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SOME HOMOLOGICAL PROPERTIES OF BOREL TYPE IDEALS

JÜRGEN HERZOG, SOMAYEH MORADI*, MASOOMEH RAHIMBEIGI AND GUANGJUN ZHU

Abstract. We study ideals of Borel type, including k-Borel ideals and t-spread Veronese ideals. We determine their free resolutions and their homological shift ideals. The multiplicity and the analytic spread of equigenerated squarefree principal Borel ideals are computed. For the multiplicity, the result is given under an additional assumption which is always satisfied for squarefree principal Borel ideals. These results are used to analyze the behaviour of height, multiplicity and analytic spread of the homological shift ideals $\mathrm{HS}_j(I)$ as functions of j, when I is an equigenerated squarefree Borel ideal.

Introduction

Let K be a field, $S = K[x_1, \dots, x_n]$ the polynomial ring in n variables over K and $I \subset S$ a graded ideal. By a famous theorems of Galligo [14] and Bayer-Stillman [5], the generic initial ideal of I is Borel-fixed, that is, it is fixed under the action of the Borel subgroups of $GL(n, K)$. Moreover, if $\mathrm{char}(K) = 0$, then the Borel-fixed ideals are precisely the strongly stable ideals (also known as Borel ideals), see [15, Proposition 4.2.4]. Applying the Kalai stretching operator ([21] and [22]) to strongly stable ideals, one obtains squarefree strongly stable ideals, which were first considered in [2] and which play an important role in algebraic shifting theory.

In this paper we focus our attention on k-Borel ideals. Let k be a positive integer. We call a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ *k-bounded*, if $a_i \leq k$ for $i = 1, \dots, n$. A monomial ideal which is generated by k-bounded monomials is said to be *k-Borel*, if for any k-bounded monomial $u \in I$, and for any $j \in \mathrm{supp}(u)$ and $i < j$, we have $x_i(u/x_j) \in I$, provided $x_i(u/x_j)$ is again k-bounded. Thus k-Borel ideals are strongly stable ideals respecting the k-boundedness. The squarefree strongly stable ideals are just the 1-Borel ideals. Other interesting restrictions of strongly stable ideals have been considered in [8] and [10].

Eliahou and Kervaire [12] gave an explicit free resolution not only for strongly stable ideals, but also for the larger class of stable ideals. This important result allowed Bigatti [6] and Hulett [20] to show that if I is a graded ideal, then the graded Betti numbers of I are bounded above by the graded Betti numbers of the corresponding lex-ideal.

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In Section 1 we state and prove some basic properties of k-Borel ideals. For any finite set $\{u_1, \dots, u_m\}$ of k-bounded monomials there exists a unique smallest k-Borel ideal containing $u_1, \dots, u_m$, which we denote by $B_k(u_1, \dots, u_m)$. For k-bounded monomials u, v of the same degree, we set $v \preceq_k u$ if $v \in B_k(u)$. This binary relation defines a partial order on the set of k-bounded monomials of the same degree. In Proposition 1.6 we show that the height of $B_k(u_1, \dots, u_m)$ is given by $\max\{\min(u_1), \dots, \min(u_m)\}$, where we set $\min(u) = \min\{i \colon x_i | u\}$ for a monomial u.

The graded Betti numbers of a k-Borel ideal I are computed in Section 2, see Corollary 2.3. This is achieved by Theorem 2.1 which provides a natural K-basis of the Koszul homology $H(x_1, \dots, x_n; S/I)$. Interestingly, the basis consists of the homology classes of monomial Koszul cycles. The proof of Theorem 2.1 also yields a K-basis of $H(x_1, \dots, x_i; S/I)$ for any initial sequence $x_1, \dots, x_i$, $1 \leq i \leq n$. By using Corollary 2.3 it is shown in Corollary 2.5 that if I is a k-Borel ideal generated in a single degree, then I has linear quotients for any order of the monomial generators of I, which extends the partial order $\prec_k$. This result is complemented by Proposition 2.6 in which we show that any k-Borel ideal (even if it is not generated in a single degree) has linear quotients with respect to the lexicographical order of its generators. The explicit resolution of a k-Borel ideal I can be given, due to the fact that I has a regular decomposition function, as shown in Proposition 2.7. This proposition together with [11, Theorem 3.10] also implies that the resolution of I is cellular and supported on a regular CW-complex.

In Section 3, we determine the multiplicity and the analytic spread of squarefree Borel ideals, which by definition are just the 1-Borel ideals. The results refer to the block decomposition of the support of a squarefree monomial. This concept was first introduced in [10]. Let $u = x_{i_1} x_{i_2} \cdots x_{i_d}$ be a squarefree monomial with $i_1 < i_2 < \cdots < i_d$. A *block* of u is a subset $\{i_l, i_{l+1}, \cdots, i_{l+k}\}$ such that $i_{l+j} = i_l + j$ for $j = 1, \dots, k$. A block of u is called *maximal* if it is not properly contained in any other block of u. Note that $\mathrm{supp}(u)$ has a unique decomposition into maximal blocks. In other words, $\mathrm{supp}(u) = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$, where each B_i is a maximal block and $\max\{j : j \in B_i\} < \min\{j : j \in B_{i+1}\} - 1$ for all i. It is shown in Theorem 3.1 that if I is a squarefree principal Borel ideal with the Borel generator u, then the multiplicity of S/I is given by $\binom{\max(B_1)}{|B_1|-1}$, where B_1 is the first block in the block decomposition of u. This result can be generalized to squarefree Borel ideals with several Borel generators, provided there is one of the Borel generators whose first block is contained in the first blocks of all the other Borel generators, see Theorem 3.2. The analytic spread of a squarefree Borel ideal I with Borel generators $u_1, \dots, u_m$ is also determined by the first blocks of the block decomposition, as shown in Theorem 3.3. If B_{i1} is the first block in the block decomposition of u_i, then the analytic spread of I is $n = \max\{\max(u_i) : 1 \leq i \leq m\}$, if $1 \notin \bigcap_{i=1}^m B_{i1}$, and is equal to $n - |\bigcap_{i=1}^m B_{i1}|$, otherwise.

Section 4 is devoted to the study of the homological shift ideals of equigenerated squarefree Borel ideals. For the moment, let I be any monomial ideal and let $\mathbb{F}$ be its minimal multigraded free S-resolution. Then $F_j = \bigoplus_{k=1}^{b_j} S(-\mathbf{a}_{jk})$ with each $\mathbf{a}_{jk}$ an integer vector in $\mathbb{Z}^n$ with non-negative entries. The monomial ideal $\mathrm{HS}_j(I)$ generated by the monomials $\mathbf{x}^{\mathbf{a}_{jk}}$, $k = 1, \dots, b_j$ is called the jth homological shift ideal of I. These ideals provide some extra information about the nature of the multigraded free resolution of I. Homological shift ideals have been studied in [3],[4] and [17]. It is conjectured that if I has linear resolution, then $\mathrm{HS}_j(I)$ has linear resolution for all j, This conjecture is still widely open, and only proved in some special cases. Here we study how the height, the analytic spread and the multiplicity of $\mathrm{HS}_j(I)$ behave as a function of j, when I is an equigenerated squarefree Borel ideal. Based on Proposition 4.1, we describe in Corollary 4.2 the minimal set of monomial generators of $\mathrm{HS}_j(I)$. From this description it can be seen that $\mathrm{HS}_j(I)$ is again a squarefree Borel ideal and that $\mathrm{HS}_1(\mathrm{HS}_j(I)) = \mathrm{HS}_{j+1}(I)$ for all $j < \operatorname{proj\,dim} S/I$. Having these information it is not so hard to see that the height and the analytic spread of $\mathrm{HS}_j(I)$ is a non-decreasing function of j, see Corollary 4.3. The multiplicity function of homological shift ideals behaves differently. Indeed, in Proposition 4.4 we show that if I is a squarefree principal Borel ideal, the multiplicity of $\mathrm{HS}_j(I)$ is a unimodal function of j.

In the last section of this paper, Section 5, we consider the homological shift ideals of t-spread Veronese ideals $I_{n,d,t}$ of degree d in n variables. These ideals naturally generalize squarefree Veronese ideals, and have been studied in several papers (see for example [13] and [1]). In particular, it is known that $I_{n,d,t}$ has linear quotients. A monomial $x_{i_1}x_{i_2}\cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called *t-spread* if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq n$, and $I_{n,d,t}$ is the ideal generated by all t-spread monomials of degree d in $K[x_1, \dots, x_n]$. Our Theorem 5.1 describes for each i the minimial set of monomial generators of $\mathrm{HS}_i(I_{n,d,t})$. This result is used to show that $\mathrm{HS}_1(I_{n,d,t})$ has linear quotients, see Theorem 5.2. We expect that is true also for the higher homological shift ideals. Actually, one could expect that for any equigenerated monomial ideal with linear quotients, all homological shift ideals have linear quotients.

1. k-Borel ideals

Throughout this paper $S = K[x_1, \dots, x_n]$ is a polynomial ring over a field K, $\mathrm{Mon}(S)$ denotes the set of monomials in S and for a monomial u, we set $\mathrm{supp}(u) = \{i :\ x_i|u\}$. For a monomial ideal I, the unique minimal set of monomial generators of I is denoted by $G(I)$. A monomial ideal I is called *k-bounded*, if it is generated by k-bounded monomials.

In this section we define k-Borel ideals as a generalization of squarefree strongly stable ideals and study their basic properties.

Definition 1.1. Let k be a positive integer and I be a k-bounded monomial ideal. We say that I is *k-Borel*, if for any k-bounded monomials $u \in I$ the following holds: if $j \in \mathrm{supp}(u)$ and $i < j$, then $x_i(u/x_j) \in I$ provided that $x_i(u/x_j)$ is k-bounded

Lemma 1.2. *Let I be a k-Borel ideal. The following conditions are equivalent:*

(i) *I is a k-Borel ideal,*
(ii) *For any $u \in G(I)$, any $j \in \mathrm{supp}(u)$ and $i < j$, if $x_i(u/x_j)$ is k-bounded, then $x_i(u/x_j) \in I$.*

Proof. (i) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i): Let $u \in I$ be a k-bounded monomial, $j \in \operatorname{supp}(u)$ and $i < j$ such that $x_i(u/x_j)$ is k-bounded. Then there exists $w \in G(I)$ such that $w \mid u$. If $x_j \nmid w$, then $w \mid x_i(u/x_j)$ and $x_i(u/x_j) \in I$. If $x_j \mid w$, then $x_i(w/x_j)$ divides $x_i(u/x_j)$. Hence $x_i(w/x_j)$ is k-bounded as well. So by our assumption it belongs to I which implies that $x_i(u/x_j) \in I$. $\square$

It is clear that the intersection of k-Borel ideals is k-Borel as well. Hence for k-bounded monomials $u_1, \ldots, u_m$, there exists a unique smallest k-Borel ideal containing $u_1, \ldots, u_m$. Indeed, the set of k-Borel ideals containing $u_1, \ldots, u_m$ is not empty, because the maximal ideal of S belongs to this set. The intersection of all k-Borel ideals containing $u_1, \ldots, u_m$ is the k-Borel ideal we are looking for. We denote this ideal by $B_k(u_1, \ldots, u_m)$ and we call the monomials $u_1, \ldots, u_m$ the *Borel generators* of this ideal. A k-Borel ideal with one Borel generator is called *principal k-Borel*.

We set $v \preceq_k u$, if v and u are k-bounded monomials of the same degree and $v \in B_k(u)$. Note that $\preceq_k$ is a partial order on the set of k-bounded monomials of the same degree. We have

$$B_k(u_1, \ldots, u_m) = (v \in \operatorname{Mon}(S) : \ v \preceq_k u_i \text{ for some } 1 \leq i \leq m).$$

In particular, it follows that $B_k(u_1, \ldots, u_m) = \sum_{i=1}^m B_k(u_i)$.

Let I be a strongly stable ideal. In the sequel we will call it a Borel ideal. Then for $k \gg 0$, I is k-Borel, since large k imposes no conditions on the exponents of the generators. It follows that for any monomials $u_1, \ldots, u_m$, there is a unique smallest Borel ideal, denoted $B(u_1, \ldots, u_m)$, which contains $u_1, \ldots, u_m$.

For the monomials $u = x_{i_1} \cdots x_{i_d}$ and $v = x_{j_1} \cdots x_{j_d}$ with $i_1 \leq \cdots \leq i_d$ and $j_1 \leq \cdots \leq j_d$, we set $v \preceq u$ if $j_\ell \leq i_\ell$ for any $1 \leq \ell \leq d$. The following easy but useful lemma explains the generators of a principal Borel ideal.

Lemma 1.3. *Let u and v be monomials of the same degree. Then $v \in B(u)$ if and only if $v \preceq u$.*

Proof. Let $u = x_{i_1} \cdots x_{i_d}$, $v = x_{j_1} \cdots x_{j_d}$ and $v \preceq u$. Then the number $\delta(v, u) = \sum_{k=1}^d (i_k - j_k)$ is non-negative and $\delta(v, u) = 0$ if and only of $u = v$. We proceed by induction on $\delta(v, u)$ to show that $v \in B(u)$. If $\delta(v, u) = 0$, then the assertion is trivial. Assume now that $\delta(v, u) > 0$. Then $i_k - j_k > 0$ for some k. Let k be the smallest such integer. Then $i_{k-1} = j_{k-1} < j_k < i_k$ and it follows that $u' = x_{i_k - 1}(u/x_{i_k}) \in B(u)$. Furthermore, $v \preceq u' \prec u$ and $\delta(v, u') < \delta(v, u)$. Our induction hypothesis implies that $v \in B(u') \subseteq B(u)$. Conversely, let $v \in B(u)$. It can be easily seen that the ideal $I = (w \in \operatorname{Mon}(S) : \ w \preceq u)$ is a Borel ideal containing u. This implies that $B(u) \subseteq I$ and hence $v \preceq u$. $\square$

For a monomial ideal I, we set $I^{\leq k} = (u \in G(I) : \ u \text{ is } k\text{-bounded})$ and call it the *k-bounded part of I*. The next result shows that a k-bounded monomial ideal I is a k-Borel ideal if and only if it is the k-bounded part of a Borel ideal.

Lemma 1.4. *Let $u_1, \ldots, u_m$ be k-bounded monomials. Then*
$$B_k(u_1, \ldots, u_m) = B(u_1, \ldots, u_m)^{\leq k}.$$

Proof. Let u be a k-bounded monomial. By [16, Lemma 2.4], $B(u)^{\leq k} = B_k(u)$. Therefore, since $B(u_1, \ldots, u_m) = \sum_{i=1}^m B(u_i)$, it follows that
$$B(u_1, \ldots, u_m)^{\leq k} = (\sum_{i=1}^m B(u_i))^{\leq k} = \sum_{i=1}^m B(u_i)^{\leq k} = \sum_{i=1}^m B_k(u_i) = B_k(u_1, \ldots, u_m).$$
□

We use the following fact repeatedly in the later proofs.

Corollary 1.5. *Let k be a positive integer and u and v be k-bounded monomials of the same degree. Then $v \preceq_k u$ if and only if $v \preceq u$.*

Proof. We have $v \preceq_k u$ if and only if $v \in B_k(u)$. Also by Lemma 1.3, $v \preceq u$ if and only if $v \in B(u)$. Then the desired result will follow from Lemma 1.4 (or even from [16, Lemma 2.4]). □

For a monomial u we set
$$\min(u) = \min\{i \colon i \in \operatorname{supp}(u)\} \quad \text{and} \quad \max(u) = \max\{i \colon i \in \operatorname{supp}(u)\}.$$
The next result describes the height of a k-Borel ideal in terms of its Borel generators.

Proposition 1.6. *Let $I = B_k(u_1, \ldots, u_m)$. Then*
$$\operatorname{height}(I) = \max\{\min(u_1), \ldots, \min(u_m)\}.$$

Proof. First we consider the case $m = 1$, and hence we may assume that $I = B_k(u)$ where $u = x_{i_1}^{a_1} x_{i_2}^{a_2} \cdots x_{i_d}^{a_d}$ with $i_1 < i_2 < \cdots < i_d$ and $a_1 > 0$. We show that $\operatorname{height}(I) = i_1$. Since I is a monomial ideal, all minimal prime ideals of I are generated by variables. Let $P = (x_1, x_2, \ldots, x_{i_1})$, and let $v \in G(I)$. By Corollary 1.5, $v \preceq u$. It follows that $1 \leq \min(v) \leq i_1$ by the definition of the partial order $\preceq$. Therefore, $v \in P$ for all $v \in I$. This shows that $I \subseteq P$ and proves that $\operatorname{height}(I) \leq i_1$.

Suppose now $\operatorname{height}(I) < i_1$. Then there exists a monomial prime ideal Q containing I with less than i_1 generators. We show that this is not possible. Indeed, we show that there exist positive integers $j_1 < j_2 < \cdots < j_d$ with $j_k \leq i_k$ for all k such that $x_{j_k} \notin Q$ for $k = 1, \ldots, d$. Then we get $v = x_{j_1}^{a_1} x_{j_2}^{a_2} \cdots x_{j_d}^{a_d} \in I$ with $v \notin Q$, which is a contradiction.

We construct $j_1 < j_2 < \cdots < j_t$ for $t = 1, \ldots, d$, inductively. Since $\mu(Q) < i_1$, there exists $j_1 \leq i_1$ such that $x_{j_1} \notin Q$. Assume now that the sequence $j_1, \ldots, j_t$ with $t < d$, has already been constructed, satisfying $j_1 < j_2 < \cdots < j_t$, $j_r \leq i_r$ and $x_{j_r} \notin Q$ for $r = 1, \ldots, t$. Since $i_{t+1} \geq i_1 + t$, it follows that $A = \{1, \ldots, i_{t+1}\} \setminus \{j_1, \ldots, j_t\}$ has at least i_1 elements. Therefore, since $\mu(Q) < i_1$, there exists $l \in A$ such that $x_l \notin Q$ and $l \leq i_{t+1}$. If $l > j_t$, then we can choose $j_{t+1} = l$. Otherwise, there exists a smallest integer $r \leq t$ such that $l < j_r$. Then we rename the elements $j_1, \ldots, j_t$, and let $j'_s = j_s$ for $s < r$, $j'_r = l$ and $j'_{s+1} = j_s$ for $s = r, \ldots, t$. This new sequence of length $t + 1$ satisfies all the requirements.

Finally, we deal with the case that $m > 1$. Let $h = \max\{\min(u_1), \dots, \min(u_m)\}$. Then from what we have seen before, it follows that $I \subseteq (x_1, \dots, x_h)$. This shows that height $(I) \leq h$. Assume that height $(I) < h$. Then there exists a monomial prime ideal Q containing I with $\mu(Q) < h$. Let r be such that $\min(u_r) = h$. Since $B_k(u_r) \subset I \subseteq Q$, we obtain a contradiction to the result we proved for $m = 1$. □

2. The resolution of k-Borel ideals

Let $I \subset S = K[x_1, \dots, x_n]$ be a graded ideal. The graded Betti numbers of I can be computed by means of the Koszul homology $H_i(\mathbf{x}; S/I)$ with $\mathbf{x} = x_1, \dots, x_n$. Indeed, there is an isomorphism $H_i(\mathbf{x}; S/I) \cong \mathrm{Tor}_i^S(K, S/I)$ of graded K-vector spaces. This isomorphism is even $\mathbb{Z}^n$-graded. In order to abbreviate notation we set $H_i(j) = H_i(x_n, x_{n-1} \dots, x_j; S/I)$. With this notation introduced we have $\beta_{i,j}(S/I) = \dim_K H_i(1)_j$. In order to compute the graded Betti numbers of S/I, we determine a K-basis of $H_i(1)_{\mathbf{a}}$ for each i and $\mathbf{a} \in \mathbb{Z}^n$. We will apply an inductive argument. For this reason, it is advisable to even determine a K-basis of the K-vector space $H_i(j)_{\mathbf{a}}$ for all i, j and $\mathbf{a} \in \mathbb{Z}^n$. The homology class of a cycle z in the Koszul complex $K_i(j) = K_i(x_n, \dots, x_j; S/I)$ is an element of $H_i(j)$ and will be denoted by $[z]_j$. When $j = 1$, we simply write $[z]$.

The Koszul complex $K(\mathbf{x}; S/I)$ is a complex of free S/I-modules. Let $e_1, \dots, e_n$ be the basis of $K_1(\mathbf{x}; S/I)$ with $\partial(e_i) = x_i$ for all i. Then the elements e_F with $F \subset [n]$ and $|F| = i$ form the basis of $K_i(\mathbf{x}; S/I)$. Here, for $F = \{j_1 < j_2 < \dots < j_i\}$, e_F denotes the wedge product $e_{j_1} \wedge \dots \wedge e_{j_i}$ (cf. [15, Appendix A.3]). In $K_i(\mathbf{x}; S/I)$ each basis element is annihilated by I. Thus for any $f \in S$, $fe_F = \bar{f}e_F$, where $\bar{f} = f + I$. We call a vector $\mathbf{a} \in \mathbb{Z}^n$, *$k$-bounded*, if $\mathbf{a} = (a_1, \dots, a_n)$ with $0 \leq a_i \leq k$ for $i = 1 \dots, n$. The multidegree of a monomial u is denoted by $\mathrm{Deg}(u)$.

Theorem 2.1. *Let $I \subset S = K[x_1, \dots, x_n]$ be a k-Borel ideal, and let $\mathbf{a} \in \mathbb{Z}^n$ be a k-bounded vector. Then for $i > 1$, $H_i(j)_{\mathbf{a}}$ has a K-basis consisting of the homology classes of the cycles*

$$u' e_F \wedge e_{m(u)}, \quad u \in G(I) \quad \textit{with}$$

$$F \subseteq [n], \quad |F| = i - 1, \quad j \leq \min(F), \quad \max(F) < m(u) \quad \textit{and} \quad \mathrm{Deg}(\mathbf{x}^F u) = \mathbf{a}.$$

Here, $m(u) = \max(u)$ and $u' = u/x_{m(u)}$. Moreover, the K-basis of $H_1(j)_{\mathbf{a}}$ consists of the homology classes of the cycles $u' e_{m(u)}$, $u \in G(I)$ with $j \leq m(u)$ and $\mathrm{Deg}(u) = \mathbf{a}$.

Proof. We first notice if $u' e_F \wedge e_{m(u)}$ is a cycle in $K_i(j)_{\mathbf{a}}$ for a monomial $u \in I$, then $u' e_F \wedge e_{m(u)} = 0$, if $u \notin G(I)$. In fact, suppose that $u \notin G(I)$. Then $u = wv$ with $v \in G(I)$ and $w \neq 1$ a monomial. If $m(w) \geq m(v)$, then $u' = w'v \in I$ and then $u' e_F \wedge e_{m(u)} = 0$. Now, let $m(w) < m(v)$. Then $u' = wv'$ and there exists $j < m(u)$ such that x_j divides w, and hence $u' = (w/x_j)(x_j v')$. Since $x_j v'$ divides u and since u is k-bounded, it follows that $x_j v'$ is also k-bounded. Finally, since I is k-Borel and $x_j v' \preceq v$, we conclude that $x_j v' \in I$, and hence $u' \in I$. Thus $u' e_F \wedge e_{m(u)} = 0$, as desired. The conclusion is that for cycles of the form $u' e_F \wedge e_{m(u)}$ whose homology class is not zero we have $u \in G(I)$.

We prove the theorem for $H_i(j)$ by backward induction starting with $j = n$. We have $H_i(n)_{\mathbf{a}} = 0$ if $i > 1$ and $H_1(n) = (I : x_n)e_n$. It follows that if $H_1(n)_{\mathbf{a}} \neq 0$, then there exist a (unique) $u \in G(I)$ with $m(u) = n$ and $\mathrm{Deg}(u) = \mathbf{a}$, and then $[u'e_n]_n$ is the K-basis of $H_1(n)_{\mathbf{a}}$. This proves the assertion for $j = n$. Now let $j < n$, and assume that the theorem holds for $j + 1$.

We proceed by induction on i to show that $H_i(j)_{\mathbf{a}}$ has the desired K-basis. In order to prove this result for $i = 1$, we consider the exact sequence of Koszul homology

$$H_2(j)_{\mathbf{a}} \to H_1(j+1)_{\mathbf{a}-\varepsilon_j} \to H_1(j+1)_{\mathbf{a}} \to H_1(j)_{\mathbf{a}} \to H_0(j+1)_{\mathbf{a}-\varepsilon_j} \to H_0(j+1)_{\mathbf{a}}.$$

Here, ε_j is the jth standard unit-vector. Let $U = \mathrm{Ker}(H_0(j+1)_{\mathbf{a}-\varepsilon_j} \to H_0(j+1)_{\mathbf{a}})$ and $V = \mathrm{Im}(H_1(j+1)_{\mathbf{a}} \to H_1(j)_{\mathbf{a}})$. Then a K-basis of V together with a preimage in $H_1(j)_{\mathbf{a}}$ of a K-basis of U establishes a K-basis of $H_1(j)_{\mathbf{a}}$.

The map $H_0(j+1) \to H_0(j+1)$ is multiplication by x_j, and $H_0(j+1) = S/(I, x_{j+1}, \ldots, x_n) = S'/I'$, where $S' = K[x_1, \ldots, x_j]$ and I' is a monomial ideal with

$$G(I') = \{u \in G(I)\colon\ m(u) \le j\}.$$

Thus, if $U \ne 0$, then U has the K-basis consisting of the residue class of u', where $u \in G(I)$ with $m(u) = j$ and $\mathrm{Deg}(u) = \mathbf{a}$. Its preimage in $H_1(j)_{\mathbf{a}}$ is $[u'e_j]_1$. By assumption, $H_1(j+1)_{\mathbf{a}}$ is generated by the elements $[u'e_{m(u)}]_{j+1}$ with $j + 1 \le m(u)$ and $\mathrm{Deg}(u) = \mathbf{a}$. Since $H_1(j+1)_{\mathbf{a}} \to H_1(j)_{\mathbf{a}}$ maps $[u'e_{m(u)}]_{j+1}$ to $[u'e_{m(u)}]_j \in H_1(j)_{\mathbf{a}}$, the proof for $i = 1$ is completed. Here we use that $H_1(j+1)_{\mathbf{a}} \to H_1(j)_{\mathbf{a}}$ is injective, because $H_2(j)_{\mathbf{a}} \to H_1(j+1)_{\mathbf{a}-\varepsilon_j}$ is surjective, see the claim below. The injectivity of the mentioned map implies that $H_1(j+1)_{\mathbf{a}}$ is isomorphic to its image V.

Now let $i > 1$, and consider the exact sequence

$$H_{i+1}(j)_{\mathbf{a}} \to H_i(j+1)_{\mathbf{a}-\varepsilon_j} \to H_i(j+1)_{\mathbf{a}} \to H_i(j)_{\mathbf{a}} \to H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j} \to H_{i-1}(j+1)_{\mathbf{a}}.$$

By induction hypothesis, if $i - 1 \ge 1$, then $H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$ is generated by the homology classes $[u'e_F \wedge e_{m(u)}]_{j+1}$, where $u \in I$ is a monomial, $|F| = i - 1$, $j + 1 \le \min(F)$, $\max(F) < m(u)$ and $\mathrm{Deg}(\mathbf{x}^F u) = \mathbf{a} - \varepsilon_j$, together with the homology classes $[u' \wedge e_{m(u)}]_{j+1}$ with $j + 1 \le m(u)$ and $\mathrm{Deg}(u) = \mathbf{a} - \varepsilon_j$, if $i - 1 = 1$.

We claim that the map $H_i(j)_{\mathbf{a}} \to H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$ is surjective for all $i - 1 \ge 1$. Indeed, given one of the generators $[u'e_F \wedge e_{m(u)}]_{j+1}$ of $H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$, we consider the element $z = u'e_G \wedge e_{m(u)}$ with $G = F \cup \{j\}$. Note that $F = \emptyset$, if $i - 1 = 1$. Applying the Koszul differential ∂ to z we get $\partial(z) = \pm e_j \wedge \partial(u'e_F \wedge e_{m(u)}) \pm x_j u'e_F \wedge e_{m(u)}$. The first summand is zero, since $u'e_F \wedge e_{m(u)}$ is a cycle representing the homology class $[u'e_F \wedge e_{m(u)}]_{j+1}$. Since $\mathrm{Deg}(\mathbf{x}^F u) = \mathbf{a} - \varepsilon_j$, and since $\mathbf{a}$ is k-bounded, it follows that the exponent of x_j in u is $< k$. The inequality $j \le m(u)$ and the fact that I is k-Borel imply that $x_j u' \in I$. Here we use again the k-Borel property. Therefore, also the second summand $x_j u'e_F \wedge e_{m(u)}$ is zero. We conclude that z is a cycle, and hence $[u'e_G \wedge e_{m(u)}]_j \in H_i(j)_{\mathbf{a}}$. Since $[u'e_G \wedge e_{m(u)}]_j$ is mapped to $\pm[u'e_F \wedge e_{m(u)}]_{j+1}$ via the map $H_i(j)_{\mathbf{a}} \to H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$, we see that $H_i(j)_{\mathbf{a}} \to H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$ is surjective, as desired.

Therefore, the previous exact sequence splits for all $i - 1 \ge 1$ into the short exact sequence

$$0 \to H_i(j+1)_{\mathbf{a}} \to H_i(j)_{\mathbf{a}} \to H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j} \to 0.$$

This implies that $H_i(j)_{\mathbf{a}}$ has K-basis consisting of the images of the basis elements of $H_i(j+1)_{\mathbf{a}}$ together with the preimages of the basis elements of $H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$. By induction hypothesis, $H_i(j+1)_{\mathbf{a}}$ has the basis elements $[u'e_F \wedge e_{m(u)}]_{j+1}$ satisfying the conditions described in the theorem. The image of $[u'e_F \wedge e_{m(u)}]_{j+1}$ in $H_i(j)_{\mathbf{a}}$ is $[u'e_F \wedge e_{m(u)}]_j$. Also by induction, the basis elements $H_{i-1}(j+1)_{\mathbf{a}-\varepsilon_j}$ are known to be $[u'e_F \wedge e_{m(u)}]_{j+1}$ with the side conditions given in the theorem. A preimage of $[u'e_F \wedge e_{m(u)}]_{j+1}$ in $H_i(j)_{\mathbf{a}}$ is $[u'e_G \wedge e_{m(u)}]_j$ with $G = F \cup \{j\}$. From this we see that $H_i(j)_{\mathbf{a}}$ has a K-basis as described in the theorem. $\square$

Corollary 2.2. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a k-Borel ideal. Then for $i > 0$, a basis for $H_i(\mathbf{x}; S/I)$ is given by the homology classes $u'e_F \wedge e_{m(u)}$ with*

$$u \in G(I), \quad |F| = i-1, \quad \max(F) < m(u) \text{ and } \operatorname{Deg}(\mathbf{x}^F u) \text{ is } k\text{-bounded.}$$

Proof. Since $H_i(\mathbf{x}; S/I) \cong \operatorname{Tor}_i(K; S/I)$, it follows that $H_i(\mathbf{x}; S/I)$ is a multigraded vector space whose graded components are $H_i(1)_{\mathbf{a}}$. Since the exponent vectors of all monomials of $G(I)$ are k-bounded, the shifts in the multigraded free resolution of S/I are also k-bounded. This implies that $H_i(1)_{\mathbf{a}} = 0$ if $\mathbf{a}$ is not k-bounded, see [7, Theorem 3.1]. Hence

$$H_i(\mathbf{x}; S/I) = \bigoplus_{\mathbf{a}} H(1)_{\mathbf{a}},$$

where the direct sum is taken over all k-bounded integral vectors $\mathbf{a}$. Therefore, the result follows from Theorem 2.1. $\square$

For $\mathbf{a} \in \mathbb{Z}^n$ with $\mathbf{a} = (a_1, \ldots, a_n)$ we set $|\mathbf{a}| = \sum_{i=1}^n a_i$. For a monomial $u = x_1^{a_1} \cdots x_n^{a_n}$ we set $\deg_{x_i}(u) = a_i$ for $1 \le i \le n$. With this notation introduced we have

Corollary 2.3. *Let $I \subset S = K[x_1, \ldots, x_n]$ be a k-Borel ideal. Then*

$$\beta_{i,i+j}(I) = \sum_{u \in G(I),\, \deg(u) = j} \binom{m(u) - L(u) - 1}{i},$$

where $m(u) = \max(u)$ and $L(u) = |\{1 \le \ell < m(u) \colon\ \deg_{x_\ell}(u) = k\}|$.

Proof. Let $\gamma_{u,i} = |\{F \subseteq [n] \colon\ |F| = i-1,\ \max(F) < m(u),\ \operatorname{Deg}(\mathbf{x}^F u) \text{ is } k\text{-bounded}\}|$. By Corollary 2.2, we have

$$\beta_{i,i+j-1}(S/I) = \sum_{u \in G(I),\, \deg(u) = j} \gamma_{u,i}.$$

Thus $\beta_{i,i+j}(I) = \beta_{i+1,i+j}(S/I) = \sum_{u \in G(I),\, \deg(u)=j} \gamma_{u,i+1}$. One can see that $\gamma_{u,i+1} = \binom{m(u)-L(u)-1}{i}$. $\square$

A monomial ideal $I \subset S$ with $G(I) = \{u_1, u_2, \ldots, u_m\}$ is said to have *linear quotients* with respect to the order $u_1, u_2, \ldots, u_m$ if for any $2 \le j \le m$, the colon ideal $(u_1, \ldots, u_{j-1}) : (u_j)$ is generated by some variables. For any j we define $\operatorname{set}(u_j) = \{i : x_i \in (u_1, \ldots, u_{j-1}) : (u_j)\}$.

We apply Corollary 2.3 to show the linear quotients property for equigenerated k-Borel ideals. For this we need

Lemma 2.4. *Let I be a monomial ideal generated in a single degree with $G(I) = \{u_1, \ldots, u_m\}$, and suppose that $I_j = (u_1, \ldots, u_j)$ has linear resolution for $j = 1, \ldots, m$. Then I has linear quotients for this order of the generators.*

Proof. Suppose that I is generated in degree d. Then $I_{j+1}/I_j \cong (S/(I_j : u_{j+1}))(-d)$. Thus we get the short exact sequence

$$0 \to I_j \to I_{j+1} \to (S/(I_j : u_{j+1}))(-d) \to 0,$$

which induces the long exact sequence

$$\mathrm{Tor}_1(K, I_{j+1})_\ell \to \mathrm{Tor}_1(K, S/(I_j : u_{j+1}))_{\ell-d} \to \mathrm{Tor}_0(K, I_j)_\ell \to \mathrm{Tor}_0(K, I_{j+1})_\ell$$

For $\ell \neq d$, $\mathrm{Tor}_0(K, I_j)_\ell = 0$, and for $\ell = d$, $\mathrm{Tor}_0(K, I_j)_\ell \to \mathrm{Tor}_0(K, I_{j+1})_\ell$, can be identified with the map $I_j/\mathfrak{m}I_j \to I_{j+1}/\mathfrak{m}I_{j+1}$. Since I_j is generated by part of a minimal system of generators of I_{j+1}, this map is injective. Hence for each ℓ we get the exact sequence $\mathrm{Tor}_1(K, I_{j+1})_\ell \to \mathrm{Tor}_1(K, S/(I_j : u_{j+1}))_{\ell-d} \to 0$. Since I_{j+1} has d-linear resolution, it follows that $\mathrm{Tor}_1(K, I_{j+1})_\ell = 0$ for $\ell \neq d + 1$. This implies that $\mathrm{Tor}_1(K, S/(I_j : u_{j+1}))_\ell = 0$ for $\ell \neq 1$, and shows that $(I_j : u_{j+1})$ is generated in degree 1, as desired. □

Corollary 2.5. *Let I be a k-Borel ideal generated in a single degree. Choose any order $u_1, \ldots, u_m$ of the elements of $G(I)$ which extends the partial order $\prec_k$, i.e. if $u_i \prec_k u_j$, then $i < j$. Then I has linear quotients with respect to this order of the generators.*

Proof. For the given order of the generators it follows that for all j, the ideal $I_j = (u_1, \ldots, u_j)$ is k-Borel. Corollary 2.3 implies that I_j has linear resolution for all j. Therefore, the result follows from the Lemma 2.4. □

In the following, it is shown that any k-Borel ideal (not necessarily generated in a single degree) has linear quotients with respect to the lexicographic order on its minimal generators. This order obviously is different from the order given in Corollary 2.5.

Proposition 2.6. *Let I be a k-Borel ideal. Then I has linear quotients. In particular, it is componentwise linear.*

Proof. Consider the lexicographic order $u_1 > \cdots > u_m$ on the minimal monomial generators of I which is induced by the order $x_1 > \cdots > x_n$. Let $1 \leq i < j \leq m$, $u_i = x_1^{a_1} \cdots x_n^{a_n}$, $u_j = x_1^{b_1} \cdots x_n^{b_n}$ and t be the smallest integer such that $x_t \mid u_i : u_j$. Since $u_i > u_j$, we have $a_r = b_r$ for any $r < t$. Also $b_t \leq a_t - 1 \leq k - 1$. There exists an integer $s > t$ such that $x_s \mid u_j$, otherwise $u_j \mid u_i$, which contradicts to $u_i, u_j \in G(I)$. Set $v = (u_j/x_s)x_t$. Then v is a k-bounded monomial, since $b_t \leq k-1$. Therefore $v \in I$. Let $u_\ell \in G(I)$ be such that $u_\ell \mid v$. Let $u_\ell = x_1^{c_1} \cdots x_n^{c_n}$. Then clearly $c_r \leq b_r$ for any $r \neq t$ and $c_t \leq b_t + 1$. If $c_t \leq b_t$, then $u_\ell \mid (u_j/x_s)$ which contradicts to $u_j \in G(I)$. Thus $c_t = b_t + 1$.

We show that $c_r = b_r$ for any $r < t$. By contradiction assume that $c_r < b_r$ for some $r < t$. Then $w = (u_\ell/x_t)x_r \in I$, since it is k-bounded. Also by comparing exponents we get $(u_\ell/x_t)x_r \mid u_j$. So $(u_\ell/x_t)x_r = u_j$, which implies that $x_t u_j = x_r u_\ell$.

Since $x_tu_j = x_sv$, we have $x_ru_\ell = x_sv$. This together with $u_\ell \mid v$ imply that $u_\ell = v$. But then $r = s$, which contradicts to $r < t < s$. So $c_r = b_r$ for any $r < t$, $u_\ell > u_j$ and $u_\ell : u_j = x_t$. This shows that $u_1, \dots, u_m$ is an order of linear quotients for I. □

Let I be a monomial ideal with linear quotients and $u_1, \dots, u_m$ be an order of linear quotients for I. We denote by $M(I)$ the set of all monomials in I. The *decomposition function of* I is defined as the map $g : M(I) \to G(I)$ given by $g(u) = u_j$, where j is the smallest number such that $u \in (u_1, \dots, u_j)$. The decomposition function of I is said to be *regular* if for each $u \in G(I)$ and every $s \in \operatorname{set}(u)$ we have

$$\operatorname{set}(g(x_su)) \subseteq \operatorname{set}(u).$$

The decomposition function of an ideal with linear quotients is not always regular. For example, consider $I = (x_2x_4, x_1x_2, x_1x_3)$. Then with respect to the given order of the generators, I has linear quotients, while $\operatorname{set}(x_1x_3) = 2$, and $\operatorname{set}(g(x_2(x_1x_3))) = 4$. It is quite obvious that stable and squarefree stable ideals have regular decomposition functions with respect to the reverse degree lexicographic order. Another class of squarefree monomial ideals with regular decomposition function is the Stanley-Reisner ideal of a matroid (see [19, Theorem 1.10]). In the following proposition we show this property for any k-Borel ideal.

Proposition 2.7. *Let I be a k-Borel ideal. Then I has a regular decomposition function.*

Proof. Consider the lexicographic order $u_1 > \cdots > u_m$ on the minimal monomial generators of I. Let $g : M(I) \to G(I)$ be the decomposition function of I. We show that $\operatorname{set}(g(x_su_j)) \subseteq \operatorname{set}(u_j)$ for any $u_j \in G(I)$ and any $s \in \operatorname{set}(u_j)$. Let $g(x_su_j) = u_i$. Clearly $i \leq j$. If $i = j$, there is nothing to prove. So we may assume that $i < j$. There exists a monomial w such that $x_su_j = u_iw$ and by [19, Lemma 1.7], $\operatorname{set}(u_i) \cap \operatorname{supp}(w) = \emptyset$. Let $u_j = x_1^{a_1} \cdots x_n^{a_n}$ and $u_i = x_1^{b_1} \cdots x_n^{b_n}$. Then $b_r \leq a_r$ for any $r \neq s$. Also note that $x_s \nmid w$, otherwise $u_i \mid u_j$, a contradiction. So $b_s = a_s + 1$. Since $u_i > u_j$, $a_1 = b_1, \dots, a_{s-1} = b_{s-1}$. In order to prove $\operatorname{set}(u_i) \subseteq \operatorname{set}(u_j)$, consider $t \in \operatorname{set}(u_i)$. Then by Proposition 2.6, we have $u_\ell : u_i = x_t$ for some $u_\ell > u_i$. Let $u_\ell = x_1^{c_1} \cdots x_n^{c_n}$. Then $c_r \leq b_r$ for any $r \neq t$ and $c_t = b_t + 1$. Also since $u_\ell > u_i$, $b_1 = c_1, \dots, b_{t-1} = c_{t-1}$. If $t = s$, then $t \in \operatorname{set}(u_j)$ and there is nothing to prove. So we may assume that $t \neq s$. We show that there exists an integer $q > t$ such that $a_q > 0$. By contradiction assume that $a_r = 0$ for any $r > t$. We consider two cases and in each case we get a contradiction.

Case 1. Let $s < t$. Since $c_r \leq b_r \leq a_r$ for any $r > t$, we have $c_r = b_r = 0$ for any $r > t$. This implies that $u_\ell = x_tu_i$, which contradicts to $u_\ell \in G(I)$.

Case 2. Let $s > t$. Since $b_r \leq a_r$ for any $r > s$, and $s > t$, we have $b_r = 0$ for any $r > s$. This implies that $u_i = x_su_j$, which contradicts to $u_i \in G(I)$.

Therefore there exists an integer $q > t$ such that $a_q > 0$. Since $t \in \operatorname{set}(u_i)$, by [19, Lemma 1.7], $x_t \notin \operatorname{supp}(w)$. Thus from the equality $x_su_j = u_iw$, we get $a_t = b_t$. Also the equality $c_t = b_t + 1$, implies that $b_t < k$, since u_ℓ is a k-bounded monomial. Thus $a_t < k$. Therefore $v = (u_j/x_q)x_t$ is a k-bounded monomial. So it belongs to

I. Let $u_m \in G(I)$ such that $u_m \mid v$. With the same argument as in the proof of Proposition 2.6, we get $u_m > u_j$ and $u_m : u_j = x_t$. Hence $t \in \text{set}(u_j)$.

In the case that I has linear quotients, Herzog and Takayama [19] showed that the mapping cone construction produces a minimal free resolution of I. If furthermore I has a regular decomposition function, then by [11, Theorem 3.10] the minimal resolution of I obtained as an iterated mapping cone is cellular and supported on a regular CW-complex. Hence as an immediate corollary of Proposition 2.7 and [11, Theorem 3.10] we have

Corollary 2.8. *Let I be a k-Borel ideal. Then the minimal free resolution of I obtained as an iterated mapping cone is cellular and supported on a regular CW-complex.*

3. Multiplicity and analytic spread of squarefree Borel ideals

Let I be a squarefree monomial ideal. Then I is called a *squarefree Borel ideal*, when I is 1-Borel. If I is a squarefree Borel ideal, then the minimal set of Borel generators of I is the set of maximal elements of $G(I)$ with respect to $\preceq$. In this section we describe some algebraic invariants of squarefree Borel ideals.

Let $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ be a squarefree monomial with $i_1 < i_2 < \cdots < i_d$. A *block* of u is a subset $\{i_l, i_{l+1}, \ldots, i_{l+k}\}$ such that $i_{l+j} = i_l + j$ for $j = 1, \ldots, k$. A block of u is called *maximal* if it is not properly contained in any other block of u. Note that $\text{supp}(u)$ has a unique decomposition into maximal blocks. In other words, $\text{supp}(u) = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$, where each B_i is a maximal block and $\max\{j : j \in B_i\} < \min\{j : j \in B_{i+1}\} - 1$ for all i. We call B_i the ith block of u and $B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$ the *block decomposition of u*.

For a set $A \subseteq [n]$, set $P_A = (x_i : i \in A)$. We use Proposition 1.6 to prove

Theorem 3.1. *Let I be a squarefree principal Borel ideal with the Borel generator u and let B_1 be the first block of u. Then P is a minimal prime ideal of I of height $h = \text{height}(I)$ if and only if $P = P_A$ for some $A \subseteq [1, \max(B_1)]$ with $|A| = h$. In particular,*

$$e(S/I) = \binom{\max(B_1)}{|B_1| - 1}.$$

Proof. By Proposition 1.6, $\text{height}(I) = \min(u) = h$. Hence $B_1 = \{h, h+1, \ldots, h+k-1\}$, where $k = |B_1|$. First we show that for any $A \subseteq [1, h+k-1]$ of cardinality h, the ideal P_A is a minimal prime ideal of I. Let A be such a set and let $v \in G(I)$, that is, $v \preceq u$ and $\deg(v) = \deg(u) = d$. Let $v = x_{i_1}\cdots x_{i_d}$, where $i_1 < i_2 < \cdots < i_d$. Then $\{i_1, \ldots, i_k\} \subseteq [1, h+k-1]$. It follows that $A \cap \{i_1, \ldots, i_k\} \neq \emptyset$, otherwise $A \cup \{i_1, \ldots, i_k\} \subseteq [1, h+k-1]$ is a set of cardinality $h+k$, which is a contradiction. Thus $v \in P_A$ for any $v \in G(I)$, and hence P_A is a minimal prime ideal of I. Clearly $\text{height}(P_A) = h$.

Now, consider an arbitrary minimal prime ideal P of I of height h. Since P is generated by variables, $P = P_A$ for some A with $A \subseteq [1, \max(u)]$ and $|A| = h$. We show that $A \subseteq [1, h+k-1]$. Let $\text{supp}(u) = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_r$ be the block decomposition of u. If $r = 1$, then $\mathrm{supp}(u) = B_1 = \{h, h+1, \ldots, h+k-1\}$ and so $m = h+k-1$. Hence $A \subseteq [1, h+k-1]$ as desired. Now suppose that $r \geq 2$ and by contradiction assume that there exists $j \in A$ with $j \geq h+k$. This implies that $|A \cap [1, h+k-1]| \leq h-1$. Let $d = \deg(u)$. Consider the monomial $v = \prod_{i=h}^{h+d-1} x_i$ of degree d. Clearly $v \prec u$ and then $v \in G(I)$. Set $J = B(v)$. Then $J \subset I \subseteq P_A$ with $\mathrm{height}(J) = \min(v) = h = \mathrm{height}(P_A)$. Hence P_A is a minimal prime ideal of J as well. Since $\mathrm{supp}(v) = [h, h+d-1]$ consists of one block, we have $A \subseteq [1, h+d-1]$. Let $x^A = \prod_{i \in A} x_i$. We show that for the monomial $w = (\prod_{i=1}^{h+d} x_i)/x^A$, we have $w \prec u$. Once we show this, since $\deg(w) = d$, we get $w \in I$, while $w \notin P_A$, a contradiction. Let $\mathrm{supp}(w) = \{j_1 < \cdots < j_d\}$ and $\mathrm{supp}(u) = \{l_1 < \cdots < l_d\}$. Then $j_t \leq h+t \leq l_t$ for any $k+1 \leq t \leq d$. It remains to show that $j_t \leq l_t = h+(t-1)$ for $1 \leq t \leq k$. Since $|A \cap [1, h]| \leq h-1$, and j_1 is the smallest integer in $[1, h+d] \setminus A$, we have $j_1 \in [1, h]$. So $j_1 \leq h = l_1$. Similarly, we have $|A \cap ([1, h+1] \setminus \{j_1\})| \leq h-1$. Thus $j_2 \in [1, h+1]$ and hence $j_2 \leq h+1 = l_2$. The same argument shows the inequality $j_t \leq l_t$ for any $1 \leq t \leq k$, as desired.

The second statement follows from the fact that the multiplicity of S/I is equal to the number of the minimal prime ideals of I of height h. $\square$

The formula for the multiplicity given in Theorem 3.1 can be generalized to squarefree Borel ideals with any number of Borel generators under some extra assumption on the first block of the Borel generators, as the next result shows. One can easily construct many examples which show that this extra assumption can not be dropped
.

Theorem 3.2. *Let I be a squarefree Borel ideal with Borel generators $u_1, \ldots, u_m$ and let B_{1i} be the first block of u_i for $1 \leq i \leq m$. Suppose there exists an integer j such that $B_{1j} \subseteq B_{1i}$ for all $1 \leq i \leq m$. Then*

$$e(S/I) = \binom{\max(B_{1j})}{|B_{1j}| - 1}.$$

Proof. Without loss of generality assume that $j = 1$ and $B_{11} \subseteq B_{1i}$ for all i. Let $B_{11} = [h, b]$. Then $h = \min(u_1) \geq \min(u_i)$ for any $1 \leq i \leq m$. Set $J = B(u_1)$. Then by Proposition 1.6, $\mathrm{height}(I) = \mathrm{height}(J) = \min(u_1) = h$. Therefore, it is enough to show that the set of minimal prime ideal of I of height h and that of J are the same. If P_A is a minimal prime ideal of I with $\mathrm{height}(P_A) = h$, since $J \subseteq I \subseteq P_A$ and $\mathrm{height}(J) = h = \mathrm{height}(P_A)$, then P_A is a minimal prime ideal of J, as well. Now, let P_A be a minimal prime ideal of J of height h. Then by Theorem 3.1, $A \subseteq [1, b]$. For any $2 \leq i \leq m$, let $J_i = B(u_i)$ and $B_{1i} = [a_i, b_i]$. Then by our assumption, $a_i \leq h \leq b \leq b_i$. Moreover, if C_i be an arbitrary subset of A with $|C_i| = a_i$, since $C_i \subseteq [1, b_i]$, by Theorem 3.1, P_{C_i} is a minimal prime ideal of J_i. Hence $u_i \in J_i \subseteq P_{C_i} \subseteq P_A$ for all i. Hence $I \subseteq P_A$ and P_A is a minimal prime ideal of I, as desired. $\square$

In the next result we describe the analytic spread of an equigenerated squarefree Borel ideal in terms of the block decomposition of its Borel generators. To prove this result we use [10, Lemma 4.3] which relates the analytic spread of an equigenerated monomial ideal I with linear relations to some combinatorial invariants of the so called linear relation graph of I.

Let I be a monomial ideal with $G(I)=\{u_1,\ldots,u_m\}$. The *linear relation graph* Γ of I is the graph with edge set

$$E(\Gamma)=\{\{i,j\}\colon \text{there exist } u_k,u_l\in G(I) \text{ such that } x_iu_k=x_ju_l\}$$

and vertex set $V(\Gamma)=\bigcup_{\{i,j\}\in E(\Gamma)}\{i,j\}$.

Theorem 3.3. *Let I be an equigenerated squarefree Borel ideal with Borel generators $u_1,\ldots,u_m$ which is not a principal ideal. For $i=1,\ldots,m$, let $B_{i1}\sqcup B_{i2}\sqcup\cdots\sqcup B_{ik_i}$ be the block decomposition of u_i and set $n=\max\{\max(u_i):1\leq i\leq m\}$. Then*

$$\ell(I)=\begin{cases} n & \text{if } 1\notin\bigcap_{i=1}^m B_{i1},\\ n-|\bigcap_{i=1}^m B_{i1}|, & \text{otherwise.}\end{cases}$$

Proof. Without loss of generality let u_m be a Borel generator with $n=\max(u_m)$. We may assume that $r\notin\operatorname{supp}(u_m)$ for some $r<n$, since otherwise $u_m=\prod_{i=1}^n x_i$ and $I=(u_m)$, which is not the case. For any $j\notin\operatorname{supp}(u_m)$, $v=x_j(u_m/x_n)\in G(I)$. Hence $x_ju_m=x_nv$, which implies that $\{j,n\}\in E(\Gamma)$, where Γ is the linear relation graph of I.

First we consider the case that $1\notin\bigcap_{i=1}^m B_{i1}$. It is enough to show that Γ is a connected graph on the vertex set $[n]$. Then by [10, Lemma 4.3], we will get $\ell(I)=n$, as desired. By our assumption $1\notin B_{r1}$ for some r. First suppose that $r=m$ and $1\notin B_{m1}$. Since $1\notin\operatorname{supp}(u_m)$, we have $\{1,n\}\in E(\Gamma)$ and $v_j=x_1(u_m/x_j)\in G(I)$ for any $j\in\operatorname{supp}(u_m)$. So $\{1,j\}\in E(\Gamma)$ for any $j\in\operatorname{supp}(u_m)$. This implies that Γ is a connected graph on $[n]$, as desired.

Now assume that $r\neq m$ and $1\in\operatorname{supp}(u_m)$. Let $B_{m1}=\{1,2,\ldots,s\}$. Then $s+1\notin\operatorname{supp}(u_m)$ and for any $j\in\operatorname{supp}(u_m)$ with $j>s$ we have $v=x_{s+1}(u_m/x_j)\in G(I)$ which implies that $\{j,s+1\}\in E(\Gamma)$. In order to show that Γ is connected, it is enough to show that for any $1\leq j\leq s$, j is connected by a path to a vertex in $\{s+1,s+2,\ldots,n\}$. Let $B_{r1}=\{t,t+1,\ldots,p\}$. If $s+1<t$, we have $\{j,s+1\}\in E(\Gamma)$ for any $j<s+1$. Indeed, $v=x_j(u_r/x_t)\in G(I)$ and $w=x_{s+1}(u_r/x_t)\in G(I)$ and $x_{s+1}v=x_jw$. So we may assume that $t\leq s+1$.

If $s+1\in\operatorname{supp}(u_r)$, since $t\leq s+1$, we have $v=x_i(u_r/x_{s+1})\in G(I)$ for any $i<t$ or any i with $t\leq i<s+1$ and $i\notin\operatorname{supp}(u_r)$. Then for such i's we have $\{i,s+1\}\in E(\Gamma)$. If $t\leq i<s+1$ and $i\in\operatorname{supp}(u_r)$, then $\{1,i\}\in E(\Gamma)$ and $i,1,s+1$ is a path in Γ and we are done in this case.

Now, we may assume that $s+1\notin\operatorname{supp}(u_r)$. Therefore $p<s+1\leq d$ and hence there exists $h>s+1$ such that $h\in\operatorname{supp}(u_r/\prod_{i=t}^p x_i)$. Then $v=x_{s+1}(u_r/x_h)\in G(I)$ and $w=x_i(u_r/x_h)\in G(I)$ for any $i<t$ or any $i\leq s$ with $i\notin\operatorname{supp}(u_r)$. Since $x_iv=x_{s+1}w$, we have $\{i,s+1\}\in E(\Gamma)$ for any $i<t$ or any $i\leq s$ with $i\notin\operatorname{supp}(u_r)$. Also for any $t\leq i\leq s$ with $i\in\operatorname{supp}(u_r)$, we have $\{1,i\}\in E(\Gamma)$, since $v=x_1(u_r/x_i)\in G(I)$. Hence $i,1,s+1$ is a path in Γ. So Γ is connected.

Now, consider the case that $1\in\bigcap_{i=1}^m B_{i1}$ and let $\bigcap_{i=1}^m B_{i1}=[t]$. Then $z=\prod_{i=1}^t x_i$ divides any minimal monomial generator of I. Hence $I=zJ$, where J is a monomial ideal in $K[x_{t+1},\ldots,x_n]$. The ideal J is isomorphic to a squarefree Borel ideal J'

such that x_1 does not divide some Borel generator of J'. Since J' is an ideal in the polynomial ring with $n-t$ variables and $J' \cong I$, by the first part of the proof $\ell(I) = \ell(J) = n-t$. $\square$

4. Homological shift ideals of equigenerated squarefree Borel ideals

In this section, for an equigenerated squarefree Borel ideal I, we obtain the Borel generators of the homological shift ideals of I. Moreover, we study the behaviour of some algebraic invariants of these shift ideals.

Let $I \subset S$ be a monomial ideal with minimal multigraded free S-resolution

$$\mathbb{F} : 0 \longrightarrow F_q \longrightarrow F_{q-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow I \longrightarrow 0,$$

where $F_i = \bigoplus_{j=1}^{b_i} S(-\mathbf{a}_{ij})$. The vectors $\mathbf{a}_{ij}$ are called the *multigraded shifts* of the resolution $\mathbb{F}$. The monomial ideal $\mathrm{HS}_i(I) = (\mathbf{x}^{\mathbf{a}_{ij}} : j = 1, \dots, b_i)$ is called the *ith homological shift ideal* of I. Note that $\mathrm{HS}_0(I) = I$.

Let u be a squarefree monomial. A positive integer i is called a *gap* of u if $i < \max(u)$ and x_i does not divide u. The set of all gaps of u is denoted by $\mathrm{gap}(u)$ and the maximal element of $\mathrm{gap}(u)$ is called the *maximal gap* of u.

Proposition 4.1. *Let $I = B_1(u_1, \dots, u_m)$ be an equigenerated squarefree Borel ideal. Then*

$$\mathrm{HS}_1(I) = B_1(x_{p_1}u_1, \dots, x_{p_m}u_m),$$

where for each i, p_i is the maximal gap of u_i.

Proof. By [4, Proposition 3.1], $\mathrm{HS}_1(I)$ is a squarefree Borel ideal and $x_{p_i}u_i \in \mathrm{HS}_1(I)$. So

$$B_1(x_{p_1}u_1, \dots, x_{p_m}u_m) \subseteq \mathrm{HS}_1(I).$$

Now, consider a generating monomial of $\mathrm{HS}_1(I)$, which by Corollary 2.2 is of the form $x_\ell v$ for some $v \in G(I)$ and some $\ell \in \mathrm{gap}(v)$. Then by Corollary 1.5, there exists $1 \le t \le m$ such that $v \preceq u_t$. We show that $x_\ell v \preceq x_{p_t}u_t$. Without loss of generality we may assume that ℓ is the maximal gap of v. Let $v = x_{j_1} \cdots x_{j_d}$ with $j_1 < \cdots < j_d$ and $u_t = x_{i_1} \cdots x_{i_d}$ with $i_1 < \cdots < i_d$, where d is the degree of the minimal monomial generators of I. We have $j_r \le i_r$ for all $1 \le r \le d$. We set $j_0 = i_0 = 0$. Let s and k be integers with $0 \le s \le d-1$ and $0 \le k \le d-1$ such that $j_s < \ell < j_{s+1}$ and $i_k < p_t < i_{k+1}$. Then $\ell = j_{s+1} - 1$ and $p_t = i_{k+1} - 1$. We may write $x_\ell v = x_{j'_1} \cdots x_{j'_{d+1}}$ and $x_{p_t}u_t = x_{i'_1} \cdots x_{i'_{d+1}}$, where $j'_1 < \cdots < j'_{d+1}$ and $i'_1 < \cdots < i'_{d+1}$. Then $j'_{s+1} = \ell = j_{s+1} - 1$ and $i'_{k+1} = p_t = i_{k+1} - 1$.

First suppose that $k = s$. Then $j'_r = j_r \le i_r = i'_r$ for any $r < k+1$. Also $j'_{k+1} = \ell = j_{k+1} - 1 \le i_{k+1} - 1 = i'_{k+1} = p_t$ and $j'_r = j_{r-1} \le i_{r-1} = i'_r$ for any $r > k+1$. Hence $x_\ell v \preceq x_{p_t}u_t$, as desired.

Now, suppose that $s < k$. Then $j'_r = j_r \le i_r = i'_r$ for any $r \le s$. Also $j'_{s+1} = \ell = j_{s+1} - 1 \le i_{s+1} - 1 = i'_{s+1} - 1 < i'_{s+1}$ and for any $s+2 \le r \le d+1$ we have $j'_r = j_{r-1} \le i_{r-1}$. Note that $i_{r-1} = i'_{r-1}$ or $i_{r-1} = i'_r$, which implies that $i_{r-1} \le i'_r$. Hence $j'_r \le i'_r$ for all r, as desired.

Finally, consider the case that $s > k$. Then $j'_r = j_r \leq i_r = i'_r$ for any $1 \leq r \leq k$. Also $j'_r = j_d - (d-r+1) \leq i_d - (d-r+1) = i'_r$ for any $s+1 \leq r \leq d+1$. Now, suppose that $k+1 \leq r \leq s$. Then $j'_r = j_r < j_d - (d-r) \leq i_d - (d-r)$. Since $i_{k+1}-1$ is the maximal gap of u_t, we have $i_d - (d-r) = i_r$ for any $k+1 \leq r \leq s$. Hence $j'_r \leq i_r - 1$ for any $k+1 \leq r \leq s$. When $k+2 \leq r \leq s$, we have $i_r - 1 = i_{r-1} = i'_r$ and hence $j'_r \leq i'_r$. Also when $r = k+1$, we have $j'_{k+1} \leq i_{k+1} - 1 = i'_{k+1}$. Therefore, $x_\ell v \preceq x_{p_t} u_t$. $\square$

As an immediate corollary of Proposition 4.1, we have

Corollary 4.2. *Let $I = B_1(u_1, \ldots, u_m)$ be an equigenerated squarefree Borel ideal. Then for any $k \geq 1$,*

(a) $\mathrm{HS}_k(I) = B_1(x_{p_{11}} \cdots x_{p_{1k}} u_1, \ldots, x_{p_{m1}} \cdots x_{p_{mk}} u_m)$, *where $p_{i1}, \ldots, p_{ik}$ are maximal possible distinct integers in* $\mathrm{gap}(u_i)$.

(b) $\mathrm{HS}_1(\mathrm{HS}_k(I)) = \mathrm{HS}_{k+1}(I)$.

Using the description of the Borel generators of homological shift ideals $\mathrm{HS}_k(I)$ in Corollary 4.2, the description for the height in Proposition 1.6 and analytic spread in Theorem 3.3 we get

Corollary 4.3. *Let $I = B_1(u_1, \ldots, u_m)$ be an equigenerated squarefree Borel ideal. Then* $\mathrm{height}(\mathrm{HS}_{k+1}(I)) \leq \mathrm{height}(\mathrm{HS}_k(I))$ *and $\ell(\mathrm{HS}_{k+1}(I)) \leq \ell(\mathrm{HS}_k(I))$ for all k.*

A similar result to Corollary 4.3 does not hold for the multiplicity of the homological shift ideal of equigenerated squarefree Borel ideals. However, we have

Proposition 4.4. *Let $I = B_1(u)$ be a squarefree principal Borel ideal. Then $e(S/\mathrm{HS}_k(I))$ is a unimodal function of k.*

Proof. Let $\mathrm{supp}(u) = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$ be the block decomposition of u with $B_1 = [a, b]$. Then by Corollary 4.2, $\mathrm{HS}_k(I) = B_1(u_k)$ for a monomial u_k for all k. Let $[a_k, b_k]$ be the first block in the block decomposition of u_k. Then there exists k_0 such that $b_k = b$ and $a_k = a$ for $1 \leq k \leq k_0$ and for $k > k_0$ we have $b_k = n = \max(u)$ and $a_k = a - (k - k_0 - 1)$. Therefore $e(S/\mathrm{HS}_k(I)) = e(S/B_1(u_k)) = \binom{b}{a} = e(S/I)$ for $1 \leq k \leq k_0$ and $e(S/\mathrm{HS}_k(I)) = \binom{n}{a-(k-k_0-1)}$ for $k > k_0$. This proves the assertion. $\square$

5. Homological shift ideals of t-spread Veronese ideals

In [13] the concept of a t-spread monomial was introduced. A monomial $x_{i_1} x_{i_2} \cdots x_{i_d}$ with $i_1 \leq i_2 \leq \cdots \leq i_d$ is called *t-spread* if $i_j - i_{j-1} \geq t$ for $2 \leq j \leq n$. We fix integers d and t. The monomial ideal in $S = K[x_1, \ldots, x_n]$ generated by all t-spread monomials of degree d is called the *t-spread Veronese ideal of degree d*. We denote this ideal by $I_{n,d,t}$. For $t = 1$ one obtains the squarefree Veronese ideals, which may also be viewed as the edge ideals of hypersimplexes. Properties of these ideals were first studied in [23]. The K-subalgebra of S generated by the monomials $v \in G(I_{n,d,t})$ is called the *t-spread Veronese algebra*. In [9] the Gorenstein property for the t-spread Veronese algebras was analyzed.

Let $t \geq 1$ be an integer $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ be a t-spreal monomial with $i_1 < i_2 < \cdots < i_d$. A *t-block* of u of size r is a subset $B \subseteq \text{supp}(u)$ such that $B = \{i_k, i_{k+1}, \ldots, i_{k+r-1}\}$ with $i_{l+1} - i_l = t$ for all $k \leq l \leq k+r-2$. A t-block of u is called *maximal* if it is not contained in any other t-block of u. The set $\text{supp}(u)$ has a a unique decomposition into maximal t-blocks, say $\text{supp}(u) = B_1 \sqcup B_2 \sqcup \cdots \sqcup B_k$, where each B_i is a maximal block and $\max\{j : j \in B_i\} < \min\{j : j \in B_{i+1}\} - t$ for all i. For $j = 0, \ldots, r-1$, the *jth gap interval* of u is the set $L_j = [\max(B_j) + t, \min(B_{j+1}) - 1]$, where $B_0 = \{-t+1\}$. The union of all gap intervals of u is denoted by $\text{gap}(u)$ and any element of $\text{gap}(u)$ is called a *gap* of u.

Let $u = x_{i_1}x_{i_2}\cdots x_{i_d}$ be a monomial with $i_1 \leq i_2 \leq \cdots \leq i_d$ and let t be a positive integer. A pair (i_k, i_{k+1}) with $1 \leq k \leq d-1$ is called a *t-irregular* pair of u if $i_{k+1} - i_k < t$.

Theorem 5.1. *Let n, d and t be positive integers with $d, t \leq n$, and $t \geq 1$ and let $I = I_{n,d,t}$. Then*

$$\text{HS}_k(I) = (x_{i_1}x_{i_2}\cdots x_{i_{d+k}} : i_1 < \cdots < i_{d+k}, \textit{ and } i_{\ell+1} - i_\ell < t \textit{ for at most } k \textit{ integers}).$$

Proof. By [19, Lemma 1.5],

$$\text{HS}_k(I) = (ux_{i_1}x_{i_2}\cdots x_{i_k} : u \in G(I),\ i_1 < \cdots < i_k,\ \{i_1, \ldots, i_k\} \subseteq \text{set}(u)).$$

Moreover, by [10, Theorem 2.1], I has linear quotients, and for any $u \in G(I)$ we have $i \in \text{set}(u)$ if and only if i belongs to some gap interval of u ([10, Lemma 2.2]). Consider $u \in \text{G}(I)$ and $\{i_1, \ldots, i_k\} \subseteq \text{set}(u)$. Let $L_1, \ldots L_m$ be the gap intervals of u, and let k_j be the number of elements in $\{i_1, \ldots, i_k\}$ which belong to the gap interval L_j. Then $k = \sum_{j=1}^m k_j$. Let $L_j = [\max(B_j) + t, \min(B_{j+1}) - 1]$ and $A_j = \{i_1, \ldots, i_k\} \cap L_j$. Then $u\prod_{i \in A_j} x_i$ has at most k_j t-irregular pairs for any $1 \leq j \leq m$, since $\min(A_j) - \max(B_j) \geq t$. Moreover, for distinct integers j and s with $j < s$ we have $\min(A_s) - \max(A_j) \geq \min(L_s) - \max(L_j) \geq t$. This implies that $ux_{i_1}x_{i_2}\cdots x_{i_k}$ has at most $\sum_{j=1}^m k_j = k$ t-irregular pairs.

Conversely, assume that $v = x_{i_1}\cdots x_{i_{d+k}}$ is a monomial with at most k t-irregular pairs and let $i_1 < \cdots < i_{d+k}$. Let $\{j_1, \ldots, j_k\} \subseteq \text{supp}(v)$ which contains the smallest element from each t-irregular pair of v. Then $u = v/x_{j_1}\cdots x_{j_k}$ is a minimal generator of I and hence $v \in \text{HS}_k(I)$. □

It is conjectured that all the homological shift ideals of t-spread Veronese ideals have linear quotients. In the next result we provide a proof only for the first shift ideal, which is still rather complicated. The expected order of linear quotients for higher shift ideals is similar to the one used in the following proof.

Theorem 5.2. *Let n, d and t be positive integers with $d, t \leq n$ and $t \geq 1$ and let $I = I_{n,d,t}$. Then $\text{HS}_1(I)$ has linear quotients.*

Proof. We set $J = \text{HS}_1(I)$. For any minimal monomial generator w of J, we have $w = x_iu$ for a monomial $u \in G(I)$ and some $i \in \text{gap}(u)$. Such a presentation is called the right presentation for w, when u is the largest monomial with respect to the lexicographic order (induced by $x_1 > x_2 > \cdots > x_n$) among all the possible monomials $v \in G(I)$ that appear in some presentation of w. We consider a total order on the set of minimal monomial generators of J as follows: Let x_iu and x_jv be two minimal monomial generators of J which are the right presentations. We set $x_iu > x_jv$ if $u >_{lex} v$ or $u = v$ and $i < j$. We show that J has linear quotients with respect to this order.

If $u = v$, then $x_iu : x_ju = x_i$ and there is nothing to prove. Let $u >_{\text{lex}} v$. Let $u = \prod_{m=1}^{d} x_{r_m}$ with $r_1 < r_2 < \cdots < r_d$ and $v = \prod_{m=1}^{d} x_{s_m}$ with $s_1 < s_2 < \cdots < s_d$. Since $u >_{\text{lex}} v$, there exists an integer $1 \leq \ell \leq d$ such that $r_1 = s_1, \ldots, r_{\ell-1} = s_{\ell-1}$ and $r_\ell < s_\ell$. By [10, Theorem 2.1], I has linear quotients with respect to the lexicographic order. Moreover, for $k = r_\ell$ and $w = (v/x_{s_\ell})x_k$ we have $w \in G(I)$, $w >_{\text{lex}} v$, and $w : v = x_k$. Also clearly $x_k | u : v$. If $\ell = d$, then $w = u$. We show that $x_ju \in J$. If $j \in \text{gap}(u)$, then the assertion is clear. If $j \notin \text{gap}(u)$, then $j > k$. Moreover, $x_ju = x_ku'$, where $u' = (x_ju/x_k)$, and $k \in \text{gap}(u')$. So $x_ju = x_ku' \in J$. Also $x_ju : x_jv = x_k$, as desired. So we may assume that $\ell < d$.

Case 1. First assume that $k \neq j$. If $j \in \text{gap}(w)$, then $x_jw \in J$. Moreover, since $w >_{\text{lex}} v$, we have $x_jw > x_jv$ and $x_jw : x_jv = x_k$, where $x_k | x_iu : x_jv$. Hence we are done. So we may assume that $j \notin \text{gap}(w)$. Since $x_j \in \text{gap}(v)$, we have either $j < s_1$ or $s_{p-1} < j < s_p$ for some $p > 1$ with $j \geq s_{p-1} + t$. If $j < s_1$, since $j \notin \text{gap}(w)$ we have $\ell = 1$ and $k < j$. Therefore $x_jw = x_kw'$, where $w' = x_jw/x_k$. We have $w' \in G(I)$, because $w' = x_jx_{s_2} \cdots x_{s_d}$ and $s_2 - j \geq s_2 - s_1 \geq t$. Moreover, since $k < j$, we have $k \in \text{gap}(w')$. So $x_kw' \in J$ and since $w' >_{\text{lex}} v$, we have $x_kw' > x_jv$. Also $x_kw' : x_jv = x_jw : x_jv = x_j$. Therefore we are done in the case that $j < s_1$. Now assume that $s_{p-1} < j < s_p$ for some $p > 1$, with $j \geq s_{p-1} + t$. If $p \neq \ell$, then $j \in \text{gap}(w)$ which is not the case. So $p = \ell$, $s_{\ell-1} < j < s_\ell$ and $j \geq s_{\ell-1} + t$. Note that $s_{\ell-1} < k < s_\ell$. If $j < k$, then j belongs to the gap interval $[s_{\ell-1} + t, k]$ of w, which contradicts to $j \notin \text{gap}(w)$. So we have $j > k$. Also since $[k + t, s_{\ell+1}]$ is a gap interval of w, j does not belong to this interval. So $j < k + t$. Then for $w' = (x_jw)/x_k$ we have $w' \in G(I)$ and $k \in \text{gap}(w')$. So $x_jw = x_kw' \in J$ and $x_kw' > x_jv$, since $w' >_{\text{lex}} v$. Moreover, $x_jw : x_jv = x_k$ as desired.

Case 2. Let $k = j$. Then $s_{\ell-1} < j < s_\ell$. First we show that $s_\ell < j + t$. Suppose in contrary that $s_\ell \geq j + t$. Since $\ell < d$, we have $s_\ell \in \text{gap}(w)$. Since $x_jv = x_{s_\ell}w$ and $w >_{\text{lex}} v$, x_jv is not a right presentation, a contradiction. So we have $s_\ell < j + t$ and hence $s_\ell \notin \text{gap}(w)$. If there exists $q \in \text{supp}(x_iu) \cap \text{gap}(w)$, then $x_qw \in J$, $x_qw : x_jv = x_q$ and $x_q | x_iu : x_jv$, since $s_\ell \notin \text{gap}(w)$. So in this situation we are done. Now, assume that $\text{supp}(x_iu) \cap \text{gap}(w) = \emptyset$ which in particular implies that $i \notin \text{gap}(w)$. Hence for the gap interval $L_1 = [j + t, s_{\ell+1} - 1]$ of w we have $L_1 \cap \text{supp}(u) = \emptyset$. This implies that $r_{\ell+1} \geq s_{\ell+1}$. So $r_{\ell+2} \geq r_{\ell+1} + t \geq s_{\ell+1} + t$. Now, consider the gap interval $L_2 = [s_{\ell+1} + t, s_{\ell+2} - 1]$ of w. Since $L_2 \cap \text{supp}(u) = \emptyset$, by the inequality $r_{\ell+2} \geq s_{\ell+1} + t$, we have $r_{\ell+2} \geq s_{\ell+2}$. By the similar argument we have $r_p \geq s_p$ for any $\ell + 1 \leq p \leq d$. Since $i \in \text{gap}(u) \setminus \text{gap}(w)$, we have $u \neq w$. So there exists p such that $r_p > s_p$. Let $b = \max\{p : \ r_p > s_p\}$. Then $r_{p+1} = s_{p+1}$ and $r_p > s_p$. Then $x_{r_p}w = x_{s_p}w'$, where $w' = (x_{r_p}w/x_{s_p})$. Since $r_p - s_{p-1} > s_p - s_{p-1} \geq t$, and $s_{p+1} - r_p = r_{p+1} - r_p \geq t$, we conclude that w' is a t-spread monomial. So $w' \in G(I)$. Also clearly $s_p \in \text{gap}(w')$. So $x_{r_p}w = x_{s_p}w' \in J$ and since $w' >_{\text{lex}} v$, we have $x_{r_p}w > x_jv$. Note that $x_{r_p}w : x_jv = x_{r_p}$, as desired. $\square$

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