is the maximum of the differential degrees of its differential monomials, it is a nonnegative integer number. Elements of $\mathfrak{A}$ will be called *differential polynomials*.

Generators of this algebra are coordinates $u_{x^j}^i$ and functions on U. Every element of $\mathfrak{A}$ can be obtained from finitely many generators using finitely many summation and multiplication operations.

The following two linear mappings will be important for us. The first one, called the *total x-derivative* and denoted by D (another standard notation used in literature is $\frac{d}{dx}$) is defined as follows. One requires that D satisfies the Leibnitz rule and then defines it on the generators of $\mathfrak{A}$, i.e., on functions $f(u)$ and coordinates (4) by setting

$$D(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial u^i} u_x^i \ , \quad D(u_{x^j}^i) = u_{x^{j+1}}^i.$$

Clearly, the operation D increases the differential degree by one at most.

Next, denote by $\tilde{\mathfrak{A}}$ the quotient algebra $\mathfrak{A}/D(\mathfrak{A})$. The tautological projection $\mathfrak{A} \to \tilde{\mathfrak{A}}$ is traditionally denoted by $\mathcal{H} \mapsto \int \mathcal{H} dx \in \tilde{\mathfrak{A}}$. In simple terms it means that we think that two differential polynomials $\mathcal{H}, \tilde{\mathcal{H}}$ are equal, if their difference is a total derivative of a differential polynomial.

Note that by construction, the operation D has the following remarkable property, which explains its name and also the notation $\frac{d}{dx}$ used for D sometimes in literature. For any curve $c : [a, b] \to U$ whose lift (3) will be denoted by $\hat{c}$ and for any element $\mathcal{H} \in \mathfrak{A}$ we have:

$$\tfrac{d}{dx} \left(\mathcal{H}(\hat{c}) \right) = \left(D\mathcal{H} \right)(\hat{c}). \tag{5}$$

The second mapping is the mapping from $\mathfrak{A}$ to an n-tuple of elements of $\mathfrak{A}$. The mapping will be denoted by δ and will be called *the variational derivative.* Its i^{th} component will be denoted by $\frac{\delta}{\delta u^i}$ and for an element $\mathcal{H} \in \mathcal{A}$ it is given by the Euler-Lagrange formula:

$$\frac{\delta \mathcal{H}}{\delta u^i} = \sum_{k=0}^{\infty} (-1)^k D^k \left(\frac{\partial \mathcal{H}}{\partial u_{x^k}^i} \right)$$

(only finitely many elements in the sum are different from zero so the result is again a differential polynomial). It is known, see e.g. [20], that for an element $\mathcal{H} \in \mathfrak{A}$ we have

$\delta\mathcal{H} = 0$ if and only if $\mathcal{H}$ is a total x-derivative. Then, we again see that the variational derivative does not depend on the choice of differential polynomial in the equivalence class $\mathcal{H} \subset \mathfrak{A}$. Then, the mapping δ induces a well-defined mapping on $\tilde{\mathfrak{A}}$, which will be denoted by the same letter δ. One can think of $\delta\mathcal{H}$ as a covector with entries from $\tilde{\mathfrak{A}}$, because the transformation rule of its entries under the change of u-coordinates is a natural generalisation of the transformation rule for (0,1)-tensors.

Following [13, 14], let us define a (homogeneous, nondegenerate) Poisson bracket of order 1. We choose a contravariant flat metric $g = g^{ij}$ of any signature whose Levi-Civita connection will be denoted by $\nabla = (\Gamma^i_{jk})$. Next, consider the following operation $\mathcal{A}_g : \tilde{\mathfrak{A}} \times \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$: for two elements $\mathcal{H}, \bar{\mathcal{H}} \in \tilde{\mathfrak{A}}$ we set

$$\mathcal{A}_g(\mathcal{H}, \bar{\mathcal{H}}) = \int \tfrac{\delta\bar{\mathcal{H}}}{\delta u^\alpha} \left(g^{\alpha\beta} D\left(\tfrac{\delta\mathcal{H}}{\delta u^\beta}\right) - \Gamma^{\alpha\beta}_{\gamma} \tfrac{\delta\mathcal{H}}{\delta u^\beta} u^\gamma_x \right) dx. \tag{6}$$

In the formula above and later in the text, we sum over repeating indexes and assume $\Gamma^{is}_j = \Gamma^s_{pj} g^{pi}$. The components Γ^{is}_j will be called *contravariant Christoffel symbols*, when we speak about different metrics we always raise the index *by the own metric*. A common way to write the operation $\mathcal{A}_g$ which we also will use in our paper assumes applying it to $\frac{\delta\mathcal{H}}{\delta u^\beta}$ and multiplication with $\frac{\delta\bar{\mathcal{H}}}{\delta u^\alpha}$ (and of course summation and projection to $\tilde{\mathfrak{A}}$):

$$\mathcal{A}_g = g^{\alpha\beta} D - \Gamma^{\alpha\beta}_{\gamma} u^\gamma_x. \tag{7}$$

It is known, see e.g. [13, 14, 15], that the operation $\mathcal{A}_g$ given by (6) defines a Poisson bracket on $\tilde{\mathfrak{A}}$, that is, it is skew-symmetric and satisfies the Jacobi identity. Moreover, one can show that the operation constructed by g and Γ via (6) defines a Poisson bracket if and only if g is flat, that is, its curvature is zero, and Γ^i_{jk} is the Levi-Civita connection of g. It is also known that the construction (6) does not depend on the coordinate system on U.

Next, let us define a (nondegenerate, homogeneous) Darboux-Poisson structure of order 3. We choose a nondegenerate contravariant flat metric $h = h^{ij}$ of arbitrary signature and define the operation $\mathcal{B}_h : \tilde{\mathfrak{A}} \times \tilde{\mathfrak{A}} \to \tilde{\mathfrak{A}}$ by the formula:

$$\begin{aligned}\mathcal{B}_h &= h^{\alpha q}\left(\delta^p_q D - \Gamma^p_{qm} u^m_x\right)\left(\delta^r_p D - \Gamma^r_{pk} u^k_x\right)\left(\delta^\beta_r D - \Gamma^\beta_{rs} u^s_x\right) = \\ &= h^{\alpha\beta} D^3 - 3h^{\alpha q}\Gamma^\beta_{qs} u^s_x D^2 + \\ &+ 3\left(h^{\alpha q}\left(\Gamma^p_{qs}\Gamma^\beta_{pr} - \frac{\partial \Gamma^\beta_{qs}}{\partial u^r}\right) u^s_x u^r_x - h^{\alpha q}\Gamma^\beta_{qs} u^s_{x^2}\right) D + \\ &+ \left(h^{\alpha q}\left(2\Gamma^a_{qs}\frac{\partial \Gamma^\beta_{ar}}{\partial u^p} + \frac{\partial \Gamma^a_{qs}}{\partial u^r}\Gamma^\beta_{ap} - \Gamma^a_{qs}\Gamma^b_{ar}\Gamma^\beta_{bp} - \frac{\partial^2 \Gamma^\beta_{qs}}{\partial u^r \partial u^p}\right) u^s_x u^r_x u^p_x + \right. \\ &\left. + h^{\alpha q}\left(2\Gamma^a_{qs}\Gamma^\beta_{ar} + \Gamma^a_{qr}\Gamma^\beta_{as} - 2\frac{\partial \Gamma^\beta_{qr}}{\partial u^s} - \frac{\partial \Gamma^\beta_{qs}}{\partial u^r}\right) u^s_x u^r_{x^2} - h^{\alpha q}\Gamma^\beta_{qs} u^s_{x^3}\right)\end{aligned} \tag{8}$$

In the formula we have used the same conventions as above, i.e., assume summation over repeating indexes. Moreover, similar to formula (7), we did not write $\mathcal{H}, \bar{\mathcal{H}}$ in the formula. They are assumed there as follows: the differential operator (8) is applied to $\frac{\delta \mathcal{H}}{\delta u^\alpha}$, the result is multiplied by $\frac{\delta \bar{\mathcal{H}}}{\delta u^\beta}$, and then we perform summation with respect to the repeating indexes α, β.

As in the case of order 1, the operation $\mathcal{B}_h$ given by (8) defines a Poisson bracket on $\tilde{\mathfrak{A}}$. The construction of this Poisson bracket is independent on the choice of coordinate system on U. However, in contrast to the case of order 1, the form (8) is not the most general form for a local Poisson bracket on $\tilde{\mathfrak{A}}$ of order 3. In fact, the word *Darboux* indicates that in a certain coordinate system (flat coordinate system for h in our case) the coefficients of the Poisson structure are constants[1]. In this *Darboux* coordinate system the Christoffel symbols Γ^i_{jk} are all zero and formula (8) reduces to[2]

$$\mathcal{B}_h(\mathcal{H}, \bar{\mathcal{H}}) = \frac{\delta \bar{\mathcal{H}}}{\delta u^\beta} h^{\alpha\beta} D^3 \left(\frac{\delta \mathcal{H}}{\delta u^\alpha} \right). \tag{9}$$

Poisson structures $\mathcal{P}_1$ of order 1 are always Darboux-Poisson, but there are examples, see e.g. [18, 19, 32], of Poisson structures $\mathcal{P}_3$ of order 3 which are not Darboux-Poisson.

Similar to the finite-dimensional case, a Poisson structure $\mathcal{P}$ and choice of a "Hamiltonian" $\mathcal{H} \in \tilde{\mathfrak{A}}$ allows one to define the Hamiltonian flow, which in our setup is a system of n PDEs on n functions $u^i(t,x)$ of two variables, t and x. It is given by:

$$\frac{\partial u^\beta}{\partial t} = \mathcal{P}^{\alpha\beta} \left(\frac{\delta \mathcal{H}}{\delta u^\alpha} \right). \tag{10}$$

For example, in the case of the Poisson structure (6) for a Hamiltonian of degree 0 (i.e., for a function H on U) the Hamiltonian flow is given by

$$\frac{\partial u^\beta}{\partial t} = g^{\beta\alpha} \frac{\partial^2 H}{\partial u^\alpha \partial u^\gamma} u^\gamma_x - \Gamma^{\beta\alpha}_\gamma \frac{\partial H}{\partial u^\alpha} u^\gamma_x = \frac{\partial u^\gamma}{\partial x} \nabla^\beta \nabla_\gamma H. \tag{11}$$

Such systems of PDEs are called *Hamiltonian systems of hydrodynamic type*.

In our paper we study compatibility of nonhomogeneous Poisson structures of type $\mathcal{P}_3 + \mathcal{P}_1$ such that the part of order 3 is Darboux-Poisson. That is, we have 4 nondegenerate Poisson structures: $\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$ constructed by flat metrics g and $\bar{g}$ by (6), and $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ constructed by flat metrics h and $\bar{h}$ by (8). We assume that $\mathcal{A}_g + \mathcal{B}_h$ and $\mathcal{A}_{\bar{g}} + \mathcal{B}_{\bar{h}}$ are (nonhomogeneous) Poisson structures and ask the question when these structures are compatible in the sense that any of their linear combinations is a Poisson structure [27]. Since it is automatically skew-symmetric, the compatibility is equivalent to the Jacobi identity for each linear combination of $\mathcal{A}_g + \mathcal{B}_h$ and $\mathcal{A}_{\bar{g}} + \mathcal{B}_{\bar{h}}$.

The meaning of the word "nondegenerate" relative to the Poisson structures under discussion is as follows: the metrics $g, \bar{g}, h, \bar{h}$ which we used to construct them are nondegenerate, i.e., they are given by matrices with nonzero determinant. Additional nondegeneracy

[1] The terminology "Darboux-Poisson" is motivated by [11].

[2] In fact, (8) is just the formula (9) rewritten in an arbitrary (not necessarily flat) coordinate system.

condition, natural from the viewpoint of mathematical physics, is as follows: the operators $R_h = \bar{h}h^{-1}$ and $R_g = \bar{g}g^{-1}$ have n different eigenvalues. Under these conditions, we solve the problem completely: we find explictly all pairs of such Poisson structures.

Let us also comment on a more physics-oriented approach to the construction above, see e.g. [11]. Physicists often view x as a space coordinate, and $(u^1, ..., u^n)$ as field coordinates. In the simplest situation, the values $u^1, ..., u^n$ at x may describe some physical values (e.g., pressure, temperature, charge, density, momenta). The total energy of the system is the integral over the x variable of some differential polynomial in $u^1, ..., u^n$, and the Hamiltonian functions $\mathcal{H} \in \mathfrak{A}$ have then the physical meaning of the density of the energy, i.e., of the integrand in the formula $\mathbf{Energy}(c) = \int \mathcal{H}(\hat{c})dx$. Further, it is assumed that the physical system is either periodic in x, or one is interested in fast decaying solutions as $x \to \pm\infty$. The integration by parts implies then that the differential polynomial is defined up to an addition of the total derivative in x which allows one to pass to $\tilde{\mathfrak{A}} = \mathfrak{A}/D(\mathfrak{A})$. The natural analog of the differential of a function in this setup is the variational derivative $\frac{\delta}{\delta u^\alpha}$, and actually the equation (10) is the natural analog of the finite-dimensional equation $\dot{u} = X_H$ (where X_H is the Hamiltonian vector field of a function H; it is given by $X_H^j = P^{ij}\frac{\partial H}{\partial u^i}$ where $P(u)^{ij}$ is the matrix of the Poisson structure; please note similarity with (10)). Generally it is useful to keep in mind the physical interpretation and the analogy with the finite-dimensional case.

1.3 Brief description of main results, structure of the paper and conventions

In this paper we address the following problems:

(A) *Description of compatible pairs, $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$, of non-homogeneous Poisson brackets in arbitrary dimension n.* In Theorems 1 and 2 we give an algebraic interpretation of this problem in terms of Frobenius algebras and reduce it to classification of Frobenius pencils, i.e. linear families of Frobenius algebras. We do it under the following nondegeneracy assumption: the (1,1)-tensor $R_h = \bar{h}h^{-1}$ (connecting h and $\bar{h}$) has n different eigenvalues.

(B) *Description and classification of Frobenius pencils.* We reduce this purely algebraic problem to a differential geometric one (explicitly formulated in Section 6.1) and completely solve it using geometric methods. The nondegeneracy assumption is that the (1,1)-tensor $R_g = \bar{g}g^{-1}$ (connecting g and $\bar{g}$) has n different eigenvalues. Namely, we show that each Frobenuis pencil in question is a subpencil of a certain *maximal* pencil. We explicitly describe all maximal pencils, see Theorems 3 and 4.

(B1) A generic in a certain sense maximal pencil corresponds to the well-known multi-Poisson structure discovered by M. Antonowitz and A. Fordy in [1] and studied by E. Ferapontov and M. Pavlov [17], see also [2, 3, 9]. We refer to it as to Antonowitz-Fordy-Frobenius pencil, *AFF-pencil*. In Theorem 3 we show that every two-dimensional Frobenius pencil with one additional genericity assumption is contained in the *AFF-pencil*.

(B2) Our main result, Theorems 4 and 5, give a complete description in the most general case. Theorem 4 constructs all the maximal Frobenius pencils using AFF-pencils as building blocks. Theorem 5 states that each Frobenuis pencil is a subpencil of a certain *maximal* pencil from Theorem 4. These maximal pencils are uniquely determined by some combinatorial data, directed rooted in-forest F with vertices labeled by natural numbers whose sum is the dimension of the manifold. The AFF-pencil corresponds to the simplest case, when F consists of a single vertex. To the best of our knowledge, the other Frobenius pencils and the corresponding bi-Poisson structures are new.

In addition, we show that common Frobenius coordinate systems admit an elegant invariant description in terms of the Nijenhuis pencil $\mathcal{L}$, see Theorem 4.

(C) *Dispersive perturbations of compatible Poisson brackets of hydrodynamic type.* The general question is as follows: given two compatible Poisson structures $\mathcal{A}_g$ and $\mathcal{A}_{\bar g}$ of the first order, can one find flat metrics h and $\bar h$ such that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g}$ are compatible Poisson structures? This passage from a Poisson bracket of hydrodynamic type to a non-homogeneous Poisson bracket of higher order is called *dispersive perturbation* in literature. We study dispersive perturbation of bi-Hamiltonian structures assuming that the third order terms $\mathcal{B}_h$ and $\mathcal{B}_{\bar h}$ are Darboux-Poisson.

We describe all such perturbations under the assumption that both $R_h = \bar h h^{-1}$ and $R_g = \bar g g^{-1}$ have n different eigenvalues, and in particular, answer a question from [17] on dispersive perturbations of the AFF-pencil (Remark 3.2).

The structure of the paper is as follows. In Section 2, we start with basic facts and constructions related to compatibility of homogeneous Poisson structures of order 1 and 3, then give description of compatible non-homogeneous structures $\mathcal{B}_g + \mathcal{A}_h$ and $\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g}$ in terms of Frobenius algebras (Theorems 1 and 2), leading us to the classification problem for the so-called Frobenius pencils. We conclude this section with an example of AFF-pencil. The AFF-pencil plays later a role of a building block in our general construction. Moreover, it provides an answer under a minor nondegeneracy assumption, see Theorem 3 in Section 3, where we also discuss a question of Ferapontov and Pavlov. Theorem 3 will be proved in Section 6.

In Section 4 we formulate the answer to the classification problem in its full generality. Theorem 5 (proved in Section 7) gives a description of Frobenius pencils in the "diagonal" coordinates for $g, \bar g$, and Theorem 4 (proved in Section 7.3) describes the corresponding Frobenius coordinates. In Section 4.2 we discuss the case of two blocks and give explicit formulas, see Theorem 6.

Finally in Section 8, we give a pure algebraic description of the Frobenius pencils from the classification theorem (Theorem 5) using the so-called warped product of pro-Frobenius algebras (as an algebraic counterpart of geometric warped product operation).

All objects in our paper are assumed to be of class C^∞; actually our results show that most of them are necessarily real-analytic.

Throughout the paper we use $\mathcal{A}_g$ and $\mathcal{B}_h$ to denote the Poisson structures of order 1 and 3 given by (9) and (8) respectively. Unless otherwise stated, the metrics we deal with (such as g, h, $\bar{g}$, $\bar{h}$, ...) are contravariant.

2 Non-homogeneous compatible brackets and Frobenius algebras

2.1 Basic facts and preliminary discussion

Recall that we study compatibility of two Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$, constructed by flat metrics $h, \bar{h}, g, \bar{g}$; our goal is to construct all of them. Recall that by definition it means that for any constants $\lambda, \bar{\lambda}$ the linear combination $\lambda(\mathcal{B}_h + \mathcal{A}_g) + \bar{\lambda}(\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}})$ is a Poisson structure. Using that $\mathcal{B}$ and $\mathcal{A}$ have different orders, one obtains (see e.g. [11])

Fact 1. *Let h, $\bar{h}$, g and $\bar{g}$ be flat metrics. If $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible Poisson structures, then the following holds:*

(i) $\mathcal{A}_g$ and $\mathcal{A}_{\bar{g}}$ are compatible,

(ii) $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ are compatible,

(iii) $\mathcal{A}_g$ and $\mathcal{B}_h$ are compatible (as well as $\mathcal{A}_{\bar{g}}$ and $\mathcal{B}_{\bar{h}}$).

This Fact naturally leads us to considering pencils (= linear combinations of metrics) $\lambda h + \bar{\lambda}\bar{h}$ and $\lambda g + \bar{\lambda}\bar{g}$. We need the following definition:

Definition 1 (Dubrovin, [15, Definition 0.5])**.** Two contravariant flat metrics g and $\bar{g}$ are said to be *Poisson compatible*, if for each (nondegenetrate) linear combination $\hat{g} = \lambda g + \bar{\lambda}\bar{g}$, $\lambda, \bar{\lambda} \in \mathbb{R}$, the following two conditions hold:

1. $\hat{g}$ is flat;
2. the contravariant Christoffel symbols for g, $\bar{g}$ and $\hat{g}$ are related as

$$\hat{\Gamma}^{\alpha\beta}_s = \lambda\Gamma^{\alpha\beta}_s + \bar{\lambda}\bar{\Gamma}^{\alpha\beta}_s. \tag{12}$$

In this case, the family of metrics $\{\lambda g + \bar\lambda \bar g\}_{\lambda,\bar\lambda\in\mathbb{R}}$ is said to be a *flat pencil* of metrics.

The next fact explains relationship between Poisson compatibility of flat metrics and compatibility of the corresponding Poisson structures.

Fact 2. *Let h, $\bar h$, g and $\bar g$ be flat metrics. Then, the following statements are true:*

(i) $\mathcal{A}_g$ and $\mathcal{A}_{\bar g}$ are compatible if and only if g and $\bar g$ are Poisson compatible.

(ii) If $\mathcal{B}_h$ and $\mathcal{B}_{\bar h}$ are compatible, then h and $\bar h$ are Poisson compatible.

(iii) If $\mathcal{B}_h$ and $\mathcal{A}_g$ are compatible, then h and g are Poisson compatible.

The (i)-part of Fact 2 is in [13], see also [16, 28, 29]. In view of formula (7), the two conditions from Definition 1 are nothing else but a geometric reformulation of the compatibility condition for Poisson structures of order one, which explains the name *Poisson compatible*. The (ii)-part is an easy corollary of [11, Theorem 3.2], see also proof of Theorem 3 below. The (iii)-part follows from [24, Theorem 2.2].

Notice that unlike the case of Poisson structures of order 1, not every pair of Poisson compatible metrics h and $\bar h$ (resp. h and g) leads to compatible Poisson structures of higher order $\mathcal{B}_h$ and $\mathcal{B}_{\bar h}$ as in (ii) (resp. $\mathcal{B}_h$ and $\mathcal{A}_g$ as in (iii)). Some extra conditions are required. These conditions will be explained below in Fact 4 (for h and g leading to compatible $\mathcal{B}_h$ and $\mathcal{A}_g$) and Theorem 3 (for h and $\bar h$ leading to compatible $\mathcal{B}_h$ and $\mathcal{B}_{\bar h}$).

Let us also recall the relation of compatible metrics to Nijenhuis geometry:

Fact 3 (see [16, 28, 29])**.** *If g and $\bar g$ are Poisson-compatible, then the (1,1)-tensor $R = \bar g g^{-1}$ is a Nijenhuis operator. Moreover, if g is flat, R is a nondegenerate Nijenhuis operator with n different eigenvalues, and $\bar g := Rg$ is flat, then $\bar g$ is compatible to g.*

As already explained, the condition that $\mathcal{B}_h + \mathcal{A}_g$ is a Poisson structure is a nontrivial geometric condition on the flat metrics h and g, stronger than their Poisson compatibility in the sense of Definition 1. This condition was studied in literature (see e.g. [4]) and it was observed that the compatibility of homogeneous Poisson structures of order 3 and 1 is sometimes related to certain algebraic structure. In our case, under the assumption that $\mathcal{B}_h$ is Darboux-Poisson, the algebraic structure which pops up naturally is *Frobenius algebra*.

Definition 2. Let $(\mathfrak{a}, \star)$ be an n-dimensional commutative associative algebra over $\mathbb{R}$ endowed with a nondegenerate symmetric bilinear form $b(\ ,\ )$. The pair $\big((\mathfrak{a}, \star), b\big)$ is called a *Frobenius algebra*, if

$$b(\xi \star \eta, \zeta) = b(\xi, \eta \star \zeta), \quad \text{for all } \xi, \eta, \zeta \in \mathfrak{a}. \tag{13}$$

The form b is then called a *Frobenius form*.

Notice that we do not assume that $\mathfrak{a}$ is unital which makes our version slightly more general than the one used in the theory of Frobenius manifolds (see e.g. [15]), or in certain branches of Algebra. The bilinear form b may have any signature.

Condition (13) is linear in b, so all Frobenius forms (if we allow them to be degenerate) on a given commutative associative algebra form a vector space.

Fix a basis $e^1, \dots, e^n$ in $\mathfrak{a}$. Below we will interpret $\mathfrak{a}$ as the dual $(\mathbb{R}^n)^*$ and for this reason we interchange lower and upper indices. Consider the structure constants a^{ij}_k defined by $e^i \star e^j = a^{ij}_k e^k$ and coefficients $b^{ij} := b(e^i, e^j)$ of the Frobenius form b. The algebra $\mathfrak{a}$ is Frobenius if and only if a^{ij}_k and b^{ij} satisfy the following conditions:

$$\begin{aligned} &a^{ij}_k = a^{ji}_k && \text{(commutativity)},\\ &a^{ij}_\alpha a^{\alpha r}_k = a^{i\alpha}_k a^{jr}_\alpha && \text{(associativity)},\\ &b^{\alpha r} a^{ij}_\alpha = b^{i\alpha} a^{jr}_\alpha && \text{(Frobenius condition)}. \end{aligned} \tag{14}$$

The dual $\mathfrak{a}^*$ has a natural structure of an affine space $\mathbb{R}^n$ with $u^i \simeq e^i$ being coordinates on $\mathfrak{a}^* \simeq \mathbb{R}^n$. Thus, on $\mathfrak{a}^*$ we can introduce the contravariant metric $g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s$ which is known to be flat (e.g. [24, Lemma 4.1]; the result also follows from [4]). What is special here is not the metric g itself, but the coordinate system $u^1, \dots, u^n$ which establishes a relationship between g and the Frobenius algebra $\mathfrak{a}$. This leads us to

Definition 3. Let g be a flat metric. We say that $u^1, \dots, u^n$ is a *Frobenuis coordinate system* for g if

$$g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s, \tag{15}$$

where $a^{\alpha\beta}_s$ are structure constants of a certain Frobenius algebra $\mathfrak{a}$ and $b = (b^{\alpha\beta})$ is a (perhaps degenerate) Frobenius form for $\mathfrak{a}$.

Frobenius coordinates possess the following important property that can be easily checked.

Fact 4 (see [4] and [24])**.** *Let g be a contravariant metric and $u^1, \dots, u^n$ a coordinate system. The following two conditions are equivalent:*

1. *In coordinates $u^1, \dots, u^n$, the contravariant Christoffel symbols $\Gamma^{\alpha\beta}_s$ of g are constant and symmetric in upper indices.*
2. *$u^1, \dots, u^n$ are Frobenius coordinates, i.e., g is given by* (15)*.*

If either of these conditions holds, then g is flat and $\Gamma^{\alpha\beta}_s = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial s}$.

The relation of Frobenius coordinate systems to our problem is established by the following remarkable and fundamental statement:

Fact 5 ([24, Theorem 2.2])**.** *Let g and h be two flat metrics. Then $\mathcal{B}_h + \mathcal{A}_g$ is Poisson if and only if there exists a coordinate system $u^1, \dots, u^n$ such that the following holds:*

1. *$g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s$, where $a^{\alpha\beta}_s$ are structure constants of a certain Frobenius algebra $\mathfrak{a}$, and b is a Frobenius form for $\mathfrak{a}$;*
2. *the entries $h^{\alpha\beta}$ of h in this coordinate system are constant;*
3. *$h = \left(h^{\alpha\beta}\right)$ is a Frobenius form for $\mathfrak{a}$, that is, $h^{\alpha q} a^{\beta\gamma}_q = h^{\gamma q} a^{\beta\alpha}_q$.*

This fact was independently obtained by P. Lorenzoni and R. Vitolo in their unpublished paper. The "if" part of the statement follows from [35] by I. Strachan and B. Szablikowski, see also [12, Theorem 5.12].

The coordinates $(u^1, \dots, u^n)$ from Fact 5 will be called *Frobenius coordinates* for the nonhomogeneous Poisson structure $\mathcal{B}_h + \mathcal{A}_g$. Of course, Frobenius coordinates are not unique; indeed, they remain to be Frobenius after any affine coordinate change. This is the only freedom since the metric h is constant in Frobenius coordinates.

2.2 Reduction of our problem to an algebraic one and Frobenius pencils

Definition 4. Let $(\mathfrak{a}, \star)$ and $(\bar{\mathfrak{a}}, \bar\star)$ be Frobenius algebras defined on the same vector space V and $h, \bar h : V \times V \to \mathbb{R}$ the corresponding Frobenius forms. We will say that $(\mathfrak{a}, h)$ and $(\bar{\mathfrak{a}}, \bar h)$ are *compatible* if the operation

$$\xi, \eta \mapsto \xi \star \eta + \xi \,\bar\star\, \eta, \qquad \xi, \eta \in V, \tag{16}$$

defines the structure of a Frobenius algebra with the Frobenuis form $h + \bar h$.

Similarly, if $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ are Frobenius algebras each of which is endowed with two Frobenius forms b, h and $\bar b, \bar h$ respectively, then we say that the triples $(\mathfrak{a}, b, h)$ and $(\bar{\mathfrak{a}}, \bar b, \bar h)$ are *compatible* if (16) defines a Frobenius algebra for which $b + \bar b$ and $h + \bar h$ are both Frobenuis forms.

Formally, the definition requires that $b + \bar b$ and $h + \bar h$ are nondegenerate. It is not essential. Indeed, if the operations $\star$ and $\bar\star$ are associative, and also the operation $\hat\star := \star + \bar\star$ given by (16) is associative, then any linear combination $\lambda\star + \bar\lambda\bar\star$ is associative. Moreover, if $\hat b := b + \bar b$, possibly degenerate, satisfies the condition (13) for $\hat\star$, then the linear combination $\lambda b + \bar\lambda \bar b$ also satisfies the condition (13) with respect to $\lambda\star + \bar\lambda\bar\star$. Thus, passing to a suitable linear combination we can make $\hat b$ and also $\hat h$ nondegenerate.

In view of Facts 4 and 5, compatible Frobenius triples $(\mathfrak{a}, b, h)$ and $(\bar{\mathfrak{a}}, \bar{b}, \bar{h})$ naturally define compatible Poisson structures $\mathcal{B}_g + \mathcal{A}_h$ and $\mathcal{B}_{\bar{g}} + \mathcal{A}_{\bar{h}}$. The next theorem shows that the converse is also true under the assumption that $R_h = \bar{h}h^{-1}$ has n different eigenvalues.

Theorem 1. *Consider two non-homogeneous Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ and suppose that $R_h = \bar{h}h^{-1}$ has n different eigenvalues.*

Then, they are compatible if and only if (g, h) and $(\bar{g}, \bar{h})$ admit a common Frobenius coordinate system $u^1, \dots, u^n$ in which

1. *$h^{\alpha\beta}$ and $\bar{h}^{\alpha\beta}$ are constant,*
2. *$g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s$ and $\bar{g}^{\alpha\beta}(u) = \bar{b}^{\alpha\beta} + \bar{a}^{\alpha\beta}_s u^s$,*
3. *$(\mathfrak{a}, b, h)$ and $(\bar{\mathfrak{a}}, \bar{b}, \bar{h})$ are compatible Frobenius triples (here $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ denote the algebras with structure constants $a^{\alpha\beta}_s$ and $\bar{a}^{\alpha\beta}_s$ respectively).*

Corollary 2.1. *In more explicit terms, compatibility of $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ such that $R_h = \bar{h}h^{-1}$ has n different eigenvalues is equivalent to reducibility of these operators, in an appropriate coordinate system $u^1, \dots, u^n$, to the following simultaneous canonical form*

$$\mathcal{B}_h + \mathcal{A}_g = h^{\alpha\beta} D^3 + b^{\alpha\beta} D + a^{\alpha\beta}_s u^s D + \tfrac{1}{2}\, a^{\alpha\beta}_s u^s_x,$$

$$\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}} = \bar{h}^{\alpha\beta} D^3 + \bar{b}^{\alpha\beta} D + \bar{a}^{\alpha\beta}_s u^s D + \tfrac{1}{2}\, \bar{a}^{\alpha\beta}_s u^s_x,$$

where $h^{\alpha\beta}, \bar{h}^{\alpha\beta}, b^{\alpha\beta}, \bar{b}^{\alpha\beta}, a^{\alpha\beta}_s, \bar{a}^{\alpha\beta}_s$ are constants symmetric in upper indices and satisfying the conditions:

$$\begin{gathered}
a^{\alpha\beta}_q a^{q\gamma}_s = a^{\gamma\beta}_q a^{q\alpha}_s, \quad \bar{a}^{\alpha\beta}_q \bar{a}^{q\gamma}_s = \bar{a}^{\gamma\beta}_q \bar{a}^{q\alpha}_s, \quad \bar{a}^{\alpha\beta}_q a^{q\gamma}_s + a^{\alpha\beta}_q \bar{a}^{q\gamma}_s = \bar{a}^{\gamma\beta}_q a^{q\alpha}_s + a^{\gamma\beta}_q \bar{a}^{q\alpha}_s, \\
h^{\alpha q} a^{\beta\gamma}_q = h^{\gamma q} a^{\beta\alpha}_q, \quad b^{\alpha q} a^{\beta\gamma}_q = b^{\gamma q} a^{\beta\alpha}_q, \quad \bar{h}^{\alpha q} \bar{a}^{\beta\gamma}_q = \bar{h}^{\gamma q} \bar{a}^{\beta\alpha}_q, \quad \bar{b}^{\alpha q} \bar{a}^{\beta\gamma}_q = \bar{b}^{\gamma q} \bar{a}^{\beta\alpha}_q, \\
\bar{h}^{\alpha q} a^{\beta\gamma}_q + h^{\alpha q} \bar{a}^{\beta\gamma}_q = \bar{h}^{\gamma q} a^{\beta\alpha}_q + h^{\gamma q} \bar{a}^{\beta\alpha}_q, \quad \bar{b}^{\alpha q} a^{\beta\gamma}_q + b^{\alpha q} \bar{a}^{\beta\gamma}_q = \bar{b}^{\gamma q} a^{\beta\alpha}_q + b^{\gamma q} \bar{a}^{\beta\alpha}_q.
\end{gathered} \tag{17}$$

Notice that the coordinates $u^1, \dots, u^n$ from Theorem 1 are just flat coordinates for h (or equivalently, for $\bar{h}$ as these metrics have common flat coordinates by Theorem 1).

We see that Theorem 1 reduces the problem of description and classification of pairs of compatible Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ such that $R_g = \bar{h}h^{-1}$ has n different eigenvalues to a purely algebraic problem. As we announced above, we will reformulate it differential geometric terms in Section 6.1, and solve it under the assumption that $R_g = \bar{g}g^{-1}$ has n different eigenvalues.

We have not succeeded in solving the problem by purely algebraic means. Like many other problems in Algebra, it reduces to a system of quadratic and linear equations (see relations (17)). For example, classification of Frobenius algebras is a problem of the same type. This problem is solved under the additional assumption that the Frobenius form is positive definite in [5], and in our opinion is out of reach otherwise. Of course, for a fixed dimension one can find complete or partial answers. In particular, in [31] it is shown, that up to dimension 6 there is a finite number of isomorphism classes of commutative associative algebras and for $n > 6$ the number of classes is infinite. In [22] the classification of nilpotent commutative associative algebras up to dimension 6 is given. See also [26, 35].

In the situation discussed in Theorem 1, consider the pencil of first order Poisson structures $\mathcal{A}_{\lambda g+\mu\bar{g}}$, which is sometimes referred to as quasiclassical limit [17] of the non-homogeneous pencil $\mathcal{B}_{\lambda h+\mu\bar{h}} + \mathcal{A}_{\lambda g+\mu\bar{g}}$. We can ask the inverse question: *Given a flat pencil $\{\lambda g + \mu\bar{g}\}$, does the corresponding Poisson pencil $\{\mathcal{A}_{\lambda g+\mu\bar{g}}\}$ admit a perturbation with nondegenerate Darboux-Poisson structures of order three of general position?*

Theorem 1 basically shows that the main condition for the related quadruple of metrics $(h, \bar{h}, g, \bar{g})$ is the existence of a common Frobenius coordinate system for g and $\bar{g}$. Indeed, if this condition holds true and this Frobenuis coordinate system is given, then the other two metrics h and $\bar{h}$ can be "reconstructed" by solving a system of linear equations. More precisely, we have the following

Theorem 2. *Let g and $\bar{g}$ be compatible flat metrics that admit a common Frobenius coordinate system $u^1, \dots, u^n$, that is*

$$g^{\alpha\beta}(u) = b^{\alpha\beta} + a_s^{\alpha\beta} u^s \quad \text{and} \quad \bar{g}^{\alpha\beta}(u) = \bar{b}^{\alpha\beta} + \bar{a}_s^{\alpha\beta} u^s,$$

where $(\mathfrak{a}, b)$ and $(\bar{\mathfrak{a}}, \bar{b})$ are Frobenius pairs (here $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ denote the algebras with structure constants $a_s^{\alpha\beta}$ and $\bar{a}_s^{\alpha\beta}$ respectively). Then

(i) the corresponding Frobenius algebras are compatible,

(ii) there exist nondegenerate metrics h and $\bar{h}$ (with $h^{\alpha\beta}$ and $\bar{h}^{\alpha\beta}$ being constant in coordinates $u^1, \dots, u^n$), such that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible Poisson structures,

(iii) in Frobenius coordinates $u^1, \dots, u^n$, the (constant) metrics h and $\bar{h}$ can always be chosen in the form

$$h^{\alpha\beta} = m^0\, b^{\alpha\beta} + a_s^{\alpha\beta} m^s \quad \text{and} \quad \bar{h}^{\alpha\beta}(u) = m^0\, \bar{b}^{\alpha\beta} + \bar{a}_s^{\alpha\beta} m^s, \quad (m^1, \dots, m^n) \in \mathbb{R}^n,\ m^0 \in \mathbb{R}. \tag{18}$$

2.3 AFF-pencil

Consider a real affine space $V \simeq \mathbb{R}^n$ with coordinates $u^1, \dots, u^n$ and define the (Nijenhuis) operator L and contravariant metric g_0 on it by:

$$L = \begin{pmatrix} u^1 & 1 & 0 & \dots & 0 \\ u^2 & 0 & 1 & \dots & 0 \\ & & & \ddots & \\ u^{n-1} & 0 & 0 & \dots & 1 \\ u^n & 0 & 0 & \dots & 0 \end{pmatrix}, \quad g_0 = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 & -u^1 \\ 0 & 0 & \dots & 1 & -u^1 & -u^2 \\ & & \iddots & & & \\ 0 & 1 & \dots & -u^{n-4} & -u^{n-3} & -u^{n-2} \\ 1 & -u^1 & \dots & -u^{n-3} & -u^{n-2} & -u^{n-1} \end{pmatrix}. \tag{19}$$

Next, introduce $n+1$ contravariant metrics $g_{\mathrm{i}} = L^i g$ for $i = 0, \dots, n$. In matrix form, we have

$$g_{\mathrm{i}} = \begin{pmatrix} a_{\mathrm{n-i}} & 0 \\ 0 & b_{\mathrm{i}} \end{pmatrix}, \tag{20}$$

where $a_{\mathrm{n-i}}$ is a $(n-i) \times (n-i)$ matrix

$$a_{\mathrm{n-i}} = \begin{pmatrix} 0 & \dots & 0 & 0 & 1 \\ 0 & \dots & 0 & 1 & -u^1 \\ 0 & \dots & 1 & -u^1 & -u^2 \\ & \dots & & & \\ 1 & \dots & -u^{n-i-3} & -u^{n-i-2} & -u^{n-i-1} \end{pmatrix}$$

and b_{i} is $i \times i$ matrix of the form

$$b_{\mathrm{i}} = \begin{pmatrix} u^{n-i+1} & u^{n-i+2} & \dots & u^{n-1} & u^n \\ u^{n-i+2} & u^{n-i+3} & \dots & u^n & 0 \\ & & \dots & & \\ u^{n-1} & u^n & \dots & 0 & 0 \\ u^n & 0 & \dots & 0 & 0 \end{pmatrix}.$$

In particular, $g_0 = a_{\mathrm{n}}$ and $g_{\mathrm{n}} = b_{\mathrm{n}}$.

The metrics $g_0, g_1, \dots, g_{\mathrm{n}}$ are flat and pairwise compatible, so that they generate an $n+1$-dimensional flat pencil with remarkable properties, see e.g. [17, 9]. We can write this pencil as

$$\{P(L)g_0\}, \quad \text{where } P(\cdot) \text{ is an arbitrary polynomial of degree} \le n \tag{21}$$

and L and g_0 are given by (19). We will refer to it as to *AFF-pencil*. This pencil was discovered, in the form (19) and (20), by M. Antonowicz and A. Fordy [1]. As we see, the components of each metric g_i are affine functions, moreover, the coordinates $(u^1, \dots, u^n)$ are common Frobenius coordinates for all of them.

The corresponding Frobenius algebras are easy to describe. Consider two well-known examples:

- the algebra $\mathfrak{a}_n$ of dimension n with basis $e_1, e_2, \ldots, e_n$ and relations

$$e_i \star e_j = \begin{cases} e_{i+j}, & \text{if } i+j \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

  Notice that $\mathfrak{a}_n$ can be modelled as the matrix algebra $\mathrm{Span}(J, J^2, \ldots, J^n)$, where J is the nilpotent Jordan block of size $(n+1) \times (n+1)$. It contains *no multiplicative unity* element.

- the algebra $\mathfrak{b}_n$ of dimension n with basis $e_1, e_2, \ldots, e_n$ and relations

$$e_i \star e_j = \begin{cases} e_{i+j-1}, & \text{if } i+j-1 \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

  This algebra can be understood as the *unital* matrix algebra $\mathrm{Span}(\mathrm{Id}, J, J^2, \ldots, J^{n-1})$ where J is the nilpotent Jordan block of size $n \times n$. The difference from the previous example is that $\mathfrak{b}_n$, by definition, contains the identity matrix. Equivalently, we can define $\mathfrak{b}_n$ as the algebra of truncated polynomials $\mathbb{R}[x]/\langle x^n \rangle$ (similarly $\mathfrak{a}_n \simeq \langle x \rangle / \langle x^{n+1} \rangle$).

It is straightforward to see that the metric $g_{\mathrm{n}} = b_{\mathrm{n}}$ is related to the Frobenius algebra $\mathfrak{b}_n$. Similarly $g_0 = a_n$ is related to the Frobenius algebra $\mathfrak{a}_n$ (this becomes obvious if we reverse the order of basis vector and mutiply each of them by -1). Hence, formula (20) shows that the Frobenius algebra associated with g_{i} is isomorphic to the direct sum $\mathfrak{a}_{n-i} \oplus \mathfrak{b}_i$.

It is interesting that a generic metric $g = P(L) g_0$ from the AFF-pencil (21), i.e. such that $P(L)$ has n distinct roots, corresponds to the direct sum $\mathbb{R} \oplus \cdots \oplus \mathbb{R} \oplus \mathbb{C} \oplus \cdots \oplus \mathbb{C}$, where each copy of $\mathbb{R}$ relates to a real root and each copy of $\mathbb{C}$ relates to a pair of complex conjugate roots of $P(\cdot)$.

It is a remarkable fact that for each g_{i} we can find a partner h_{i} such that $\mathcal{B}_{h_{\mathrm{i}}} + \mathcal{A}_{g_{\mathrm{i}}}$ is a Poisson structure and all these structures are pairwise compatible. The (constant) metrics h_{i} take the form

$$h_{\mathrm{i}} = (g_{\mathrm{i}})_{\bar{m}, m^0}, \quad \bar{m} = (m^1, \ldots, m^n) \in \mathbb{R}^n, m^0 \in \mathbb{R}, \tag{22}$$

where $(g_{\mathrm{i}})_{\bar{m}, m^0}$ is obtained from the matrix $g_{\mathrm{i}}(u)$ by replacing u^s with m^s and all 1's with m^0. In this way, we obtain an $(n+1)$-dimensional pencil of non-homogeneous Poisson structures generated by $\mathcal{B}_{h_{\mathrm{i}}} + \mathcal{A}_{g_{\mathrm{i}}}$:

$$\left\{ \sum_{i=0}^{n} c_i \big(\mathcal{B}_{h_{\mathrm{i}}} + \mathcal{A}_{g_{\mathrm{i}}} \big) \right\}_{c_i \in \mathbb{R}} \tag{23}$$

Alternatively, the pencil (23) can be described as follows. Fix $\bar m = (m^1, \dots, m^n) \in \mathbb{R}^n$, $m^0 \in \mathbb{R}$ and let $L(\bar m)$ denote the operator with constant entries obtained from $L = L(u)$ by replacing u^i with constants $m^i \in \mathbb{R}$. Similarly, $g_0(\bar m)$ denotes the metric with constant coefficients obtained from g_0 by replacing u^i with the same constants $m^i \in \mathbb{R}$.

Then for $g = P(L)g_0$ we can define its partner h (metric with constant entries) as

$$h = m^0 P\left(L\left(\tfrac{1}{m^0}\,\bar m\right)\right) g_0\left(\tfrac{1}{m^0}\,\bar m\right)$$

It can be easily checked that the correspondence $(m^0, m^1, \dots, m^n) \mapsto h$ defined by this formula is linear so that it makes sense for $m_0 = 0$ (the denominators cancel out). Then the pencil (23) can be, equivalently, defined as

$$\left\{\mathcal{B}_{m^0 P\left(L\left(\frac{1}{m^0}\,\bar m\right)\right) g_0\left(\frac{1}{m^0}\,\bar m\right)} + \mathcal{A}_{P(L)g_0}\right\}_{\deg P(\cdot)\le n}. \tag{24}$$

Notice that such a pencil is not unique, as the above construction depends on $n+1$ arbitrary parameters $m^0, m^1, \dots, m^n$. In other words, in (24), the polynomial $P(\cdot)$ serves as a parameter of the bracket within the AFF-pencil, whereas $(m_0, \bar m)$ parametrise dispersive perturbations of this pencil.

Remark 2.1. For our purposes below it will be convenient to rewrite this pencil in another coordinate system by taking the eigenvalues of L as local coordinates $x^1, \dots, x^n$. In these coordinates, g_0 and L from (19) take the following diagonal form (see e.g. [17, p. 214] or [6, §6.2])[3]:

$$g_{\mathsf{LC}} = \sum_{i=1}^{n} \left(\prod_{s\neq i}(x^i - x^s)\right)^{-1} \left(\tfrac{\partial}{\partial x^i}\right)^2, \quad L = \mathrm{diag}(x^1, ..., x^n), \tag{25}$$

so that the AFF pencil (21) becomes diagonal too:

$$\{P(L)g_{\mathsf{LC}}\}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree} \le n. \tag{26}$$

We also notice that the transition from the diagonal coordinates x to Frobenius coordinates u is quite natural: the coordinates u^i are the coefficients σ_i of the characteristic polynomial $\chi_L(t) = \det(t\cdot \mathrm{Id} - L) = t^n - \sigma_1 t^{n-1} - \sigma_2 t^{n-2} - \dots - \sigma_n$, so that, up to sign, u_i are elementary symmetric polynomials in $x^1, \dots, x^n$.

The AFF pencil provides a lot of examples of compatible flat metrics g and $\bar g$ that admit a common Frobenius coordinate system: one can take any two metrics from the pencil (21) or, equivalently, (26).

[3]The letters LC in g_{LC} refer to Levi-Civita. The metric g_{LC} played the key role in his classification of geodesically equivalent metrics [25]. See also [9] for discussion on the relationship between projectively equivalent and Poisson-compatible metrics.

3 Compatible flat metrics with a common Frobenius coordinate system: generic case

Theorems 1 and 2 reduce the compatibility problem for two Poisson structures of the form $\mathcal{B}_h + \mathcal{A}_g$ to a classification of all pairs of metrics g and $\bar g$ admitting a common Frobenius coordinate system. The next theorem solves this problem under the standard assumption that $R_g = \bar g g^{-1}$ has n different eigenvalues and one minor additional condition.

Theorem 3. *Let g and $\bar g$ be compatible flat metrics that admit a common Frobenius coordinate system. Assume that the eigenvalues of the operator $R_g = \bar g g^{-1}$ are all different and in the diagonal coordinates (such that R_g is diagonal) every diagonal component of g depends on all variables. Then the flat pencil $\lambda g + \mu \bar g$ is contained in the AFF-pencil, in other words, there exists a coordinate system $(x^1, \dots, x^n)$ such that*

$$g = P(L) g_{\mathsf{LC}} \quad \textit{and} \quad \bar g = Q(L) g_{\mathsf{LC}}.$$

for some polynomials $P(\cdot)$ and $Q(\cdot)$ of degree $\leq n$ and g_{LC} and L defined by (25).

Moreover, if $n \geq 2$ and $P(\cdot)$ and $Q(\cdot)$ are not proportional, then the common Frobenius coordinate system for $g = P(L) g_{\mathsf{LC}}$ and $\bar g = Q(L) g_{\mathsf{LC}}$ is unique up to an affine coordinate change.

Theorem 3 will be proved in Section 6. The uniqueness part will be explained in Section 7.3, see Remark 7.2.

Remark 3.1. In Theorem 3 we allow some of the eigenvalues R_g to be complex. In this case, we think that a part of the *diagonal* coordinates $(x^1, ..., x^n)$ is also complex-valued. For example, the coordinates $x^1, ..., x^k$ may be real-valued, and the remaining coordinates $x^{k+1} = z^1, x^{k+2} = \bar z^1, ..., x^{n-1} = z^{\frac{n-k}{2}}, x^n = \bar z^{\frac{n-k}{2}}$, where " $-$ " means complex conjugation, are complex-valued. In this case (26) gives us a well-defined (real) metric g_{LC} and a (real) Nijenhuis operator L.

The genericity condition in Theorem 3 is that every diagonal component of g depends on all variables. In Theorems 5, 4 below we will solve the problem in full generality, without assuming this or any other genericity condition.

Remark 3.2. In [17, §5] E. Ferapontov and M. Pavlov asked whether dispersive perturbations of the pencil (21) with g_0 and L given by (25) other than those described in Section 2.3 are possible. Theorem 3 leads to a negative answer under the additional assumption that the dispersive perturbation is in the class of nondegenerate Darboux-Poisson structures of order 3. Indeed, according to Theorem 1 every dispersive perturbation $\lambda(\mathcal{B}_h + \mathcal{A}_g) + \mu(\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g})$ of the pencil $\lambda \mathcal{A}_g + \mu \mathcal{A}_{\bar g}$ can be reduced to a simple normal form in a common Frobenius coordinate system for g and $\bar g$ (assuming that $R_h = \bar h h^{-1}$

has different eigenvalues). Moreover, in this coordinate system h and $\bar{h}$ are constant and represent Frobenius forms for the corresponding Frobenius algebras $\mathfrak{a}$ and $\bar{\mathfrak{a}}$. Since by Theorem 3, such a coordinate system is unique, it remains to solve a Linear Algebra problem of choosing suitable forms h and $\bar{h}$, satisfying three conditions (cf. (17)):

$$\begin{aligned} h(\xi \star \eta, \zeta) &= h(\xi, \eta \star \zeta), \\ \bar{h}(\xi \mathbin{\bar{\star}} \eta, \zeta) &= \bar{h}(\xi, \eta \mathbin{\bar{\star}} \zeta), \\ \bar{h}(\xi \star \eta, \zeta) + h(\xi \mathbin{\bar{\star}} \eta, \zeta) &= \bar{h}(\xi, \eta \star \zeta) + h(\xi, \eta \mathbin{\bar{\star}} \zeta), \end{aligned} \tag{27}$$

It is straightforward to show for a generic pair g, $\bar{g}$ of metrics from the AFF-pencil, the forms h and $\bar{h}$ are defined by $n+1$ parameters $m^0, m^1, \dots, m^n$ as in (24). No other solutions exist. In particular, formula (24) describes all possible dispersive perturbations of the AFF-pencil by means of nondegenerate Darboux-Poisson structures of order 3. Moreover, this conclusion holds for any generic two-dimensional subpencil.

4 Compatible flat metrics with a common Frobenius coordinate system: general case

4.1 General multi-block Frobenius pencils

Let us now discuss the general case without assuming that in diagonal coordinates, every diagonal component of g depends on all variables.

Similar to Theorem 3, the metrics g and $\bar{g}$ will belong to a large Frobenius pencil built up from several blocks each of which has a structure of an (extended) AFF pencil. We start with constructing a series of such pencils.

We first divide our diagonal coordinates into B blocks of positive dimensions $n_1, ..., n_B$ with $n_1 + ... + n_B = n$:

$$(\underbrace{x_1^1, ..., x_1^{n_1}}_{X_1}, ..., \underbrace{x_B^1, ..., x_B^{n_B}}_{X_B}). \tag{28}$$

Next, we consider a collection of n_α-dimensional Levi-Civita metrics g_α^{LC} and n_α-dimensional operators L_α (as in Theorem 3 but now for each block separately):

$$g_\alpha^{\mathsf{LC}} = \sum_{s=1}^{n_\alpha} \left(\prod_{j \neq s} (x_\alpha^s - x_\alpha^j) \right)^{-1} \left(\frac{\partial}{\partial x_\alpha^s} \right)^2 \quad , \quad L_\alpha = \operatorname{diag}(x_\alpha^1, ..., x_\alpha^{n_\alpha}). \tag{29}$$

Then we introduce a new block-diagonal metric $\hat g$

$$\hat g = \operatorname{diag}(\hat g_1, \dots, \hat g_B) \quad \text{with } \hat g_\alpha = \prod_{s<\alpha} \left(\frac{1}{\det L_s}\right)^{c_{s\alpha}} g_\alpha^{\mathsf{LC}}, \tag{30}$$

where $c_{s\alpha} = 0$ or 1. The values of the discrete parameters $c_{s\alpha}$ are determined by some combinatorial data as explained below.

Finally, we consider the pencil of (contravariant) metrics of the form

$$\{\hat L \, \hat g \mid \hat L \in \mathcal{L}\} \tag{31}$$

where $\mathcal{L}$ is a family (pencil) of block-diagonal operators of the form

$$\hat L = \operatorname{diag}\big(P_1(L_1), P_2(L_2), \dots, P_B(L_B)\big).$$

where $P_\alpha(\cdot)$ are polynomials with $\deg P_\alpha \leq n_\alpha + 1$ treated as parameters of this family. The coefficients of the polynomials P_α are not arbitrary but satisfy a collection of linear relations involving coefficients from different polynomials so that this pencil, in general, is not a direct sum of blocks (although, direct sum is a particular example). Notice that $\mathcal{L}$ is a Nijenhuis pencil whose algebraic structure is quite different from that of the pencil $\{P(L)\}$ from Theorem 3.

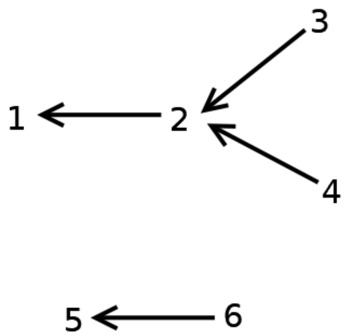
The numbers $c_{s\alpha}$ and relations on the coefficients of P_α's are determined by a combinatorial object, an oriented graph F with special properties, namely, a directed rooted in-forest (see [38] for a definition). This graph may consists of several connected components, each of which is a rooted tree whose edges are oriented from its *leaves* to the *root*. An example is shown in Figure 1.

Each vertex of F is associated with a certain block of the above decomposition (28) and labelled by an integer number $\alpha \in \{1, ..., B\}$. The structure of a directed graph defines a natural strict partial order (denoted by $\prec$) on the set $\{1, ..., B\}$: for two numbers $\alpha \neq \beta \in \{1, ..., B\}$ we set $\alpha \prec \beta$, if there exists an oriented way from β to α. For instance, for the graph shown on Fig. 1, we have $1 \prec 3$, $2 \prec 4$, $5 \prec 6$. Without loss of generality we can and will always assume that the vertices of F are labeled in such a way that $\alpha \prec \beta$ implies $\alpha < \beta$.

Notice that the vertices of degree one are of two types, roots and leaves: α is a *root* if there is no β such that $\beta \prec \alpha$ and, conversely, β is a *leaf* if there is no β such that $\alpha \prec \beta$. We say $\alpha = \operatorname{next}(\beta)$, if $\alpha \prec \beta$ and there is no γ with $\alpha \prec \gamma \prec \beta$. In the upper tree of Fig. 1 the root is 1, the leaves are 3 and 4 and we have: $1 = \operatorname{next}(2)$ and $2 = \operatorname{next}(3)$, $2 = \operatorname{next}(4)$.

The numbers $c_{s\alpha}$ in (30) are now defined from F as follows:

$$c_{s\alpha} = \begin{cases} 1, & \text{if } s \prec \alpha, \\ 0, & \text{otherwise.} \end{cases} \tag{32}$$



$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 1: A 6×6 matrix $c_{\alpha\beta}$ and the corresponding in-forest. The upper tree corresponds to the upperleft 4×4-block, the lower tree corresponds to the downright 2×2-block.

Notice that in our assumptions, $s \prec \alpha$ implies $s < \alpha$ so that the $B \times B$-matrix $c_{s\alpha}$ is uppertriangular with zeros on the diagonal, see Figure 1.

Finally, for a vertex α we denote the coefficients of the corresponding polynomial P_α by $P_\alpha(t) = \overset{\alpha}{a}_0 + \overset{\alpha}{a}_1 t + ... + \overset{\alpha}{a}_{n_\alpha+1} t^{n_\alpha+1}$. Then the conditions on the coefficients $P_\alpha(t)$ are

(i) If α is a root, then $\overset{\alpha}{a}_{n_\alpha+1} = 0$, i.e., $\deg P_\alpha \leq n_\alpha$.

(ii) If $n_\alpha = 1$, then $\overset{\alpha}{a}_{n_\alpha+1} = 0$, i.e. $\deg P_\alpha \leq 1$.

(iii) If $\alpha = \mathrm{next}(\beta)$, then $\overset{\alpha}{a}_0 = 0$ and $\overset{\alpha}{a}_1 = (-1)^{n_\alpha} \overset{\beta}{a}_{n_\beta+1}$.

(iv) If $\alpha = \mathrm{next}(\beta)$ and $\alpha = \mathrm{next}(\gamma)$ with $\beta \neq \gamma$, then $\overset{\beta}{a}_{n_\alpha+1} = \overset{\gamma}{a}_{n_\gamma+1} = 0$ (and, hence, $\overset{\alpha}{a}_1 = 0$ in view of (iii)).

This completes the description of the pencil (31) of (contravariant) metrics and we can state our next result.

Theorem 4. *The pencil* (31) (*with* $c_{\alpha\beta}$ *defined by* (32) *and coefficients of* P_α *satisfying* (i)-(iv)) *is Frobenius. In other words, all the metrics*

$$g = \hat{L}\,\hat{g} = \mathrm{diag}\big(P_1(L_1)\hat{g}_1, \dots, P_B(L_B)\hat{g}_B\big) \quad \textit{with } \hat{g}_\alpha = \prod_{s<\alpha} \left(\frac{1}{\det L_s}\right)^{c_{s\alpha}} g_\alpha^{\mathsf{LC}}, \tag{33}$$

are flat, Poisson compatible and admit a common Frobenius coordinate system

$$(u^1, ..., u^n) = (\underbrace{u_1^1, ..., u_1^{n_1}}_{U_1}, ..., \underbrace{u_B^1, ..., u_B^{n_B}}_{U_B}) \tag{34}$$

which is defined as follows. Let $\sigma^1_\alpha, \dots, \sigma^{n_i}_\alpha$ denote the coefficients of the characteristic polynomial of L_α

$$\chi_{L_\alpha}(t) := \det\left(t\,\mathrm{Id}_{n_\alpha\times n_\alpha} - L_\alpha\right) = t^{n_\alpha} - \sigma^1_\alpha t^{n_\alpha-1} - \sigma^2_\alpha t^{n_\alpha-2} - \dots - \sigma^{n_\alpha}_\alpha, \quad \alpha = 1,\dots,B.$$

Then

$$\begin{array}{lll} u^k_1 & = \sigma^k_1, & k = 1,\dots,n_1,\\ u^k_2 & = (\det L_1)^{c_{12}}\ \sigma^k_2, & k = 1,\dots,n_2,\\ u^k_3 & = (\det L_1)^{c_{13}}(\det L_2)^{c_{23}}\ \sigma^k_3, & k = 1,\dots,n_3,\\ & \dots & \\ u^k_B & = \prod_{s<B}(\det L_s)^{c_{sB}}\ \sigma^k_B, & k = 1,\dots,n_B. \end{array} \tag{35}$$

Theorem 4 will be proved in Section 8.

The advantage of the formulas for Frobenius coordinates in Theorem 4 is that they are invariant in the sense they do not depend on the choice of coordinates in blocks, but use coefficients of the characteristic polynomials of blocks L_i.

Let us explain how one can use this property to check algorithmically (say, using computer algebra software) that the coordinates in Theorem 4 are indeed Frobenius for the metric g.

In each block (with number α), we change from diagonal coordinates $X_\alpha = (x^1_\alpha, \dots, x^{n_\alpha}_\alpha)$ to the coordinates $Y_\alpha = (y^1_\alpha, \dots, y^{n_\alpha}_\alpha)$ given as follows:

$$\chi_{L_\alpha}(t) = t^{n_\alpha} - y^1_\alpha\, t^{n_\alpha-1} - y^2_\alpha\, t^{n_\alpha-2} - \dots - y^{n_\alpha}_\alpha. \tag{36}$$

Note that in the coordinates Y_α, the metric g^{LC}_α and the operator L_α have the form (19) with $u^1, \dots, u^n$ replaced by $y^1_\alpha, \dots, y^{n_\alpha}_\alpha$. Since in these coordinates we have $(-1)^{n_\alpha}\det L_\alpha = y^{n_\alpha}_\alpha$, the iterated warped product metric $g = (g^{ij})$ is given by the following easy algebraic formula

$$g = g_1 + \left(\frac{(-1)^{n_1}}{y^{n_1}_1}\right)^{c_{12}} g_2 + \left(\frac{(-1)^{n_1}}{y^{n_1}_1}\right)^{c_{13}} \left(\frac{(-1)^{n_2}}{y^{n_2}_2}\right)^{c_{23}} g_3 + \dots,$$

with $g_\alpha = P_\alpha(L_\alpha) g^{\mathsf{LC}}_\alpha$ and g^{LC}_α and L_α explicitly given by (19).

In order to check whether the coordinates u given by (35) are Frobenius, one needs to perform the multiplication

$$J g J^\top,$$

where $J = \left(\frac{\partial u^i}{\partial y^j}\right)$ is the Jacobi matrix of the coordinate transformation[4] $(y^1, \dots, y^n) \to (u^1, \dots, u^n)$ and check whether the entries of the resulting matrix $JgJ^\top$ are affine functions in u^i and conditions (17) are fulfilled. All these operations can be realised by standard computer algebra packages.

[4]This transformation is given by (35) as $y^i_\alpha = \sigma^i_\alpha$ and $\det L_\alpha = (-1)^{n_\alpha} y^{n_\alpha}_\alpha$.

The next result gives a description of two-dimensional Frobenius pencils in the general case.

Theorem 5. *Let g and $\bar g$ be compatible flat metrics that admit a common Frobenius coordinate system. If the eigenvalues of the operator $R_g = \bar g g^{-1}$ are all different at a point p, then in a neighbourhood of this point the pencil $\lambda g + \mu \bar g$ is isomorphic to a two-dimensional subpencil of the Frobenius pencil* (31)*, i.e., in a certain coordinate system these metrics take the form*

$$g = \operatorname{diag}\left(P_1(L_1)\hat g_1, \dots, P_B(L_B)\hat g_B\right) \quad \text{and} \quad \bar g = \operatorname{diag}\left(Q_1(L_1)\hat g_1, \dots, Q_B(L_B)\hat g_B\right) \tag{37}$$

(*with parameters $c_{\alpha\beta}$ defined by* (32) *and coefficients of P_α and Q_α satisfying* (i)-(iv)).

Theorem 5 will be proved in Section 7.

Remark 4.1. In Theorem 5 we allow complex eigenvalues of R_g. The corresponding part of diagonal coordinates is then complex. Moreover, the polynomials P_α and Q_α may have complex coefficients. The only condition is that the metrics given by (30) should be well-defined real metrics. It is easy to see that this condition implies in particular that every block $(g^{\mathsf{LC}}_\alpha, L_\alpha)$ is either real or pure complex (= all coordinates are complex; the coefficients of the polynomials P_α and Q_α may be complex as well), and that a pure complex block comes together with a complex-conjugate one. See also [6, §3] for discussion on Nijenhuis operators some of whose eigenvalues are complex.

In certain special cases, a common Frobenius coordinate system for g and $\bar g$ is not unique (up to affine transformations). This is the case when $n_\alpha = 1$, $c_{\alpha\beta} = 0$ for all β (i.e., this block represents a leaf of the corresponding in-forest) and the diagonal component of $R_g = \bar g g^{-1}$ corresponding to this block is constant, in other words, the (linear) polynomials P_α and Q_α are proportional. The restrictions g_α and $\bar g_\alpha$ onto this blocks are then as follows

$$g_\alpha = f \cdot (a x^\alpha + b)\left(\tfrac{\partial}{\partial x^\alpha}\right)^2 \quad \text{and} \quad \bar g_\alpha = c\, g_\alpha = c\, f \cdot (a x^\alpha + b)\left(\tfrac{\partial}{\partial x^\alpha}\right)^2, \quad c \in \mathbb{R},$$

where f is some function of the remaining coordinates and $L_\alpha = (x^\alpha)$ (diagonal 1×1 matrix). However, we can do coordinate transformation $x^\alpha \mapsto \tilde x^\alpha = \tilde x^\alpha(x^\alpha)$ that kills the factor $a x^\alpha + b$ and reduces the metrics to the form

$$g_\alpha = f \cdot \left(\tfrac{\partial}{\partial \tilde x^\alpha}\right)^2 \quad \text{and} \quad \bar g_\alpha = c\, g_\alpha = c\, f \cdot \left(\tfrac{\partial}{\partial \tilde x^\alpha}\right)^2, \quad c \in \mathbb{R},$$

Hence, with a new operator $L_\alpha^{\mathsf{new}} = (\tilde x^\alpha)$ and new polynomials $P_\alpha^{\mathsf{new}}(t) = 1$, $Q_\alpha^{\mathsf{new}}(t) = c$, we still remain in the framework of our construction and (37) still holds. This non-affine transformation will lead to another Frobenius coordinate system. In Section 7.3 we explain that only this situation allows ambiguity in the choice of Frobenius coordinates up to affine transformations.

Remark 4.2. In [17, Theorem 2] it was claimed that *under some general assumptions* for $n > 2$, there is only one equivalence class of $(n+1)$-Hamiltonian hydrodynamic systems (in the sense of [17]) and $n+1$ is the best possible. The corresponding multi-Hamiltonian structure comes from the $(n+1)$-dimensional AFF-pencil. In this view, it is interesting to notice that multi-block pencils from Theorem 5 also provide such a structure, which may have even higher dimension.

4.2 Case of two blocks

In the case of two blocks, i.e., $B = 2$, the construction explained in the previous section gives a natural and rather simple answer. We have two cases: $c_{12} = 0$ and $c_{12} = 1$. The first case is trivial being a direct product of two blocks (possibly complex conjugate) each of which is as in Theorem 3; in (33) we set $\hat{g}_i = g_i^{\mathsf{LC}}$ and take P_i to be arbitrary polynomials of degrees $\le n_i$ $(i = 1, 2)$.

Theorem below is a special case of Theorem 4 in the non-trivial case $c_{12} = 1$.

Theorem 6. *Suppose $B = 2$, $c_{12} = 1$ and consider the metric g given by the construction from Section 4.1:*

$$g = g_1 + \frac{1}{\det L_1}\, g_2, \quad \textit{with } g_i = P_i(L_i) g_i^{\mathsf{LC}}. \tag{38}$$

Following this construction, assume that the polynomials P_1 and P_2 have degrees no greater than n_1 and n_2+1 respectively: $P_1 = \sum_{s=0}^{n_1} a_s t^s$ and $P_2 = \sum_{s=0}^{n_2+1} b_s t^s$; moreover, if $n_2 = 1$ then $b_{n_2+1} = 0$. Then the coordinates from Theorem 4 are Frobenius for g if and only if $a_0 = 0$ and $(-1)^{n_1} a_1 = b_{n_2+1}$.

Example 4.1. In Theorem 6, take $n_1 = n_2 = 2$. In diagonal coordinates x^1, x^2, x^3, x^4, the metric $g = (g^{ij})$ is as follows:

$$g = \operatorname{diag}\left(\frac{P_1(x^1)}{x^1 - x^2}, \frac{P_1(x^2)}{x^2 - x^1}, \frac{P_2(x^3)}{x^1 x^2 (x^3 - x^4)}, \frac{P_2(x^4)}{x^1 x^2 (x^4 - x^3)} \right),$$

where $P_1(t) = a_1 t + a_2$ and $P_2(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3$ with $b_3 = a_1$. Recall that $L = L_1 \oplus L_2$ with $L_1 = \operatorname{diag}(x^1, x^2)$, $L_2 = \operatorname{diag}(x^3, x^4)$, and the relation between the diagonal coordinates x^i and the Frobenius coordinates u^i given by Theorem 4 are as follows:

$$\begin{aligned}
u^1 &= \operatorname{tr} L_1 = x^1 + x^2, \\
u^2 &= -\det L_1 = -x^1 x^2, \\
u^3 &= \det L_1 \cdot \operatorname{tr} L_2 = x^1 x^2 (x^3 + x^4), \\
u^4 &= -\det L_1 \cdot \det L_2 = -x^1 x^2 x^3 x^4 = -\det L.
\end{aligned}$$

In these Frobenius coordinates, the metric $g = (g^{ij})$ has the following form:

$$g = \begin{pmatrix} a_2u^1 + a_1 & a_2u^2 & a_2u^3 & a_2u^4 \\ a_2u^2 & a_1u^2 & a_1u^3 & a_1u^4 \\ a_2u^3 & a_1u^3 & -a_1u^4 - b_1u^2 - b_2u^3 & -b_0u^2 - b_2u^4 \\ a_2u^4 & a_1u^4 & -b_0u^2 - b_2u^4 & b_0u^3 - b_1u^4 \end{pmatrix}.$$

This formula defines a 5-dimensional pencils of metrics (with parameters a_1, a_2, b_0, b_1, b_2). For any choice of the parameters such that g is nondegenerate, the coordinates u^i are Frobenius for it in the sense of Definition 3.

From the algebraic viewpoint, we may equivalently think of this formula as 5-parametric family (pencil) of Frobenius algebras $(\mathfrak{a}, b)$. The entries of g define the structure constants of $\mathfrak{a}$. For instance, $g^{11} = a_2u^1 + a_1$ and $g^{34} = -b_0u^2 - b_2u^4$ imply

$$e^1 \star e^2 = a_2e^1 \quad \text{and } e^3 \star e^4 = -b_0e^2 - b_2e^4$$

for a basis e^1, e^2, e^3, e^4 of $\mathfrak{a}$. The matrix (b^{ij}) of the corresponding Frobenius form b is obtained from that of g by assigning to u^i any constant values $u^i = m^i \in \mathbb{R}$ (such that b is non-degenerate for generic choice of a_1, a_2, b_0, b_1, b_2). To get a Frobenius pencil the constants m^i should be the same for all parameters a_1, a_2, b_0, b_1, b_2.

In the coordinates $(u^1, ..., u^4)$ the operators L_1 and L_2 are given by the matrices

$$L_1 = \begin{pmatrix} u^1 & 1 & 0 & 0 \\ u^2 & 0 & 0 & 0 \\ u^3 & 0 & 0 & 0 \\ u^4 & 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{-u^4u^2-(u^3)^2}{(u^2)^2} & \frac{u^3}{u^2} & 1 \\ 0 & -\frac{u^4u^3}{(u^2)^2} & \frac{u^4}{u^2} & 0 \end{pmatrix}.$$

The matrices of g_1^{LC} and g_2^{LC} are

$$g_1^{\mathsf{LC}} = \begin{pmatrix} 0 & 1 & \frac{u^3}{u^2} & \frac{u^4}{u^2} \\ 1 & -u^1 & -\frac{u^3u^1}{u^2} & -\frac{u^1u^4}{u^2} \\ \frac{u^3}{u^2} & -\frac{u^3u^1}{u^2} & -\frac{(u^3)^2u^1}{(u^2)^2} & -\frac{u^1u^4u^3}{(u^2)^2} \\ \frac{u^4}{u^2} & -\frac{u^1u^4}{u^2} & -\frac{u^1u^4u^3}{(u^2)^2} & -\frac{u^1(u^4)^2}{(u^2)^2} \end{pmatrix}, \quad g_2^{\mathsf{LC}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -u^2 \\ 0 & 0 & -u^2 & u^3 \end{pmatrix}.$$

5 Proof of Theorems 1 and 2

Proof of Theorem 1. We assume that $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar{h}} + \mathcal{A}_{\bar{g}}$ are compatible with the additional condition that eigenvalues of $R_h = \bar{h}h^{-1}$ are pairwise different. We also assume that $h + \bar{h}$ is nondegenerate.

Recall that Theorem 7.1 from [11] implies that $\mathcal{B}_h$ and $\mathcal{B}_{\bar{h}}$ are compatible Poisson structures (item (i) in Fact 1). Let $\Gamma_s^{\alpha\beta}$ and $\bar{\Gamma}_s^{\alpha\beta}$ denote the contravariant Levi-Civita connections of h and $\bar{h}$. From Theorem 3.2 in [11] applied to $\mathcal{B}_h + \mathcal{B}_{\bar{h}}$, it follows that the connection $\hat{\Gamma}_{qs}^{\beta}$ defined from

$$\Gamma_s^{\alpha\beta} + \bar{\Gamma}_s^{\alpha\beta} = (h + \bar{h})^{\alpha q} \hat{\Gamma}_{qs}^{\beta}$$

is symmetric and flat.

By direct computation $\hat{\nabla}(h+\bar{h}) = \nabla h + \bar{\nabla}\bar{h} = 0$, so that $\hat{\Gamma}$ is the Levi-Civita connection for $h + \bar{h}$ and moreover, $h + \bar{h}$ is flat. According to Theorem 6.2 in [11], this implies that $\mathcal{B}_h + \mathcal{B}_{\bar{h}}$ is Darboux-Poisson (i.e., is given by (8)). Hence, in our notations, we obtain the formula

$$\mathcal{B}_h + \mathcal{B}_{\bar{h}} = \mathcal{B}_{h+\bar{h}}. \tag{39}$$

Setting $\hat{\Gamma}_s^{\alpha\beta} = (h + \bar{h})^{\alpha q} \hat{\Gamma}_{qs}^{\beta}$ to be the contravariant Levi-Civita connection of $h + \bar{h}$, we get

$$\Gamma_s^{\alpha\beta} + \bar{\Gamma}_s^{\alpha\beta} = \hat{\Gamma}_s^{\alpha\beta}, \tag{40}$$

and conclude that h and $\bar{h}$ are Poisson compatible in the sense of Definition 1 (in particular, this proves the (ii)-part of Fact 2). Hence, $R_h = \bar{h}h^{-1}$ is a Nijenhuis operator (Fact 3).

For a pair of flat metrics h and $\bar{h}$, introduce the so-called obstruction tensor

$$S_{rq}^{\beta} = \Gamma_{rq}^{\beta} - \bar{\Gamma}_{rq}^{\beta}.$$

It vanishes if and only if $h, \bar{h}$ can be brought to constant form simultaneously (thus, the name). It is obviously symmetric in lower indices. Condition (40) can be written in equivalent form ([28], Lemma 3.1 and Theorem 3.2) in terms of only $\Gamma_s^{\alpha\beta}, \bar{\Gamma}_s^{\alpha\beta}, h, \bar{h}$

$$\bar{\Gamma}_q^{\alpha\beta} h^{q\gamma} - \bar{\Gamma}_q^{\gamma\beta} h^{q\alpha} + \Gamma_q^{\alpha\beta} \bar{h}^{q\gamma} - \Gamma_q^{\gamma\beta} \bar{h}^{q\alpha} = 0 \tag{41}$$

After lowering both indices with h and rearranging the terms we get

$$S_{pq}^{\beta} R_s^q - R_p^q S_{qs}^{\beta} = 0. \tag{42}$$

For a given metric h and its Levi-Civita connection, define

$$c_{rs}^{\alpha\beta} = h^{\alpha q} \left(\Gamma_{qr}^a \Gamma_{as}^{\beta} - \frac{\partial \Gamma_{qs}^{\beta}}{\partial u^r} \right).$$

The $\bar{c}_{rs}^{\alpha\beta}, \hat{c}_{rs}^{\alpha\beta}$ for $\bar{h}$ and $h + \bar{h}$ are defined in a similar way. This formula is one “half” of the formula for Riemann curvature tensor and the flatness of the metrics implies that $c_{rs}^{\alpha\beta} = c_{sr}^{\alpha\beta}$ (and similarly for metrics $\bar{h}, h + \bar{h}$). Using this symmetry in lower indices, we apply the general formula (8) to the Poisson structures in (39) and collect coefficients in front of D^2 to get

$$3c^{\alpha\beta}_{rs}u^r_x u^s_x - 3\Gamma^{\alpha\beta}_s u^s_{xx} + 3\bar{c}^{\alpha\beta}_{rs}u^r_x u^s_x - 3\bar{\Gamma}^{\alpha\beta}_s u^s_{xx} = 3\hat{c}^{\alpha\beta}_{rs}u^r_x u^s_x - 3\hat{\Gamma}^{\alpha\beta}_s u^s_{xx}.$$

Collecting all the terms with $u^r_x u^s_x$ in this differential polynomial, in turn, implies

$$\hat{c}^{\alpha\beta}_{rs} - c^{\alpha\beta}_{rs} - \bar{c}^{\alpha\beta}_{rs} = 0.$$

Using the characteristic property of the Levi-Civita connection

$$\frac{\partial h^{\alpha\beta}}{\partial u^s} + h^{\alpha q}\Gamma^{\beta}_{qs} + \Gamma^{\alpha}_{qs}h^{q\beta} = 0$$

we rewrite $c^{\alpha\beta}_{rs}$ as

$$c^{\alpha\beta}_{rs} = h^{\alpha q}\left(\Gamma^a_{qr}\Gamma^\beta_{as} - \frac{\partial \Gamma^\beta_{qs}}{\partial u^r}\right) = -\frac{\partial}{\partial u^r}\left[h^{\alpha q}\Gamma^\beta_{qs}\right] - h^{pq}\Gamma^\alpha_{qr}\Gamma^\beta_{ps} \tag{43}$$

Applying (40), (41) and (43) yields

$$\begin{aligned}
0 &= \left(\hat{c}^{\alpha\beta}_{rs} - c^{\alpha\beta}_{rs} - \bar{c}^{\alpha\beta}_{rs}\right)(h+\bar{h})^{s\gamma} = \left(h^{pq}\Gamma^\alpha_{qr}\Gamma^\beta_{ps} + \bar{h}^{pq}\bar{\Gamma}^\alpha_{qr}\bar{\Gamma}^\beta_{ps} - (h+\bar{h})^{pq}\hat{\Gamma}^\alpha_{qr}\hat{\Gamma}^\beta_{ps}\right)(h+\bar{h})^{s\gamma} = \\
&= h^{pq}\Gamma^\alpha_{qr}\Gamma^\beta_{ps}h^{s\gamma} + h^{pq}\Gamma^\alpha_{qr}\Gamma^\beta_{ps}\bar{h}^{s\gamma} + \bar{h}^{pq}\bar{\Gamma}^\alpha_{qr}\bar{\Gamma}^\beta_{ps}h^{s\gamma} + \bar{h}^{pq}\bar{\Gamma}^\alpha_{qr}\bar{\Gamma}^\beta_{ps}\bar{h}^{s\gamma} - \left(h^{pq}\Gamma^\alpha_{qr} + \bar{h}^{pq}\bar{\Gamma}^\alpha_{qr}\right)\left(\Gamma^\beta_{ps}h^{s\gamma} + \bar{\Gamma}^\beta_{ps}\bar{h}^{s\gamma}\right) = \\
&= \Gamma^\alpha_{qr}\left(h^{pq}\Gamma^\beta_{ps}\bar{h}^{s\gamma} - h^{pq}\bar{\Gamma}^\beta_{ps}\bar{h}^{s\gamma}\right) - \bar{\Gamma}^\alpha_{qr}\left(\bar{h}^{pq}\Gamma^\beta_{ps}h^{s\gamma} - \bar{h}^{pq}\bar{\Gamma}^\beta_{ps}h^{s\gamma}\right) = \\
&= \left(S^\alpha_{qr} + \bar{\Gamma}^\alpha_{qr}\right)\left(h^{pq}\Gamma^\beta_{ps}\bar{h}^{s\gamma} - h^{pq}\bar{\Gamma}^\beta_{ps}\bar{h}^{s\gamma}\right) - \bar{\Gamma}^\alpha_{qr}\left(\bar{h}^{pq}\Gamma^\beta_{ps}h^{s\gamma} - \bar{h}^{pq}\bar{\Gamma}^\beta_{ps}h^{s\gamma}\right) = \\
&= S^\alpha_{qr}h^{pq}S^\beta_{ps}\bar{h}^{s\gamma} + \bar{\Gamma}^\alpha_{qr}\left(h^{pq}\Gamma^\beta_{ps}\bar{h}^{s\gamma} - h^{pq}\bar{\Gamma}^\beta_{ps}\bar{h}^{s\gamma} - \bar{h}^{pq}\Gamma^\beta_{ps}g^{s\gamma} + \bar{h}^{pq}\bar{\Gamma}^\beta_{ps}h^{s\gamma}\right) = \\
&= S^\alpha_{qr}h^{pq}S^\beta_{ps}\bar{h}^{s\gamma}.
\end{aligned}$$

Now consider the coordinate system in which the Nijenhuis operator R_h is diagonal. As R_h by definition is self-adjoint with respect to both h and $\bar{h}$, we get that both contravariant metrics are also diagonal. Condition (42) implies that for given β the only non-zero elements of S^β_{pq} are the ones that stand on the diagonal. The previous calculation yields

$$S^\alpha_{qr}h^{pq}S^\beta_{ps}\bar{h}^{s\gamma} = 0$$

which, for fixed α and β, is just the product of four diagonal matrices, two of which are nondegenerate. Taking $\alpha = \beta$ we see that the matrix S^α_{qr} must be zero. As α is arbitrary, this implies that the obstruction tensor vanishes and $h, \bar{h}$ have common Darboux coordinates.

Fix the coordinates in which both h and $\bar{h}$ are flat. Applying Fact 5, we see that these coordinates are Frobenius for both g and $\bar{g}$. Using (39) and applying Fact 5 to the sum of our Poisson structures, we get that $a^{\alpha\beta}_s + \bar{a}^{\alpha\beta}_s$ define a commutative associative algebra, while $b^{\alpha\beta}_s + \bar{b}^{\alpha\beta}_s$ and $h^{\alpha\beta}_s + \bar{h}^{\alpha\beta}_s$ are Frobenius forms for this algebra.

The inverse statement immediately follows from Facts 4 and 5. $\square$

Proof of Theorem 2. Consider a pair of compatible flat metrics $g, \bar g$ in common Frobenius coordinates $u^1, \dots, u^n$

$$g^{\alpha\beta}(u) = b^{\alpha\beta} + a^{\alpha\beta}_s u^s \quad \text{and} \quad \bar g^{\alpha\beta}(u) = \bar b^{\alpha\beta} + \bar a^{\alpha\beta}_s u^s,$$

Fact 4 implies that $-\frac{1}{2} a^{\alpha\beta}_s$ and $-\frac{1}{2}\bar a^{\alpha\beta}_s$ are the contravariant Christoffel symbols for g and $\bar g$ respectively.

Compatibility of g and $\bar g$ means that the contravariant Levi-Civita symbols for flat metric $g + \bar g$ are the sum of the corresponding symbols for g and $\bar g$, that is, $-\frac{1}{2} a^{\alpha\beta}_s - \frac{1}{2}\bar a^{\alpha\beta}_s$. At the same time these symbols are constant and symmetric in upper indices. Hence the coordinates $u^1, \dots, u^n$ are Frobenius for $g + \bar g$ (Fact 4).

This, in turn, implies that $a^{\alpha\beta}_s + \bar a^{\alpha\beta}_s$ are the structure constants of a commutative associative algebra and $b^{\alpha\beta}_s + \bar b^{\alpha\beta}_s$ is one of its Frobenius form. Thus, the corresponding Frobenius algebras are compatible.

As g and $\bar g$ are both nondegenerate metrics, this implies that for a generic collection of constants $m^0, m^1, \dots, m^n$, the bilinear forms

$$h^{\alpha\beta} = m^0 b^{\alpha\beta} + a^{\alpha\beta}_s m^s \text{ and } \bar h^{\alpha\beta} = m^0 \bar b^{\alpha\beta} + \bar a^{\alpha\beta}_s m^s$$

are both nondegenerate too. At the same time, each of them is the sum of a Frobenius form ($m^0 b^{\alpha\beta}$ and resp. $m^0 \bar b^{\alpha\beta}$) and trivial form ($a^{\alpha\beta}_s m^s$ and resp. $\bar a^{\alpha\beta}_s m^s$), which corresponds to $m \in \mathfrak{a}^*$ with coordinates $m^1, \dots, m^n$ and, thus, is also Frobenius[5]. As a result, h and $\bar h$ lead us to Frobenius triples $(h, b, \mathfrak{a})$ and $(\bar h, \bar b, \bar{\mathfrak{a}}.)$.

By construction, $h + \bar h$ defines (if nondegenerate) a Frobenius form for the sum of the algebras. Thus, we get compatible Frobenius triples, which yield compatible non-homogeneous Poisson structures $\mathcal{B}_h + \mathcal{A}_g$ and $\mathcal{B}_{\bar h} + \mathcal{A}_{\bar g}$. $\square$

6 Proof of Theorem 3

6.1 Rewriting the existence of Frobenius coordinates in a differential-geometric form

We start with the following observation related to Frobenius coordinate systems (Fact 4): $(u^1, \dots, u^n)$ is a Frobenius coordinate system for a metric g if and only if the contravariant

[5]Here we use a well known fact for any $m \in \mathfrak{a}^*$, the form $\xi, \eta \mapsto \langle \xi \star \eta, m\rangle$ is Frobenius, perhaps degenerate. If $\mathfrak{a}$ has a unity element, then every Frobenius form is of this kind. Otherwise, there might exist other (nontrivial) Frobenius forms.

Christoffel symbols $\Gamma^{ij}_k = \sum_s g^{si}\Gamma^j_{sk}$ in this coordinate system are constant and symmetric in upper indices.

We denote by $\Gamma, \bar\Gamma$ the Levi-Civita connections of g and $\bar g$. Assuming that a common Frobenius coordinate system $u^1, \dots, u^n$ exists, we let $\hat\Gamma$ be the flat connection whose Christoffel symbols identically vanish in this coordinate system. Let $R^i{}_{jk\ell}$, $\bar R^i{}_{jk\ell}$, and $\hat R^i{}_{jk\ell}$ denote the corresponding curvature tensors. We assume $n \geq 2$, the case $n = 1$ is trivial.

Consider the tensors

$$\begin{aligned} S^{ij}{}_k &:= \sum_s g^{si}\left(\Gamma^j_{sk} - \hat\Gamma^j_{sk}\right) \\ \bar S^{ij}{}_k &:= \sum_s \bar g^{si}\left(\bar\Gamma^j_{sk} - \hat\Gamma^j_{sk}\right). \end{aligned}$$

In terms of these tensors, the necessary and sufficient conditions that the connection $\hat\Gamma$ determines Frobenius coordinates are:

$$0 = \hat R^i{}_{jk\ell} = R^i{}_{jk\ell} = \bar R^i{}_{jk\ell} \tag{44}$$

$$S^{ij}{}_k = S^{ji}{}_k \tag{45}$$

$$\bar S^{ij}{}_k = \bar S^{ji}{}_k \tag{46}$$

$$0 = \hat\nabla_m S^{ij}{}_k = \hat\nabla_m \bar S^{ij}{}_k. \tag{47}$$

Indeed, if $(u^1, \dots, u^n)$ is a common Frobenius coordinate system for g and $\bar g$, then in these coordinates $\hat\Gamma^j_{sk} = 0$, and $\Gamma^{ij}_k = g^{is}\Gamma^j_{sk}$ and $\bar\Gamma^{ij}_k = \bar g^{is}\bar\Gamma^j_{sk}$ are both constant and symmetric in upper indices by Fact 4. Hence, (44)-(47) obviously follow.

Conversely, if (44)-(47) hold, then in the flat coordinates for $\hat\Gamma^j_{sk}$ we see that $\Gamma^{ij}_k = S^{ij}{}_k$ and $\bar\Gamma^{ij}_k = \bar S^{ij}{}_k$ are both symmetric in upper indices due to (45) and (46) and are also constant due to (47). Therefore, by Fact 4, $(u^1, \dots, u^n)$ are Frobenius coordinates for both g and $\bar g$.

6.2 General form of the metric in diagonal coordinates

We work in the coordinates $(x^1, ..., x^n)$ such that

$$R_g = \bar g g^{-1} = \mathrm{diag}(\ell_1(x^1), ..., \ell_n(x^n)) \ , \quad g_{ij} = \mathrm{diag}(\varepsilon_1 e^{g_1}, ..., \varepsilon_n e^{g_n}), \tag{48}$$

where g_i are local functions on our manifold and $\varepsilon_i \in \{-1, 1\}$. It follows from Facts 2 and 3 that R_g is a Nijenhuis operator and therefore, according to Haantjes theorem, is diagonalisable and ℓ_i depends on x^i only (see also various versions of diagonalisability theorems in [6] which, in particular, allows us to include the case of complex eigenvalues too). We assume that all $\ell_i(x^i)$ are different and never vanish.

Remark 6.1. We allow some of the *diagonal* variables x^i to be complex. Note that if a variable x^i is complex then by [6, §3] we may assume that the corresponding eigenvalue ℓ_i is a holomorphic function of x^i. In the first read, we recommend to think of all the eigenvalues as real and then to carefully check that our proofs are based on algebraic manipulations and differentiations, which are perfectly defined over complex coordinates, so that generalisation of the proofs to complex eigenvalues requires no change in formulas. See also a discussion at the end of [9, §7].

Note that the results we use (e.g. [33, 34]) are also based on algebraic manipulations (essentially, on a careful calculation of the curvature tensor and connection coefficients) and are applicable if a part of eigenvalues is complex.

Let us first consider the conditions (45, 46). We view them as linear (algebraic) system of equations with unknown $\hat\Gamma^i_{jk}$'s (satisfying also $\hat\Gamma^i_{jk} = \hat\Gamma^i_{kj}$) whose coefficient matrix is constructed from the entries of g and L and the free terms are constructed from $g, \bar g, \Gamma, \bar\Gamma$. Being rewritten in such a way that unknowns are on the left hand side and free terms are on the right hand side, it has the following form:

$$\begin{aligned} e^{-g_i}\hat\Gamma^j_{ik} - e^{-g_j}\hat\Gamma^i_{jk} &= e^{-g_i}\Gamma^j_{ik} - e^{-g_j}\Gamma^i_{jk}\,, \\ \ell_i e^{-g_i}\hat\Gamma^j_{ik} - \ell_j e^{-g_j}\hat\Gamma^i_{jk} &= \ell_i e^{-g_i}\bar\Gamma^j_{ik} - \ell_j e^{-g_j}\bar\Gamma^i_{jk}\,. \end{aligned} \tag{49}$$

We see that for fixed $i = j = k$, the system bears no information. For fixed $i \neq j$, the coefficient matrix $\begin{pmatrix} e^{-g_i} & -e^{-g_j} \\ \ell_i e^{-g_i} & -\ell_j e^{-g_j} \end{pmatrix}$ of the linear system (49) is nondegenerate, since the eigenvalues ℓ_i are all different, and therefore the system has a unique solution.

The entries of the connections Γ and $\bar\Gamma$ of the diagonal metrics g_{ij} and $\bar g_{ij} := gL^{-1}$ were calculated many times in the literature, see e.g. [9, Lemma 7.1], and are given by the following formulas:

- $\Gamma^k_{ij} = \bar\Gamma^k_{ij} = 0$ for pairwise different i, j and k,
- $\Gamma^k_{kj} = \frac{1}{2}\frac{\partial g_k}{\partial x^j}$ for arbitrary k, j,
- $\Gamma^k_{jj} = -\varepsilon_j \frac{e^{g_j - g_k}}{2}\frac{\partial g_j}{\partial x^k}$ for arbitrary $k \neq j$,
- $\bar\Gamma^k_{kj} = \Gamma^k_{kj}$ for arbitrary $k \neq j$,
- $\bar\Gamma^i_{ii} = \Gamma^i_{ii} - \frac{\ell_i'}{2\ell_i}$
- $\bar\Gamma^k_{jj} = \frac{\ell_k}{\ell_j}\Gamma^k_{jj}$ for arbitrary $k \neq j$.

By direct calculations using these formulas we obtain that the solution of the system (49) is as follows:

(A) $\hat{\Gamma}^i_{ii} = u_i$ for all i

(B) $\hat{\Gamma}^i_{ij} = \hat{\Gamma}^i_{ji} = \frac{\partial g_i}{\partial x^j}$ for $i \neq j$

(C) $\hat{\Gamma}^i_{jk} = 0$ for all $i \neq j$ and $k \neq i$ (we allow the case $k = j$).

Here u_i should be viewed as local functions on the manifold.

Combining these with the formulas for ${S^{ij}}_k$ we obtain:

- ${S^{ii}}_i = \varepsilon_i e^{-g_i} \left(u_i - \frac{1}{2}\frac{\partial g_i}{\partial x^i}\right)$ for all i,
- ${S^{ii}}_j = \frac{\varepsilon_i}{2} e^{-g_i} \frac{\partial g_i}{\partial x^j}$, for all $i \neq j$,
- ${S^{ij}}_i = {S^{ji}}_i = \frac{\varepsilon_j}{2} e^{-g_j} \frac{\partial g_i}{\partial x^j}$ for all $i \neq j$,
- ${S^{ij}}_k = 0$ for all $i \neq j \neq k \neq i$.

By direct calculations we see that for any $i \neq j \neq k \neq i$ we have $\hat{\nabla}_k {S^{ij}}_j = \frac{\varepsilon_i}{2} \frac{\partial^2 g_i}{\partial x^j \partial x^k}$ implying

$$0 = \frac{\partial^2 g_i}{\partial x^j \partial x^k}. \tag{50}$$

Next, consider the terms of the form $\hat{\nabla}_j {S^{ii}}_i$ and $\hat{\nabla}_i {S^{ii}}_j$ with $i \neq j$. They are given by

$$\hat{\nabla}_j {S^{ii}}_i = \varepsilon_i \frac{e^{-g_i}}{2} \left(\frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^i} + \frac{\partial^2 g_i}{\partial x^i \partial x^j} \right),$$

$$\hat{\nabla}_i {S^{ii}}_j = -\varepsilon_i \frac{e^{-g_i}}{2} \left(\frac{\partial g_i}{\partial x^j} \frac{\partial g_j}{\partial x^i} + \frac{\partial^2 g_i}{\partial x^i \partial x^j} - 2\frac{\partial u_i}{\partial x^j} \right).$$

Since $\hat{\nabla}_j {S^{ii}}_i = \hat{\nabla}_i {S^{ii}}_j = 0$, the formulas above imply

$$\frac{\partial u_i}{\partial x^j} = 0 \tag{51}$$

so each u_i is a function of x^i only.

Next, we prove the following Lemma. Denote by $U_i = U_i(x^i)$ and $U_j = U_j(x^j)$ the primitive functions for $e^{\tilde{u}_i}$ and $e^{\tilde{u}_j}$, where $\tilde{u}_i$ and $\tilde{u}_j$ are primitive functions for u_i and u_j. By their definition, $U_i' \neq 0$ and $U_j' \neq 0$.

Lemma 6.1. *There exist constants C_{ij} such that for the constants $\alpha_{ij} \in \{0,1\}$ given by the formula*

$$\alpha_{ij} = \alpha_{ji} = \begin{cases} 1 & \text{if } C_{ij} = C_{ji} = 0 \\ 0 & \text{otherwise} \end{cases}$$

and for any $i \neq j$ the function

$$g_i - \ln\left(|C_{ij}U_i - C_{ji}U_j|^{\alpha_{ij}}\right)$$

does not depend on x^j (we use the convention that $0^0 = 1$).

Proof. We consider the curvature tensor $\hat{R}^i{}_{jk\ell}$ of the connection $\hat{\Gamma}$. To compute it, we need to substitute $\hat{\Gamma}$ given by (A,B,C) above into the standard formula for the curvature

$$\hat{R}^\ell{}_{ijk} = \tfrac{\partial}{\partial x^j}\hat{\Gamma}^\ell_{ik} - \tfrac{\partial}{\partial x^k}\hat{\Gamma}^\ell_{ij} + \sum_s \left(\hat{\Gamma}^\ell_{js}\hat{\Gamma}^s_{ik} - \hat{\Gamma}^\ell_{ks}\hat{\Gamma}^s_{ij}\right). \tag{52}$$

We obtain for $i \neq j$:

$$0 = \hat{R}^i{}_{iji} = -\frac{\partial g_i}{\partial x^j}\frac{\partial g_j}{\partial x^i} - \frac{\partial^2 g_i}{\partial x^j \partial x^i} \tag{53}$$

$$0 = \hat{R}^i{}_{jji} = -\frac{\partial g_i}{\partial x^j}u_j + \left(\frac{\partial g_i}{\partial x^j}\right)^2 + \frac{\partial^2 g_i}{\partial {x^j}^2} \tag{54}$$

$$0 = \hat{R}^j{}_{jji} = \frac{\partial g_i}{\partial x^j}\frac{\partial g_j}{\partial x^i} + \frac{\partial^2 g_j}{\partial x^i \partial x^j} \tag{55}$$

$$0 = \hat{R}^j{}_{iji} = -\left(\frac{\partial g_j}{\partial x^i}\right)^2 + u_i\frac{\partial g_j}{\partial x^i} - \frac{\partial^2 g_j}{\partial {x^i}^2}. \tag{56}$$

We view 4 equations above a system of PDEs on the unknown functions

$$a = \frac{\partial g_i}{\partial x^j} \text{ and } b = \frac{\partial g_j}{\partial x^i}. \tag{57}$$

The condition (51) implies that the coefficients of the system depend on x^i and x^j and we may temporarly "forget" all other variables. The system then has the following form:

$$\frac{\partial a}{\partial x^i} = -ab\ ,\ \frac{\partial a}{\partial x^j} = -a^2 + au_j\ ,\ \frac{\partial b}{\partial x^i} = -b^2 + bu_i\ ,\ \frac{\partial b}{\partial x^j} = -ab. \tag{58}$$

This system is of Cauchy-Frobenius type (in the sense that all first derivatives of unknown functions are explicit expressions of the unknown functions and variables). By direct computation we check that its integrability conditions hold identically. Then, its solution depends on an arbitrary choice of the values of a and b at one arbitrarily chosen point p_0. Note that if $a(p_0) = 0$ then a is identically 0, the same is true for b.

By direct substitution we see that for any functions C_i, C_j the pair of functions

$$a = \frac{C_j U_j'}{C_i U_i + C_i U_j}, b = \frac{C_i U_i'}{C_i U_i + C_j U_j} \tag{59}$$

satisfies the equation. In the case $C_i = C_j = 0$ we think that a and b given by (59) vanish identically. The functions $U_i = U_i(x^i)$ and $U_j = U_j(x^j)$ used in (59) are as defined before Lemma 6.1. By varying the constants C_i, C_j one can get any nonzero initial values $a(p_0), b(p_0)$ so this is indeed a general solution.

In the case $C_i \neq 0$ or $C_j \neq 0$, using (57), we obtain

$$g_i = \ln(|C_i U_i + C_j U_j|) + D_i \quad \text{and} \quad g_j = \ln(|C_i U_i + C_j U_j|) + D_j \tag{60}$$

with D_i independent of x^j and D_j independent of x^i. In the case $C_i = 0 = C_j$ we obtain that g_i is independent of x^j and g_j is independent of x^i automatically.

Note that if we 'remember' all the coordinates, then C_i, D_i may also depend on all other variables x^k, $k \notin \{i, j\}$.

Let us study the dependence of C_i and C_j on the variable x^k with $k \notin \{i, j\}$. We first consider the case when $C^i \neq 0$ and $C^j \neq 0$. We observe that by (50) we have that (we assume $i \neq j \neq k \neq i$)

$$0 = \frac{\partial^2 g_i}{\partial x^j \partial x^k} \overset{(57,59)}{=} \frac{\partial}{\partial x^k} \frac{C_j U_j'}{C_i U_i + C_j U_j} = -\frac{U_j' U_i \frac{\partial}{\partial x^k} \frac{C_i}{C_j}}{(\frac{C_i}{C_j} U_i + U_j)^2}$$

implying that the ratio C_i/C_j is a constant. Note that $U_j' \neq 0$ since it is a primitive function for a nonvanishing function. Then, we may assume that C_i and C_j are constants, since in the formula (60) the dependence of C_i and C_j on other variables can be hidden in D_i and D_j.

In the cases $C_i = 0$ or $C_j = 0$ (but not $C_i = 0 = C_j$ both) the dependence of C_i and C_j on other variables can trivially be hidden in D_i and D_j. In the remaining case, when $C_i = 0 = C_j$, we already know that g_i is independent of x^j and g_j is independent of x^i.

Thus, if $C_i \neq 0$ or $C_j \neq 0$, $g_i - \ln(|C_i U_i + C_j U_j|)$ does not depend on x^j and $g_j - \ln(|C_i U_i + C_j U_j|)$ does not depend on x^i. Note that the constants C_i and C_j are constructed by fixed i and j, let us call them C_{ij} and C_{ji}. In the case $C_{ij} = C_{ji} = 0$, the function $g_i - \ln(|C_{ij} U_i + C_{ji} U_j|^0)$ does not depend on x^j (recall that in our convention $0^0 = 1$). $\square$

Consequently applying the Lemma, we see that

$$g_i - \sum_{s \neq i} \ln\left(|C_{ij} U_i + C_{is} U_s|^{\alpha_{is}}\right)$$

depends on x^i only.

Therefore, the ith diagonal component g_{ii} of the metric g is as follows:

$$g_{ii} = \varepsilon_i e^{g_i} = h_i(x^i)\left(\prod_{s\neq i}\left(C_{is}U_i(x^i) + C_{si}U_s(x^s)\right)^{\alpha_{is}}\right) \tag{61}$$

for some functions h_i of one variable.

6.3 Last step of the proof: making all C_{ij} equal ± 1

In the previous section we have proved that the the metric g is given by (61). Observe that by the assumptions of Theorem 3, the diagonal coordinates depend on all variables, so all $\alpha_{ij} = 1$ and all $C_{ij} \neq 0$ for $i \neq j$. First note that in the case when all $C_{ij} = 1$ for $i < j$ and $C_{ij} = -1$ for $i > j$, the diagonal metric g is in the so-called *Levi-Civita* form:

$$g_{ii} = \left(\prod_{j\neq i}(U_i(x^i) - U_j(x^j))\right) h_i(x^i). \tag{62}$$

In this section we show that one can bring the metric to the form (62) by certain 'admissible' operations which include only coordinate transformations and renaming of functions. Combining this with a result of A. Solodovnikov (Fact 6) will prove Theorem 3.

We will use the condition $\hat{\nabla}_i S^{ij}{}_k = 0$. Assuming $i \neq j \neq k \neq i$, this condition reads

$$-\frac{\varepsilon_j}{2}e^{-g_j}\left(\frac{\partial g_i}{\partial x^j}\frac{\partial g_i}{\partial x^k} - \frac{\partial g_i}{\partial x^j}\frac{\partial g_j}{\partial x^k} - \frac{\partial g_i}{\partial x^k}\frac{\partial g_k}{\partial x^j}\right) = 0. \tag{63}$$

Substituting (61) there , we see that the following condition should be satisfied:

$$0 = C_{ik}C_{ji}C_{kj} - C_{ij}C_{jk}C_{ki} = \det\begin{pmatrix} 0 & C_{ij} & C_{ik} \\ -C_{ji} & 0 & C_{jk} \\ -C_{ki} & -C_{kj} & 0 \end{pmatrix}. \tag{64}$$

Let us now use the condition (64) and 'make' $C_{ij} = 1$ for $i < j$ and $C_{ij} = -1$ for $i > j$. We will use the following operations for it:

(a) We can multiply the factor $(C_{ij}U_i + C_{ji}U_j)$ in the ith and jth diagonal components of g (given by (61)) by a nonzero constant and correspondingly change h_i and h_j by dividing them by the same constant.

(b) For every fixed pair $i \neq j$ if $C_{ij} \neq 0$ we can rename $C_{ij}U_i$ by U_i (which we will do if $i < j$) or by $-U_i$ (if $i > j$).

By applying the operation (a), we make $C_{1i} = 1$ for all $i \neq 1$. By applying the operation (b), we make and $C_{i1} = -1$ for all $i \neq 1$. In addition, by applying operation (a) we make $C_{2i} = 1$ with $i > 2$. Then, the first two rows and the first column of the matrix C_{ij} are as we want. Then, the condition (64) with $i = 1, j = 2$ arbitrary $k > 2$ reads $C_{k2} = -1$ so the second column is automatically as we want. Then, applying operation (a) we make all C_{3k} with $k > 3$ equal to 1. The condition (64) with $i = 1, j = 3$ arbitrary $k > 3$ reads $C_{k3} = -1$ and implies that the third column is as we want. Repeating the procedure, we bring the metrics in the Levi-Civita form (62).

Let us now take $U_i(x^i)$ as a local coordinate system: $x^i_{\mathsf{new}} := U_i(x^i_{\mathsf{old}})$. We can do it because the derivative of U_i is not zero. In the new coordinates R_g is still diagonal and the i^{th} diagonal component depend on the variable i only. In these coordinates, the metric (62) is diagonal with

$$g_{ii} = \left(\prod_{j\neq i}^{n}(x^i - x^j)\right)\frac{1}{H_i(x^i)}. \tag{65}$$

Therefore, the metric $\bar{g}$ is also diagonal with similar diagonal elements of the form

$$\bar{g}_{ii} = \left(\prod_{j\neq i}^{n}(x^i - x^j)\right)\frac{1}{\bar{H}_i(x^i)} \tag{66}$$

Fact 6. *The diagonal metric of form* (65) (*resp.* (66)) *in dimension at least two has constant curvature if and only if there exists a polynomial P* (*resp. Q*) *of degree $\leq n+1$ such that $H_i(x^i) = P(x^i)$* (*resp. $\bar{H}_i(x^i) = Q(x^i)$*)*. Moreover, the curvature of the metric vanishes if and only if the polynomial P* (*resp. Q*) *has degree $\leq n$.*

Fact 6 was proved in [34, §5] and easily follows from calculations in [9, §7].

Taking $L = \mathrm{diag}(x^1, ..., x^n)$ and the (contravariant) Levi-Civita metric

$$g_{\mathsf{LC}} = \sum_i \left(\prod_{j\neq i}^{n}(x^i - x^j)\right)^{-1}\left(\tfrac{\partial}{\partial x^i}\right)^2, \tag{67}$$

we see that $g = P(L)g_{\mathsf{LC}}$ and $\bar{g} = Q(L)g_{\mathsf{LC}}$, which completes the proof of Theorem 3. (The "uniqueness" part of Theorem 3 will be explained in Remark 7.2).

7 Proof of Theorem 5

7.1 Upperblockdiagonal structure of the matrix C_{ij}

We assume that the metric g is diagonal and its diagonal elements have the form (61). We view C_{ij} as entries of an $n \times n$-matrix. For cosmetic reasons we assume that all diagonal elements C_{ii} of the matrix C_{ij} are zero. We can do it because these elements do not come into the formula for g.

Let us show that by rearranging the coordinates $x^1, ..., x^n$ one can make the matrix C upperblockdiagonal. Moreover, in every diagonal block all nondiagonal entries are different from zero. We will need the following Lemma:

Lemma 7.1. *If $C_{ji} = 0$ for certain different $i, j \in \{1, ..., n\}$, then for any $k \in \{1, ..., n\}$ we have $C_{jk}C_{ki} = 0$. Moreover, if in addition $C_{ij} = 0$, then for any $k \in \{1, ..., n\}$ we have $C_{ik}C_{jk} = 0$.*

Proof. For $k = i$ or $k = j$ the statement follows from our convention $C_{ii} = C_{jj} = 0$, further we assume $i \neq k \neq j (\neq i)$.

We consider the equation (63): under the assumption $C_{ji} = 0$ the terms $\frac{\partial g_i}{\partial x^j}\frac{\partial g_i}{\partial x^k}$ and $\frac{\partial g_i}{\partial x^j}\frac{\partial g_j}{\partial x^k}$ vanish. Then, the equation reads $\frac{\partial g_i}{\partial x^k}\frac{\partial g_k}{\partial x^j} = 0$ and implies that $\frac{\partial g_i}{\partial x^k} = 0$ (which in turn implies $C_{ki} = 0$) or $\frac{\partial g_k}{\partial x^j} = 0$ (which in turn implies $C_{jk} = 0$). This proves the first statement of the lemma. Next, observe that under the assumption $C_{jk} = C_{kj} = 0$ the equation (63) reads $\frac{\partial g_i}{\partial x^j}\frac{\partial g_i}{\partial x^k} = 0$ implying $C_{ki}C_{ji} = 0$. Renaming $i \leftrightarrow k$ finishes the proof. □

Next, consider $i \in \{1, ..., n\}$ such that the i^{th} column of the matrix C_{ij} contains the maximal number of zero entries. We assume without loss of generality that $i = 1$, that the elements $C_{21}, ..., C_{d1}$ are not zero and the other elements of the first column are zero. Applying Lemma 7.1 to the element $C_{d'1}$ with $d' > d$, we obtain that $C_{d'k}C_{k1} = 0$. Since $C_{k1} \neq 0$ for $k \leq d$, we obtain $C_{d'k} = 0$ for such k.

Thus, all elements of the matrix C_{ij} staying under the upper left $d \times d$ block are zero. If $C_{ij} = 0$ with $i \neq j \in \{1, ..., d\}$, we obtain a contradiction with the assumption that the i^{th} column of the matrix C_{ij} contains the maximal number of zeros. Thus, all C_{ij} with $i \neq j \in \{1, ..., d\}$ are not zero. Thus, the first d columns of C_{ij} are as in the upperblockdiagonal matrix with the first block of dimension $d \times d$. We further have that all nondiagonal components of the first block are different from zero.

Next, consider the index $i \in \{d + 1, ..., n\}$ such that the number of zero entries in the columns of lower right $(n - d) \times (n - d)$ block is maximal. We may assume without loss of generality that $i = d + 1$, that the components $C_{d+1\ d+2}, ..., C_{d+1\ d+d'}$ are not zero and the components $C_{d+1\ d+d'+1}, ..., C_{d+1\ n}$ are zero. Arguing as above, using Lemma 7.1, we obtain that for any $k \in \{d + 1, ..., d + d'\}$ the components $C_{d+k \quad d+d'+1}, ..., C_{d+k \quad n}$ are zero. Thus, the first $d + d'$ columns of C_{ij} are as in the upperblockdiagonal matrix with the first block of dimension $d \times d$ and the second block of dimension $d' \times d'$. Moreover, by the 'maximality' condition in our choice of the first column of the second block, all nondiagonal elements of the second block are nonzero.

We can repeat the procedure further and further and obtain that the matrix C_{ij} is as we claimed: it is upperblockdiagonal and in every block all nondiagonal entries are different from zero. Let us explain now that by the operations (a,b) from Section 6.3 we can make C_{ij} in every block equal 1 for $i < j$ and -1 for $j > i$. Indeed, by applying the operations (a) and (b) we can make the first two rows and the first column of every block to be as we claimed. The condition (64) automatically implies that the second column of the block is as we want. Next, applying operation (a) we make the third row as we want. Then, (64) implies that the third column is as we want and so on. Note that these operations with one block do not affect other blocks.

We will denote by $B_{\alpha\beta}$ the blocks of the matrix C (corresponding to the decomposition $n = n_1 + ... + n_B$), the block $B_{\alpha\beta}$ has dimension $n_\alpha \times n_\beta$. Above we have shown that if $\alpha \neq \beta$, then either all its entries are zero or are equal to 1. In the first case we put $c_{\alpha\beta} = 0$, in the second case $c_{\alpha\beta} = 1$. If $\alpha > \beta$ then all entries of the block $B_{\alpha\beta}$ are zero, so such $c_{\alpha\beta} = 0$. We put $c_{\alpha\alpha} = 0$.

Similarly, one shows that if such a block $B_{\alpha\beta}$ with $\alpha < \beta$ is zero and the block $B_{\alpha\alpha'}$ is not, then all the blocks $B_{\alpha'\beta}$ with $\alpha' \neq \beta$ are also zero. In order to do it, we take an element C_{ij} of this block. By Lemma 7.1, if $c_{\alpha\beta} = 0$ with $\alpha \neq \beta$, then for any $s \notin \{\alpha, \beta\}$ we have $c_{\alpha s} c_{s\beta} = 0$ implying the claim. Analogously one shows, using the second statement of Lemma 7.1, that $c_{\alpha\beta} = 0$ with $\alpha < \beta$ implies $c_{\alpha s} c_{\beta s} = 0$.

Let us summarise the properties of the $B \times B$ matrix $(c_{\alpha\beta})$:

(a) $c_{\alpha\beta} = 0$ for $\alpha \geq \beta$, i.e., the matrix is upper triangular with zeros on the diagonal.

(b) If $c_{\alpha\beta} = 0$, then for every $s \in \{1, ..., B\}$ we have $c_{s\beta} c_{\alpha s} = 0$.

(c) If $c_{\alpha\beta} = 0$ for certain $\alpha < \beta$, then for every $s \in \{1, ..., B\}$ we have $c_{\beta s} c_{\alpha s} = 0$.

Let us show that any such matrix can be constructed from a directed rooted in-forest by a procedure described in Section 4.1. To see this, we introduce the relation $\prec$ on the set $\{1, ..., B\}$: we define $\alpha \prec \beta$ if and only if $c_{\alpha\beta} = 1$. Clearly, $\alpha \prec \beta$ implies that the number α is smaller than the number β. The relation $\prec$ is a strict partial order. Indeed, $\alpha \not\prec \alpha$ because of (a), so the relation is irreflexive. If $\alpha \prec \beta$, then $\alpha < \beta$ by (a) implying

$\beta \not\prec \alpha$, so the relation is asymmetric. If $\alpha \prec \beta$ and $\beta \prec \gamma$ then by (b) we have $\alpha \prec \gamma$, so the relation is transitive.

Moreover, for every $s \in \{1, ..., B\}$ the set $S_s := \{\alpha \mid \alpha \prec s\}$ is a *chain*, i.e., is totally ordered. Indeed, for $\alpha < \beta \in S_s$ we have $c_{\alpha s} = c_{\beta s} = 1$ implying $c_{\alpha\beta} = 1$ in view of (c).

Next, it is easy to see that every strict partially ordered finite set such that every S_s is a chain can be described by a directed rooted in-forest. The vertices of the forest are the numbers $1, ..., B$, and two vertices α, γ are connected by the oriented edge $\vec{\gamma\alpha}$ if $\alpha \prec \gamma$ and if there is no β such that $\alpha \prec \beta \prec \gamma$, each connected component of this oriented graph is a directed rooted in-tree. The forest clearly reconstructs the order "$\prec$" and therefore the matrix $(c_{\alpha\beta})$: for two numbers $\alpha \neq \beta \in \{1, ..., B\}$ we have $\alpha \prec \beta$ if there exists an oriented way from β to α, see example on Fig. 1.

The converse is also true: every directed in-forest (with appropriately labeled vertices) defines a matrix $c_{\alpha\beta}$ with properties (a), (b) and (c).

Example 7.1. If $B = 3$, the matrix $c_{\alpha\beta}$ is one of the following:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The first 4 cases correspond to the direct product situations. For example, in case 3 the metric g is the direct product of the metrics $g_1 + \frac{1}{\det(L_1)} g_3$ and g_2 with $g_1 = Q_1(L_1) g_1^{\mathsf{LC}}$, $g_2 = Q_2(L_2) g_2^{\mathsf{LC}}$ and $g_3 = Q_3(L_3) g_3^{\mathsf{LC}}$. Notice that the metric $g_1 + \frac{1}{\det(L_1)} g_3$ depends on the coordinates X_1, X_3 only and the metric g_2 depends on the coordinates g_2 only.

Finally, plugging these C_{ij} into (61) and passing to the new coordinate system $x^i_{\mathsf{new}} = U_i(x^i_{\mathsf{old}})$, we obtain the form from Theorem 5 with the only exception that the diagonal factors $H_i(x^i) := \frac{1}{h_i(x^i)}$ of the metric are not necessarily $P_i(x^i)$ for polynomials P_i of degrees $\leq n_i + 1$. In order to show this, we notice that the metric has the iterated warped product structure:

$$g_{ij} = g_1 + \sigma_1(X_1) g_2 + \sigma_2(X_1, X_2) g_2 + ... + \sigma_{B-1}(X_1, ..., X_{B-1}) g_B.$$

Since g is flat, g_1 must be flat and $g_2, ..., g_B$ of constant curvature. Applying the result of [34, §5] (see Fact 6 above) shows that blocks g_α of dimension greater than one are given by $P_\alpha(L_\alpha) g_\alpha^{\mathsf{LC}}$, where P_α is a polynomial of degree $\leq n_\alpha + 1$ (in Section 7.2 we will show that coefficients of these polynomials satisfy the conditions (i–iv) from Section 4.1 and also consider 1-dimensional blocks).

Remark 7.1. We also see that if for a certain i at least one $C_{ij} \neq 0$, then $\hat{\Gamma}^i_{ii} = 0$. In the case all $C_{ij} = 0$ for a fixed i, the corresponding block is one-dimensional and the other blocks do not depend on the coordinate x^i of this block. In this case, the only component from $\hat{\Gamma}^i_{jk}$ which is possibly not zero is $u_i := \hat{\Gamma}^i_{ii}$. By a coordinate change of the coordinate x^i we may achieve $u_i = 0$. Then, substituting the formulas for S^{ij}_k and $\hat{\Gamma}$ from Section 6.2 in the condition $\hat{\nabla}_i S^{ii}_i = 0$ we obtain $\frac{d^2}{(dx^i)^2}\left(\frac{1}{h_i(x^i)}\right) = 0$ implying that the function $H_i := \frac{1}{h_i(x^i)}$ is a polynomial in x^i of degree ≤ 1 as we claimed in Theorem 5.

7.2 Conditions on the coefficients of P_i

To complete the proof of Theorem 5, it remains to explain conditions (i)-(iv) on the coefficients of P_i stated before Theorem 4. We will need some facts and preliminary work.

Fact 7. *Consider the n-dimensional metric $g = P(L)g_{\mathsf{LC}}$, where L and g_{LC} are as in* (29), *and P is a polynomial $P(t) = a_1 t + ... + a_{n+1}t^{n+1}$ (with zero free term a_0). Then*

$$g\left(\mathrm{d}\sqrt{\det L}, \mathrm{d}\sqrt{\det L}\right) = \tfrac{(-1)^{n+1}}{4}a_1 + \tfrac{1}{4}a_{n+1}\det L. \tag{68}$$

Proof. The metric g_{LC} and operator L are explicitly given so the proof is an exercise in the Vandermonde identities and is left to the reader. □

Recall that a *Casimir* of a Poisson structure is defined by the property that Poisson structure applied to it gives zero. In the case of a first order Poisson structure coming from a (flat) metric g, Casimir (of the lowest order) can be viewed as a function f satisfying $\nabla_i\nabla_j f = 0$. Of course, any constant is a Casimir and n functionally independent Casimirs give us flat coordinates for g in which the components g^{ij} are all constants. For a metric g of constant curvature K we define *Casimir* as a function $\hat{f}$ satisfying the equation[6] $\nabla_i\nabla_j\hat{f} + K\hat{f}g_{ij} = 0$. The space of Casimirs of a constant curvature metric (on a simply-connected manifold) is a vector space of dimension $n+1$.

Below, we work with warped product metrics for which the "covariant language" is more convenient. For this reason, starting from Fact 8 and till the end of the current Section 7.2, g and g_i will denote covariant metrics. For the corresponding contravariant metrics we use g^* and g_i^*.

Fact 8. *Suppose a warped product metric $g = g_1 + f(X_1)^2 g_2$ has constant curvature. Then g_1 and g_2 have constant curvatures. Moreover, the following statements hold:*

1. *If g is flat, then g_1 is flat.*

2. *If g is flat, then $K_2 = g_1^*(\mathrm{d}f, \mathrm{d}f)$, where K_2 is the curvature of g_2.*

[6] The functions satisfying this equation are indeed Casimirs of the (nonlocal) Poisson structure corresponding to the constant curvature metric g, see e.g. [9, §2] and references therein.

3. $f(X_1)$ *is* g_1*-Casimir.*

Proof. The first statement is well-known and immediately follows from geometric arguments. The second statement follows from the second formula in the first line of [33, (4.2)]. The third statement follows, under the additional assumption that the curvature is zero, from the second line of [33, (4.2)]. If the curvature is not zero, we may assume that it is equal to 1. Then, we employ the conification construction: we consider the $(n+1)$-dimensional metric $\hat{g} = (dx^0)^2 + (x^0)^2\big(g_1 + f(X_1)^2 g_2\big)$. The metric $\hat{g}$ is flat and can be viewed as a warped product metric with base $(dx^0)^2 + (x^0)^2 g_1$ and warping function $\hat{f}^2 := \big(x^0 f(X_1)\big)^2$. Then, the function $\hat{f} := x^0 f(X_1)$ is a Casimir of $\hat{g}_1 := (dx^0)^2 + (x^0)^2 g_1$ implying that $f(X_1)$ is a Casimir of g_1. □

Next, we need the following technical lemma:

Lemma 7.2. *Suppose the warped product metric* $g_1 + f(X_1)^2 g_2$ *is flat and* $f(X_1)$ *is* g_1*-Casimir. Then, the following holds:*

1. *Every* g_1*-Casimir* $F(X_1)$ *such that* $g_1^*(\mathrm{d}\, f, \mathrm{d}\, F) = 0$ *is a Casimir of* g*.*
2. *If* g_2 *has constant nonzero curvature, then for any function* $\phi(X_2)$ *satisfying* $\nabla^{g_2}\nabla^{g_2}\phi + K\phi g_2 = 0$ *the function* $f\phi$ *is a* g*-Casimir.*
3. *If* g_2 *is flat, then for any function* $\phi(X_2)$ *satisfying* $\nabla^{g_2}\nabla^{g_2}\phi(X_2) = \mathrm{const}\, g_2$ *and for any* g_1*-Casimir* $\tilde{f}$ *such that* $g_1^*(\mathrm{d}\, f, \mathrm{d}\, \tilde{f}) = 1$ *we have that* $f\phi - \mathrm{const}\, \tilde{f}$ *is a* g*-Casimir.*

Notice that the g-Casimirs described in Lemma 7.2 span a $n+1$-dimensional vector space and, therefore, form a basis of the space of g-Casimirs. Indeed, the first statement gives an n_1-dimensional space. For a metric g_2 of nonzero constant curvature K, the space of solutions to the equation $\nabla^{g_2}\nabla^{g_2}\phi + K\phi g_2 = 0$ is known to be (n_2+1)-dimensional, see e.g. [37]. Clearly, the Casimirs described in the first and second statements are linearly independent, so that we get $n_1 + n_2 + 1 = n + 1$ independent functions, as required.

Now, if the metric g_2 is flat, then the space of Casimirs described by the third statement is $n_2 + 2$, since this is the dimension of the space of solutions of $\nabla\nabla\phi = \mathrm{const}\cdot g_2$. But the spaces of g-Casimirs constructed using statements 1 and 3 of the Lemma clearly have precisely two-dimensional intersection, namely the subspace generated by constants an by the function f. We see that also in this case we achieved the whole dimension $n+1$ of all Casimirs of g.

Proof. By direct calculations (done many times in the literature, see e.g. [33, (4.1)]) one sees that the Christoffel symbols of the warped product metric g are given by the following formulas:

$$\Gamma^a_{bc} = \overset{1}{\Gamma}{}^a_{bc}\ ,\ \Gamma^\alpha_{\beta\gamma} = \overset{2}{\Gamma}{}^\alpha_{\beta\gamma}\ , \Gamma^a_{\beta\gamma} = -f\sum_s f_{,s}\overset{1}{g}{}^{sa}\overset{2}{g}_{\beta\gamma}\ , \Gamma^\alpha_{a\beta} = \frac{1}{f}f_{,a}\delta^\alpha_\beta\ , \Gamma^a_{\alpha b} = \Gamma^a_{b\alpha} = \Gamma^\alpha_{ab} = 0.$$

Here $\overset{1}{\Gamma}$, $\overset{2}{\Gamma}$ relate to the Christoffel symbols of the metrics $g_1 = \overset{1}{g}$ and $g_2 = \overset{2}{g}$ respectively; a, b, c, s run from 1 to n_1 and α, β, γ from $n_1 + 1$ to n. The notation $f_{,s}$ means $\frac{\partial f}{\partial x^s}$.

For any function $F(X_1)$ we have:

$$\nabla_a\nabla_b F = \nabla^{g_1}_a\nabla^{g_2}_b F\ , \quad \nabla_\alpha\nabla_a F = 0\ , \quad \nabla_\alpha\nabla_\beta F = f g_1^*(dF, df)\overset{2}{g}_{\alpha\beta}. \tag{69}$$

In particular, if F is a g_1-Casimir, then $\nabla^{g_1}_i\nabla^{g_1}_j F = f g_1^*(dF, df)\overset{2}{g}_{ij}$ which implies the first statement. Also, if g_2 is flat, then $\mathrm{grad}_{g_1} f$ is light-like (Fact 8, item 2), so f is a g-Casimir.

Next, for any function $\phi(X_2)$ we have

$$\nabla_a\nabla_b\phi = 0\ , \quad \nabla_a\nabla_\beta\phi = -\frac{1}{f}f_{,a}\phi_{,\beta}\ , \quad \nabla_\alpha\nabla_\beta\phi = \nabla^{g_2}_\alpha\nabla^{g_2}_\beta\phi.$$

In particular, if ϕ satisfies $\nabla^{g_2}_\alpha\nabla^{g_2}_\beta\phi = \mathrm{const}\,\overset{2}{g}_{\alpha\beta}$ and g_2 is flat, then

$$\nabla_a\nabla_b(\phi f) = 0\ , \quad \nabla_\alpha\nabla_b(\phi f) = 0, \quad \nabla_\alpha\nabla_\beta(\phi f) = \mathrm{const}\,\overset{2}{g}_{\alpha\beta}f.$$

Combining this with (69), we see that $\phi f - \mathrm{const}\,\tilde{f}$ is a Casimir.

If g_2 is of constant nonzero curvature K and ϕ satisfies $\nabla^{g_2}\nabla^{g_2}\phi + K\phi g_2 = 0$, then $\nabla_\alpha\nabla_\beta(\phi f) = -K\phi\overset{2}{g}_{\alpha\beta}f - \phi f g_1^*(df, df)\overset{2}{g}_{\alpha\beta} = 0.$ □

Now, we are able to describe conditions on the polynomials P_α that are necessary for the flatness of the metric g from Theorem 5 given by (37), and also to finish the case of one-dimensional blocks (we need to show that if an α-block is one-dimensional and there exists at least one nonzero $c_{\alpha\beta}$, then the corresponding function $H_j(x^j) = \frac{1}{h_j(x^j)}$ from (65) is a polynomial of degree ≤ 1).

First we consider the most important case, when $c_{\alpha\beta} = 1$ for all $1 \leq \alpha < \beta \leq B$. The corresponding graph in this case is just a "path" $1 \longleftarrow 2 \longleftarrow \cdots \longleftarrow B$ from leaf B to root 1, so that the metric g_{ij} is given by the warped product of the form

$$g = g_1 + f_1(X_1)^2 g_2 + f_1(X_1)^2 f_2(X_2)^2 g_3 + ... + \left(\prod_{s=1}^{B-1} f_s(X_s)^2\right) g_B. \tag{70}$$

Suppose that a g_α-component with $\alpha < B$ is one-dimensional[7] and denote the corresponding coordinate by x^j. Consider the metric $g_\alpha + f(X_\alpha)^2 g_{\alpha+1} + ... + \left(\prod_{s=\alpha}^{B-1} f_s(X_s)^2\right) g_B$. It has constant curvature and is a warped product metric with base $g_\alpha = h_j(x^j)(dx^j)^2$ and warping factor $f(X_\alpha)^2 = \det L_\alpha = x^j$. From Fact 8 it follows that the function $f(x^j) = \sqrt{x^j}$ is a Casimir of g_α. The equation $\nabla^{g_\alpha}\nabla^{g_\alpha} x^j + K g_\alpha = 0$ implies then that $K \neq 0$ and that $H_j := \frac{1}{h(x^j)}$ is a polynomial of degree ≤ 1.

Thus, our metric g is given by (70) with each $g_\alpha = g_\alpha^{\mathsf{LC}}\big(P_\alpha(L_\alpha)\big)^{-1}$ and $f_\alpha = \sqrt{\det L_\alpha}$. We already know that for every α, the degree of P_α is at most $n_\alpha + 1$. Moreover, for those α such that $n_\alpha = 1$ the degree of P_α is at most 1. We denote the coefficients of the polynomials by

$$P_1(t) = a_0 + a_1 t + ... + a_{n_1+1} t^{n_1+1}\ ,\ \ P_2(t) = b_0 + b_1 t + ... + b_{n_2+1} t^{n_2+1}\ , P_3(t) = c_0 + c_1 t + ... + c_{n_3+1} t^{n_3+1}\ ...\ ,$$

$$P_{B-1}(t) = d_0 + d_1 t + ... + d_{n_{B-1}+1} t^{n_{B-1}+1}\ , P_B(t) = e_0 + e_1 t + ... + e_{n_B+1} t^{n_B+1}.$$

Lemma 7.3. *In the above notation, the flatness of g implies the following relations:*

$$\begin{array}{l} a_{n_1+1} = 0 \\ a_0 = b_0 = c_0 = ... = d_0 = 0\ , \\ a_1 = (-1)^{n_1} b_{n_2+1},\ \ b_1 = (-1)^{n_2} c_{n_2+1}, ..., d_1 = (-1)^{n_{B-1}} e_{n_B+1}. \end{array} \tag{71}$$

To avoid misunderstanding, let us mention that the free term e_0 of the polynomial P_B may be non-zero (see Example 4.1 with $B = 2$). Note also that in view of Theorem 4, conditions (71) are sufficient for the flatness of g and existence of Frobenius coordinates.

Proof. We view g as a warped product metric over the $n_1 + n_2$ dimensional base equipped with the metric $g_1 + f_1(X_1)^2 g_2$. Then, the metric $g_1 + f_1(X_1)^2 g_2$ is flat. Combining Facts 7 and 8, we obtain $a_{n_1+1} = 0$ implying the first line of (71). Next, from Fact 8 we know that $f_1(X_1)$ is a g_1-Casimir implying $a_0 = 0$ in view of Fact 7.

Let us now show that $a_1 = (-1)^{n_1} b_{n_2+1}$. Since $g_1 + f_1(X_1)^2 g_2$ is flat, by Fact 8 we have

$$\tfrac{1}{4} g^*\big(\mathrm{d}\,(f_1), \mathrm{d}\,(f_1)\big) = K_2. \tag{72}$$

Combing (72) with Fact 7, we obtain

$$b_{n_2+1} = (-1)^{n_1} a_1. \tag{73}$$

Next, we view g as a warped product metric over the $n_1 + n_2 + n_3$ dimensional base equipped with the metric $g_1 + f_1(X_1)^2 g_2 + f_1(X_1)^2 f_2(X_2)^2 g_3$. Then, the metric $g_1 + f_1(X_1)^2 g_2 +$

[7] The case $\alpha = B$ follows from Remark 7.1.

$f_1(X_1)^2 f_2(X_2)^2 g_3$ is flat. But it is itself a warped product metric over the n_1+n_2 dimensional base with the metric $g_1 + f_1(X_1)^2 g_2$. By Fact 8, this implies that $f_1 f_2$ is a Casimir of this metric. By Lemma 7.2, f_2 is a Casimir of g_2 so $b_0 = 0$. Moreover, by Fact 8 we have

$$\tfrac{1}{4} g^*\big(\mathrm{d}\,(f_1 f_2), \mathrm{d}\,(f_1 f_2)\big) = K_3 \ \ (= \ -\tfrac{1}{4} c_{n_3+1}). \tag{74}$$

On the other hand

$$\begin{aligned} g^*\big(\mathrm{d}\,(f_1 f_2), \mathrm{d}\,(f_1 f_2)\big) \ &= {f_2}^2 g_1^*(\mathrm{d}\, f_1, \mathrm{d}\, f_1) + g_2^*(\mathrm{d}\, f_2, \mathrm{d}\, f_2) \\ &\overset{(68)}{=} \tfrac{(-1)^{n_1+1}}{4} f_2^2 a_1 + \tfrac{(-1)^{n_2+1}}{4} b_1 + b_{n_2+1} f_2^2. \end{aligned} \tag{75}$$

Combining this with (74) and (73) we obtain $c_{n_3+1} = (-1)^{n_2} b_1$, as claimed. Iterating this procedure we obtain (71). Note that at the last step of the iteration, the condition $e_0 = 0$ does not appear. □

Lemma 7.3 completes the proof of Theorem 5 under the additional assumption that for every $\alpha < \beta$ we have $c_{\alpha\beta} = 1$. We now reduce the general case to this situation.

We assume without loss of generality that the combinatorial data are given by a directed rooted in-tree with B vertices, i.e. the graph F is connected. Otherwise, we have the direct product situation, i.e., the metric and all other relevant objects are direct products of lower-dimensional metrics and relevant lower dimensional objects.

We denote by $1, 2, ..., B$ the vertices of the in-tree F in such a way that $\alpha \prec \beta$ implies $\alpha < \beta$; of course, the vertex 1 is then the root. Other vertices of degree one are called *leaves*. Recall that $\alpha = \mathrm{next}(\beta)$, if $\alpha \prec \beta$ and there is no γ with $\alpha \prec \gamma \prec \beta$.

For every leaf β we define the *chain* S_β (oriented path to the root) as the sub-tree with vertices β, $\mathrm{next}(\beta)$, $\mathrm{next}\,(\mathrm{next}(\beta))$,...,1. For example, the upper tree of Fig. 1 has two chains, one with vertices $3, 2, 1$ and another with vertices $4, 2, 1$.

Next, for the chain S_β and for any fixed point p we consider the following submanifold M_β passing through p: in the coordinates $(X_1, ..., X_B) = (x^1, ..., x^n)$ it is defined by the system of equations

$$X_\alpha = X_\alpha(p) \ \text{ for every } \alpha \notin S_\beta\ .$$

This is a totally geodesic submanifold with respect to the connections Γ, $\bar\Gamma$ and $\hat\Gamma$. For $\hat\Gamma$ this follows from formulas (A,B,C) of Section 6.2. For Γ and $\bar\Gamma$ it follows from (61).

Therefore, the restriction of g and $\bar g$ onto M_β satisfies the assumptions of Theorem 5. Moreover, the components $c_{\alpha\beta}$ corresponding to this restriction to 1 for $\alpha < \beta$.

For example for $\beta = 3$, the metric g corresponding to the upper tree of Fig. 1 is given by

$$g = g_1 + \det L_1 \cdot g_2 + \det L_1 \det L_2 \cdot (g_3 + g_4) \tag{76}$$

and the restriction of the metric g onto M_3 is

$$g_1 + \det L_1 \cdot g_2 + \det L_1 \det L_2 \cdot g_3.$$

The case when $c_{\alpha\beta} = 1$ for all $\alpha < \beta$ have been completely understood above and it has been proved that the metric is as in Theorem 5. This implies that the metrics g and $\bar{g}$ are constructed as in Section 4.1 and the coefficients of P_α and Q_α satisfy conditions (i–iii) from Section 4.1. It remains to show that they also satisfy condition (iv). In order to do this, suppose that $\alpha = \text{next}(\beta) = \text{next}(\gamma)$ with $\beta \neq \gamma$. We consider the sub-tree with vertices α, β, $\gamma := \text{next}(\alpha) = \text{next}(\beta)$ and the corresponding warped product metric with the base metric g_α and fibre metric $g_\beta + g_\gamma$. For example, for $\alpha = 2$ in the case of the upper tree of Fig. 1, we consider the warped product metric $g_2 + \det L_2 \cdot (g_3 + g_4)$.

We know that is must be of constant curvature which implies that the direct product metric $g_\beta + g_\gamma$ must be of constant curvature which in turn implies that it is flat. Then, by Fact 7 the coefficients $\overset{\beta}{a}_{n_\beta+1}$ and $\overset{\gamma}{a}_{n_\gamma+1}$ vanish implying $\overset{\alpha}{a}_1 = 0$. Theorem 5 is proved.

7.3 On the uniqueness of Frobenius coordinates for a pair of metrics

We consider two flat metrics $g, \bar{g}$ possessing a common Frobenius coordinate system, and discuss the uniqueness of this coordinate system. As before we assume that $R_g = \bar{g}g^{-1}$ has n different eigenvalues. We know that $g, \bar{g}$ are as described in Theorem 5. In particular, in the corresponding coordinates, the connection $\hat{\Gamma}$ defining the Frobenius coordinate system (i.e., the flat connection that vanishes in Frobenius coordinates) is given by the formulas from Section 6.2. That is,

- For every $i \neq j$ and $k \neq i$, $\hat{\Gamma}^i_{jk} = 0$.
- For the indexes $i \neq j$ from one block, $\hat{\Gamma}^i_{ij} = \frac{1}{x^j - x^i}$.
- For the index i from the block number α and j from the block number $\beta \neq \alpha$, we have $\hat{\Gamma}^j_{ji} = \hat{\Gamma}^j_{ij} = \frac{c_{\alpha\beta}}{x^i}$.
- For every i, the component $\Gamma^i_{ii} := u_i$ depends on x^i only.

Moreover, the function u_i is necessary zero unless the only component of the metric g which may depend on x^i is the component g_{ii}, see Remark 7.1.

Clearly, the coordinate system is determined, up to affine coordinate changes, by its flat connection. Therefore, the freedom in choosing Frobenius coordinate system is possible if some of u_i are not zero. This happens if certain blocks are one-dimensional and other blocks do not depend on the coordinates of these blocks. It is easy to see that in this case the freedom is as discussed in Section 4.1 after Theorem 5.

Remark 7.2. Coming back to Theorem 3 (uniqueness part), we notice that for the metrics $g = P(L)g_{\mathsf{LC}}$ and $\bar{g} = Q(L)g_{\mathsf{LC}}$, every diagonal component g_{ii} depends on all variables $x^1, ..., x^n$. Then, the connection $\hat{\Gamma}$ is unique implying that Frobenius coordinates are unique up to an affine coordinate change, as required.

8 Pro-Frobenius algebras and multi-block Frobenius pencils. Proof of Theorem 4

8.1 Extended AFF-pencils and pro-Frobenius algebras

As seen from Section 4.1, the main ingredients in general multi-block Frobenius pencils (Theorem 5) are metrics of the form

$$P(L)g_{\mathsf{LC}}, \quad \text{where } P(\cdot) \text{ is a polynomial of degree } n+1. \tag{77}$$

If $\deg P \leq n$, then such metrics are flat and form the AFF-pencil (26). However, if $\deg P = n+1$, i.e., $P(t) = a_{n+1}t^{n+1} + \dots$, then $g = P(L)g_{\mathsf{LC}}$ has constant curvature $K = -\frac{1}{4}a_{n+1}$ (see Fact 8). All together, the metrics (77) form a pencil of compatible constant curvature metrics, which can be thought of as one-dimensional extension of the AFF-pencil (26), see details in [17], [9]. In Frobenuis coordinates $u^1, \dots, u^n$ from Section 2.3, the coefficients $g^{\alpha\beta}$ of the metric $g = P(L)g_{\mathsf{LC}}$ with $\deg P = n+1$ are not affine functions anymore. In particular, in the notation from Section 2.3, for $P(t) = t^{n+1}$ we get:

$$g_{n+1} = L^{n+1}g_0 = \begin{pmatrix} u^2 & u^3 & \dots & u^n & 0 \\ u^3 & \dots & u^n & 0 & 0 \\ \vdots & ⋰ & ⋰ & \vdots & \vdots \\ u^n & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} + \begin{pmatrix} u^1u^1 & u^1u^2 & \dots & u^1u^n \\ u^2u^1 & u^2u^2 & \dots & u^2u^n \\ u^3u^1 & u^3u^2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ u^nu^1 & u^nu^2 & \dots & u^nu^n \end{pmatrix}.$$

All the other metrics from the extended AFF pencil (77), in coordinates $u^1, \dots, u^n$, take the form

$$g^{\alpha\beta} = b^{\alpha\beta} + a^{\alpha\beta}_s u^s - 4Ku^\alpha u^\beta,$$

which looks as a *quadratic perturbation* of (15). These metrics still possess remarkable properties, similar to those from Fact 4 and related to the following generalisation of Frobenius algebras.

Definition 5. A *pro-Frobenius* algebra is a triple $(\mathfrak{a}, b, K)$, where $\mathfrak{a}$ is a real finite-dimensional commutative algebra with operation $\star$, b is a symmetric bilinear form on $\mathfrak{a}$, and $K \in \mathbb{C}$ satisfying the following two properties

$$\begin{aligned} &1. \quad b(\xi \star \eta, \zeta) = b(\xi, \eta \star \zeta), \\ &2. \quad \xi \star (\eta \star \zeta) - (\xi \star \eta) \star \zeta = -4K\big(b(\eta, \zeta)\xi - b(\xi, \eta)\zeta\big). \end{aligned} \tag{78}$$

For $K = 0$ (or $b = 0$), we obtain an associative algebra. Also if $K = 0$ and b is nondegenerate, then $(\mathfrak{a}, b)$ is a Frobenius algebra.

This definition is motivated by the following

Proposition 8.1. *Let g be a (contravariant) metric of arbitrary signature and $(u^1, \dots, u^n)$ be a coordinate system. The following two conditions are equivalent:*

1. *In coordinates $u^1, \dots, u^n$, the contravariant Christoffel symbols $\Gamma^{\alpha\beta}_s$ of g are symmetric in upper indices and take the form*

$$\Gamma^{\alpha\beta}_s = -\frac{1}{2} a^{\alpha\beta}_s + 2K u^\alpha \delta^\beta_s + 2K \delta^\alpha_s u^\beta, \tag{79}$$

   where $a^{\alpha\beta}_s \in \mathbb{R}$ are constants.

2. *In coordinates $u^1, \dots, u^n$, the metric g is in the form*

$$g^{\alpha\beta} = b^{\alpha\beta} + a^{\alpha\beta}_s u^s - 4K u^\alpha u^\beta, \tag{80}$$

   where the triple $(a^{\alpha\beta}_s, b^{\alpha\beta}, K)$ defines a pro-Frobenius algebra (in this case we will say that $u^1, \dots, u^n$ are *pro-Frobenius coordinates* for g)*.*

A metric satisfying either of these equivalent conditions has constant curvature K.

This statement generalises Fact 4 that relates to the particular case of $K = 0$.

Proof. Fix a basis $e^1, \dots, e^n$ of $\mathfrak{a}$ (for the sake of consistency we assume that vectors have lower indices). In coordinates, conditions (78) can be written as

$$\begin{aligned} &1. \quad a^{\alpha\beta}_q b^{q\gamma} = b^{\alpha q} a^{\beta\gamma}_q, \\ &2. \quad a^{\alpha q}_s a^{\beta\gamma}_s - a^{\alpha\beta}_q a^{q\gamma}_s = -4K\Big(\delta^\alpha_s b^{\beta\gamma} - b^{\alpha\beta}\delta^\gamma_s\Big). \end{aligned} \tag{81}$$

First, assume that the contravariant Christoffel symbols are given by (79). By definition we have

$$\nabla_s g^{\alpha\beta} = \frac{\partial g^{\alpha\beta}}{\partial u^s} + \Gamma^{\alpha\beta}_s + \Gamma^{\beta\alpha}_s = \frac{\partial g^{\alpha\beta}}{\partial u^s} + 2\Gamma^{\alpha\beta}_s = 0.$$

This implies that the metric is of the form (80).

Now we use the condition that *usual* Christoffel symbols Γ^{β}_{pq} are symmetric in lower indices. First, we calculate

$$
\begin{aligned}
-\Gamma^{\alpha\beta}_{q} g^{q\gamma} &= \Big(\frac{1}{2}a^{\alpha\beta}_{q} - 2Ku^{\alpha}\delta^{\beta}_{q} - 2Ku^{\beta}\delta^{\alpha}_{q}\Big)\Big(b^{q\gamma} + a^{q\gamma}_{s}u^{s} - 4Ku^{q}u^{\gamma}\Big) = \\
&= \frac{1}{2}a^{\alpha\beta}_{q}b^{q\gamma} + \Big(- 2Ku^{\alpha}b^{\beta\gamma} - 2Ku^{\beta}b^{\alpha\gamma} + \frac{1}{2}a^{\alpha\beta}_{q}a^{q\gamma}_{s}u^{s}\Big) + \\
&+ \Big(- 2Ku^{\alpha}a^{\beta\gamma}_{s}u^{s} - 2Ku^{\beta}a^{\alpha\gamma}_{s}u^{s} - 2Ka^{\alpha\beta}_{s}u^{s}u^{\gamma}\Big) + 16K^{2}u^{\alpha}u^{\beta}u^{\gamma}.
\end{aligned}
$$

Applying this to $\Gamma^{\gamma\beta}_{q}g^{q\alpha} - \Gamma^{\alpha\beta}_{q}g^{q\gamma} = \Gamma^{\gamma\beta}_{q}g^{q\alpha} - \Gamma^{\alpha\beta}_{p}g^{p\gamma} = (\Gamma^{\beta}_{pq} - \Gamma^{\beta}_{qp})g^{p\gamma}g^{q\alpha} = 0$ we get

$$
\begin{aligned}
0 &= \Gamma^{\gamma\beta}_{q}g^{q\alpha} - \Gamma^{\alpha\beta}_{q}g^{q\gamma} = \\
&= \frac{1}{2}\Big(a^{\alpha\beta}_{q}b^{q\gamma} - a^{\gamma\beta}_{q}b^{q\alpha}\Big) + \Big(- 2Ku^{\alpha}b^{\beta\gamma} + 2Kb^{\alpha\beta}u^{\gamma} + \frac{1}{2}a^{\alpha\beta}_{q}a^{q\gamma}_{s}u^{s} - \frac{1}{2}a^{\gamma\beta}_{q}a^{q\alpha}_{s}u^{s}\Big).
\end{aligned}
\tag{82}
$$

Thus, we get exactly conditions (81) and therefore $(a^{\alpha\beta}_{s}, b^{\alpha\beta}, K)$ defines a pro-Frobenius algebra, as stated.

Conversely, assume that g is of the form (80) so that $u^1, \dots, u^n$ are pro-Frobenius coordinates. Consider the connection $\bar{\Gamma}^{\beta}_{rs} = g_{rq}\left(-\frac{1}{2}a^{q\beta}_{s} + 2Ku^{q}\delta^{\beta}_{s} + 2K\delta^{q}_{s}u^{\beta}\right)$. Relation (82) implies that this connection is symmetric (w.r.t. r and s). At the same time

$$
\bar{\nabla}_{s}g^{\alpha\beta} = a^{\alpha\beta}_{s} - 4Ku^{\alpha}\delta^{\beta}_{s} - 4K\delta^{\alpha}_{s}u^{\beta} - \frac{1}{2}a^{\alpha\beta}_{s} + 2Ku^{\alpha}\delta^{\beta}_{s} + 2Ku^{\beta}\delta^{\alpha}_{s} - \frac{1}{2}a^{\beta\alpha}_{s} + 2Ku^{\alpha}\delta^{\beta}_{s} + 2Ku^{\beta}\delta^{\alpha}_{s} = 0.
$$

Hence, $\bar{\Gamma}^{\beta}_{rs}$ is the Levi-Civita connection of g. Thus, Conditions 1 and 2 from Proposition 8.1 are equivalent.

Finally, we compute the curvature tensor of g in terms of contravariant Christoffel symbols [15, formula 0.9]:

$$
R^{\beta\gamma\alpha}_{s} = -\frac{\partial \Gamma^{\gamma\alpha}_{p}}{\partial u^{s}}g^{p\beta} + \frac{\partial \Gamma^{\gamma\alpha}_{s}}{\partial u^{p}}g^{p\beta} + \Gamma^{\beta\gamma}_{q}\Gamma^{q\alpha}_{s} - \Gamma^{\beta\alpha}_{q}\Gamma^{q\gamma}_{s}. \tag{83}
$$

For contravariant Christoffel symbols given by (79) we have

$$
\frac{\partial \Gamma^{\alpha\beta}_{p}}{\partial u^{s}} = \frac{\partial \Gamma^{\alpha\beta}_{s}}{\partial u^{p}} = 2K\Big(\delta^{\alpha}_{s}\delta^{\beta}_{p} + \delta^{\alpha}_{p}\delta^{\beta}_{s}\Big).
$$

This yields

$$
R^{\beta\gamma\alpha}_{s} = \Gamma^{\beta\gamma}_{q}\Gamma^{q\alpha}_{s} - \Gamma^{\beta\alpha}_{q}\Gamma^{q\gamma}_{s}. \tag{84}
$$

Next, we calculate (here we use the symmetry of $\Gamma^{\alpha\beta}_s$ in upper indices)

$$\begin{aligned}\Gamma^{\alpha\beta}_q\Gamma^{q\gamma}_s &= \Big(-\frac{1}{2}\,a^{\alpha\beta}_q + 2Ku^\alpha\delta^\beta_q + 2Ku^\beta\delta^\alpha_q\Big)\Big(-\frac{1}{2}\,a^{q\gamma}_s + 2Ku^q\delta^\gamma_s + 2Ku^\gamma\delta^q_s\Big) = \\ &= \frac{1}{4}\,a^{\alpha\beta}_q a^{q\gamma}_s - \Big(Ku^\alpha a^{\beta\gamma}_s + Ku^\beta a^{\alpha\gamma}_s + Ku^\gamma a^{\alpha\beta}_s + Ka^{\alpha\beta}_q u^q\delta^\gamma_s\Big) + \\ &+ 4\Big(K^2u^\alpha u^\beta\delta^\gamma_s + K^2u^\alpha u^\gamma\delta^\beta_s + K^2u^\beta u^\gamma\delta^\alpha_s\Big).\end{aligned}$$

Substituting this into (84) we get

$$\begin{aligned}R^{\beta\gamma\alpha}_s &= \frac{1}{4}\,(a^{\gamma\beta}_q a^{q\alpha}_s - a^{\alpha\beta}_q a^{q\gamma}_s) + \delta^\alpha_s\Big(-Ka^{\beta\gamma}_q u^q + 4K^2u^\beta u^\gamma\Big) - \Big(-Ka^{\alpha\beta}_q u^q + 4K^2u^\alpha u^\beta\Big)\delta^\gamma_s = \\ &= -K\delta^\alpha_s b^{\beta\gamma} + Kb^{\alpha\beta}\delta^\gamma + \delta^\alpha_s\Big(-Ka^{\beta\gamma}_q u^q + 4K^2u^\beta u^\gamma\Big) - \Big(-Ka^{\alpha\beta}_q u^q + 4K^2u^\alpha u^\beta\Big)\delta^\gamma_s = \\ &= K\Big(g^{\alpha\beta}\delta^\gamma_s - \delta^\alpha_s g^{\beta\gamma}\Big).\end{aligned}$$

Thus, the metric has constant curvature K. □

8.2 Algebraic interpretation of warped product

To a pair of Riemannian metrics of constant curvature we can naturally apply the warped product operation which, under some additional conditions (see Fact 8), leads to a constant curvature metric again. Having in mind the relationship between constant curvature metrics and pro-Frobenius algebras explained in Proposition 8.1, we now describe an algebraic analog of warped product for pro-Frobenius algebras.

Take two pro-Frobenius algebras $(\mathfrak{a}, b, K)$ and $(\hat{\mathfrak{a}}, \hat{b}, \hat{K})$ with the following additional properties:

(i1) there is a distinguished element $m \in \mathfrak{a}$ which generates a one-dimensional ideal so that $m \star u = \alpha(u)m$, $\alpha(u) \in \mathbb{R}$, for all $u \in \mathfrak{a}$;

(i2) $m \in \operatorname{Ker} b$, i.e. $b(m, u) = 0$ for all $u \in \mathfrak{a}$;

(i3) $\alpha(m) = 4\hat{K}$.

Notice that (i2) and (i3) are conditions on the first algebra $\mathfrak{a}$ only, whereas (i3) should be understood as an intertwining relation between $\mathfrak{a}$ and $\hat{\mathfrak{a}}$.

If (i1), (i2) and (i3) are fulfilled, we introduce the following commutative multiplication[8] $\star_{\mathsf{w}}$ on the direct product $\mathfrak{a} \times \hat{\mathfrak{a}}$ (here we use natural inclusions $\mathfrak{a}, \hat{\mathfrak{a}} \subset \mathfrak{a} \times \hat{\mathfrak{a}}$):

[8] Here the index w stands for *warped* product

- if $u_1, u_2 \in \mathfrak{a}$, then $u_1 \star_{\mathsf{w}} u_2 = u_1 \star u_2$, in other words, the product remains unchanged so that $\mathfrak{a}$ is a subalgebra of $(\mathfrak{a} \times \hat{\mathfrak{a}}, \star_{\mathsf{w}})$;
- if $u \in \mathfrak{a}$ and $v \in \hat{\mathfrak{a}}$, then $u \star_{\mathsf{w}} v = \alpha(u)v$;
- finally if $v_1, v_2 \in \hat{\mathfrak{a}}$, then $v_1 \star_{\mathsf{w}} v_2 = v_1 \star v_2 + \hat{b}(v_1, v_2)m$.

We will denote the algebra so obtained by $\mathfrak{a} \times_{\mathsf{w}} \hat{\mathfrak{a}}$.

Proposition 8.2. *Under the above conditions* (i1), (i2) *and* (i3)*, the triple* $(\mathfrak{a} \times_{\mathsf{w}} \hat{\mathfrak{a}}, b, K)$ *is a pro-Frobenius algebra. (Here by* b *we denote the bilinear form on* $\mathfrak{a} \times \hat{\mathfrak{a}}$ *which coincides with* b *on* $\mathfrak{a}$ *and has* $\hat{\mathfrak{a}}$ *as a kernel.)*

Proof. Straightforward verification. □

To the geometric language, this construction can be translated as follows.

Proposition 8.3. *Let* g *and* $\hat{g}$ *be two (contravariant) metrics written in pro-Frobenius coordinates* $u^1, \dots, u^{n_1}$ *and* $v^1, \dots v^{n_2}$*:*

$$\begin{aligned} g^{\alpha\beta} &= b^{\alpha\beta} + a_s^{\alpha\beta} u^s - 4K u^\alpha u^\beta, \\ \hat{g}^{ij} &= \hat{b}^{ij} \ + \ \hat{a}_r^{ij} v^r - 4\hat{K} v^i v^j, \end{aligned}$$

(recall that g *and* $\hat{g}$ *then automatically have constant curvatures* K *and* $\hat{K}$ *respectively). Assume that the following conditions are satisfied:*

$$\begin{aligned} &(\mathrm{i1}^*): \quad a_s^{in_1} = 0 \ \textit{for}\ s \neq n_1; \\ &(\mathrm{i2}^*): \quad b^{in_1} = 0 \ \textit{for}\ i = 1, \dots, n_1; \\ &(\mathrm{i3}^*): \quad a_{n_1}^{n_1 n_1} = 4\hat{K} \ \textit{(intertwining condition).} \end{aligned}$$

Then the warped product metric $g_{\mathsf{warp}} = g(u) + \frac{1}{u^{n_1}} \hat{g}(v)$ *has constant curvature and the coordinates* $(y^1, \dots, y^n)$*,* $n = n_1 + n_2$*, defined by*

$$\begin{aligned} y^1 = u^1, \dots, y^{n_1} = u^{n_1}, \\ y^{n_1+1} = u^{n_1} v^1, \dots, y^{n_1+n_2} = u^{n_1} v^{n_2}, \end{aligned}$$

are pro-Frobenius for g_{warp}*.*

Proof. The statement can be proved by straightforward verification. Alternatively, one can argue that Proposition 8.3 is a geometric counterpart of Proposition 8.2 with appropriate adjustments. □

We can naturally define a pro-Frobenius pencil as a family of metrics $\{g_\lambda\}$ having a common pro-Frobenius coordinate system $u^1, \dots, u^n$ so that

$$g_\lambda^{\alpha\beta} = b^{\alpha\beta}(\lambda) + a_s^{\alpha\beta}(\lambda)u^s - 4K(\lambda)u^\alpha u^\beta, \tag{85}$$

where for each λ, the triple $\big(b^{\alpha\beta}(\lambda), a_s^{\alpha\beta}(\lambda), K(\lambda)\big)$ defines a pro-Frobenius algebra and $g_{\lambda_1+\lambda_2}^{\alpha\beta} = g_{\lambda_1}^{\alpha\beta} + g_{\lambda_2}^{\alpha\beta}$ (in other words, $b^{\alpha\beta}(\lambda)$, $a_s^{\alpha\beta}(\lambda)$ and $K(\lambda)$ are linear in λ). It is straightforward to see from Proposition 8.1 that each pro-Frobenius pencil consists of Poisson compatible constant curvature metrics.

Equivalently, in algebraic language, we can talk about a pencil of pro-Frobenius algebras $\mathcal{P} = \{(\mathfrak{a}_\lambda, b_\lambda, K_\lambda)\}$. Observe two simple properties of pro-Frobenius pencils.

First of all notice that any affine transformation $u \mapsto Au + a$ preserves the form of the metric (85). More precisely, linear transformations preserve the triple $(\mathfrak{a}_\lambda, b_\lambda, K_\lambda)$ as an invariant algebraic object, whereas under shifts ($u \mapsto u + a$), the ingredients of any pro-Frobenius algebra $(\mathfrak{a}, b, K)$ change according to the following rule:

$$\begin{aligned} \xi \star_{\mathsf{new}} \eta &= \xi \star \eta - 4K\big(\xi \cdot a(\eta) + \eta \cdot a(\xi)\big), \\ b_{\mathsf{new}}(\xi, \eta) &= b(\xi, \eta) + a(\xi \star \eta) - 4K a(\xi) a(\eta), \\ K_{\mathsf{new}} &= K, \end{aligned}$$

leading to a new pro-Frobenius pencil $\mathcal{P}_{\mathsf{new}} = \big\{\big((a_\lambda)_{\mathsf{new}}, (b_\lambda)_{\mathsf{new}}, (K_\lambda)_{\mathsf{new}}\big)\big\}$. Since this transformation is quite simple and controllable, we will think of pencils $\mathcal{P}$ and $\mathcal{P}_{\mathsf{new}}$ related in this way as equivalent.

The second observation is that every pro-Frobenius pencil contains a Frobenius pencil of codimension one. In geometric language, this means that every pencil $\mathcal{P}$ of constant curvature metrics of type (85) contains a codimension-one subpencil of flat metrics $\mathcal{P}^{\text{flat}} \subset \mathcal{P}$. Indeed, the metrics g_λ from $\mathcal{P}^{\text{flat}}$ is defined by a single linear relation $K(\lambda) = 0$. For example, the (flat) AFF pencil (26) is a codimension one subpencil of the extended AFF pencil (77).

8.3 Geometric construction for general multi-block Frobenius pencils

The above discussion leads us to a rather natural, alternative geometric description of Frobenius pencils from Theorem 4. This description will automatically imply that metrics (31) admit a common Frobenius coordinate system given by (35) so that Theorem 4 follows.

We now explain how, using the *directed rooted in-forest* structure (see Section 4.1), one can construct the pencil (31) from elementary building blocks related to vertices of this in-forest graph.

This step-by-step construction works as follows. Without loss of generality, we assume that the graph F (see Section 4.1) related to a given pencil is connected, i.e., the in-forest consists of a single tree. We start with its leaves and move from each of them towards the root to construct branches and then the whole tree. At each step of this construction we obtain a collection of pro-Frobenius pencils $\mathcal{P}_1, \dots, \mathcal{P}_s$. The starting point is the collection of extended AFF-pencils related to all the leaves.

Geometrically, the (re)construction reduces to two simple operations, namely *flattened direct product* and *warped product*. Let us describe them.

- Flattened direct product.

  We choose some pencils $\mathcal{P}_{i_1}, \mathcal{P}_{i_2}, \dots, \mathcal{P}_{i_s}$, $s \geq 2$, from our collection and combine them into direct product $\mathcal{P}_{i_1} \times \mathcal{P}_{i_2} \times \dots \times \mathcal{P}_{i_s}$. The dimension of this new pencil is $\sum \dim \mathcal{P}_{i_k}$. However, in general, this pencil is not suitable for our purposes as direct product of constant curvature metrics is not a constant curvature metric unless all of them are flat. We know, however, that each pro-Frobenius pencil contains a Frobenius pencil $\mathcal{P}^{\mathsf{flat}}_{i_k} \subset \mathcal{P}_{i_k}$ of codimension one (unless $\mathcal{P}_{i_k}$ is Frobenius itself, i.e. consists of flat metrics so that $\mathcal{P}^{\mathsf{flat}}_{i_k} = \mathcal{P}_{i_k}$). We replace each $\mathcal{P}_{i_k}$ by $\mathcal{P}^{\mathsf{flat}}_{i_k}$ and take the direct product

  $$\mathcal{P}^{\mathsf{flat}}_{i_1} \times \mathcal{P}^{\mathsf{flat}}_{i_2} \times \dots \times \mathcal{P}^{\mathsf{flat}}_{i_s} = \left\{ \begin{pmatrix} g_{i_1} & & & \\ & g_{i_2} & & \\ & & \ddots & \\ & & & g_{i_s} \end{pmatrix}, \ g_{i_k} \in P^{\mathsf{flat}}_{i_k} \right\},$$

  which is still a Frobenius pencil (flattened direct product). Notice that the dimension of this pencil is now smaller than $\sum \dim \mathcal{P}_{i_k}$ due to flattening. In terms of coefficients of the polynomials $P_{i_k}(\cdot)$ (see Section 4.1), this operation corresponds to condition (iv): the highest order coefficients a_{n+1} must vanish.

- Warped product.

  Take an extended AFF pencil $\mathcal{F}$ and an arbitrary pro-Frobenuis pencil $\mathcal{P}$. Let $u^1, \dots, u^n$ be a common pro-Frobenius coordinate system for all $g_\lambda \in \mathcal{F}$ so that

  $$g^{\alpha\beta}_\lambda = b^{\alpha\beta}(\lambda) + a^{\alpha\beta}_s(\lambda) u^s - 4K(\lambda) u^\alpha u^\beta,$$

  Here $\lambda \in \mathbb{R}^{n+2}$ is a linear parameter of the extended AFF pencil $\mathcal{F}$ so that $b^{\alpha\beta}(\lambda)$, $a^{\alpha\beta}_s(\lambda)$ and $K(\lambda)$ are linear in λ. Recall that the space of parameters $\mathbb{R}^{n+2}$ for the extended AFF-pencil $\mathcal{F}$ is naturally identified with the space of polynomials $\mathbb{R}_{n+1}[t]$ so that we naturally set $\lambda = P(t) = a_{n+1}t^{n+1} + a_n t^n + \dots a_1 t + a_0$.

  Following Proposition 8.3, we now consider all warped product metrics of the form

  $$g_\lambda + \frac{1}{u^n}\,\hat{g}_\mu, \qquad g_\lambda \in \mathcal{F}, \ \hat{g}_\mu \in \mathcal{P}.$$

However, we need to ensure that Conditions (i1*), (i3*) and (i3*) are met, that is:

$$a_s^{in}(\lambda) = 0 \text{ for } s \neq n, \quad b^{in}(\lambda) = 0, \quad \text{and} \quad a_n^{nn}(\lambda) = 4\hat{K}.$$

The first two conditions are linear equations on λ. They both amount to the condition $a_0 = 0$ (cf. the first part of Condition (iii) in Section 4.1). Finally the third is an intertwining linear relation[9] between the parameters of the two pencils (as $\hat{K}$ depends linearly on the parameter μ of $\mathcal{P}$).

Imposing these two conditions, we obtain a new pro-Frobenius pencil (Proposition 8.3), which we will understand as *warped product* $\mathcal{F} \times_{\mathsf{w}} \mathcal{P}$ (recall that $\mathcal{F}$ is an extended AFF-pencil whereas $\mathcal{P}$ is an arbitrary pro-Frobenius pencil).

We can use these two operations to construct more and more complicated pro-Frobenius pencils starting from simple ones (in our case, we start from extended AFF-pencils). Moreover, at each step we construct a pro-Frobenius coordinate system for the new pencil from those of the initial pencils by using Proposition 8.3.

All the pencils from Theorem 4 can be obtained in this way. Indeed, if the tree F (defining a pencil from Theorem 4) is given, then the reconstruction procedure is quite natural. We start with the leaves and then, moving towards the root, at each step add one more vertex, say i. The result of this step will be a pro-Frobenius pencil $\mathcal{P}_i$ corresponding to this vertex (and keeping information from all the vertices k located above i, i.e. such that $i \prec k$). There are two essentially different cases:

1) If i is not a branching point, i.e., there is only one vertex j such that $i = \operatorname{next}(j)$, then we use the warped product operation. More precisely, for this vertex j, we have already a certain pro-Frobenius pencil $\mathcal{P}_j$ constructed previously. We take the extended AFF-pencil $\mathcal{F}_i$ and apply the warped product procedure for these two pencils

$$\mathcal{F}_i, \mathcal{P}_j \quad \overset{\mathsf{warp}}{\longrightarrow} \quad \mathcal{P}_i = \mathcal{F}_i \times_{\mathsf{w}} \mathcal{P}_j$$

to get a new pro-Frobenius pencil $\mathcal{P}_i$ that corresponds to the vertex i.

2) If i is a branching point, i.e., there are several vertices $j_1, \dots, j_k$ such that $i = \operatorname{next}(j_s)$, then we first apply the flattened direct product operation to the pro-Frobenius pencils $\mathcal{P}_{j_s}$ and then take the warped product with the extended AFF pencil $\mathcal{F}_i$:

$$\mathcal{F}_i, \mathcal{P}_{j_1}^{\mathsf{flat}} \times \dots \times \mathcal{P}_{j_s}^{\mathsf{flat}} \quad \overset{\mathsf{warp}}{\longrightarrow} \quad \mathcal{P}_i = \mathcal{F}_i \times_{\mathsf{w}} \left(\mathcal{P}_{j_1}^{\mathsf{flat}} \times \dots \times \mathcal{P}_{j_s}^{\mathsf{flat}}\right).$$

At the very end, when we come to the root of the whole tree, say $i = 1$, and obtain a pro-Frobenuis pencil $\mathcal{P}_1$, we need to perform one more operation, namely, making it flat

[9] If $\mathcal{P}$ is an extended Frobenius pencil in dimension $\hat{n}$, then in terms of polynomials $\lambda = P$ and $\hat{\lambda} = \hat{P}$, this relation takes the form $a_1 = \pm \hat{a}_{\hat{n}+1}$, cf. the second part of Condition (iii) from Section 4.1. Also, if $\mathcal{P}$ is flat, i.e. $\hat{K} = 0$, then this relation is $a_1 = 0$, cf. Condition (iv) from Section 4.1.

(cf. Condition (i) from Section 4.1):

$$\mathcal{P}_1 \quad \longrightarrow \quad \mathcal{P}_1^{\mathsf{flat}}.$$

This final, already Frobenius, pencil $\mathcal{P}_1^{\mathsf{flat}}$ is the one which corresponds to the given tree F in the sense of Theorem 4.

It is important to notice that each time as we perform a warped product operation, we simultaneously construct a common pro-Frobenius coordinate system for the new pencil $\mathcal{P}_i$. According to Proposition 8.3 we simply need to multiply some coordinates by u^n, which in our case is $\pm \det L_i$. In other words, the above construction leads us, step-by-step, to the Frobenius coordinate system (35). Theorem 4 is proved.

8.4 Algebraic reformulation of the classification Theorem 5

The above construction can be naturally reformulated in purely algebraic language. We now work with pencils of pro-Frobenius algebras, i.e., with linear families $\{(\mathfrak{a}_\lambda, b_\lambda, K_\lambda)\}_{\lambda\in\mathbb{R}^k}$. The building blocks of our construction are extended AFF-pencils which we will still denote by $\mathcal{F}_1, \dots, \mathcal{F}_B$ (we use a natural correspondence between pro-Frobenius pencils of metrics and algebras and basically identify them).

Our goal is to construct a (multi-block) pencil of Frobenius algebras starting from pro-Frobenuis pencils $\mathcal{F}_1, \dots, \mathcal{F}_B$. The below construction is an almost literal translation of the previous *geometric* Section 8.3 into *algebraic* language. This translation is straightforward due to one-to-one relationship between (pro-)Frobenius pencils of metrics and (pro-)Frobenius pencils of algebras explained above in Sections 8.1 and 8.2.

We start with two basic operations which we will apply to pro-Frobenius pencils. The first operation (we call it *flattening*) can be applied to any pro-Frobenius pencil $\mathcal{F} = \{(\mathfrak{a}_\lambda, b_\lambda, K_\lambda)\}_{\lambda\in\mathbb{R}^k}$ and is as follows:

$$\mathcal{F} = \{(\mathfrak{a}_\lambda, b_\lambda, K_\lambda)\}_{\lambda\in\mathbb{R}^k} \quad \mapsto \quad \mathcal{F}^{\mathsf{flat}} = \{(\mathfrak{a}_\lambda, b_\lambda)\}_{\lambda\in\mathbb{R}^k, K(\lambda)=0}$$

In other words, $\mathcal{F}^{\mathsf{flat}}$ is a subpencil of $\mathcal{F}$ defined by the relation $K(\lambda)=0$. As $K(\lambda)$ is a linear function in λ, we will get a subpencil of codimension one (unless $K(\lambda)\equiv 0$ and then $\mathcal{F}^{\mathsf{flat}} = \mathcal{F}$ because $\mathcal{F}$ itself is already *flat*[10]).

The second operation, *warped product*, can be applied to a pair of pro-Frobenius pencils $\mathcal{F}, \mathcal{P}$, the first of which $\mathcal{F} = (\mathfrak{a}_\lambda, b_\lambda, K_\lambda)\}_{\lambda\in\mathbb{R}^k}$ is the extended AFF-pencil, while the second one $\mathcal{P} = (\hat{\mathfrak{a}}_\mu, \hat{b}_\mu, \hat{K}_\mu)\}_{\mu\in\mathbb{R}^m}$ is arbitrary:

$$\mathcal{F}, \mathcal{P} \quad \mapsto \quad \mathcal{F} \times_{\mathsf{w}} \mathcal{P}$$

[10]We use "flat" to emphasise the relation with Riemannian metrics. If $K=0$, then the corresponding metric is flat.

This pencil will consists of warped products $(\mathfrak{a}_\lambda \times_{\mathsf{w}} \hat{\mathfrak{a}}_\mu, b_\lambda, K_\lambda)$ with the parameters $\lambda \in \mathbb{R}^k$ and $\mu \in \mathbb{R}^m$ appropriately chosen (in order for this operation to make sense, see Conditions (i1), (i2), (i3)). More precisely, recall that in the case of the extended AFF-pencil, the parameter $\lambda \in \mathbb{R}^{n+2} \simeq \mathbb{R}_{n+1}[t]$ is identified with a polynomial of degree $n+1$ (where $n = \dim \mathfrak{a}_\lambda$):

$$\lambda = P(t) = a_0 + a_1 t + \cdots + a_{n+1} t^{n+1}.$$

For each polynomial we construct a metric $g = P(L) g_0$ with L and g_0 given by (19). In coordinates $u^1, \dots, u^n$, this metric takes the form

$$g^{\alpha\beta} = b^{\alpha\beta} + a_s^{\alpha\beta} u^s - 4K u^\alpha u^\beta$$

leading to a certain pro-Frobenius algebra $(\mathfrak{a}, b, K)$ whose ingredients depend on $\lambda = P(t)$. Following the definition of the warped product of pro-Frobenius algebras, we set $m = e^n$ (the last basis vector in our standard presentation for the AFF-pencil). To fulfil Conditions (i1), (i2), (i3), we set $a_0 = 0$ (this gives (i1) and (i2) for each $\mathfrak{a}_\lambda$). To guarantee (i3) for $(\mathfrak{a}_\lambda, b_\lambda, K_\lambda)$ and $(\hat{\mathfrak{a}}_\mu, \hat{b}_\mu, \hat{K}_\mu)$, we need to set $a_1 = -4\hat{K}_\mu$.

Thus finally we get:

$$\mathcal{F} \times_{\mathsf{w}} \mathcal{P} = \{\mathfrak{a}_\lambda \times_{\mathsf{w}} \hat{\mathfrak{a}}_\mu, b_\lambda, K_\lambda\}_{a_0(\lambda)=0,\ a_1(\lambda)=-4\hat{K}_\mu,\ \lambda \in \mathbb{R}^{n+2},\ \mu \in \mathbb{R}^k}$$

Notice that the dimension of the underlying algebras in the pencil $\mathcal{F} \times_{\mathsf{w}} \mathcal{P}$ equals $\dim \mathfrak{a}_\lambda + \dim \hat{\mathfrak{a}}_\mu$, the sum of the dimensions of the corresponding algebras from $\mathcal{F}$ and $\mathcal{P}$. However, the dimension of the pencil $\mathcal{F} \times_{\mathsf{w}} \mathcal{P}$ itself equals $\dim \mathcal{F} + \dim \mathcal{P} - 2$ because of two additional linear relations $a_0 = 0$ and $a_1 = -4\hat{K}_\mu$.

Finally, we notice that for two (or several) Frobenius pencils $\mathcal{P}_1 = \{\mathfrak{a}_\lambda, b_\lambda\}_{\lambda \in \mathbb{R}^k}$ and $\mathcal{P}_2 = \{\hat{\mathfrak{a}}_\mu, \hat{b}_\mu\}_{\mu \in \mathbb{R}^m}$ we can naturally define the direct product

$$\mathcal{P}_1 \times \mathcal{P}_2 = \{\mathfrak{a}_\lambda \times \hat{\mathfrak{a}}_\mu, b_\lambda \times \hat{b}_\mu\}_{(\lambda,\mu) \in \mathbb{R}^{k+m}}$$

where $b_\lambda \times \hat{b}_\mu\big((u, \hat{u}), (v, \hat{v})\big) = b_\lambda(u, v) + \hat{b}_\mu(\hat{u}, \hat{v})$.

We now apply the above two operations, flattened direct product and warped product, to our building blocks.

For instance, we can take several "building blocks" $\mathcal{F}_{i_1}, \dots, \mathcal{F}_{i_s}$ and take the flattened direct product

$$\mathcal{P} = \mathcal{F}_{i_1}^{\mathsf{flat}} \times \cdots \times \mathcal{F}_{i_s}^{\mathsf{flat}}$$

Next, we can take an extended AFF-pencil $\mathcal{F}_\alpha$ and consider a warped product $\mathcal{F}_\alpha \times_{\mathsf{w}} \mathcal{P}$, and then repeat this operation with another extended AFF-pencil, i.e., $\mathcal{F}_\beta \times_{\mathsf{w}} (\mathcal{F}_\alpha \times_{\mathsf{w}} \mathcal{P})$. We can continue in this way, applying any combination of these two operations. If we end up with a certain pro-Frobenius pencil, we should not forget to make it flat (at the very final step) to get a desired Frobenius pencil. It is easy to notice that the whole process is governed by a graph, which is exactly the in-forest F discussed in Section 4.1.

The general classification theorem for Frobenius pencils (Theorem 5), in algebraic language, takes now the following form.

Theorem 5B. *Let $(\mathfrak{a}, b)$ and $(\bar{\mathfrak{a}}, \bar{b})$ be two compatible Frobenius algebras satisfying the following genericity condition: the operator R defined from the identity $b(Ru, v) = \bar{b}(u, v)$ has different eigenvalues. Then the corresponding Frobenius pencil $\{\lambda\mathfrak{a}+\mu\bar{\mathfrak{a}}, \lambda b+\mu\bar{b}\}_{(\lambda,\mu)\in\mathbb{R}^2}$ is a subpencil of a multi-block Frobenius pencil constructed from extended AFF-pencils by means of two operations, warped product and flattened direct product, as explained above.*

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