Mathematisches Forschungsinstitut Oberwolfach

Algebraic Geometry: Moduli Spaces, Birational Geometry and Derived Aspects

Organized by
Christopher Hacon, Salt Lake City
Daniel Huybrechts, Bonn
Richard Thomas, London
Chenyang Xu, Princeton

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ABSTRACT. The workshop covered recent developments in algebraic geometry in a broad sense with a special emphasis on various moduli spaces. Problems related to mirror symmetry phenomena were discussed in a number of talks as well as singularity theory in the context of the MMP in positive characteristic. Derived categories and algebraic cycles, as well as rationality questions figured prominently in the talks and the discussions.

Introduction by the Organizers

We roughly kept the format of the last workshops with 21 talks of 50 min each. For the first time, we included an after-dinner 'gong show' of 6 min presentations plus 3 min questions in order to give all early career researchers visiting the MFO for the first time the chance to present themselves and their work. This new feature was widely appreciated and has led to more interactions and interesting discussions during the rest of the week. One of the presenters at the gong show was subsequently asked to give a full talk later.

For the 2022 installment 13 female algebraic geometers accepted the invitation (compared to only 9 in 2017). Unfortunately, for various mostly Covid related reasons, not all could attend in the end. Our target of 20-25% female participants was reached, but it needs further effort to stabilize and possibly increase

this percentage. Despite the ongoing effects of the corona pandemic, most of the participants could come to Oberwolfach (even from Japan!) and a few followed the talks online. The hybrid format worked well, but the general feeling among the participants certainly was that the full 'Oberwolfach experience' includes the formal and informal discussions during the breaks and after dinner, the meals, the hikes and the music. Only in this in person format the workshop is likely to lead to stimulating exchanges and possibly to new collaborations.

The mix of people worked well. The topics represented by the participants were quite broad but still sufficiently close to make an immediate interaction possible. Although not intended by the organizers originally, it was remarked by the participants that something of a generation shift has taken place over the last few installments. The percentage of early and middle career researchers was notably higher, which probably was one further reason for the open and stimulating atmosphere.

The variety of subjects covered by the talks was very much appreciated. Some talks served well as general introductions (for example, Popa's talk on various conjectures concerning the Kodaira dimension, in particular the Iitaka conjecture) and others treated specific and concrete problems (for example, Fu's talk on algebraic cycles on GM varieties). The majority of talks (Gross, Liu, Castravet, DeVleming, Blum, Belmans) were related to moduli spaces in a broad sense in one way or the other, e.g. moduli spaces of K3 surfaces, Fano varieties, (log)Calabi–Yau varieties, bundles, complexes, curves. Other topics included foliations (Druel), rationality of hypersurfaces (Ottem), Chow and Brauer groups (Charles), non-archimedean geometry (Mazzon), and quotient singularities (Martin, Tevelev).

Workshop: Algebraic Geometry: Moduli Spaces, Birational Geometry and Derived Aspects

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Abstracts

Compactifications of moduli spaces of K3 surfaces inspired by mirror symmetry

MARK GROSS

(joint work with Paul Hacking, Sean Keel, Bernd Siebert)

This talk constituted an update on the paper [1]. The motivation for this work goes back to a paper of David Morrison [4], which suggested, based on arguments from mirror symmetry, that given a mirror pair Y, X of Calabi-Yau manifolds, one should be able to describe toroidal compactifications of the moduli space of complex structures on X via decompositions of the Kähler cone of Y into a rational polyhedral fan invariant under the action of the automorphism group of Y. This should at least give compactification data in a neighbourhood of the large complex structure limit point of the moduli space of X corresponding to the mirror Y.

In [1], we wish to apply this philosophy and build (partial) compactifications of F_g, the moduli space of polarized K3 surfaces with hyperplane sections genus g, and hence degree $2n$ with $n = g - 1$. We consider the case that n is square-free, because in this case there is, up to equivalence, a unique choice of "large complex structure limit" in the mirror moduli space, hence a unique choice of the basic data needed in our construction. The mirror moduli space is one-dimensional, and this basic data is a one-parameter simple normal crossings family $\mathcal{Y} \to D$ of degenerating K3 surfaces in this mirror moduli space. Here D is a disk centered at this large complex structure limit point. In particular, $\mathcal{Y} \to D$ is a type III degeneration, i.e., $\mathcal{Y}_0$ is a union of rational surfaces with dual intersection complex a two-sphere. Further, the generic fibre is a K3 surface with a rank 19 Picard group.

Given a choice σ of rational polyhedral cone contained in the relatively nef cone $\mathrm{NEF}(\mathcal{Y}/D)$, the general mirror construction of [2, 3] yields a mirror family $\mathcal{X} \to \widehat{X(\sigma)}$, where $\widehat{X(\sigma)}$ denotes the completion of the toric variety $X(\sigma)$ corresponding to the cone σ at the zero-dimensional toric stratum of $X(\sigma)$.

The first task is to prove the birational invariance of the construction, which is phrased as follows. There are many possible choices of relatively minimal models of $\mathcal{Y} \to D$, and, roughly speaking, their nef cones fit together to create the *Mori fan* $\mathrm{Mori}(\mathcal{Y}/D)$. This yields a toric variety $X(\mathrm{Mori}(\mathcal{Y}/D))$, and cones in this fan contained in two different maximal cones corresponding to relatively minimal models $\mathcal{Y}_1, \mathcal{Y}_2 \to D$ describe the flopping contraction which passes between these two models. A priori, the construction of [2, 3] only gives mirror families over formal neighbourhoods of the zero dimensional strata of $X(\mathrm{Mori}(\mathcal{Y}/D))$. We show that these mirror constructions extend in a compatible way over formal neighbourhoods of strata corresponding to such cones. Thus we obtain a well-defined mirror family over a formal neighbourhood of a union of strata of $X(\mathrm{Mori}(\mathcal{Y}/D))$.

The second task is to reduce the dimension of the family. In fact the toric variety $X(\mathrm{Mori}(\mathcal{Y}/D))$ is $19 + g$ dimensional, whereas F_g is 19 dimenisional. We use the

birational geometry of $\mathcal{Y} \to D$ and the Mori fan to obtain the data necessary to describe a toroidal compactification of F_g, and then, using the family built in the first step, obtain a family over a formal completion of (part of) the boundary of the corresponding toroidal compactification of F_g.

The third and final task is to show that the family constructed over this formal completion then glues to the universal family over F_g (viewed as a stack). This involves a calculation of the periods of our family, which is accomplished using [5].

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Moduli of log Calabi-Yau pairs

Yuchen Liu

(joint work with Kenneth Ascher, Dori Bejleri, Harold Blum, Kristin DeVleming, Giovanni Inchiostro, Xiaowei Wang)

This talk is a report on an ongoing work joint with K. Ascher, D. Bejleri, H. Blum, K. DeVleming, G. Inchiostro, and X. Wang.

According to the Minimal Model Program, there are three classes of building blocks for all varieties: canonically polarized varieties, Fano varieties, and Calabi-Yau varieties[1]. Thus constructing and studying moduli spaces for each of these three classes of varieties are important problems. For canonically polarized varieties and Fano varieties, the theories of Kollár-Shepherd-Barron stability and K-stability have been successful in constructing their projective moduli spaces, respectively. However, the case of moduli for Calabi-Yau varieties remains less well understood.

Instead of studying moduli of Calabi-Yau varieties directly, we focus on moduli of log Calabi-Yau pairs that are Fano varieties with complements, i.e. pairs (X, D) consisting of a Fano variety X and an effective $\mathbb{Q}$-divisor $D \sim_{\mathbb{Q}} -K_X$ with mild singularities. Many Calabi-Yau varieties are naturally associated to log Calabi-Yau pairs. For instance, an elliptic curve and a general quartic K3 surface are anti-canonical divisors in $\mathbb{P}^2$ and $\mathbb{P}^3$ respectively; a general degree 2 K3 surface is a double cover of the log Calabi-Yau pair $(\mathbb{P}^2, \frac{1}{2}C_6)$ where the branch divisor C_6 is a smooth plane sextic curve. Thus studying the moduli problem for log Calabi-Yau pairs has direct applications to moduli of Calabi-Yau varieties.

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[1] Here we mean K-trivial varieties, including abelian varieties and hyperkähler varieties.

In this report, we discuss a new approach for the moduli compactification of log Calabi-Yau pairs $(\mathbb{P}^2, \frac{3}{d}C_d)$ where C_d is a plane curve of degree $d \geq 3$. For $0 < \epsilon \ll 1$, let P_d^{H} be the KSBA moduli space that compactifies $(\mathbb{P}^2, (\frac{3}{d}+\epsilon)C_d)$ studied by Hacking in [5]. Let P_d^{K} be the K-moduli space compactifies $(\mathbb{P}^2, (\frac{3}{d} - \epsilon)C_d)$ studied in [3]. Let $\mathcal{P}_d^{\mathrm{H}}$ and $\mathcal{P}_d^{\mathrm{K}}$ be the moduli stacks corresponding to P_d^{H} and P_d^{K} respectively. We want to construct a moduli space of log Calabi-Yau pairs that bridges the KSBA-moduli space and the K-moduli space.

Definition 1. We define the moduli stack $\mathcal{P}_d^{\mathrm{CY}}$ whose $\mathbb{C}$-points correspond to pairs (X, D) such that

- (X, D) is a log Calabi-Yau pair, i.e. $K_X + D \sim_{\mathbb{Q}} 0$;
- X is a Fano variety, i.e. $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor;
- (X, D) has semi-log-canonical (slc) singularities;
- (X, D) admits a $\mathbb{Q}$-Gorenstein smoothing to $(\mathbb{P}^2, \frac{3}{d}C_d)$.

It is not hard to show that $\mathcal{P}_d^{\mathrm{CY}}$ is an Artin stack locally of finite type with affine diagonal over $\mathbb{C}$. Moreover, as both KSBA stability and K-semistability imply slc singularities, there are open immersions of stacks

$$\mathcal{P}_d^{\mathrm{H}} \hookrightarrow \mathcal{P}_d^{\mathrm{CY}} \hookleftarrow \mathcal{P}_d^{\mathrm{K}}.$$

In order to construct the moduli space P_d^{CY}, we need to identify points in the stack $\mathcal{P}_d^{\mathrm{CY}}$ by S-equivalence. Here two log Calabi-Yau pairs (X_1, D_1) and (X_2, D_2) are called S-equivalent if they admit a common degeneration in the sense of test configurations.

Theorem 2. *There exists a projective scheme P_d^{CY} that parametrizes S-equivalence classes of $\mathcal{P}_d^{\mathrm{CY}}$. Moreover, we have a diagram*

$$P_d^{\mathrm{H}} \to P_d^{\mathrm{CY}} \leftarrow P_d^{\mathrm{K}},$$

and the Hodge line bundle on P_d^{CY} is ample.

Here the Hodge line bundle of a log Calabi-Yau fibration $\pi : (\mathcal{X}, \mathcal{X}) \to B$ is the $\mathbb{Q}$-line bundle λ on B such that $K_{\mathcal{X}/B} + \mathcal{X} = \pi^*\lambda$. One can show that the Hodge line bundle on the stack $\mathcal{P}_d^{\mathrm{CY}}$ for its universal family descends to P_d^{CY}.

The idea of proving the existence of P_d^{CY} goes as follows. First of all, we prove that the stack $\mathcal{P}_d^{\mathrm{CY}}$ satisfies Θ-reductivity and S-completeness, two valuative criteria over surfaces introduced in [1]. Next, we show that $\mathcal{P}_d^{\mathrm{CY}}$ is bounded if 3 does not divide d. This implies the existence of P_d^{CY} as the good moduli space of $\mathcal{P}_d^{\mathrm{CY}}$ by [1]. The difficult case is when $3 \mid d$, where P_d^{CY} is never bounded as it contains $(\mathbb{P}(a^2, b^2, c^2), (xyz = 0))$ for all (infinitely many) Markov triples (a, b, c). To resolve this issue, we consider the bounded open substack

$$\mathcal{P}_d^{\mathrm{CY}, m} := \{(X, D) \in \mathcal{P}_d^{\mathrm{CY}} \mid \text{the Cartier index of } K_X \text{ is at most } m\}.$$

Then we show that $\mathcal{P}_d^{\mathrm{CY}, m}$ admits a good moduli space for every m. Moreover, these good moduli spaces stablize as $m \to \infty$, which implies the existence of P_d^{CY}.

The ampleness of the Hodge line bundle follows from a detailed analysis of S-equivalence classes in $\mathcal{P}_d^{\mathrm{CY}}$ and a positivity result of Ambro [2]. Finally we remark

that Theorem 2 together with [4] confirms the **b**-semiampleness Conjecture of Prokhorov-Shokurov when the geometric generic fiber is a rational surface.

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Exceptional collections on moduli spaces of stable rational curves

ANA-MARIA CASTRAVET

(joint work with Jenia Tevelev)

Let $\overline{M}_{0,n}$ be the Deligne-Mumford-Knudsen moduli space of stable, genus 0 curves with n ordered, marked points. The symmetric group S_n acts on $\overline{M}_{0,n}$ by permuting the marked points. The main result is the following:

Theorem. [3, 4, 5] *The derived category of the moduli space $\overline{M}_{0,n}$, of stable, rational curves with n marked points, admits a full, exceptional collection which is invariant under the action of the symmetric group S_n.*

This answers affirmatively a question of Orlov and Manin. Here is an example:

Example. The derived category of $\overline{M}_{0,5}$ admits the following full, exceptional collection which is invariant under the action of the symmetric group S_5:

$$\mathcal{O}, \quad \{\pi_i^*\mathcal{O}(1)\}_{i=1,2,3,4,5}, \quad \Omega_{\overline{M}_{0,5}}(\log).$$

Here $\Omega_{\overline{M}_{0,5}}(\log)$ is the sheaf of log-differentials with poles along boundary, while

$$\pi_i : \overline{M}_{0,5} \to \overline{M}_{0,4} = \mathbb{P}^1$$

is the map that forgets the i-th marked point. The 5 line bundles $\{\pi_i^*\mathcal{O}(1)\}_{i=1,2,3,4,5}$ are permuted by S_5, while $\Omega_{\overline{M}_{0,5}}(\log)$ is a rank 2 vector bundle which is invariant under the action of S_5.

A consequence of the Theorem is the following:

Corollary. *The Grothendieck group $K(\overline{M}_{0,n})$ is an S_n-permutation lattice. In particular, the cohomology $H^*(\overline{M}_{0,n})$ has a basis which is permuted by S_n.*

The character of the S_n-representation given by the cohomology $H^*(\overline{M}_{0,n}; \mathbb{Q})$ has been previously studied by Getzler [6] and Bergstrom-Minabe [1]. The result on the cohomology $H^*(\overline{M}_{0,n}; \mathbb{Q})$ being a permutation representation was not

known previously. A very recent article of Choi, Kiem and Lee [2] gives a new proof using different ideas.

An ordered sequence of objects $E_1, \ldots, E_r$ in the bounded derived category $D^b(X)$ of a smooth projective variety X over the complex numbers, is called *exceptional* if $R\operatorname{Hom}(E_i, E_i) = \mathbb{C}$ for all i and $R\operatorname{Hom}(E_i, E_j) = 0$ for all $i > j$. An exceptional collection is called *full* if the smallest full triangulated subcategory containing all E_i is equivalent to $D^b(X)$. If a full, exceptional collection exists, then the K-group of X with integer coefficients is freely generated by the classes of $E_1, \ldots, E_r$. A classical result of Beillinson is that $\mathcal{O}, \mathcal{O}(1), \mathcal{O}(2), \ldots, \mathcal{O}(n)$ is a full, exceptional collection on $\mathbb{P}^n$.

By a theorem of Kapranov, for each $i \in \{1, \ldots, n\}$, the tautological line bundles $\psi_i{}^1$ induce birational morphisms $\overline{M}_{0,n} \to \mathbb{P}^{n-3}$ that are iterated blow-ups of $(n-1)$ points in linearly general position, proper transforms of all $\binom{n-1}{2}$ lines spanned by any two points, all $\binom{n-1}{3}$ planes spanned by any three points, etc. For example, for $\overline{M}_{0,5}$, the Kapranov maps give all the different ways of identifying a del Pezzo surface of degree 5 as a blow-up of $\mathbb{P}^2$ at four points.

A theorem of Orlov gives explicit full exceptional collections on smooth blow-ups, if one has full exceptional collections on the base and the center of the blow-up. Hence, Kapranov's blow-up description of $\overline{M}_{0,n}$ – combined with Orlov and Beilinson's theorems – gives full, exceptional collections on $\overline{M}_{0,n}$. However, the Kapranov blow-up map is not S_n-equivariant (only S_{n-1}-equivariant) and therefore, the exceptional collections obtained in this way are not S_n-invariant.

An alternative approach is inspired by the work of Bergrstrom-Minabe [1]. There exists a tower of S_n-equivariant, birational morphisms between moduli spaces of *weighted*, stable, rational curves with n markings:

$$\overline{M}_{0,n} = \overline{M}_{(1,\ldots,1)} \to \overline{M}_{(\frac{1}{2},\ldots,\frac{1}{2})} \to \ldots \to \overline{M}_{(\frac{1}{r},\ldots,\frac{1}{r})}$$

Here for $\mathcal{A} = (a_1, \ldots, a_n)$, with $a_i \in \mathbb{Q}$, $\sum a_i > 2$, we denote by $\overline{M}_{\mathcal{A}}$ the moduli space of $\mathcal{A}$-stable rational curves [7], parametrizing data $(C, p_1, \ldots, p_n)$ such that

- C is a tree of projective lines, $p_1, \ldots, p_n$ smooth points on C,
- The log-canonical $\mathbb{Q}$-line bundle $\omega_C(a_1 p_1 + \ldots + a_n p_n)$ is ample, and
- If $\{p_i\}_{i \in I}$ coincide, then $\sum_{i \in I} a_i \leq 1$.

At each step in the tower, we blow-up loci isomorphic to the moduli spaces $\overline{M}_{(1,\ldots,1,b\ldots,b)}$ (with $p \geq 2$ weights of 1 and $q \geq 0$ weights of b, for some $0 < b < 1$). The blown-up loci intersect transversely. The group $S_p \times S_q$ acts on the moduli spaces $\overline{M}_{(1,\ldots,1,b\ldots,b)}$ by permuting the two sets of markings of equal weight. Using an enhanced version of Orlov's theorem, we first prove that it suffices to find full, invariant, exceptional collections on the following types of Hassett moduli spaces:

(A) Losev-Manin moduli spaces $\overline{M}_{(1,1,\epsilon\ldots,\epsilon)}$, where there are m weights $0 < \epsilon \ll 1$ (for all $m \geq 2$, construct $S_2 \times S_m$-invariant full, exceptional collections),

(B) The spaces $\overline{M}_{p,q} := \overline{M}_{(a+\epsilon,\ldots,a+\epsilon,\eta\ldots,\eta)}$, where there are p weights $a+\epsilon$ and q weights η, subject to the condition that $pa + q\eta = 2$ and $0 < \epsilon, \eta \ll 1$

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[1]The fiber of ψ_i at $(C, p_1, \ldots, p_n) \in \overline{M}_{0,n}$ is the cotangent bundle $(T_{p_i} C)^\vee$.

(for all $p \geq 2$, all $q \geq 0$, construct $S_p \times S_q$-invariant full, exceptional collections).

The difficult part is to construct these collections; for (A) we did this in [3] and for (B) in [4, 5]. In the talk I discussed how do the exceptional collections on the spaces $\overline{M}_{p,q}$ look like, which is the most involved part in the whole project.

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Stable pairs on elliptically fibered threefolds

GEORG OBERDIECK

(joint work with Maximilian Schimpf)

1. PANDHARIPANDE-THOMAS THEORY

Let X be a smooth projective threefold. A *stable pair* (F, s) on X consists of

- a pure 1-dimensional sheaf F,
- a section $s \in H^0(X, F)$ with zero-dimensional cokernel.

Let $P_{n,\beta}(X)$ be the fine projective moduli space of stable pairs with numerical data

$$\mathrm{ch}_2(F) = \beta \in H_2(X, \mathbb{Z}), \quad \chi(F) = n \in \mathbb{Z}.$$

The moduli space carries naturally the descendent classes

$$\mathrm{ch}_k(\gamma) = \pi_{P*}(\mathrm{ch}_k(\mathcal{O} - \mathbb{F}) \cup \pi_X^*(\gamma)), \quad k \geq 0, \quad \gamma \in H^*(S, \mathbb{Q}),$$

where π_P, π_X are the projections of $P_{n,\beta}(X) \times X$ to the factors and $\mathcal{O} \to \mathbb{F}$ is the universal stable pair. Pandharipande-Thomas invariants of X are defined by integrating these classes over the virtual fundamental class of the moduli space:

$$\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_n}(\gamma_n) \rangle_{n,\beta}^X = \int_{[P_{n,\beta}(X)]^{\mathrm{vir}}} \prod_i \mathrm{ch}_{k_i}(\gamma_i).$$

The invariants reflect the rich enumerative geometry of algebraic curves on the threefolds and have relationships to Donaldson-Thomas invariants, Gromov-Witten

invariants. We refer to [5] for an introduction to these invariants, and an overview over the central role they play in the field of enumerative geometry.

2. Elliptic threefolds

Let B be a smooth projective surface. We assume that the threefold X admits an elliptic fibration

$$\pi : X \to B,$$

by which we mean a a flat morphism with ω_π trivial on all fibers, such that:

- π is equipped with a section $\iota : B \to X$,
- $\pi : X \to B$ is a Weierstraß model.

In this talk we are interested in understanding how the Pandharipande-Thomas invariants of X are constrained by the existence of this elliptic fibrations. For example, can we find connections to modular forms in the invariants, which reflect the existence of non-trivial derived auto-equivalences coming from the relative Poincaré bundle of the fibration $\pi : X \to B$? For elliptic Calabi-Yau threefold this has been intensively studied in the physics literature and culminated in a precise conjecture by Huang-Katz-Klemm that the natural generating series of Pandharipande-Thomas invariants of X are meromorphic Jacobi forms, see [1]. On the mathematical side this has been studied and partially proven in [3, 4]. In particular, in [3] the connection to derived auto-equivalences was made.

In this talk I have described recent joint work with Maximilian Schimpf in which we go beyond the Calabi-Yau case, and allow the total space X to be non-Calabi-Yau. Before stating the main conjectures we need to define the relevant generating series: Consider the divisor class

$$W = [\iota(B)] - \frac{1}{2}\pi^*(c_1(N_{B/X})) \ \in H^2(X).$$

For any fixed class $\beta \in H_2(B, \mathbb{Z})$ we define the generating series:

$$\langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_n}(\gamma_n) \rangle_\beta^X = \sum_{\substack{\widetilde{\beta} \in H_2(X,\mathbb{Z}) \\ \pi_* \widetilde{\beta} = \beta}} \sum_{m \in \frac{1}{2}\mathbb{Z}} i^{2m} p^m q^{W \cdot \widetilde{\beta}} \langle \mathrm{ch}_{k_1}(\gamma_1) \cdots \mathrm{ch}_{k_n}(\gamma_n) \rangle_{m+\frac{1}{2}d_{\widetilde{\beta}}, \widetilde{\beta}}^X$$

where $i = \sqrt{-1}$ and $d_\beta = \int_{\widetilde{\beta}} c_1(T_X)$.

Let QJac be the algebra of quasi-Jacobi forms, see for example [2]. It is bigraded, $\mathsf{QJac} = \oplus_{k,m}\mathsf{QJac}_{k,m}$, by weight k and index m with finite-dimensional summands $\mathsf{QJac}_{k,m}$. The algebra QJac admits an embedding into the free polynomial ring of two generators A and G_2 over the algebra of Jacobi forms,

$$\mathsf{QJac} \subset \mathsf{Jac}[A, G_2].$$

Our main conjectures are the following:

Conjecture 1 (O.-Schimpf). *We have*

$$\langle 1 \rangle_0^X = \prod_{m \geq 1} (1 - q^m)^{-e(B) - c_1(N) \cdot (c_1(T_B) + c_1(N))} \prod_{\ell, m \geq 1} (1 - p^\ell q^m)^{-\ell \cdot c_3(T_X \otimes \omega_X)}$$

where $N = N_{B/X}$ is the normal bundle of the section.

Define the normalized series

$$Z_\beta\left(\mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_n}(\gamma_n)\right) := \frac{\langle\mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_n}(\gamma_n)\rangle^X_\beta}{\langle 1\rangle^X_0}$$

Conjecture 2 (O.-Schimpf). *Assume that $c_3(T_X\otimes\omega_X)=0$. Then the normalized series $Z_\beta(\widetilde{\mathrm{ch}}_{k_1}(\gamma_1)\cdots\widetilde{\mathrm{ch}}_{k_n}(\gamma_n))$ is a meromorphic quasi-Jacobi form of*

$$\text{weight}\quad K_X\cdot\iota_*\beta+\sum_{i=1}^n(k_i-1+\mathrm{wt}(\gamma_i))$$

$$\text{index}\quad \frac{1}{2}\beta\cdot(\beta+c_1(N_{B/X})).$$

It is of the form

$$Z_\beta\left(\mathrm{ch}_{k_1}(\gamma_1)\cdots\mathrm{ch}_{k_n}(\gamma_n)\right)=\Delta(q)^{\frac{1}{2}c_1(N)\cdot\beta}\sum_{\alpha=(\beta_1,\ldots,\beta_\ell)}\frac{\varphi_\alpha(p,q)}{\prod_{i=1}^\ell\Theta(p^{\mathrm{div}(\beta_i)},q)^2}$$

where α runs over all decompositions $\beta=\beta_1+\ldots\beta_k$ into effective classes $\beta_i\in H_2(B,\mathbb{Z})$ which are of divisibility $\mathrm{div}(\beta_i)$ in $H_2(B,\mathbb{Z})$, and $\varphi_\alpha\in\mathsf{QJac}$. Moreover,

- $\widetilde{\mathrm{ch}}_k(\gamma)=\mathrm{ch}_k(\gamma)+\frac{1}{24}\mathrm{ch}_{k-2}(\gamma\cdot c_2(T_X))$
- $\mathrm{wt}=$ *grading by eigenvalues of $[W,\pi^*\pi_*]:H^*(X)\to H^*(X)$,*
- $\Delta(q)=q\prod_{n\ge 1}(1-q^n)^{24}$.

Conjecture 3 (O.-Schimpf). *Assume again $c_3(T_X\otimes\omega_X)=0$ and the previous conjecture. Then we have the holomorphic anomaly equations:*

$$\frac{d}{dA}Z_\beta\left(\widetilde{\mathrm{ch}}_{k_1}(\gamma_1)\cdots\widetilde{\mathrm{ch}}_{k_n}(\gamma_n)\right)=\sum_{i=1}^n Z_\beta\left(\cdots\widetilde{\mathrm{ch}}_{k_i-1}(\gamma_i\Delta_{B,1})\widetilde{\mathrm{ch}}_2(\Delta_{B,2})\cdots\right)$$

$$+\sum_{i=1}^n Z_\beta\left(\cdots\widetilde{\mathrm{ch}}_{k_i+1}(\pi^*\pi_*(\gamma_i))\cdots\right).$$

$$\frac{d}{dG_2}Z_\beta\left(\widetilde{\mathrm{ch}}_{k_1}(\gamma_1)\cdots\widetilde{\mathrm{ch}}_{k_n}(\gamma_n)\right)_\beta=$$

$$-2\sum_{i<j}Z_\beta\left(\cdots\widetilde{\mathrm{ch}}_{k_i-1}(\gamma_i\Delta_{B,1})\cdots\widetilde{\mathrm{ch}}_{k_j-1}(\gamma_j\Delta_{B,2})\cdots\right)-\sum_{i=1}^n Z_\beta\left(\cdots\widetilde{\mathrm{ch}}_{k_i-2}(\gamma_i\cdot c_2(B))\cdots\right)$$

$$+\sum_{\substack{i=1,\ldots,n\\m_1+m_2=k_i}}Z_\beta\left(..\frac{(-1)^{(1+\mathrm{wt}^L)(1+\mathrm{wt}^R)}(m_1-1+\mathrm{wt}^L)!(m_2-1+\mathrm{wt}^R)!}{(k_i-2+\mathrm{wt}(\gamma_i))!}\widetilde{\mathrm{ch}}_{m_1}\widetilde{\mathrm{ch}}_{m_2}(\mathcal{E}(K_X\gamma_i))..\right)$$

$$-2\sum_{i<j}\sigma_{ij}\binom{k_i+k_j-4+\mathrm{wt}(\gamma_i)+\mathrm{wt}(\gamma_j)}{k_i-2+\mathrm{wt}(\gamma_i),\ k_j-2+\mathrm{wt}(\gamma_j)}Z_\beta\left(\widetilde{\mathrm{ch}}_{k_i+k_j-2}(\mathcal{E}(\gamma_i\boxtimes\gamma_j))\prod_{\ell\ne i,j}\widetilde{\mathrm{ch}}_{k_\ell}(\gamma_\ell)\right)$$

where σ_{ij} is the sign obtained by permuting the i-th and j-th entry of $(\gamma_1,\ldots,\gamma_n)$ to the left-most position, and $\mathcal{E}\in H^*(X\times X\times X)$ is defined by

$$\mathcal{E}:=\Delta_{X,12}\cdot\pi^*\Delta_{B,13}+\Delta_{X,13}\cdot\pi^*\Delta_{B,12}+\Delta_{X,23}\cdot\pi^*\Delta_{B,12}$$

$$-\pi^*\Delta_{B,123}\left(\mathrm{pr}_1^*W+\mathrm{pr}_2^*W+\mathrm{pr}_3^*W\right)$$

For elliptic Calabi-Yau threefolds these conjectures specialize precisely to the predictions made by Huang-Katz-Klemm using physics methods [1]. We also recover some known cases in non-Calabi-Yau situations. For once, a relative version of the above conjectures in the case $K3 \times \mathbb{P}^1$ relative to fibers was first studied and proven in [2]. Further evidence was found for $\mathbb{P}^2 \times E$ in [6]. Our conjectures serve also as a (so far conjecturally) powerful tool to compute Pandharipande-Thomas invariants in elliptic geometries. In particular, this will lead to new explicit conjectures about Pandharipande-Thomas invariants for K3 geometries in forthcoming work.

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Categorical MacMahon formula

Yukinobu Toda

(joint work with Tudor Pădurariu)

For a variety X, the Hilbert scheme of points $\mathrm{Hilb}(X, d)$ parametrizes zero-dimensional closed subschemes in X with length d. The derived category of $\mathrm{Hilb}(X, d)$ has been also extensively studied when $\dim X \leq 2$. In the case of $X = \mathbb{C}^3$, the Hilbert schemes $\mathrm{Hilb}(\mathbb{C}^3, d)$ define DT invariants $\mathrm{DT}(d) \in \mathbb{Z}$ counting zero-dimensional closed subschemes in $\mathbb{C}^3$. It is well-known that the generating series of $\mathrm{DT}(d)$ is equal to $M(-q)$, where $M(q)$ is the series

$$M(q) = \prod_{k \geq 1} \frac{1}{(1 - q^k)^k} = 1 + q + 3q^2 + 6q^3 + 13q^4 + 24q^5 + \cdots .$$

The series $M(q)$ is called *MacMahon function*, whose coefficient $p_3(d)$ of q^d is the number of 3D partitions of d. The purpose of this talk is to give a categorical and K-theoretic analogue of the MacMahon formula for $\mathrm{Hilb}(\mathbb{C}^3, d)$, based on the joint work with Tudor Pădurariu [1].

In general $\mathrm{Hilb}(\mathbb{C}^3, d)$ is highly singular and the usual derived categories are not well-behaved. Instead we use the following critical locus description of $\mathrm{Hilb}(\mathbb{C}^3, d)$.

Let NHilb(d) be the non-singular quasi-projective variety, called the *non-commutative Hilbert scheme*, defined by

$$\text{NHilb}(d) := \left(V \oplus \text{Hom}(V, V)^{\oplus 3} \right)^{\text{ss}} / GL(V),$$

where V is a d-dimensional vector space and $(-)^{\text{ss}}$ is the GIT semistable locus consisting of (v, X, Y, Z) such that V is generated by v under the action of (X, Y, Z). Then $\text{Hilb}(\mathbb{C}^3, d)$ is the critical locus of the regular function

$$W \colon \text{NHilb}(d) \to \mathbb{C}, \ (v, X, Y, Z) \mapsto \text{Tr}(Z[X, Y]).$$

The *DT category* is defined by

$$\mathcal{DT}(d) := \text{MF}(\text{NHilb}(d), W),$$

where the right hand side is the category of matrix factorizations on $\text{NHilb}(d)$ with respect to the function W. A dg-enhancement of the DT category recovers the DT invariant $\text{DT}(d)$ up to sign by taking the Euler characteristic of its periodic cyclic homology.

The first main result is the existence of semi-orthogonal decompositions on DT categories of the form:

$$(1) \qquad \mathcal{DT}(d) = \left\langle \otimes_{i=1}^{k} \mathbb{S}(d_i)_{w_i} : 0 \leq \frac{v_1}{d_1} < \cdots < \frac{v_k}{d_k} < 1 \right\rangle,$$

where the right hand side consists of tuplets $A = (d_i, w_i)_{i=1}^{k}$ satisfying $d_1 + \mathcal{X}ots + d_k = d$, and v_i is defined by

$$v_i = w_i + d_i \left(\sum_{j > i} d_j - \sum_{j < i} d_j \right).$$

For $(d, w) \in \mathbb{N} \times \mathbb{Z}$, the category $\mathbb{S}(d)_w$ is defined to be a certain triangulated subcategory

$$\mathbb{S}(d)_w \subset \text{MF}(\mathcal{X}(d), W), \ \mathcal{X}(d) = \left[\text{Hom}(V, V)^{\oplus 3} / GL(V) \right]$$

consisting of matrix factorizations with factors coherent sheaves on $\mathcal{X}(d)$ whose weights with respect to the maximal torus of $GL(d)$ are contained in a certain polytope in the weight lattice. The subcategory similar to $\mathbb{S}(d)_w$ first appeared in the work of Špenko-Van den Bergh in their construction of non-commutative crepant resolutions. The semi-orthogonal decomposition in (1) is regarded as a categorification of wall-crossing formula of DT invariants, and the category $\mathbb{S}(d)_w$ may be regarded as a categorification of BPS sheaves by Davison-Meinhardt (though not precise). So we call $\mathbb{S}(d)_w$ as *quasi-BPS category*.

The second main result is a construction of objects in quasi-BPS categories which, together with (1), give a basis of torus localized K-theory of the DT category. Here the torus T is the subtorus of $(\mathbb{C}^*)^3$

$$T = \{ (t_1, t_2, t_3) \in (\mathbb{C}^*)^{\times 3} \mid t_1 t_2 t_3 = 1 \}.$$

It acts on $\text{NHilb}(d)$ in a natural way which preserves the function W. Let $\mathcal{C}(d)$ be the derived stack of commuting matrices of size d, or equivalently the derived

moduli stack of zero-dimensional coherent sheaves on $\mathbb{C}^2$ with length d. The Koszul duality equivalence says that there is an equivalence

$$\Phi \colon D^b(\mathcal{C}(d)) \xrightarrow{\sim} \mathrm{MF}^{\mathrm{gr}}(\mathcal{X}(d), W).$$

Here the grading on the right hand side is given by the weight two $\mathbb{C}^*$-action on the last factor of $\mathrm{Hom}(V, V)$. Then there is a subcategory $\mathbb{T}(d)_w \subset D^b(\mathcal{C}(d))$ corresponding to $\mathbb{S}^{\mathrm{gr}}(d)_w$ under the above equivalence. For each $(d, v) \in \mathbb{N} \times \mathbb{Z}$, we can construct an object $\mathcal{E}_{d,v} \in \mathbb{T}(d)_v$ explicitly, using the derived stack of filtrations of zero-dimensional coherent sheaves on $\mathbb{C}^2$. It turns out that, up to a constant in $\mathbb{K} := K(BT) = \mathbb{Z}[q_1^{\pm 1}, q_2^{\pm 1}]$, the images of the above objects under a natural map

$$\iota_* \colon G_T(\mathcal{C}(d)) \to \mathbb{K}[z_1^{\pm 1}, \ldots, z_d^{\pm 1}]^{\mathfrak{S}_d}$$

are part of a basis of the Elliptic Hall algebra considered by Negut. Using the result of Negut, we can show the following: let $(d, v) \in \mathbb{N} \times \mathbb{Z}$ be coprime integers, let $n \in \mathbb{N}$, and set $\mathbb{F}$ to be the fraction field of $\mathbb{K}$. The $\mathbb{F}$-vector space $K_T(\mathbb{T}(nd)_{nv}) \otimes_{\mathbb{K}} \mathbb{F}$ has a basis $\mathcal{E}_{n_1 d, n_1 v} * \cdots * \mathcal{E}_{n_k d, n_k v}$, where $n_1, \ldots, n_k \geq 1$ and $\sum_{i=1}^k n_i = n$. Here $*$ is the categorical Hall product on $D^b(\mathcal{C}(d))$.

Let $\mathcal{F}_{d,w} \in \mathbb{S}_{d,w}$ be the object corresponding to $\mathcal{E}_{d,w}$ under the Koszul duality equivalence and forgetting the grading. By combining with (1), we obtain the following basis of the torus localized K-theory of the DT category:

$$(2) \qquad K(\mathcal{DT}_T(d)) \otimes_{\mathbb{K}} \mathbb{F} = \bigoplus_{\substack{d_1 + \cdots + d_k = d \\ 0 \leq v_1/d_1 \leq \cdots \leq v_k/d_k < 1}} \mathbb{F} \cdot [\mathcal{F}_{d_1, w_1}] * \cdots * [\mathcal{F}_{d_k, w_k}]$$

where tuples $(d_i, w_i)_{i=1}^k$ in the right hand side are unordered for subtuplets $(d_i, w_i)_{i=a}^b$ with $v_a/d_a = \cdots = v_b/d_b$. In particular, the dimension of the left hand side in (2) is $p_3(d)$, giving the K-theoretic MacMahon formula.

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Conjectures on the Kodaira dimension
Mihnea Popa

The talk was devoted to a series of "superadditivity" conjectures on the behavior of the Kodaira dimension of fibrations, made recently in [2], and complementing Iitaka's well-known subadditivity conjecture.

Let $f \colon X \to Y$ be an algebraic fiber space (i.e. a surjective morphism with connected fibers), with X and Y smooth and projective complex varieties. Iitaka's Subadditivity (or $C_{n,m}$) conjecture predicts that

$$\kappa(X) \geq \kappa(F) + \kappa(Y),$$

where F denotes the general fiber of f. This conjecture is known in various important cases: when F is a variety of general type (Kollár) or more generally one that admits a good minimal model (Kawamata), when Y is of general type (Viehweg) or a curve (Kawamata), and more recently when Y is an abelian variety (Cao-Păun), a variety of maximal Albanese dimension (Hacon-Popa-Schnell) or a surface (Cao). While it is not currently known in general (for instance when $\dim Y = 3$ or $\kappa(Y) = 0$), it does follow from the conjectures of the Minimal Model Program (MMP) according to a result of Kawamata.

In [2] I made some conjectures to the effect that $\kappa(X)$ can also be bounded from above in terms of natural data associated to f, and moreover that one should always have addtivity in the case of smooth morphisms. These involve the notion of log Kodaira dimension; concretely, if V is a smooth quasi-projective variety and one chooses a smooth projective compactification Z such that the boundary $D = Z \setminus V$ is a simple normal crossing divisor,[1] then the log Kodaira dimension of V is defined as

$$\kappa(V) := \kappa(Z, \omega_Z(D)),$$

i.e. the Iitaka dimension of the line bundle $\omega_Z(D)$. (This is easily seen to be independent of the choice of compactification.)

The main conjecture from [2] regarding fibrations of projective varieties is the following:

Conjecture 1. *Let $f \colon X \to Y$ be an algebraic fiber space, with X and Y smooth projective varieties, and with general fiber F. If $V \subseteq Y$ is the open set over which f is a smooth morphism, then*

$$\kappa(F) + \kappa(V) \geq \kappa(X).$$

Note that when f is smooth we have $V = Y$, hence in combination with Iitaka's conjecture, this becomes

Conjecture 2. *With the same assumptions, if f is smooth, then*

$$\kappa(X) = \kappa(F) + \kappa(Y).$$

This last statement is in fact the prototype for the most general conjecture one can make, which essentially amounts to considering $\kappa(U)$ instead of $\kappa(X)$ in Conjecture 1, where $U = f^{-1}(V)$. More precisely:

Conjecture 3. *Let $f \colon U \to V$ be a smooth projective algebraic fiber space, with U and V smooth quasi-projective varieties, and with general fiber F. Then*

$$\kappa(U) = \kappa(F) + \kappa(V).$$

This implies Conjecture 1, since if $U \subseteq X$ is a compactification, one has $\kappa(U) \geq \kappa(X)$.

While connected to Iitaka's conjecture, these conjectures are also very much related to hyperbolicity in the sense of the well-known Shafarevich-Viehweg conjectures for families of varieties. Therefore the recent proofs of various cases in [1]

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[1]This can always be done by Hironaka's theorem

and [4] rely on Hodge module techniques considered in [3], as well as on previous methods developed by Kovács and Viehweg-Zuo among others, for the study of hyperbolicity. Moreover, when the base Y has Kodaira dimension 0, analytic techniques (in the sense of singular metrics on direct images of pluricanonical bundles, as in the work of Berndtsson and Păun-Takayama) become essential. Another main ingredient is an important theorem of Campana-Păun, detecting varieties of general type.

Besides the obvious cases, some of the results I described in the talk are:

- Conjectures 1 and 3 hold when Y is an abelian variety or a curve, or when V compactifies to an abelian variety; see [1].
- Conjecture 3 holds when the fiber F is of general type; see [4].
- Conjecture 2 holds when X is of general type, and so does Conjecture 1 if one assumes that a compactification Y of V is not uniruled (so in particular when $\kappa(Y) \geq 0$); see [4].

We also showed in [4] that Conjecture 2 is implied by the conjectures of the MMP. During the conference I received a preprint from F. Campana, claiming the same for Conjecture 3.

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A question of Mori and families of plane curves

Kristin DeVleming

(joint work with David Stapleton)

In this talk, we discussed smooth projective limits of complex plane curves. I focused on the following question:

Question. *For which degrees $d > 1$ is every smooth compact limit of complex degree d plane curves a degree d plane curve?*

It is easy to see that d must be prime. For example, a nonhyperelliptic genus 3 curve is a canonically embedded degree 4 plane curve, but the canonical map for a hyperelliptic genus 3 curve gives a double cover of a conic. Similar cyclic cover constructions occur whenever the degree d is composite: it is straightforward to construct a degeneration of a degree ab hypersurface in any dimension to a degree a cover of a degree b hypersurface ([1, Ex. 1.59]) so d being prime is necessary.

In higher dimensions, Mori asked if being prime is also sufficient.

Question. ([2, p. 642]) *If $n \geq 3$, is every smooth projective limit of prime degree hypersurfaces of dimension n in $\mathbb{P}^{n+1}_{\mathbb{C}}$ also a hypersurface in $\mathbb{P}^{n+1}_{\mathbb{C}}$?*

This has been proven for the primes 2 [11, 6, 4], 3 [7], and 5 [3] in all dimensions, and for the prime 7 in dimension 3 [3]. Interestingly, the statement is false if the dimension is 1 or 2. The purpose of this project is to develop and provide evidence for an analogous conjecture in the case of plane curves.

Griffin gave an example [8] of a family of smooth plane quintics such that the limit is hyperelliptic and consequently nonplanar. We generalize this fact by proving that many prime degrees admit non-planar limits:

Theorem A. *For any Markov number $d > 2$, there is family of smooth plane curves of degree d with a smooth projective non-planar limit. In particular, for any Markov number $p > 2$ that is prime there is a smooth family of prime degree p plane curves with a non-planar central fiber.*

Recall that a *Markov number* is a natural number that appears as a solution to the equation:
$$a^2 + b^2 + c^2 = 3abc.$$
The first few Markov numbers are
$$1, 2, 5, 13, 29, 34, 89, ...$$

There are infinitely many Markov numbers, the Markov triples are naturally organized into a binary tree, and every Markov number is obtained from (1,1,1) by repeating a standard mutation process. In this paper, the relation between Markov numbers and plane curves is the following: the only log terminal $\mathbb{Q}$-Gorenstein degenerations of $\mathbb{P}^2$ are the weighted projective spaces $\mathbb{P}(a^2, b^2, c^2)$ where (a, b, c) is a Markov triple, or a partial smoothing of one of these weighted projective spaces. To prove Theorem A, we construct smooth Cartier divisors in such a degeneration and bound the gonality of the curves. Motivated by this construction and general results from moduli of stable pairs compactifying the space of pairs $(\mathbb{P}^2, C)$, we conjecture the following:

Conjecture B. *Any smooth limit of a family of plane curves of prime degree is a Cartier divisor in a log-terminal $\mathbb{Q}$-Gorenstein degeneration of $\mathbb{P}^2$.*

Conjecture B implies the following.

Conjecture C. *Let p be a prime number that is not a Markov number. Any smooth limit of plane curves of degree p is a plane curve.*

As the smooth limit of any family of curves of degree 2 or 3 is also planar, the first primes to verify the conjecture are $p = 5$ and $p = 7$. Our main result is to prove the conjectures in these cases. In the Theorem below, $M(5)$ is a degeneration of $\mathbb{P}^2$ with an isolated $\frac{1}{25}(1, 4)$ singularity.

Theorem D. *Every smooth projective limit of a family of degree 5 plane curves is either planar or a Cartier divisor in $M(5)$. Furthermore, the smooth limits in $M(5)$ are all hyperelliptic.*

Theorem E. *Every smooth projective limit of a degree 7 family of plane curves is a plane curve.*

Verifying Conjectures B and C has strong consequences for the intersection of various loci in M_g, the moduli space of smooth genus $g := g(d)$ curves, where $g(d)$ is the genus of a smooth degree d plane curve. In particular, Conjecture B places bounds on the gonality of the curves that can be in the closure of the locus of planar curves, and Conjecture C implies that for p not a Markov number, the closure of the locus of degree p plane curves is itself, so curves of gonality less than $p-1$ cannot be written as limits of plane curves of degree p. The following Corollaries are consequences of Theorems D and E. The first uses that every hyperelliptic genus 6 curve can be written as a limit of a family of plane quintics.

Corollary. *Inside the moduli space of smooth genus 6 curves M_6, the closure of the plane quintic locus P_5 is $P_5 \cup H_6$, where H_6 is the hyperelliptic locus.*

Corollary. *Inside the moduli space of smooth genus 15 curves M_{15}, the closure of the plane septic locus P_7 is P_7. In particular, a plane septic cannot degenerate to a curve with gonality less than 6.*

We proposed a general approach to the conjecture by using Hacking's work [9] on limits of pairs $(\mathbb{P}^2, C)$. Let $D \to T$ be a smooth family of compact complex curves over a smooth complex curve T. Assume that the general curve D_t is a plane curve of degree d. To this family (after a possible base change of T) there is an associated threefold pair $(X^{\mathrm{CY}}, D^{\mathrm{CY}}) \to T$ that we call Hacking's Calabi-Yau limit; see [9, Defn. 2.4], satisfying:

(1) for a general fiber $t \in T$ we have $X_t^{\mathrm{CY}} \cong \mathbb{P}^2$ and $D_t \cong D_t^{\mathrm{CY}}$, and
(2) over the central fiber $0 \in T$, the fiber X_0^{CY} is a log terminal limit of $\mathbb{P}^2$ such that $(X_0^{\mathrm{CY}}, \frac{3}{d} D_0^{\mathrm{CY}})$ is log canonical.

On the other hand, the pair $(\mathbb{P}^2, D_t)$ has a limit as a KSBA stable pair, i.e. there is a threefold (X_T, D) with a map to T such that the general fiber is $(\mathbb{P}^2, D_t)$ and the central fiber is an slc pair (X_0, D_0). Here D_0 is (as the notation suggests) the original smooth central fiber. There is a MMP that interpolates between (X, D) and $(X^{\mathrm{CY}}, D^{\mathrm{CY}})$. Roughly speaking, to prove Conjecture B it would suffice to prove that nothing happens in this MMP.

While the talk focused primarily on the cases of curves, there is a generalization of Hacking's work to higher dimensional pairs $(\mathbb{P}^n, D)$ in [10]. The general approach above applies in this situation, and the associated Hacking Calabi-Yau limits are log terminal degenerations X_0 of $\mathbb{P}^n$ containing ample divisors D_0. If D_0 is an ample smooth Cartier divisor, X_0 necessarily has isolated singularities. As in the proof of Theorem A where log terminal degenerations of $\mathbb{P}^2$ with isolated singularities are used to construct non-planar limits of prime degree $p > 2$ curves, it is of interest to construct log terminal $\mathbb{Q}$-Gorenstein degenerations X_0 of $\mathbb{P}^n$ with isolated singularities. Certainly, a cone over a Fano hypersurface of degree b in $\mathbb{P}^n$ is such an example, but for a Cartier divisor D_0 on X_0, projection away from the vertex realizes D_0 as a degree a cover of a degree b hypersurface. If $a > 1$, this is the limit of a family of degree ab hypersurfaces, which is not prime, and if $a = 1$,

D_0 is isomorphic to a degree b hypersurface. In particular, cones are not examples that can contain smooth non-hypersurface limits of prime degree hypersurfaces. Therefore, we pose the following question, as answers are potential candidates for constructing non-hypersurface limits of prime degree hypersurfaces:

Question. *Other than cones, what are examples of log terminal $\mathbb{Q}$-Gorenstein degenerations of $\mathbb{P}^n$ with isolated singularities?*

At least one example is known and interesting in the prime degree case: by work of Horikawa [5], there are smooth limits of quintic surfaces in $\mathbb{P}^3$ that do not embed in $\mathbb{P}^3$. However, by [10, Ex. 5.2], these limits do embed in a log terminal degeneration of $\mathbb{P}^3$ with isolated singularities.

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A global Weinstein splitting theorem for holomorphic Poisson manifolds

Stéphane Druel

(joint work with Jorge Vitório Pereira, Brent Pym, Frédéric Touzet)

A holomorphic Poisson structure on a complex manifold X is a bivector field $\pi \in H^0(X, \wedge^2 T_X)$, such that the bracket

$$\{f, g\} := \langle \pi, df \wedge dg \rangle$$

defines a Lie algebra structure on $\mathcal{O}_X$. A Poisson structure defines a skew-symmetric map

$$\pi^\sharp : \Omega^1_X \to T_X;$$

the rank of π at a point $x \in X$ is the rank of $\pi^\sharp(x)$. It is even because $\pi^\sharp$ is skew-symmetric. Every holomorphic function f on some open subset $U \subseteq X$ has a

Hamiltonian vector field $H_f := \pi^\sharp(df)$ which acts as a derivation on holomorphic functions g on U by $H_f(g) = \{f, g\}$. The sheaf $\mathcal{F} := \mathrm{Im}(\pi^\sharp)$ locally generated by Hamiltonian vector fields is involutive and defines a (possibly singular) foliation on X. A leaf of π is a maximal connected and immersed holomorphic submanifold $L \subseteq X$ such that

$$T_L = \mathrm{Im}(\pi^\sharp|_L) \subseteq T_X|_L.$$

Then $\pi|_L \in H^0(L, \wedge^2 T_L) \subseteq H^0(L, \wedge^2 T_X|_L)$ is nondegenerate and hence L carries a canonical symplectic form $(\pi|_L)^{-1} \in H^0(L, \Omega_L^2)$. It follows that a Poisson manifold is naturally partitioned into regularly immersed symplectic manifolds (of possibly different dimensions).

Example 1.

(1) Every complex manifold admits a Poisson structure for which the Poisson bracket vanishes on all functions.

(2) The data of a Poisson structure of rank $\dim X$ at every point is equivalent to that of a (holomorphic) symplectic structure.

(3) Let $\mathfrak{g}$ be a complex Lie algebra, and set $X := \mathfrak{g}^\vee$. There is a Poisson bracket $\{-, -\}$ on X such that, for any $f \in \mathfrak{g} \subset \mathcal{O}_X(X)$, $\{f, -\} \colon \mathcal{O}_X \to \mathcal{O}_X$ extends $\mathrm{ad}_f(-) = [f, -] \colon \mathfrak{g} \to \mathfrak{g}$. The symplectic leaves are the orbits of the adjoint group G acting on X through the coadjoint representation. More precisely, each coadjoint orbit carries a canonical G-invariant symplectic structure (called the Kostant-Kirillov symplectic structure).

(4) If (X, π_X) and (Y, π_Y) are Poisson manifolds, then $(X \times Y, \mathrm{pr}_X^* \pi_X + \mathrm{pr}_Y^* \pi_Y)$ is a Poisson manifold. Every leaf of $\mathrm{pr}_X^* \pi_X + \mathrm{pr}_Y^* \pi_Y$ is the product of leaves of π_X and π_Y.

Weinstein proved in [2] that if x is a point in a Poisson manifold (X, π) at which the Poisson bracket has rank $2k$, then x has a neighborhood that decomposes as a product of a symplectic manifold of dimension $2k$, and a Poisson manifold for which the Poisson bracket vanishes at the marked point. A natural question is then the following.

Question 2. *Under which conditions this splitting is global, so that X decomposes as a product of Poisson manifolds, having a symplectic leaf as a factor (perhaps after passing to a suitable covering space)?*

If X is compact, an obvious necessary condition is that the symplectic leaf is also compact. The result I presented in this talk is proved in [1].

Theorem 3. *Let (X, π) be a compact Kähler Poisson manifold, and suppose that $L \subseteq X$ is a compact leaf of π whose fundamental group is finite. Then there exist a compact Kähler Poisson manifold Y, and a finite étale Poisson morphism*

$$M \times Y \to X,$$

where M is the universal cover of L equipped with the symplectic structure induced from that of L.

Recall that a map $\phi\colon (X, \pi_X) \to (Y, \pi_Y)$ is a Poisson morphism if $\phi^*\{f, g\}_Y = \{\phi^*f, \phi^*g\}$ for any (local) holomorphic functions f and g.

Corollary 4. *Let (X, π) be a compact Kähler Poisson manifold with $h^{2,0}(X) = 0$. If $L \subseteq X$ is a compact leaf of π, then L is a point.*

Remark 5. The conclusion of the theorem may fail if the complex Poisson manifold X is not compact, if X is non-Kähler, or if the fundamental group of L is infinite.

Moreover, it is easy to construct examples of smooth real Poisson manifolds where both X and its leaf are simply connected, but X does not decompose as a product.

Problem 6. *Describe compact Kähler Poisson manifolds having a compact leaf with infinite fundamental group.*

In fact, Problem 6 reduces to the special case where the leaf is a finite étale quotient of a torus.

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Tropical Degenerations and Stable Rationality

JOHN CHRISTIAN OTTEM

(joint work with J. Nicaise)

The paper [2] gives a general technique for studying rationality problems of hypersurfaces and complete intersections in toric varieties, based on the motivic volume formula of Nicaise–Shinder [4]. Here are two of the main applications:

Theorem 1. *A very general complex quartic fivefold is not stably rational.*

Theorem 2. *A very general complex complete intersection of a quadric and a cubic in $\mathbb{P}^6$ is not stably irrational.*

These theorems fit into a long list of results on the rationality problem for hypersurfaces and complete intersections, going back to the works of Clemens–Griffiths, Iskovskih–Manin in the 1970s, and more recently, Colliot-Thelene–Pirutka, Kollár, Schreieder, Totaro, Voisin. In these works, the approaches typically combine specialization arguments with various types of birational invariants (Brauer groups, differential forms, decomposition of the diagonal, etc). The method of [2] however, seems to be the first instance where one specializes to a union of several components, and deduces irrationality from that of lower-dimensional varieties.

1.1. **Background.** Two varieties X and Y are said to be *stably birational* if $X \times \mathbb{P}^m$ and $Y \times \mathbb{P}^n$ are birational for some positive integers m, n. A variety X is *stably rational* if it is stably birational to a projective space.

For a field F, we let SB_F denote the set of stable birational equivalence classes of integral F-varieties. For a variety X, we write $[X]_{\mathrm{sb}}$ for its equivalence class.

We will work in $\mathbb{Z}[\mathrm{SB}_F]$, the free abelian group on the set SB_F. Elements of $\mathbb{Z}[\mathrm{SB}_F]$ are formal sums of the form

$$a_1[X_1]_{\mathrm{sb}} + \ldots + a_r[X_r]_{\mathrm{sb}}$$

for integers $a_1, \ldots, a_r$. In fact, $\mathbb{Z}[\mathrm{SB}_F]$ is a *ring*, with multiplication defined via the fiber product, i.e., $[X]_{\mathrm{sb}} \cdot [Y]_{\mathrm{sb}} = [X \times_F Y]_{\mathrm{sb}}$.

The motivic volume. We work over the field of Puiseux series

$$K = \mathbb{C}\{\{t\}\} = \bigcup_{m>0} \mathbb{C}((t^{1/m}))$$

and its valuation ring

$$R = \bigcup_{m>0} \mathbb{C}[[t^{1/m}]].$$

In short, we consider families $\mathcal{X} \to \mathrm{Spec}\, R$, and want to compare the rationality properties of the generic fiber $\mathcal{X}_K$, to that of the special fiber $\mathcal{X}_\mathbb{C}$. Note however that $\mathcal{X}_\mathbb{C}$ may have several irreducible components, so it makes most sense to perform this comparison in $\mathbb{Z}[\mathrm{SB}_\mathbb{C}]$. Indeed, the *motivic volume* will be a map $\mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_\mathbb{C}]$.

It suffices to define the motivic volume on proper R-schemes $\mathcal{X}$ which are *strictly semi-stable*, i.e., $\mathcal{X}_\mathbb{C}$ is a reduced simple normal crossing divisor on $\mathcal{X}$. In the formula (2) below, $\mathcal{X}$ will be a proper strictly semi-stable R-scheme, and we decompose special fiber into irreducible components

$$(1) \qquad \mathcal{X}_\mathbb{C} = \sum_{i \in I} X_i.$$

Theorem (Nicaise–Shinder)**.** *There exists a unique ring homomorphism*

$$\mathrm{Vol} : \mathbb{Z}[\mathrm{SB}_K] \to \mathbb{Z}[\mathrm{SB}_\mathbb{C}]$$

such that, for any $\mathcal{X}$ as above,

$$(2) \qquad \mathrm{Vol}([\mathcal{X}_K]_{\mathrm{sb}}) = \sum_{\emptyset \neq J \subseteq I} (-1)^{|J|-1} [X_J]_{\mathrm{sb}}$$

where $X_J = X_{j_1} \cap \ldots \cap X_{j_r}$.

Note that Vol sends $[\mathrm{Spec}\, K]_{\mathrm{sb}}$ to $[\mathrm{Spec}\, \mathbb{C}]_{\mathrm{sb}}$. This simple observation gives an obstruction to stable rationality: If $\mathcal{X}/R$ is a family such that the alternating sum (2) *does not cancel out to* $[\mathrm{Spec}\, \mathbb{C}]$ *in* $\mathbb{Z}[\mathrm{SB}_\mathbb{C}]$, *then the generic fiber $\mathcal{X}_K$ is not stably rational.*

Example 3. Suppose the special fiber $\mathcal{X}_{\mathbb{C}}$ consists of two components, X_0 and X_1, intersecting along X_{01}. The motivic volume takes the form

$$\text{(3)} \qquad \text{Vol}(\mathcal{X}_K) = [X_0]_{\text{sb}} + [X_1]_{\text{sb}} - [X_{01}]_{\text{sb}}.$$

From this, we deduce that either of the following conditions guarantee that the generic fiber $\mathcal{X}_K$ is not stably rational:

   i) Exactly one of X_0, X_1, X_{01} is stably irrational.
   ii) X_0 and X_1 are both stably irrational.
   iii) X_0 and X_{01} are stably irrational, but they are not stably birational to each other.
   iv) X_0, X_1, X_{01} are all stably irrational.

Remark 4. The condition of strict semi-stability is quite restrictive, and producing a semi-stable model often leads to many blow-ups which which are hard to analyze. An important point is that the formula (2) also applies when $\mathcal{X}$ is *strictly toroidal* (see [3]). This condition is much more flexible, and reduces the computations substantially.

1.2. **Quartic fivefolds.** We are now in position to prove Theorem 1. Let $F \in \mathbb{C}[x_0, \dots, x_6]$ be a very general homogeneous polynomial of degree 4. Consider the following R-scheme

$$\text{(4)} \qquad \mathcal{X} = \text{Proj}\, R[x_0, \dots, x_6, y]/(x_5 x_6 - ty, y^2 - F)$$

where the variable y has weight 2. Note that the generic fiber $\mathcal{X}_K$ is isomorphic to a smooth quartic hypersurface in $\mathbb{P}_K^6$ (inverting t allows us to eliminate y using the first equation). Moreover, $\mathcal{X}$ is strictly toroidal.

The special fiber has two components:

$$\begin{aligned} X_0 &= \text{Proj}\,\mathbb{C}[x_0, \dots, x_6, y]/(x_5, y^2 - F) \\ X_1 &= \text{Proj}\,\mathbb{C}[x_0, \dots, x_6, y]/(x_6, y^2 - F). \end{aligned}$$

Note that these are both very general quartic double fivefolds. We do not know whether these are stably rational or not. However, their intersection,

$$X_{01} = \text{Proj}\,\mathbb{C}[x_0, \dots, x_4, y]/(y^2 - F),$$

is a very general quartic double fourfold, and thus stably irrational by Hassett–Pirutka–Tschinkel [1]. In any event, we are in either case i) or iv) in Example 3, so we conclude also that $\mathcal{X}_K$, and hence the also the very general quartic fivefold, is stably irrational.

1.3. **Further results.** The paper [2] gives many other applications to rationality problems. For instance, we give logarithmic bounds for irrationality of complete intersections à la Schreieder, and study hypersurfaces in products of projective spaces.

While the main idea is simple, the challenge is now to write down suitable degenerations where one can apply the obstruction. For this, we utilize the theory of tropical degenerations. This has the advantage that degenerations can be found

and studied using combinatorial methods, i.e., finding regular subdivisions of polytopes. For instance, the combinatorial picture for the 2-dimensional analogue of the above degeneration is shown in Figure 1.
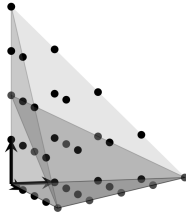

FIGURE 1. Degenerating a quartic surface into a union of two quartic double surfaces intersecting along a quartic double curve.

In fact, while the family (4) is completely explicit, most of the applications in [2] require degenerations which are much more involved and harder to write down concretely. For instance, proving irrationality of a $(2,3)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^4$ involves degenerating $\mathbb{P}^1 \times \mathbb{P}^4$ into a union of 26 toric varieties.

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Vanishing theorems for Fano's in positive characteristics

EMELIE ARVIDSSON

(joint work with Fabio Bernasconi, Justin Lacini)

The Kodaira and Kawamata–Viehweg vanishing theorems are important theorems in algebraic geometry over the complex numbers. However, these theorems fail to be true over fields of positive characteristic. The first counter–examples where constructed by Raynauld already in the 70's [7]. These first examples, as well as many of those that followed, where for varieties of general type. It is more difficult to find examples of varieties of Fano–type for which these theorems fail. Even though Totaro [8] has constructed such examples in every positive characteristic, these examples are such that in a fixed dimension they only work in small characteristics (depending on the dimension). It has therefore been conjectured that these theorems may hold true for Fanos in large enough characteristic $p = p(n)$ depending on the dimension n. In dimension two this conjecture was proven by

Cascini–Tanaka–Witaszek [4]. However, it was not possible to get an explicit integer $p(2)$ from their work. In joint work with Bernasconi–Lacini we proved that $p(2) = 5$ is both sufficient and necessary [1].

Theorem 1 (Kawamata–Viehweg for del Pezzo surfaces in char. $p > 5$)**.** *Let X be a surface of del Pezzo type over a perfect field of characteristic $p \neq 2, 3, 5$. If (X, Δ) is a Kawamata log terminal pair and D a $\mathbb{Z}$-divisor such that $D - (K_X + \Delta)$ is big and nef then*

$$H^i(X, \mathcal{O}_X(D)) = 0 \text{ for all } i > 0.$$

The strategy of proof of the above theorem is to show that del Pezzo surfaces in characteristic $p > 5$ admits a log resolution that lifts to characteristic zero (to the ring of *Witt vectors* $W(k)$). By the seminal work of Deligne–Illusie [5], such liftability implies Kodaira vanishing. To conclude, we show that these surfaces do not only lift themselves, but also together with a boundary satisfying the Kawamata–Viehweg vanishing condition. This implies the full Kawamata–Viehweg vanishing theorem for log del Pezzo surfaces in characteristic $p > 5$. More precisely, Theorem 1 is directly implied by the following theorem [1].

Theorem 2. *If X is a surface of del Pezzo type over an algebraically closed field of characteristic $p > 5$ and D is a $\mathbb{Z}$-divisor such that there exists a boundary Δ such that (X, Δ) is klt and $D - K_X - \Delta$ is big and nef then there exists a boundary $\Delta' \leq \Delta$ such that $D - K_X - \Delta'$ is big and nef and such that (X, Δ') has a log resolution that lifts to $W(k)$.*

Theorem 1 has an important application to the understanding of Kawamata log terminal singularities. By work of Hacon–Witaszek [6] the question if Kawamata log terminal threefold singularities are Cohen–Macaulay or not in positive characteristic was reduced to the question if vanishing theorems are valid on log del Pezzo's in that same characteristic. The reason that such a relationship exists is that by considering *a plt blow-up* of the singularity the unique irreducible exceptional divisor, called the *Kollár* component, is a surface of del Pezzo type.

Theorem 3. [1, 6] *Kawamata log terminal threefold singularities over a perfect field of characteristic $p > 5$ are Cohen–Macaulay.*

Converserly, Theorems 1, 2 and 3 can all fail in characteristic $p = 2, 3, 5$, see [2], [3] and [1] respectively.

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Non-archimedean SYZ fibrations for Calabi–Yau hypersurfaces
ENRICA MAZZON

Mirror symmetry is a fast-moving research area at the boundary between mathematics and theoretical physics. Originated from observations in string theory, it suggests the existence of a duality between Calabi–Yau (CY) manifolds, complex manifolds with a nowhere vanishing holomorphic form of maximal degree. It predicts that every CY manifold X has a mirror partner $\check{X}$, such that the complex geometry of $\check{X}$ is equivalent to the symplectic geometry of X, in some appropriate sense, and vice versa.

Various approaches have been developed to find a rigorous definition of a mirror pair $(X, \check{X})$, and methods to construct mirror partners; a geometric explanation was proposed by Strominger, Yau and Zaslow (SYZ) in [13]. In its current formulation, the SYZ conjecture concerns CY manifolds in certain degenerating families rather than individual manifolds. More precisely, consider a projective family $(X_t)_t$ of CY varieties of dimension n over a punctured disk, such that the family is maximally degenerate, i.e. the monodromy operator on the degree n cohomology of X_t has a Jordan block of maximal size, that is $n + 1$.

Conjecture 1 (SYZ conjecture). *For all sufficiently small t, X_t admits a fibration $\pi : X_t \to B$, whose fibres are special Lagrangian tori, away from a locus Δ of codimension 2 in B. Moreover, the mirror partner $\check{X}_t$ of X_t is obtained by dualizing the special Lagrangian toric fibres of π and by suitably compactifying the resulting space.*

While some examples of special Lagrangian torus fibrations can be produced, dealing with the general case seems very difficult. The insight of Kontsevich and Soibelman is to replace the above conjecture by an analogous one in the non-archimedean world, and to interpret the latter as an asymptotic limit of the complex phenomenon when $t \to 0$.

More precisely, one can associate to the degenerating family $X = (X_t)_t$ the Berkovich non-archimedean space X^{an}, whose points are valuations defined locally on X. Given a degeneration $\mathcal{X}$ of X, we say that $\mathcal{X}$ is snc (respectively dlt) if the pair $(\mathcal{X}, \mathcal{X}_0)$ is strict normal crossing (respectively divisorially log terminal), where $\mathcal{X}_0$ is fiber over $t = 0$; see [4] for more details. Given any snc or dlt degeneration, the dual intersection complex $\mathcal{D}(\mathcal{X}_0)$ is a simplicial complex encoding the combinatorics of the multiple intersections of the components of $\mathcal{X}_0$. It admits a canonical embedding in X^{an}, whose image is called the skeleton of $\mathcal{X}$ and denoted

Sk($\mathcal{X}$), and a retraction $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to$ Sk($\mathcal{X}$). Among various degenerations, minimal (in the sense of MMP) dlt models $\mathcal{X}$ of X determine a canonical skeleton Sk(X) = Sk($\mathcal{X}$), called the essential skeleton of X and independent of the choice of the minimal model; see [9, 10] for more details. The essential skeleton and the retractions $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to$ Sk($\mathcal{X}$), which do depend on $\mathcal{X}$, are of particular relevance in the non-archimedean reformulation of the SYZ conjecture. Indeed,

Theorem 2. [11, Theorem 6.1] *Let $\mathcal{X}$ be a minimal dlt model of X. The retraction $\rho_{\mathcal{X}} : X^{\mathrm{an}} \to$ Sk(X) is an n-dimensional affinoid torus fibration away from the union Z of the simplices in Sk(X) of codimension ≥ 2.*

The condition of being an affinoid torus fibration can be seen as the non-archimedean analogue of being a smooth torus fibration. The conjecture by Kontsevich and Soibelman predicts that the base B in Conjecture 1 and Sk(X) coincide as metric integral affine manifolds, away from the singular loci Δ and Z respectively.

We focus on the case of CY hypersurfaces, i.e.

$$X_t = \{z_0 z_1 \ldots z_{n+1} + t f_{n+2}(z) = 0\} \subset \mathbb{P}^{n+1}$$

where f_{n+2} is a generic homogeneous polynomial of degree $n+2$. In collaboration with Léonard Pille-Schneider, we consider quintic 3-folds CY, so when $n = 3$, and provide evidence for the Kontsevich and Soibelman conjecture. Indeed, we construct new types of non-archimedean retractions and prove that they induce on Sk(X) the same singular integral affine structures that arise on B, where the base B of SYZ fibrations is obtained as dual complex of toric degenerations in the Gross–Siebert program, as well as the Gromov–Hausdorff limit of the family. More precisely,

Theorem 3. [8, Theorem A] *For $n = 3$, there exists a continuous retraction $\rho : X^{\mathrm{an}} \to$ Sk(X) such that*

- *ρ is an affinoid torus fibration outside a piecewise-linear locus Γ that has codimension 2 and is contained in the 2-skeleton of Sk(X);*
- *ρ induces an integral affine structure on Sk(X) $\setminus \Gamma$ which is isomorphic to the ones constructed in [1], [2], [6] and [12].*

The main idea behind our construction is to generalize Theorem 2, consider retractions $\rho_{\mathcal{X}_i}$ associated with several minimal dlt models $\mathcal{X}_i$ of X, adapted to different regions of Sk(X) and glued together.

The sigularities of the model $\mathcal{X} = \{z_0 z_1 \ldots z_{n+1} + t f_{n+2}(z) = 0\}$ of X are not $\mathbb{Q}$-factorial, so that the retraction $\rho_{\mathcal{X}}$ is only well-defined over the union U of open maximal faces of Sk(X). Nevertheless, this is enough to prove a conjecture by Li which relates the SYZ conjecture to the non-archimedean SYZ fibration $\rho_{\mathcal{X}}$ over U. Indeed, in [7, Theorem 1.3] Li proved that SYZ fibrations exist on large open regions of CY manifolds, provided that a comparison property holds for the solution to the non-archimedean Monge-Ampère equation over U; see [7, Proposition 3.7] and [14] for more details.

In a work in progress with Jakob Hultgren, Mattias Jonsson and Nick Mc-Cleerey, we prove that the comparison property holds for generic CY hypersurfaces of dimension n. Thus, we conclude that

Corollary 4. [3] *Given $\delta > 0$, for all sufficiently small t there exist a special Lagrangian torus fibration on an open subset of X_t of normalized CY volume at least $1 - \delta$.*

We show that the solution to the non-archimedean equation can be derived from the solution to a tropical Monge-Ampère equation, and in particular is the restriction of a continuous semipositive toric metric on projective space $\mathbb{P}^{n+1}$.

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Algebraic cycles on Gushel–Mukai varieties

LIE FU

(joint work with Ben Moonen)

I report on my recent joint work with Ben Moonen [14] [15] on the study of cycle theoretic aspects of a special class of Fano varieties, called Gushel–Mukai (GM) varieties.

1. Gushel–Mukai varieties

We briefly recall the definition of GM varieties; see [7] and the references therein for more details. Let $\mathbf{k}$ be the base field. We always assume that the characteristic of $\mathbf{k}$ is $p \neq 2$. GM varieties are Fano varieties of dimension $n = 3, 4, 5$ or 6, defined as a smooth dimensionally transverse intersection in $\mathbb{P}(\wedge^2 V_5 \oplus \mathbf{k})$:

$$X = \mathrm{CGr}(2, V_5) \cap \mathbb{P}(W) \cap Q,$$

where V_5 is a 5-dimensional $\mathbf{k}$-vector space, $\mathrm{CGr}(2, V_5)$ is the projective cone over the Grassmannian $\mathrm{Gr}(2, V_5)$, W is a linear subspace of $\wedge^2 V_5 \oplus \mathbf{k}$ of dimension $n+5$, and Q is a smooth quadric hypersurface.

There are two types of GM varieties, depending on whether $\mathbb{P}(W)$ passes through the vertex O of the cone over the Grassmannian:

- (Mukai-type GM varieties). If $O \notin \mathbb{P}(W)$, then by projecting to $\mathbb{P}(\wedge^2 V_5)$ from O, we see that such a GM variety is isomorphic to a smooth dimensionally transverse intersection in $\mathbb{P}(\wedge^2 V_5)$:

$$X = \mathrm{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q,$$

  where W is a subspace of $\wedge^2 V_5$ of dimension $n + 5$ and Q is a smooth quadric hypersurface.
- (Gushel-type GM varieties). If $O \in \mathbb{P}(W)$, then by projecting to $\mathbb{P}(\wedge^2 V_5)$ from O, we see that such a GM variety is a double cover of a smooth dimensionally transverse intersection in $\mathbb{P}(\wedge^2 V_5)$:

$$M = \mathrm{Gr}(2, V_5) \cap \mathbb{P}(\bar{W}),$$

  branched along a Mukai-type GM variety of dimension $n - 1$, where $\bar{W}$ is the image of W under projection and is a subspace of $\wedge^2 V_5$ of dimension $n + 4$.

Clearly, by pertubing W, Gushel-type GM varieties are specializations of Mukai-type GM varieties. We denote by $\gamma \colon X \to \mathrm{Gr}(2, V_5)$ the natural morphism, called the Gushel map, which is an embedding if X is of Mukai type, and is of degree two to its image if X is of Gushel type.

Arguably, the most important feature of even-dimensional GM varieties is that they are *K3-like*. To make it more precise, let us restrict to the complex numbers $\mathbf{k} = \mathbb{C}$.

- The middle cohomology $H^n(X, \mathbb{Z})$ carries a weight-2 Hodge structure of K3 type, in the sense that $h^{\frac{n}{2}-1, \frac{n}{2}+1}(X) = 1$ and $h^{p,q}(X) = 0$ if $|p-q| > 2$.
- Its deformation theory is unobstructed and admits period maps to Shimura varieties of orthogonal type [11] [12].
- Its bounded derived category of coherent sheaves contains a so-called Kuznetsov component (defined as the semi-orthogonal complement of an exceptional collection), which is a non-commutative K3 surface, in the sense that the Serre functor is [2]. See Kuznetsov–Perry [22].

- Certain geometric construction [20] [10] relates them to O'Grady's double EPW sextics [28], which are hyper-Kähler fourfolds deformation to Hilbert squares of K3 surfaces.

In those regards, GM varieties are very similar to cubic fourfolds.

The purpose of our work is to explore the analog for cycle theoretic aspects.

2. Integral Chow groups

As a piece of basic information, we compute all the integral Chow groups of complex GM varieties, except for CH_1 of GM 4-folds and the CH_2 of GM 6-folds, the only two infinite dimensional ones (in the sense of Mumford). Here is a summary of results:

- In all dimensions $n \in \{3, 4, 5, 6\}$, $CH_0(X) = \mathbb{Z}$ (since X is Fano and hence rationally chain connected by Kollár–Miyaoka–Mori) and $CH^1(X)$ is freely generated by the pullback of the Plücker polarization on $\mathrm{Gr}(2, V_5)$ via γ (the Gushel map). Thanks to Bloch–Srinivas [3], we can also handle $CH^2(X)$ easily: it is the extension of its image in $H^4(X, \mathbb{Z})$ via the cycle class map, by the intermediate jacobian associated to the Hodge structure $H^3(X, \mathbb{Z})(1)$.
- ($n = 3$). $CH^2(X)$ is the extension of $H^4(X, \mathbb{Z}) \simeq \mathbb{Z}$ by the intermediate jacobian $J^3(X)$.
- ($n = 4$). $CH^2(X)$ is isomorphic, via the cycle class map, to its image in $H^4(X, \mathbb{Z})$, which is the subgroup of integral Hodge classes (by Perry [29]).
- ($n = 5$). $CH^2(X) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. Using techniques similar to [16] (more precisely, unramified cohomology [6] [30] and decomposition of the diagonal [24]), Lin Zhou recently showed in [32] that $CH_1(X) \simeq \mathbb{Z}$ (generated by a line) and $CH^3(X)$ is the extension of $H^6(X, \mathbb{Z}) \simeq \mathbb{Z}$ by the intermediate jacobian $J^5(X)$.
- ($n = 6$). $CH^2(X) \simeq H^4(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}$. A similar analysis as in [32] gives that $CH_1(X) \simeq \mathbb{Z}$ (generated by a line) and $CH^3(X)$ is isomorphic to its image in $H^6(X, \mathbb{Z})$ via the cycle class map.

3. Mumford–Tate conjecture

Thanks to the refined decomposition of the diagonal [24], the Hodge conjecture for GM varieties readily follows from the computation of Chow groups in the previous section.

The work of André [2] was presented in an axiomatic way and it allows us to simply verify a few conditions (the most significant one being the image of the peirod map contains an open subset of the period domain) to conclude the following results for even dimensional GM varieties defined over a finitely generated field of characteristic zero:

- the Mumford–Tate conjecture, which says that under the identification $GL(H^n(X_{\overline{\mathbf{k}}}, \mathbb{Q}_\ell)) \simeq GL(H^n(X(\mathbb{C}), \mathbb{Q}) \otimes \mathbb{Q}_\ell)$ by Artin comparison, the

identity component of the Zariski closure of the image of the Galois representation is identified with the Mumford–Tate group tensored with $\mathbb{Q}_\ell$.

- that their André motives (as is defined in [1]) are abelian. This in fact leads to a stronger result known as the *motivated* Mumford–Tate conjecture, saying that the motivic Galois group is also identified with the two algebraic groups in the usual Mumford–Tate conjecture.
- the Tate conjecture: this is a formal consequence of the combination of the Hodge conjecture and the Mumford–Tate conjecture.

4. Generalized partners and duals

We are over the field of complex numbers in this section. We defined GM varieties as certain intersections. From such a datum (called a *GM data set* in [10]), one can extract the so-called *Lagrangian data set* consisting of a triple (V_6, V_5, A) where V_6 is a 6-dimensional vector space, V_5 is a hyperplane and A is a Lagrangian subspace of $\wedge^3 V_6$, with respect to the natural $\wedge^6 V_6$-valued symplectic form. Such an association is canonical and one often writes $V_6(X), V_5(X), A(X)$ etc. for the data attached to a GM variety X.

Following [10] and [22], two GM varieties X, X' of dimension n and n' with the same parity are called *generalized partners* (resp. *generalized duals*) if there is an isomorphism between $V_6(X)$ and $V_6(X')$ (resp. $V_6(X')^\vee$) which identifies $A(X)$ and $A(X')$ (resp. $A(X')^\perp$). Generalized partners share the same period points (i.e. they have isomorphic Hodge structures of middle degree, up to a possible Tate twist).

For X and X' two GM varieties which are generalized partners or duals, on the level of derived categories [23], Kuznetsov and Perry showed that their Kuznetsov components are equivalent. We showed the analog on the level of rational Chow motives: their middle Chow motives are isomorphic:

$$\mathfrak{h}^n(X) \simeq \mathfrak{h}^{n'}(X')(\frac{n'-n}{2}).$$

This is the analog of [19] [17] (for K3 surfaces) and [18] (for cubic fourfolds).

5. Tate conjecture

The main result of our work is the Tate conjecture for even-dimensional GM varieties in positive characteristics. More precisely, let $\mathbf{k}$ be a finitely generated field of characteristic $p \geq 5$. Let X be an even-dimensional GM variety defined over $\mathbf{k}$. Then we proved the Tate conjecture which says that for any integer i and any prime number $\ell \neq p$,

- The Galois representation $\rho \colon \mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k}) \to H^{2i}(X_{\overline{\mathbf{k}}}, \mathbb{Q}_\ell(i))$ is completely reducible.
- The cycle class map

$$\mathrm{CH}^i(X) \otimes \mathbb{Q}_\ell \to H^{2i}(X_{\overline{\mathbf{k}}}, \mathbb{Q}_\ell(i))^{\mathrm{Gal}(\overline{\mathbf{k}}/\mathbf{k})}$$

is surjective.

Only the case $i = \frac{n}{2}$ is interesting.

For the proof, we followed the strategy of Madapusi Pera [25] for the Tate conjecture for K3 surfaces, which builds on the theory of integral canonical model of Shimura varieties of orthogonal type as is developed by himself in [26] (based on the work of Kisin [21]). Roughly speaking, for a given Tate class on a GM variety, the idea is to lift the GM variety together with this Tate class to characteristic zero. To this end, we perform the Kuga–Satake construction to relate the lifting problem to that for abelian varieties, where there are well developed results [25] [31]. In this process, the subtle point arises when one tries to relate the Tate classes on a GM variety and the Tate classes on its Kuga–Satake variety: this is where the integral canonical model comes in.

There are certainly other approaches to the Tate conjecture for K3 like varieties (also uses Kuga–Satake constructions), e.g. the work of Charles [4] [5]. It will be interesting to deduce our results from the Tate conjecture for the associated hyper-Kähler variety (double EPW sextic). Unfortunately, so far we have not succeeded in doing so.

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Ranks of polynomials

ALEXANDER POLISHCHUK

(joint work with David Kazhdan, Chen Wang)

The talk is based on the works [2] and [3]. Recall that the slice rank $\mathrm{srk}(f)$ of a homogeneous polynomial f over a field k is the minimal number r such that $f = l_1 f_1 + \ldots + l_r f_r$, where l_i are linear forms. Equivalently, this is a minimal codimension (in the ambient affine space) of a linear subspace contained in the hypersurface $f = 0$.

Adapting the techniques from Derksen's work [1], we prove the following result.

Theorem A [2] *Assume that the field k is perfect. Then for a homogeneous polynomial over k of degree $d \geq 2$, one has*

$$\mathrm{srk}_k(f) \leq d \cdot \mathrm{srk}_{\overline{k}}(f).$$

Geometrically, the statement is that if a hypersurface $X \subset \mathbb{P}^N$ of degree d, defined over k, contains a linear subspace L of codimension r in $\mathbb{P}^N$, defined over $\overline{k}$, then X contains a linear subspace L_0 of codimension $\leq dr$ in $\mathbb{P}^N$, defined over

k. One can ask for an explicit construction of L_0 from L and its Galois conjugates. The simplest answer would be that one can just take L_0 to be the intersection of all the Galois conjugates of L. The following result shows that for cubics this works, but with a worse estimate for the codimension. Set

$$L_f := \cap_{L \subset X, \operatorname{codim}_{\mathbb{P}^n} L = r} L.$$

Theorem B [3] *Let f be a cubic of slice rank r. Then* $\operatorname{codim}_{\mathbb{P}^n} L_f \leq r^2 + \frac{(r+1)^2}{4} + r$.
Considering the cubics

$$f_n = \sum_{1 \leq i < j \leq n} x_i x_j y_{ij},$$

of slice rank $n - 1$, one sees that $\operatorname{codim}_{\mathbb{P}^n} L_f$ grows quadratically with the slice rank. The proof of Theorem B is based on the following result about linear ideals.

Theorem C [3] *Let $(P_s)_{s \in S}$ be any collection of subspaces of linear forms, such that $\dim P_s \leq r$ and $\cap P_s = 0$. Consider the intersection of the corresponding linear ideals $I = \cap_{s \in S}(P_s)$. Then $\dim I_2 \leq r^2$.*

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Degenerations of unstable Fano varieties and Kähler-Ricci soliton

HAROLD BLUM

(joint work with Yuchen Liu, Chenyang Xu, and Ziquan Zhuang)

A central problem in complex geometry is to find canonical metrics on complex projective varieties. On a smooth Fano variety X, the natural metrics to consider are Kähler-Einstein metrics, which are Kähler metrics $\omega \in c_1(X)$ such that

$$\operatorname{Ric}(\omega) = \omega.$$

While such a metric does not always exist, the Yau-Tian-Donaldson Conjecture, which is a theorem by [4, 16], states that a smooth Fano variety admits a Kähler-Einstein metric if and only if it is K-polystable (see also [14] for the singular case). The term K-stability appearing above is an algebraic criterion introduced by Tian and Donaldson and is defined in terms of degenerations [15, 7]. Recently, the notion has received interest from algebraic geometers due to its use in constructing compact moduli spaces of Fano varieties.

To understand Fano varieties that are K-unstable and hence, do not admit Kähler-Einstein metrics, it is necessary to look at more general metrics. One class of metric to consider are Kähler-Ricci solition, which are the data of a Kähler metric $\omega \in c_1(X)$ and a vector field $\xi \in H^0(T_X)$ such that

$$\operatorname{Ric}(\omega) = \omega + L_\xi \omega,$$

where L_ξ denotes the Lie derivative. Similar to the case of Kähler-Einstein metrics, the existence of a Kähler-Ricci soliton is equivalent to a version of K-stability for Fano varieties with vector fields (X, ξ) [5, 8].

To produce Kähler-Ricci soliton, one can fix an initial metric $\omega_0 \in c_1(X)$ and study the long term behavior of the normalized Kähler-Ricci flow given by

$$\frac{\partial \omega_t}{\partial t} = -\mathrm{Ric}(\omega_t) + \omega_t.$$

The Tian-Hamilton Conjecture, now a theorem by [1, 6], states that, up to taking a subsequence, the Gromov-Hausdorff limit of (X, ω_t) as $t \to \infty$ is naturally a klt Fano variety with a Kähler-Ricci soliton (Y, ω_Y). Chen, Sun, and Wang observed that the degeneration $X \rightsquigarrow Y$ can be achieved in two steps

$$X \rightsquigarrow (Z, \xi_Z) \rightsquigarrow (Y, \xi_Y),$$

and conjectured that this degeneration process is uniquely determined by X, and independent of the choice of initial metric [5]. The latter was recently confirmed in [11].

A natural problem is to try to construct the two step degeneration algebraically and for all singular Fano varieties. This is achieved in recent joint work with Liu, Xu, and Zhuang that builds on work of Han and Li.

Main Theorem. [2, 11] *Any klt Fano variety X admits a canonical two step degeneration*

$$X \rightsquigarrow (Z, \xi_Z) \rightsquigarrow (Y, \xi_Y)$$

Furthermore, (Y, ξ_Y) admits a Kähler-Ricci soliton.

The first degeneration in the above theorem is the unique $\mathbb{R}$-degeneration minimizing the H-functional of Dervan and Szekelyhidi [9] and the pair (Z, ξ_Z) is K-semistable. The pair (Y, ξ_Y) is the unique K-polystable degeneration of (Z, ξ_Z).

Note that the two step degeneration in the above theorem was previously constructed for smooth Fano varieties using deep analytic results. The techniques of [2, 11] are algebraic and apply to all Fano varieties with at worst klt singularities.

The approach to proving the theorem is through valuations and their relation to degenerations, which has been particularly effective in the algebraic study of K-stability, see e.g. [3, 10, 13]. In particular, Han and Li introduced a function

$$H : \mathrm{Val}_X \to \mathbb{R} \cup \{+\infty\},$$

where Val_X is the space of real valuations of X [11]. Constructing the first degeneration $X \rightsquigarrow (Z, \xi_Z)$ and proving it is canonical amounts to showing

(1) There exists a quasimonomial valuation v minimizing H [11],
(2) The minimizer v is unique [2, 11], and
(3) the associated graded ring of $\mathrm{gr}_v R$ of the ring $R := \oplus_{m \in \mathbb{N}} H^0(X, -mK_X)$ is finitely generated [2].

Step (3) uses recent powerful finite generation results of Liu, Xu, and Zhuang in [14]. With these steps complete, $Z := \mathrm{Proj}(\mathrm{gr}_v R)$ and there is a natural torus action $\mathbb{T}$ on Z and vector field $\xi \in N_\mathbb{R} := \mathrm{Hom}(\mathbb{G}_m, \mathbb{T}) \otimes_\mathbb{Z} \mathbb{R}$.

We expect that the above theorem has applications toward the moduli theory of Fano varieties. Indeed, K-polystable Fano varieties with vector fields (X, ξ) are expected to admit a good moduli theory (this is known for smooth Fano varieties by [12]). In future work, we seek to understand the two step degeneration process in families and in the process provide a moduli theory for K-unstable Fano varieties.

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Isolated quotient singularities in positive characteristic

Gebhard Martin

(joint work with Christian Liedtke, Yuya Matsumoto)

A complex isolated quotient singularity X is the quotient of $Y \coloneqq \operatorname{Spec} \mathbb{C}[\![u_1, ..., u_d]\!]$ by a *very small* action of a finite group (scheme) G, that is, by an action which fixes 0 and is free on $Y - \{0\}$. Such singularities satisfy a number of good properties:

(1) By a lemma of Cartan [3], the G-action on Y is *linearizable*, that is, it is conjugate to an action of a finite subgroup of $\operatorname{GL}_{d,\mathbb{C}}$ acting linearly on Y.

(2) Since G acts freely outside the closed point of Y and Y is simply connected, the group G is isomorphic to the (étale) fundamental group of the punctured singularity $X - x$. Moreover, by a result of Prill [12, Theorem 2], the conjugacy class of G in $\mathrm{GL}_{d,\mathbb{C}}$ is uniquely determined by X.

(3) Therefore, the problem of classifying complex isolated quotient singularities boils down to classifying finite and very small subgroups of $\mathrm{GL}_{d,\mathbb{C}}$. The finite subgroups of $\mathrm{SL}_{2,\mathbb{C}}$ have been classified by Klein in [7]. Later, Brieskorn [2] classified the finite very small subgroups of $\mathrm{GL}_{2,\mathbb{C}}$. The classification of finite very small subgroups of $\mathrm{GL}_{d,\mathbb{C}}$ was completed by Wolf [16] in his solution to the *spherical space form problem.*

(4) In the terminology of the MMP, isolated quotient singularities are rational, Cohen–Macaulay, and even klt. Conversely, in dimension 2, all klt singularities are quotient singularities.

(5) By Schlessinger's work [15], isolated quotient singularities of dimension at least 3 are infinitesimally rigid.

(6) In dimension 2, Riemenschneider [13] conjectured that quotient singularities deform to quotient singularities. This conjecture was later proven by Esnault and Viehweg [4].

Moreover, for an isolated surface singularity X, Flenner's extension [6] of Mumford's theorem [10] states that X is smooth if and only if $X - x$ is étale simply connected. In particular, X is a quotient singularity if and only if the étale fundamental group of $X - x$ is finite.

All of these results are known to fail for general quotient singularities in positive characteristic. To deal with this problem, there seem to be two sensible approaches:

- *Either* we put enough restrictions on the group scheme G so that most pathologies disappear,
- *or* we accept these pathologies as part of the theory of singularities in positive characteristic and develop tools to understand these phenomena.

The first approach leads to the notion of *linearly reductive quotient singularities* studied in [8]. In the second approach, we apply these tools to *torsors over the rational double points* in [9].

1. Linearly reductive quotient singularities

Let k be an algebraically closed field of positive characteristic p. A finite group scheme G over k is called *linearly reductive* if all its linear representations are completely reducible. By a result typically attributed to Nagata [11] (but see also [1]), a finite group scheme G is linearly reductive if and only if in its connected-étale decomposition $G = G^0 \rtimes G^{\text{ét}}$, the connected part G^0 is a product of $\boldsymbol{\mu}_{p^n}$'s, the group scheme of p^n-th roots of unity, and $G^{\text{ét}}$ is the constant group scheme associated to a finite group of order prime to p.

Definition 1. An *isolated linearly reductive quotient singularity* (*LRQ-singularity* for short) is the quotient of $\mathrm{Spec}\, k[\![u_1, \ldots, u_d]\!]$ by a very small action of a finite and linearly reductive group scheme G.

The main result of [8] is that most of the good properties of complex isolated quotient singularities carry over to the class of LRQ-singularities:

(1) Very small actions of finite linearly reductive group schemes can be linearized. In fact, a very small action of a finite group scheme G on Y can be linearized *if and only if* G is linearly reductive.

(2) The group scheme G is uniquely determined by X. More precisely, $G^{\text{ét}}$ is the local étale fundamental group of X, and G^0 is the Cartier dual of the class group of the integral closure of the universal cover of $X - x$.

(3) The classification of finite very small subgroup schemes of $\text{GL}_{d,k}$ can be reduced to the classification over the complex numbers. More precisely, every linearly reductive group scheme G arises via base change from a flat group scheme G' over $\mathbb{Z}$, and G occurs as a very small subgroup scheme of $\text{GL}_{d,k}$ if and only if $G'_{\mathbb{C}}$ embeds as a very small subgroup into $\text{GL}_{d,\mathbb{C}}$. The finite very small subgroups of $\text{GL}_{d,\mathbb{C}}$ of the form $G'_{\mathbb{C}}$ are precisely those with a unique cyclic p-Sylow subgroup.

(4) In the terminology of the MMP in positive characteristic, LRQ-singularities are rational, Cohen–Macaulay, klt, and even strongly F-regular. In dimension 2, all strongly F-regular singularities are LRQ-singularities.

(5) In dimension at least 4, LRQ-singularities are infinitesimally rigid. In dimension 3, they are rigid, but not necessarily infinitesimally rigid, and we calculate their miniversal deformation spaces over the Witt ring $W(k)$.

(6) In dimension 2, LRQ-singularities deform to LRQ-singularities. While we proved this only for $\mathbb{Q}$-Gorenstein deformations, this assumption was later removed by Sato and Takagi [14].

2. Torsors over the rational double points

In [5], Esnault and Viehweg introduced the *local Nori fundamental group scheme* $\pi_{\text{loc}}^{\text{N}}(X)$ of X which, roughly speaking, measures the difference between torsors over $X - x$ and torsors over X. In particular, $\pi_{\text{loc}}^{\text{N}}(X)$ is trivial if and only if every torsor over $X - x$ extends to X. Unfortunately, computing $\pi_{\text{loc}}^{\text{N}}(X)$ seems to be very hard: Even if X is a quotient singularity by a finite group G, the local fundamental group scheme $\pi_{\text{loc}}^{\text{N}}(X)$ can be larger than G.

In [9], we introduce the (loc,loc)-part $\text{Picloc}_{X/k}^{\text{loc,loc}}$ of the local Picard sheaf of an isolated singularity X. This is an ind-group scheme that receives all morphisms from finite group schemes of (loc,loc)-type. As such, it measures the extendability of (loc,loc)-torsors from $X - x$ to X. In contrast to $\pi_{\text{loc}}^{\text{N}}(X)$, the group scheme $\text{Picloc}_{X/k}^{\text{loc,loc}}$ is actually computable, since its Dieudonné-module is related to the Frobenius action on the second local cohomology of X with Witt vector coefficients. For surface singularities, $\text{Picloc}_{X/k}^{\text{loc,loc}}$ is trivial if and only if X is F-injective.

We calculate the group scheme $\text{Picloc}_{X/k}^{\text{loc,loc}}$ for all rational double points (RDPs for short). This allows us to give the following positive characteristic analogue of Mumford's criterion, strengthening a previous result of Esnault and Viehweg [5]:

Theorem 2. *Let X be an isolated singularity over k. Then, X is smooth if and only if $\pi^{\mathrm{N}}_{\mathrm{loc}}(X)$ is trivial.*

Moreover our description of $\mathrm{Picloc}^{\mathrm{loc,loc}}_{X/k}$ allows us to study the question whether RDPs are quotient singularities by finite group schemes.

Theorem 3. *Let k be an algebraically closed field of characteristic $p > 0$.*

(1) If $p \geq 5$, then every RDP over k is a quotient singularity.
(2) If $p \leq 3$, there are RDPs over k which are not quotient singularities.

The precise structure of $\pi^{\mathrm{N}}_{\mathrm{loc}}(X)$, however, remains unknown.

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Mirror symmetry for moduli of rank 2 bundles on a curve

PIETER BELMANS

(joint work with Sergey Galkin, Swarnava Mukhopadhyay)

1. MIRROR SYMMETRY FOR FANO VARIETIES

Mirror symmetry refers to identifications between certain geometric objects, where properties of the complex geometry on one side are identified with properties of the symplectic geometry on the other. For Calabi–Yau threefolds its origins lie in a symmetry for Hodge diamonds which was then enhanced to a relation between Gromov–Witten invariants and periods [1], and finally an equivalence between the derived category and the Fukaya category [9]. In (almost) all of these settings the mirror to a Calabi–Yau threefold is again a Calabi–Yau threefold.

For Fano varieties the origin story is slightly different, in that the mirror to a Fano variety X is not another variety, rather it is a *Landau–Ginzburg model*: a pair (Y, f) where Y is a smooth quasiprojective variety and $f: Y \to \mathbb{A}^1$ is a regular function. The origins of this can be traced back to [7]. Again, mirror symmetry can take on different manifestations, and we will discuss two.

A strong form of mirror symmetry for Fano varieties and Landau–Ginzburg models is *homological mirror symmetry*, which posits that X and (Y, f) are mirror if there are equivalences of triangulated categories as in the table.

A-side (symplectic)	B-side (complex)
FS(Y, f)	$\mathbf{D}^{\mathrm{b}}(\operatorname{coh} X)$
Fukaya–Seidel category	derived category of coherent sheaves
Fuk(X)	MF(Y, f)
Fukaya category	matrix factorisation category

A weaker form of mirror symmetry, similar to the equality between Gromov–Witten invariants and Hodge-theoretic periods for Calabi–Yau varieties, is *enumerative mirror symmetry*. For us this will take the form of an equality between the quantum period (an A-side invariant for X) and the classical period (a B-side invariant for (Y, f)). The (regularised) *quantum period* is defined as the power series

$$(1) \qquad \widehat{G}_X(t) := 1 + \sum_{d=2}^{+\infty} d! \langle [\mathrm{pt}] \cdot \psi^{d-2} \rangle_{0,1,d}^X t^d$$

giving a slice of the (descendant) Gromov–Witten invariants of X. On the mirror side we consider a torus $\mathbb{G}_{\mathrm{m}}^{\dim X}$ (Y is expected to be glued together from different tori, so this restriction makes sense) so that $f \in \mathbb{C}[x_1^{\pm}, \ldots, x_n^{\pm}]$ is a Laurent polynomial in $n = \dim X$ variables, and we can define its classical period as the power

series

$$(2) \qquad \pi_f(t) := \frac{1}{2\pi \mathrm{i}} \int_{(\mathrm{S}^1)^n} \frac{1}{1 - tf(x_1, \ldots, x_n)} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_n}{x_n}.$$

When we are only given a Laurent polynomial, we will say that it is a *weak mirror* if this equality holds. This is a discrete invariant of Fano varieties, in that it does not depend on the complex structure of X, which forms the basis of the Fanosearch program [2].

These two aspects of mirror symmetry are well-studied in certain situations. One setting which is out of reach of the existing methods is that of the moduli space $\mathrm{M}_C(2, \mathcal{L})$ of stable vector bundles of rank two and fixed odd determinant on a curve of genus $g \geq 2$. The construction of a candidate weak Landau–Ginzburg mirror, the verification of mirror symmetry hypotheses, and the application to a conjectural semiorthogonal decomposition form the subject of the works I'm reporting on.

2. Graph potentials

In [6] we construct a family of Laurent polynomials from the datum of a trivalent graph $\gamma = (V, E)$ of genus g (so that $\#V = 2g - 2$ and $\#V = 3g - 3$) with a coloring $c \colon V \to \mathbb{Z}/2\mathbb{Z}$. For a vertex v with coloring $c(v) \in \mathbb{Z}/2\mathbb{Z}$ and the incident edges labelled as x, y, z we define the *vertex potential* as

$$(3) \qquad W_{v,c(v)} := \begin{cases} xyz + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} & c(v) = 0 \\ \frac{1}{xyz} + \frac{xy}{z} + \frac{xz}{y} + \frac{yz}{x} & c(v) = 1 \end{cases}$$

and if we denote by $x_1, \ldots, x_{3g-3}$ the edges in γ we define the *graph potential* as

$$(4) \qquad W_{\gamma,c} := \sum_{v \in V} W_{v,c(v)}.$$

There are many different choices of γ and c for a fixed genus, and using elementary operations between different trivalent graphs and colorings we can relate the classical periods: the periods only depend on the genus and the parity of the number of colored vertices. We can in fact prove the following theorem, by considering all genera simultaneously.

Theorem 1 (B–Galkin–Mukhopadhyay). *There are two topological quantum field theories in the functorial sense of Atiyah (depending on the parity of the coloring) $\mathrm{Z}_{\mathrm{gp},0}(t)$ and $\mathrm{Z}_{\mathrm{gp},1}(t)$ for every sufficiently small value of t which allow us to compute efficiently compute the value at t of (the inverse Laplace transform of) the classical period $\pi_{W_{\gamma,c}}(t)$ from $\mathrm{Z}_{g,\epsilon}(t)(\Sigma_g)$ if $g(\gamma) = g$ and $\epsilon(c) := \sum_{v \in V} c(v) = \epsilon$.*

3. Enumerative mirror symmetry

In [5] we prove the form of enumerative mirror symmetry introduced above relating graph potentials to the moduli space $M_C(2, \mathcal{L})$ of rank 2 stable vector bundles with fixed determinant $\mathcal{L}$ of odd degree, a smooth projective Fano variety of dimension $3g - 3$, Picard rank 1, and index 2 (so that $\omega^\vee_{M_C(2,\mathcal{L})} \cong \mathcal{O}_{M_C(2,\mathcal{L})}(2)$). We have the following.

Theorem 2 (B–Galkin–Mukhopadhyay). *We have*

$$(5) \qquad \widehat{G}_{M_C(2,\mathcal{L})}(t) = \pi_{W_{\gamma,c}}(t)$$

where C is a curve of genus $g \geq 2$, γ is a trivalent graph of genus g, and c is an odd coloring, i.e. $\sum_{v \in V} c(v) = 1$.

The proof is somewhat involved, and combines

- toric degenerations for $M_C(2, \mathcal{L})$ introduced by Manon [10] (using vector bundles of conformal blocks on $\overline{\mathcal{M}}_{g,n}$);
- monotone Lagrangian tori defined by integrable systems coming from toric degenerations, an idea introduced by Nishinou–Nohara–Ueda [11] and Bondal–Galkin and Mikhalkin (unpublished) and suitably generalized in op. cit.;
- an identification of the period of the Landau–Ginzburg potential for a monotone Lagrangian with the quantum period, due to Tonkonog [13];
- an identification of the Landau–Ginzburg potential for these specific monotone Lagrangians with the graph potential, so that the Newton polytope of the graph potential agrees with the fan polytope of the toric degeneration.

4. On (semi)orthogonal decompositions and homological mirror symmetry

In 2018 we proposed the following conjecture (independently formulated by Narasimhan around the same time) [3].

Conjecture 3 (B–Galkin–Mukhopadhyay, Narasimhan). *Let C be a smooth projective curve of genus g. Then there exists a semiorthogonal decomposition*

$$(6) \qquad \begin{aligned} \mathbf{D}^{\mathrm{b}}(M_C(2, \mathcal{L})) = \langle \mathcal{O}_{M_C(2,\mathcal{L})}, \mathbf{D}^{\mathrm{b}}(C), \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^2 C), \dots, \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-1} C), \\ \mathbf{D}^{\mathrm{b}}(\mathrm{Sym}^{g-2} C), \dots, \mathbf{D}^{\mathrm{b}}(C), \mathcal{O}_{M_C(2,\mathcal{L})}(1) \rangle \end{aligned}$$

In [4] we give various types of evidence for this conjecture, most importantly an identity in the Grothendieck ring of categories. The state-of-the-art for this conjecture is achieved in [12, 14], which both exhibit the presence of all components in the semiorthogonal decomposition (and their semiorthogonality), whilst the fullness is still open.

We can also study it from the point-of-view of homological mirror symmetry:

- by computing decompositions of the Fukaya–Seidel category of the (so far hypothetical) mirror to obtain evidence for the conjecture;

- taking the conjecture at face value and considering the decompositions on the mirror side as evidence for the fact that graph potentials are indeed building blocks of the homological mirror.

In [4] we show that the critical locus of an appropriate graph potential has exactly the right shape for the Fukaya–Seidel category of the (yet-to-be-constructed) mirror to have a semiorthogonal decompositions with pieces as in (6), and also for the orthogonal decompositions that should exist for the Fukaya category and matrix factorisation category respectively.

In the spirit of Dubrovin's conjecture, and a generalisation to the case of non-semisimple quantum cohomology (see also [8]), we also describe the identification of the eigenvalue decomposition (due to Muñoz) for quantum multiplication with $c_1(M_C(2, \mathcal{L}))$ to the critical value decomposition of the graph potential. The different graph potentials can thus be seen as restrictions of the Landau–Ginzburg potential on the true mirror to various cluster charts, and the mutation rules are the gluing procedure. It would be interesting to understand more of gluing and the global picture that emerges from it.

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Categorical aspects of the Kollár-Shepherd-Barron correspondence

Jenia Tevelev

(joint work with Giancarlo Urzúa)

Let $P \in \overline{W}$ be a germ of a cyclic quotient singularity $\frac{1}{\Delta}(1, \Omega)$ embedded in a projective surface $\overline{W}$ satisfying several global assumptions [7, Assumptions 1.4] (these compactifications always exist). The Hirzebruch–Jung continued fractions

$$\frac{\Delta}{\Omega} = b_1 - \cfrac{1}{\ddots - \cfrac{1}{b_s}} \qquad \text{and} \qquad \frac{\Delta}{\Delta - \Omega} = a_1 - \cfrac{1}{\ddots - \cfrac{1}{a_\ell}}$$

encode the self-intersections $-b_1, \ldots, -b_s$ in the chain of rational curves in the minimal resolution W^{min} and the equations: $\hat{\mathcal{O}}_{P,\overline{W}} = \mathbb{C}[[z_0, \ldots, z_{\ell+1}]]/\langle L_{ij} - R_{ij} \rangle$. Expressions L_{ij}, R_{ij} (for $i \le j$) are computed using the Riemenschneider's matrix

$$\begin{bmatrix} z_0 & & z_1 & & \cdots & & z_{\ell-1} & & z_\ell^{a_\ell-1} \\ & 1 & & z_2^{a_2-2} & \cdots & z_{\ell-1}^{a_{\ell-1}-2} & & 1 & \\ z_1^{a_1-1} & & z_2 & & \cdots & & z_\ell & & z_{\ell+1} \end{bmatrix}$$

by the following formulas: for a matrix $\begin{bmatrix} A_1 & & A_2 & \ldots & A_{n-1} & & A_n \\ & B_1 & & \ldots & & B_{n-1} & \\ C_1 & & C_2 & \ldots & C_{n-1} & & C_n \end{bmatrix}$, we let $L_{ij} = A_i C_j$ and $R_{ij} = A_j B_{j-1} \ldots B_i C_i$.

The non-commutative analogue of $\hat{\mathcal{O}}_{P,\overline{W}}$ was defined by Kalck and Karmazyn [3]. It is the algebra $\overline{R} = \mathbb{C}\langle z_0, \ldots, z_{\ell+1} \rangle / \langle L_{ij}, R_{ij}, z_0, z_{\ell+1} \rangle$ of dimension Δ.

The versal deformation space $\mathrm{Def}_{P \in \overline{W}}$ contains the Artin component of deformations that can be lifted to a deformation of W^{min} (after a finite base change). The equations of the family are given by $\hat{\mathcal{O}}_{P,\mathcal{W}} = B[[z_0, \ldots, z_{\ell+1}]]/\langle L_{ij} - R_{ij} \rangle$, where the quantities L_{ij}, R_{ij} are computed as above using a deformed matrix

$$\begin{bmatrix} z_0 & & z_1 & & \cdots & & z_{\ell-1} & & Z_\ell^{(a_\ell-1)} \\ & 1 & & Z_2^{(a_2-2)} & \cdots & Z_{\ell-1}^{(a_{\ell-1}-2)} & & 1 & \\ Z_1^{(a_1-1)} & & z_2 - t_2 & \cdots & z_{\ell-1} - t_{\ell-1} & & z_\ell & & z_{\ell+1} \end{bmatrix}.$$

Here $Z_j^{(a)} = z_j^a - s_j^1 z_j^{a-1} - \ldots - s_j^a$ and $B = \mathbb{C}[[t_\alpha, s_\beta^\gamma]]$. The non-commutative analogue of $\hat{\mathcal{O}}_{P,\mathcal{W}}$ is the algebra $\mathcal{R} = B\langle z_0, \ldots, z_{\ell+1} \rangle / \langle L_{ij}, R_{ij}, z_0, z_{\ell+1} \rangle$, which is a flat deformation of $\overline{R}$ over $\mathrm{Spec} B$. It has a remarkable property: its general fiber $\hat{R}$ is a path algebra of a quiver, in particular $\hat{R}$ does not deform further.

The same result holds for all irreducible components $\mathrm{Def}_{P \in \overline{W}}^W \subset \mathrm{Def}_{P \in \overline{W}}$, which by Kollár–Shepherd-Barron [6] are parametrized by M-resolutions $W \to \overline{W}$, i.e. partial resolutions with singularities of type $\frac{1}{n^2}(1, na - 1)$ (Wahl singularities) and relatively nef canonical divisor. The corresponding deformations of $\overline{W}$ are dominated (after a finite base change) by $\mathbb{Q}$-Gorenstein deformations of W.

Theorem 1. *The versal deformation space* $\mathrm{Def}_{P\in\overline{W}}$ *of the singularity embeds in the versal deformation space* $\mathrm{Def}_{\overline{R}}$ *of its Kalck-Karmazyn algebra. A general deformation in an irreducible component* $\mathrm{Def}_{P\in\overline{W}}^{W} \subset \mathrm{Def}_{P\in\overline{W}}$ *gives a deformation* $\hat{R}$ *that is a hereditary algebra. Algebras isomorphic to* $\hat{R}$ *form a dense open subset in an irreducible component* $\mathrm{Def}_{\overline{R}}^{\hat{R}} \subset \mathrm{Def}_{\overline{R}}$.

Example 2. By a classical theorem of Pinkham, $\mathrm{Def}_{\frac{1}{4}(1,1)}$ has two components, a 3-dimensional Artin component and a 1-dimensional component of $\mathbb{Q}$-Gorenstein deformations. The algebra $\overline{R}$ is a 4-dimensional fat point $\mathbb{C}[z_1, z_2, z_3]/(z_1, z_2, z_3)^2$. Its deformations are known by a classical theorem of Gabriel. Under the correspondence of Theorem 1, generic deformations of $P \in \overline{W}$ from the Artin component correspond to deformations of $\overline{R}$ to the path algebra of the Kronecker quiver and $\mathbb{Q}$-Gorenstein deformations to deformations of $\overline{R}$ to the 2×2 matrix algebra.

Every M-resolution of $\overline{W}$ has an associated N-resolution with a relatively nef *anti*-canonical divisor. Both M- and N- resolutions are examples of Wahl resolutions. $\mathbb{Q}$-Gorenstein smoothings of arbitrary Wahl resolutions carry exceptional collections of Hacking vector bundles [1]. On the other hand, $\overline{W}$ carries the Kawamata vector bundle $\overline{F}$ of rank Δ [4] which deforms to a vector bundle F on any deformation of $\overline{W}$, which we also call the Kawamata vector bundle.

Choose an M-resolution W^{+} and let Y be its sufficiently general $\mathbb{Q}$-Gorenstein smoothing. Let $E_r, \ldots, E_0$ be an exceptional collection of Hacking vector bundles on Y coming from W^{+}. The same component of the versal deformation space of $\overline{W}$ also contains a $\mathbb{Q}$-Gorenstein smoothing Y' of the N-resolution W^{-} associated to W^{+}. Let $\bar{E}_r, \ldots, \bar{E}_0$ be a unique deformation to Y of an exceptional collections of Hacking vector bundles on Y' associated to W^{-}.

Theorem 3. *The Kawamata vector bundle* F *on* Y *is isomorphic to* $\bigoplus_{i=0}^{r} \bar{E}_i^{n_{r-i}}$.

The Kalck–Karmazyn algebra $\bar{R} = \mathrm{End}(\bar{F})$ *deforms to the algebra* $\hat{R} = \mathrm{End}(F)$, *which is Morita-equivalent to the hereditary algebra* $\mathrm{End}(\bar{E}_r \oplus \ldots \oplus \bar{E}_0)$.

We have an admissible subcategory

$$D^b(\hat{R} - \mathrm{mod}) \simeq \langle E_r, \ldots, E_0 \rangle = \langle \bar{E}_r, \ldots, \bar{E}_0 \rangle \subset D^b(Y)$$

where

- $\bar{E}_r, \ldots, \bar{E}_0$ *is a strong exceptional collection.*
- *In contrast,* $\mathrm{Ext}^k(E_i, E_j) = 0$ *for* $k \neq 1$, $i > j$.

The admissible subcategory $D^b(\hat{R} - \mathrm{mod}) \subset D^b(Y)$ is a deformation of the admissible subcategory $D^b(\overline{R} - \mathrm{mod}) \subset D^b(\overline{W})$. We view it as a categorification of the Milnor fiber of the smoothing of $P \in \overline{W}$. It is a local construction in a sense that $\hat{R}$ is rigid, i.e. does not change with Y. The quiver of $\hat{R}$ is connected unless $\hat{R}$ is semisimple, which is equivalent to the smoothing of $\overline{W}$ being $\mathbb{Q}$-Gorenstein. This case was studied by Kawamata [5].

Example 4. Let $(P \in \overline{W}) = \frac{1}{19}(1, 7)$. The singularity admits three M-resolutions (see [6, Example 3.15]), where $\binom{n}{a}$ denotes the Wahl singularity $\frac{1}{n^2}(1, na - 1)$:

$$(3) - (4) - (2) \qquad \binom{2}{1} - (1) - \binom{3}{1} \qquad (3) - \binom{2}{1} - (2).$$

The corresponding N-resolutions and quivers are:

$$\binom{8}{3} - (1) - \binom{8}{3} - (1) - \binom{2}{1} - (1) \qquad \binom{5}{2} - (1) - \binom{2}{1} \qquad \binom{8}{3} - (1) - \binom{5}{2} - (1)$$
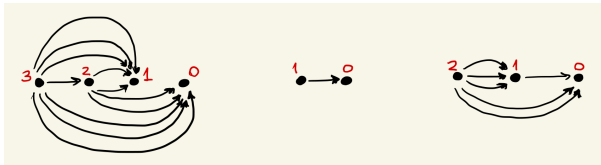
The N-resolution is connected to the M-resolution via a geometric action [7] of the braid group on all Wahl resolutions constructed using the universal family of k1A and k2A extremal neighborhoods [2]. The action is illustrated below starting with an M-resolution with three Wahl singularities. All of these Wahl resolutions admit the same $\mathbb{Q}$-Gorenstein smoothing Y and the braid group action on central fibers corresponds to the braid group action on exceptional collections of Hacking vector bundles on the general fiber. The geometric braid group action on Wahl resolutions is essential for the proof that $\hat{R}$ is hereditary.
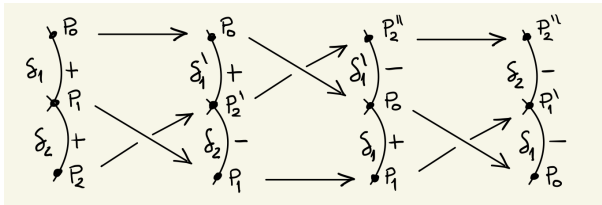
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A global variant of Kunz's theorem

Zsolt Patakfalvi

(joint work with Javier Carvajal-Rojas)

We work over an algebraically closed field k of characteristic $p \geq 0$. Let us consider two important dichotomies in algebraic geometry: local vs global and characteristic zero vs positive characteristics. For instance, the local study of coherent sheaves surrounds the notion of freeness/flatness whereas globally it focuses on positivity (e.g. ampleness). Likewise, characteristic zero geometry is governed by differentials Ω^1 whereas on positive characteristic geometry the Frobenius endomorphisms must be taken into account. Thus, with respect to the above two dichotomies, there are four scenarios in which one may do algebraic geometry. We claim that there is a theorem on three of these scenarios and an analogy between them but the analogous theorem is missing in the fourth scenario. The situation is summarized as follows:

	Local (singularities)	Global (projective geometry)
Differentials	Jacobian criterion	Mori–Hartshorne's theorem
Frobenius ($p > 0$)	Kunz's theorem	?

For the reader's convenience, we briefly recall these three prominent theorems. Let us start with Kunz's theorem [Kun69] and assume that $p > 0$. Kunz's theorem establishes that a variety X/k is smooth if and only if $F_* \mathcal{O}_X$ is locally free of rank (necessarily) $p^{\dim X}$, where $F = F_X : X \to X$ denotes the (absolute) Frobenius endomorphism of X. Equivalently, let us consider the exact sequence

$$0 \to \mathcal{O}_X \xrightarrow{F^\#} F_* \mathcal{O}_X \to \mathcal{B}^1_X \to 0$$

defining $\mathcal{B}^1_X$ as the *cokernel of Frobenius*. Then, Kunz's theorem can be rephrased by saying that X/k is smooth if and only if $\mathcal{B}^1_X$ is locally free of rank (necessarily) $p^{\dim X} - 1$.[1]

Compare this to the jacobian criterion: a variety X/k is smooth if and only if $\Omega^1_{X/k}$ is locally free of rank $\dim X$. Thus, the smoothness of a variety X/k can be determined by using either $\Omega^1_{X/k}$ or $\mathcal{B}^1_X$.

On the other hand, we may consider Mori's theorem (originally called Hartshorne's conjecture) characterizing the projective spaces among smooth projective varieties by the ampleness of the tangent sheaf [Mor79], *cf.* [Har70, Mab78, MS78]. Concretely, a d-dimensional smooth projective variety X/k has anti-ample tangent sheaf $\Omega^1_{X/k}$ if and only if $X \cong \mathbb{P}^d_k := \operatorname{Proj} k[x_0, \ldots, x_d]$. We refer to this as the Mori–Hartshorne's theorem. See [Kol96, V, Corollary 3.3] for a treatment in arbitrary (equal) characteristic.

Further, a direct graded-algebra computation shows that

$$(1) \qquad F_* \mathcal{O}_{\mathbb{P}^d_k} \cong \mathcal{O}_{\mathbb{P}^d_k} \oplus \mathcal{O}_{\mathbb{P}^d_k}(-1)^{\oplus a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^d_k}(-d)^{\oplus a_d},$$

---

[1]Technically speaking, Kunz's theorem characterizes the regularity of X rather than the smoothness of X/k.

where the integers $a_1, \ldots, a_d$ are uniquely determined by such isomorphism. Moreover,

$$\mathcal{B}^{1,\vee}_{\mathbb{P}^d_k} \cong \mathcal{O}_{\mathbb{P}^d_k}(1)^{\oplus a_1} \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^d_k}(d)^{\oplus a_d}$$

is ample.

In view of all the above, it is inevitable to wonder:

Question 1. *Let X/k be a d-dimensional smooth projective variety such that the locally free sheaf $\mathcal{B}^1_X$ is anti-ample. Is X isomorphic to $\mathbb{P}^d_k$?*

If Question 1 were to have have an affirmative answer, we may think of it as both a projective Kunz's theorem and a Frobenius-theoretic Mori–Hartshorne's criterion. That is, we could tell the projective spaces apart among smooth projective varieties by the ampleness of a locally free sheaf naturally defined via its Frobenius. Unfortunately, Question 1 has a negative answer for all $d \geq 3$. Indeed, using the description in [Lan08, Ach12], $\mathcal{B}^1_X$ can be seen to be anti-ample already for quadrics of dimension $d \geq 3$ and $p \geq 3$.

On the positive side, we show that if $\mathcal{B}^1_X$ is anti-ample then X is a Fano variety. In particular, 1 has affirmative answer for $d = 1$. We are able to verify this for surfaces as well. For threefolds, we manage to reduce the class of Fano threefolds for which $\mathcal{B}^1_X$ is anti-ample via an extremal contraction analysis. We obtain the following result.

Theorem 2. *Let X/k be a d-dimensional smooth projective variety such that $\mathcal{B}^1_X$ is anti-ample and $d \leq 3$. Then, X is a Fano variety of Picard rank 1.*

Its converse, however, seems to be rather tricky. Except for the projective space and the quadric, we do not know whether or not $\mathcal{B}^1_X$ is anti-ample for Fano threefolds of Picard rank 1 (i.e. for those of index 1 or 2). However, if X is the quadric, $\mathcal{B}^1_X$ is anti-ample if and only if $p \neq 2$. This leads to:

Question 3. *For which Fano threefolds of Picard rank 1 is $\mathcal{B}^1_X$ anti-ample? Does this depend on the characteristic as it does for quadrics?*

We end with one more question. For that recall the definition

$$\mathcal{B}^i_X = \operatorname{im}\left(F_*(d) : F_*\Omega^{i-1}_{X/k} \to F_*\Omega^i_{X/k} \right)$$

for a smooth projective variety X over k. For $i = 1$, the $\mathcal{B}^i_X$ agrees with the previously defined $\mathcal{B}^1_X$. Then, it is natural to wonder that if the anti-ampleness of all the $\mathcal{B}^i_X$ distinguishes $\mathbb{P}^n_k$ out of all smooth projective varieties. The first natural question in this direction is:

Question 4. *Can we use the ampleness of $\mathcal{B}^2$ to distinguish between $\mathbb{P}^3_k$ and the threefold quadric $\mathbb{Q}^3/k$?*

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A family of Calabi-Yau 3-folds in characteristic 0 and characteristic 3

NICOLAS ADDINGTON

(joint work with Daniel Bragg)

In [8], Gross and Popescu studied many examples of complex Calabi–Yau three-folds fibered in non-principally polarized Abelian surfaces. One example [ibid., §4] was a small resolutions of an intersections of two cubics in $\mathbb{P}^5$ with 72 nodes, with a fibration $X \to \mathbb{P}^1$ in $(1,6)$-polarized Abelian surfaces. They computed the Hodge numbers of X as

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & 6 & & 0 & \\
1 & & 6 & & 6 & & 1 \\
 & 0 & & 6 & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1. & & &
\end{array}
$$

Bragg and I [3] have constructed a dual Abelian fibration $M \to \mathbb{P}^1$ and a derived equivalence $D^b(X) \cong D^b(M)$, using arguments of Schnell [10] and a theorem of Bridgeland and Maciocia [7] as amended by Bridgeland and Iyengar [6]. Schnell studied another Abelian-fibered Calabi–Yau threefold from [8], but where that one had integral fibers, the one we use has some reducible fibers, so the construction of the dual fibration as a compactified relative Jacobian is more delicate. If we just let M be the moduli space of sheaves L on the fibers $F \subset X$ with the same Hilbert polynomial as $\mathcal{O}_F$, then semi-stable sheaves would appear no matter what ample line bundle we chose, so we could not get a universal sheaf on $X \times_{\mathbb{P}^1} M$ to

induce the derived equivalence. But by choosing a section $s \subset X$, we can resolve this issue in one of three ways:

(1) Let M be a moduli space of *framed* sheaves, consisting of sheaves L on fibers F as above, together with a surjection $L \twoheadrightarrow \mathcal{O}_{s \cap F}$.

(2) Let M be the moduli space of sheaves L as above such that $L \otimes I_{s/X}$ is stable, where $I_{s/X}$ is the ideal sheaf of s in X.

(3) Let M be a moduli space of sheaves on the blow-up $\mathrm{Bl}_s(X)$, and prove that the universal sheaf descends to $X \times_{\mathbb{P}^1} M$.

All three get at the same idea; for technical reasons we use the third approach.

This yields another example of a derived equivalence over $\mathbb{C}$ where the two varieties have different fundamental groups and Brauer groups, as in [10] and [2]: we have $\pi_1(X) = 0$ and $\pi_1(M) = (\mathbb{Z}/3)^2$, hence an exact sequence

$$0 \to (\mathbb{Z}/3)^2 \to \mathrm{Br}(X) \to \mathrm{Br}(M) \to 0.$$

To put it another way, the free parts of $H^*(X, \mathbb{Z})$ and $H^*(M, \mathbb{Z})$ are the same, but the torsion parts appear in different degrees.

We then switch from $\mathbb{C}$ and $\bar{\mathbb{F}}_3$, where the same construction still gives smooth threefolds and a derived equivalence, but the Hodge numbers jump:

```
              X                              M

              1                              1
          0       0                      1       1
      0       7       1              1       7       0
  1       8       8       1      1       6       6       1
      1       7       0              0       7       1
          0       0                      1       1
              1                              1
```

This is surprising in view of the fact, first proved by Popa and Schnell [9] and later by Abuaf [1] using a different argument, that Hodge numbers of threefolds are derived invariants in characteristic zero. Note the sums of the columns of the Hodge diamonds are equal on both sides, consistent with [5], and the Euler characteristics $\chi(\Omega^p)$ are equal, consistent with [4, Thm. 5.33].

Our M raises questions about the right definition of Calabi–Yau threefolds in positive characteristic: of course they should have trivial canonical bundle, but some authors ask for $h^*(\mathcal{O}) = 1, 0, 0, 1$, while we think one should only ask for $b_1 = 0$. Recall that Betti numbers in positive characteristic are the ranks of ℓ-adic or crystalline cohomology groups, and need not be equal to sums of Hodge numbers. Our example also sheds light on the possibilities for $h^0(\Omega^p)$ of Calabi–Yau threefolds in positive characteristic.

An interesting point discussed in the last section of the paper, which I failed to mention in the talk, is that the free parts of $H^*_{\mathrm{crys}}(X/W)$ and $H^*_{\mathrm{crys}}(M/W)$ are the same, but the torsion parts appear in different degrees.

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Some uniform bounds for Chow groups and Brauer groups

François Charles

(joint work with Anna Cadoret, Alena Pirutka)

The following theorem is the culmination of work of Manin, Mazur, etc.

Theorem 1 (Mérel, [6]). *Let d be a positive integer. Then there exists a positive integer C, depending only on d, such that for any number field k of degree at most d over $\mathbb{Q}$, and any elliptic curve E over k, then the torsion subgroup of $E(k)$ has cardinal at most C.*

While the conjectural extension of this result to higher-dimensional abelian varieties seem out of reach, Cadoret and Tamagawa have obtained the following variant by considering only ℓ-torsion subgroups:

Theorem 2 ([2]). *Let $\pi : \mathcal{A} \to S$ be a family of abelian varieties over a quasi-projective smooth curve over $\mathbb{Q}$. Let ℓ be a prime number, and let d be a positive integer. Then there exists a positive integer C, depending only on π and d, such that for any number field k of degree at most d over $\mathbb{Q}$, and any k-point s of S, the ℓ-primary torsion subgroup of $\mathcal{A}_s(k)$ has cardinal at most C.*

The goal of the talk was to investigate in what sense some of these results hold when considering invariants other that torsion subgroups of abelian varieties. Of course, the torsion subgroup of the group of rational points of an abelian variety A may be identified with the torsion subgroup of the Picard group of A – equivalently, to the torsion subgroup of $H^1(A, \mathbb{G}_m)$.

As advocated in [7], we may consider the torsion subgroup of $H^2(X, \mathbb{G}_m)$, where X is a smooth projective variety, i.e., the torsion subgroup of the Brauer group.

Given a prime number ℓ, the ℓ-primary torsion subgroup of the Brauer group of X is finite if and only if X satisfies the ℓ-adic Tate conjecture for divisors. We prove the following uniform boundedness result.

Theorem 3 ([1]). *Let $\pi : \mathcal{X} \to S$ be a family of smooth projective varieties over a quasi-projective smooth curve over $\mathbb{Q}$. Let ℓ be a prime number, and let d be a positive integer. Then there exists a positive integer C, depending only on π and d, such that for any number field k of degree at most d over $\mathbb{Q}$, and any k-point s of S such that $\mathcal{X}_s$ satisfies the Tate conjecture for divisors, the ℓ-primary torsion subgroup of $Br(\mathcal{X}_s(k))$ has cardinal at most C.*

Another way to investigate extensions of uniform boundedness results for torsion subgroups of Picard groups is to consider Chow groups of higher-codimension cycles. In this setting, the main general finiteness result is due to Colliot-Thélène-Raskind and Salberger.

Theorem 4 ([4]). *Let X be a smooth projective variety over a number field. Assume that $H^2(X, \mathcal{O}_X) = 0.$. Then the torsion group $CH^2(X)_{\mathrm{tors}}$ is finite.*

The question of finding uniform bounds – when considering cubic surfaces for instance – has been considered in various forms by Manin and has a negative answer in general: even in one-parameter families, results of Colliot-Thélène-Sansuc-Swinnerton-Dyer imply that the 2-torsion part of the Chow group of codimension 2 cycles might be unbounded in one-parameter families. However, combining K-theoretic methods of Colliot-Thélène-Raskind, Bloch, Saito, etc. and the aforementioned results of Cadoret-Tamagawa, we prove the following theorem showing that, while uniform bounds for ℓ-primary torsion do not hold, there are uniform bounds on the *exponent* of ℓ-primary torsion.

Theorem 5 ([3]). *Let $\pi : \mathcal{X} \to S$ be a family of smooth projective varieties over a quasi-projective smooth curve over $\mathbb{Q}$ with $R^2\pi_*\mathcal{O}_{\mathcal{X}} = 0$. Let ℓ be a prime number, and let d be a positive integer. Then there exists a positive integer C, depending only on π and d, such that for any number field k of degree at most d over $\mathbb{Q}$, and any k-point s of S, the ℓ-primary torsion subgroup of $CH^2(\mathcal{X}_s)$ has exponent at most C.*

We offer the following question.

Question 6. *Let $\pi : \mathcal{X} \to S$ be a family of smooth projective varieties over a quasi-projective scheme over $\mathbb{Q}$. Let d be a positive integer. Does there exist a positive integer C, depending only on π and d, such that for any number field k of degree at most d over $\mathbb{Q}$, any positive integer i, and any k-point s of S, we have*

$$C \, CH^i(X_s)_{\mathrm{tors}} = 0 \,?$$

Of course, it seems more reasonable to consider variants of the question above over a one-dimensional base and looking only at ℓ-primary torsion for a single prime ℓ.

Finally, basic computations on Châtelet surfaces suggest that the failure of uniform boundedness for ℓ-primary torsion is related to the number of places of bad reduction. We offer the following result in this direction.

Theorem 7 ([3]). *Let $\pi : \mathcal{X} \to S$ be a family of smooth projective varieties over a quasi-projective scheme over $\mathbb{Z}$ with $R^2\pi_*\mathcal{O}_\mathcal{X} = 0$. Let k be a number field with ring of integers $\mathcal{O}_k$ and let U be an open subscheme of $\mathrm{Spec}\,\mathcal{O}_k$. Then there exists a positive integer C, depending only on π and U, such that for any k-point s of S that extends to a U-point of S, the torsion subgroup of $CH^2(\mathcal{X}_s)$ has cardinal at most C.*

The proof combines effective versions of the methods of Colliot-Thélène-Raskind and Somekawa together with some arguments in the geometry of numbers.

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Summary of other activities:
The "Gong Show" – An hour of short talks

As an after-dinner event on Tuesday evening, all early career researchers visiting the MFO for the first time were invited to give a six minute talk to present their research. Here we list the names of the speakers and the titles of their presentations in chronological order:

- Fei Xie
  Derived categories and semiorthogonal decompositions
- Fabio Bernasconi
  Lifting globally F-split surfaces to characteristic 0
- Chuyu Zho
  Calabi-Yau moduli sitting between K-moduli and KSBA-moduli

- Gebhard Martin
  On the non-degeneracy of Enriques surfaces
- Yajnaseni Dutta
  Why I like Hyperkähler varieties
- Denis Nesterov
  The square
- Thorsten Beckmann
  Hyper-Kähler manifolds and their Mukai lattice

Reporter: Gebhard Martin

Participants

Prof. Dr. Nicolas Addington
Department of Mathematics
University of Oregon
Eugene 97403-1222
UNITED STATES

Dr. Emilie Arvidsson
School of Mathematics
Institute for Advanced Study
1 Einstein Drive
Princeton NJ 08540
UNITED STATES

Thorsten Beckmann
Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY

Prof. Dr. Pieter Belmans
Université du Luxembourg
Campus Belval, Maison du Nombre
6, Avenue de la Fonte
4364 Esch-sur-Alzette
LUXEMBOURG

Dr. Fabio Bernasconi
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND

Prof. Dr. Harold Blum
Department of Mathematics
University of Utah
84112 Salt Lake City
UNITED STATES

Prof. Dr. Paolo Cascini
Department of Mathematics
Imperial College of Science, Technology
and Medicine
Huxley Building
180 Queen's Gate
London SW7 2BZ
UNITED KINGDOM

Prof. Dr. Ana-Maria Castravet
Laboratoire de Mathématiques
Université Paris-Saclay, Versailles
Bâtiment Fermat 3304
45 Avenue des États Unis
78035 Versailles Cedex
FRANCE

Dr. Francois Charles
DMA
École Normale Supérieure
45, rue d'Ulm
75005 Paris
FRANCE

Dr. Kristin DeVleming
University of Massachusetts Amherst
Department of Mathematics
710 N Pleasant St.
Amherst, MA 01003
UNITED STATES

Prof. Dr. Stephane Druel
Département de Mathématiques
Université Claude Bernard de Lyon I
LaPCS Bat. 101
43, Bd. du 11 Novembre 1918
69622 Villeurbanne Cedex
FRANCE

Dr. Yajnaseni Dutta
Mathematisches Institut
Universität Bonn
Hausdorff Center for Mathematics
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Soheyla Feyzbakhsh
Department of Mathematics
Imperial College London
614 Huxley Building
South Kensington Campus
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Dr. Stefano Filipazzi
Section de Mathématiques
Station 8
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND

Dr. Lie Fu
IMAPP, Radboud University
Heyendaalseweg 135
P.O. Box 9010
6525 AJ Nijmegen
NETHERLANDS

Prof. Dr. Angela Gibney
Department of Mathematics
David Rittenhouse Laboratory
University of Pennsylvania
209 South 33rd Street
Philadelphia, PA 19104-6395
UNITED STATES

Dr. Yoshinori Gongyo
Graduate School of Mathematical
Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Prof. Dr. Antonella Grassi
Dipartimento di Matematica
Universita degli Studi di Bologna
Piazza Porta S. Donato, 5
40127 Bologna
ITALY

Prof. Dr. Mark Gross
Department of Pure Mathematics
and Mathematical Statistics
University of Cambridge
Wilberforce Road
Cambridge CB3 0WB
UNITED KINGDOM

Prof. Dr. Christopher D. Hacon
Department of Mathematics
College of Sciences
University of Utah
155 South 1400 East, JWB 233
Salt Lake City, UT 84112-0090
UNITED STATES

Prof. Dr. Daniel Huybrechts
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Anne-Sophie Kaloghiros
Department of Mathematics and
Statistics
Brunel University
Kingston Lane
Uxbridge Middlesex UB8 3PH
UNITED KINGDOM

Prof. Dr. Sándor J. Kovács
Department of Mathematics
University of Washington
Padelford Hall
Box 354350
Seattle WA 98195-4350
UNITED STATES

Dr. John Lesieutre
Department of Mathematics
Penn State University
204 McAllister Building
University Park PA 16802
UNITED STATES

Dr. Yuchen Liu
Department of Mathematics
Lunt Hall
Northwestern University
Evanston, IL 60208-2730
UNITED STATES

Dr. Gebhard Martin
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Mirko Mauri
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor MI 48109-1043
UNITED STATES

Dr. Enrica Mazzon
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor MI 48109-1043
UNITED STATES

Prof. Dr. Mircea Mustata
Department of Mathematics
University of Michigan
530 Church Street
Ann Arbor, MI 48109-1043
UNITED STATES

Denis Nesterov
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. Johannes Nicaise
Department of Mathematics
Imperial College London
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Georg Oberdieck
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Prof. Dr. John Christian Ottem
Department of Mathematics
University of Oslo
Blindern
P.O. Box 1053
0316 Oslo
NORWAY

Prof. Dr. Zsolt Patakfalvi
EPFL SB MATH CAG
MA C3 635
Station 8
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND

Dr. Laura Pertusi
Dipartimento di Matematica
"Federigo Enriques"
Università degli Studi di Milano
Via Cesare Saldini 50
20133 Milano
ITALY

Prof. Dr. Alexander Polishchuk
Department of Mathematics
University of Oregon
Eugene OR 97403-1222
UNITED STATES

Prof. Dr. Mihnea Popa
Department of Mathematics
Harvard University
Science Center
One Oxford Street
Cambridge MA 02138-2901
UNITED STATES

Dr. Jorgen Rennemo
Department of Mathematics
University of Oslo
P. O. Box 1053 - Blindern
0316 Oslo
NORWAY

Dr. Giulia Saccà
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Karl Schwede
Department of Mathematics
University of Utah
155 South 1400 East
Salt Lake City 84112-0090
UNITED STATES

Dr. Roberto Svaldi
Institut de Mathématiques
École Polytechnique Fédérale de
Lausanne
Bâtiment MA, B1 497
Station 8
1015 Lausanne
SWITZERLAND

Prof. Dr. Jenia Tevelev
Department of Mathematics
University of Massachusetts
Lederle Graduate Research Tower
710 North Pleasant Street
Amherst, MA 01003-9305
UNITED STATES

Dr. Richard P. W. Thomas
Department of Mathematics
Imperial College of Science and
Technology
Huxley Building
180 Queen's Gate
London SW7 2AZ
UNITED KINGDOM

Prof. Dr. Yukinobu Toda
Kavli Institute for the Physics &
Mathematics
of the Universe (Kavli IPMU)
University of Tokyo
5-1-5 Kashiwanoha, Chiba
Kashiwa 277-8583
JAPAN

Dr. Fei Xie
School of Mathematics
University of Edinburgh
James Clerk Maxwell Bldg.
Peter Guthrie Tait Road
Edinburgh EH9 3FD
UNITED KINGDOM

Prof. Dr. Chenyang Xu
Department of Mathematics
Princeton University
Fine Hall
Washington Road
Princeton, NJ 08544-1000
UNITED STATES

Prof. Dr. Qizheng Yin
Beijing International Center for
Mathematical Research (BICMR)
Peking University
5 Yiheyuan Road, Haidian
Beijing 100 871
CHINA

Dr. Chuyu Zhou
Section de Mathématiques
Station 8
École Polytechnique Fédérale de
Lausanne
1015 Lausanne
SWITZERLAND