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Mini-Workshop: Variable Curvature Bounds, Analysis and Topology on Dirichlet Spaces (hybrid meeting)

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Abstract. A Dirichlet form $\mathcal{E}$ is a densely defined bilinear form on a Hilbert space of the form $L^2(X,\mu)$, subject to some additional properties, which make sure that $\mathcal{E}$ can be considered as a natural abstraction of the usual Dirichlet energy $\mathcal{E}(f_1,f_2)=\int_D(\nabla f_1,\nabla f_2)$ on a domain D in $\mathbb{R}^m$. The main strength of this theory, however, is that it allows also to treat nonlocal situations such as energy forms on graphs simultaneously. In typical applications, X is a metrizable space, and the theory of Dirichlet forms makes it possible to define notions such as curvature bounds on X (although X need not be a Riemannian manifold), and also to obtain topological information on X in terms of such geometric information.

Mathematics Subject Classification (2020): 31C25.

Introduction by the Organizers

A *Dirichlet space* is a triple $(X,\mu,\mathcal{E})$, where X is a metrizable space, μ is a Radon measure on X and $\mathcal{E}$ is a (symmetric) *Dirichlet form* on $L^2(X,\mu)$, meaning that $\mathcal{E}$ is a densely defined, closed nonnegative bilinear form on $L^2(X,\mu)$ which satisfies the Markovian property. As such, $\mathcal{E}$ canonically induces a self-adjoint nonnegative operator H on $L^2(X,\mu)$, and the *heat semigroup* $P_t := e^{-tH}$, $t \geq 0$, canonically induces a contraction semigroup in $L^p(X,\mu)$ for all $p \in [1,\infty]$. Fundamental results by Fukushima and Ma/Röckner assert that under a weak additional regularity assumption on $\mathcal{E}$, which is satisfied in most applications, one can associate an essentially uniquely determined Hunt process $\mathbf{X}$ on (the one-point compactification of) X to P such that

$$P_t f(x) = \mathbb{E}^x[1_{\{t<\zeta\}} f(\mathbf{X}_t)],$$

where $\zeta > 0$ denotes the possibly finite lifetime of $\mathbf{X}$. The natural concept of convergence for a sequence of Dirichlet spaces is the so called *Mosco convergence*, which in particular implies a natural convergence of the associated heat semigroups and typically also of the spectra. A natural, however subtle, construction principle for new Dirichlet forms out of given ones is the so called *trace construction*, which under certain assumptions allows to consider $\mathcal{E}$ as a form on $L^2(X, \mu')$ where μ' is another Radon measure on X which in typical applications is not absolutely continuous with respect to μ.

The prototypes of examples one should have in mind are:

Weighted Riemannian manifolds: here, one takes X to be a smooth Riemannian manifold possibly with boundary, μ a measure which has a sufficiently regular (but not necessarily smooth) density $\psi : X \to (0, \infty)$ with respect to the Riemannian volume measure, and $\mathcal{E}$ to be the form

$$\mathcal{E}(f_1, f_2) := \int_X (\nabla f_1, \nabla f_2) d\mu,$$

in the Hilbert space $L^2(X, \mu)$, with domain of definition either $W_0^{1,2}(\mathrm{int}(X))$ or $W^{1,2}(\mathrm{int}(X))$. In case μ is the Riemannian volume measure (that is, $\psi \equiv 1$), the first choice of $\mathcal{E}$ corresponds to the Dirichlet-Laplacian and $\mathbf{X}$ to a Brownian motion which is killed at the boundary of X, while the second one corresponds to the Neumann-Laplacian and $\mathbf{X}$ to a Brownian motion which is reflected on the boundary of X.

Metric measure spaces: here, one takes X to be a metric space (X, d). Then the metric measure space (X, d, μ) canonically induces a first order Sobolev space $W^{1,2}(X, d, \mu)$, which in general is a Banach space. In case this space is actually a Hilbert space (which excludes Finsler-type geometries), following Gigli, one calls (X, d, μ) a Hilbertian metric measure space. In this case, one canonically gets a quasi-regular Dirichlet form $\mathcal{E}$ in $L^2(X, \mu)$, the so called *Cheeger energy*. This construction includes the previous one (that is, if one takes X to be a Riemannian manifold with boundary and d to be the geodesic distance and Neumann boundary conditions), but it also includes many other (possibly very singular) spaces, such as n-dimensional Alexandrov spaces with their n-dimensional Hausdorff measure, and also many infinite dimensional examples, such as the configuration space over a metric space with the L^2-transportation distance and μ the law of a point process with values in the given metric space. An important class of Hilbertian metric measure spaces is given by the class $\mathsf{RCD}(K, \infty)$ of *Riemannian metric measure spaces with a synthetic Ricci curvature* $\geq K$ *in the sense of Sturm and Lott/Villani*, where $K \in \mathbb{R}$ is a constant: here, (X, d, μ) is assumed to be Hilbertian and to satisfy an abstract lower Ricci bound, which can be formulated in terms of a K-convexity property of an entropy type functional on the space $W_2(X, d, \mu)$ given by all probability measures on (X, d) with finite second moments (which is the

Wasserstein space of (X, d)) that are absolutely continuous with respect to μ. One important feature of the class of RCD-spaces is its stability under natural (measured variants of) Gromov-Hausdorff type convergence concepts for metric spaces (which typically imply Mosco convergence).

While the previous examples are all instances of so called *strongly local Dirichlet spaces*, the prototype of a nonlocal Dirichlet space is provided by:

Weighted graphs: here, one takes X to be any countable set with the discrete topology and μ to be any function $\mu : X \to (0, \infty)$. Then for every symmetric function $b : X \times X \to (0, \infty)$ which is zero on the diagonal and summable, one can define $\mathcal{E}$ as the closure in $\ell^2(X, \mu)$ of

$$C_c(X) \ni (f_1, f_2) \longmapsto \sum_{x,y \in X} b(x, y)(f_1(x) - f_1(y))(f_2(x) - f_2(y)) \in \mathbb{R}.$$

In this case, $\mathbf{X}$ is a pure jump process, and current research focuses (among many other things) on useful concepts of curvature, keeping in mind that the above nonlocal $\mathcal{E}$ cannot be realized as a (necessarily local) Cheeger energy, and so the theory of RDC-space does not apply.

The Oberwolfach Mini-Workshop *Variable curvature bounds, analysis and topology on Dirichlet spaces* has presented some of the most important recent results concerning the theory of Dirichlet forms. The given talks can be roughly categorized into the following groups:

- Generalizations of the concept of an RDC-space, with the aim of allowing variable distributional lower Ricci curvature bounds, with measures (or more generally: distributions) that are not necessarily bounded from below by a constant. These so called *tamed spaces* have been defined recently by Erbar/Rigoni/Sturm/Tamanini, and allow to define and to study e.g. lower Ricci bounds in the so called *Kato class* of the underlying Dirichlet form.
- The study of topological data and concepts (such as orientability or Betti numbers) in terms of the geometry induced by the underlying Dirichlet form.
- Regularity results for Gromov-Hausdorff type limits of Riemannian manifolds with uniform lower Ricci bounds in the Kato class.
- Curvature concepts for weighted graphs and other discrete spaces.
- Functional inequalities for Dirichlet spaces.
- Regularity results for abstract Dirichlet forms, in particular, on infinite dimensional spaces.

The mini-workshop has taken place in a hybrid format and has brought together about 20 scientists from the following countries:

- Austria
- China
- France

- Germany
- Japan
- Luxembourg
- Tunisia.

A total of 15 talks has been given, with 9 talks given at the MFO and 6 online, and each talk has lead to a fruitful discussion among the participants.

We thank the MFO for creating a very stimulating and inspiring atmosphere.

Gilles Carron, Batu Güneysu, Matthias Keller, and Kazuhiro Kuwae.

Mini-Workshop (hybrid meeting): Variable Curvature Bounds, Analysis and Topology on Dirichlet Spaces

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Abstracts

Spectral gap estimates for Brownian motion on domains with sticky-reflecting boundary diffusion

Max von Renesse
(joint work with Vitalii Konarovskyi, Victor Marx)

Introducing an interpolation method we estimate the spectral gap for Brownian motion on general domains with sticky-reflecting boundary diffusion associated to the first nontrivial eigenvalue for the Laplace operator with corresponding Wentzell-type boundary condition. In the manifold case our proofs involve novel applications of the celebrated Reilly formula.

1. Introduction and Problem Setting

Brownian motion on smooth domains with sticky-reflecting diffusion along the boundary has a long history, dating back at least to Wentzell [15]. As a prototype consider a diffusion on the closure $\overline{\Omega}$ of a smooth domain Ω with Feller generator $(\mathcal{D}(A), A)$

$$\mathcal{D}(A) = \{f \in C_0(\overline{\Omega}) \,|\, Af \in C_0(\overline{\Omega})\}$$
$$Af = \Delta f \mathbb{I}_\Omega + (\beta \Delta^\tau f - \gamma \frac{\partial f}{\partial \nu}) \mathbb{I}_{\partial\Omega} \tag{1}$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative, Δ^τ is the Laplace-Beltrami operator on the boundary $\partial\Omega$ and $\beta > 0, \gamma \in \mathbb{R}$. The case of pure sticky reflection but no diffusion along the boundary corresponds to the regime $\beta = 0$; models with $\beta > 0$ have appeared recently in interacting particle systems with singular boundary or zero-range pair interaction [1, 4, 8]. An efficient process construction in symmetric cases was given by Grothaus and Voßhall via Dirichlet forms in [10]. Qualitative regularity properties of the associated semigroups were studied e.g. in [9].

In this note we address the problem of estimating the spectral gap for such processes, which is a natural question also in algorithmic applications. For $\beta = 0$ this question was studied beofore by Kennedy [11] and Shouman [14]. However, for $\beta > 0$ the properties of the process change significantly, which is indicated by the fact that the energy form of A now also contains a boundary part and which also constitutes the main difference to the closely related work [12].

We treat the case when $\gamma > 0$ which corresponds to an inward sticky reflection at $\partial\Omega$ using a simple interpolation between two extremal cases. To this aim assume that Ω and $\partial\Omega$ have finite (Hausdorff) measure so that we may choose $\alpha \in (0, 1)$ for which

$$\frac{\alpha}{1-\alpha} \frac{|\partial\Omega|}{|\Omega|} = \gamma.$$

Introducing λ_Ω and λ_∂ as normalized volume and Hausdorff measures on Ω and $\partial\Omega$ and setting

$$\lambda_\alpha = \alpha \lambda_\Omega + (1 - \alpha)\lambda_\partial,$$

we find that $-A$ is λ_α-symmetric with first nonzero eigenvalue/spectral gap characterized by the Rayleigh quotient

$$\sigma_{\alpha,\beta} = \inf_{\substack{f\in C^1(\overline{\Omega})\\ \mathrm{Var}_{\lambda_\alpha}(f)>0}} \frac{\mathcal{E}_{\alpha,\beta}(f)}{\mathrm{Var}_{\lambda_\alpha} f},$$

where

$$\mathrm{Var}_{\lambda_\alpha} f = \int_\Omega f^2 d\lambda_\alpha - \left(\int_\Omega f d\lambda_\alpha\right)^2$$

and

$$\mathcal{E}_{\alpha,\beta}(f) = \alpha \int_\Omega \|\nabla f\|^2 d\lambda_\Omega + (1-\alpha)\int_{\partial\Omega} \beta \|\nabla^\tau f\|^2 d\lambda_\partial,$$

and ∇^τ denotes the tangential derivative operator on $\partial\Omega$.

2. Results

When $\Omega = B_1 \subset \mathbb{R}^2$ is a 2-dimensional unit ball and $\beta = 1$, for instance, we obtain the estimate

$$\sigma_\alpha \geq \frac{8(1+\alpha)\sigma_\Omega}{8(1-\alpha)\sigma_\Omega + 16\alpha + 3\alpha(1-\alpha)\sigma_\Omega} \text{ with } \alpha = \frac{\gamma}{2+\gamma},$$

where $\sigma_0 \approx 3.39$ is the spectral gap for the Neumann Laplacian on the 2-dimensional unit ball. In case when Ω is a d-dimensional manifold with Ricci curvature bounded from below by $k_R > 0$ and with boundary $\partial\Omega$ whose second fundamental form $\mathrm{II}_{\partial\Omega}$ is bounded from below by $k_2 > 0$ we obtain (again with $\beta = 1$, for simplicity) that

$$\sigma_\alpha \geq \min\left(\frac{dk_R}{C_\Omega dk_R + (1-\alpha)(d-1)}, \frac{dk_R}{C_{\partial\Omega}} \frac{2(1-\alpha)+\alpha k_2 C_{\partial\Omega}}{2(1-\alpha)dk_R + \alpha d k_2 k_R C_\Omega + \alpha(1-\alpha)(d-1)k_2}\right),$$

where C_Ω and $C_{\partial\Omega}$ are the usual (Neumann) Poincaré constants of Ω and $\partial\Omega$ respectively. To derive this result we combine Escobar's lower bound [6] on the first Steklov eigenvalue of Ω with a novel estimate on the optimal zero mean trace Poincaré constant of Ω, for which we obtain that

$$\int_\Omega f^2 dx \leq \frac{d-1}{dk_R} \int_\Omega |\nabla f|^2,$$

for all $f \in C^1(\Omega)$ with $\int_{\partial\Omega} f dS = 0$, and which is of independent interest. The proof is based on a novel application of Reilly's formula [13] which is also used for a complementary lower bound of σ independent of the interpolation approach stating that

$$\sigma_\alpha \geq \min\left(\frac{dk_2}{3d-1}\frac{\alpha}{1-\alpha}\frac{|\partial\Omega|}{|\Omega|}, \frac{d}{d-1}k_R\right),$$

but which is generally weaker for small values of α.

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Metric-measure-space orientations and nonconvex boundaries

Sebastian Boldt

(joint work with Batu Güneysu and Stefano Pigola)

For a smooth Riemannian manifold (M^m, g) with possibly nonempty boundary, we consider the notion of a *metric-measure-space orientation (mmso)*, i.e. $\omega \in \Omega^m_{L^\infty}(M, g)$ such that (1) $|\omega| = 1$ a.e. and (2) $(\omega, \mathrm{d}f_1 \wedge \ldots \wedge f_m) \in H^{1,2}(M, g)$ for all *test functions* $f_1, \ldots, f_m$, where we call f a test function if $f \in \mathrm{Dom}(\Delta_g) \cap \mathrm{Lip}_b(M, g)$ with $\Delta_g f \in \mathrm{Dom}(\Delta_g)$. and where Δ_g denotes the Neumann-Laplacian.

This definition was given by Honda [3] in the context of Ricci limit spaces and in light of [2] the results therein remain valid in the larger class of finite-dimensional RCD-spaces.

Since the Riemannian metric is immanent in this definition we are lead to ask how an mmso relates to a smooth orientation $\omega \in \Omega^m_{C^\infty}(M)$, $\omega_p \neq 0$, $p \in M$, which is of course independent of any metric. To this end, we consider two classes of Riemannian metrics on M. The first one, $\mathcal{M}_2(M;K) = \{g \,|\, g \text{ complete}, \mathrm{Ric}_g \geq K, \partial M \text{ is } g - \text{convex}\}$, $K \in \mathbb{R}$, is precisely the class of Riemannian metrics with which M is a finite-dimensional RCD-space. For the second class, for any metric g on M denote by $D(M,g) := \big\{\phi \in C^2(M) \,|\, \sup \phi < \infty, \inf \phi = 1, \Pi_g \geq -N_g \log \phi\big\}$, where N_g is the inward-pointing unit normal vector field and Π_g the second fundamental form of ∂M w.r.t. g. Then $\mathcal{M}_3(M;K)$ consists of those complete Riemannian metrics g, such that for some $\phi \in D(M,g)$ one has (3) $\mathrm{Ric}_g + \Delta_g \log \phi - 2|\mathrm{d}\phi|^2_g \geq K$ and $\mathrm{Ric}_{\phi^{-2}g} \geq K$. Condition (3) implies in particular that the conformal metric $g' := \phi^{-2} g$ lies in $\mathcal{M}_2(M;K)$, although (M,g) itself might not have convex boundary. In [1] it was shown that in case ∂M is compact, any complete metric with bounded sectional curvature lies in $\mathcal{M}_3(M;K)$, which in this case implies that $\mathcal{M}_3(M;K)$ is nonempty.

With these definitions, our first result reads as follows.

Theorem (A). *i) Let $\widetilde{\omega}$ be a smooth orientation on M. Then for every $g \in \mathcal{M}_3(M;K)$ with $\mathrm{Ric}_g \geq K$, the form $\omega := \widetilde{\omega}/|\widetilde{\omega}|_g$ is an mmso on (M,g).*

ii) Assume g is a Riemannian metric on M such that (M,g) admits an mmso ω. Then ω is automatically a smooth orientation (in particular, uniquely determined mod $\mathbb{Z}_2$). In particular, the existence of such a Riemannian metric imposes a topological obstruction on M.

The proof of this theorem follows a strategy outlined in [3] and depends crucially on our second result, which should be of independent interest, and in which P^g_t denotes the heat semigroup associated with Δ_g.

Theorem (B). *i) Assume $g \in \mathcal{M}_3(M;K)$. Then for any $\phi \in D(M,g)$ satisfying (3) one has the* modified Bakry-Emery estimate

$$|\mathrm{d}P^g_t f|^2_g \leq \|\phi\|_\infty \, \mathrm{e}^{-Kt} P^g_t(|\mathrm{d}f|^2_g) \quad \textit{for all } f \in H^{1,2}(M,g) \cap L^\infty(M,g).$$

ii) Assume g is complete with $\mathrm{Ric}_g \geq -K^2$ for some $K \geq 0$ and let $\epsilon > 0$. Then one has the L^2-Calderon-Zygmund inequality*,*

$$\|\mathrm{Hess}_g(f)\|^2_{2;g} \leq \big(1 + K^2/(2\epsilon^2)\big) \, \|\Delta_g f\|^2_{2;g} + (K\epsilon^2)/2 \, \|f\|^2_{2;g} < \infty \tag{1}$$

for all $f \in \mathrm{Dom}(\Delta_g)$.

Lastly, we are currently working on the following

Problem. *For any smooth manifold M with nonempty boundary there exists $K \in \mathbb{R}$ such that $\mathcal{M}_2(M;K)$ is nonempty.*

By Theorem (A) above this would imply that a smooth manifold is smoothly orientable if and only if it is orientable in the sense of metric-measure spaces as above.

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The Hessian of heat semigroups and geometric applications

Anton Thalmaier

(joint work with Jun Cao, Li-Juan Cheng)

We discuss recent formulas for the Hessian of a heat semigroup generated by the Laplace-Beltrami operator on a Riemannian manifold and describe some geometric applications such as new versions of log-Sobolev and transportation inequalities connecting relative entropy, Stein discrepancy and Fisher information on Riemannian manifolds [8], as well as local and global quantitative C^1 estimates for functions on Riemannian manifolds [7]. The main part of the talk concentrates on Calderón-Zygmund type inequalities on complete Riemannian manifolds [5].

The L^p-Calderón-Zygmund inequality is one of the cornerstones in the regularity theory of elliptic equations. It allows to control the Hessian operator which contains all information of second order derivatives by the much simpler Laplace operator Δ, see the survey [12] for background and its wide applications particularly in the regularity theory of elliptic equations.

On Euclidean space $\mathbb{R}^n$, the Calderón-Zygmund inequality states that for any $p \in (1,\infty)$ and $u \in C_c^\infty(\mathbb{R}^n)$, it holds

$$\|\operatorname{Hess} u\|_{L^p(\mathbb{R}^n)} \le C\|\Delta u\|_{L^p(\mathbb{R}^n)},$$

where $C = C(n,p) > 0$ is a constant depending only on n and p. This inequality has been first proved by Calderón and Zygmund [4] via their seminal theory of singular integral operators based on the explicit representation of the Green kernel of the Laplacian. The inequality extends to second order uniformly elliptic operators $L = -\operatorname{div} A\nabla$ with variable coefficients A on domains $\Omega \subset \mathbb{R}^n$ without any boundary conditions [9]. In this setting, one has the following local Calderón-Zygmund inequality that given any domains $\Omega_1 \Subset \Omega$, $p \in (1,\infty)$ and $u \in C_c^\infty(\Omega)$, it holds

$$\|\operatorname{Hess} u\|_{L^p(\Omega_1)} \le C\Big(\|u\|_{L^p(\Omega)} + \|Lu\|_{L^p(\Omega)}\Big),$$

where $C = C(\Omega_1,\Omega,n,p,A) > 0$ is a constant depending on Ω_1,Ω,p,n and the elliptic coefficients of A. The proof is based on a perturbation argument of A. In particular, if A is a constant matrix, one can get rid of the term $\|u\|_{L^p(\Omega)}$ on the right of the last inequality.

A further step was taken by Güneysu and Pigola in [11] where they considered the following global Calderón-Zygmund inequality on Riemannian manifolds M of the form that for any $p \in (1,\infty)$ and $u \in C_c^\infty(M)$, it holds

$$\mathbf{CZ}(p) \qquad \|\operatorname{Hess} u\|_{L^p(M)} \leq C_1\|u\|_{L^p(M)} + C_2\|\Delta u\|_{L^p(M)},$$

where Δ is the Laplace-Beltrami operator on M and C_1, C_2 are two positive constants. It is known that in general such an inequality $\mathbf{CZ}(p)$ can hold or fail depending on p and the geometry of M.

For $p = 2$, if there is a bound $\operatorname{Ric} \geq -K$ of the Ricci curvature of M for some constant $K > 0$, then it is well-known that $\mathbf{CZ}(2)$ is a straightforward consequence of Bochner's identity. The extension of $\mathbf{CZ}(p)$ from $p = 2$ to an arbitrary $p \in (1,\infty)$ is however an intriguing problem. Inequality $\mathbf{CZ}(p)$ is usually reduced to the existence of positive constants C and σ such that

$$\big\| \, |\operatorname{Hess}(\Delta+\sigma)^{-1}u| \, \big\|_p \leq C\,\|u\|_p,$$

which is equivalent to

$$\big\| \, |\nabla(\Delta_1+\sigma)^{-1/2} \circ d(\Delta_0+\sigma)^{-1/2}u| \, \big\|_p \leq C\,\|u\|_p.$$

Here we write $\|\cdot\|_p = \|\cdot\|_{L^p(M)}$; the Laplacian Δ acting on functions and forms, is understood as self-adjoint positive operator on $L^2(M)$. The problem is thus reduced to the study of conditions for boundedness of the classical Riesz transform $d(\Delta_0+\sigma)^{-1/2}$ on functions and boundedness of the covariant Riesz transform $\nabla(\Delta_1+\sigma)^{-1/2}$ on one-forms.

Recently, Baumgarth, Devyver and Güneysu [3] studied the covariant Riesz transform on k-forms. Their results can be applied to $\mathbf{CZ}(p)$ when $1 < p < 2$ requiring the same curvature conditions as in [11] but without the local volume doubling assumption (LD) made in [11]. This comes from the fact that (LD) already holds if $\operatorname{Ric} \geq K$ for some constant K, as can be seen by the Bishop-Gromov comparison theorem and the well-known formula for the volume of balls in hyperbolic space. However, it seems difficult to establish $\mathbf{CZ}(p)$ for $p > 2$ in this way, as when trying to extend the machinery of [2] to L^p-boundedness of covariant Riesz transform for $p > 2$, the local Poincaré inequality, used extensively in [2], does not make sense on differential forms. These observations raise the following questions:

1. In the case $1 < p < 2$, is it possible to weaken the assumptions on the Riemann curvature tensor $\|R\|_\infty$ and $\|\nabla R\|_\infty$, e.g., replacing them by Ricci curvature bounds?
2. Without a lower control on the injectivity radius, under which conditions on the manifold M, the inequality $\mathbf{CZ}(p)$ holds on M for $p > 2$?

In order to answer these two questions, we develop in [5] a new functional analytic approach to $\mathbf{CZ}(p)$ that works for all $p \in (1,\infty)$. Unlike the procedure described above, our method makes use of the observation that $\mathbf{CZ}(p)$ is equivalent to the L^p-boundedness of the operator

$$\operatorname{Hess}(\Delta+\sigma)^{-1} = \int_0^\infty e^{-\sigma t} \operatorname{Hess} P_t \, dt$$

where P_t denotes the heat semigroup generated by $-\Delta$. Based on this observation, our strategy is to first establish some Hessian heat kernel estimates and then to bridge from these estimates to $\mathbf{CZ}(p)$.

Theorem 1. [8] *Let (M, g) be a complete Riemannian manifold satisfying* $\mathrm{Ric} \geq K$ *for some constant K. Let $1 < p < 2$ be fixed. Then there exists a constant $\sigma > 0$ such that the operator* $\mathrm{Hess}(\Delta + \sigma)^{-1}$ *is bounded in L^p, i.e.* $\mathbf{CZ}(p)$ *holds.*

The lower Ricci curvature bound gives rise to local Gaussian upper bounds of the heat kernel and the local volume doubling condition which enables us to deduce a series of L^2 weighted off-diagonal estimates of the Hessian heat kernel. These estimates together with the classical argument of Calderón-Zygmund decomposition allow to derive Theorem 1 which gives an affirmative answer to the first question. Compared with existing results on $\mathbf{CZ}(p)$, it should be pointed out first that the result is valid without any injectivity radius assumptions and secondly, instead of boundedness of $\|R\|_\infty$ and $\|\nabla R\|_\infty$, only a lower bound of Ric is needed. In view of the argument for $p = 2$, the result seem close to optimal.

For $p > 2$, the situation is quite different. It is well known that a lower Ricci curvature bound is not enough for $\mathbf{CZ}(p)$ to hold.

Condition (H) Assume that

(i) $\mathrm{Ric} \geq K$ for some constant K;

(ii) there exist functions K_1 and K_2 in the Kato class to M [10] such that for the curvature tensor R the following estimates hold:

$$|R|^2(x) \leq K_1(x) \quad \text{and} \quad \left|\nabla \mathrm{Ric}^\sharp + d^* R\right|^2(x) \leq K_2(x), \quad x \in M,$$

where for $x \in M$ and $v_1, v_2, v_3 \in T_xM$,

$$|R|(x) = \sup\Big\{|R^{\#,\#}(v_1, v_2)|_{\mathrm{HS}}(x) \colon\ v_1, v_2 \in T_xM,\ |v_1| \leq 1,\ |v_2| \leq 1\Big\},$$

$\langle d^* R(v_1, v_2), v_3\rangle = \langle(\nabla_{v_3} \mathrm{Ric}^\sharp)(v_1), v_2\rangle - \langle(\nabla_{v_2} \mathrm{Ric}^\sharp)(v_3), v_1\rangle$ with $R^{\#,\#}(v_1, v_2) = R(\cdot, v_1, v_2, \cdot)$ and $\mathrm{Ric}^\sharp(v) = \mathrm{Ric}(\cdot, v)^\sharp$ for $v \in T_xM$.

Under condition **(H)** we establish in [5] the following two key pointwise inequalities for the Hessian of the semigroup that for any $t > 0$, $f \in C_c^\infty(M)$ and $x \in M$, it holds

$$|\operatorname{Hess} P_t f|(x) \leq e^{2Kt} P_t |\operatorname{Hess} f|(x) + C e^{(2K+\theta)t} (P_t |\nabla f|^2)^{1/2}(x)$$

and

$$t\, |\operatorname{Hess} P_t f|(x) \leq C(1 + \sqrt{t})\, e^{(2K+\theta)t} \left(P_t |f|^2\right)^{1/2}(x)$$

for some constants $C, \theta > 0$. Both estimates are proved by using probabilistic tools from Stochastic Analysis [1, 6].

The inequalities above for $|\operatorname{Hess} P_t f|$ play a key role in the proofs of our results for $p > 2$. They allow to overcome the difficulties from the non-availability of the local Poincaré inequality and then to adapt the techniques from [2], in particular, the sharp maximal function and good-λ inequalities. Our main result for $p > 2$ reads as follows.

Theorem 2. [8] *Let M be a complete Riemannian manifold satisfying* **(H)***. Let $p > 2$ be fixed. Then there exists a constant $\sigma > 0$ such that the operator* $\operatorname{Hess}(\Delta + \sigma)^{-1}$ *is bounded in L^p, i.e.* $\mathbf{CZ}(p)$ *holds.*

Theorem 2 gives in particular an answer to the question in [11] about sufficient conditions for $\mathbf{CZ}(p)$ when $p > 2$ in the absence of control of the injectivity radius. It is worth mentioning that compared with the sufficient conditions even in the case $1 < p < 2$ ([11, Theorem D], [3, Corollary 1.8]), the geometric quantities $|R|$ and $|\nabla \operatorname{Ric}|$ do not need to be uniformly bounded on M; it is sufficient to have bounds in the Kato class which is a kind of integral condition [10].

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Limits of manifolds with a Kato bound on the Ricci curvature

Ilaria Mondello

(joint work with Gilles Carron, David Tewodrose)

Gromov's pre-compactness theorem states that a sequence of complete Riemannian manifolds with Ricci curvature uniformly bounded from below always admits a sub-sequence, converging in the pointed Gromov-Hausdorff topology to a metric space, that we refer to as a Ricci limit (see [7]). Understanding the geometry and regularity of this limit space has been a wide subject of study in the last three decades. The most significant results have been obtained in the non-collapsing case, that is when the sequence carries a uniform positive lower bound on the volume of the unit ball. Thanks to the work of M. Anderson, J. Cheeger, T.H. Colding, A. Naber, W. Jiang, there is now a good understanding of non-collapsed Ricci limits: we refer to [10] for a presentation of the state of art and precise references. However, in many situations in geometric analysis, one needs to study limits of manifolds without having a uniform lower bound on the Ricci curvature. This happens for example in the study of geometric flows (see [1, 2, 13, 14] in the case of Ricci and Kähler-Ricci flow) or of critical metrics [15]. It is then important to weaken the assumption on the Ricci curvature, and to understand the regularity of limit spaces under less restrictive hypothesis. One possible direction is to consider L^p bounds on the negative part of the Ricci curvature (see for example [11]). Together with G. Carron and D. Tewodrose, we studied a new weaker integral bound, inspired by Kato potentials in $\mathbb{R}^n$, and we obtained regularity results that recover those coming from Cheeger-Colding theory for Ricci limits.

The negative part $\mathrm{Ric}_- : M \to \mathbb{R}_+$ of the Ricci curvature is defined as follows: it is equal to zero whenever $\mathrm{Ric} \geq 0$, to minus the lowest eigenvalue of Ric otherwise. The Kato constant of a closed manifold (M^n, g) is given by

$$\mathrm{k}_t(M,g) = \sup_{x\in M} \int_0^t \int_M H(t,x,y)\mathrm{Ric}_-(y)dv_g(y)ds,$$

where $H : \mathbb{R}_+ \times M \times M \to \mathbb{R}_+$ is the heat kernel associated to the Laplacian Δ_g. It is known that, roughly, smallness of $\mathrm{k}_t(M,g)$ means that the operator $\Delta_g - \mathrm{Ric}_-$ is a small perturbation of the Laplacian. Moreover, a bound on the Kato constant ensures many geometric and analytic results (see for instance [3, 5]). In particular, if $\mathrm{k}_1(M^n,g)$ is less than or equal to $1/16n$, G. Carron showed that there exists a constant only depending on the dimension n such that (M^n,g) is doubling. This ensures pre-compactness for the set of closed manifolds with $\mathrm{k}_1(M^n,g) \leq 1/16n$. However, in order to understand the local geometry of the limit space (X,d,o), one needs to study tangent cones: for $x \in X$, those are blow up limits of the dilated spaces $(X, r_\alpha^{-1}d, o)$, for a positive sequence $\{r_\alpha\}$ tending to zero. Tangent cones can be expressed as limits of re-scaled manifolds $(M_\alpha, r_\alpha^{-2}g_\alpha, x_\alpha)$. In the case of Ricci limits, the scale invariance of Ricci curvature implies that $(M_\alpha, r_\alpha^{-2}g_\alpha)$ satisfies an almost non-negative Ricci lower bound, in the sense that $\mathrm{Ric}_{r_\alpha^{-2}g} \geq -Kr_\alpha^2$, where the right-hand side tends to 0 as i goes to infinity. If we only assume $\mathrm{k}_1(M^n,g) \leq 1/16n$, for the re-scaled manifold $(M_\alpha, r_\alpha^{-2}g_\alpha, x_\alpha)$ we have

$$\mathrm{k}_t(M_\alpha^n, r_\alpha^{-2}g) = \mathrm{k}_{r_\alpha^2 t}(M^n,g) \leq \frac{1}{16n}.$$

This is not enough to obtain information on tangent cones. We then need a stronger uniform control on the Kato constant, and in particular on the way it tends to 0 with t. We say that a sequence of closed manifolds (M_α^n, g_α) satisfies a *strong Kato bound* if there exists a non decreasing function $f : [0,1] \to \mathbb{R}_+$ such that for all $t \in [0,1]$

$$\text{(SK)} \qquad \mathrm{k}_t(M^n_\alpha, g_\alpha) \le f(t) \le \frac{1}{16n}, \quad \int_0^1 \frac{\sqrt{f(t)}}{t} dt < +\infty.$$

In this case, for the re-scaled manifolds $(M_\alpha, r_\alpha^{-2} g_\alpha)$ and all $t \in [0,1]$, $\mathrm{k}_t(M_\alpha, r_\alpha^{-2} g_\alpha) \le f(r_\alpha^2 t) \to 0$ as α tends to infinity. The strong Kato bound also implies a Euclidean volume bound: for all $x \in M$ and r, R such that $0 < r \le R \le 1$, the ratio between the volumes of the balls centered at x respectively of radius R and r is controlled by a constant, only depending on the dimension, times $(R/r)^n$. Moreover, a strong Kato bound is implied by conditions that were previously considered in the literature. For example, a smallness condition on the L^p-norm of Ric_- [12], or the L^p-bound on the Ricci curvature considered in [14], both guarantee the above bound (SK). In even dimension higher than 4, a strong Kato bound holds whenever the scalar curvature is bounded and the Q-curvature is non-negative.

In the non-collapsed case and under the strong Kato bound (SK), we obtain the following regularity results.

Theorem 1 (Carron-M-Tewodrose [4])**.** *Let $(M^n_\alpha, g_\alpha, o_\alpha)$ be a sequence of closed manifolds satisfying the strong Kato bound* (SK)*, such that for $v > 0$*

$$\mathrm{vol}_g(B_1(o_\alpha)) \ge v,$$

converging in the pointed Gromov-Hausdorff topology to (X, d, o). Then we have:

(1) *The Hausdorff dimension of X is equal to n.*
(2) ***Volume convergence**: for all $r > 0$ and $x_\alpha \in M_\alpha$ with $x_\alpha \to x \in X$,*

$$\mathrm{vol}_{g_\alpha}(B_r(x_\alpha)) \to \mathcal{H}^n(B_r(x)).$$

(3) *For all $x \in X$, all tangent cones are RCD(0,n) metric measure cones.*
(4) ***Stratification**: for all $k \in \{0, \dots, n-1\}$, set*

$$\mathcal{S}^k = \{x \in X, \text{ all tangent cone at } x \text{ do not split a factor } \mathbb{R}^{k+1}\}.$$

The Hausdorff dimension of $\mathcal{S}^k$ is less than or equal to k.
(5) ***Codimension 2**: The set $\mathcal{S}^{n-1} \setminus \mathcal{S}^{n-2}$ is empty.*
(6) *Let $\mathcal{R} = X \setminus \mathcal{S}^{n-2}$. For all $\alpha \in (0,1)$ there exists an open set $\mathcal{U}$ carrying the structure of a C^α-manifold, such that $\mathcal{R} \subset \mathcal{U}$.*

We plan to make the proof of the last point available online in the first half of 2022. One key point in our proofs consists in using the appropriate monotone quantities. On a closed manifold (M^n, g) with $\mathrm{Ric}_g \ge 0$, one can define different monotone quantities: for all $x \in M$, Bishop-Gromov inequality implies that the volume ratio $r \mapsto \mathrm{vol}_g(B_r(x))/\omega_n r^n$ is non-increasing. Besides, Li-Yau inequality ensures for all $x \in M$ that

$$(1) \qquad t \mapsto (4\pi t)^{\frac{n}{2}} H(t,x,x) \text{ is non-decreasing.}$$

Inspired by Huisken's entropy, we define a map "interpolating" between the volume ratio and this latter quantity: we define the function $U(t,x,y)$ so that

$$(4\pi t)^{\frac{n}{2}} H(t,x,y) = \exp\left(-\frac{U(t,x,y)}{4t}\right),$$

and the θ-volume

$$\theta_x^M(s,t) = (4\pi s)^{-\frac{n}{2}} \int_M \exp\left(-\frac{U(t,x,y)}{4s}\right) dv_g(y).$$

We show that if $\mathrm{Ric}_g \geq 0$, then for all $x \in M$, the map $\lambda \mapsto \theta_x^M(\lambda s, \lambda t)$ is monotone non-increasing for $t \leq s$, non-decreasing for $t \geq s$. In the case of a strong Kato bound (SK), Bishop-Gromov inequality does not hold, but we have a perturbed Li-Yau inequality [3]. This allows us to prove that if (M^n, g) satisfies a strong Kato bound, then for all $x \in M$ the map

$$\lambda \mapsto \exp\left(c_n \left(\int_0^{\lambda t} \frac{\sqrt{f(\tau)}}{\tau} d\tau\right)\left(\frac{t}{s} - \frac{s}{t}\right)\right) \theta_x^M(\lambda s, \lambda t), \tag{2}$$

is monotone non-increasing for $t \leq s$, non-decreasing for $t \geq s$.

By adapting previous results of [8, 9], we also prove that a sequence of manifolds satisfying a strong Kato bound converges in the *Mosco-Gromov-Hausdorff* sense to its limit, that is the limit X can be endowed with a limit measure μ and a limit Dirichlet energy $\mathcal{E}$ such that $(X, d, \mu, \mathcal{E})$ is a PI-Dirichlet space (doubling with a local Poincaré inequality). This ensures that the θ-volume is well-defined on X. Moreover, the θ-volume is continuous with respect to Mosco-Gromov-Hausdorff convergence of PI-Dirichlet spaces. As a consequence, (2) passes to the limit and we obtain the analog monotone map on X.

To show how we exploit this monotone quantity, we briefly sketch the proof of the fact that tangent cones are metric cones, when assuming that they are RCD(0,n) metric measure spaces. Let (X, d, o) be as in Theorem 1 and for $x \in X$, $s, t > 0$, define

$$\vartheta_x(s,t) = \lim_{\lambda \to 0} \exp\left(c_n \left(\int_0^{\lambda t} \frac{\sqrt{f(\tau)}}{\tau} d\tau\right)\left(\frac{t}{s} - \frac{s}{t}\right)\right) \theta_x^X(\lambda s, \lambda t) = \lim_{\lambda \to 0} \theta_x^X(\lambda s, \lambda t).$$

Then ϑ_x^X is 0-homogeneous in s, t. Now consider a tangent cone (Y, d, μ_Y, x) of X at x. This is a limit of the re-scaled spaces $X_\alpha = (X, \varepsilon_\alpha^{-1} d, \varepsilon_\alpha^{-n} \mu, x)$ as $\varepsilon_\alpha \to 0$. By the continuity of the θ-volumes and the re-scaling properties of the heat kernel, we have

$$\theta_x^Y(s,t) = \lim_\alpha \theta_x^{X_\alpha}(s,t) = \lim_\alpha \theta_x^X(\varepsilon_\alpha s, \varepsilon_\alpha t) = \vartheta_x(s,t).$$

As a consequence, when passing to the limit as t tends to zero, in the right-hand side of the last inequality we obtain a constant $c > 0$, so that

$$\lim_{t \to 0} \theta_x^Y(s,t) = c.$$

This leads to $\mu_Y(b_s(x)) = c\omega_n r^n$ for all $s > 0$. By a result of [6] for weakly non-collapsed RCD(0,n) spaces, the homogeneity of the measure implies that (Y, d_Y, μ_Y, x) is a metric measure cone at x.

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Distribution-valued Ricci Bounds

Karl-Theodor Sturm

Uniform lower Ricci bounds of the form $\mathsf{CD}(K,\infty)$ on mm-spaces

- are preserved for Neumann Laplacian on convex subsets, but
- never hold for Neumann Laplacian on non-convex subsets.

The goal thus is

- to find appropriate modification for non-convex subsets
- to replace constant K, by function k, measure κ, distribution, ...

To do so, for any given mm-space $(\mathsf{X},\mathsf{d},\mathsf{m})$, define the spaces $W^{1,p}(\mathsf{X})$ for $p \in [1,\infty]$, put $W^{1,\infty}_*(\mathsf{X}) := \{f \in W^{1,2}_{loc}(\mathsf{X}) : \||\nabla f|\|_{L^\infty} < \infty\}$, and denote by $W^{-1,\infty}(\mathsf{X})$ the topological dual of

$$W^{1,1+}(\mathsf{X}) := \big\{f \in L^1(\mathsf{X}) :\ f_n := f \wedge n \vee (-n) \in W^{1,2}(\mathsf{X}),\ \sup_n \big\||\nabla f_n|\big\|_{L^1} < \infty\big\}.$$

Definition 1. *Given $\kappa \in W^{-1,\infty}(\mathsf{X})$, we say that the Bochner inequality $\mathsf{BE}_1(\kappa,\infty)$ holds iff $|\nabla f| \in W^{1,2}$ for all $f \in \mathrm{D}(\Delta)$, and*

$$-\int_X \langle \nabla|\nabla f|, \nabla\phi\rangle + \frac{1}{|\nabla f|}\langle \nabla f, \nabla \Delta f\rangle \phi \, dm \geq \big\langle |\nabla f|\, \phi, \kappa \big\rangle_{W^{1,1},W^{-1,\infty}} \tag{1}$$

for all $f \in \mathrm{D}(\Delta)$ with $\Delta f \in W^{1,2}$ and all nonnegative $\phi \in W^{1,2}$.

Given $\kappa \in W^{-1,\infty}(X)$, we define a closed, lower bounded bilinear form $\mathcal{E}^\kappa$ on $L^2(X)$ by

$$\mathcal{E}^\kappa(f,g) := \mathcal{E}(f,g) + \langle f\,g, \kappa\rangle_{W^{1,1+},W^{-1,\infty}}$$

for $f, g \in \mathrm{Dom}(\mathcal{E}^\kappa) := W^{1,2}(\mathsf{X})$. Associated to it, there is a strongly continuous, positivity preserving semigroup $(P_t^\kappa)_{t\geq 0}$ on $L^2(\mathsf{X})$.

Theorem 2 ([11])**.** *The Bochner inequality $\mathsf{BE}_1(\kappa,\infty)$ is equivalent to the gradient estimate*

$$|\nabla P_t f| \leq P_t^\kappa\big(|\nabla f|\big). \tag{2}$$

To gain a better understanding of the semigroup $(P_t^\kappa)_{t\geq 0}$, assume that $\kappa = -\underline{\Delta}\psi$ for some $\psi \in W^{1,\infty}$.

Theorem 3 ([5, 11])**.** *Then $\mathcal{E}^\kappa(f,g) = \mathcal{E}(f,g) + \mathcal{E}(fg,\psi)$ and*

$$P_{t/2}^\kappa f(x) = \mathbb{E}_x\big[e^{N_t^\psi} f(B_t)\big] \tag{3}$$

where $(\mathbb{P}_x, (B_t)_{t\geq 0})$ denotes Brownian motion starting in $x \in X$, and N^ψ is the zero energy part in the Fukushima decomposition, i.e. $N_t^\psi = \psi(B_t) - \psi(B_0) - M_t^\psi$.

If $\psi \in \mathrm{Dom}(\Delta)$ then $N_t^\psi = \frac{1}{2}\int_0^t \Delta\psi(B_s)ds$ — in consistency with Braun–Habermann–St. [1].

Remark 4. The concept of *tamed spaces* proposed by Erbar–Rigoni–St.–Tamanini [7] generalizes the previous approach to distribution-valued lower Ricci bounds in various respects:

- ∗ the objects under consideration are strongly local, quasi-regular Dirichlet spaces $(\mathsf{X}, \mathcal{E}, \mathsf{m})$ (rather than infinitesimally Hilbertian mm-spaces $(\mathsf{X}, \mathsf{d}, \mathsf{m})$);
- ∗ the Ricci bounds are formulated in terms of distributions $\kappa \in W^{-1,2}_{qloc}(\mathsf{X})$ (rather than $\kappa \in W^{-1,\infty}(\mathsf{X})$); for such distributions κ which lie quasi locally in the dual of $W^{1,2}(\mathsf{X})$, the previous ansatz for defining the semigroup $(P_t^\kappa)_{t>0}$ still works with appropriate sequences of localizing stopping times;
- ∗ in addition, the distributions κ are assumed to be moderate in the sense that

$$\sup_{t\leq 1, x\in \mathsf{X}} P_t^\kappa 1(x) < \infty.$$

This reminds of the Kato condition but is significantly more general since it does not require any decomposition of κ into positive and negative parts. It always holds if $\kappa = -\underline{\Delta}\psi$ for some $\psi \in \mathrm{Lip}_b(X)$.

Example 5. The prime examples of *tamed spaces* are provided by:

(1) ground state transformation of Hamiltonian for molecules [8, 2]; yields curvature bounds in terms of unbounded functions in the Kato class;
(2) Riemannian Lipschitz manifolds with lower Ricci bound in the Kato class [10, 3, 4];
(3) time change of $\mathsf{RCD}(K,N)$-spaces with $W \in \mathrm{Lip}_b(\mathsf{X})$; typically yields curvature bounds κ which are not signed measures;
(4) restriction of $\mathsf{RCD}(K,N)$-spaces to (convex or non-convex) subsets $\mathsf{Y} \subset \mathsf{X}$ or, in other words, Laplacian with Neumann boundary conditions; yields curvature bounds in terms of signed measures $\kappa = k\,m + \ell\,\sigma$, see below.

Assume that $(\mathsf{X},\mathsf{d},\mathsf{m})$ satisfies an $\mathsf{RCD}(k,N)$-condition with variable $k : \mathsf{X} \to \mathbb{R}$ and finite N. Let a closed subset $\mathsf{Y} \subset \mathsf{X}$ be given which can be represented as sub-level set $\mathsf{Y} = \{V \le 0\}$ for some semiconvex function $V : \mathsf{X} \to \mathbb{R}$ with $|\nabla V| = 1$ on $\partial\mathsf{Y}$. Typically, V is the signed distance functions $V = \mathsf{d}(\,.\,,\mathsf{Y}) - \mathsf{d}(\,.\,,\mathsf{X}\setminus\mathsf{Y})$.

A function $\ell : \mathsf{X} \to \mathbb{R}$ is regarded as "generalized lower bound for the curvature (or second fundamental) form of $\partial\mathsf{Y}$" iff it is a synthetic lower bound for the Hessian of V.

Example 6. Assume that X is an Alexandrov space with sectional curvature ≥ 0 and that $\mathsf{Y} \subset \mathsf{X}$ satisfies an exterior ball condition: $\forall z \in \partial\mathsf{Y} : \exists$ ball $B_r(x) \subset \mathsf{X}\setminus\mathsf{Y}$ with $z \in \partial B_r(x)$. Then $\ell(z) := -\frac{1}{r(z)}$ is a lower bound for the curvature of $\partial\mathsf{Y}$.

Under weak regularity assumptions, the distributional Laplacian $\sigma_{\mathsf{Y}} := \underline{\Delta} V^+$ is a (nonnegative) measure which then will be regarded as "the surface measure of $\partial\mathsf{Y}$".

Theorem 7 ([11])**.** *Under weak regularity assumptions on V and ℓ, the restricted space $(\mathsf{Y},\mathsf{d}_{\mathsf{Y}},\mathsf{m}_{\mathsf{Y}})$ satisfies a Bakry–Émery condition $\mathsf{BE}_1(\kappa,\infty)$ with a signed measure valued Ricci bound*

$$\kappa = k\cdot m_{\mathsf{Y}} + \ell\cdot\sigma_{\mathsf{Y}}. \tag{4}$$

Thus the Neumann heat semigroup on Y satisfies

$$|\nabla P_t^{\mathsf{Y}} u|(x) \quad\le\quad \mathbb{E}_x\Big[|\nabla u|(B_t)\cdot e^{-\int_0^t k(B_s)ds}\cdot e^{-\int_0^t \ell(B_s)dL_s}\Big] \tag{5}$$

where $(B_{s/2})_{s\ge0}$ denotes Brownian motion in Y and $(L_s)_{s\ge0}$ the continuous additive functional associated with σ_{Y}.

For smooth subsets in Riemannian manifolds, this kind of gradient estimate — with $(L_s)_{s\ge0}$ being the *local time* of the boundary — has been firstly derived by Hsu [9], cf. also [12, 6].

Let us illustrate the power of the above estimates with two simple examples: the ball and its complement.

Corollary 8. *Let $(\mathsf{X},\mathsf{d},\mathsf{m})$ be an N-dimensional Alexandrov space ($N \ge 3$) with* $\mathrm{Ric} \ge -1$ *and* $\mathsf{sec} \le 0$. *Then for $\mathsf{Y} := \mathsf{X}\setminus B_r(z)$,*

$$\big|\nabla P_{t/2}^{\mathsf{Y}} f\big|(x) \quad\le\quad \mathbb{E}_x^{\mathsf{Y}}\Big[e^{t/2+\frac{1}{2r}L_t^{\partial\mathsf{Y}}}\cdot\big|\nabla f(B_t^{\mathsf{Y}})\big|\Big].$$

In particular, $\mathrm{Lip}(P^{\mathsf{Y}}_{t/2}f) \le \sup_x \mathbb{E}^{\mathsf{Y}}_x\big[e^{t/2+\frac{1}{2r}L^{\partial\mathsf{Y}}_t}\big]\cdot \mathrm{Lip}(f)$ *and*

$$\big|\nabla P^{Y}_{t/2}f\big|^2(x) \le e^{Ct+C'\sqrt{t}}\cdot P^{\mathsf{Y}}_{t/2}\big|\nabla f\big|^2(x). \tag{6}$$

Upper and lower bound of curvature (here 0 and -1, resp) can be chosen to be any numbers. Note that *no estimate* of the form

$$\big|\nabla P^{\mathsf{Y}}_{t/2}f\big|^2(x) \le e^{Ct}\cdot P^{\mathsf{Y}}_{t/2}\big|\nabla f\big|^2(x)$$

can hold true due to the non-convexity of Y. Thus it is *necessary* to take into account the singular contribution arising from the negative curvature of the boundary.

In the next example, the singular contribution arising from the positive curvature of the boundary can be ignored. However, taking it into account will significantly *improve* the gradient estimate.

Corollary 9. *Let* $(\mathsf{X},\mathsf{d},\mathsf{m})$ *be an* N*-dimensional Alexandrov space with* $\mathrm{Ric}\ge 0$ *and* $\mathsf{sec}\le 1$. *Then for* $\mathsf{Y} := \overline{B}_r(z)$ *for some* $z\in\mathsf{X}$ *and* $r\in(0,\pi/4)$.

$$\big|\nabla P^{Y}_{t/2}f\big|(x) \quad\le\quad \mathbb{E}^{\mathsf{Y}}_x\Big[e^{-\frac{\cot r}{2}L^{\partial\mathsf{Y}}_t}\cdot\big|\nabla f(B^Y_t)\big|\Big].$$

In particular, $\mathrm{Lip}(P^{Y}_{t/2}f) \le \sup_x \mathbb{E}^{\mathsf{Y}}_x\big[e^{-\frac{\cot r}{2}L^{\partial\mathsf{Y}}_t}\cdot \mathrm{Lip}(f)\big]$ *and*

$$\big|\nabla P^{\mathsf{Y}}_{t/2}f\big|^2(x) \le e^{-t\frac{N-1}{2}\cot^2 r+1}\cdot P^{\mathsf{Y}}_{t/2}\big|\nabla f\big|^2(x). \tag{7}$$

Taking into account the curvature of the boundary, allows us to derive a positive lower bound for the spectral gap (without involving any diameter bound and despite possibly vanishing Ricci curvature in the interior).

Corollary 10. *In the previous setting,* $\lambda_1 \ge \frac{N-1}{2}\cot^2 r$.

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Weak Hardy/Poincaré inequalities and criticality theory

Marcel Schmidt

Let $q\colon L^2(X,\mu) \to [0,\infty]$ be a closed quadratic form satisfying the first Beurling-Deny criterion, i.e., $q(|f|) \leq q(f)$ for all $f \in L^2(X,\mu)$. Typical examples are quadratic forms of (discrete) Schrödinger operators, see e.g. [6, 10, 3] or Dirichlet forms, see e.g. [2]. This talk is about inequalities of the form

$$(\diamond) \qquad \int_X f^2 w d\mu \leq \alpha(r) q(f) + r\Phi(f), \quad f \perp_w \ker q,\, r > 0.$$

Here, $w\colon X \to (0,\infty)$, $\alpha\colon (0,\infty) \to (0,\infty)$ is decreasing, $\Phi\colon L^2(X,\mu) \to [0,\infty]$ satisfies $\Phi(\lambda f) = |\lambda|^2 \Phi(f)$ for $\lambda \in \mathbb{R}$, $f \in L^2(X,\mu)$, and the symbol $\perp_w$ indicates that orthogonality is considered in $L^2(X, w\mu)$. If $\ker q = \{0\}$, Inequality ($\diamond$) is called *weak Hardy inequality* and if $\ker q \neq \{0\}$, Inequality ($\diamond$) is called *weak Poincaré inequality*. In the case $\Phi = 0$, the function α becomes a constant and Inequality ($\diamond$) is referred to as *Hardy inequality* if $\ker q = \{0\}$, respectively *Poincaré inequality* if $\ker q \neq \{0\}$.

We discuss abstract criteria for (weak) Hardy/Poincaré inequalities with respect to the functionals $\Phi = 0$ and $\Phi(f) = \|f/h\|_\infty^2$, where $h\colon X \to (0,\infty)$. We study which w are eligible but do not aim at giving explicit bounds for α. As an application we show how these inequalities can be employed to establish a criticality theory for general forms satisfying the first Beurling-Deny criterion.

It turns out that if one replaces $\int f^2 w d\mu$ on the left side of Inequality ($\diamond$) by $(\int f w d\mu)^2$, then for generic Φ there exists a function α such that the modified inequality holds. For forms with the first Beurling-Deny criterion and $\ker q = \{0\}$, this always leads to a weak Hardy inequality with respect to $\Phi(f) = \|f/h\|_\infty^2$ as long as $h \in L^2(X,\mu)$ and $w \in L^2(X, h^2\mu)$. Thus, one can say that weak Hardy inequalities hold generically for forms with the first Beurling-Deny criterion and trivial kernel.

Forms satisfying a Hardy inequality for some $w\colon X \to (0,\infty)$ are called *subcritical* and there is a large amount of literature on them, see e.g. [4, 5, 7] (and references therein) for elliptic operators with real coefficients, [1, 10, 11] for generalized Schrödinger forms (Dirichlet form plus potential term) and [3] for discrete Schrödinger operators. It turns out that weak Hardy inequalities can be employed to study subcriticality. We give a comprehensive characterization which should cover most previous results but also gives some new insights.

If q is irreducible, in the situations where subcriticality is well understood (see the mentioned references on subcriticality), there is a dichotomy. Either q is subcritical or there exists a sequence (φ_n) with $q(\varphi_n) \to 0$ that converges pointwise to a strictly positive function h. In the second case q is called *critical*, and the function h is unique up to multiplication by a constant and called *Agmon ground state* of q. Our discussion shows that this dichotomy may fail in general. It becomes a trichotomy and the third case (besides criticality and subcriticality) happens if and only if the form q does not possess an excessive functions. So far we do not have a concrete example for the third case and the reason is the following: In the situations where subcriticality was studied previously, the corresponding semigroups are irreducible semigroups of kernel operators and we can prove that such semigroups always admit excessive functions (which are then excessive functions for the form q). With the same arguments we obtain a partial answer to a question of Schep [9] related to Schur tests on the continuity of positive operators on L^p-spaces.

Weak Poincaré inequalities were introduced in [8] for conservative Dirichlet forms on finite measure spaces to study the rate of convergence of their semigroups to equilibrium when there is no spectral gap, i.e., when they do not satisfy a Poincaré inequality with $w = 1$. Similarly, the weak Hardy inequalities mentioned above can be employed to establish the rate of convergence to 0 of semigroups coming from forms with trivial kernel.

A closed quadratic form q on $L^2(X, \mu)$ with the first Beurling-Deny criterion can be extended to a lower semicontinuous quadratic form q_e on $L^0(X, \mu)$, the so-called extended form. With this at hand the Agmon ground state (if it exists) is an element of the kernel of q_e and criteria for subcriticality can be formulated conveniently in terms of q_e. One of the observations is that subcriticality is equivalent to the domain of q_e being a Hilbert space, a fact which is well-known for Dirichlet forms (where subcriticality is called transience and the domain of the extended form is the extended Dirichlet space), see e.g. [2, Section 1.6]. Moreover, it is known for critical Dirichlet forms (usually called recurrent Dirichlet forms) that the quotient of the extended Dirichlet space modulo constants is a Hilbert space if a Poincaré inequality holds, see e.g. [2, Section 4.8]. We show that some sort of converse holds in our setting: In the critical case completeness of the domain of q_e modulo the kernel of q_e is equivalent to a weak Poincaré inequality. This observation seems to be new. An example of a Dirichlet form on configuration space from [8] shows that there are irreducible conservative Dirichlet forms without weak Poincaré inequality and as a consequence we obtain that their extended Dirichlet space modulo constants (i.e. the domain of the extended form modulo its kernel) is not complete. To the best of our knowledge it is a new observation that such forms exist.

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Tamed spaces – Dirichlet spaces with distribution-valued Ricci bounds

Chiara Rigoni

(joint work with Matthias Erbar, Karl-Theodor Sturm, Luca Tamanini)

Synthetic lower Ricci bounds have proven to be a powerful concept for analyzing the geometry of singular spaces. The most prominent versions of such synthetic Ricci bounds are the Eulerian formulation in the setting of Dirichlet spaces by Bakry–Émery and the Lagrangian formulation in the setting of metric measure spaces by Lott–Villani and Sturm.

Bakry and Émery, in their seminal paper [5], characterized synthetic lower Ricci bounds $K \in \mathbb{R}$ for a given strongly local Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ in terms of the generalized Bochner inequality

$$\Gamma_2(f) \geq K \cdot \Gamma(f).$$

Here Γ denotes the carré du champ associated with $\mathcal{E}$ and Γ_2 the iterated carré du champ. For the canonical Dirichlet space with $\mathrm{X} = \mathsf{M}$, $\mathfrak{m} = \mathrm{vol}_{\mathsf{g}}$, and $\mathcal{E}(f) = \frac{1}{2}\int_{\mathsf{M}} |\nabla f|^2 \, \mathrm{d}\mathfrak{m}$ on a Riemannian manifold (M, g) this reads as

$$\frac{1}{2}\Delta|\nabla f|^2 - \langle \nabla f, \nabla \Delta f \rangle \geq K \cdot |\nabla f|^2, \tag{1}$$

which in turn is well known – due to Bochner's equality – to be equivalent to $\mathrm{Ric}_{\mathsf{g}} \geq \mathrm{K} \cdot \mathsf{g}$.

A synthetic notion of lower Ricci curvature bounds in the setting of metric measure spaces based on optimal transport has been developed by Lott, Villani and Sturm in [8, 9, 10], leading to a fruitful study of analytic and geometric properties of these structures. In particular, Ambrosio, Gigli and Savaré in a series of seminal papers [1, 2] developed a powerful first order calculus on such spaces leading to natural notions of (modulus of the) gradient, energy functional (called Cheeger energy), and heat flow. For so-called infinitesimally Hilbertian spaces the Cheeger energy is quadratic and defines a Dirichlet form and (under minimal assumptions) the Eulerian and Langrangian approaches to synthetic Ricci bounds have been shown to be equivalent [3, 6, 4], providing in particular a Bochner inequality for metric measure spaces.

In this talk, we develop a generalization of the concept of synthetic lower Ricci bounds that goes far beyond the framework of uniform bounds. Indeed, many important properties and quantitative estimates which typically are regarded as consequences of uniform lower Ricci bounds also hold true in much more general settings.

Our notion of **tamed spaces** will refer to Dirichlet spaces $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ which admit a distribution-valued lower Ricci bound, formulated as a canonical generalization of (1). Roughly speaking, we are going to replace the constant K in (1) by a distribution κ and to consider the inequality in distributional sense, that is, as

$$\int_{\mathrm{X}} \varphi\, \Gamma_2(f)\, \mathrm{d}\mathfrak{m} \geq \big\langle \kappa, \varphi\, \Gamma(\mathrm{f}) \big\rangle$$

for all sufficiently regular f and $\varphi \geq 0$. The distributions κ to be considered will lie in the class $\mathcal{F}^{-1}_{\mathrm{qloc}}$. Here $\mathcal{F}^{-1}$ denotes the dual space of the form domain $\mathcal{F} = \mathsf{D}(\mathcal{E})$ and $\mathcal{F}^{-1}_{\mathrm{qloc}}$ denotes the class of κ's for which there exists an exhaustion of X by quasi-open subsets $G_n \nearrow \mathrm{X}$ such that κ coincides on each G_n with some element in $\mathcal{F}^{-1}_{G_n}$.

Already in the case of Riemannian manifolds, our new setting contains plenty of important examples which are not covered by any of the concepts of "spaces with uniform lower Ricci bounds", among them we have

(i) *"Singularity of Ricci at ∞"*: Smooth Riemannian manifolds with Ricci curvature bounded from below in terms of a continuous – but unbounded – function which globally lies in the Kato class.

(ii) *"Local singularities of Ricci"*: Riemannian manifolds with (synthetic) Ricci curvature bounded from below in terms of a locally unbounded function which lies in L^p for some $p > n/2$. Such "singular" manifolds for instance are obtained from smooth manifolds by ground state transformations (see e.g. [7]), conformal transformations, or time changes with singular weight functions.

(iii) *"Singular Ricci induced by the boundary"*: Riemannian manifolds with boundary for which the second fundamental form is bounded from below in terms of a (possibly unbounded) function which lies in L^p w.r.t. the boundary measure for some $p > n-1$. Such manifolds with boundaries in particular appear as closed subsets of manifolds without boundaries.

(iv) *"Singular Ricci at the rim"*: Doubling of a Riemannian surface with boundary leads to a (nonsmooth) Riemannian surface which admits a uniform (synthetic) lower Ricci bound if and only if the initial surface has convex boundary.

Given a Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ and a distribution $\kappa \in \mathcal{F}^{-1}_{\mathrm{qloc}}$, the crucial quantities to formulate our synthetic lower Ricci bound will be the **taming energy** $\mathcal{E}^\kappa$ – a singular zero-order perturbation of $\mathcal{E}$ – and the **taming semigroup** $(P_t^\kappa)_{t\geq 0}$. The latter allows for a straightforward definition via the Feynman-Kac formula as

$$P_t^\kappa f(x) := \mathbb{E}_x\Big[e^{-A_t^\kappa}\, f(B_t)\Big]$$

in terms of the stochastic process $\big(\mathbb{P}_x, B_t\big)_{x\in\mathrm{X},t\geq 0}$ properly associated with $(\mathrm{X},\mathcal{E},\mathfrak{m})$ and in terms of the local continuous additive functional $(A_t^\kappa)_{t\geq 0}$ associated with κ. We say that the distribution κ is **moderate** if

$$\sup_{t\in[0,1]}\sup_{x\in\mathrm{X}}\mathbb{E}_x\Big[e^{-A_t^\kappa}\Big] < \infty.$$

In this case, $(P_t^\kappa)_{t\geq 0}$ defines a strongly continuous, exponentially bounded semigroup on $L^2(\mathrm{X}, \mathfrak{m})$ and thus it generates a lower bounded, closed quadratic form $\big(\mathcal{E}^\kappa, \mathsf{D}(\mathcal{E}^\kappa)\big)$. The latter indeed can be identified with the relaxation of the quadratic form

$$\dot{\mathcal{E}}^\kappa(f) := \mathcal{E}(f) + \mathcal{E}_1(\psi_n, f^2)$$

defined on a suitable subset of $\bigcup_n \mathcal{F}_{G_n}$ where $(G_n)_n$ denotes an exhaustions of X by quasi-open sets G_n such that $\kappa \in \mathcal{F}^1_{G_n}$ and where $\psi_n := (-\mathsf{L}_{\mathsf{G_n}} + 1)^{-1}\kappa$. We also provide a condition on κ which guarantees that $\dot{\mathcal{E}}^\kappa$ is closable, in which case $\mathcal{E}^\kappa$ is its closure.

At this point, we say that a Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$ is **tamed** if there exists a moderate distribution $\kappa \in \mathcal{F}^{-1}_{\mathrm{qloc}}$ such that the following *Bochner inequality* holds true:

$$\mathcal{E}^{\kappa/2}\big(\varphi, \Gamma(f)^{1/2}\big) + \int \varphi\, \frac{1}{\Gamma(f)^{1/2}}\Gamma\big(f, \mathsf{L}\mathsf{f}\big)\,\mathrm{d}\mathfrak{m} \leq 0 \tag{2}$$

for all f and $\varphi \geq 0$ in appropriate functions spaces. In this case, κ will be called **distribution-valued lower Ricci bound** or **taming distribution** for the Dirichlet space $(\mathrm{X}, \mathcal{E}, \mathfrak{m})$.

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Vector calculus for tamed Dirichlet spaces

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In the sequel, we present the results achieved in the preprint [2].

Summary. Let $(M,\mathscr{E},\mathfrak{m})$ be a quasi-regular, strongly local Dirichlet space with form domain $\mathcal{F}$. For simplicity, we assume that $\mathscr{E}$ comes with a carré du champ $\Gamma\colon \mathcal{F}^2 \to L^1(M)$ w.r.t. $\mathfrak{m}$. Let $(\mathsf{P}_t)_{t\geq 0}$ be the heat flow associated with $\mathscr{E}$ whose generator, in turn, is denoted by Δ. In particular, $\mathsf{P}_t = \mathrm{e}^{\Delta t}$.

Framework. Recently, partly motivated by the results gathered in [4, 13, 14], the theory of *tamed spaces* has been introduced in [6]. The inherent taming condition $\mathrm{BE}_2(\kappa,\infty)$ gives a meaning to the Ricci curvature of $(M,\mathscr{E},\mathfrak{m})$ being bounded from below by a distribution κ. Its synthetic Bakry–Émery-type formulation is based on a weak version of Bochner's inequality in terms of Schrödinger operators.

This setting widely extends the theory of BE spaces for uniform bounds, which already *strictly* contains RCD spaces. Moreover, it includes situations with unbounded "Ricci" curvature, interior singularities, or boundary irregularities, and arises after time change, drift transformation, or conformal transformation of non-convex subsets of RCD spaces [14].

Objective. Inspired by and following the work [7] for RCD spaces, we construct a first and second order calculus over a $\mathrm{BE}_2(\kappa,\infty)$ space $(M,\mathscr{E},\mathfrak{m})$ if κ is a signed measure in the extended Kato class $\mathcal{K}_{1-}(M)$. In turn, this will allow us to define e.g. Hessians, covariant and exterior derivatives, and Ricci curvature.

Extended Kato class. The Kato class originates in the seminal works [1, 11] within the study of molecular Schrödinger operators. Recently, it has attracted increasing interest in connection with singular lower Ricci bounds, see e.g. [3, 5, 8, 9].

Technically, our reason for considering $\kappa \in \mathcal{K}_{1-}(M)$ instead of more general distributions as treated in [6] is that the Schrödinger operator $\Delta^{q\kappa}$ with potential $q\kappa$, $q \in [1,2]$, can be understood using form techniques, a fine analysis of which, in turn, yields the self-improvement property for $\mathrm{BE}_2(\kappa,\infty)$ [6]. Moreover, the associated Feynman–Kac semigroup has good L^p-properties [12].

First order differential structure. A first order calculus is well-known to exist for general Dirichlet spaces with carré du champ w.r.t. $\mathfrak{m}$, see e.g. [10]. It relies on the construction of measurable Hilbert fields $(\mathscr{H}_x)_{x\in M}$ playing the role of a (co-)tangent bundle, from which 1-forms and vector fields — hence differentials and gradients of functions — can be axiomatized in a *fiberwise* manner.

In [2] we provide an equivalent construction of spaces of L^2-1-forms and L^2-vector fields based on the concept of L^p-normed L^∞-modules [7]. Roughly speaking, the latter are Banach spaces $\mathscr{M}$ endowed with a *pointwise norm* $|\cdot|\colon \mathscr{M} \to L^p(M)$ satisfying certain locality properties such that $\|\cdot\|_{\mathscr{M}} = \big\||\cdot|\big\|_{L^p(M)}$. The function $|\cdot|$ allows us to speak about $\mathfrak{m}$-a.e. properties of tensor fields without having a (co-)tangent bundle at our disposal a priori.

Theorem 1. *There exists a tuple $(L^2(T^*M),\mathrm{d})$, unique in a certain sense, consisting of an L^2-normed L^∞-module $L^2(T^*M)$ with pointwise norm $|\cdot|\colon L^2(T^*M) \to L^2(M)$ as well as a linear map $\mathrm{d}\colon \mathcal{F} \to L^2(T^*M)$, such that $L^2(T^*M)$ is locally generated by $\mathrm{d}\,\mathcal{F}$ in a certain sense, and for every $f \in \mathcal{F}$,*

$$|\mathrm{d}f|^2 = \Gamma(f) \quad \mathfrak{m}\text{-a.e.}$$

This *cotangent module* $L^2(T^*M)$ naturally induces further objects and spaces, e.g. the *tangent module* $L^2(TM)$, L^2-spaces of pointwise tensor and exterior products, L^p-spaces of tensor fields, divergence, etc.

Second order differential structure. In the absence of any smooth structure and since Dirichlet spaces are first order in nature, a higher order calculus might be out of reach in general. Oriented by [7], however, the key that a second order calculus can be developed in our situation is precisely the presence of a lower (albeit possibly quite singular) Ricci bound.

Hessian, covariant derivative, exterior differential. Our second order calculus in [2] is grounded upon making sense of the three named differential operators. In turn, their axiomatization is motivated by the subsequent identities from Riemannian geometry, for suitable objects involved:

$$\begin{aligned}
2\,\mathrm{Hess}\, f(\nabla g_1, \nabla g_2) &= \big\langle \nabla\langle\nabla f, \nabla g_1\rangle, \nabla g_2\big\rangle + \big\langle \nabla\langle\nabla f, \nabla g_2\rangle, \nabla g_1\big\rangle \\
&\quad - \big\langle \nabla f, \nabla\langle\nabla g_1, \nabla g_2\rangle\big\rangle, \\
\big\langle \nabla_{\nabla g_1} X, \nabla g_2\big\rangle &= \big\langle \nabla\langle X, \nabla g_1\rangle, \nabla g_2\big\rangle - \mathrm{Hess}\, g_2(X, \nabla g_1), \\
\mathrm{d}\omega(X_1, X_2) &= \mathrm{d}\big[\omega(X_2)\big](X_1) - \mathrm{d}\big[\omega(X_1)\big](X_2) \\
&\quad - \omega(\nabla_{X_1} X_2 - \nabla_{X_2} X_1).
\end{aligned}$$

The first identity characterizes the Hessian only in terms of Γ, the second characterizes the covariant derivative only in terms of Γ and Hess, and the third characterizes the exterior differential only in terms of Γ and ∇. Hence, these differential operators are defined by the respective r.h.s.'s in an integration by parts manner.

Indeed, many functions have a Hessian, which results from the following integrated Bochner identity. Compared to [7], however, more work has to be done in verifying this, since κ is not necessarily absolutely continuous w.r.t. $\mathfrak{m}$.

Theorem 2. *Every function $f \in \mathscr{D}(\Delta)$ belongs to $\mathscr{D}(\mathrm{Hess})$ with*

$$\int_M \big|\mathrm{Hess}\, f\big|_{\mathrm{HS}}^2 \,\mathrm{d}\mathfrak{m} \leq \int_M (\Delta f)^2 \,\mathrm{d}\mathfrak{m} - \int_M |\nabla f|_\sim^2 \,\mathrm{d}\kappa.$$

Ricci curvature. The second main result of [2] is an appropriate definition of a measure-valued Ricci curvature for tamed spaces. It is defined in terms of the expected vector Bochner identity, the r.h.s. of which indeed makes sense for so-called *regular vector fields* [2] which are dense in $L^2(TM)$.

Theorem 3. *There exists a unique continuous, symmetric, $\mathbf{R}$-bilinear map* **Ric** *from $H^{1,2}_\sharp(TM)$ taking values in the space of finite signed Borel measures on M charging no $\mathscr{E}$-polar sets such that for every regular vector field X,*

$$\mathbf{Ric}(X, X) = \mathbf{\Delta}\frac{|X|^2}{2} + \big\langle X, (\vec{\Delta} X^\flat)^\sharp\big\rangle\,\mathfrak{m} - \big|\nabla X\big|_{\mathrm{HS}}^2\,\mathfrak{m}.$$

Here $H^{1,2}_\sharp(TM)$ denotes the closure of the space of regular vector fields w.r.t. a specific covariant $H^{1,2}$-norm, $\mathbf{\Delta}$ is the measure-valued Laplacian — which, albeit being given distributionally a priori, is signed-measure-valued [2, 6] — and $\vec{\Delta}$ is the Hodge Laplacian. Moreover, $\mathbf{Ric}(X, X) \geq |X|_\sim^2\,\kappa$ for every $X \in H^{1,2}_\sharp(TM)$.

Further results. Our work [2] comes with many further results, some of which we outline here. First, a first order calculus on 1-forms and vector fields allows us to *define* heat flows on the respective objects in a standard way. In [2], we prove first functional inequalities for the latter. Second, we show a version of the Hodge theorem. Third, we concisely incorporate the finite-dimensional taming condition $\mathrm{BE}_2(\kappa, N)$, $N \in [1, \infty)$, from [6] in our discussion. This leads to first insights into the structure of tamed spaces — for instance, the local dimension, in a certain sense, of $L^2(TM)$ is $\mathfrak{m}$-a.e. no larger than $\lfloor N \rfloor$ — and an "N-Ricci tensor".

Open questions. We regard [2] as the starting point of a thorough study of tamed spaces in various directions, equally influential as [7] for RCD spaces. Possible problems we will devote ourselves to in the near future are the following.

- Define and study nonsmooth covariant Schrödinger semigroups [8].
- Prove rigidity results for and certain properties, e.g. structural ones, of finite-dimensional tamed spaces.
- Develop foundations of a nonsmooth stochastic differential geometry — which is missing even for RCD spaces to date —, yielding e.g. versions of the Feynman–Kac formula for the heat flow on 1-forms, or the Bismut–Elworthy–Li formula for the functional heat flow.

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Construction and convergenve for traces of quadratic forms

Ali BenAmor
(joint work with H. BelHadjAli, C. Seifert and A. Thabet)

Traces of quadratic forms in Hilbert spaces are generalizations of Dirichlet-to-Neumann operators. Constructions of such objects and their analysis are given in [1, 4, 5]. In case where the considered quadratic form is a Dirichlet form the obtained quadratic form is the one associated with the time changed process. Here we present a new method for constructing traces of forms and discuss continuity property of the trace map. Our method relies on an approximation procedure together with the use of Kato–Simon monotone convergence theorem for quadratic forms. Our contribution is mainly based on the paper [3].

Let $\mathcal{H}, \mathcal{H}_{\rm aux}$ be two Hilbert spaces. Let $(\cdot,\cdot)$ and $(\cdot,\cdot)_{\rm aux}$ denote the scalar products on $\mathcal{H}$ and $\mathcal{H}_{\rm aux}$, respectively. Let $\mathcal{E}$ be a closed positive quadratic form with domain $\mathcal{D} \subseteq \mathcal{H}$. For $u \in \mathcal{D}$ we abbreviate $\mathcal{E}[u] := \mathcal{E}(u,u)$ and for every $\lambda > 0$ set

$$\mathcal{E}_\lambda[u] := \mathcal{E}[u] + \lambda \|u\|^2.$$

Assume we are given a linear operator $J\colon \operatorname{dom} J \subseteq \mathcal{D} \to \mathcal{H}_{\rm aux}$ with dense range such that J is closed in $(\mathcal{D}, \mathcal{E}_1^{1/2})$. For $\lambda > 0$ we define $J_\lambda\colon \operatorname{dom} J \subseteq (\mathcal{D}, \mathcal{E}_\lambda^{1/2}) \to \mathcal{H}_{\rm aux}$ by $J_\lambda u := Ju$. Let $(\ker J_\lambda)^{\perp_{\mathcal{E}_\lambda}}$ be the $\mathcal{E}_\lambda$-orthogonal complement of $\ker J_\lambda$ and let P_λ the $\mathcal{E}_\lambda$-orthogonal projection onto $(\ker J_\lambda)^{\perp_{\mathcal{E}_\lambda}}$. For $\lambda > 0$ we construct a new family of closed positive densely defined quadratic forms as follows (see [2, Theorem 1.1])

$$\operatorname{dom} \check{\mathcal{E}}_\lambda := \operatorname{ran} J, \quad \check{\mathcal{E}}_\lambda[Ju] := \mathcal{E}_\lambda[P_\lambda u] \quad \text{for all } u \in \operatorname{dom} J. \tag{1}$$

Let $\check{H}_\lambda$ be the positive self-adjoint operator associated with $\check{\mathcal{E}}_\lambda$.

Theorem 1 (Dirichlet principle)**.** *Let $\lambda > 0$, $u \in \operatorname{dom} J$.*

$$\check{\mathcal{E}}_\lambda[Ju] = \inf\{\mathcal{E}_\lambda[v] :\ v \in \operatorname{dom} J,\ Jv = Ju\}.$$

It follows that $\check{\mathcal{E}}_\lambda \le \check{\mathcal{E}}_\mu$ for $\lambda \le \mu$.

The Dirichlet principle is the main input towards proving:

Theorem 2. *There exists a positive selfadjoint operator $\check{H}$ in $\mathcal{H}_{\rm aux}$ such that*

$$\lim_{\lambda \downarrow 0} (\check{H}_\lambda + 1)^{-1} = (\check{H} + 1)^{-1} \text{ strongly.}$$

Furthermore, defining $\check{\mathcal{E}}_0$ in $\mathcal{H}_{\rm aux}$ by

$$\operatorname{dom} \check{\mathcal{E}}_0 := \operatorname{ran} J, \quad \check{\mathcal{E}}_0[Ju] := \lim_{\lambda \downarrow 0} \check{\mathcal{E}}_\lambda[Ju] \quad \text{for all } u \in \operatorname{dom} J,$$

then $\check{H}$ is the self-adjoint operator associated with the closure of $(\check{\mathcal{E}}_0)_{\rm reg}$. In particular, if $\check{\mathcal{E}}_0$ is closable then $\check{H}$ is the self-adjoint operator associated with the closure of $\check{\mathcal{E}}_0$.

We set $\check{\mathcal{E}}$ the densely defined positive closed quadratic form associated to $\check{H}$ and we call it the *trace* of $\mathcal{E}$ with respect to J. For explicit computation of $\check{\mathcal{E}}$ as well as its properties we refer the reader to [3].

It is an open problem whether this method still applies for nonsymmetric forms. Let us turn our attention to discuss continuity of the trace operation. Let X be a locally compact separable metric space, m a positive Radon measure on X with full support X and $\mathcal{E}$ a regular Dirichlet form having domain $\mathcal{D} \subseteq L^2(X,m)$. Let μ be a positive Radon measure on X charging no set of zero capacity. We consider a sequence $(\mathcal{E}^n)$ of regular Dirichlet forms with $\operatorname{dom}\mathcal{E}^n = \mathcal{D}$ for all $n \in \mathbb{N}$.

We make the following three assumptions. First, assume there exists a constant $c > 0$ such that

$$\text{(A.1)} \qquad c^{-1}\mathcal{E}[u] \leq \mathcal{E}^n[u] \leq c\mathcal{E}[u] \quad \text{for all } u \in \mathcal{D}, n \in \mathbb{N}.$$

Assumption (A.1) implies in particular that $\mathcal{E}$ and $\mathcal{E}^n$ induce equivalent capacities. Hence we shall use deliberately the abbreviations “q.e.” and “q.c.” to mean with respect to any of these capacities. The second assumption is

$$\text{(A.2)} \qquad J_1\colon (\mathcal{D} \cap L^2(X,\mu), \mathcal{E}_1) \to L^2(X,\mu), \quad u \mapsto u \quad \text{is continuous.}$$

Note that since J_1 is densely defined, we can then extend J_1 to $\mathcal{D}$.

For $n \in \mathbb{N}$ we define as before

$$J_1^n\colon (\mathcal{D} \cap L^2(X,\mu), \mathcal{E}_1^n) \to L^2(X,\mu), \quad u \mapsto u.$$

By (A.1) and (A.2) also J_1^n is continuous and can be extended to $\mathcal{D}$. For the third assumption, for $n \in \mathbb{N}\cup\{\infty\}$ let H^n be the positive self-adjoint operator associated with $\mathcal{E}^n$ and $K^n := (H^n + 1)^{-1}$. Then we assume that for all $u \in L^2(X,m)$ we have

$$\text{(A.3)} \qquad J_1^n K^n u \to J_1^\infty K^\infty u \quad \text{in } L^2(X,\mu).$$

For $n \in \mathbb{N} \cup \{\infty\}$ and $\lambda > 0$ we denote by $\check{\mathcal{E}}_\lambda^n$ the trace of the Dirichlet form $\mathcal{E}_\lambda^n$ w.r.t. the measure μ.

Theorem 3. *Assume* (A.1)*,* (A.2) *and* (A.3)*. Let* $(\mathcal{E}^n)$ *be Mosco-convergent to* $\mathcal{E}^\infty$*. Then:*

(a) *The sequence of trace forms* $(\check{\mathcal{E}}_\lambda^n)$ *Mosco-converges to the corresponding trace form* $\check{\mathcal{E}}_\lambda^\infty$ *for every* $\lambda > 0$*.*

(b) *For every sequence* (λ_j) *in* $(0,\infty)$ *such that* $\lambda_j \downarrow 0$ *there exists a sequence* (n_j) *in* $\mathbb{N}$ *with* $n_j \to \infty$ *such that* $(\check{\mathcal{E}}_{\lambda_j}^{n_j})$ *Mosco-converges to the trace form* $\check{\mathcal{E}}^\infty$*.*

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Bounds on the first Betti number - an approach via Schatten norm estimates on semigroup differences

Peter Stollmann
(joint work with Marcel Hansmann, Christian Rose)

1. Introduction-Motivation

The talk was about compact Riemannian manifolds (M, g) of dimension n, no boundary.

Theme: Mostly nonnegative Ricci curvature implies small $b_1(M)$.

- View $M \ni x \mapsto \mathrm{Ric}_x$, and Ric_x is the Ricci tensor interpreted as an endomorphism of the cotangent space $\Omega^1_x(M) := (T_xM)^*$.
- Set
$$\rho : M \to \mathbb{R}, \rho(x) := \inf \sigma(\mathrm{Ric}_x),$$
the lowest eigenvalue of $\mathrm{Ric}_\bullet$.
- The **Laplace–Beltrami operator** is a non-negative selfadjoint operator in $L^2(M)$:
$$\Delta = \delta d \geq 0,$$
- The **Hodge–Laplacian** is acting on 1-forms:
$$\mathbf{\Delta^1} = \delta d + d\delta \geq 0$$
is selfadjoint in $L^2(M; \Omega^1) = L^2(M; \mathbb{R}^n)$.
- The first Betti number is
$$b_1(M) = \dim(\mathrm{Ker}(\mathbf{\Delta^1}))$$

A short history from a personal perspective.

- Bochner '46
- Elworthy and Rosenberg '91: Manifolds with wells of negative curvature. Idea: use the **Kato class**
- Starting point for PhD thesis of Christian Rose
- Gallot, Bérard, Besson ... L^p–conditions '88 –
- Petersen & Sprouse '98
- Güneysu-Pallara '13
- Carron '16, Rose-PS '16
- Carron-Rose '18, ...

After the talk, Prof. X.D. Li kindly informed me, that he had used a general form of Kato's condition (known as gaugeability) in his work on Riesz transforms as early as 2006, see [18].

Today's message from Hansmann-Rose-S '18: Use a different tool from Mathematical Physics, the Birman–Schwinger principle.

The main result.

Theorem (See [14]). *For $\rho_0 > 0$ and $t_0 < 0$:*

$$b_1(M) \leq 4n\rho_0^{-2} \left\|(\mathrm{Ric}_\bullet - \rho_0)_-\right\|_{2,HS}^2 \left\|e^{-t_0(\Delta+\rho)}\right\|_{2,\infty}^2 \qquad (*)$$

2. The Birman–Schwinger principle

After M.Sh. Birman (1928-2009) and J.S. Schwinger (1918–1994)

Set-up:

- selfadjoint operators H, H' on a Hilbert space $\mathcal{H}$,
- $H \geq 0$ and $H' \geq \rho_0$ for some $\rho_0 > 0$.
- **Hilbert-Schmidt norm** of K

$$\|K\|_{\mathcal{H}S}^2 = \sum_{\alpha \in A} \|K\varphi_\alpha\|^2,$$

where $(\varphi_\alpha)_{\alpha \in A}$ is any orthonormal basis of $\mathcal{H}$.

Proposition. *Let $D_t := e^{-tH} - e^{-tH'}, t > 0$. Then the following holds:*

(i) $\ker(H) = \ker((I - e^{-tH'})^{-1} D_t - I)$, *where $I \in \mathcal{L}(\mathcal{H})$ denotes the identity.*

(ii) *If $D_{t_0} \in \mathcal{HS}$ and $t_0 > 0$, then*

$$\dim \ker(H) \leq \|(I - e^{-t_0 H'})^{-1} D_{t_0}\|_{\mathcal{H}S}^2.$$

In particular,

$$\dim \ker(H) \leq (1 - e^{-\rho_0 t_0})^{-2} \|D_{t_0}\|_{\mathcal{H}S}^2.$$

We now specialize to L^2–spaces:

- $(X, \mathcal{F}, m)$ a σ-finite measure space
- $L^2(X) = L^2(X; \mathbb{K})$ for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$,
- **vector-valued version** $L^2(X; \mathbb{K}^n)$ for $n \in \mathbb{N}$.
- $H \geq 0$ in $L^2(X; \mathbb{K}^n)$, semigroup $(e^{-tH}; t \geq 0)$
- $V : X \to \mathbb{K}^{n \times n}$ measurable and $V \succeq 0$.
- $H' = H + V$.

This gives the following Hilbert–Schmidt norm estimate for semigroup differences, see [14]:

Proposition. *Assume that $T \in \mathcal{L}(L^2(X; \mathbb{K}^n), L^\infty(X; \mathbb{K}^n))$ and $V \in L^2(X; \mathbb{K}^{n \times n})$. Then the operator $VT \in \mathcal{L}(L^2(X; \mathbb{K}^n))$ is Hilbert-Schmidt and*

$$\|VT\|_{HS} \leq \sqrt{n} \cdot \|V\|_{2,HS} \cdot \|T\|_{2,\infty}.$$

For the connection with **the little Grothendieck theorem**, [10], III.F and III:G, see the discussion in [26] and [6] for an easier approach used here. The Duhamel principle gives:

Theorem. *Assume that $H \geq 0$ is a selfadjoint operator in $L^2(X;\mathbb{K}^n)$ and $V \in L^2 \cap L^\infty(X;\mathbb{K}^{n\times n})$. Moreover, suppose that for some $t_0 > 0$ we have $e^{-t_0 H}, e^{-t_0(H+V)} \in \mathcal{L}(L^2(X;\mathbb{K}^n), L^\infty(X;\mathbb{K}^n))$. Then*

$$\|e^{-2t_0 H} - e^{-2t_0(H+V)}\|_{HS} \leq$$
$$\sqrt{n}\,\|V\|_{2,HS}\left(\|e^{-t_0 H}\|_{2,\infty} + \|e^{-t_0(H+V)}\|_{2,\infty}\right) \cdot \int_0^{t_0} \|e^{-s(H+V)}\|_{2,2}\,ds.$$

Semigroup domination, Kato's inequality. Let H_0 be a selfadjoint lower-semibounded operator in $L^2(X)$. We say that its semigroup $(e^{-tH_0})_{t\geq 0}$ **dominates** $(e^{-tH})_{t\geq 0}$ if the following relation is satisfied for all $t > 0$:

$$|e^{-tH} f|(x) \leq e^{-tH_0}|f|(x), \quad (x \in X, f \in L^2(X;\mathbb{K}^n)).$$

Proposition (Hess, Schrader, Uhlenbrock '80)**.** *For the operators defined above, we have the following domination of the corresponding semigroups:*

(1) *For all $\omega \in L^2(M;\Omega^1)$ and $t \geq 0$:*
$$\left|e^{-t\nabla^*\nabla}\omega\right| \leq e^{-t\Delta}|\omega|.$$

(2) *For all $\omega \in L^2(M;\Omega^1)$ and $t \geq 0$:*
$$\left|e^{-t\mathbf{\Delta^1}}\omega\right| \leq e^{-t(\Delta+\rho)}|\omega|.$$

Putting things together. We put

$$H' := \mathbf{\Delta^1} + (\mathrm{Ric}_x - \rho_0)_-$$

Note that
$$V := (\mathrm{Ric} - \rho_0)_- \succeq 0$$
in the sense of the previous section. Moreover

$$\begin{aligned} H' &= \mathbf{\Delta^1} + (\mathrm{Ric} - \rho_0)_- \\ &= \nabla^*\nabla + \mathrm{Ric} - \rho_0 + \rho_0 + (\mathrm{Ric} - \rho_0)_- \\ &= \nabla^*\nabla + (\mathrm{Ric} - \rho_0)_+ + \rho_0 \\ &\geq \rho_0. \end{aligned}$$

This readily implies:

Corollary. *For $\rho_0 > 0$, $t_0 > 0$ and $p \geq 1$:*

$$b_1(M) \leq \left(1 - e^{-2\rho_0 t_0}\right)^{-p} \left\|e^{-2t_0\mathbf{\Delta^1}} - e^{-2t_0(\mathbf{\Delta^1} + (\mathrm{Ric} - \rho_0)_-)}\right\|_{\mathcal{S}_p}^p. \tag{1}$$

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Weakly non-collapsed RCD spaces are strongly non-collapsed

Shouhei Honda
(joint work with Camillo Brena, Nicola Gigli, Xingyu Zhu)

Let us consider a complete separable metric space (X, d) with a Borel measure m satisfying that the $H^{1,2}$-Sobolev space is a Hilbert space and that the Ricci curvature is bounded below by K and the dimension is bounded above by N for some $K \in \mathbb{R}$ and some $N \in [1, \infty)$, in a synthetic sense (see [10, 11, 12]). Such a metric measure space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is called an RCD(K, N) space.

An RCD(K, N) space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ is said to be *non-collapsed* if m coincides with the N-dimensional Hausdorff measure $\mathcal{H}^N$. It is known in [6] that non-collapsed RCD(K, N) spaces have fine properties rather than that of general RCD(K, N) spaces. For example any non-collapsed RCD(K, N) space is a topological N-manifold except for a closed m-null set (see [5, 9]). Thus it is natural to ask whether we can easily check the coincidence between m and $\mathcal{H}^N$ for given RCD(K, N) space $(\mathsf{X}, \mathsf{d}, \mathsf{m})$.

In [6] it is conjectured that if m is absolutely continuous with respect to $\mathcal{H}^N$, then m coincides with $\mathcal{H}^n$ up to multiplication by a positive constant to m. In this talk we give a proof of this conjecture given in [3].

The key idea is borrowed from [8] which proved the conjecture in the case when X is compact. Namely as the first step, we consider a time-dependent map $\Phi_t : \mathsf{X} \to L^2(\mathsf{X}, \mathsf{m})$ defined by

$$\Phi_t(x) = (y \mapsto p(x, y, t)), \tag{1}$$

where p denotes the heat kernel of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$. Then consider the pull-back $\mathsf{g}_t = \Phi_t^* \mathsf{g}_{L^2}$ by Φ_t of the flat Riemannian metric g_{L^2} of $L^2(\mathsf{X}, \mathsf{m})$;

$$\mathsf{g}_t(x) = \Phi_t^* \mathsf{g}_{L^2}(x) = \int_{\mathsf{X}} d_x p(x, y, t) \otimes d_x p(x, y, t) \mathrm{d}\mathsf{m}(y). \tag{2}$$

The second step for the proof of the conjecture is to prove as $t \to 0^+$;

$$t\mathsf{m}(B_{\sqrt{t}}(x))\mathsf{g}_t(x) \to c_n \mathsf{g} \tag{3}$$

in the L^q_{loc}-sense for any $q \in [1, \infty)$, where n denotes the essential dimension of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$ (see [4]), g denotes the canonical Riemannian metric of $(\mathsf{X}, \mathsf{d}, \mathsf{m})$, and c_n is a positive constant depending only on n (see also [1, 2]).

The third step is to prove the formula;

$$\nabla^* g_t(x) = -\frac{1}{4} d_x \Delta_x p(x, x, 2t). \tag{4}$$

Let us emphasize that it is not hard to check the formula in the case when X is compact because of the expansion of p by eigenfunctions. In the case when X is non-compact, we need a new idea in order to justify the formula.

Finally the fourth (final) step is, under assuming that m is absolutely continuous with respect to $\mathcal{H}^N$, letting $t \to 0^+$ in (4) after multiplying $t^{(N+2)/2}$ with (3) allows us to get;

$$\int_{\mathsf{X}} \mathsf{g}(\nabla f_1, \nabla f_2) \mathrm{d}\mathcal{H}^N = -\int_{\mathsf{X}} f_1 \mathrm{tr}(\mathrm{Hess}_{f_2}) \mathrm{d}\mathcal{H}^N \tag{5}$$

for all compactly supported functions $f_i (i = 1, 2)$ satisfying some regularities. Then since a result of [7] shows $\Delta f_2 = \mathrm{tr}(\mathrm{Hess}_{f_2})$ in (5), it is not so hard to check that (5) implies $\mathsf{m} = c\mathcal{H}^N$ for some positive constant c.

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W-entropy and Langevin deformation on Wasserstein space over Riemannian manifolds

Xiang-Dong Li
(joint work with Songzi Li)

1. Introduction

Let (M, g) be a complete Riemannian manifold equipped with a weighted volume measure $d\mu = e^{-f}d\nu$, where $f \in C^2(M)$ and $d\nu$ denotes the volume measure on (M, g). The Witten Laplacian, which is self-adjoint and non-positive definite on $L^2(M, \mu)$, is defined as follows

$$L := \Delta - \nabla f \cdot \nabla.$$

Let $m \in [n, \infty]$. The m-dimensional Bakry-Emery Ricci curvature associated with L is defined by

$$\mathrm{Ric}_{m,n}(L) := \mathrm{Ric} + \nabla^2 f - \frac{\nabla f \otimes \nabla f}{m - n}.$$

Here we make the convention that $m = n$ if and only if f is a constant, $L = \Delta$ and $\mathrm{Ric}_{n,n}(\Delta) = \mathrm{Ric}$. Moreover, we simply use $\mathrm{Ric}(L)$ to denote $\mathrm{Ric}_{\infty,n}(L)$, i.e., $\mathrm{Ric}(L) = \mathrm{Ric} + \nabla^2 f$.

Let $P_2(M, \mu)$ (resp. $P_2^\infty(M, \mu)$) be the Wasserstein space (reps. the smooth Wasserstein space) of all probability measures $\rho(x)d\mu(x)$ with density function (resp. with smooth density function) ρ on M such that $\int_M d^2(o, x)\rho(x)d\mu(x) < \infty$, where $d(o, \cdot)$ denotes the distance function from a fixed point $o \in M$. By Otto [7], the tangent space $T_{\rho d\mu}P_2^\infty(M, \mu)$ is identified as follows

$$T_{\rho d\mu}P_2^\infty(M, \mu) = \{s = \nabla_\mu^*(\rho\nabla\phi) : \phi \in C^\infty(M), \quad \int_M |\nabla\phi|^2 \rho d\mu < \infty\},$$

where ∇_μ^* denotes the L^2-adjoint of the Riemannian gradient ∇ with respect to the weighted volume measure $d\mu$ on (M, g). For $s_i = \nabla_\mu^*(\rho\nabla\phi_i) \in T_{\rho d\mu}P_2^\infty(M, \mu)$, $i = 1, 2$, Otto [7] introduced the following infinite dimensional Riemannian metric on $T_{\rho d\mu}P_2^\infty(M, \mu)$

$$\langle\langle s_1, s_2\rangle\rangle := \int_M \langle\nabla\phi_1, \nabla\phi_2\rangle \rho d\mu.$$

Let $T_{\rho d\mu}P_2(M, \mu)$ be the completion of $T_{\rho d\mu}P_2^\infty(M, \mu)$ with Otto's Riemannian metric. Then $P_2(M, \mu)$ is a formal infinite dimensional Riemannian manifold.

Let $m_i \in P_2(M, \mu)$, $i = 0, 1$. The Wasserstein distance between m_0 and m_1 is defined by

$$W_2(m_0, m_1) = \inf_\pi \left(\int_{M\times M} d^2(x, y)d\pi(x, y)\right)^{1/2},$$

where $\pi \in \prod(m_0, m_1)$, i.e., π is a probability measure on $M \times M$ such that

$$\int_M \pi(\cdot, dy) = m_0, \quad \int_M \pi(dx, \cdot) = m_1.$$

Suppose that $m_i = \rho_i \mu$, $i = 0, 1$, where ρ_0 and ρ_1 are two compactly supported smooth probability densities on M. By Benamou and Brenier [1], we have

$$W_2(m_0, m_1) = \inf_{(\rho,\phi)} \left(\int_M \int_0^1 |\nabla \phi(x,t)|^2 \rho(x,t) d\mu(x) dt \right)^{1/2},$$

where $(\rho, \phi) : M \times [0,1] \to [0, \infty) \times \mathbb{R}$ satisfies the transport equation

$$\partial_t \rho - \nabla_\mu^* (\rho \nabla \phi) = 0, \tag{1}$$

with the boundary condition $\rho(\cdot, 0) = \rho_0$ and $\rho(\cdot, 1) = \rho_1$. Moreover, the infimum is achieved if ϕ further satisfies the Hamilton-Jacobi equation

$$\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 = 0. \tag{2}$$

In view of this, we can regard (ρ, ϕ) as the geodesic flow on the tangent bundle $TP_2(M, \mu)$ over the Wasserstein space $P_2(M, \mu)$.

By Otto [7], the heat equation of the Witten Laplacian

$$\partial_t \rho = L\rho \tag{3}$$

can be regarded as the gradient flow of the Boltzmann-Shannon entropy Ent on the Wasserstein space $P_2^\infty(M, \mu)$, where

$$\mathrm{Ent}(\rho) := \int_M \rho \log \rho d\mu.$$

2. Langevin deformation on Wasserstein space

In [3, 4], we introduce the Langevin deformation of geometric flows $(\rho, \phi) : [0, T] \to TP_2(M, \mu)$ as the solution to the following equations on $TP_2(M, \mu)$ (the tangent bundle over the Wasserstein space $P_2(M, \mu)$)

$$\partial_t \rho - \nabla_\mu^* (\rho \nabla \phi) = 0, \tag{4}$$

$$c^2 \left(\partial_t \phi + \frac{1}{2} |\nabla \phi|^2 \right) = -\phi - \log \rho - 1, \tag{5}$$

where $c \geq 0$ is the parameter.

The Langevin deformation of flows has a close connection with the compressible Euler equation with damping. Using the Kato-Majda theory of the hyperbolic quasi-linear systems, we can prove that, for any given $c > 0$, there exists $T = T_c > 0$ such that the Cauchy problem of the system (4) and (5) has a unique smooth solution $(\rho, \phi) \in C^1([0, T], C^\infty(M, \mathbb{R}^+) \times C^\infty(M))$ with given initial data $(\rho_0, \phi_0) \in C^\infty(M, \mathbb{R}^+) \times C^\infty(M)$. If the initial data is smooth and has small energy, it has global smooth solution. Moreover, it is proved in [4] that when $c \to 0$ (respectively, $c \to \infty$), the Langevin deformation of flows converges in a proper sense to the heat equation (3) (respectively, and the geodesic flow (1) and (2)). For the precise statement of this result, see [4].

The following result gives us the dissipation formulae for the Hamiltonian and Lagrangian along the Langevin deformation of flows on $TP_2(M, \mu)$. For its proof, see [4].

Theorem 2.1. *Let M be a compact Riemannian manifold. Define*

$$\begin{aligned} H(\rho,\phi) &= \frac{c^2}{2}\int_M |\nabla\phi|^2\rho d\mu + \int_M \rho\log\rho d\mu, \\ L(\rho,\phi) &= \frac{c^2}{2}\int_M |\nabla\phi|^2\rho d\mu - \int_M \rho\log\rho d\mu. \end{aligned}$$

Then

$$\begin{aligned} \frac{d}{dt}H(\rho,\phi) &= -\int_M |\nabla\phi|^2\rho d\mu, \\ \frac{d^2}{dt^2}L(\rho,\phi) &= 2\int_M \left[c^{-2}|\nabla\phi+\nabla\log\rho|^2 + |\text{Hess }\phi|^2 + \text{Ric}(L)(\nabla\phi,\nabla\phi)\right]\rho d\mu. \end{aligned}$$

In particular, if the $CD(0,\infty)$-condition holds, i.e., $\text{Ric}(L) = \text{Ric} + \nabla^2 f \geq 0$, *then* $L(\rho,\phi)$ *is convex along the Langevin deformation* (ρ,ϕ) *defined by* (4) *and* (5).

Let $u : (0,T) \to (0,\infty)$ be a smooth solution to the ODE

$$c^2u'' + u' = \frac{1}{2u}$$

where $T > 0$ is the lifetime of the solution u. Let $\alpha(t) = \frac{u'(t)}{u(t)}$, and let $\beta(t) \in C((0,T),\mathbb{R})$ be the unique solution to the ODE

$$c^2\dot\beta(t) = -\beta(t) - m\log u(t) - \frac{m}{2}\log(4\pi) + 1,$$

with any given initial data $\beta(0) \in \mathbb{R}$. For $x \in \mathbb{R}^m$ and $t \in (0,T)$, define

$$\rho_m(x,t) = \frac{1}{(4\pi u^2(t))^{m/2}}e^{-\frac{\|x\|^2}{4u^2(t)}}, \qquad \phi_m(x,t) = \frac{\alpha(t)}{2}\|x\|^2 + \beta(t).$$

Then (ρ_m,ϕ_m) is a smooth solution of (4) and (5) on $(\mathbb{R}^m, dx)$.

3. W-entropy for Langevin deformation

For any $c \in (0,\infty)$, we define the W-entropy for the Langevin deformation of flows as follows

$$\frac{d}{dt}W_c(\rho(t)) = \frac{d^2}{dt^2}\text{Ent}(\rho(t)) + \left(2\alpha(t) + \frac{1}{c^2}\right)\frac{d}{dt}\text{Ent}(\rho(t)) + \frac{1}{c^2}\|\nabla\text{Ent}(\rho(t))\|^2,$$

where $\|\nabla\text{Ent}(\rho(t))\|^2 = \int_M \frac{|\nabla\rho(t)|^2}{\rho(t)}d\mu$. By direct calculation, we have

$$\frac{d}{dt}W_c(\rho_m(t)) = -m\alpha^2(t).$$

The following result can be viewed as an analogue of the W-entropy formula for the Langevin deformation of geometric flows on $TP_2^\infty(M,\mu)$. For its proof, see [4].

Theorem 3.1. *Let M be a compact Riemannian manifold. Then*

$$\frac{d}{dt}(W_c(\rho(t)) - W_c(\rho_m(t))) = \int_M |\text{Hess } \phi - \alpha(t)g|^2 \rho d\mu + \int_M \text{Ric}_{m,n}(L)(\nabla\phi, \nabla\phi)\rho d\mu + \frac{1}{m-n}\int_M |\nabla f \cdot \nabla\phi + (m-n)\alpha(t)|^2 \rho d\mu.$$

In particular, if $\text{Ric}_{m,n}(L) \geq 0$*, then for all* $t > 0$*, we have the comparison inequality*

$$\frac{d}{dt}W_c(\rho(t)) \geq \frac{d}{dt}W_c(\rho_m(t)).$$

Remark 3.2. *When* $c \to 0$ *and* $c \to \infty$*, Theorem 3.1 allows us to recapture the* W*-entropy formula for the heat equation of the Witten Laplacian on* (M, μ) *and the* W*-entropy formula for the geodesic flow on* $TP_2^\infty(M, \mu)$ *respectively. For details, see* [3, 4]*. In particular, we can recapture a previous result due to Lott and Villani* [5, 6]*, which states that: Let* M *be a compact Riemannian manifold with non-negative Ricci curvature. Then* $t\text{Ent}(\rho(t)) + nt\log t$ *is convex in time* t *along the geodesic on* $P_2(M, \nu)$*. Moreover, we can extend Theorem 3.1 to a class of smooth solutions* (ρ, ϕ) *with natural growth condition on weighted complete manifolds* (M, g, f) *with bounded geometry condition. In view of this, we can therefore expect that the following rigidity theorem holds: Let* M *be a complete Riemannian manifold with bounded geometry condition and with* $CD(0, m)$*-condition, i.e.,* $Ric_{m,n}(L) \geq 0$*. Then* $\frac{d}{dt}W_c(\rho(t)) = \frac{d}{dt}W_c(\rho_m(t))$ *holds at some* $t = t_0 > 0$ *if and only if* M *is isometric to* $\mathbb{R}^n$*,* $m = n$*,* f *is a constant, and* $(\rho, \phi) = (\rho_m, \phi_m)$*. When* $c = 0$ *and* $c = \infty$*, this has been proved in* [2, 3, 4]*.*

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Harnack estimates on path space

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(joint work with Robert Haslhofer, Aaron Naber)

1. Lower bounds for Ricci curvature

The classical Bochner inequality is the starting point of the analysis of manifolds with lower Ricci curvature. For solutions to the heat equation $H_t f$ and $\mathrm{Ric} \geq -\kappa$ it may be written as

$$(\partial_t - \Delta)|\nabla H_t f|^2 \leq -|\nabla^2 H_t f|^2 + \kappa |\nabla H_t f|^2.$$

This is equivalent to the gradient estimate

$$|\nabla H_t f| \leq e^{\kappa t} H_t |\nabla f|,$$

which vice versa characterizes the lower Ricci curvature by $-\kappa$.

2. Bounded Ricci curvature

In order to characterize bounded Ricci curvature Naber considers in [5] the path space over the manifold. He showed that $|\mathrm{Ric}| \leq \kappa$ if and only if

$$|\nabla_x \int_{PM} F \, d\Gamma_x| \leq \int_{PM} (|\nabla_0 F| + \int_0^\infty \kappa e^{\kappa s} |\nabla_s F| \, ds) \, d\Gamma_x$$

holds for all test functions $F\colon PM \to \mathbb{R}$. In the simplest case of one-point cylinder functions this reduces to the gradient estimate.

3. Harnack inequalities on manifolds

The classical Harnack estimate tells us that if $\mathrm{Ric} \geq 0$ and f_t is a nonnegative solution to the heat equation

$$f_{t_2}(x_2) \geq \left(\tfrac{t_1}{t_2}\right)^{n/2} e^{-\frac{d(x_1,x_2)^2}{4(t_2-t_1)}} f_{t_1}(x_1)\,.$$

It has been found by Li and Yau in [4] that this can be derived from the differential Harnack estimate

$$\frac{\Delta f_t}{f_t} - \frac{|\nabla f_t|^2}{f_t^2} + \frac{n}{2t} \geq 0\,.$$

Another differential Harnack estimate is given by Hamilton's Matrix Harnack inequality in [2], which implies the estimate by Li and Yau through tracing. However their result is not fully contained since Hamilton assumes the stronger geometric constraints $\nabla \mathrm{Ric} = 0$ and $\sec \geq 0$. The Hessian version is given by

$$\frac{\nabla^2 f_t}{f_t} - \frac{\nabla f_t \otimes \nabla f_t}{f_t^2} + \frac{g}{2t} \geq 0\,.$$

4. Differential Harnack on Euclidean path space

We consider the Euclidean path space

$$P_0\mathbb{R}^n = \{\gamma \in C^0([0,\infty),\mathbb{R}^n) : \gamma(0) = 0\}.$$

endowed by the Wiener measure Γ_0. The standard deviation is given by the Cameron-Martin space:

$$\mathcal{H} = \left\{ h \in P_0(\mathbb{R}^n) \, : \, ||h||^2 \equiv \int_0^\infty |\dot{h}|^2 dt < \infty \right\}.$$

The generalized Matrix Harnack inequality in the path space setting is then cf. Theorem 1.11 in [3]:

Theorem 1. *If $F : P_0\mathbb{R}^n \to \mathbb{R}^+$ is a positive integrable function, then the associated functional*

$$\Phi_F : \mathcal{H} \to \mathbb{R}, \quad \Phi_F(h) = \ln\left(\int_{P_0\mathbb{R}^n} F(\gamma + h)\, d\Gamma_0(\gamma)\right) + \frac{1}{4}||h||^2$$

is convex.

For the simplest function $F(\gamma) = f(\gamma_t)$ and linear curve $h(s) = \frac{1}{t}x$ this yields precisely the Matrix Harnack estimate.

5. Differential Harnack on P_xM for Ricci-flat manifolds

In order to show a differential Harnack inequality in the manifold case we first show that a specific quadratic form is nonnegative for all vector fields V on P_xM, which consists of all $v \in \mathcal{H}$ transported parallel along the paths. The following is our Halfway Harnack inequality for n-dimensional Ricci-flat manifolds M, cf. Theorem 4.8 in [3]:

Theorem 2. *Let M be a Ricci-flat manifold, and let $F\colon P_xM \to \mathbb{R}^+$ be a nonnegative cylinder function. Then, the quadratic form*

$$Q_F[V,V] := \frac{\mathbb{E}_x[D_V(D_VF)]}{\mathbb{E}_x[F]} - \frac{\mathbb{E}_x[D_VF]^2}{\mathbb{E}_x[F]^2} + \frac{\mathbb{E}_x[D_{\nabla_V V}F]}{\mathbb{E}_x[F]} + \frac{1}{2}\frac{\mathbb{E}_x[F||V||^2]}{\mathbb{E}_x[F]},$$

is nonnegative for every vector field V. Here, ∇ denotes the Markovian connection.

The Markovian connection has been introduced in [1] and makes use of the full Riemannian curvature tensor. This connection gives rise to a more geometric meaning of the nonnegativity of Q_F. Namely, we can define the Hessian $\mathrm{Hess}\, F$ for functions on the path space, which gives rise to a finite dimensional Laplacian

$$\Delta_\varphi F = \sum_{a=1}^n \mathrm{Hess} F(V^a, V^a),$$

for each $\varphi\colon [0,\infty) \to \mathbb{R}$ with bounded H^1-norm $||\varphi||$ and $v_t = \varphi_t e_a$, where $e_1, \ldots, e_n$ is an orthonormal basis of $\mathbb{R}^n$. Then our differential Harnack estimate looks the following, cf. Theorem 1.37 in [3]:

Theorem 3. *Let M be a Ricci-flat manifold, and let $F : P_xM \to \mathbb{R}$ be a nonnegative function. Then, for all $\varphi \in H_0^1(\mathbb{R}^+)$ we have the inequality*

$$\frac{\mathbb{E}_x\left[\Delta_\varphi F\right]}{\mathbb{E}_x[F]} - \frac{\left|\mathbb{E}_x\left[\nabla_\varphi F\right]\right|^2}{\mathbb{E}_x[F]^2} + \frac{n}{2}||\varphi||^2 \geq 0.$$

This reduces to the Li-Yau inequality if we choose $F(\gamma) = f(\gamma_t)$ and $\varphi(s) = \frac{s}{t}$ if $s \leq t$.

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Characterizations of Forman curvature

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(joint work with Jürgen Jost)

In 2003, Forman introduced a Ricci curvature notion on cell complexes and derived diameter bounds and trivial first homology in case of uniformly positive curvature, see [1]. The generalized Forman curvature of a cell complex $X = \bigcup X_k$ with coboundary operator δ and Hodge Laplacian $H := \delta\delta^* + \delta^*\delta$ is defined as

$$F(x) := Hx(x) - \sum_{y \neq x} \frac{\omega(y)}{\omega(x)} Hy(x).$$

for cells $x \in X$ with the abuse of notation $x = 1_x$, and cell weights $\omega : X \to (0, \infty)$, see [2]. We characterize lower bounds of the Forman curvature via contraction of the Hodge semigroup. More precisely if $\omega = 1$, the following statements are equivalent for a given curvature bound $K \in \mathbb{R}$ and dimension $k \in \mathbb{N}$,

(1) $F(x) \geq K$ for all $x \in X_k$,
(2) $\|e^{-Ht} f\|_p \leq e^{-Kt}\|f\|_p$ for all $p \in [1, \infty]$ and all $f : X_k \to \mathbb{R}$,
(3) $\|e^{-Ht} f\|_\infty \leq e^{-Kt}\|f\|_\infty$ for all $f : X_k \to \mathbb{R}$.

A similar statement holds true for general ω.

We compare Forman curvature with the Ollivier curvature which was introduced in [4] via optimal transport theory and extended to weighted graphs in [3]. While the Ollivier curvature is classically defined on edges, we generalize the definition to cells of arbitrary dimension. The generalized Ollivier curvature of a cell $x \in X_k$ is defined as

$$\kappa(x) := \inf_{\substack{|\delta f| \leq \omega \\ \delta f(x) = \omega(x)}} \delta\delta^* \delta f(x).$$

The curvature $\kappa(x)$ on edges $x \in X_1$ coincides with the standard Ollivier curvature when taking the path distance with respect to ω. While the Forman curvature on X_k depends on both, the coboundary operator $\delta : X_{k-1} \to X_k$ and $\delta : X_k \to X_{k+1}$, the Ollivier curvature only depends on $\delta : X_{k-1} \to X_k$. We prove that on X_1, Forman and Ollivier curvature coincide when maximizing the Forman curvature over $\delta : X_k \to X_{k+1}$ and the inner product on X_{k+1}. More precisely, for a graph G,

$$\kappa(x) = \max F(x)$$

for all $x \in X_1$ where the maximum is taken over cell complexes having G as 1-skeleton. A similar result under strong restrictions for the underlying graph was established in [5]. The proof of our result uses that the maximization is a linear program with the dual

$$\max F(x) = \min_{\substack{|h| \leq \omega \\ h(x) = \omega(x) \\ \delta h \cdot \delta x \leq 0}} \frac{\delta\delta^* h(x)}{\omega(x)}.$$

This holds true for arbitrary dimension of the cell x. It is still open however, whether the coincidence of $\kappa(x)$ and $\max F(x)$ also holds true for arbitrary dimension of x.

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Configuration Spaces over Singular Spaces

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(joint work with Kohei Suzuki)

Background. The configuration space $\Upsilon = \Upsilon(X)$ over a proper metric space (X, d) is the space of all $\mathbb{N}_0$-valued Radon measures on X. As configurations effectively describe a variety of objects, they appear in different areas of mathematics:

(A) *Algebraic geometry/topology:* braid groups arise as fundamental groups of configuration spaces.

(B) *Statistical physics:* (random) dynamical systems on Υ describe the (stochastic) evolution of systems of infinitely many interacting particles on X driven by infinite systems of ODEs (SDEs). The laws μ of many point processes arise as invariant measures of these systems, e.g. Poisson, canonical Gibbs, Sine, Bessel, Airy, Ginibre, determinantal/permanental random measures, [1, 2, 7].

(C) *Random geometry:* (random) configurations naturally describe a large variety of geometric objects. For instance: a Euclidean *discrete affine hyperplane arrangement* is a configuration over an affine Grassmannian; a *discrete ϵ-proximity graph* consists of the configuration of its vertices in X together with the adjacency rule $x \sim y$ iff $\mathsf{d}(x, y) < \epsilon$.

(D) *Infinite-dimensional geometry:* the distance d induces an extended distance on Υ, namely the L^2-transportation distance D (an extension of the classical L^2-Kantorovich–Rubinshtein distance). Together with a measure μ on Υ as above, the extended-metric measure space $(\Upsilon, \mathsf{D}, \mu)$ is an interesting object in its own right, inheriting its metric/geometric properties from the base space (X, d), [6, 3].

(E) *Representation theory:* when $X = M$ is a Riemannian manifold, the configuration space Υ plays a central role in the study of representations of the group of diffeomorphisms $\mathrm{Diff}(M)$ on M, [8]. Indeed, if μ is quasi-invariant w.r.t. the action of $\mathrm{Diff}(M)$ by push-forward of measures, then a quasi-regular representation of $\mathrm{Diff}(M)$ is induced on $L^2(\mu)$.

For this and other reasons, configuration spaces drew great interest in disciplines ranging from the mathematical foundation of statistical physics to infinite-dimensional analysis and algebraic geometry.

Different structures. We consider four different structures on Υ, corresponding to one the settings (A)–(B) above:

(a) a topological structure $(\Upsilon, \tau_{\mathrm{v}})$, with τ_{v} the vague topology;

(b) a measure structure $(\Upsilon, \mathcal{B}_{\tau_{\mathrm{v}}}, \mu)$, with $\mathcal{B}_{\tau_{\mathrm{v}}}$ the Borel σ-field of τ_{v} and μ a diffuse probability measure;

(c) an extended metric measure structure structure $\mathbb{M} := (\Upsilon, \mathsf{D}, \mu)$;

(d) a differential structure $\mathbb{D} := (\Upsilon, \Gamma^{\Upsilon})$ consisting of a local square field operator Γ^{Υ} 'lifted' from the base space X, and defined on a suitable class of cylinder functions.

In the case of a finite-dimensional Riemannian manifold (M, g), the Riemannian metric g alone determines every other structure: it generates the given topology, induces the correct volume measure vol_g, the geodesic distance d_g, and the gradient ∇^g. Compared to a Riemannian manifold, the configuration space Υ is less 'consistent', and the interplay of the four structures (a)–(d) listed above is not clear. For instance, D-balls of finite radius are typically μ-negligible, the topology generated by D is strictly finer than the vague topology τ_{v}, and D-Lipschitz functions are in general not measurable.

Main results. In [5], we develop a single 'consistent' framework for (a)–(d) by a robust approach that does not require X to be smooth nor μ to satisfy the quasi-invariance in (E) nor any integral identity, in order to apply it to a wide range of problems in (A)–(B). Furthermore, based on this framework, we aim to construct bridges among (A)–(B) by which one can apply techniques arising in one context to problems arising from a different one.

More in detail, we establish the foundations for a systematic approach to analysis, geometry, and stochastic analysis on configuration spaces over non-smooth spaces. We do so in the broadest possible generality, e.g., without relying on the well-known theory of synthetic Ricci curvature lower bounds for X, and restricting the necessary assumptions to an irreducible minimum, in such a way to sensibly generalize known results in the theory and to extend them also beyond the smooth setting. This level of generality provides great technical insight even to the case of smooth base manifolds. It is furthermore necessary to applications in (C), since countable random graphs are generally not isometrically embeddable in $\mathbb{R}^d$; to applications in (D), allowing to deal with $(\Upsilon, \mathsf{D}, \mu)$ with the same techniques used for a metric measure space $(X, \mathsf{d}, \mathsf{m})$; to applications in (B), allowing to treat particle systems in singular spaces (e.g. with edges, bottlenecks, etc.) or degenerate dynamics (e.g. 'single file' dynamics).

Interplay. We prove that the metric measure structure $\mathbb{M}$ and the differential structure $\mathbb{D}$ are 'consistent' in the following sense:

- the Dirichlet form $\mathcal{E}^\mu$ defined by integration of Γ^Υ w.r.t. μ coincides with the Cheeger energy of $(\Upsilon, \mathsf{D}, \mu)$;
- the intrinsic distance of $\mathcal{E}^\mu$ coincides with the L^2-transportation distance D.

That is

$$\mathbb{M} \iff \mathbb{D}\,. \tag{$*$}$$

As a consequence of this identification, the structure $\mathbb{M}$ completely characterizes the analytic and stochastic-analytic structures based on $\mathbb{D}$. This allows us to apply the metric geometric analysis (D) to stochastic dynamical systems as in (B), granting, e.g.:

- the existence and the quasi-uniqueness of the Brownian motion $\mathbf{B}_\bullet$ on Υ, i.e. the (singularly) interacting diffusion process associated to $\mathcal{E}^\mu$, as a consequence of the Rademacher property for $\mathbb{M}$;

- a Gaussian upper heat kernel estimate w.r.t. D;
- the integral Varadhan short-time asymptotic for $\mathbf{B}_\bullet$ w.r.t. D.

Furthermore, $(*)$ contributes to a metric measure property in (D) by using the ergodicity of the associated interacting diffusions in (B):

- the ergodicity of $\mathbf{B}_\bullet$ implies that $\mathsf{D}(\Lambda_1, \Lambda_2) < +\infty$ whenever $\mu\Lambda_i > 0$.

Since the distance D is 'very often' infinite, as discussed above, it is a fundamental geometric problem to know exactly when $\mathsf{D}(\Lambda_1, \Lambda_2) < +\infty$ occurs. The above statement gives a verifiable sufficient condition for this geometric problem by a statistical physical approach, since the ergodicity of μ-invariant diffusions on Υ has been intensively studied in [1, 2].

Curvature. In forthcoming work [4], we undertake a complete study of synthetic Ricci curvature bounds for the space $(\Upsilon, \mathsf{D}, \pi)$, i.e. in the case where $\mu = \pi$ is a Poisson random measure. In particular we show:

- Bakry–Émery gradient estimates, based on the identification of the heat kernel of $\mathcal{E}^\pi$ with an infinite-product heat kernel on $X^{\times\infty}$.
- Bochner, Wasserstein-contractivity, and Evolution-Variation (EVI) estimates, based on $(*)$;
- the D-regularizing property of the heat semigroup on Υ.

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