Mathematisches Forschungsinstitut Oberwolfach

Report No. 26/2022

DOI: 10.4171/OWR/2022/26

Deterministic Dynamics and Randomness in PDE

Organized by
Andrea Nahmod, Amherst
Gigliola Staffilani, Cambridge MA
Hendrik Weber, Bath
Sijue Wu, Ann Arbor

22 May – 28 May 2022

Abstract. Over the last few years there has been spectacular progress in the study of parabolic SPDE, of nonlinear dispersive and wave equations and of probabilistic methods in PDE. An important direction connecting these three fields is the general question of how randomness affects the behavior of solutions to PDE. Research in recent years has been driven by the study of randomness in nonlinear evolution equations with a focus on the question of how to quantify the transport of such randomness under the nonlinear flow.

Mathematics Subject Classification (2020): 35xx, 60xx, 76xx.

Introduction by the Organizers

The workshop titled *Deterministic Dynamics and Randomness in PDE*, organized by Andrea Nahmod (University of Massachusetts Amherst), Gigliola Staffilani (MIT), Hendrik Weber (University of Bath, UK) and Sijue Wu (University of Michigan) was a great success. It was attended by a total of 53 mathematicians, of which 37 in person and 16 virtually, eight of them were women. There was broad geographic representation from most continents and participants' career level spanned over all different stages, from graduate students to full professors. We want to remark that having two graduate students from the start assigned to handle and manage all of the virtual component of the workshop turned out to be a fantastic and necessary set up. Everything went smoothly during the talks and the interactions between the in house and the virtual audience.

When the workshop was proposed we anchored its intellectual merit on the fruitful interaction across three topics in the general area of Partial Differential

Equations: Stochastic Parabolic PDE, Dispersive and Wave PDE in a deterministic and probabilistic setting and fluid dynamics. In our proposal in fact we wrote: *We envision the Oberwolfach workshop to be a venue where experts and junior researchers on nonlinear dispersive and wave equations (deterministic and nondeterministic), on nonlinear stochastic parabolic equations, and on the intersections of these two major fields of research, converge to discuss in a synergistic fashion recent results and explore the deep connections between approaches and the development of new integrative methods to tackle some of the challenging questions that remain unanswered in these fields.* We believe that our workshop fully delivered along the directions we had set for it, and ultimately the secret to its success can be captured by two elements: the interdisciplinary character of the talks and the dynamic atmosphere provided by a great balance of junior and senior participants.

Every day we scheduled five talks, except for Wednesday and Friday, when only three talks were given in the morning. The speakers were given guidelines in advance on how to prepare their talks. These included the following five strongly suggested elements:

(1) A very general overview of the theme of your talk that includes central/relevant definitions for non-experts.
(2) An outline of a theory/a strategy/group of ideas, that you think are relevant.
(3) A sketch of a proof that you think is most insightful to present.
(4) A description of some of the most salient open questions related to your talk.
(5) When possible indicate connections/ideas that are fertile for cross pollination among SPDE/Prob/PDE.

We are happy to report that the speakers in general adhered to our suggestions and as a result there was a lively engagement by the audience during the talks, and an eagerness to continue the conversation during the breaks.

We also scheduled two sessions on Tuesday and Wednesday evenings, during each of which six junior researchers gave short talks (15 min). These short talks served two purposes: facilitating the communication between junior and senior researchers and showcasing the wonderful recent work of so many talented junior people who could not be accommodated in the schedule with a full length talk. We are aware of at least one case in which a short talk generated great interest among two senior participants and a collaboration was started.

We highlight some of the scientific aspects of the workshop in the next section. We would like to mention that a large number of participants, and in particular the junior ones, specifically mentioned that during the whole workshop one could enjoy an extremely relaxed, cooperative, fun and stimulating environment. In this atmosphere conducive to the easy exchange of ideas, new collaborations started, fresh conjectures were made, horizons were broadened and hopefully a new generation of researchers was inspired.

1. Highlights of the workshop

As mentioned above, the 21 talks were all of great quality, well delivered and mathematically of broad interest. Among them we highlight a few which reported on new and important results, explained cutting edge ideas and novel techniques. The order we give below is that found in the section with the extended abstracts, which in turn reflects the order in which the talks were delivered.

1.1. *DiPerna-Lions for dispersive PDEs: quasi-invariance and global wellposedness for fractional NLS in negative regularity*, **by Leonardo Tolomeo.** In this talk, a proof of global well-posedness for a fractional non-linear Schrödinger equations using the quasi-invariance of certain Gaussian measures on distributions was presented. This work connects two different active branches of research, both key to the workshop, namely: establishing global well-posedness using the exact invariance of a given measure (e.g. the Gibbs measure) and showing quasi-invariance of Gaussian probability measures. The former is by now a well-established technique that has been applied to many equations. A limitation of this technique is that the Gibbs measures in general fixes the regularity or realisations to a specific value, thereby limiting the scope of situation one can consider. On the other hand, a lot of recent activity has been dedicated to showing that whole families of Gaussian measures of different regularities, while not exactly preserved under the flow of a PDE still remain quasi-invariant. Up to Tolomeo's work, this was however limited to high-regularity situations, where global well-posedness was clear by different (easier) techniques. In his talk, Leonardo provided the first example of a global well-posedness result which is *based on* the quasi-invariance and genuinely goes beyond the deterministic theory.

1.2. *Invariant Gibbs measures for the three-dimensional cubic nonlinear wave equation, parts I and II,* **by Bjoern Bringmann and Yu Deng.** These two talks presented a highly impressive recent work by Bringmann, Deng, Nahmod and Yue, which established the invariance of the Euclidean Φ^4 measure on the three-dimensional torus under the dynamics of the cubic non-linear wave equation. Establishing the invariance of Gibbs measures under the flow of Hamiltonian PDE is a by-now well-established research direction and as mentioned above, many results had been obtained previously. The work presented here however, pushes these results to a completely different level of difficulty. Realisations of the three-dimensional Φ^4 measure are much more irregular than what had been treated in this context so far. Technically, the talks showcased an impressive way of dealing with multi-linear estimates and cancellations in a large number perturbative terms, mixed with some ideas from the community of (parabolic) SPDEs, thereby underlining the potential benefit of collaboration between the communities brought together in this workshop. At the end of Bringmann's talk M. Gubinelli pointed out that a mysterious cancellation that is key in the work presented (named the 1533-cancellation) was reminiscent to a similar phenomenon in the KPZ equation.

1.3. *Stochastic analysis of Euclidean fields via the variational method,* **by Massimiliano Gubinelli.** This talk surveyed a series of seminal works by Gubinelli with Barashkov where a powerful and original variational approach was introduced to study the construction of Euclidean QFT. This approach was first introduced as an alternative to stochastic quantization for the Φ^4_3 model. This point of view allowed Barashkov and Gubinelli to construct a novel measure via a random shift of the Gaussian free field and proving that the Φ^4_3 measure can be constructed as an absolutely continuous perturbation of it. They later studied the infinite volume limit of the variational description of Euclidean quantum fields. In his talk, Gubinelli reviewed this method, its uses and conveyed important connections to renormalisation group approach and studies of transport of measures. His talk was very well received not only as very instructive but also quite stimulating.

1.4. *Renormalization and stochastic estimates without Feynman diagrams,* **by Felix Otto.** In this talk, Otto presented recent joint work with Linares, Tempelmayr, and Tsatsoulis where the authors give an innovative approach to construct and stochastically estimate the renormalized model in the work of Chandra and Hairer (2016). More precisely under a spectral gap assumption on the noise ensemble, it provides within regularity structures an inductive mechanism to renormalize and estimate the centered model while avoiding the use of Feynman diagrams. In his talk Otto explained this new method and presented how to naturally carry it out for a class of quasi-linear parabolic PDEs driven by noise in the full singular range.

1.5. *On the wave turbulence theory for a stochastic KdV type equation,* **by Minh-Binh Tran,** and *The mathematical theory of wave turbulence,* **by Zaher Hani.** In these talks the speakers presented the first mathematically rigorous derivation of the wave kinetic equations (WKE), at the kinetic time, for a KdV type dispersive equation and for the cubic NLS respectively. In a sense one should view these derivations as the equivalent of the derivation of the Boltzmann equation, but for waves instead of particles interactions. The WKE give the effective dynamics for the energy spectrum of the dispersive equation at hand, when a weak nonlinearity is considered. In his talk Tran explained his work with Staffilani on the derivation of the WKE for a multidimensional KdV type equation with an added stochastic term, which conserves the energy and does not regularize the nonlinear effects. The stochastic term though is necessary to offset the badly behaved dispersive relation of the KdV that is defined on a multidimensional lattice. Hani's talk was centered also on the derivation of the WKE, but for a continuum cubic NLS equation, a joint work with Deng. He explained how the Duhamel expansions translate into *decorated trees*, and how different type of key cancellations were necessary to control the combinatorics in the higher order *trees.* Hani pointed out that one of these cancellations was similar to the one already mentioned above in the description of the talks of Bringmann and Deng and remarked upon by Gubinelli.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-1641185, "US Junior Oberwolfach Fellows".

Workshop: Deterministic Dynamics and Randomness in PDE

Table of Contents

Abstracts

On asymptotic stability of solitons in classical 1D scalar field theories

Jonas Lührmann
(joint work with Wilhelm Schlag, Yongming Li)

We consider the asymptotic stability properties of solitons in classical (1 + 1)-dimensional scalar field theories

$$(\partial_t^2 - \partial_x^2)\phi + W'(\phi) = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{1}$$

where $W : \mathbb{R} \to \mathbb{R}$ is a scalar interaction potential. Prime examples include the ϕ^4 model and the sine-Gordon model, as well as the quadratic and cubic 1D Klein-Gordon equations. The main goal of this talk is to survey recent advances in the study of the (conditional) asymptotic stability of the solitons in these classical models, and to explain the role that threshold resonances of the linearized operators play for the long-time dynamics of perturbations of these solitons.

More specifically, we discuss two types of scalar potentials:

- single-well potentials $W(\phi) = \frac{1}{2}\phi^2 - \frac{1}{p+1}|\phi|^{p+1}$, $p > 1$, leading to the 1D focusing Klein-Gordon equations (KG_p),
- double-well potentials $W(\phi)$ featuring two consecutive global minima $\phi_- < \phi_+$ with $W(\phi_\pm) = 0$. The minima $\phi_\pm$ are sometimes referred to as vacuum states. The classical examples are the sine-Gordon model with $W(\phi) = 1 - \cos(\phi)$ and the ϕ^4 model with $W(\phi) = \frac{1}{4}(1 - \phi^2)^2$.

Both types of models admit non-trivial static solutions $-\partial_x^2 Q + W'(Q) = 0$, $x \in \mathbb{R}$, satisfying $\lim_{x\to\pm\infty} Q(x) = 0$ for the single-well potentials and $\lim_{x\to\pm\infty} Q(x) = \phi_\pm$ for the double-well potentials. In the latter case these static solutions are called *kinks* since they connect the two distinct vacuum states $\phi_\pm$.

Due to the invariance of the models under translations and Lorentz transformations, arbitrary perturbations may cause the solitons $Q(x)$ to start moving. To reduce the complexity of the asymptotic stability problems for them, in a first step one imposes symmetry assumptions about the perturbations to prevent the translational mode from entering the dynamics. Then the evolution equation for a small perturbation $u(t,x) := \phi(t,x) - Q(x)$ is of the schematic form

$$(\partial_t^2 - \partial_x^2 + V(x) + m^2)u = \alpha(x)u^2 + \beta_0 u^3, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \tag{2}$$

where $V(x)$ is a smooth localized potential, $m > 0$ is a mass parameter, $\alpha(x)$ is a smooth (bounded, possibly localized) variable coefficient, and $\beta_0 \in \mathbb{R}$ is a constant coefficient. Depending on the specific asymptotic stability problem, the linearized operator $-\partial_x^2 + V(x) + m^2$ has additional spectral features such as threshold resonances (sG, ϕ^4, KG_2, KG_3), positive gap eigenvalues (ϕ^4, KG_2), and negative eigenvalues (KG_2, KG_3). The zero eigenvalue (translational mode) is not relevant due to the symmetry assumptions on the perturbations.

The (conditional) asymptotic stability problems for the solitons of the ϕ^4 model, the sine-Gordon model, and for $\mathrm{KG}_{2/3}$ are deeply related by the fact that the potentials in the linearized operators all belong to the hierarchy of reflectionless Pöschl-Teller potentials [20] given by $V(x) = \ell(\ell+1)\mathrm{sech}^2(x)$, $\ell \in \mathbb{N}_0$ ($\ell = 1$: sG; $\ell = 2$: ϕ^4, KG_3; $\ell = 3$: KG_2).

Proving the (conditional) asymptotic stability of $Q(x)$ then consists in proving the decay to zero of small solutions to (2). Key difficulties in the analysis of the long-time behavior of small solutions to (2) are the slow dispersive decay of 1D Klein-Gordon waves, the low power nonlinearities (leading to modified scattering), and the consequences of threshold resonances or internal modes exhibited by the linearized operators.

A main goal of this talk is to explain the new phenomena and outstanding challenges caused by threshold resonances in the analysis of the long-time behavior of small solutions to (2). Formally, the linearized operator $L = -\partial_x^2 + V(x) + m^2$ exhibits a threshold resonance if there exists a bounded non-trivial function $\varphi \neq 0$ such that $L\varphi = m^2\varphi$. The significance of the presence of a threshold resonance for the dynamics of perturbations of a soliton is primarily that the corresponding Klein-Gordon waves only have slow local decay. In fact, the bulk of the Klein-Gordon waves still have improved local decay, only a projection to the threshold resonance exhibits the slow local decay, see the refined local decay estimate (2.62) in [14, Corollary 2.17].

One generally distinguishes the notion of *local asymptotic stability*, referring to the convergence to zero of small solutions to (2) locally in the energy space, and the notion of *full asymptotic stability*, referring to sharp $L_x^\infty(\mathbb{R})$ decay estimates and asymptotics for small solutions to (2).

An approach to proving (conditional) local asymptotic stability via virial-type (or positive commutator) methods has been pioneered in [9, 11, 12, 8, 10]. See also [17, 4] for related recent contributions. We highlight that [9, 10] establish the local asymptotic stability of the ϕ^4 kink under odd perturbations. An outstanding difficulty for this local asymptotic stability approach is that integrated local energy decay estimates for the (dispersive part of) perturbations of the soliton do not seem to be possible due to the failure of L_t^2-integrability in time of the contributions of the threshold resonances to the local decay of the corresponding Klein-Gordon waves.

Several approaches towards proving full asymptotic stability results have been put forth over the last few years: using the distorted Fourier transform, see e.g. [6, 7, 3], using the wave operator, see e.g. [5], or using super-symmetric factorization properties of the linearized operators [18]. The key difficulty that the presence of a threshold resonance poses for full asymptotic stability results is the formation of a singular quadratic source term due to the slow local decay of the Klein-Gordon waves. This type of source term is highly problematic for (distorted) vector field based methods to derive decay, and can potentially lead to a slow-down of the decay rate of the perturbations [16, 14].

In [14, 18] a remarkable non-resonance property of the corresponding quadratic nonlinearity in (2) for perturbations of the sine-Gordon kink was uncovered, which suppresses the worst effects of the source term. This led to a perturbative proof via super-symmetry by the author and W. Schlag [18] of the full asymptotic stability of the sine-Gordon kink under odd perturbations, for which the proof ideas are discussed in more detail in this talk. See [2] for a proof using inverse scattering techniques for the completely integrable sine-Gordon model.

If no favorable cancellation structures are present, further ideas are needed at this point to deal with the quadratic source term caused by the threshold resonances to obtain full (conditional) asymptotic stability results. This is for instance the main difficulty to establish the full conditional asymptotic stability of the soliton for KG_3 under even perturbations. Interestingly, an internal mode creates a related (but even worse) source term in the nonlinear Klein-Gordon equation for the dispersive part of perturbations of the ϕ^4 kink. It is the main challenge towards establishing a full asymptotic stability result for odd perturbations of the ϕ^4 kink, which remains a major open problem. See [5] for long-time decay estimates up to times ε^{-4+c}.

We conclude by referring to the sample of very recent works [6, 5, 2, 8, 14, 18, 1, 4, 19, 13, 10] for further references and perspectives on the study of the asymptotic stability of solitons, or solitary waves, for 1D wave-type or 1D Schrödinger models.

References

[1] G. Chen, *Long-time dynamics of small solutions to 1d cubic nonlinear Schrödinger equations with a trapping potential*, Preprint arXiv:2106.10106.

[2] G. Chen, J. Liu, B. Lu, *Long-time asymptotics and stability for the sine-Gordon equation*, Preprint arXiv:2009.04260.

[3] G. Chen, F. Pusateri, *On the 1d cubic NLS with a non-generic potential*, Preprint arXiv:2205.01487.

[4] S. Cuccagna, M. Maeda, *Asymptotic stability of kink with internal modes under odd perturbation*, Preprint arXiv:2203.13468.

[5] J.-M. Delort, N. Masmoudi, *On the stability of kink solutions of the* Φ^4 *model in* $1+1$ *space time dimensions*, Preprint hal-02862414.

[6] P. Germain, F. Pusateri, *Quadratic Klein-Gordon equations with a potential in one dimension*, Preprint arXiv:2006.15688.

[7] P. Germain, F. Pusateri, Z. Zhang, *On 1d quadratic Klein-Gordon equations with a potential and symmetries*, Preprint arXiv:2202.13273.

[8] M. Kowalczyk, Y. Martel, C. Muñoz, H. Van Den Bosch, *A sufficient condition for asymptotic stability of kinks in general* $(1+1)$*-scalar field models*, Ann. PDE **7** (2021), no. 1, Paper No. 10, 98.

[9] M. Kowalczyk, Y. Martel, C. Muñoz, *Kink dynamics in the* ϕ^4 *model: asymptotic stability for odd perturbations in the energy space*, J. Amer. Math. Soc. **30** (2017), no. 3, 769–798.

[10] M. Kowalczyk, Y. Martel, *Kink dynamics under odd perturbations for* $(1+1)$*-scalar field models with one internal mode*, Preprint arXiv:2203.04143.

[11] M. Kowalczyk, Y. Martel, C. Muñoz, *Nonexistence of small, odd breathers for a class of nonlinear wave equations*, Lett. Math. Phys. **107** (2017), no. 5, 921–931.

[12] M. Kowalczyk, Y. Martel, C. Muñoz, *Soliton dynamics for the 1D NLKG equation with symmetry and in the absence of internal modes*, to appear in J. Eur. Math. Soc.

[13] T. Léger, F. Pusateri, *Internal modes and radiation damping for quadratic Klein-Gordon in 3D*, Preprint arXiv:2112.13163.

[14] H. Lindblad, J. Lührmann, W. Schlag, A. Soffer, *On modified scattering for 1D quadratic Klein-Gordon equations with non-generic potentials*, to appear in Int. Math. Res. Not.

[15] H. Lindblad, J. Lührmann, A. Soffer, *Decay and asymptotics for the 1D Klein-Gordon equation with variable coefficient cubic nonlinearities*, SIAM J. Math. Anal. **52** (2020), no. 6, 6379–6411.
[16] H. Lindblad, J. Lührmann, A. Soffer, *Asymptotics for 1D Klein-Gordon equations with variable coefficient quadratic nonlinearities*, Arch. Ration. Mech. Anal. **241** (2021), no. 3, 1459–1527.
[17] J. Lührmann, Y. Li, *Soliton dynamics for the 1D quadratic Klein-Gordon equation with symmetry*, Preprint arXiv:2203.11371.
[18] J. Lührmann, W. Schlag, *Asymptotic stability of the sine-Gordon kink under odd perturbations*, Preprint arXiv:2106.09605.
[19] Y. Martel, *Asymptotic stability of solitary waves for the 1D cubic-quintic Schrödinger equation with no internal mode*, Preprint arXiv:2110.01492.
[20] G. Pöschl, E. Teller, *Bemerkungen zur Quantenmechanik des anharmonischen Oszillators*, Zeitschrift für Physik **83** (1933), no. 3-4, 143–151.

DiPerna-Lions for dispersive PDEs: quasi-invariance and global well-posedness for fractional NLS in negative regularity

Leonardo Tolomeo
(joint work with Justin Forlano)

In this talk, we consider the Cauchy problem for the fractional NLS with cubic nonlinearity,

$$iu_t + (-\Delta)^\alpha u \pm |u|^2 u = 0, \qquad \text{(FNLS)}$$

posed on the one-dimensional torus $\mathbb{T}$, with a gaussian random initial of the form

$$u_0 = \sum \frac{g_n}{\langle n \rangle^s} e^{in\cdot x},$$

where g_n are i.i.d. complex-valued normal random variables. In particular, we focus our attention to the *global well-posedness theory* when $s \le \frac{1}{2}$, which implies that $u_0 \notin L^2(\mathbb{T})$.

The study of Hamiltonian PDEs with random initial data was initiated by McKean and Vaninsky for the cubic wave equation, and by Bourgain for the mass-critical Schrödinger equation, both posed on $\mathbb{T}$, in [9, 3]. These works were concerned with showing the invariance of the Gibbs measure associated to the Hamiltonian; the measures having been rigorously studied earlier by Lebowitz, Rose and Speer [8].

In particular, Bourgain in [3] introduced the so-called Bourgain's invariant measure argument. The idea is to use the formal invariance of the Gibbs measure, as a replacement for a conservation law, to obtain almost sure global well-posedness on the support of the Gibbs measure. This scheme was applied in [3] to the one-dimensional NLS

$$i\partial_t u - \Delta^2 u + |u|^{p-1} u = 0,$$

where $p \in 2\mathbb{N}+1$, for almost every initial data (distributed according to the Gibbs measure) belonging to the L^2-based Sobolev space $H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$, where $\varepsilon > 0$. The local-in-time dynamics had been constructed earlier in another seminal paper by Bourgain [2].

These results by Bourgain generated a lot of interest in the study of dispersive/Hamiltonian PDEs with random initial data, especially in the situation where it is possible to prove invariance of the Gibbs measure. This is typically a very difficult problem as in many cases of interest, such as in [4], the Gibbs measure is supported on a space of functions which are too rough for deterministic well-posedness theory to apply (if it even exists at such regularities). This issue necessitates a *probabilistic* (local) well-posedness theory, that goes beyond deterministic results by exploiting cancellations due to random oscillations.

Since the original papers by Bourgain, the field has had a tremendous development, and to this day, there is an abundance of probabilistic local well posedness results for dispersive equations with random initial data, even in situations where no invariant measure is available. However, the corresponding global well posedness results are particularly lacking. Due to the very low regularity of the solutions, the standard techniques are often not applicable, and the only technique that has been applied with some success is showing energy estimates for a smooth part of the solution, see [5].

In this talk, we present a new approach for the global well posedness theory of (FNLS) with gaussian initial data. Inspired by the celebrated work of DiPerna-Lions [6] and subsequently of Ambrosio [1], we focus our attention to the solution of the Liouville equation

$$\mu_t = (\Phi_t)_\# \mu_0,$$

where μ_0 denotes the law of the initial data u_0. By exploiting the formal expression for the solution of this equation

$$\mu_t = \exp\Big(\int_0^t \mathcal{Q}(\Phi_{-t'}(u_0))\Big)\mu_0,$$

and exploiting the *probabilistic* local well posedness theory, we can show that the Liouville equation is globally well posed in the Orlicz space $\exp\big((\log L^\beta)\big)(\mu_0)$, for some appropriate $\beta(s,\alpha) > 1$, as long as $s > s_*(\alpha)$. We can then use Bourgain's invariant measure argument to extend these bounds to the solution of (FNLS) emanating from almost every initial data.

When

$$1 < \alpha < \tfrac{1}{20}(17 + 3\sqrt{21}) \approx 1.537,$$

this global well posedness holds for initial data which is rougher than what is allowed by the deterministic local well posedness theory.

The result presented in this talk can be found in [7].

References

[1] L. Ambrosio, *Transport equation and Cauchy problem for BV vector fields*, Invent. Math. 158 (2004), 227–260.

[2] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I: Schrödinger equations*, Geom. Funct. Anal. 3 (1993), 107–15.

[3] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. 166 (1994), no. 1, 1–26.
[4] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176(2): 421-445 (1996).
[5] N. Burq, N. Tzvetkov, *Random data Cauchy theory for supercritical wave equations. II. A global existence result*, Invent. Math. 173 (2008), no. 3, 477–496.
[6] R. J. DiPerna and P.-L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. 98 (1989), 511–547.
[7] J. Forlano, L. Tolomeo, *Quasi-invariance of Gaussian measures of negative regularity for fractional nonlinear Schrödinger equations*, arXiv:2205.11453 [math.AP].
[8] J. Lebowitz, H. Rose, E. Speer, *Statistical Mechanics of the Nonlinear Schrödinger Equation*, J. Stat. Phys. 50 (1988), no.3, 657–687.
[9] H.P. McKean, K.L. Vaninsky, K.L., *Statistical Mechanics of Nonlinear Wave Equations*, in Sirovich, L. (eds) *Trends and Perspectives in Applied Mathematics*, Applied Mathematical Sciences, vol 100. Springer, New York, NY (1994).

The Euclidean ϕ^4_2 theory as a limit of an interacting Bose gas

Vedran Sohinger
(joint work with Jürg Fröhlich, Antti Knowles, Benjamin Schlein)

A *Euclidean field theory* of a scalar field $\Lambda \subset \mathbb{R}^d$ is given by a formally-defined probability measure on the space of fields $\phi : \Lambda \to \mathbb{R}^N$ of the form

$$\mu(\mathrm{d}\phi) = \frac{1}{c}\,\mathrm{e}^{-S(\phi)}\,\mathrm{D}\phi\,, \tag{1}$$

where $\mathrm{D}\phi = \prod_{x\in\Lambda}$ denotes the formal Lebesgue measure on the space of fields, S is the action and c is a normalisation constant. In the N-component *Euclidean ϕ^4_d theory*, the action in (2) is given by

$$S(\phi) := -\int_\Lambda \mathrm{d}x\,\phi(x)\,\cdot\,(\theta + \Delta/2)\,\phi(x) + \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\,|\phi(x)|^4\,. \tag{2}$$

In (2), θ and λ are constants, Δ is the Laplacian on Λ with appropriate boundary conditions, and $|\cdot|$ is the Euclidean norm on $\mathbb{R}^N$. We consider throughout the regime $N = 2$ and identify $\mathbb{R}^2 \equiv \mathbb{C}$, in which case (1) is formally invariant under the flow of the *nonlinear Schrödinger equation (NLS)*

$$\mathrm{i}\partial_t\phi = \frac{1}{2}\nabla S(\phi) = -(\theta + \Delta/2)\phi + \lambda|\phi|^2\phi\,.$$

In this context, one refers to (1) as the *Gibbs measure.* The study of these measures and their invariance in the framework of random initial data with low regularity was pioneered in the work of Bourgain [1, 2], with many subsequent developments.

A difficulty in the analysis that arises when $d > 1$ is that the interaction term $V(\phi) := \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\,|\phi(x)|^4$ in (2) is ill-defined due to the roughness of ϕ. We rewrite (1) by introducing an $\mathbb{R}^2$-valued Gaussian free field on Λ with law $\mathbb{P}$, given by the

Gaussian measure on the space of fields with mean zero and covariance $(2\kappa-\Delta)^{-1}$ for $\kappa>0$ a constant. For a suitable normalisation constant ζ, we write (1) as

$$\mu(\mathrm{d}\phi) = \frac{1}{\zeta}\,\mathrm{e}^{-V(\phi)}\,\mathbb{P}(\mathrm{d}\phi)\,. \tag{3}$$

Throughout the sequel, we consider $d=2$, in which case to define (3), we need to Wick-order the interaction as

$$\begin{aligned} V(\phi) &= \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\ :|\phi(x)|^4: \\ &= \frac{\lambda}{2}\int_\Lambda \mathrm{d}x\left(|\phi(x)|^4 - 4\mathbb{E}\big[|\phi(x)|^2\big]\,|\phi(x)|^2 + 2\mathbb{E}\big[|\phi(x)|^2\big]^2\right). \end{aligned} \tag{4}$$

The quantity (4) is non-positive, yet is is still possible to show that $\mathrm{e}^{-V(\phi)}\in L^1(\mathbb{P})$ [8, 9].

Our goal is to establish a relationship between Euclidean field theories of the form (1), (3) and interacting Bose gases with repulsive two-body interactions in two dimensions. In particular, we show that the Euclidean ϕ^4_2 theory describes the limiting behaviour of an interacting Bose gas at positive temperature. The limiting regime we consider is a high-density limit in a box of fixed size with the range of the interaction tending to zero in a suitable way.

In the quantum setting, we work on the bosonic Fock space $\mathcal{F}=\bigoplus_n \mathfrak{H}^{(n)}$, where $\mathfrak{H}^{(n)}$ denotes the symmetric subspace of $L^2(\Lambda)^{\otimes n}$. Given $\nu,\epsilon>0$, we define the *quantum many-body Hamiltonian* on $\mathcal{F}$

$$\mathbb{H}_{\nu,\epsilon} = \bigoplus_{n\in\mathbb{N}} H^{(n)}_{\nu,\epsilon}\,,$$

where

$$H^{(n)}_{\nu,\epsilon} = \nu\sum_{i=1}^n\Big(-\frac{\Delta_i}{2}+\kappa\Big) + \frac{\nu^2}{2}\sum_{i,j=1}^n v^\epsilon(x_i-x_j) - a_{\nu,\epsilon}n + b_{\nu,\epsilon}\,. \tag{5}$$

In (5), we take the rescaled interaction potential

$$v^\epsilon(x) = \sum_{n\in\mathbb{Z}^2}\frac{1}{\epsilon^2}\,v\Big(\frac{x-n}{\epsilon}\Big)\,,$$

where $v:\mathbb{R}^d\to\mathbb{R}$ is an even, smooth, compactly supported function of positive type (i.e. $\hat{v}\geq 0$) whose integral is equal to one. Moreover, $a_{\nu,\epsilon}$ and $b_{\nu,\epsilon}$ are suitable mass and energy renormalisations respectively. The *grand canonical ensemble* is the sequence $(\rho_n)_n$ given by

$$\rho_n \equiv \rho_{\nu,\epsilon,n} = \frac{1}{Z_{\nu,\epsilon}}\,\mathrm{e}^{-H^{(n)}_{\nu,\epsilon}}\,, \qquad Z_{\nu,\epsilon} := \sum_{n\in\mathbb{N}}\mathrm{Tr}_{\mathfrak{H}^{(n)}}\mathrm{e}^{-H^{(n)}_{\nu,\epsilon}}\,. \tag{6}$$

For $p\in\mathbb{N}$, one defines the *p-particle reduced density matrix* $\gamma_{\nu,\epsilon,p}$ by its operator kernel

$$(\gamma_{\nu,\epsilon,p})_{x_1,\dots,x_p;y_1,\dots,y_p} := \sum_{n\geq p}\frac{n!}{(n-p)!}\,\mathrm{Tr}_{p+1,\dots,n}(\rho_{\nu,\epsilon,n})\,, \tag{7}$$

where $\mathrm{Tr}_{p+1,\ldots,n}$ denotes the partial trace in $x_{p+1},\ldots,x_n$. We compare (7) with the *classical p-particle correlation functions*

$$(\gamma_p)_{x_1,\ldots,x_p;y_1,\ldots,y_p} := \mathbb{E}_\mu\left[\overline{\phi}(y_1)\cdots\overline{\phi}(y_1)\,\phi(x_1)\cdots\phi(x_p)\right]. \tag{8}$$

We then show the following result.

Theorem 1 (Theorem 2.1 in [6]). *Suppose that* $\epsilon \equiv \epsilon(\nu)$ *satisfies*

$$\epsilon \geq \exp\bigl(-(\log \nu^{-1})^{1/2-c}\bigr) \tag{9}$$

for some $c > 0$. *Then, with notation as in* (3), (6), (7), (8), *as* $\epsilon, \nu \to 0$ *satisfying* (9), *we have convergence of the partition function*

$$\mathcal{Z}_{\nu,\epsilon} := \frac{Z_{\nu,\epsilon}}{Z_\nu^{(0)}} \to \zeta \tag{10}$$

and for all $p \in \mathbb{N}$ *and* $r \in [1,\infty)$

$$\gamma_{\nu,\epsilon,p} \xrightarrow{L^r} \gamma_p\,. \tag{11}$$

In (10), $Z_\nu^{(0)}$ *denotes the free partition function, which is obtained by setting* $v^\epsilon = 0$ *and* $a_{\nu,\epsilon}, b_{\nu,\epsilon} = 0$ *in* (5). *Furthermore, we can improve the convergence in* (11) *to that in the* L^∞ *norm under a suitable Wick-ordering procedure of the reduced density matrices and correlation functions.*

In our analysis, we compare the field theory obtained from the interaction (4) with that obtained from a regularised interaction

$$V^\epsilon = \frac{1}{2}\int_{\Lambda^2} \mathrm{d}x\,\mathrm{d}y\ :|\phi(x)|^2:\,v^\epsilon(x-y)\,:|\phi(y)|^2: - \tau^\epsilon \int_\Lambda \mathrm{d}x\ :|\phi(x)|^2: - E^\epsilon\,, \tag{12}$$

where $:\cdot:$ denotes Wick ordering with respect to $\mathbb{P}$ and $\tau_\epsilon, E_\epsilon$ are suitably chosen diverging counterterms. The proof of the theorem consists of three ingredients.

(1) A quantitative analysis of the convergence of the quantum problem (with nonlocal interaction) to the classical problem with interaction (12) for fixed $\epsilon > 0$. This step relies on a precise analysis of the functional integral formulation we had previously set up in [4]. The details of this step are given in [6, Section 5].
(2) A comparison of the classical field theory with interaction (12) and (4). This is based on a version of *Nelson's argument* [8] for nonlocal interactions. This comparison is done in [6, Sections 4.1–4.2].
(3) A Gaussian integration by parts that allows us to obtain uniform control on the Wick-ordered correlation functions. This method is reminiscent of Malliavin calculus. The details are given in [6, Section 4.3].

The methods applied above allow us to study the defocusing nonlocal cubic NLS for interaction potentials in optimal L^q classes [6, Theorem 5.23]. We note that an alternative approach to the results for the nonlocal problem proved in [4] was independently given by Lewin, Nam, and Rougerie in [7]. A summary of these related results is given in [5].

References

[1] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Comm. Math. Phys. **166** (1994), no. 1, 1–26.
[2] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. **176** (1996), no. 8, 421–445.
[3] J. Bourgain, *Invariant measures for the Gross-Pitaevskii equation*, J. Math. Pures Appl. (9) **76** (1997), no. 8, 649–702.
[4] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger. *The mean-field limit of quantum Bose gases at positive temperature*, Preprint arXiv: 2001.01546 (2020). To appear in J. Amer. Math. Soc.
[5] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger. *A path-integral analysis of interacting Bose gases and loop gases*, J. Stat. Phys. **180** (2020), no. 1–6, 810–831.
[6] J. Fröhlich, A. Knowles, B. Schlein, and V. Sohinger. *The Euclidean Φ^4_2 theory as a limit of an interacting Bose gas*, Preprint arXiv: 2201.07632 (2022).
[7] M. Lewin, P.-T. Nam, N. Rougerie, *Classical field theory limit of many-body quantum Gibbs states in 2D and 3D*, Invent. Math. **224** (2021), no. 2, 315–444.
[8] E. Nelson *The free Markoff field*, J. Functional Anal. **12** (1973), no. 2, 211–227.
[9] E. Nelson, *Probability theory and Euclidean field theory*, Constructive quantum field theory, Springer, 1973, pp. 94–124.

Transport of Gaussian measures with exponential cut-off for Hamiltonian PDEs

Renato Lucà
(joint work with Giuseppe Genovese, Nikolay Tzvetkov)

1.1. **Introduction.** The goal here is to study the the transport of Gaussian measures under the flow of non integrable Hamiltonian PDEs, in particular their quasi-invariance. The results have been obtained in [5].

The main contribution here is the introduction of a suitable exponential cut-off that helps to prove quasi-invariance of Gaussian measures for a class of Hamiltonian PDEs. As example for the method we consider the fractional Benjamin-Bona-Mahony (BBM) equation and the one dimensional quintic defocussing nonlinear Schrödinger equation (NLS), both in the one dimensional periodic setting. For the BBM equation the situation is particularly simple, as the candidate for a quasi-invariant measure is simply the Gaussian measure with a cut-off on the energy (3), while for the NLS equation we consider additional weights based on the modified energies introduced in [13], [14]. For BBM we also show almost sure global well-posedness for data in $C^\alpha(\mathbb{T})$ for arbitrarily small $\alpha > 0$ and invariance of the Gaussian measure associated with the $H^{\beta/2}(\mathbb{T})$ norm, following the seminal approach developed by Bourgian in [1].

Definition 1. A measure μ is invariant under a (reversible) flow map $\{\Phi_t\}_{t\in\mathbb{R}}$ if $\mu \circ \Phi_t = \mu$ for any $t \in \mathbb{R}$ and it is quasi-invariant if $\mu \circ \Phi_t$ is absolutely continuous w.r.t. μ.

1.2. **The BBM equation.** For $\beta > 1$, we consider the fractional BBM equation, posed on the one dimensional flat torus $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$:

$$\partial_t u + \partial_t |D_x|^\beta u + \partial_x u + \partial_x(u^2) = 0, \quad u(0,x) = u_0(x)\,, \tag{1}$$

where u is real valued and

$$|D_x|^\beta(u)(x) := \sum_{n\neq 0} |n|^\beta \hat{u}(n) e^{inx}\,.$$

The parameter β measure the dispersion, the case $\beta = 1$ is the dispersionless.

Let $\{h_n\}_{n>0}$, $\{l_n\}_{n>0}$ be two independent sequences of independent standard Gaussian random variables. Let g_0 be a standard Gaussian random variable independent on anything else and set

$$g_n := \begin{cases} \frac{1}{\sqrt{2}}(h_n + il_n) & n \in \mathbb{N} \\ \frac{1}{\sqrt{2}}(h_n - il_n) & -n \in \mathbb{N}\,. \end{cases}$$

Let $\beta > 1$, $s \geq 0$ and denote by γ_s the Gaussian measure on H^s induced by the random Fourier series

$$\varphi_s(x) = \sum_{n\in\mathbb{Z}} \frac{g_n}{(1+|n|^{2s+\beta})^{\frac{1}{2}}}\, e^{inx}\,. \tag{2}$$

The measure γ_0 is special because we expect that it is invariant under Φ_t thanks to the $H^{\beta/2}$ conservation. More precisely

Theorem 2. [5]. *Let $\beta > 1$. Equation* (1) *is globally well-posed for γ_0-almost all initial data. Moreover the measures γ_0 is invariant under the resulting flow.*

The proof exploits : 1) the standard characterisation of the support of γ_0, namely that $\bigcap_{\alpha<\frac{\beta-1}{2}} C^\alpha$ is a full γ_0-measure set.; 2) the (simple) fact that equation (1) is locally wellposed on C^α, for all $\alpha \geq 0$. Thus combining Theorem 2 and the Poincaré recurrence theorem, we have for all $\beta > 1$ recurrence of the solutions with respect to the C^α topology, $\alpha < \frac{\beta}{2} - \frac{1}{2}$, almost surely with respect to γ_0.

When $s > 0$ we study the quasi invariance of γ_s introducing suitable weights. First, we introduce a rigid cut-off on

$$\int |u|^2 + |D^{\beta/2}u|^2, \tag{3}$$

namely on the $H^{\beta/2}$ norm, which is a conserved quantity. Moreover, and this is the mail novelty of our approach, we introduce an exponential weight the H^s Sobolev norm. More precisely, given some $R > 0$, we define

$$\rho_s(du) := 1_{\{\|u\|_{H^{\beta/2}}\leq R\}}(u) \exp(-\|u\|_{H^s}^{2r})\gamma_s(du)\,, \quad r > 2. \tag{4}$$

We focus on the case $\beta \in (1,2]$, however, the case $\beta > 2$ may be deduced by a classical result of Ramer [15]; see [16].

Theorem 3. [5]. *Let $\beta \in (1,2]$, $s > \frac{\beta}{2}$ such that $s + \beta/2 > 3/2$. Let also $r > 2$. The measures ρ_s are quasi-invariant along the flow of* (1). *The densities $f_s(t,u)$*

of the transported measures are in $L^p(\rho_s)$ for all $t \in \mathbb{R}$ and $p < \infty$. Moreover if $s > \frac{3}{2}$

$$f_s(t,u) := \exp\Big(- \|\Phi_t u\|_{H^s}^{2r} - \frac{1}{2}\|\Phi_t u\|^2_{H^{s+\frac{\beta}{2}}} + \|u\|_{H^s}^{2r} + \frac{1}{2}\|u\|^2_{H^{s+\frac{\beta}{2}}}\Big) . \tag{5}$$

This result extends and refine (at least in some aspect) the one in [16]. We remark that the difference

$$-\frac{1}{2}\|\Phi_t u\|^2_{H^{s+\frac{\beta}{2}}} + \frac{1}{2}\|u\|^2_{H^{s+\frac{\beta}{2}}} \tag{6}$$

has to be interpreted as a single object, since both terms are a.s. infinite in the support of γ_s, while their difference is finite.

The restriction $s > \beta/2$ is in order to take advantage of the exponential cut-off, since it gives no additional help for $s \le \beta/2$ than the rigid cut-off on the $H^{\beta/2}$ norm. The assumption $s > \frac{3}{2}$ for the densities is technical.

1.3. **The NLS equation.** We prove similar results for the defocusing quintic NLS on $\mathbb{T}$:

$$i\partial_t u + \partial_x^2 u = |u|^4 u, \quad u(0,x) = u_0(x). \tag{7}$$

The L^2 norm of the solution $\|u\|_{L^2}$ and the energy

$$\mathcal{E}_1(u) = \frac{1}{2}\|u\|^2_{H^1} + \frac{1}{6}\|u\|^6_{L^6} \tag{8}$$

are formally conserved by the flow. In [13], [14] a countable family of *modified energies*

$$\mathcal{E}_{2k} := \|u\|_{H^{2k}} + R_{2k}, \qquad k \ge 1,$$

has been introduced. The derivative along the flow of the modified energies is not zero, but it presents however some smoothing, which makes them still useful in order to control the growth in time of the Sobolev norms of the solutions. We will not specify here the form of the remainder R_{2k}.

Since here we have complex solutions, we take a sequence of complex standard Gaussian random variables $\{g_n\}_{n\in\mathbb{Z}}$ and for integers $k \ge 2$ the Fourier series

$$\varphi_{2k}(x) = \sum_{n\in\mathbb{Z}} \frac{g_n}{(1+|n|^{4k})^{\frac{1}{2}}}\, e^{inx} . \tag{9}$$

We indicate by γ_{2k} the induced measure on $H^{2k-\frac{1}{2}-} := \bigcap_{\varepsilon>0} H^{2k-\frac{1}{2}-\varepsilon}$. We study the transport of the Gaussian measure γ_{2k} introducing : 1) a rigid cut-off on the conserved quantities introduced above, *i.e.* mass and energy, 2) suitable weights based on the modified energies of [14], 3) an exponential cut-off on the H^{2k-1} norm. More precisely, given some $R > 0$

$$\mu_{2k}(du) := 1_{\{\|u\|_{L^2}+\mathcal{E}_1(u)\le R\}}(u) \exp(-R_{2k}(u) - \|u\|^{2r}_{H^{2k-1}})\gamma_{2k}(du)\,, \tag{10}$$

We prove

Theorem 4. [5]. *Let $k \geq 2$ be an integer. There exists $r(k) > 0$ sufficiently large such that for all $r > r(k)$ the measures μ_{2k} are quasi-invariant along the flow of* (7). *For all $t \in \mathbb{R}$, there exists $p = p(|t|) > 1$ such that the densities $f_{2k}(t, u)$ of the transported measures are in $L^p(\mu_{2k})$. Moreover*

$$f_{2k}(t,u) := \exp\Big(- \|\Phi_t u\|^{2r}_{H^{2k-1}} - \mathcal{E}_{2k}(\Phi_t u) + \|u\|^{2r}_{H^{2k-1}} + \mathcal{E}_{2k}(u)\Big) . \tag{11}$$

This refines the analogous result from [14].

Remark 5. The result of [16] was extended to more involved models in [2, 4, 7, 8, 3, 9, 10, 12, 6]. We believe that, beyond the BBM and NLS equation, the idea of an exponential cut-off introduced in the present paper may be relevant in the context of some of these works and, more generally, in the study of quasi-invariant measures for Hamiltonian PDEs.

References

[1] J. Bourgain, *Periodic nonlinear Schrödinger equation and invariant measures*, Commun. Math. Phys. 166 (1994), 1–26.

[2] A. Debussche, Y. Tsustumi, *Quasi-Invariance of Gaussian Measures Transported by the Cubic NLS with Third-Order Dispersion on* $\mathbb{T}$, arXiv:2002.04899 (2020).

[3] J. Forlano, W. Trenberth, *On the transport of Gaussian measures under the one-dimensional fractional nonlinear Schrödinger equation*, Ann. Inst. Henri Poincare, Anal. Non Lineaire, 36 (2019), 1987–2025.

[4] G. Genovese, R. Lucà, N. Tzvetkov, *Quasi-invariance of low regularity Gaussian measures under the gauge map of the periodic derivative NLS*, J. Funct. Anal., 282(1), (2022), https://doi.org/10.1016/j.jfa.2021.109263.

[5] G. Genovese, R. Lucà, N. Tzvetkov, *Transport of Gaussian measures with exponential cut-off for Hamiltonian PDEs*, arXiv:2103.04408.

[6] G. Genovese, R. Lucà, N. Tzvetkov, *Quasi-invariance of Gaussian measures for the periodic Benjamin-Ono-BBM equation*, Stoch. PDE: Anal. Comp. (2022), https://doi.org/10.1007/s40072-022-00240-2

[7] G. Genovese, R. Lucà, D, Valeri, *Gibbs Measures Associated to the Integrals of Motion of the Periodic DNLS*, Sel. Math. New Ser., 22(3), 1663–1702, (2016).

[8] G. Genovese, R. Lucà, D, Valeri, *Invariant measures for the periodic derivative nonlinear Schrödinger equation*, Mathematische Annalen 374 (3-4), 1075-1138, 2019.

[9] T. Oh, N. Tzvetkov, *Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation*, Probab. Theory Relat. Fields 169 (2017), 1121-1168.

[10] T. Oh, N. Tzvetkov, *Quasi-invariant Gaussian measures for the two-dimensional defocusing cubic nonlinear wave equation*, JEMS 22 (2020) 1785–1826.

[11] T. Oh, P. Sosoe, N. Tzvetkov, *An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation*, J. Ec. Polytechnique, Math., 5 (2018), 793–841.

[12] T. Oh, Y. Tsutsumi and N. Tzvetkov, *Quasi-invariant Gaussian measures for the cubic nonlinear Schrödinger equation with third-order dispersion*, C. R. Acad. Sci. Paris, Ser. I, 357 (2019), 366–381.

[13] F. Planchon, N. Tzvetkov, N. Visciglia, On the growth of Sobolev norms for NLS on 2-and 3-dimensional manifolds, Anal. PDE 10 (2017), 1123-1147.

[14] F. Planchon, N. Tzvetkov and N. Visciglia, *Transport of Gaussian measures by the flow of the nonlinear Schrödinger equation*, Math. Ann. 378 (2020) 389–423.

[15] R. Ramer, *On nonlinear transformations of Gaussian measures*, Journal of Functional Analysis 15 (1974), 166-187.
[16] N. Tzvetkov, *Quasi-invariant Gaussian measures for one dimensional Hamiltonian PDEs*, Forum Math. Sigma 3 (2015), e28, 35 pp.

Resonance-based schemes for dispersive equations via decorated trees

Yvain Bruned
(joint work with Katharina Schratz)

We consider nonlinear dispersive equations of the form

$$
\begin{aligned}
&i\partial_t u(t,x) + \mathcal{L}u(t,x) = p\left(u(t,x), \overline{u}(t,x)\right) \\
&u(0,x) = v(x), \quad (t,x) \in \mathbb{R}_+ \times \mathbf{T}^d
\end{aligned} \tag{1}
$$

where $\mathcal{L}$ is a differential operator and p is a polynomial nonlinearity. We assume local wellposedness of the problem on the finite time interval $]0,T]$, $T < \infty$ for $v \in H^n$. Our main aim is to give a numerical approximation of u at low regularity when n is small. Interesting examples are the cubic nonlinear Schrödinger (NLS) equation ($\mathcal{L} = \Delta$, $p(u,\overline{u}) = |u|^2 u$) and the Korteweg–de Vries (KdV) equation ($\mathcal{L} = i\partial_x^3$, $p(u,\overline{u}) = i\partial_x u^2$). In [9], a new numerical framework for (1) embeds the underlying resonance structure into the discretisation. This allows an approximation with high order accuracy under low regularity assumptions that classical techniques require. The scheme is built using a tailored decorated tree formalism that takes its inspiration from algebraic structures developed for singular SPDEs with Regularity Structures (see [14, 6, 3, 7]). We introduce a novel class of decorations encoding the dominant frequencies. In order to construct such a scheme, one iterates Duhamel's formula in Fourier space given by:

$$
\hat{u}_k(\tau) = e^{i\tau P(k)}\hat{v}_k + e^{i\tau P(k)}(-i\int_0^\tau e^{-i\xi P(k)} p_k\left(u(\xi), \overline{u}(\xi)\right) d\xi) \tag{2}
$$

where $P(k)$ and $p_k(u,\bar{u})$ are given for NLS by

$$
P(k) = -k^2, \quad p_k(u,\bar{u}) = \sum_{k=-k_1+k_2+k_3} \bar{\hat{u}}_{k_1}\hat{u}_{k_2}\hat{u}_{k_3}.
$$

Then, one can write a truncated B-series type expansion U_k^r of the solution $\hat{u}_k$:

$$
U_k^r(\tau, v) = \sum_{T \in \mathcal{V}_k^r} \frac{\Upsilon^p(T)(v)}{S(T)} \left(\Pi T\right)(\tau) \tag{3}
$$

where $\mathcal{V}_k^r$ is a finite set of decorated trees, $S(T)$ is the symmetry factor associated to the tree T, $\Upsilon^p(T)(v)$ is the coefficient appearing in the iteration of Duhamel's formulation and $(\Pi T)(\tau)$ represents a Fourier iterated integral. The exponent r in $\mathcal{V}_k^r$ means that we consider iterated integrals with $r+1$ integrals. Then, one has to replace each $(\Pi T)(\tau)$ by its resonance approximation $(\Pi^{r,n} T)(\tau)$ where n is the regularity assumed a priori on the initial value v. We illustrate this resonance discretisation on the following integral:

$$(\Pi T)(\tau) = \int_0^\tau e^{i\xi\left(k^2+k_1^2-k_2^2-k_3^2\right)} d\xi, \quad k = -k_1 + k_2 + k_3 \tag{4}$$

described by an appropriate decorated tree T. One has

$$k^2 + k_1^2 - k_2^2 - k_3^2 = \mathcal{L}_{\text{dom}} + \mathcal{L}_{\text{low}}, \quad \mathcal{L}_{\text{dom}} = 2k_1^2, \quad \mathcal{L}_{\text{low}} = -2(k_2 + k_3)k_1 + 2k_2k_3.$$

Then, we Taylor-expand the lower part $\mathcal{L}_{\text{low}}$ that asks only one derivative on the initial value in comparison two derivatives coming from the dominant part $\mathcal{L}_{\text{low}}$. This dominant part is integrated exactly and mapped back to physical space. For our example above, one gets:

$$(\Pi T)(\tau) = (\Pi^{n,r} T)(\tau) + \mathcal{O}\Big(\tau \mathcal{L}_{\text{low}}\Big), \quad (\Pi^{n,r} T)(\tau) = \frac{e^{2i\tau k_1^2} - 1}{2ik_1^2}. \tag{5}$$

Here, we assumed that $n \leq 1$. If $n = 2$, one can proceed with a full Taylor expansion. The resonance scheme $U_k^{n,r}$ of order r with regularity n is defined by

$$U_k^{n,r}(\tau, v) = \sum_{T \in \mathcal{V}_k^r} \frac{\Upsilon^p(T)(v)}{S(T)} \left(\Pi^{n,r} T\right)(\tau). \tag{6}$$

The main idea of the local error analysis is to single out oscillations. Indeed, the exact integration of a polynomial $P(k)$ provides two resonances $e^{i\tau P(k)}$ and 1:

$$\int_0^\tau e^{i\xi P(k)} d\xi = \frac{e^{i\tau P(k)} - 1}{iP(k)} \tag{7}$$

Then, they will interact with other oscillations in a big iterated integrals. The combinatorial difficulty of the problem can be handled by a deformed Butcher-Connes-Kreimer coproduct (see [10, 11, 12, 9]) and the use of a Birkhoff factorisation (see [9, 5]). This leads to one of the main results in [9], for every $T \in \mathcal{V}_k^r$

$$\left(\Pi T - \Pi^{n,r} T\right)(\tau) = \mathcal{O}\left(\tau^{r+2} \mathcal{L}_{\text{low}}^r(T, n)\right). \tag{8}$$

where $\mathcal{L}_{\text{low}}^r(T, n)$ involves all lower order frequency interactions. Let us mention some recent extensions of this scheme as a conclusion. This scheme has been extended to more general nonlinearities in [1] where it is written directly in physical space via the use of nested commutators. A discretisation of the second moment of the solution is performed via this resonance scheme for PDEs with random initial data in [2]. In the future, one may expect connections between the decorated trees combinatorial approach of this scheme and some other works presented in this workshop in Wave turbulence theory (see [13, 15]) and in dispersive equations with random initial data (see [4]) where the same type of combinatorics is needed.

Acknowledgements. YB thanks the Max Planck Institute for Mathematics in the Sciences (MiS) in Leipzig for supporting his research via a long stay in Leipzig from January to June 2022.

References

[1] Y. Alama Bronsard, Y. Bruned, K. Schratz. *Low regularity integrators via decorated trees.* arxiv:2202.01171.

[2] Y. Alama Bronsard, Y. Bruned, K. Schratz. *Approximation of dispersive PDEs in the presence of low-regularity randomness.* arxiv:2205.02156.

[3] Y. Bruned, A. Chandra, I. Chevyrev, M. Hairer. *Renormalising SPDEs in regularity structures.* J. Eur. Math. Soc. (JEMS), **23**, no. 3, (2021), 869-947. doi:10.4171/JEMS/1025.

[4] B. Bringmann, Y. Deng, A. R. Nahmod, H. Yue. *Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation.* arXiv:2205.03893.

[5] Y. Bruned, K. Ebrahimi-Fard. *Bogoliubov type recursions for renormalisation in regularity structures.* arXiv:2006.05284.

[6] Y. Bruned, M. Hairer, L. Zambotti. *Algebraic renormalisation of regularity structures.* Invent. Math. **215**, no. 3, (2019), 1039–1156. doi:10.1007/s00222-018-0841-x.

[7] Y. Bruned, M. Hairer, L. Zambotti. *Renormalisation of Stochastic Partial Differential Equations.* EMS Newsletter **115**, no. 3, (2020), 7–11. doi: 10.4171/NEWS/115/3.

[8] Y. Bruned, D. Manchon. *Algebraic deformation for (S)PDEs.* To appear in J. Math. Soc. Japan. arXiv:2011.05907.

[9] Y. Bruned, K. Schratz. *Resonance based schemes for dispersive equations via decorated trees.* Forum of Mathematics, Pi, 10, E2. doi:10.1017/fmp.2021.13.

[10] J. C. Butcher. *An algebraic theory of integration methods.* Math. Comp. **26**, (1972), 79–106. doi:10.2307/2004720.

[11] A. Connes, D. Kreimer. *Hopf algebras, renormalization and noncommutative geometry.* Comm. Math. Phys. **199**, no. 1, (1998), 203–242. doi:10.1007/s002200050499.

[12] A. Connes, D. Kreimer. *Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem.* Commun. Math. Phys. **210**, (2000), 249–73. doi:10.1007/s002200050779.

[13] Y. Deng, Z. Hani. *Full derivation of the wave kinetic equation.* arXiv:2104.11204.

[14] M. Hairer. *A theory of regularity structures.* Invent. Math. **198**, no. 2, (2014), 269–504. doi:10.1007/s00222-014-0505-4.

[15] G. Staffilani, M.-B. Tran. *On the wave turbulence theory for a stochastic KdV type equation.* arXiv:2106.09819.

On the hard phase fluid with free boundary in relativity

Shuang Miao

(joint work with Sohrab Shahshahani, Sijue Wu)

The hard phase model describes a relativistic barotropic and irrotational fluid with sound speed equal to the speed of light. In the framework of general relativity, the fluid, as a matter field, affects the geometry of the background spacetime. Therefore the motion of the fluid must be coupled to the Einstein equations which describe the structure of the underlying spacetime. This model is an idealized model for the physical situation where, during the gravitational collapse of the degenerate core of a massive star, the mass-energy density exceeds the nuclear saturation density. See [3, 4, 8, 15, 25, 29].

In [16] we study Einstein's equations coupled to a hard phase fluid with free boundary surrounded by vacuum. The fluid domain is a priori unknown and has to be determined by the boundary conditions. The first boundary condition simply says that the fluid pressure vanishes on the boundary, which is the analogue of the vanishing of surface tension in the Newtonian setting. The second condition states that a fluid particle on the boundary will remain on the boundary for all time. The restriction to the hard phase state is both because of the independent interest of this model as discussed above, and because, as described in [17], this model captures the important mathematical features of more general barotropic equations of state.

Following Choquet-Bruhat's existence theorem in [7] for Einstein's equations in vacuum, well-posedness results have been established for the Einstein equations coupled to many different matter fields. Indeed, much progress has been made beyond the local theory in the past few decades. Despite all the progress in this direction, in the presence of isolated bodies, especially with free boundaries, our understanding of even the local theory is very limited. A well-posedness theory for isolated bodies is of central importance as it is naturally the first step in any further analysis of the motion and interaction of gravitating bodies. With the exception of [2, 14, 22], which consider special solutions under symmetries or where the motion of the boundary is not tracked, the only work on well-posedness for isolated bodies in general relativity that we are aware of is [1], which considers the very different case of solid elastic bodies. The free boundary problem for fluid bodies is, however, very different and includes many new analytical challenges even in the Newtonian setting. To the best of our knowledge the current work is the first to prove well-posedness for a free-boundary fluid equation in the setting of general relativity. The type of fluid model considered in this work, where the energy density has a jump across the fluid boundary, is sometimes referred to as a liquid model. Well-posedness for such free boundary fluid models is already subtle in the Newtonian case, and the first satisfactory local theory in Sobolev spaces was developed only in the mid 1990s in [26, 27] for the case of water waves.[1] More recently, well-posedness was established for relativistic liquids with free boundary in [18, 20, 21, 17], but in the case of a fixed background, that is, without coupling to Einstein's equations. See also [19] for the case of two spatial dimensions, [10] for a priori estimates under smallness assumptions, [24] for an existence results using Nash-Moser iteration, and [11]. For related developments in the gaseous case (where the energy density vanishes at the fluid boundary) on a fixed background see [6, 12, 13].

The work [16] is a continuation of our earlier work [17]. The idea is to use the general setup developed in [17] to treat the relativistic fluid. However, a main difficulty is that in the presence of Einstein's equations one has to guarantee that geometric quantities do not break the regularity structure necessary to close the estimates for the fluid quantities. This is achieved by working in a frame that is parallel transported along the fluid velocity. The idea of using frames to derive estimates in the study of Einstein's equations has a long history, and has proved especially useful in long time analysis of the dynamics. See for instance [5]. Some

[1] The interested reader is referred to these works and for instance [17, 28] for more on the history of the well-posedness theory in the Newtonian case.

of the choices we have made are inspired by the work [9] which studies various formulations of the Bianchi equations as a first order hyperbolic system. However, our final formulation is different from the ones considered in [9], especially because the free boundary fluid is analyzed differently based on [17].

References

[1] L. Andersson, and T. A. Oliynyk, and B. G. Schmidt, *Dynamical compact elastic bodies in general relativity*, Archive for Rational and Mechanical Analysis, **220**(2): 849–887, 2016.

[2] Y. Choquet-Bruhat, and H. Friedrich, *Motion of isolated bodies*, Classical Quantum Gravity, **23**(20): 5941–5949, 2006.

[3] D. Christodoulou, *https://www.youtube.com/watch?v=FFMSteogq40*

[4] D. Christodoulou, *Self-gravitating relativistic fluids: a two-phase model*, Archive for Rational and Mechanical Analysis, **130** (4): 343-400, 1995.

[5] D. Christodoulou, and S. Klainerman. *The global nonlinear stability of the Minkowski space*, volume 41 of Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1993.

[6] M. M. Disconzi, and M. Ifrim, and D. Tataru. *The relativistic Euler equations with a physical vacuum boundary: Hadamard local well-posedness, rough solutions, and continuation criterion*, arXiv:2007.05787, July 2020.

[7] Y. Fourès-Bruhat, *Théorème d'existence pour certain systèmes d'équations aux dérivées partielles non linéaires*, Acta Math., **88**: 141-225, 1952

[8] B. Friedman, and V. R. Pandharipande, *Hot and cold, nuclear and neutron matter*, Nuclear Physics, **A361**: 502–520, 1981.

[9] H. Friedrich, *Hyperbolic reductions for Einstein's equations. Classical Quantum Gravity*, **13**(6):1451–1469, 1996.

[10] D. Ginsberg, *A priori estimates for a relativistic liquid with free surface boundary*, Journal of Hyperbolic Differential Equations, **16**(3):401–442, 2019.

[11] D. Ginsberg, and H. Lindblad, *On the local well-posedness for the relativistic Euler equations for a liquid body*, arXiv:2109.01899, September 2021.

[12] M. Hadzic, S. Shkoller, and J. Speck, *A priori estimates for solutions to the relativistic Euler equations with a moving vacuum boundary*, Comm. Partial Differential Equations, **44**(10):859–906, 2019.

[13] J. Jang, P. G. LeFloch, and N. Masmoudi. Lagrangian formulation and a priori estimates for relativistic fluid flows with vacuum. J. Differential Equations, 260(6):5481–5509, 2016.

[14] S. Kind, and J. Ehlers, *Initial-boundary value problem for the spherically symmetric Einstein equations for a perfect fluid*, Classical Quantum Gravity, **10**(10): 2123–2136, 1993.

[15] A. Lichnerowicz, *Relativistic Hydrodynamics and Magnetohydrodynamics: Lectures on the Existence of Solutions*, W. A. Benjamin, Inc., 1967.

[16] S. Miao, and S. Shahshahani, *Well-posedness for the free boundary hard phase model in general relativity*, arXiv preprint, 2021.

[17] S. Miao, S. Shahshahani, and S. Wu, *Well-posedness of free boundary hard phase fluids in Minkowski background and their Newtonian limit*, Cambridge Journal of Mathematics, **9** no.2, 269–350, 2021.

[18] T. A. Oliynyk. *On the existence of solutions to the relativistic Euler equations in two spacetime dimensions with a vacuum boundary*, Classical Quantum Gravity, **29**(15): 155013, 28, 2012.

[19] T. A. Oliynyk. *Dynamical relativistic liquid bodies I: constraint propagation*, arXiv:1707.08219, Jul 2017.

[20] T. A. Oliynyk. *A priori estimates for relativistic liquid bodies*, Bull. Sci. Math., **141**(3): 105–222, 2017.

[21] T. A. Oliynyk. *Dynamical relativistic liquid bodies*, arXiv e-prints, page arXiv:1907.08192, Jul 2019.

[22] A. D. Rendall, *The initial value problem for a class of general relativistic fluid bodies*, Journal of Mathematical Physics., **33** (3):1047–1053, 1992.
[23] L. Rezzolla, and O. Zanotti, *Relativistic Hydrodynamics*, Oxford University Press, 2013.
[24] Y. Trakhinin, *Local existence for the free boundary problem for nonrelativistic and relativistic compressible Euler equations with a vacuum boundary condition*, Communications on Pure and Applied Mathematics, **62**(11):1551–1594, 2009.
[25] J. D. Walecka, *A theory of highly condensed matter*, Annals of Physics, **83**: 491–569, 1974.
[26] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 2-D*, Inventiones Mathematicae, **130**(1): 39–72, 1997.
[27] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, Journal of the American Mathematical Society **12**(2): 445–495, 1999.
[28] S. Wu, *Wellposedness and singularities of the water wave equations*, Lectures on the theory of water waves, volume 426 of London Math. Soc. Lecture Note Ser., pages 171–202. Cambridge Univ. Press, Cambridge, 2016.
[29] Ya. B. Zel'dovich, *The equation of state at ultrahigh densities and its relativistic limitations*, J. Exp. Theor. Phys. (U.S.S.R.), **41**:1609–1615, 1961.

Paracontrolled calculus and regularity structures

Masato Hoshino
(joint work with Ismael Bailleul)

The two pathwise approaches for the renormalizations of singular stochastic PDEs have been developed in the last decades: the theory of *regularity structures* [8] and *paracontrolled calculus* [7]. The former is highly general and strong enough to establish the so-called "black box theorem" by the continuing papers [8, 5, 6, 4]. On the other hand, the paracontrolled calculus developed in [7] is not sufficient to do the same thing as [8]. Indeed, in some situations (e.g. 3d generalized parabolic Anderson model), we need more decompositions of the solutions as in [1]. The first motivation of our work is that whether we can do the same thing as regularity structures within the paracontrolled calculus. As a first step, we tried to prove that the deterministic parts of the both approaches are same at least in the Euclidean space $\mathbb{R}^d$.

The main results of our work consists of two parts: to construct an interpretation map of the languages in the regularity structures to the paracontrolled counterparts, and to show the existence of the inverse map. In the following, we describe the statements of [2, 3] roughly. We call a pair (T^+, T) a *concrete regularity structure* if $T^+ = \bigoplus_{\alpha\in A^+} T^+_\alpha$ is a graded Hopf algebra with the countable and locally finite index set $A^+ \subset [0,\infty)$ and $T = \bigoplus_{\beta\in A} T_\beta$ is a graded linear space with the countable and locally finite index set $A \subset \mathbb{R}$ bounded from below and with the right comodule structure $\Delta : T \to T \otimes T^+$. We assume that each $T^{(+)}_\alpha$ is finite dimensional and fix a basis $\mathcal{B}^{(+)}_\alpha$ of $T^{(+)}_\alpha$. We set $\mathcal{B}^{(+)} := \bigcup_{\alpha\in A^{(+)}} \mathcal{B}^{(+)}_\alpha$. For any $\tau, \sigma \in \mathcal{B}^{(+)}$, we can define the element $\tau/\sigma \in T^+$ by the expansion formula $\Delta^{(+)}\tau = \sum_{\sigma\in\mathcal{B}^{(+)}} \sigma \otimes (\tau/\sigma)$ of the coproduct $\Delta^{(+)}\tau$, where Δ^+ is the coproduct operator defined on T^+. A *model* (g, Π) is a pair of two objects: g is a continuous function from $\mathbb{R}^d$ to the character group G on T^+ with the regularity conditions

$$\mathsf{g}_{yx}(\tau) := (\mathsf{g}_y \otimes \mathsf{g}_x^{-1})\Delta^+\tau = O(|y-x|^\alpha)$$

for any $\tau \in \mathcal{B}_\alpha^+$. Moreover, $\mathsf{\Pi}$ is a continuous linear operator from T to $\mathcal{S}'(\mathbb{R}^d)$ with the regularity conditions

$$\mathsf{\Pi}_x^{\mathsf{g}}\sigma := (\mathsf{\Pi} \otimes \mathsf{g}_x^{-1})\Delta\sigma = O(|y-x|^\beta)$$

for any $\sigma \in \mathcal{B}_\beta$ (this should be understood in a distributional sense). For any given model $(\mathsf{g}, \mathsf{\Pi})$ and a parameter $\gamma \in \mathbb{R}$, a *modelled distribution* f of class $\mathcal{D}^\gamma(\mathsf{g})$ is a function from $\mathbb{R}^d$ to T such that the regularity conditions

$$\mathcal{Q}_\alpha\{f(y) - \mathsf{g}_{yx} \cdot f(x)\} = O(|y-x|^{\gamma-\alpha})$$

hold, where $\mathsf{g}_{yx} \in G$ is identified with the linear operator on T by $\mathsf{g}_{yx} \cdot \tau := (\mathrm{id} \otimes \mathsf{g}_{yx})\Delta\tau$, and $\mathcal{Q}_\alpha : T \to T_\alpha$ is the canonical projection map.

The first part of the main results is stated as follows. We fix a Littlewood-Paley decomposition $\{\Delta_i\}$ and define the Hölder-Besov space C^α and the Bony's paraproduct $\mathsf{P}_f g := \sum_{i<j-1} \Delta_i f \Delta_j g$.

Theorem 1 ([2])**.** *For any given model* $(\mathsf{g}, \mathsf{\Pi})$*, we define the continuous linear maps* $[\cdot] : T^+ \to C(\mathbb{R}^d)$ *and* $[\cdot] : T \to \mathcal{S}'(\mathbb{R}^d)$ *by the formulas*

$$(1) \qquad \mathsf{g}(\tau) = \sum_{\eta \in \mathcal{B}_{\alpha'}^+,\, \alpha' < \alpha} \mathsf{P}_{\mathsf{g}(\tau/\eta)}[\eta] + [\tau], \qquad \mathsf{\Pi}\sigma = \sum_{\zeta \in \mathcal{B}_{\beta'},\, \beta' < \beta} \mathsf{P}_{\mathsf{g}(\sigma/\zeta)}[\zeta] + [\sigma]$$

for any $\tau \in \mathcal{B}_\alpha^+$ *and* $\sigma \in \mathcal{B}_\beta$*. Then* $[\tau] \in C^\alpha$ *for any* $\tau \in \mathcal{B}_\alpha^{(+)}$*, and the mapping from* $(\mathsf{g}, \mathsf{\Pi})$ *to* $[\cdot]$ *is continuous.*

Moreover, we have similar decompositions for the modelled distributions: for any $f = \sum_{\sigma \in \mathcal{B}_\beta,\, \beta<\gamma} f_\sigma \sigma \in \mathcal{D}^\gamma(\mathsf{g})$*, we define the functions* $[f_\sigma]$ *by*

$$(2) \qquad f_\sigma = \sum_{\mu \in \mathcal{B}_\alpha,\, \beta < \alpha} \mathsf{P}_{f_\mu}[\mu/\sigma] + [f_\sigma]$$

for any $\sigma \in \mathcal{B}_\beta$*. Then* $[f_\sigma] \in C^{\gamma-\beta}$ *for any* $\sigma \in \mathcal{B}_\beta$*, and the mapping from* $\mathcal{D}^\gamma(\mathsf{g})$ *to* $[\cdot]$ *is continuous.*

The next problem is that, if the remainders $[\tau] \in C^\alpha$ and $[f_\sigma] \in C^{\gamma-\beta}$ are given, then whether the formulas (1) and (2) define a model and a modelled distribution, respectively. We need more inductive assumptions on the base structure (T^+, T), which are (essentially) satisfied by the specific algebra introduced in [5]. The second part of the main results is stated as follows. We actually consider Hölder spaces with polynomial weights in this part.

Theorem 2 ([3])**.** *We assume that* $\mathcal{B}^+$ *(resp.* $\mathcal{B}$*) is generated from a basis* $\{X^k\}_{k \in \mathbb{N}^d}$ *of the polynomial regularity structure and a finite generating set* $\mathcal{G}_\circ^+$ *(resp.* $\mathcal{B}_\bullet$*) (see Assumptions (A)-(C) in* [3] *for the precise meaning). The the subfamilies* $\{[\tau]\,;\, \tau \in \mathcal{G}_\circ^+\}$ *and* $\{[\sigma]\,;\, \sigma \in \mathcal{B}_\bullet \cap \mathcal{B}_\beta,\, \beta < 0\}$ *determine a unique model* $(\mathsf{g}, \mathsf{\Pi})$ *by the formulas* (1)*.*

Moreover, if we further assume that the expansion of $\Delta\tau$ *for any* $\tau \in \mathcal{B}_\bullet$ *does not contain the term* $\sigma \otimes X^k$ *with* $k \neq 0$ *(see Assumption (D) in* [3]*), then the subfamily* $\{[f_\sigma]\,;\, \sigma \in \mathcal{B}_\bullet\}$ *determine a unique modelled distribution* f *by the formula* (2)*.*

References

[1] I. Bailleul and F. Bernicot, *High order paracontrolled calculus*, Forum Math. Sigma **7** (2019), e44.
[2] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures I*, J. Math. Soc. Japan **73** (2021), 553–595.
[3] I. Bailleul and M. Hoshino, *Paracontrolled calculus and regularity structures II*, J. Ec. polytech. Math. **8** (2021), 1275–1328.
[4] Y. Bruned, A. Chandra, I. Chevyrev, and M. Hairer, *Renormalising SPDEs in regularity structures*, J. Europ. Math. Soc. **23** (2021), 869–947.
[5] Y. Bruned, M. Hairer, and L. Zambotti, *Algebraic renormalization of regularity structures*, Invent. Math. **215** (2019), 1039–1156.
[6] A. Chandra and M. Hairer, *An analytic BPHZ theorem for Regularity Structures*, arXiv:1612.08138, (2016).
[7] M. Gubinelli, P. Imkeller, and N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi **3** (2015), e6.
[8] M. Hairer, *A theory of regularity structures*, Invent. Math. **198** (2014), 269–504.

Codimension one stability of catenoids under the hyperbolic vanishing mean curvature flow with respect to non-symmetric perturbations

Sohrab Shahshahani
(joint work with Jonas Lührmann and Sung-Jin Oh)

In this talk we discuss the stability of the Lorentzian catenoid as a solution of the hyperbolic vanishing mean curvature flow (HVMCF), with respect to non-symmetric perturbations. Solutions of the HVMC flow are embeddings

$$\Phi : M \mapsto (\mathbb{R}\times\mathbb{R}^{n+1}, \eta),$$

where M is an $n+1$ dimensional manifold and η is the standard Minkowski metric

$$\eta = \begin{pmatrix} -1 & 0 \\ 0 & I_{(n+1)\times(n+1)} \end{pmatrix},$$

such that $\Phi^*\eta$ is Lorentzian and $\Box_{\Phi^*_\eta}\Phi = 0$. Here $\Box_{\Phi^*\eta}$ denotes the Laplace-Beltrami operator with respect to the pull back metric $\Phi^*\eta$, which, since $\Phi^*\eta$ is Lorentzian, is the wave operator on $(M, \Phi^*\eta)$. The HVMCF is the hyperbolic analogue of the classical minimal surface equation, or the parabolic mean curvature flow. Formally, the equation can be viewed as the Euler-Lagrange equation for the action (area functional)

$$\mathcal{A} = \int_M \sqrt{|\det \Phi^*\eta|}.$$

Local well-posedness for this equation for sufficiently regular data was proved in the late 1970s by Aurilia and Christodoulou [2, 1]. In general, critical points of the Riemannian area functional for maps

$$\Phi_0 : \underline{M} \to (\mathbb{R}^{n+1}, e),$$

where e denotes the Euclidean metric and $\underline{M}$ is an n dimensional manifold, satisfy the equation

$$\Delta_{\Phi_0^* e}\Phi_0 = 0.$$

Any such Φ_0 gives rise to a solution of the HVMCF from $M = \mathbb{R} \times \underline{M}$ defined by

$$\Phi(t, x) = (t, \Phi_0(x)).$$

We refer to these solutions as *stationary* solutions, or *solitons* of the HVMCF. A natural question is the stability of such solutions.

The simplest example of such Φ_0 is a linear embedding of a hyperplane. In this case, after a suitable formulation of the equation (that is, a choice of gauge) the problem reduces to proving decay estimates for solutions to a quasilinear wave equation with small initial data on Minkowski space. Stability of the hyperplane solution (in a suitable topology) was proved by Brendle [3] when $n \geq 3$ and by Lindblad [6] for $n = 2$. Another natural choice for Φ_0 is the embedding of a Riemannian catenoid as a surface of revolution, which gives rise to the Lorentzian catenoid as a stationary solution of the HVMCF. This amounts to proving the stability of a non-trivial stationary solution to a quasilinear wave equation. It is known from the work of Fischer-Colbrie and Schoen [5] that the Catenoid is an index one minimal surface. This means that the second variation of the area functional has a growing mode. Consequently, the best result one can hope for in terms of stability is the codimension one stability in an appropriate topology for the initial data. Moreover, isometries of the ambient Minkowski space $(\mathbb{R} \times \mathbb{R}^{n+1}, \eta)$ map the Lorentzian catenoid into other Lorentzian catenoid which are close to the original one (even in spatially weighted L^2 topologies for the metrics). Therefore, one should consider the stability not of just one Lorentzian catenoid, but of a $2n$ dimensional family, $\mathcal{F}$, of Lorentzian catenoids generated by translations and Lorentz boosts of the ambient space[1]. For radially symmetric perturbations Donninger, Krieger, Szeftel, and Wong [4] proved codimension one stability of the Lorentzian catenoid for[2] $n = 2$. Note that the restriction of radial symmetry precludes translations and Lorentz boosts, and therefore one does not need to consider the family $\mathcal{F}$ in this case. In this work we consider arbitrary perturbations without assumption of symmetry. When n is greater than or equal to five, we find a codimension one set of initial perturbations of the Lorentzian catenoid, for which the solution converges to a member of the family $\mathcal{F}$. The convergence is tracked by ODEs (known as modulation equations) for the translation and Lorentz boost parameters which track which member of $\mathcal{F}$ the solution converges to.

Compared to the more classical semilinear case there are a number new difficulties. Some of the key challenges in our context are: (1) the quasilinearity of the equation, (2) slow (polynomial) decay of the stationary solution at infinity,

[1] Translations and Lorentz boosts in the direction of the axis of symmetry, as well as scalings, yield perturbations which are far from the original catenoid at spatial infinity in weighted L^2 topologies.

[2] $n = 2$ is analytically the most difficult case, and the methods in [4] extends to higher dimensions.

and (3) lack of symmetry assumptions. To address these challenges, we introduce several new ideas, such as a geometric construction of modulated profiles, smoothing of modulation parameters, and a robust framework for proving decay for the radiation part, which we hope will be useful in the broader context of stability analysis of stationary solutions of quasilinear wave equations in the presence of modulation.

References

[1] A. Aurilia and D. Christodoulou. Theory of strings and membranes in an external field. I. General formulation. *J. Math. Phys.*, 20(7):1446–1452, 1979.
[2] A. Aurilia and D. Christodoulou. Theory of strings and membranes in an external field. II. The string. *J. Math. Phys.*, 20(8):1692–1699, 1979.
[3] S. Brendle. Hypersurfaces in Minkowski space with vanishing mean curvature. *Comm. Pure Appl. Math.*, 55(10):1249–1279, 2002.
[4] R. Donninger, J. Krieger, J. Szeftel, and W. Wong. Codimension one stability of the catenoid under the vanishing mean curvature flow in Minkowski space. *Duke Math. J.*, 165(4):723–791, 2016.
[5] Fischer-Colbrie, D. and Schoen, R., The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature, *Communications on Pure and Applied Mathematics*, 33, 1980, 2, 199–211,
[6] H. Lindblad. A remark on global existence for small initial data of the minimal surface equation in Minkowskian space time. *Proc. Amer. Math. Soc.*, 132(4):1095–1102, 2004.

Growth of energy density for the Schrödinger map

Valeria Banica
(joint work with Luis Vega)

We will first introduce a chain of connected equations: the binormal flow, which is a model for vortex filament dynamics in 3D fluids, the 1D Schrödinger map with values in the 2D sphere, which is the classical continuous Heisenberg model in ferromagnetism, and the 1D cubic NLS, which arises in many physical models.

Vortex filaments appear in 3-D fluids when vorticity is large and concentrated in a thin tube around a curve in $\mathbb{R}^3$. The binormal flow (BF) is the oldest, simpler and richer formally derived model for one vortex filament dynamics. More precisely, if the vorticity $\omega(t)$ is concentrated along an arclength-parametrized curve $\chi(t)$ in $\mathbb{R}^3$, its evolution in time, after a time rescaling, is modeled by:

$$\chi_t = \chi_x \times \chi_{xx}. \tag{1}$$

This goes back first to Da Rios in 1906 ([9]), by using formal asymptotics starting from Biot-Savart formula. The latest result concerning the rigorous justification of the model is due to Jerrard and Smets in 2017 ([14]), which, still under the hypothesis of persistence of concentration of the vorticity, give a proof driven by the Hamilton-Poisson structures of Euler and BF.

Considering the tangent vector $T(t,x) = \partial_x \chi(t,x)$ it is easy to see that it solves the 1D Schrödinger map to $\mathbb{S}^2$:

$$T_t = T \times T_{xx}. \tag{2}$$

Moreover, Hasimoto introduced in 1972 ([13]) the construction of a function u in terms of the curvature and torsion of the curve that satisfies the 1D cubic nonlinear Schrödinger equation:

$$iu_t + u_{xx} + (|u|^2 - A(t))u = 0, \tag{3}$$

for some real space-independent function A. Moreover, T and its complexified parallel frame normal vector N satisfy

$$T_x = \Re(\overline{u}N), \ \ N_x = -uT. \tag{4}$$

Conversely, from a regular enough solution of (3) Hasimoto's transform ensures the existence of a solution of (2) and of (1), and the resulting curves and tangent vectors are independent of the function A.

This methods allowed first to prove well-posedness for (1) and (2) for regular enough data. For instance (1) is well-posed when curvature and torsion are in high Sobolev spaces ([13],[20],[16]). Also, the Schrödinger map was proved to be well-posed for data in H^2 and for some data in H^1 ([7],[19]). Recently it has also been used to study the (1)-evolution of rough data curves with one or several corners and except at the corners having curvature in weighted space ([12],[10],[1],[2]).

There are also several other completely different methods that have been used for studying (1) and (2): geometric measure theory methods using integral currents in the sense of Federer-Fleming 60 ([15]), integrable system methods ([6, 17, 11, 18]), probabilistic methods using geometric rough paths in the sense of Lyons 98 ([5]).

We report here on the result in [4] for the Schrödinger map (2), obtained at low regularity. We consider the following rough situation: $T_0(x)$ to be constant except at $x \in \{-1, 1\}$ where it has a jump of angle θ. It is known from [2] that there exists a smooth solution of (2) on $t \in (0, 1)$ that has T_0 as a trace at time $t = 0$. These solutions are constructed using Hasimoto's method explained above, starting from solutions of (3) constructed at critical regularity, and that are superpositions of fundamental solutions of the Schrödinger equation:

$$\sum_{j\in\mathbb{Z}} A_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}. \tag{5}$$

It turns out that the (2) solutions constructed have Schrödinger map interaction energy $\int |T_x(t,x)|^2 dx$ infinite as $|T_x(t,x)|$ is $2\pi t-$periodic. The Schrödinger map interaction energy $\int |T_x(t,x)|^2 dx$ is infinite as $|T_x(t,x)|$ is $2\pi t-$periodic. However it was proved in [3] that we have a finite energy:

$$\Xi(t) := \lim_{n\to\infty} \int_n^{n+1} |\widehat{T_x}(t,\xi)|^2 d\xi,$$

conserved for $t > 0$ with a discontinuity at $t = 0$. The result we present here is that the energy density blows up pointwise, and this (only) at frequencies that go to infinity:

- there exists $C_\theta > 0$ such that

$$\sup_{\xi \in B(\pm\frac{1}{t},\sqrt{t})} |\widehat{T_x}(t,\xi)| = C_\theta |\log t|.$$

- for $\xi \notin B(\frac{1}{t},\frac{3}{4t}) \cup B(-\frac{1}{t},\frac{3}{4t})$ we have an upper-bound of $\widehat{T_x}(t,\xi)$ depending only on θ.

The proof of the growth of $\widehat{T_x}(t,\xi)$ is done by using the equations (4), the oscillatory nature of u in (5), and IBP.

We end this report with a few remarks. First, it is notable that the growth is in term of the critical Fourier-Lebesgue norm $\mathcal{F}L^\infty$. Also, energy cascade has been observed previously for non-integrable equations as the linear Schrödinger equation with potential, 2D cubic Schrödinger equation, systems of 1D cubic Schrödinger equations (Bourgain 95-97, Kuksin 97, Colliander-Keel-Staffilani-Takaoka-Tao 10, Carles-Foau 12, Grébert-Paturel -Thomann 13, Delort 14, Hani 14, Hani-Pausader-Tzvetkov-Visciglia 15, Guardia-Kaloshin 15, Bambusi-Grébert-Maspero-Robert 18, Carles-Gallagher 18, Thomann 20, Faou-Raphaël 20, Bambusi-Langella - Montalto 20,...), or for abstract integrable models as Szëgo's (Gérard-Grellier 12-17, Pocovnicu 11-13, Gérard-Lenzmann-Pocovnicu-Raphaël 18,...). Here we have Fourier modes growth for a 1D integrable equation, that is connected with a turbulent phenomena in fluids. Indeed, aside the fact that the binormal flow is derived from fluids presenting a vortex filament, the vector T_x describes through this model the variations of the direction of the vorticity, that plays a central role in Constantin-Fefferman-Majda's criterium [8].

References

[1] V. Banica and L. Vega, *The initial value problem for the binormal flow with rough data,* Ann. Sci. Éc. Norm. Supér. **48** (2015), 1421–1453.

[2] V. Banica and L. Vega, Evolution of polygonal lines by the binormal flow, *Ann. PDE* **6** (2020), Paper No. 6, 53 pp.

[3] V. Banica and L. Vega, On the energy of critical solutions of the binormal flow, *Comm. PDE* **45** (2020), 820–845.

[4] V. Banica and L. Vega, Unbounded growth of the energy density associated to the Schrödinger map and the binormal flow, *Ann. I. H. Poincaré, An. Non Lin., to appear.*

[5] Z. Brzeźniak, M. Gubinelli and M. Neklyudov, Global solutions of the random vortex filament equation, *Nonlinearity* **26** (2013) 2499–2514.

[6] A. Calini and T. Ivey, Stability of Small-amplitude Torus Knot Solutions of the Localized Induction Approximation. *J. Phys. A: Math. Theor.* **44** (2011) 335204.

[7] N.-H. Chang, J. Shatah and K. Uhlenbeck, Schrödinger maps, *Comm. Pure Appl. Math.* **53** (2000) 590–602.

[8] P. Constantin, C. Fefferman and A. J. Majda, Geometric constraints on potentially singular solutions for the 3-D Euler equations, *Comm. Partial Differential Equations* **21** (1996), 559–571.

[9] L.S. Da Rios, On the motion of an unbounded fluid with a vortex filament of any shape, *Rend. Circ. Mat. Palermo* **22** (1906), 117–135.

[10] F. de la Hoz and L. Vega, Vortex filament equation for a regular polygon, *Nonlinearity* **27** (2014), 3031–3057.

[11] P. G. Grinevich and M. U. Schmidt, Closed curves in $\mathbf{R}^3$ and the nonlinear Schrödinger equation, *Proc. of the Workshop on Nonlinearity, Integrability and All That: Twenty Years after NEEDS '79 (Gallipoli, 1999)*, (2000),139–145.
[12] S. Gutiérrez, J. Rivas and L. Vega, Formation of singularities and self-similar vortex motion under the localized induction approximation, *Commun. PDE* **28** (2003) 927–968.
[13] H. Hasimoto, A soliton in a vortex filament, *J. Fluid Mech.* **51** (1972), 477–485.
[14] R. L. Jerrard et C. Seis, *On the vortex filament conjecture for Euler flows,* Arch. Ration. Mech. Anal. **224** (2017), 135–172.
[15] R. L. Jerrard and D. Smets, On the motion of a curve by its binormal curvature, *J. Eur. Math. Soc.* **17** (2015), 1148–1515.
[16] N. Koiso, Vortex filament equation and semilinear Schrödinger equation. *Nonlinear Waves, Hokkaido University Technical Report Series in Mathematics* **43** (1996), 221–226.
[17] S. Lafortune, Stability of solitons on vortex filaments, *Phys. Lett. A* **377** (2013), 766–769.
[18] F. Maggioni, S. Z. Alamri, C. F. Barenghi, and R. L. Ricca, Velocity, energy and helicity of vortex knots and unknots. *Phys. Rev. E* **82** (2010), 26309–26317.
[19] A. Nahmod, J. Shatah, L. Vega and C. Zeng, Schrödinger maps and their associated frame systems. *IMRN* **21** (2007), 29pp.
[20] T. Nishiyama and A. Tani, Solvability of the localized induction equation for vortex motion, *Comm. Math. Phys.* **162** (1994), 433–445.

Analysis of the Anderson operator

Ismaël Bailleul
(joint work with V.N. Dang & A. Mouzard)

Let $\mathcal{S}$ be a two dimensional closed Riemannian manifold with metric g and associated volume measure μ. White noise on $\mathcal{S}$ is a $\mathcal{D}'(\mathcal{S})$-valued random variable ξ with Gaussian law with null mean and covariance

$$\mathbb{E}\big[\xi(\varphi_1)\,\xi(\varphi_2)\big] = \int_{\mathcal{S}} \varphi_1\varphi_2\,d\mu,$$

for φ_1, φ_2 smooth functions on $\mathcal{S}$. Almost surely it takes values in the Besov space $\mathcal{B}^{\alpha-2}_{\infty\infty}(\mathcal{S})$, for any $\alpha < 1$, a distribution space, and its law depends only on the metric g on $\mathcal{S}$. Let $h \in C^\infty(\mathcal{S})$ be a smooth function. Denote by $M_{h\xi}$ the multiplication operator by $h\xi$, and by Δ the Laplace-Beltrami operator associated with the Riemannian metric on $\mathcal{S}$. The Anderson Hamiltonian is the random operator

$$H := \Delta + M_{h\xi}, \tag{1}$$

perturbation of the Laplace-Beltrami operator by a distribution-valued potential. The smooth function h plays the role of a modulator for the noise, a position dependent coupling constant. The operator H arises naturally as the scaling limit of a number of microscopic discrete operators of interest in statistical physics. The study of the Anderson Hamiltonian presents an additional difficulty compared to its discrete counterparts. Unlike what happens for the Laplace-Beltrami operator Δ or its perturbations by smooth potentials, the low regularity of ξ prevents a straightforward definition of H as a continuous operator from the Sobolev space $H^2(\mathcal{S})$ into $L^2(\mathcal{S})$ since

$$M_{h\xi}(f) = fh\xi$$

is not an element of $L^2(\mathcal{S})$ for a generic $f \in H^2(\mathcal{S})$. One had to wait for the recent development of the theory of paracontrolled calculus and regularity structures before appropriate functional settings were introduced for the study of the Anderson Hamiltonian – corresponding to $h = 1$. Let $\mathbb{T}^2$ stand for the two dimensional flat torus. Allez and Chouk [1] first used paracontrolled calculus to define a random domain for H and proved that one can define H as an unbounded self-adjoint operator on $L^2(\mathbb{T}^2)$, with discrete spectrum $\lambda_n(\widehat{\xi}\,)$ tending to $+\infty$ and eigenvalues $\lambda_n(\widehat{\xi}\,)$ that are continuous functions of a measurable functional $\widehat{\xi}$ of ξ taking values in a Banach space. The basic mechanics at work in [1] was improved in Gubinelli, Ugurcan & Zachhuber's recent work [6] in which a similar result on the three dimensional torus was proved, amongst others. Labbé was also able in [7] to use the tools of regularity structures to get similar results. We refer to these works for detailed accounts of related matters and extensive references to the litterature. All these works are set in the torus. The very recent work of Mouzard [8] used the tools of the high order paracontrolled calculus developed by Bailleul & Bernicot in [2, 3, 4] to study Anderson Hamiltonian on a two dimensional manifold, simplifying a number of technical points compared to [1, 6] and proving that the random spectrum of H satisfies the same Weyl asymptotic law as the spectrum of the Laplace-Beltrami operator.

∘ *Anderson operator.* One can give a self-contained functional analytic construction of the Anderson operator that is different from the previous constructions. It relies on the direct construction of the resolvent operator via a fixed point equation where the analytic Fredholm theory can be used efficiently. Given a positive regularization parameter r let $\xi_r = e^{-r\Delta}(\xi)$ stand for the heat regularized white noise. The family of operators $\Delta + M_{h\xi_r} - \frac{|\log r|}{4\pi}\, h^2$ converges in probability as r goes to 0 to a limit random unbounded self-adjoint operator H which has a discrete spectrum $\sigma(H)$ tending to $+\infty$. This random operator is called *Anderson operator.*

A detailed description of the solution to the parabolic Anderson equation with singular initial conditions gives back in particular the heat kernel $p_t(x, y)$ of H. This fine description of $p_t(x, y)$ actually contains a lot of information on the operator H itself. As a direct illustration one can recover Mouzard's Weyl law for the spectrum of H from a Tauberian point of view. Information on different norms of the eigenfunctions or quasi-modes of H can also be recovered from a good control of the heat semigroup. Denote by $(u_n)_{n\geq 0}$ the sequence of L^2 normalized eigenfunctions of H with corresponding eigenvalues $\lambda_n(\widehat{\xi}\,)$. Recall $\alpha - 2 < -1$ stands for the almost sure Hölder regularity of white noise ξ.

Theorem 1. *For every $\beta' > 1$ there exists a positive random variable C such that the following two facts hold true almost surely.*

- *One has for all $n \geq 0$ such that $|\lambda_n(\widehat{\xi}\,)| \geq 1$ the n-uniform estimate*

$$\|u_n\|_{C^{2\alpha-1}} \leq C\big|\lambda_n(\widehat{\xi}\,)\big|^{\frac{\beta'}{2}}. \tag{2}$$

- *For every* $\Lambda \in \mathbb{R}$ *and every* $u \in \mathrm{span}\big(u_n\,;\,\lambda_n(\widehat{\xi}\,) \leq \Lambda\big)$ *with unit* L^2 *norm one has*
$$\|u\|_{H^\alpha} \leq C\Lambda^{1/2}.$$

We are able to obtain lower and upper Gaussian bounds for $p_t(x,y)$, which imply an interesting parabolic Harnack estimate for $(\partial_t + H)$-harmonic functions. Somewhat independently of the good control on the heat kernel we are also able to quantify the spectral gap of H in terms of some isoperimetric constant of the Riemannian manifold $(\mathcal{S}, g)$ generalizing Cheeger's Poincaré inequalities to our setting and also under the assumption that the Riemannian volume form μ satisfies a log-Sobolev inequality. The eigenfunction u_0 – the ground state, is associated with the smallest eigenvalue $\lambda_0(\widehat{\xi}\,)$ of H.

Theorem 2. *One has the following two almost sure estimates on the spectral gap of* H*.*

- *Denote by* $C(\mathcal{S}, g) > 0$ *the Cheeger constant of the Riemannian manifold* $(\mathcal{S}, g)$*. Then one has the spectral gap estimate*
$$\lambda_1(\widehat{\xi}\,) - \lambda_0(\widehat{\xi}\,) \geq \left(\frac{\min u_0}{\max u_0}\right)^4 \frac{C(\mathcal{S}, g)^2}{4} > 0.$$
- *Assume that the Riemannian volume measure* μ *satisfies a* log*-Sobolev inequality with constant* C_{LS}*. Then one has the spectral gap estimate*
$$\lambda_1(\widehat{\xi}\,) - \lambda_0(\widehat{\xi}\,) \geq \left(\frac{\min u_0}{\max u_0}\right)^2 \frac{\big(\max u_0^4 + \max u_0^{-4}\big)^{-1}}{2\,C_{\mathrm{LS}}} > 0.$$

◦ *Anderson Gaussian free field.* We introduce and study the Anderson Gaussian free field. This doubly random field ϕ on $\mathcal{S}$ is defined from the L^2 spectral decomposition of the random operator H in the same way as Gaussian free field is defined from the L^2 spectral decomposition of Δ. It thus has two layers of randomness. Like the usual Gaussian free field it is almost surely of regularity 0^-. One can define the Wick square $:\!\phi^2\!:$ of ϕ as a doubly random variable; its distribution $\mathcal{L}(:\!\phi^2\!:)$ depends on H so it is random. The following result is a qualitative version of more precise statements.

Theorem 3. *The law of the random spectrum of* H *is characterized by the law of* $\mathcal{L}(:\!\phi^2\!:)$*.*

◦ *The polymer measure.* The polymer measure provides a mathematical model for the random motion of a particle subject to a thermal motion in an extremely disordered potential modeled by white noise. From Feynman-Kac representation formula it is the non-negative measure Q formally defined at a generic point $w \in C([0,1], \mathcal{S})$ by its density
$$\exp\left(\int_0^1 \xi(w_t)dt\right)$$
with respect to the Wiener measure $P_{\mathscr{W}}$ on path space over $\mathcal{S}$, up to a multiplicative normalization constant. The pointwise evaluation of the distribution ξ is however

meaningless, which motivates a definition of the polymer measure Q as a limit as $r > 0$ goes to 0 of the measures $Q^{(r)}$ obtained from a regularized noise ξ_r setting

$$\frac{dQ^{(r)}}{dP_{\mathscr{W}}}(w) \sim \exp\left(\int_0^1 \left(\xi_r + \frac{|\log r|}{4\pi}\right)(w_t)dt\right). \tag{3}$$

Note that the measures $Q^{(r)}$ and the limit measure Q are random, as the white noise environment is random. (Both $Q^{(r)}$ and Q depend implicitly on the starting point of the path w, that may be fixed or random, possibly independently of the environment.) This measure was first constructed in the flat setting of the two dimensional torus by Cannizzaro & Chouk in [5] using the then newly developed tools of paracontrolled calculus. Their method of proof is not easily adapte to a manifold setting. We give here the first construction of this measure on a closed Riemannian manifold. Our construction is different from that of Cannizzaro & Chouk and we construct the random measure Q as the law of a Markov process with transition probability $e^{-t(H-\lambda_0(\widehat{\xi}))}$. The sharp small time asymptotic that we obtain on the kernel of that operator, or the Gaussian bound proved for that kernel, allow for a straightforward use of Kolmogorov's criterion to construct the polymer measure on a space of Hölder paths. It is singular with respect to Wiener measure on $C([0,1],\mathcal{S})$ although it has support in all the spaces $C^\gamma([0,1],\mathcal{S})$, for $\gamma < 1/2$ like the law of Brownian motion. Following a long tradition going back to the work of Symanzik on constructive quantum field theory in the 60's, we can relate the distribution of the square of the Anderson Gaussian free field and the distribution of the renormalized occupation measure $\mathcal{O}_{1/2}$ of a certain Poisson point process of polymer loops in $\mathcal{S}$.

Theorem 4. *The renormalized occupation measure $\mathcal{O}_{1/2}$ has the same distribution as the Wick square $:\phi^2:$ of the Anderson Gaussian free field.*

Finally one can prove that the polymer measure on free and fixed endpoints paths satisfies the same large deviation principle as Wiener measure and the rate function does not see the effect of the white noise potential.

References

[1] R. Allez and K. Chouk, *The continuous Anderson Hamiltonian in dimension two.* arXiv:1511.02718, (2015).

[2] I. Bailleul and F. Bernicot, *Heat semigroup and singular PDEs.* J. Funct. Anal., **270**:3344–3452, (2016).

[3] I. Bailleul and F. Bernicot and D. Frey, *Spacetime paraproducts for paracontrolled calculus, 3d-PAM and multiplicative Burgers equations.* Ann. Sc. Éc. Norm. Sup., **51**:1399–1457, 2018.

[4] I. Bailleul and F. Bernicot, *High order paracontrolled calculus.* Forum Math., Sigma, **7**(e-44):1–93, (2019).

[5] G. Cannizzaro and K. Chouk, *Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential.* Ann. Probab. ,**46**:1710–1763, (2018).

[6] M. Gubinelli and B. Ugurcan and I. Zachhuber, *Semilinear evolution equations for the Anderson Hamiltonian in two and three dimensions.* Stoch. and Part. Diff. Eq.: Analysis and Computations. **8**:82–149, (2020).

[7] C. Labbé, *The continuous Anderson hamiltonian in $d \leq 3$*. J. Funct. Anal., **227**(9):3187–3225, (2019).
[8] A. Mouzard, *The Anderson Hamiltonian on a two-dimensional Riemannian manifold.* arXiv:2009.03549, to appear in Ann. Institut H. Poincaré, (2022).

Invariant Gibbs measures for the three-dimensional cubic nonlinear wave equation (I)

Bjoern Bringmann
(joint work with Y. Deng, A. Nahmod, H. Yue)

In this talk, we discuss the invariance of the Gibbs measure for the three dimensional cubic nonlinear wave equation. In the beginning of this talk, we discuss the connections of this problem with various problems in constructive quantum field theory, singular SPDEs, and random dispersive equations, which concern different Φ^4_d-models. The starting point of our discussion is the energy

$$E(\phi) = \int_{\mathbb{T}^d} \mathrm{d}x \bigg(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^4}{4} \bigg). \tag{1}$$

In (1), we omit the possible renormalization of the potential energy term, since its precise form depends on the dimension d. Equipped with (1), we formally define the Φ^4_d-measure as

$$\text{“}\mathrm{d}\Phi^4_d(\phi) = \mathcal{Z}^{-1} \exp\bigg(- \int_{\mathbb{T}^d} \mathrm{d}x \bigg(\frac{|\phi|^2}{2} + \frac{|\nabla \phi|^2}{2} + \frac{|\phi|^4}{4} \bigg) \bigg) \mathrm{d}\phi.\text{”} \tag{2}$$

In addition to the Φ^4_d-measure, the energy in (1) induces three different evolution equations, which are called dynamical Φ^4_d-models:

(i) A Langevin equation, which is given by the cubic heat equation with space-time white noise,
(ii) a real-valued Hamiltonian equation, which is given by the cubic wave equation,
(iii) and a complex-valued Hamiltonian equation, which is given by the cubic Schrödinger equation.

The evolution equations described in (i), (ii), and (iii) are also known as the parabolic, hyperbolic, and Schrödinger Φ^4_d-models, respectively. The main problems then concern the construction of the Φ^4_d-measure, the probabilistic well-posedness of the three evolution equations, as well as the invariance of (minor variants of) the Φ^4_d-measure under the three evolution equations. The extensive literature on Φ^4_d-models, which spans over six decades, is illustrated in Figure 1.

Dimension	Measure	Heat	Wave	Schrödinger
$d=1$		[Iwa87]	[Zhi94]	[Bou94]
$d=2$	[Nel66]	[DPD03]	[Bou99]	[Bou96]
$d=3$	[GJ73]	[Hai14]	*This talk*	*Open*
$d=4$	[ADC21]			
$d\geq 5$	[Aiz81, Fro82]			

FIGURE 1. Existence and invariance of the Gibbs measure for the cubic stochastic heat, wave, and Schrödinger equations.

We now state the main result of this talk. To this end, let $N \geq 1$ be a frequency-truncation parameter. The frequency-truncated and renormalized three dimensional cubic nonlinear wave equation is given by

$$\begin{cases} (\partial_t^2 + 1 - \Delta) u_{\leq N} = -P_{\leq N}\Big[:(P_{\leq N} u_{\leq N})^3: + \gamma_{\leq N} \cdot P_{\leq N} u_{\leq N} \Big] \\ (u_{\leq N}, \langle \nabla \rangle^{-1} \partial_t u_{\leq N})\big|_{t=0} = (\phi^{\cos}, \phi^{\sin}). \end{cases} \tag{3}$$

Here, $P_{\leq N}$ denotes a (sharp) frequency-truncation, the dots $:$ indicate the Wick-ordering, and $\gamma_{\leq N}$ denotes a (further) renormalization constant. Furthermore, let $\mu_{\leq N}$ be the corresponding frequency-truncated Gibbs measure, whose first marginal is given by a frequency-truncated Φ^4_3-measure. It is known that $(\mu_{\leq N})_N$ weakly converges to a unique limit, which is denoted by μ.

Theorem 1 (Global well-posedness and invariance, rigorous version)**.** *For any frequency-scale $N \geq 1$ and $(\phi^{\cos}, \phi^{\sin}) \in \mathscr{H}_x^{-1/2-\epsilon}(\mathbb{T}^3)$, let $u_{\leq N}$ be the solution of the frequency-truncated cubic wave equation* (3) *with initial data $u_{\leq N}[0] = (\phi^{\cos}, \phi^{\sin})$. In addition, let μ be the Gibbs measure from above. Then, for μ-almost every $(\phi^{\cos}, \phi^{\sin})$ and all $T \geq 1$, the limiting dynamics*

$$u[t] = \lim_{N\to\infty} u_{\leq N}[t] \tag{4}$$

exists in $C_t^0 \mathscr{H}_x^{-1/2-\epsilon}([-T,T]\times\mathbb{T}^3)$. Furthermore, the Gibbs measure is invariant under the limiting dynamics, i.e.,

$$\mathrm{Law}_\mu\big(u[t]\big) = \mu \tag{5}$$

for all $t \in \mathbb{R}$.

In later parts of this talk, we discuss the following aspects of our proof: First, we discuss a caloric representation of the Gibbs measure. This caloric representation is inspired by Tao's caloric gauge [Tao04] and obtained using the parabolic Φ^4_3-model, i.e., the three-dimensional cubic stochastic heat equation.

Then, we discuss the para-controlled Ansatz for the solution $u_{\leq N}$, which takes the form

(6) $$u_{\leq N} = \mathsf{I}_{\leq N} - \mathsf{Y}_{\leq N} - \mathsf{I}_{\leq N} + 3\,\mathsf{Y}_{\leq N} + X^{(1)}_{\leq N} + X^{(2)}_{\leq N} + Y_{\leq N}.$$

The first four summands in (6) are explicit stochastic objects, the fifth and sixth summands are para-controlled components, and the last summand is a nonlinear remainder. At the end of this talk, we discuss further aspects regarding explicit stochastic objects, such as a hidden cancellation between sextic stochastic objects.

References

[Aiz81] M. Aizenman. Proof of the triviality of φ^4_d field theory and some mean-field features of Ising models for $d > 4$. *Phys. Rev. Lett.*, 47(1):1–4, 1981.

[ADC21] M. Aizenman and H. Duminil-Copin. Marginal triviality of the scaling limits of critical 4D Ising and ϕ^4_4 models. *Ann. of Math. (2)*, 194(1):163–235, 2021.

[Bou94] J. Bourgain. Periodic nonlinear Schrödinger equation and invariant measures. *Comm. Math. Phys.*, 166(1):1–26, 1994.

[Bou96] J. Bourgain. Invariant measures for the 2D-defocusing nonlinear Schrödinger equation. *Comm. Math. Phys.*, 176(2):421–445, 1996.

[Bou99] J. Bourgain. Nonlinear Schrödinger equations. In *Hyperbolic equations and frequency interactions (Park City, UT, 1995)*, volume 5 of *IAS/Park City Math. Ser.*, pages 3–157. Amer. Math. Soc., Providence, RI, 1999.

[DPD03] G. Da Prato and A. Debussche. Strong solutions to the stochastic quantization equations. *Ann. Probab.*, 31(4):1900–1916, 2003.

[Fro82] J. Froehlich. On the triviality of $\lambda\varphi^4_d$ and the approach to the critical point in $d_{(-)} > 4$ dimensions. *Nucl. Phys. B*, 200:281–296, 1982.

[GJ73] J. Glimm and A. Jaffe. Positivity of the ϕ^4_3 Hamiltonian. *Fortschr. Physik*, 21:327–376, 1973.

[Hai14] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.

[Iwa87] K. Iwata. An infinite-dimensional stochastic differential equation with state space $C(\mathbf{R})$. *Probab. Theory Related Fields*, 74(1):141–159, 1987.

[Nel66] E. Nelson. A quartic interaction in two dimensions. In *Mathematical Theory of Elementary Particles (Proc. Conf., Dedham, Mass., 1965)*, pages 69–73. M.I.T. Press, Cambridge, Mass., 1966.

[Tao04] T. Tao. Geometric renormalization of large energy wave maps. In *Journées "Équations aux Dérivées Partielles"*, pages Exp. No. XI, 32. École Polytech., Palaiseau, 2004.

[Zhi94] P. E. Zhidkov. An invariant measure for a nonlinear wave equation. *Nonlinear Anal.*, 22(3):319–325, 1994.

Invariant Gibbs measures for the three-dimensional cubic nonlinear wave equation (II)

Yu Deng

(joint work with Bjoern Bringmann, Andrea R. Nahmod and Haitian Yue)

This talk presents the recent proof [2] of the invariance of Gibbs measure for the cubic nonlinear wave equation on the three-dimensional torus, which is joint work with Bjoern Bringmann, Andrea R. Nahmod and Haitian Yue.

The Gibbs measure for the 3D cubic nonlinear wave equation, commonly known as the Φ^4_3 measure, has been a subject of extensive study since the 70's. In the seminal work of Hairer [7], it has been established that the Φ_3^4 measure is invariant under the *stochastic heat equation*

$$(\partial_t - \Delta)u + u^3 - \infty \cdot u + \xi = 0, \tag{1}$$

where ξ is the spacetime white noise. The result presented in this talk can be viewed as the *hyperbolic* counterpart of the result in [7], namely we establish the invariance of the Φ_3^4 measure under the *wave* dynamics.

More precisely, consider the suitably renormalized cubic nonlinear wave equation

$$(\partial_t^2 - \Delta)u + u^3 - \infty \cdot u = 0 \tag{2}$$

with the initial data

$$u[0] = (u(0), \partial_t u(0))$$

distributed according to the measure

$$u[0] \sim \mathrm{d}\mu \otimes \mathrm{d}\rho,$$

where $\mathrm{d}\mu$ is the Φ_3^4 measure on the torus $\mathbb{T}^3$, and $\mathrm{d}\rho$ is the white noise measure again on the torus.

Then, with probability 1, the equation (2) has a unique global solution with initial data $u[0]$ (this should be understood as the sequence of truncated solutions with suitable renormalizations converging to the unique limit), and at each time t, the data $u[t] = (u(t), \partial_t u(t))$ is again distributed according to the measure

$$u[t] \sim \mathrm{d}\mu \otimes \mathrm{d}\rho.$$

The proof of this result combines the major technologies developed for the probabilistic theory of PDEs in the last few years. The most important aspects include: the "caloric" representation of data, interplay of heat and wave objects, para-controlled calculus, random tensor theory, a hidden cancellation that has its roots in wave turbulence, and a novel bilinear estimate. Below we will briefly explain the last four aspects, which are closely linked to the local theory.

The para-controlled calculus is what the main ansatz of our solutions is based on. This idea first appeared in the parabolic context due to Gubinelli-Imkeller-Perkowski [5] and was then extended to the wave setting by Gubinelli-Koch-Oh [6] (see also recent works such as [1, 9, 10]). The main idea is to identify the high-low interactions where the high frequency component is explicitly Gaussian or multi-Gaussian, and view such interactions as appropriate "shifts" of the exact Gaussians or multi-Gaussians. Then they should inherit the important cancellation and large deviation properties from these explicit Gaussian expressions, which provide improved estimates when they are involved in otherwise troublesome interactions.

The random tensor theory is developed in the work of Deng-Nahmod-Yue [4], following its precursor, the random averaging operators in [3]. The main idea is to unravel more layers of randomness structure than what is done in the para-controlled calculus by (i) exhibiting the probabilistic independence between the high and low frequency components, and (ii) exploiting also the randomness of the low frequency components by performing an inductive on frequency argument. This has been applied to nonlinear Schödinger equations, while para-controlled calculus has been quite successful in nonlinear heat and wave equations, due to the different level of smoothing in both cases.

The equation (2) is a semilinear wave equation, which has adequate smoothing; as such, we have chosen to use the para-controlled ansatz rather than the random tensor ansatz in [4]. However, the random tensor *estimates* in [4], which were developed for justifying the ansatz in that paper, turns out to be extremely useful also in our setting. In fact, they provide estimates for the various norms of general random tensors that are sharp for most purposes. As such, they can be used to deduce the estimates we need in almost all scenarios - except for one special case, in which a bilinear improvement is required.

Another main ingredient is a hidden cancellation, which we refer to as the *1533* cancellation. This has to do with the linear term $\mathcal{M}^{(1)}$, the cubic term $\mathcal{M}^{(3)}$ and the quintic term $\mathcal{M}^{(5)}$, which are zeroth, first and second iterates starting from the linear Gaussian input. In fact, a logarithmic divergence occurs when one tries to calculate

$$\mathbb{E}|\mathcal{M}^{(3)}|^2 = \lim_{N\to\infty} \mathbb{E}|\mathcal{M}_N^{(3)}|^2,$$

where $M_N^{(3)}$ is the quantity $\mathcal{M}^{(3)}$ for the approximate solution with frequency truncation N. However, it turns out that, this term is exactly cancelled (up to lower order terms) by the correlation

$$\mathbb{E}\big(\mathcal{M}^{(1)}\mathcal{M}^{(5)}\big),$$

more precisely we have

$$\mathbb{E}\big(|\mathcal{M}^{(3)}|^2 + 6\mathcal{M}^{(1)}\mathcal{M}^{(5)}\big) = \lim_{N\to\infty} \mathbb{E}\big(|\mathcal{M}_N^{(3)}|^2 + 6\mathcal{M}_N^{(1)}\mathcal{M}_N^{(5)}\big), \tag{3}$$

where the limit does converge. This cancellation is quite mysterious, however it seems to root from the calculations of the *wave kinetic equation*, because (3) is essentially the first order coefficient in the time Taylor expansion of the solution to the wave kinetic equation, however this solution should be independent of time because the Gibbs measure corresponds to a stationary solution to the wave kinetic equation, so the leading contribution of (3) should vanish.

Finally, we mention a new bilinear estimate developed in this paper, which is needed due to the inefficiency of the linear estimates in [4] in one specific scenario. Thus, we have to exploit the true bi-linearity of the norms involved instead of naively controlling it by linear norms. This results in a key lemma (Lemma 8.1 in [2]) which brings a major improvement upon [4]. We have not found any similar results in the literature, and believe that such results should be of independent interest, as they go beyond the regime of *random matrices* and involve genuinely bilinear (or multilinear) phenomena. We believe this will lead to many new possibilities, and it remains to be seen what are expected to the optimal estimates in these settings.

References

[1] B. Bringmann. Invariant Gibbs measures for the three-dimensional wave equation with a Hartree nonlinearity II: Dynamics. *JEMS* to appear.

[2] B. Bringmann, Y. Deng, A. R. Nahmod, and H. Yue. Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation. arXiv:2205.03893, May 2022.
[3] Y. Deng, A. R. Nahmod, and H. Yue. Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two. arXiv:1910.08492, October 2019.
[4] Y. Deng, A. R. Nahmod, and H. Yue. Random tensors, propagation of randomness, and nonlinear dispersive equations. *Invent. Math.*, (228):539–686, 2022.
[5] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.
[6] M. Gubinelli, H. Koch, and T. Oh. Paracontrolled approach to the three-dimensional stochastic nonlinear wave equation with quadratic nonlinearity. arXiv:1811.07808, November 2018.
[7] M. Hairer. A theory of regularity structures. *Invent. Math.*, 198(2):269–504, 2014.
[8] M. Hairer and K. Matetski. Discretisations of rough stochastic PDEs. *Ann. Probab.*, 46(3):1651–1709, 2018.
[9] T. Oh, M. Okamoto, and L. Tolomeo. Focusing Φ^4_3 model with a Hartree-type nonlinearity. arXiv:2009.03251, September 2020.
[10] T. Oh, M. Okamoto, and L. Tolomeo. Stochastic quantization of the Φ^3_3-model. arXiv:2108.06777, August 2021.

What is stochastic quantisation?

Massimiliano Gubinelli

In my talk I gave a personal perspective of the recent progress on the construction of Euclidean quantum field theories (EQFTs) via stochastic quantisation. These progresses were made possible by novel techniques in the analysis of singular stochastic partial differential equations (SPDEs) which spurred a renewed interest in using stochastic and PDE techniques in constructive EQFTs. Stochastic quantisation has been introduced by Nelson and Parisi-Wu from different perspectives. After early rigorous work of Jona-Lasinio and Mitter in the '80s and Da Prato–Debbusche in the early 2000s, a breakthrough came from the invention of regularity structures by Hairer (2014) and paracontrolled distributions by Perkowski, Imkeller and myself. These tools allowed to construct and renormalise solutions of evolutionary non-linear stochastic partial differential equations whose invariant measures were EQFTs in dimension two and three. A series of progress allowed also the passage to the infinite volume limit, most notably thanks to work of Mourrat and Weber in the case of Φ^4_2 and then Gubinelli and Hofmanova which constructed Φ^4_3 in the full space in a fully non-perturbative fashion. In the talk I stressed more recent work my my collaborators and myself which tries to articulate the question in the title: what is (really) stochastic quantisation? In particular we currently know various different realisations of the idea of stochastic quantisation which rely on various kinds of equations:

- Parabolic equations [1];
- Elliptic equations [3, 4, 2, 8];
- Stochastic control problems [5, 6, 7];
- Wave equations [10, 9].

All these different methods have strong similarities which hints to the fact that stochastic quantisation is really a kind of stochastic analysis of EQFTs, in the sense of Ito's approach to the theory of Markov processes, see e.g. the introduction to [12]. The key aspects of these approach is that the EQFT measure is constructed as the push-forward of a Gaussian measure via the solution map of a well-behaved singular non-linear equation. As a byproduct, this construction automatically provides a coupling of the interacting field with a free field (possibly on an extended space) which is an important technical tool to overcome the possible singularity of the interacting measure with respect to the Gaussian free field itself. PDE techniques can then be used to analyse this push-forward and provide useful results. It is useful to note that recent literature, see e.g. [11] develops similar idea in the context of functional inequalities, and it is not difficult to imagine that fruitful interactions between these two topics will be possible in the near future.

References

[1] M. Gubinelli and M. Hofmanová, *A PDE construction of the Euclidean ϕ^4_3 quantum field theory*, Comm. Math. Phys. **384** (2021), no. 1, 1–75.

[2] M. Gubinelli and M. Hofmanová, *Global solutions to elliptic and parabolic Φ^4 models in Euclidean space*, Comm. Math. Phys. **368** (2019), no. 3, 1201–1266.

[3] S. Albeverio, F. C. De Vecchi and M. Gubinelli, *Elliptic stochastic quantization*, Ann. Probab. **48** (2020), no. 4, 1693–1741.

[4] S. Albeverio, F. C. De Vecchi and M. Gubinelli, *The elliptic stochastic quantization of some two dimensional Euclidean QFTs*, Ann. Inst. Henri Poincaré Probab. Stat. **57** (2021), no. 4, 2372–2414.

[5] N. Barashkov and M. Gubinelli, *A variational method for Φ^4_3*, Duke Math. J. **169** (2020), no. 17, 3339–3415.

[6] N. Barashkov and M. Gubinelli, *On the Variational Method for Euclidean Quantum Fields in Infinite Volume*, ArXiv:2112.05562 (2021).

[7] N. Barashkov, *A Stochastic Control Approach to Sine Gordon EQFT*, ArXiv:2203.06626 (2022).

[8] Barashkov, Nikolay, and Francesco C. De Vecchi, *Elliptic Stochastic Quantization of Sinh-Gordon QFT*. ArXiv:2108.12664 [Math-Ph] (2021).

[9] M. Gubinelli, H. Koch and T. Oh, *Paracontrolled Approach to the Three-Dimensional Stochastic Nonlinear Wave Equation with Quadratic Nonlinearity*, ArXiv:1811.07808 (2018).

[10] M. Gubinelli, H. Koch and T. Oh, *Renormalization of the two-dimensional stochastic nonlinear wave equations*, Trans. Amer. Math. Soc. **370** (2018), no. 10, 7335–7359.

[11] D. Mikulincer, Y. Shenfeld, *The Brownian Transport Map*, ArXiv:2111.11521 (2021).

[12] D. W. Stroock, *Markov processes from K. Itô's perspective*, Annals of Mathematics Studies, 155, Princeton University Press, Princeton, NJ, 2003. MR1980149

A stochastic analysis approach to lattice Yang–Mills

Rongchan Zhu

(joint work with Hao Shen, Scott Smith, Xiangchan Zhu)

In this talk, we consider the lattice Yang–Mills model based on stochastic analysis approach. We first recall the basic setup and definitions of the model.

Let $\Lambda_L = \mathbb{Z}^d \cap L\mathbb{T}^d$ be a finite d dimensional lattice with side length L and unit lattice spacing. We will sometimes write $\Lambda = \Lambda_L$ for short. Let E^+ (resp. E^-) be the set of positively (resp. negatively) oriented edges, and denote by $E^+_{\Lambda_L}$, $E^-_{\Lambda_L}$

the corresponding subsets of edges with both beginning and ending points in Λ_L. Define $E := E^+ \cup E^-$.

We write G for the Lie group $SO(N)$ or $SU(N)$ and $\mathfrak{g}$ for the associated Lie algebra $\mathfrak{so}(N)$ or $\mathfrak{su}(N)$.

A plaquette is a closed path of length four which traces out the boundary of a square. Also, let $\mathcal{P}_{\Lambda_L}$ be the set of plaquettes whose vertices are all in Λ_L, and $\mathcal{P}^+_{\Lambda_L}$ be the subset of plaquettes $p = e_1e_2e_3e_4$ such that the beginning point of e_1 is lexicographically the smallest among all the vertices in p and the ending point of e_1 is the second smallest.

The lattice Yang-Mills theory (or lattice gauge theory) on Λ_L for the structure group G, with $\beta \in \mathbb{R}$ the inverse coupling constant, is the probability measure $\mu_{\Lambda_L,N,\beta}$ on the set of all collections $Q = (Q_e)_{e\in E^+_{\Lambda_L}}$ of G-matrices, defined as

$$d\mu_{\Lambda_L,N,\beta}(Q) := Z^{-1}_{\Lambda_L,N,\beta} \exp\Big(\mathcal{S}(Q)\Big) \prod_{e\in E^+_{\Lambda_L}} d\sigma_N(Q_e)\,,$$

with

$$\mathcal{S}(Q) := N\beta \mathrm{Re} \sum_{p\in\mathcal{P}^+_{\Lambda_L}} \mathrm{Tr}(Q_p),$$

where $Z_{\Lambda_L,N,\beta}$ is the normalizing constant, $Q_p := Q_{e_1}Q_{e_2}Q_{e_3}Q_{e_4}$ for a plaquette $p = e_1e_2e_3e_4$, and σ_N is the Haar measure on G.

In the first part, we give a new derivation of the finite N master loop equation for lattice Yang-Mills theory with structure group $SO(N)$, $U(N)$ or $SU(N)$. The $SO(N)$ case was initially proved by Chatterjee in [1], and $SU(N)$ was analyzed in a follow-up work by Jafarov [2]. Our approach is based on the Langevin dynamic, an SDE on the manifold of configurations, and yields a simple proof via Itô's formula. More precisely, given a loop $l = e_1e_2\cdots e_n$, the Wilson loop variable W_l is defined as

$$W_l = \mathrm{Tr}(Q_{e_1}Q_{e_2}\cdots Q_{e_n})\,.$$

For any non-null loop sequence s with minimal representation $(l_1,\dots,l_m)$ such that each l_i is contained in Λ, define

$$W_s = W_{l_1}W_{l_2}\cdots W_{l_m}, \quad \phi(s) \;:=\; \mathbf{E}\frac{W_s}{N^m}\,.$$

The master loop equation is a recursion which expresses $\phi(s)$ in terms of a linear combination of $\phi(s')$, where s' is a loop sequence obtained by performing an operation on s. The operations are called splitting, twisting, merger, deformation, and expansion; each being further divided into a positive or negative type. We reprove the master loop equation using a simple Langevin dynamic and Itô calculus. For more details we refer to [3].

In the second part we develop a new stochastic analysis approach to the lattice Yang–Mills model at strong coupling in any dimension $d > 1$, with t' Hooft scaling βN for the inverse coupling strength. We study their Langevin dynamics, ergodicity, functional inequalities, large N limits, and mass gap.

Assuming $|\beta| < \frac{N-2}{32(d-1)N}$ for the structure group $SO(N)$, or $|\beta| < \frac{1}{16(d-1)}$ for $SU(N)$, we prove the following results. The invariant measure for the corresponding Langevin dynamic is unique on the entire lattice, and the dynamic is exponentially ergodic under a Wasserstein distance. The finite volume Yang–Mills measures $\mu_{\Lambda_L,N,\beta}$ converge to this unique invariant measure in the infinite volume limit, for which Log-Sobolev and Poincaré inequalities hold. These functional inequalities imply that the suitably rescaled Wilson loops for the infinite volume measure has factorized correlations and converges in probability to deterministic limits in the large N limit, and correlations of a large class of observables decay exponentially, namely the infinite volume measure has a strictly positive mass gap. Our method improves earlier results or simplifies the proofs, and provides some new perspectives to the study of lattice Yang–Mills model.

We develop new methods based on stochastic analysis and give new proofs to these results. In these methods, the curvature properties of the Lie groups are better exploited via the verification of the Bakry–Émery condition. In particular, this allows us to perform more delicate calculations and obtain more explicit smallness condition on inverse coupling. As another novelty we study the Langevin dynamics (or stochastic quantization) and we prove uniqueness of the infinite volume measures by showing that the dynamic on the entire $\mathbb{Z}^d$ has a unique invariant measure. To this end we employed coupling methods for our stochastic dynamics, which is a variant of Kendall–Cranston's coupling. Such stochastic coupling arguments were used earlier in the stochastic analysis on manifolds, but to our best knowledge this appears to be the first time that such coupling arguments are used in the setting of statistical physics or lattice quantum field theory models with manifold target spaces. For our coupling arguments we will also need to introduce suitable weighted distances on the product manifolds, and in our calculations a subtle comparison between the weight parameter and the curvature plays a key role in order to obtain ergodicity. For more details on this part we refer to [4].

References

[1] S. Chatterjee. *Rigorous solution of strongly coupled SO(N) lattice gauge theory in the large N limit.* Comm. Math. Phys., 366(1):203–268, 2019.

[2] J. Jafarov. *Wilson loop expectations in $SU(N)$ lattice gauge theory.* arXiv preprint arXiv:1610.03821, 2016.

[3] H. Shen, S. A. Smith, and R. Zhu, *A new derivation of the finite n master loop equation for lattice Yang–Mills*, arXiv preprint arXiv:2202.00880, 2022.

[4] H. Shen, R. Zhu, and X. Zhu, *A stochastic analysis approach to lattice Yang–Mills at strong coupling*, arXiv preprint arXiv:2204.12737, 2022.

A diagram-free approach to the stochastic estimates in regularity structures

Felix Otto

(joint work with Pablo Linares, Markus Tempelmayr, Pavlos Tsatsoulis)

We propose an alternative to the work [1] by Chandra and Hairer in the sense of obtaining "automated" stochastic estimates for the centered model in regularity structures introduced in [2]. The main result is the existence of a centered model Π_x and recentering maps Γ_{xy} for any space-time points $x, y \in \mathbb{R}^2$, satisfying the postulates of regularity structures and

$$
\begin{aligned}
(1) \qquad & \big(\mathbb{E}|\Pi_{x\beta}(y)|^p\big)^{\frac{1}{p}} \lesssim |y-x|^{|\beta|}, \\
& \big(\mathbb{E}|(\Gamma_{xy})_\beta^\gamma|^p\big)^{\frac{1}{p}} \lesssim |y-x|^{|\beta|-|\gamma|},
\end{aligned}
$$

for all $0 < |y-x| < \infty$ (UV and IR) and $p < \infty$, where β, γ are multi-indices and $|\beta|$ denotes the homogeneity of a multi-index defined below.

We implement this for a prototypical parabolic stochastic quasilinear equation

$$
(2) \qquad (\partial_2 - \partial_1^2)u = \xi + a(u)\partial_1^2 u - \big(h + \tilde{h}\,\partial_1 u + \tilde{\tilde{h}}\,(\partial_1 u)^2 + \dots\big),
$$

where we think of 1 as being the space direction and 2 the time direction. The last expression on the r. h. s. denotes the counterterm that we introduce for the following reason: we think of ξ as a random Schwartz distribution with realizations a.s. in $C^{\alpha-2}$ (space-time white noise corresponds to $\alpha = \frac{1}{2}-$); then classical Schauder theory suggests that $u \in C^\alpha$, hence $a(u)\partial_1^2 u \in C^\alpha \cdot C^{\alpha-2}$ which is ill-defined for $\alpha \geq 1$ and makes the equation *singular* in the sense of [2] for $\alpha < 1$. However, the equation is *subcritical* in the sense of [2] as long as $\alpha > 0$.

For simplicity, we make the following assumption on the noise:

Assumption 1: The law of ξ is invariant under translation and spatial reflection.

Under this assumption, and by postulating that the counterterm is of lower order, the counterterm takes the simpler form of a deterministic function $h = h[a](u(x))$. Moreover, we make the crucial postulate $h[a(\cdot + v)](u) = h[a](u + v)$ for $v \in \mathbb{R}$, which implies that $h[a](u) = c[a(\cdot + u)]$ for some deterministic functional c on the space of a's. It is convenient to adopt a more algebraic point of view and formally expand

$$
a(u) = \sum_{k \geq 0} u^k z_k, \quad \text{i. e.} \quad z_k[a] = \frac{1}{k!}\frac{d^k a}{du^k}(0),\ k \in \mathbb{N}_0.
$$

Since c is a functional of a only, it takes the form of a power series in the variables $\{z_k\}_{k\geq 0}$,

$$
\mathbb{R}[[z_k]] \ni c[a] = \sum_\beta c_\beta z^\beta, \quad \text{where} \quad z^\beta = \prod_{k\geq 0} z_k^{\beta(k)},
$$

so that for the counterterm h we obtain

$$h(u)=\sum_{k\geq 0}\frac{1}{k!}u^k(D^{(0)})^k c, \tag{3}$$

where $D^{(0)}\in \mathrm{End}(\mathbb{R}[[z_k]])$ is the infinitesimal generator of u-shift.

We formally extend the coordinates $\{z_k\}_{k\geq 0}$ on the space of a's to coordinates on the manifold of solutions u (up to constants). In case of $a\equiv 0$, the manifold of solutions is an affine space over space-time functions p with $(\partial_2-\partial_1^2)p=0$; those functions are analytic. It is convenient to free oneself from the constraint $(\partial_2-\partial_1^2)p=0$ by relaxing to the manifold of all space-time functions u that satisfy (2) up to a space-time analytic function. In view of the Cauchy-Kovalevskaya theorem, one expects that for analytic a, the space of analytic space-time functions p (modulo constants) still provides a parametrization of the nonlinear solution manifold – at least for sufficiently small a and locally near a base-point $x\in\mathbb{R}^2$. We think of p as providing a germ at x via the space-time shift $p(\cdot-x)$ so that

$$z_{\mathbf{n}}[p]=\frac{1}{\mathbf{n}!}\frac{\partial^{\mathbf{n}}p}{\partial x^{\mathbf{n}}}(0),\ \ \mathbf{n}\in\mathbb{N}_0^2\setminus\{(0,0)\}$$

are natural coordinates for the parameterization near x. This allows to think of the solution u as a formal power series in the variables $\{z_k,z_{\mathbf{n}}\}_{k\geq 0,\mathbf{n}\neq\mathbf{0}}$, cf. B-series,

$$u(y)=\sum_{\beta}\Pi_{x\beta}(y)z^{\beta}\in\mathbb{R}[[z_k,z_{\mathbf{n}}]],\quad\text{where}\quad z^{\beta}=\prod_{k\geq 0}z_k^{\beta(k)}\prod_{\mathbf{n}\neq\mathbf{0}}z_{\mathbf{n}}^{\beta(\mathbf{n})}.$$

In case of an ODE, this index set of multi-indices is greedier than the one for branched rough paths, but less greedy than the one for geometric rough paths. In case of a PDE, the index set is more lavish than polynomial decorations. From $(\partial_2-\partial_1^2)u=\xi+a(u)\partial_1^2u-h(u)$ we obtain via $\Pi_{x\beta}=\frac{1}{\beta!}\partial^{\beta}u$ and Leibniz' rule

$$(\partial_2-\partial_1^2)\Pi_x=\xi 1+\big(\sum_{k\geq 0}\Pi_x^k z_k\big)\partial_1^2\Pi_x-\sum_{k\geq 0}\frac{1}{k!}\Pi_x^k(D^{(0)})^k c \text{ modulo polynomials,}$$

which together with $\Pi_{xe_{\mathbf{n}}}=(\cdot-x)^{\mathbf{n}}$ serves as a rigorous definition of the centered model.

Working on the whole space-time plane $\mathbb{R}^2$ instead of the torus allows us to exploit scaling, which we shall do in the sequel to motivate the homogeneity of a multi-index $|\beta|$. By rescaling $x_1=\lambda\hat{x}_1$, $x_2=\lambda^2\hat{x}_2$ and $\xi=_{\mathrm{law}}\lambda^{\alpha-2}\hat{\xi}$, we obtain a scale invariance of the solution manifold in law, $u=_{\mathrm{law}}\lambda^{\alpha}\hat{u}$, provided we rescale the nonlinearities accordingly, $\hat{a}(\hat{u})=a(\lambda^{\alpha}\hat{u})$ and $\hat{h}(\hat{u})=\lambda^{\alpha-2}h(\lambda^{\alpha}\hat{u})$. On the level of the coordinates and the model we thus have $z_{\mathbf{n}}[u]=\lambda^{\alpha-|\mathbf{n}|}z_{\mathbf{n}}[\hat{u}]$, $z_k[a]=\lambda^{-k\alpha}z_k[\hat{a}]$ and $\Pi_{x\beta}(y)=_{\mathrm{law}}\lambda^{|\beta|}\Pi_{\hat{x}\beta}(\hat{y})$. Here, we made use of the parabolic length $|\mathbf{n}|=n_1+2n_2$ and the homogeneity of a multi-index defined by

$$|\beta|=\alpha-\sum_{\mathbf{n}\neq\mathbf{0}}(\alpha-|\mathbf{n}|)\beta(\mathbf{n})+\sum_{k\geq 0}\alpha k\beta(k)$$

Let us now comment on the choice of the constants c_β that fix the counterterm h via (3). For $|\beta| < 2$, c_β is inductively determined by estimate (1) which yields

$$\lim_{R\uparrow\infty} \fint_{B_R(x)} dy\, \mathbb{E}(\partial_2 - \partial_1^2)\Pi_{x\beta}(y) = 0,$$

via

$$c_\beta = (\partial_2 - \partial_1^2)\Pi_{x\beta} + \text{ terms depending only on } c_\gamma \text{ with } |\gamma| < |\beta|.$$

This amounts to a BPHZ-choice of renormalization (however, anchored on the infrared side). For $|\beta| > 2$, we just set $c_\beta = 0$. This renormalization, which takes care of the mean, complements well with the following spectral gap assumption on the noise, which estimates the variance of a functional by its Malliavin derivative.

Assumption 2: The law of ξ satisfies a spectral gap inequality, i. e.

$$\mathbb{E}|F - \mathbb{E}F|^2 \leq \mathbb{E}\|\frac{\partial F}{\partial \xi}\|_*^2,$$

for all cylindrical funcionals F of ξ, where $\|\frac{\partial F}{\partial \xi}\|_* = \sup_{\delta\xi} \frac{\delta F}{\|\delta\xi\|}$ and $\|\cdot\|$ denotes the homogeneous Sobolev norm of order $(\alpha - 2) + \frac{\dim_{\text{eff}}=3}{2} = \alpha - \frac{1}{2}$ for some $\alpha > \frac{1}{4}$. Taking the Malliavin derivative crucially helps in the reconstruction of the singular product $\Pi_x^k \partial_1^2 \Pi_x \in C^\alpha \cdot C^{\alpha-2}$ in

$$\Pi_x^- = \xi 1 + \big(\sum_{k\geq 0} \Pi_x^k z_k\big)\partial_1^2 \Pi_x - \sum_{k\geq 0} \frac{1}{k!} \Pi_x^k (D^{(0)})^k c,$$

as we shall see in the following. By Leibniz' rule we obtain that $\delta\Pi_x^-$ equals an expression in Π_x, $\delta\Pi_x$ and c, which infers a subtle increase in regularity by $\frac{\dim_{\text{eff}}=3}{2}$:

$$\delta\Pi_{x\beta} = \delta\Pi_{x\beta}(z) + \sum_\gamma (\mathrm{d}\Gamma_{xz})_\beta^\gamma \Pi_{z\gamma} + \mathcal{O}(|\cdot - z|^{\frac{3}{2}+\alpha})$$

for some modelled (w. r. t. the model itself) distribution $\mathrm{d}\Gamma_{xz}$. This higher order vanishing close to z yields

$$(\delta\Pi_x^- - \mathrm{d}\Gamma_{xz}\Pi_z^-)(z) = \delta\xi(z)1 + \sum_{k\geq 0} z_k \underbrace{\Pi_x^k}_{\in C^\alpha}(z)\, \partial_1^2 \underbrace{(\delta\Pi_x - \delta\Pi_x(z) - \mathrm{d}\Gamma_{xz}\Pi_z)}_{\in \mathcal{D}^{\alpha+\frac{3}{2}}}(z),$$

where, crucially, the divergent c drops out (i. e. no overlapping subdivergences). It is this identity that allows for reconstruction as long as $\alpha > \frac{1}{4}$, and yields an estimate for $\delta\Pi_x^-$, which together with the choice of c estimates Π_x^- via the spectral gap inequality.

Our last assumption makes it possible to propagate estimates via integration (i. e. Schauder theory) from Π_x^- to Π_x. As is typical for Schauder theory, integer exponents have to be avoided, which translates in our setting into:

Assumption 3: $\alpha \notin \mathbb{Q}$.

By breaking scaling and splitting (1) into two separate estimates for the ultraviolet and the infrared regime, this assumption may be seen w. l. o. g., however due to avoiding rational α we do not (yet) capture logarithmic infrared divergences.

References

[1] A. Chandra and M. Hairer, *An analytic BPHZ theorem for Regularity Structures*, Preprint arXiv:1612.08138, 2016.

[2] M. Hairer, *A theory of regularity structures*, Invent. Math., 198(2):269–504, 2014.

[3] P. Linares, F. Otto, M. Tempelmayr and P. Tsatsoulis, *A diagram-free approach to the stochastic estimates in regularity structures*, Preprint arXiv:2112.10739, 2021.

Stochastic 3D Navier-Stokes equations via convex integration

Xiangchan Zhu

(joint work with Martina Hofmanová, Rongchan Zhu)

In this talk, we talk about global-in-time existence and non-uniqueness of probabilistically strong solutions to the three dimensional Navier–Stokes system driven by space-time white noise. In this setting, solutions are expected to have space regularity at most $-1/2-\kappa$ for any $\kappa > 0$. Consequently, the convective term is ill-defined analytically and probabilistic renormalization is required. Up to now, only local well-posedness has been known. With the help of paracontrolled calculus we decompose the system in a way which makes it amenable to convex integration. By a careful analysis of the regularity of each term, we develop an iterative procedure which yields global non-unique probabilistically strong paracontrolled solutions. Our result applies to any divergence free initial condition in $L^2 \cup B^{-1+\kappa}_{\infty,\infty}$, $\kappa > 0$, and implies also non-uniqueness in law.

More precisely, we consider the three dimensional Navier–Stokes system with periodic boundary conditions driven by a space-time white noise

$$
\begin{aligned}
du + \mathrm{div}(u \otimes u)\, dt + \nabla p\, dt &= \Delta u\, dt + dB, \\
\mathrm{div} u &= 0, \\
u(0) &= u_0,
\end{aligned}
\tag{1}
$$

where B is a cylindrical Wiener process on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. The time derivative of B is the space-time white noise. The main result of this talk is given as follows:

For any given divergence free initial condition $u_0 \in L^2 \cup B^{-1+\kappa}_{\infty,\infty}$ $\mathbf{P}$-a.s., $\kappa > 0$, there exist infinitely many global-in-time probabilistically strong solutions solving (1) in a paracontrolled sense.

In two space dimensions, the problem was solved locally in time in the seminal paper by Da Prato, Debussche [1]. Furthermore, using the properties of the Gaussian invariant measure, it was possible to obtain global-in-time existence for a.e. initial condition with respect to the invariant measure.

The more irregular three dimensional setting remained open for much longer as substantially new ideas were required. These came in a parallel development with the theory of regularity structures by Hairer [3] and with the paracontrolled distributions introduced by Gubinelli, Imkeller and Perkowski [2]. They led to a local well-posedness theory for the Navier–Stokes system (1) in three dimensions by Zhu, Zhu [6].

The question of global existence is even more challenging. Roughly speaking, in the field of singular SPDEs the only available global existence results rely either on a strong drift present in the system or a particular transform for certain non-linearities or on properties of an invariant measure. No such results are available for the 3D Navier–Stokes system with space-time white noise:

- There is no strong drift helping to stabilize the evolution.
- Due to the appearance of the divergence free condition and the corresponding pressure term, it is impossible to apply maximum principle or Cole–Hopf's transform.
- The existence of an invariant measure is an open problem.
- No global energy (or other) estimates are available due to irregularity of solutions.

Our idea is to apply the method of convex integration in order to construct global-in-time solutions. This is an iterative procedure which permits to construct solutions explicitly scale by scale. It makes an essential use of the form of the nonlinearity which propagates oscillations and reduces an error term, the so-called Reynolds stress, in order to approach a solution as one proceeds through the iteration. As typical for the convex integration constructions, the same method gives raise to infinitely many solutions.

Compared to the classical uniform estimates and the compactness argument, convex integration provides a new way of constructing solutions. This turns out to be particularly useful in the stochastic setting as uniqueness of Leray solutions is unknown and there has been no result of existence of global probabilistically strong solutions before. In [4], we proved such a result for a trace-class noise by convex integration. Furthermore, there are no alternative globally defined solutions whatsoever (neither probabilistically strong nor probabilistically weak). In this talk, we use convex integration to construct global probabilistically strong solutions in this setting when the energy inequality is out of reach.

We introduce a decomposition of the Navier–Stokes system (1), which makes also this singular setting amenable to convex integration. The common idea in the field of singular SPDEs is to prescribe a particular form of a solution u so that the nonlinearity can be made sense of. In the first step, we write

$$u = z + z_1 + h.$$

The first term z solves the stochastic heat equation

$$dz + \nabla p^z \, dt = \Delta z \, dt + dB, \qquad \mathrm{div} z = 0,$$

and permits to isolate the most irregular part of u, the rest being more regular.

The above decomposition and paracontrolled ansatz for h is sufficient to prove local well-posedness as done in [6]. However, a much more refined analysis is indispensable to apply convex integration. Therefore, we split further $h = v^1 + v^2$ where v^1 represents the irregular part and v^2 the regular one. In addition, the equation for v^1 is linear whereas the one for v^2 contains the quadratic nonlinearity. Even with this decomposition into $v^1 + v^2$, it is not possible to derive global estimates via the energy method. Our idea is instead to apply convex integration on the level of v^2. However, the equation for v^2 is coupled with the equation for v^1. Therefore, we put forward a joint iterative procedure approximating both equations at once. The Reynolds stress $\mathring{R}_q$ is only included in the equation for v_q^2, where $q \in \mathbb{N}_0$ is the iteration parameter. Consequently, the construction of the new iteration v_{q+1}^2 relies only on the previous stress $\mathring{R}_q$. As the next step, we solve the equation for v_{q+1}^1 exactly by a fixed point argument.

In order to make this strategy possible, it is necessary to find the decomposition of the equation for h into the system for v^1 and v^2 and to define the corresponding equations for the iterations v_q^1 and v_q^2. This together with the construction of each approximate velocity v_{q+1}^2 through the intermittent jets has to be done in a way to decrease the corresponding Reynolds stress $\mathring{R}_{q+1}$ as $q \to \infty$. Especially the control of $\mathring{R}_{q+1}$ requires a careful analysis of each of the terms appearing in the equation for h. We have to balance various competing factors such as regularity, integrability, blow-up as $t \to 0$ and blow-up as $q \to \infty$ of various terms. The divergencies need to be compensated by small quantities. We rely on a decomposition of each product into the two paraproducts and the resonant term, because each of these parts behaves differently and requires a different treatment. Roughly speaking, irregular terms are included into v^1 while regular ones into v^2, but the precise splitting is delicate. For more details we refer to [5].

References

[1] G. Da Prato and A. Debussche. *Two-Dimensional Navier–Stokes Equations Driven by a Space–Time White Noise*. Journal of Functional Analysis, 196, 180–210, 2002.

[2] M. Gubinelli, P. Imkeller, N. Perkowski, *Paracontrolled distributions and singular PDEs*, Forum Math. Pi 3 no. 6, 2015.

[3] M. Hairer, *A theory of regularity structures.* Invent. Math. 198(2), 269–504, 2014.

[4] M. Hofmanová, R. Zhu, X. Zhu. *Global-in-time probabilistically strong and Markov solutions to stochastic 3D Navier–Stokes equations: existence and non-uniqueness.* arXiv:2104.09889, 2021.

[5] M. Hofmanová, R. Zhu, X. Zhu. *Global existence and non-uniqueness for 3D Navier–Stokes equations with space-time white noise.* arXiv:2112.14093, 2021.

[6] R. Zhu, X. Zhu, *Three-dimensional Navier-Stokes equations driven by space-time white noise*, Journal of Differential Equations, 259, 9, 5, 4443–4508, 2015.

Invariant manifolds and soliton resolution for evolution PDEs

Wilhelm Schlag

This talk will be a survey of results pertaining to the long-term dynamics of evolution partial differential equations. The emphasis lies on basic notions and results from dynamical systems, such as invariant manifolds and omega limit set. While these notions are directly applicable to dissipative PDEs, Hamiltonian PDEs do not fall under the scope of classical convergence theorems. Nevertheless, recent results on the soliton resolution problem for wave maps draw on ideas from dynamical systems in the form of one-pass type theorems. We will discuss some of these developments.

Stable and unstable periodic water waves

Massimiliano Berti

We present an overview of results proved in the last years concerning the long time dynamics of 1-dimensional space periodic water waves, regarding

(1) KAM: bifurcation of time quasi-periodic solutions [6, 5, 9, 1];
(2) Long time stability results [2, 4, 3];
(3) Benjamin-Feir instability of the Stokes waves [7, 8].

We consider the Euler equations for a 2-dimensional incompressible and inviscid fluid with constant vorticity γ, under the action of gravity. The fluid occupies the region $\mathcal{D}_{\eta,\mathtt{h}} := \{(x, y) \in \mathbb{T} \times \mathbb{R} : -\mathtt{h} < y < \eta(t, x)\}$ where $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$. The unknowns of the problem are the free surface $y = \eta(t, x)$ and the divergence free velocity field $(u(t, x, y), v(t, x, y))$. If the fluid has constant vorticity $v_x - u_y = \gamma$, the velocity field is the sum of the Couette flow $(-\gamma y, 0)$ and an irrotational field, expressed as the gradient of a harmonic function Φ. Denoting $\psi(t, x) := \Phi(t, x, \eta(t, x))$, one recovers Φ by solving the elliptic problem $\Delta\Phi = 0$ in $\mathcal{D}_{\eta,\mathtt{h}}$, $\Phi = \psi$ at $y = \eta(t, x)$ and $\Phi_y = 0$ at $y = -\mathtt{h}$. The water waves equations then are

$$\begin{cases} \eta_t = G(\eta)\psi + \gamma\eta\eta_x \\ \psi_t = -g\eta - \frac{\psi_x^2}{2} + \frac{(\eta_x\psi_x + G(\eta)\psi)^2}{2(1+\eta_x^2)} + \gamma\eta\psi_x + \gamma\partial_x^{-1}G(\eta)\psi \end{cases} \tag{1}$$

where g is the gravity and $G(\eta)\psi := (-\Phi_x\eta_x + \Phi_y)|_{y=\eta(x)}$ is the Dirichlet-Neumann operator. It results $G(0) = |D|\tanh(\mathtt{h}|D|)$. Capillarity effects may be included by adding $\kappa\Big(\frac{\eta_x}{\sqrt{1+\eta_x^2}}\Big)_x$ in the second equation in (1). The water waves system (1) is Hamiltonian and reversible. The variable η belongs to $H_0^s(\mathbb{T})$ and $\psi \in \dot{H}^s(\mathbb{T})$.

1) <u>KAM</u>. The linearized equations (1) at $(\eta, \psi) = (0, 0)$ possess the solutions

$$\begin{pmatrix} \eta(t,x) \\ \psi(t,x) \end{pmatrix} = \sum_{n\in\mathbb{N}} \begin{pmatrix} M_n\rho_n\cos(nx - \Omega_n(\gamma)t) \\ P_n\rho_n\sin(nx - \Omega_n(\gamma)t) \end{pmatrix} + \begin{pmatrix} M_n\rho_{-n}\cos(nx + \Omega_{-n}(\gamma)t) \\ P_{-n}\rho_{-n}\sin(nx + \Omega_{-n}(\gamma)t) \end{pmatrix},$$

which are the linear superposition of plane waves, traveling either to the right or to the left, where $\rho_n \geq 0$, $M_j := \Big(\frac{G_j(0)}{g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}}\Big)^{1/4}$, $j \in \mathbb{Z}\setminus\{0\}$, and $P_{\pm n} := \frac{\gamma}{2}\frac{M_n}{n} \pm M_n^{-1}$, $n \in \mathbb{N}$. The frequencies $\Omega_{\pm n}(\gamma)$ are $\Omega_j(\gamma) := \sqrt{\Big(g + \frac{\gamma^2}{4}\frac{G_j(0)}{j^2}\Big)G_j(0)} + \frac{\gamma}{2}\frac{G_j(0)}{j}$, $j \in \mathbb{Z}\setminus\{0\}$. Do such solutions persist for the nonlinear equations (1)?

Theorem 1. [6] *Consider finitely many integers $\mathbb{S}^+ := \{\bar{n}_1, \ldots, \bar{n}_\nu\} \subset \mathbb{N}$, $1 \leq \bar{n}_1 < \ldots < \bar{n}_\nu$, and signs $\Sigma := \{\sigma_1, \ldots, \sigma_\nu\}$, $\sigma_a \in \{-1, 1\}$, $a = 1, \ldots, \nu$. Fix a subset $[\gamma_1, \gamma_2] \subset \mathbb{R}$. Then there exist $\bar{s} > 0$, $\varepsilon_0 \in (0, 1)$ such that, for any $|\xi| \leq \varepsilon_0^2$, $\xi := (\xi_{\sigma_a\bar{n}_a})_{a=1,\ldots,\nu} \in \mathbb{R}_+^\nu$, the following hold:*

1) There exists a Borel set $\mathcal{G}_\xi \subset [\gamma_1, \gamma_2]$ *with* $\lim_{\xi\to 0} |\mathcal{G}_\xi| = \gamma_2 - \gamma_1$*;*

2) For any $\gamma \in \mathcal{G}_\xi$*, the gravity water waves equations* (1) *have a quasi-periodic traveling wave solution of the form*

$$\begin{aligned}\begin{pmatrix}\eta(t,x)\\ \psi(t,x)\end{pmatrix} &= \sum_{a\in\{1,\dots,\nu\}:\, \sigma_a=+1} \begin{pmatrix} M_{\bar n_a}\sqrt{\xi_{\bar n_a}}\cos(\bar n_a x - \widetilde{\Omega}_{\bar n_a}(\gamma)t)\\ P_{\bar n_a}\sqrt{\xi_{\bar n_a}}\sin(\bar n_a x - \widetilde{\Omega}_{\bar n_a}(\gamma)t)\end{pmatrix} \\ &+ \sum_{a\in\{1,\dots,\nu\}:\, \sigma_a=-1} \begin{pmatrix} M_{\bar n_a}\sqrt{\xi_{-\bar n_a}}\cos(\bar n_a x + \widetilde{\Omega}_{-\bar n_a}(\gamma)t)\\ P_{-\bar n_a}\sqrt{\xi_{-\bar n_a}}\sin(\bar n_a x + \widetilde{\Omega}_{-\bar n_a}(\gamma)t)\end{pmatrix} + r(t,x)\end{aligned} \tag{2}$$

where $r(t,x) = \breve{r}(\widetilde{\Omega}_{\sigma_1\bar n_1}(\gamma)t - \sigma_1\bar n_1 x, \dots, \widetilde{\Omega}_{\sigma_\nu\bar n_\nu}(\gamma)t - \sigma_\nu\bar n_\nu x)$*, for* $\breve{r} \in H^{\bar s}(\mathbb{T}^\nu, \mathbb{R}^2)$*, satisfying* $\lim_{\xi\to 0} \frac{\|\breve{r}\|_{\bar s}}{\sqrt{|\xi|}} = 0$*, with a Diophantine frequency vector denoted by* $\widetilde{\Omega} := (\widetilde{\Omega}_{\sigma_a\bar n_a})_{a=1,\dots,\nu}$ *in* $\mathbb{R}^\nu$*, depending on* γ, ξ*, and satisfying*

$$\lim_{\xi\to 0} \widetilde{\Omega} = (\Omega_{\sigma_a\bar n_a}(\gamma))_{a=1,\dots,\nu}.$$

In addition these quasi-periodic solutions are linearly stable.

The solutions (2) are the nonlinear superposition of multiple Stokes traveling waves with rationally independent speeds, and can not be reduced to steady solutions in any moving frame. A similar KAM result with surface tension is proved in [5]. Quasi-periodic standing wave solutions for irrotational fluids had been previously obtained in [1] and with surface tension in [9].

2) Long time dynamics. A complementary almost global existence result for irrotational capillary-gravity water waves with even initial data is the following.

Theorem 2. [2] *There is a zero measure subset* $\mathcal{N}$ *in* $]0,+\infty[^2$ *such that, for any* (g,κ) *in* $]0,+\infty[^2 - \mathcal{N}$*, for any* N *in* $\mathbb{N}$*, there is* $s_0 > 0$ *and, for any* $s \geq s_0$*, there are* $\varepsilon_0 > 0, c > 0, C > 0$ *such that, for any* $\varepsilon \in]0, \varepsilon_0[$*, any even function* (η_0, ψ_0) *in* $H_0^{s+\frac14}(\mathbb{T},\mathbb{R}) \times \dot H^{s-\frac14}(\mathbb{T},\mathbb{R})$ *with* $\|\eta_0\|_{H_0^{s+\frac14}} + \|\psi_0\|_{\dot H^{s-\frac14}} < \varepsilon$*, the gravity-capillary water waves equations* (1) *with* $\gamma = 0$ *have a unique classical solution* (η,ψ) *defined on* $]-T_\varepsilon, T_\varepsilon[\times\mathbb{T}$ *with* $T_\varepsilon \geq c\varepsilon^{-N}$*, belonging to* $C^0\big(]-T_\varepsilon, T_\varepsilon[, H_0^{s+\frac14}(\mathbb{T},\mathbb{R}) \times \dot H^{s-\frac14}(\mathbb{T},\mathbb{R})\big)$*, satisfying the initial condition* $\eta|_{t=0} = \eta_0, \psi|_{t=0} = \psi_0$*. Moreover, this solution is even in space and it stays at any time in the ball of center 0 and radius* $C\varepsilon$ *of* $H_0^{s+\frac14}(\mathbb{T},\mathbb{R}) \times \dot H^{s-\frac14}(\mathbb{T},\mathbb{R})$*.*

The local well-posedness of the water waves was first proved by S. Wu. In particular, for initial data of size ε, the solutions exist for times of order ε^{-1}. In [3] we proved that $T_\varepsilon \geq c\varepsilon^{-2}$ for *any* value of $(g, \kappa, \mathtt{h})$ and in [4] that $T_\varepsilon \geq c\varepsilon^{-3}$ for pure gravity water waves in infinite depth, proving an old conjecture of Zhakarov-Dyachenko. A tool to extend the lifespan of solutions is normal form theory. The idea is to make changes of variables which diminish iteratively the size of the nonlinearity, when possible, and to prove that the "resonant" terms which are left in the normal form do not contribute to the growth of the solutions. Here the Hamiltonian or reversible structure of the PDE plays a key role.

3) <u>Benjamin-Feir instability of Stokes waves</u>. A classical problem in fluid dynamics, pioneered by Stokes in 1847, concerns the spectral stability/instability of periodic traveling water waves, called Stokes waves. Benjamin-Feir discovered in the sixties, through experiments and formal arguments, that (pure gravity and irrotational) Stokes waves in deep water are unstable, proposing a heuristic mechanism which leads to the disintegration of wave trains. The problem is mathematically formulated as follows: consider a 2π-periodic Stokes wave with amplitude $0 < \varepsilon \ll 1$. The linearized water waves equations at the Stokes wave are, in the inertial frame moving with the speed c_ε of the Stokes wave, a linear time independent system of the form $h_t = \mathcal{L}_\varepsilon h$ where $\mathcal{L}_\varepsilon$ is a linear operator with 2π-periodic coefficients which possesses the eigenvalue 0 with algebraic multiplicity four. The problem is to prove that $h_t = \mathcal{L}_\varepsilon h$ has solutions of the form $h(t,x) = \mathrm{Re}\left(e^{\lambda t} e^{\mathrm{i}\mu \mathrm{x}} v(x)\right)$ where $v(x)$ is a 2π-periodic function, μ in $\mathbb{R}$ (called Floquet exponent) and λ has positive real part, thus $h(t,x)$ grows exponentially in time. By Bloch-Floquet theory, such λ is an eigenvalue of the operator $\mathcal{L}_{\mu,\varepsilon} := e^{-\mathrm{i}\mu\mathrm{x}}\,\mathcal{L}_\varepsilon\, e^{\mathrm{i}\mu\mathrm{x}}$ acting on 2π-periodic functions. The main result in [7] provides the full description of the eigenvalues with non-zero real part close to zero of the operator $\mathcal{L}_{\mu,\varepsilon}$ for ε and μ small. We denote by $r(\varepsilon^{m_1}\mu^{n_1}, \ldots, \varepsilon^{m_p}\mu^{n_p})$ a real analytic function fulfilling for some $C > 0$ and ε, μ small, $|r(\varepsilon^{m_1}\mu^{n_1}, \ldots, \varepsilon^{m_p}\mu^{n_p})| \leq C \sum_{j=1}^{p} |\varepsilon|^{m_j} |\mu|^{n_j}$.

Theorem 3. [7] *There exist $\varepsilon_1, \mu_0 > 0$ and an analytic function $\underline{\mu} : [0,\varepsilon_1) \to [0,\mu_0)$, of the form $\underline{\mu}(\varepsilon) = 2\sqrt{2}\varepsilon(1 + r(\varepsilon))$, such that, for any $\varepsilon \in [0,\varepsilon_1)$, the operator $\mathcal{L}_{\mu,\varepsilon}$ has two eigenvalues $\lambda_1^{\pm}(\mu,\varepsilon)$ of the form*

$$\begin{cases} \frac{\mathrm{i}\mu}{2} + \mathrm{i}\mathrm{r}(\mu\varepsilon^2, \mu^2\varepsilon, \mu^3) \pm \frac{\mu}{8}\sqrt{8\varepsilon^2\big(1 + \mathrm{r}_0(\varepsilon,\mu)\big) - \mu^2\big(1 + \mathrm{r}_0'(\varepsilon,\mu)\big)}, & \mu \in [0, \underline{\mu}(\varepsilon)), \\ \frac{\mathrm{i}}{2}\underline{\mu}(\varepsilon) + \mathrm{i}\mathrm{r}(\varepsilon^3), & \mu = \underline{\mu}(\varepsilon), \\ \frac{\mathrm{i}\mu}{2} + \mathrm{i}\mathrm{r}(\mu\varepsilon^2, \mu^2\varepsilon, \mu^3) \pm \frac{\mathrm{i}\mu}{8}\sqrt{\mu^2\big(1 + \mathrm{r}_0'(\varepsilon,\mu)\big) - 8\varepsilon^2\big(1 + \mathrm{r}_0(\varepsilon,\mu)\big)}, & \mu \in (\underline{\mu}(\varepsilon), \mu_0). \end{cases}$$

The function $8\varepsilon^2\big(1 + r_0(\varepsilon,\mu)\big) - \mu^2\big(1 + r_0'(\varepsilon,\mu)\big)$ is > 0, respectively < 0, provided $0 < \mu < \underline{\mu}(\varepsilon)$, respectively $\mu > \underline{\mu}(\varepsilon)$.

The proof relies on a symplectic and reversible version of Kato perturbation theory. The more complex finite depth case is considered in [8].

References

[1] Baldi P., Berti M., Haus E., Montalto R. *Time quasi-periodic gravity water waves in finite depth*, Inventiones Math. **214**, 2018, no. 2, 739–911.

[2] Berti M., Delort J.-M. *Almost global solutions of capillary-gravity water waves equations on the circle.* UMI Lecture Notes 2018, ISBN 978-3-319-99486-4.

[3] Berti M., Feola R., Franzoi L. *Quadratic life span of periodic gravity-capillary water waves.* Water Waves 3, no. 1, 85-115, 2021.

[4] Berti M., Feola R., Pusateri F. *Birkhoff normal form and long time existence for periodic gravity water waves.* https://doi.org/10.1002/cpa.2204, Comm. Pure Applied Math, 2022.

[5] Berti M., Franzoi L., Maspero A. *Traveling quasi-periodic water waves with constant vorticity.* Arch. Rat. Mech. Anal. **240**, 2021, 99–202.

[6] Berti M., Franzoi L., Maspero A. *Pure gravity traveling quasi-periodic water waves with constant vorticity*, `arXiv:2101.12006`, 2021, to appear in Comm. Pure Appl. Math.

[7] Berti M., Maspero A., Ventura P. *Full description of Benjamin-Feir instability of Stokes waves in deep water*, `arXiv:2109.11852`, 2021, to appear in Inventiones Math.
[8] Berti M., Maspero A., Ventura P. *Benjamin-Feir instability of Stokes waves in finite depth*, `https://arxiv.org/abs/2204.00809`, 2022.
[9] Berti M., Montalto R. *Quasi-periodic standing wave solutions of gravity-capillary water waves.* Memoirs AMS, **263**, 2020, no. 1273.

Wave turbulence theory for a stochastic KdV type equation

Minh-Binh Tran
(joint work with Gigliola Staffilani)

The question concerning the migration of the energy of a periodic global solution to a certain dispersive equation from low to high frequencies (forward cascade) has been an important question in the theory of PDEs. The first approach, introduced by Bourgain in [3], is centered on the asymptotic analysis of the norm $H^s, s \gg 1$, of the solution itself. The second approach is based on finding an effective equation, referred to as the wave kinetic equation, for the expectation of the square of the norm of the solution.

The second approach is related to the so-called wave turbulence theory in continuum mechanics (see [23]). Wave turbulence theory has the origin in the works of Peierls [20, 21], Brout-Prigogine [4], Zaslavskii-Sagdeev [25], Hasselmann [17, 18], Benney-Saffman-Newell [1, 2], Zakharov [24]. Denoting $\lambda > 0$ the parameter that describes the weak interactions of the wave system under consideration, it is expected that the associated wave kinetic equation can be derived at the *van Hove limit*

$$t = \mathcal{O}(\lambda^{-2}). \tag{1}$$

In rigorously deriving wave kinetic equations, the work of Lukkarinen and Spohn [19] for the lattice cubic nonlinear Schrödinger equation (NLS) is pioneering plays the pioneering role. Different from the work of Lukkarinen and Spohn [19], the work of Buckmaster-Germain-Hani-Shatah [5, 6], Deng-Hani [9], and Collot-Germain [7, 8] give rigorous derivations of the 4-wave kinetic equations from the out of statistical equilibrium cubic NLS equation at limits closed to (1), in the continuum setting, with random initial data. Works that derive the 4-wave kinetic equation from the stochastic NLS in the continuum setting are done by by Dymov, Kuksin and collaborators in [12, 13, 14, 15]. In [10] and [22], Deng-Hani and Staffilani-Tran provided rigorous derivations at the kinetic limit (1) of the homogeneous 4-wave and 3-wave kinetic equations from the random NLS equation in the continuum setting and the stochastic ZK equation in the lattice setting, out of equilibrium. A propagation of chaos result was also obtained in [11]. Our starting point of [22], inspired by the work of Faou [16], is the ZK equation in d-dimension ($d \geq 2$),

$$\begin{aligned} \mathrm{d}\psi(x,t) &= -\Delta\partial_{x_1}\psi(x,t)\mathrm{d}t + \lambda\partial_{x_1}\Big(\psi^2(x,t)\Big)\mathrm{d}t + \sqrt{2c_r}\partial_{x_1}\psi \odot \mathrm{d}W(t), \\ \psi(x,0) &= \psi(x), \end{aligned} \tag{2}$$

where $1 > \lambda > 0$ is a real constant, and

(3) $$c_r = \mathfrak{C}_r \lambda^{\theta_r},$$

for some universal constants $\mathfrak{C}_r > 0$ and $1 \geq \theta_r > 0$. The constant λ is the parameter describing the weak interactions of the nonlinear wave system and as mentioned above we will send λ to 0. In the identity

$$U \odot \mathrm{d}W(x,t) \ = \ \int_\Lambda U(x-x',t) \circ \mathrm{d}W(x',t)\mathrm{d}x'.$$

the symbol $\circ$ is the Stratonovich product and the symbol $\odot$ represent the combination of the Stratonovich product and the convolution. The lattice system can be rewritten in the Fourier space as

$$\begin{aligned}
\mathrm{d}\hat{\psi}(k,t) \ &= \ \mathbf{i}\omega(k)\hat{\psi}(k,t)\mathrm{d}t \ + \ \mathbf{i}\bar{\omega}(k)\sqrt{2c_r}\hat{\psi}(k,t) \circ \mathrm{d}W(t) \\
&\quad + \mathbf{i}\lambda\bar{\omega}(k)\frac{1}{|\Lambda_*|^2} \sum_{k=k_1+k_2; k_1,k_2 \in \Lambda_*} \hat{\psi}(k_1,t)\hat{\psi}(k_2,t), \\
\hat{\psi}(k,0) \ &= \ \hat{\psi}_0(k).
\end{aligned}$$

The mesh and the dispersion relation are written as follows

$$\Lambda_* = \Lambda_*(L) = \left\{ -\frac{L}{2L+1}, \cdots, 0, \cdots, \frac{L}{2L+1} \right\}^d,$$

$$\omega_k \ = \ \omega(k) \ = \ \sin(2\pi k^1)\Big[\sin^2(2\pi k^1) + \cdots + \sin^2(2\pi k^d)\Big], \quad \bar{\omega}(k) \ = \sin(2\pi k^1),$$

with $k = (k^1, \cdots, k^d)$.

By setting $a_k \ = \ \frac{\hat{\psi}(k)}{\sqrt{|\bar{\omega}(k)|}}$, we find

$$\begin{aligned}
\mathrm{d}a_k \ &= \mathbf{i}\omega(k)a_k\mathrm{d}t \ + \ \mathbf{i}\sqrt{2c_r}a_k \circ \mathrm{d}W_k(t) \\
&\quad + \ \mathbf{i}\lambda \int_{\Lambda^*} \mathrm{d}k_1 \int_{\Lambda^*} \mathrm{d}k_2 \mathrm{sign}(k^1)\sqrt{|\bar{\omega}(k)\bar{\omega}(k_1)\bar{\omega}(k_2)|}\delta(k-k_1-k_2)a_{k_1}a_{k_2}\mathrm{d}t.
\end{aligned}$$

If we considerr the two-point correlation function

(4) $$f(k,t) \ = \ \langle \alpha_t(k,-1)\alpha_t(k,1) \rangle,$$

in the limit of $L \to \infty$, $\lambda \to 0$ and $t = \lambda^{-2}\tau = \mathcal{O}(\lambda^{-2})$, the two-point correlation function $f(k,t)$ has the limit

$$\lim_{\lambda \to 0, D \to \infty} f(k, \lambda^{-2}\tau) = f^\infty(k,\tau)$$

which is the solution of the 3-wave equation

(5) $$\frac{\partial}{\partial \tau} f^\infty(k,\tau) = \mathcal{C}\big(f^\infty\big)(k,\tau)$$

where
(6)

$$\mathcal{C}(f^\infty)(k_1) = \int_{(\mathbb{T}^d)^2} \mathrm{d}k_2 \mathrm{d}k_3 |\mathcal{M}(k_1,k_2,k_3)|^2 \frac{1}{\pi}\delta(\omega(k_3)+\omega(k_2)-\omega(k_1))$$
$$\times\, \delta(k_2+k_3-k_1)\Big(f_2^\infty f_3^\infty - f_1^\infty f_2^\infty \mathrm{sign}(k_1^1)\mathrm{sign}(k_3^1) - f_1^\infty f_3^\infty \mathrm{sign}(k_1^1)\mathrm{sign}(k_2^1)\Big),$$

in which $\mathbb{T}$ is the periodic torus $[-1/2, 1/2]$. Here we have introduced the shorthand notation $f_j^\infty = f^\infty(k_j), j = 1,2,3$. We also set $\mathbb{T}^d_+ = \{k = (k^1,\cdots,k^d) \in \mathbb{T}^d \mid k^1 \geq 0\}$.

References

[1] D. J. Benney and A. C. Newell. Random wave closures. *Studies in Applied Mathematics*, 48(1):29–53, 1969.

[2] D. J. Benney and P. G. Saffman. Nonlinear interactions of random waves in a dispersive medium. *Proc. R. Soc. Lond. A*, 289(1418):301–320, 1966.

[3] J. Bourgain. On the growth in time of higher sobolev norms of smooth solutions of hamiltonian pde. *International Mathematics Research Notices*, 1996(6):277–304, 1996.

[4] R. Brout and I. Prigogine. Statistical mechanics of irreversible processes part viii: general theory of weakly coupled systems. *Physica*, 22(6-12):621–636, 1956.

[5] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. On the kinetic wave turbulence description for nls. *Quarterly of Applied Mathematics, to appear*, 2019.

[6] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. Onset of the wave turbulence description of the longtime behavior of the nonlinear schrödinger equation. *Submitted*, 2019.

[7] C. Collot and P. Germain. On the derivation of the homogeneous kinetic wave equation. *arXiv preprint arXiv:1912.10368*, 2019.

[8] C. Collot and P. Germain. Derivation of the homogeneous kinetic wave equation: longer time scales. *arXiv preprint arXiv:2007.03508*, 2020.

[9] Y. Deng and Z. Hani. On the derivation of the wave kinetic equation for nls. *arXiv preprint arXiv:1912.09518*, 2019.

[10] Y. Deng and Z. Hani. Full derivation of the wave kinetic equation. *arXiv preprint arXiv:2104.11204*, 2021.

[11] Y. Deng and Z. Hani. Propagation of chaos and the higher order statistics in the wave kinetic theory. *arXiv preprint arXiv:2110.04565*, 2021.

[12] A. Dymov and S. Kuksin. Formal expansions in stochastic model for wave turbulence 1: kinetic limit. *arXiv preprint arXiv:1907.04531*, 2019.

[13] A. Dymov and S. Kuksin. Formal expansions in stochastic model for wave turbulence 2: method of diagram decomposition. *arXiv preprint arXiv:1907.02279*, 2019.

[14] A. Dymov and S. Kuksin. On the zakharov-l'vov stochastic model for wave turbulence. In *Doklady Mathematics*, volume 101, pages 102–109. Springer, 2020.

[15] A. Dymov, S. Kuksin, A. Maiocchi, and S. Vladuts. The large-period limit for equations of discrete turbulence. *arXiv preprint arXiv:2104.11967*, 2021.

[16] E. Faou. Linearized wave turbulence convergence results for three-wave systems. *Communications in Mathematical Physics*, 378(2):807–849, 2020.

[17] K. Hasselmann. On the non-linear energy transfer in a gravity-wave spectrum part 1. general theory. *Journal of Fluid Mechanics*, 12(04):481–500, 1962.

[18] K. Hasselmann. On the spectral dissipation of ocean waves due to white capping. *Boundary-Layer Meteorology*, 6(1-2):107–127, 1974.

[19] J. Lukkarinen and H. Spohn. Weakly nonlinear Schrödinger equation with random initial data. *Invent. Math.*, 183(1):79–188, 2011.

[20] R. Peierls. Zur kinetischen theorie der warmeleitung in kristallen. *Annalen der Physik*, 395(8):1055–1101, 1929.

[21] R. E. Peierls. Quantum theory of solids. In *Theoretical physics in the twentieth century (Pauli memorial volume)*, pages 140–160. Interscience, New York, 1960.
[22] G. Staffilani and M.-B. Tran. On the wave turbulence theory for a stochastic kdv type equation. *arXiv preprint arXiv:2106.09819*, 2021.
[23] T. Tao. Weakly turbulent solutions for the cubic defocusing nonlinear schrödinger equation. *https://terrytao.wordpress.com/*, 2008.
[24] V. E. Zakharov, V. S. L'vov, and G. Falkovich. *Kolmogorov spectra of turbulence I: Wave turbulence*. Springer Science & Business Media, 2012.
[25] G. M. Zaslavskii and R. Z. Sagdeev. Limits of statistical description of a nonlinear wave field. *Soviet physics JETP*, 25:718–724, 1967.

The mathematical theory of wave turbulence

Zaher Hani
(joint work with Yu Deng)

The subject falls under the umbrella of Hilbert's sixth problem, which asks for a rigorous derivation of the laws of physics starting from first principles. Most notably, one is interested in rigorously deriving the laws of statistical mechanics from the laws of dynamics. This features a justification of irreversible (in time) laws (like the second law of thermodynamics) starting from reversible ones (like Newton's or Hamilton's equations). Explaining this apparent paradox is the essence of Hilbert's sixth problem.

The classical instance of such a result is the justification of Boltzmann's kinetic theory for particle collisions starting from first principles given by Newton's equation. This was undergone starting from Lanford's seminal work in 1975 and continued by works of Cercignani-Illner-Pulvirenti, Pulvirenti-Saffirio-Simonella, and completed by works of Gallagher-Saint-Raymond-Texier. There one justifies the appearance of Boltzmann's equation in the limit where the particle number $N \to \infty$ and their radius $r \to 0$ under the so-called Boltzmann-Grad scaling $Nr^{d-1} = O(1)$.

We are interested in the analogous problem for waves, in which the microscopic system itself is given by a nonlinear PDE that governs the interaction of waves. Rather than thinking of particles colliding, we think of many waves interacting in a large system whose size goes to infinity. The strength of the interaction ϵ goes to zero in the kinetic limit, which is the analog of the limit $r \to 0$ in the particle setting described above [17, 19].

To make the discussion concrete, let us fix our emblematic microscopic wave system given by the nonlinear Schrödinger equation

$$(i\partial_t + \Delta)u = \epsilon|u|^2u, \qquad t \in \mathbb{R}, x \in \mathbb{T}_L^d \tag{1}$$

where the spatial domain $\mathbb{T}_L^d$ is a periodic box of size L and $\epsilon > 0$ denotes the strength of the nonlinear interaction. The initial data u_{in} is taken to be randomly distributed as follows: Denoting by $\widehat{u}_{\text{in}}(k)$ the Fourier modes of u_{in} at k defined by

$$\widehat{u}(k) = L^{-d/2}\int_{\mathbb{T}_L^d} u(x)e^{-ik\cdot x}dx,$$

where $k \in \mathbb{Z}_L^d = L^{-1}\mathbb{Z}^d$ belongs to a lattice of mesh size L^{-1}, we let

$$\widehat{u}_{\text{in}}(k) = \sqrt{n_{\text{in}}(k)}\,\eta_k^\omega.$$

Here $n_{\text{in}}(k)$ is a sufficiently smooth and decaying non-negative function on $\mathbb{R}^d$, and η_k^ω are i.i.d. random variables of mean zero and variance 1. As such, one has that $\mathbb{E}|\widehat{u}_{\text{in}}(k)|^2 = n_{\text{in}}(k)$, and we call such data well-prepared.

The aim of the wave kinetic theory is to study the longtime behavior of the distribution of the Fourier modes a_k starting with the central quantity $\mathbb{E}|\widehat{u}(k,t)|^2$, whose effective dynamics are conjectured to satisfy the *wave kinetic equation* (WKE)

$$\text{(WKE)} \qquad \begin{cases} \partial_t n &= \mathcal{K}(n,n,n) \\ n(0) &= n_{\text{in}}, \end{cases}$$

where

$$\mathcal{K}(n,n,n)(\xi) = 2\int_{\substack{\xi_1,\xi_2,\xi_3 \in \mathbb{R}^d \\ \xi_1-\xi_2+\xi_3=\xi}} \delta_{\mathbb{R}}(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi|^2)$$
$$n(\xi_1)n(\xi_2)n(\xi_3)n(\xi)\left(\frac{1}{n(\xi_1)} - \frac{1}{n(\xi_2)} + \frac{1}{n(\xi_3)} - \frac{1}{n(\xi)}\right)$$

The (WKE) is a wave analog of Boltzmann's equation. Note that this kinetic approximation features a passage from the time reversible NLS equation into the time-irreversible wave kinetic equation. There is also an inhomogeneous version thereof in which n is also space-dependent and the LHS of the (WKE) has a transport term. We restrict ourselves here to the homogeneous setting.

After recounting various previous contributions on the topic, most notably [15, 2, 6, 3, 4, 10, 11, 12, 13], we state our main result:

Theorem 1 (Deng-H., 2021)**.**

- *Consider (NLS) on the periodic box* $\mathbb{T}_L^d$ *with* $d \geq 3$.
- *Take* $n_{\text{in}} \geq 0$ *to be a sufficiently smooth and decaying function on* $\mathbb{R}^d$, *and* u_{in} *to be well-prepared, i.e.* $\widehat{u}_{\text{in}}(k) = \sqrt{n_{\text{in}}(k)}\eta_k(\omega)$, *and suppose that the law of* $\eta_k(\omega)$ *is rotationally symmetric and has exponential tails (e.g. Gaussian).*
- *Scaling laws: Let* $\alpha \sim L^{-\gamma}$ *for* $\gamma \in [1 - \frac{1}{20d}, 1]$, *and recall that* $T_{\text{kin}} = \alpha^{-2}$. *For* $\gamma = 1$, *we assume suitable genericity conditions on the aspect ratios of the box.*

THEN, there exists $\delta < 1$ fixed, *and an absolute constant* $\nu > 0$ *such that for* L *large enough it holds that*

$$\mathbb{E}|\widehat{u}(t,k)|^2 = n(\frac{t}{T_{\text{kin}}}, k) + O(L^{-\nu})$$

uniformly in (t,k) *for* $t \in [0, \delta \cdot T_{\text{kin}}]$. *Here* $n(t,k)$ *solves the wave kinetic equation with data* n_{in}.

Moreover, suppose that $k_1, \dots, k_r$ are distinct, then

(1) Propagation of Chaos: *The random variables $\widehat{u}(t, k_j)$ $(1 \le j \le r)$ retain their independence in the kinetic limit $L \to \infty$.*

(2) Limiting law: *The law of $\widehat{u}(t, k)$ converges to the density function $\rho_k(t, v)$ (with $v \in \mathbb{R}^2$) which evolves according to the linear PDE*

$$\partial_t \rho_k = \frac{\sigma_k(t)}{4} \Delta \rho_k - \frac{\gamma_k(t)}{2} \nabla \cdot (v \rho_k),$$

where $\sigma_k(t) > 0$ and $\gamma_k(t)$ are functions constructed from the solution $n(t, k)$ to the wave kinetic equation.

(3) Propagation of Gaussianity: *In particular, if $\eta_k(\omega)$ are Gaussian, then $\rho_k(t, v)$ is Gaussian with variance $n(t, k)$ for any $t > 0$.*

The precise statements, particularly about propagation of chaos and Gaussianity, can be found in [7, 8] on which this talk is based. A larger range of scaling laws than that cited in the above theorem is treated in [9] with similar outcomes. We also mention several results that also came out in the past year on similar questions, most notably [18, 1, 16, 5, 14]

References

[1] I. Ampatzoglou, C. Collot, P. Germain. Derivation of the kinetic wave equation for quadratic dispersive problems in the inhomogeneous setting. *Preprint.* arXiv:2107.11819.

[2] T. Buckmaster, P. Germain, Z. Hani, and J. Shatah. Onset of the wave turbulence description of the longtime behavior of the nonlinear Schrödinger equation. *Inventiones Mathematicae* (2021). Published online. (arXiv:1907.03667).

[3] C. Collot and P. Germain. On the derivation of the homogeneous kinetic wave equation. *Preprint*, arXiv:1912.10368.

[4] C. Collot and P. Germain. Derivation of the homogeneous kinetic wave equation: longer time scales. *Preprint*, arXiv:2007.03508.

[5] G. Dubach, P. Germain, B. Harrop-Griffiths. On the derivation of the homogeneous kinetic wave equation for a nonlinear random matrix model. arXiv:2203.13748.

[6] Y. Deng and Z. Hani. On the derivation of the wave kinetic equation for NLS. *Forum of Math. Pi* 9 (2021), e6, 1–37.

[7] Y. Deng and Z. Hani. Full derivation of the wave kinetic equation. arXiv:2104.11204.

[8] Y. Deng and Z. Hani. Propagation of chaos and the higher order statistics in the wave kinetic theory. arXiv:2110.04565.

[9] Y. Deng and Z. Hani. Rigorous Wave Kinetic Theory with general scaling laws. *In preparation.*

[10] A. Dymov, S. Kuksin. Formal expansions in stochastic model for wave turbulence 1: kinetic limit. arXiv:1907.04531.

[11] A. Dymov, S. Kuksin. Formal expansions in stochastic model for wave turbulence 2: method of diagram decomposition. arXiv:1907.02279

[12] A. Dymov, S. Kuksin, A. Maiocchi, S. Vladuts. The large-period limit for equations of discrete turbulence. arXiv:2104.11967.

[13] E. Faou, Linearized wave turbulence convergence results for three-wave systems. arXiv: 1805.11269.

[14] R. Grande and Z. Hani. On the wave turbulence description for a stochastic NLS. *Preprint.*

[15] J. Lukkarinen, H. Spohn, Weakly nonlinear Schrödinger equation with random initial data, *Invent. Math.* 183 (2011), pp. 79–188.

[16] X. Ma. Almost sharp wave kinetic theory of multidimensional KdV type equations with $d \geq 3$. arXiv:2204.06148.
[17] S. Nazarenko. *Wave turbulence*, volume 825 of *Lecture Notes in Physics*. Springer, Heidelberg, 2011.
[18] G. Staffilani and M.-B. Tran. On the wave turbulence theory for the stochastic and random multidimensional KdV type equations. arXiv:2106.09819v4.
[19] V.E. Zakharov, V.S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence: I Wave Turbulence*. Springer, Berlin, 1992.

Stochastic Quantization of Yang Mills

Ajay Chandra
(joint work with Ilya Chevyrev, Martin Hairer, and Hao Shen)

I report on the results obtained in the recent works [3, 4] which give the first steps of a stochastic quantization approach to the construction of continuum, finite volume, non-abelian Yang-Mills Euclidean quantum field theories (EQFTs) in two and three dimensions.

Fixing a compact Lie group G and writing $\mathfrak{g}$ for the associated Lie algebra, the corresponding Yang-Mills EQFT on $\mathbb{T}^d$ is often *formally* written as a probability measure on $\mathfrak{g}$-valued 1-forms $A(x) = A_1(x)\mathrm{d}x_1 + \cdots A_d(x)\mathrm{d}x_d \in \Omega^1_{\mathfrak{g}}$ given by

$$\mathrm{d}\mu_{\mathrm{YM}}(A) = \mathcal{Z}^{-1} \exp\big[- S_{\mathrm{YM}}(A)\big]\mathrm{d}A \,, \tag{1}$$

where $\mathrm{d}A$ is a formal "Lebesgue measure" on $\Omega^1_{\mathfrak{g}}$, $\mathcal{Z}$ is a normalization constant, and the YM action is given by

$$S_{\mathrm{YM}}(A) = \int_{\mathbb{T}^d} |F_A(x)|^2 \mathrm{d}x \,, \tag{2}$$

where $F_A = dA + A \wedge A$ is the curvature 2-form of A and the norm $|F_A|$ is given by an Ad-invariant inner product on the Lie algebra $\mathfrak{g}$ of G.

In addition to the problems around ultraviolet renormalisation that appear in the analysis of other EQFTs, the Yang-Mills measure has a more fundamental issue in that (1) in invariant under the infinite dimensional group $\mathcal{G} = \{g : \mathbb{T}^d \to G\}$ of gauge transformations under the action $A \mapsto g \cdot A = gAg^{-1} - (\mathrm{d}g)g^{-1}$, in particular $S_{\mathrm{YM}}(g \cdot A) = S_{\mathrm{YM}}(A)$. This means one would not expect the measure (1) to giving rise to a finite measure on a space of 1-forms, the more natural state space to support the Yang-Mills measure as a probability measure would be some sort of quotient space $\Omega^1_{\mathfrak{g}}/\mathcal{G}$ - we say 1-forms A and B are gauge equivalent if $B = g \cdot A$ for some $g \in \mathcal{G}$.

While it is well understood how to use explicit formulae to construct the Yang-Mills measure for $d = 2$, the situation in $d = 3$ remains open.

The starting point of the stochastic quantization approach to the construction of EQFTs (see the contribution by M. Gubinelli for an introduction to stochastic quantization) is to introduce a new "fictitious" time variable t and to obtain (1) as the long-time invariant measure of an associated stochastic gradient descent dynamic which in our setting reads

$$\partial_t A = -\nabla S_{\mathrm{YM}}(A) + \xi \;, \tag{3}$$

where $\xi = (\xi_i)_{i=1}^d$ is a $\mathfrak{g}^d$-valued space-time white noise. Formally, solutions to (3) are gauge covariant in law under the action of constant in time gauge transformations. The equation (3) is an example of a singular stochastic partial differential equation, but it is not fully parabolic - intuitively the deterministic part of (3) only moves transversely to $\mathcal{G}$-orbits. In order to apply the theory of regularity structures we study a variant of (3) where we add a DeTurck–Zwanziger term $-\mathrm{d}_A \mathrm{d}^* A$ which gives full parabolicity, in coordinates this new dynamic is given by

$$\partial_t A_i = \xi_i + \sum_{j=1}^{d} [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] \;, \quad \text{for } 1 \leq i \leq d \;. \tag{4}$$

We call (4) the *stochastic Yang-Mills heat flow.* Formally it satisfies a gauge covariance in law under a class of time-evolving gauge transformations. Moreover, when $d = 2$ or 3 local well-posedness of (4) can be obtained by applying the theory of regularity structures, this gives us a finite dimensional family of solutions obtained as the $\epsilon \downarrow 0$ limit of regularized, renormalized versions of (4) given by

$$\partial_t A_i = \xi_i^\epsilon + C^\epsilon A_i + \sum_{j=1}^{d} [A_j, 2\partial_j A_i - \partial_i A_j + [A_j, A_i]] \;, \quad \text{for } 1 \leq i \leq d \;. \tag{5}$$

where $\xi_i^\epsilon = \xi_i * \rho_\epsilon$ with ρ_ϵ a smooth space-time mollifier converging to a Dirac delta as $\epsilon \downarrow 0$. The renormalization C^ϵ in principle could diverge as $\epsilon \downarrow 0$ and the finite dimensional family of solutions referenced above is explored by shifting $C^\epsilon \mapsto C + C^\epsilon$ for a fixed constant C.

However our aim is more involved than the local well-posedness described above, we want to obtain a "quotient dynamic" associated to (4) which has as its state space a "1-forms modulo gauge transformations" as described earlier. The Yang-Mills EQFT then should be the the long-time invariant measure of this quotient dynamic.

Two major difficulties appear here:

- the regularization and renormalization procedure appear to destroy the gauge covariance in law of the dynamic (4), this prevents us from associating a quotient dynamic to (4), and
- the solutions to (4) will actually be distributional 1-forms, which makes it far from obvious how to construct a reasonable quotient state space.

We overcome the first problem by identifying a special choice in our finite dimensional family of solutions (i.e., a special value of C) such that has gauge covariance in law is restored in the $\epsilon \downarrow 0$ limit. When $d = 2$ this can be argued by using brute-force techniques, but these same techniques become impractical in the $d = 3$ case – there we instead use the stabilty of our solution theory in small noise limits in order and the gauge covariance of the non-renormalized deterministic dynamics.

Regarding the second point, in $d = 2$ one can use probabilistic arguments to show such a quotienting procedure makes sense for realizations of the two dimensional Gaussian Free Field (GFF). Our local solution theory for (4) coming from the theory of regularity structures tells us that our solutions are perturbations of the GFF by more regular objects, and our quotienting procedure can be extended to such a setting.

In $d = 3$ the GFF itself is too singular for the probabilistic arguments above to apply, so we instead define a weaker notion of gauge equivalence that first evolves/smoothens 1-forms under the deterministic Yang-Mills heat flow ((4) without the noise term ξ) and then asks if these regularized 1-forms are gauge equivalent. This allows us to define a quotienting procedure in a space of distributions that includes realizations of the $d = 3$ GFF and perturbations thereof, such as the solutions to (4). However, actually proving this is quite technically involved since evolving the $d = 3$ GFF under the deterministic Yang-Mills heat flow itself requires probabilistic arguments. We remark that this use of the Yang-Mills heat flow to define a state space for the Yang-Mills measure was independently investigated along similar lines in the recent works [1, 2].

References

[1] Sky Cao, Sourav Chatterjee, *The Yang-Mills heat flow with random distributional initial data*, Preprint 2021, arXiv:2111.10652

[2] Sky Cao, Sourav Chatterjee, *A state space for 3D Euclidean Yang-Mills theories*, Preprint 2021, arXiv:2111.12813

[3] Ajay Chandra, Ilya Chevyrev, Martin Hairer, Hao Shen, *Langevin dynamic for the 2D Yang-Mills measure*, To appear in Publications mathématiques de l'IHÉS. arXiv:2006.04987

[4] Ajay Chandra, Ilya Chevyrev, Martin Hairer, Hao Shen, *Stochastic quantisation of Yang-Mills-Higgs in 3D*, Preprint 2022, arXiv:2201.03487

Reporters: Daniel Eceizabarrena and Guilherme de Lima Feltes

Participants

Dr. Siddhant Agrawal
C/ Nicolás Cabrera, nº13-15 (Campus Cantoblanco)
Instituto de Ciencias Matemáticas (ICMAT-CSIC)
Calle Nicolás Cabrera
28049 Madrid
SPAIN

Dr. Ismael Bailleuil
Département de Mathématiques
Université de Rennes I
35042 Rennes Cedex
FRANCE

Prof. Dr. Valeria Banica
Laboratoire Jacques-Louis Lions
Sorbonne Université
4, Place Jussieu
75005 Paris
FRANCE

Prof. Dr. Massimiliano Berti
S.I.S.S.A.
Via Bonomea 265
34136 Trieste
ITALY

Dr. Bjoern Bringmann
School of Mathematics
Institute for Advanced Study
1 Einstein Dr
Princeton NJ 08540
UNITED STATES

Dr. Yvain Bruned
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
Peter Guthrie Tait Road
EH9 3FD Edinburgh
UNITED KINGDOM

Nicolas Camps
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 425
91405 Orsay Cedex
FRANCE

Prof. Dr. Ana Patrícia Carvalho Goncalves
Instituto Superior Técnico
Universidade de Lisboa
Avenida Rovisco Pais, 1
Lisboa 1049-001
PORTUGAL

Dr. Ajay Chandra
Imperial College London
Department of Mathematics
SW19 2LZ London
UNITED KINGDOM

Dr. Ilya Chevyrev
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Guilherme de Lima Feltes
Dept. of Mathematical Sciences
University of Bath
Bath BA2 7AY
UNITED KINGDOM

Dr. Yu Deng
Department of Mathematics
University of Southern California
3620 South Vermont Ave., KAP 104
Los Angeles, CA 90089-2532
UNITED STATES

Dr. Daniel Eceizabarrena
Department of Mathematics and Statistics
University of Massachusetts Amherst
Amherst, MA 01003-9305
UNITED STATES

Chenjie Fan
Academy of Mathematics
and Systems Science, CAS
Beijing 100080
CHINA

Dr. Louise Gassot
Departement Mathematik und Informatik
Universität Basel
Spiegelgasse 1
4051 Basel
SWITZERLAND

Dr. Ricardo Grande
ENS Paris
45, rue d'Ulm
75005 Paris
FRANCE

Prof. Dr. Massimiliano Gubinelli
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Trishen Gunaratnam
Université de Genève
Section de Mathématiques
UNI DUFOUR
24, Rue du Général Dufour
Case Postale 64
1211 Genève 4
SWITZERLAND

Dr. Zaher Hani
Department of Mathematics
University of Michigan
Ann Arbor 48109
UNITED STATES

Amirali Hannani
ICERM, Brwon University
121 Main St.
02903 Providence
UNITED STATES

Dr. Masato Hoshino
Graduate School of Engineering Science
Osaka University
Machikaneyama 1-3, Toyonaka
Osaka 560-8531
JAPAN

Prof. Dr. Alexandru D. Ionescu
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Dean Katsaros
Department of Mathematics
University of Massachusetts
Amherst, MA 01003-9305
UNITED STATES

Prof. Dr. Joachim Krieger
Section de Mathématiques
École Polytechnique Fédérale de Lausanne
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Yao Li
Department of Mathematics and Statistics
University of Massachusetts
710 N Pleasant St
Amherst, MA 01003-9305
UNITED STATES

Dr. Renato Lucà
BCAM
Basque Center for Applied Mathematics
Mazarredo Zumarkalea, 14
48009 Bilbao, Bizkaia
SPAIN

Dr. Jonas Lührmann
Department of Mathematics
Texas A&M University
College Station TX 77843-3368
UNITED STATES

Dr. Alberto Maspero
SISSA
International School for Advanced Studies
via Bonomea 265
34136 Trieste
ITALY

Dr. Shuang Miao
School of Mathematics and Statistics
Wuhan University
299 Ba-Yi Rd
Wuhan 430072
CHINA

Joseph Miller
Department of Mathematics
The University of Texas at Austin
2515 Speedway Stop C1200
Austin, TX 78712-1082
UNITED STATES

Prof. Dr. Andrea R. Nahmod
Department of Mathematics and Statistics
University of Massachusetts Amherst
710 North Pleasant Street
Amherst MA 01003-9305
UNITED STATES

Prof. Dr. Felix Otto
Max-Planck-Institut für Mathematik in den Naturwissenschaften
Inselstraße 22 - 26
04103 Leipzig
GERMANY

Dr. Neel Patel
ICMAT
Instituto de Ciencias Matematicas
Campus de Cantoblanco, UAM
Nicolás Cabrera, nº 13-15
28049 Madrid
SPAIN

Dr. Benoit Pausader
Department of Mathematics
Brown University
151 Thayer Street
02912 Providence
UNITED STATES

Prof. Dr. Natasa Pavlovic
Department of Mathematics
The University of Texas at Austin
2515 Speedway Stop C1200
Austin, TX 78712-1082
UNITED STATES

Dr. Fabio Pusateri
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Dr. Matthew Rosenzweig
Room 2-180
Department of Mathematics
Massachusetts Institute of
Technology
77 Massachusetts Avenue
Cambridge, MA 02139-4307
UNITED STATES

Prof. Dr. Svetlana Roudenko
Department of Mathematics and
Statistics
Florida International University
112000 SW 8th St, DM-430
Miami FL 33199
UNITED STATES

Prof. Dr. Wilhelm Schlag
Department of Mathematics
Yale University
Box 208 283
New Haven, CT 06520
UNITED STATES

Dr. Sohrab Shahshahani
Department of Mathematics
University of Massachusetts
Amherst, MA 01003-9305
UNITED STATES

Changyang Shao
Department of Mathematics
Massachusetts Institute of Technology
77 Massachusetts Avenue
Cambridge MA 02139-4307
UNITED STATES

Prof. Dr. Jalal Shatah
Courant Institute of Mathematical
Sciences
New York University
251, Mercer Street
New York, NY 10012-1110
UNITED STATES

Harprit Singh
Imperial College London
Department of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Vedran Sohinger
Mathematics Institute
University of Warwick
Gibbet Hill Road
Coventry CV4 7AL
UNITED KINGDOM

Prof. Dr. Gigliola Staffilani
Department of Mathematics
Massachusetts Institute of Technology
Room 2-251
77 Massachusetts Avenue
02139 Cambridge MA 02139-4307
UNITED STATES

Dr. Qingtang Su
Dept. of Mathematics, DRB 155
University of Southern California
3620 South Vermont Avenue
Los Angeles, CA 90089-1113
UNITED STATES

Prof. Dr. Catherine Sulem
Department of Mathematics
University of Toronto
40 St George Street
Toronto ON M5S 2E4
CANADA

Dr. Chenmin Sun
LAMA (UMR 8050)
Université Paris-Est Créteil
2 Avenue, Adolphe Chauvin
75005 Cergy-Pontoise
FRANCE

Dr. Leonardo Tolomeo
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Minh-Binh Tran
Department of Mathematics
Southern Methodist University
Clements Hall
Dallas, TX 75275-0156
UNITED STATES

Prof. Dr. Nikolay Tzvetkov
Département de Mathématiques
CY Cergy Paris University
Site Saint-Martin, BP 222
2, avenue Adolphe Chauvin
95302 Cergy-Pontoise
FRANCE

Prof. Dr. Luis Vega
Departamento de Matematicas
Facultad de Ciencias
Universidad del Pais Vasco
Apartado 644
48080 Bilbao, Bizkaia
SPAIN

Prof. Dr. Nicola Visciglia
Dipartimento di Matematica
Universita di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Prof. Dr. Hendrik Weber
Dept. of Mathematical Sciences
University of Bath
Claverton Down
Bath BA2 7AY
UNITED KINGDOM

Prof. Dr. Bobby L.E. Wilson
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Prof. Dr. Sijue Wu
Department of Mathematics
University of Michigan
2074 East Hall
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Dr. Xueying Yu
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle, WA 98195-4350
UNITED STATES

Dr. Haitian Yue
ShanghaiTech
199 Huanke Road, Apt. 702, Unit 6
Shanghai 201210
CHINA

Dr. Hui Zhu
Department of Mathematics
University of Michigan
2074 East Hall
530 Church Street
Ann Arbor MI 48109-1043
UNITED STATES

Prof. Dr. Rongchan Zhu
School of Mathematics & Statistics
Beijing Institute of Technology
5 South Zhongguancun Street
Beijing 100081
CHINA

Prof. Dr. Xiangchan Zhu
Chinese Academy of Science
Beijing 100044
CHINA