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Non-Commutative Geometry and Cyclic Homology

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Abstract. The meeting displayed the cyclic theory as a fundamental mathematical tool with applications in diverse domains such as analysis, algebraic K-theory, algebraic geometry, arithmetic geometry, solid state physics and quantum field theory.

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Introduction by the Organizers

Cyclic cohomology, since its discovery forty years ago in noncommutative differential geometry, has developed as a fundamental mathematical tool with applications in domains as diverse as analysis, algebraic K-theory, algebraic geometry, arithmetic geometry, solid state physics and quantum field theory. The meeting provided a user friendly introduction to sophisticated domains of applications such as topological Hochschild and cyclic theory for ring spectra, Hopf algebra symmetries, de Rham-Witt complex, quantum physics etc, in which cyclic homology plays the role of a unifying theme. In analysis where cyclic cohomology index formulas are capturing the higher invariants of manifolds and where the group symmetries are extended to Hopf algebra actions and where Lie algebra cohomology is greatly extended to the cyclic cohomology of Hopf algebras which becomes the natural receptacle for characteristic classes. In algebraic topology where the cyclotomic structure obtained using the cyclic subgroups of the circle action on topological

Hochschild homology gives rise to remarkably significant arithmetic structures intimately related to crystalline cohomology through the de Rham Witt complex as well as to the Fontaine theory and the Fargues-Fontaine curve.

The main goal of the meeting was to create an interaction between the two domains where the theory is most active: noncommutative geometry and analysis on one hand and on the other the highly successful topological cyclic homology (TC) and topological Hochschild homology (THH) introduced by Bokstedt, Hsiang and Madsen in the world of ring spectra.

Already in the first talk by A. Efimov on the K-theory of inverse limits of DG categories, appeared an obvious link between his treatment of projective limits allowing one to go beyond the traditional finiteness conditions and the known analysis notions of nuclear operators and of the Calkin algebra in infinite dimensional Hilbert space.

The meeting served one important purpose which is to remove the language barrier between the homotopy theorists and the analysts. Talks by members of the two groups were alternating and hopefully understandable by both groups. For instance the talk by M. Land (joint work with Nikolaus and Schlichting) on the L-theory of C^*-algebras was easy to grasp by both groups. Besides analysis talks dealing with pseudo-differential operator algebras such as talks by V. Nistor and E. Schrohe, there was for instance a user friendly talk by W. van Suijlekom on the role of cyclic homology in quantum field theory explaining his breakthrough contribution with T. Nuland showing that the perturbations of the spectral action are amazingly encoded by an entire cocycle whose cyclic properties survive at the one loop level and are intimately related to the Ward identities of perturbative expansions of gauge theories. Also L. Hesselholt gave an high level introduction to some of the spectacular recent results on topological cyclic homology of Nikolaus-Scholze, Bhatt–Morrow–Scholze, and Antieau–Mathew–Morrow–Nikolaus and explained how the Fargues-Fontaine curve and its decomposition into a punctured curve and the formal neighborhood of the puncture naturally appears from various forms of topological cyclic homology and maps between them. E. Getzler's talk was the prototype of result understandable by both groups, he showed that a cosimplicial generalization of the Chern character in the negative cyclic complex yields an explicit formula for the Chern character in the completed de Rham complex of the derived stack of perfect complexes of a compact CY algebra A.

Another domain where the interaction between the two groups was maximal is the use of the Γ-rings of G. Segal as the framework of algebra over the absolute base. There was in particular a long discussion on the Wednesday evening in order to improve the terminology: while in the book of Dundas-Goodwillie and McCarthy on the local structure of algebraic K-theory the identity functor from pointed sets to themselves is denoted by $\mathbb{S}$ for "sphere spectrum" (see 2.1.2 page 67), the use of this terminology seemed to conflict with the standard use of the term in homotopy theory and it was suggested that the notation $\mathbb{F}_1$ (for the field with one element) would be more appropriate for this absolute base. Algebra over this base is highly successful in that it provides a natural extension of the structure sheaf of $\mathrm{Spec}\mathbb{Z}$ to its Arakelov compactification $\overline{\mathrm{Spec}\mathbb{Z}}$, as a subsheaf of the constant sheaf $\mathbb{Q}$ using the generalization of abelian groups obtained by viewing an abelian group A as a covariant functor HA from the category of finite pointed sets to the category of pointed sets which assigns to X the pointed set of A-valued divisors on X which are 0 on the base point. The functoriality is obtained by taking the sum over the preimage of a point. The power of this construction is that it gives a Riemann-Roch theorem for Arakelov divisors (see the report of A. C. on joint work with C. Consani). It allows to envisage as a far reaching goal that cyclic homology (in the guise of the de-Rham Witt or crystalline cohomology) should give access to the sought for cohomology of $\overline{\mathrm{Spec}\mathbb{Z}}$ displaying the zeros of the Riemann zeta function as eigenvalues of the Frobenius.

The titles and abstracts of the talks below display the wide variety of topics covered in the meeting.

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Workshop: Non-Commutative Geometry and Cyclic Homology

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Abstracts

K-theory of inverse limits

Alexander I. Efimov

We prove a general result on the commutation of K-theory with inverse limits. Before formulating a general statement, we formulate its application to formal affine schemes.

Let R be a (commutative) noetherian ring and $I \subset R$ an ideal. Clausen and Scholze [2] defined a category $\mathrm{Nuc}(R_{\hat{I}})$ of nuclear solid modules over $R_{\hat{I}}$, which is a dualizable presentable category. Below we always consider the non-connective K-theory spectra.

Theorem 1 (E.). *We have an equivalence (of spectra)*

$$K^{\mathrm{cont}}(\mathrm{Nuc}(R_{\hat{I}})) \simeq \varprojlim_n K(R/I^n). \tag{1}$$

Here by K^{cont} denotes the continuous K-theory, which is defined for dualizable presentable categories in [1, 3].

Note that the naive K-theory of perfect complexes on formal schemes is not a reasonable invariant. The main reason is that the restriction functor to an open subscheme is not a localization in the categorical sense. This is because the ring-theoretic localization does not commute with completion, e.g. $\mathbb{Z}[x][[y]][x^{-1}] \not\cong \mathbb{Z}[x, x^{-1}][[y]]$. For the same reason the naive K-theory of formal schemes does not satisfy Zariski descent.

On the other hand, if one defines K-theory of (say, affine) formal schemes as in the RHS of (1), then the issue with the localization disappears and Zariski descent works fine. It turns out that considering the category Nuc instead of Perf also provides a solution to the same problem: the restriction to an open subset now becomes a categorical localization. The basic properties of continuous K-theory hence also imply that $K^{\mathrm{cont}}(\mathrm{Nuc})$ satisfies Zariski descent.

Theorem 1 states that the two approaches to the definition of K-theory of formal schemes give the same result. Below we provide some details and formulate a more general result about K-theory of inverse limits.

1. K-theory of large categories

It is well-known that the usual K-theory vanishes for stable ∞-categories with countable direct sums due to Eilenberg swindle argument applied to the identity functor (it is sufficient to have countable sums of copies of any object). However, it turns out that one can define a reasonable version of K-theory for presentable cocomplete ∞-categories under additional assumption of being dualizable (equivalently, compactly assembled). For simplicity we consider DG (differential graded) categories (although the $\mathbb{Z}$-linear structure is irrelevant here).

One of the definitions of a presentable DG category is the following. $\mathcal{C}$ is presentable if there exists a DG ring A and an A-module M such that

$$\mathcal{C} \simeq \mathcal{D}_{(A,M)} := M^{\perp} = \{X \in \text{Mod-}A \mid \mathbf{R}\mathrm{Hom}_A(M, X) = 0\} \subset \text{Mod-}A.$$

We call a functor $F : \mathcal{C} \to \mathcal{D}$ between such DG categories *continuous* if it commutes with small direct sums. We denote by $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{cont}}$ the ∞-category of presentable DG categories and continuous functors.

There is a natural symmetric monoidal structure on $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{cont}}$ – the Lurie tensor product, which we denote by $-\hat{\otimes}-$. The internal Hom is given by $\mathrm{Fun}^{\mathrm{cont}}(\mathcal{C}, \mathcal{D})$ – the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ formed by continuous functors.

Definition 2. A presentable DG category $\mathcal{C}$ is dualizable if it is a dualizable object in the symmetric monoidal category $(\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{cont}}, \hat{\otimes})$.

The following is due to Lurie.

Theorem 3 (Lurie). *Let $\mathcal{C}$ be a presentable DG category. TFAE:*
(i) $\mathcal{C}$ is dualizable.
(ii) There is a short exact sequence

$$0 \to \mathcal{C} \to \text{Mod-}A \xrightarrow{-\otimes_A B} \text{Mod-}B \to 0,$$

where $A \to B$ is a homological epimorphism of DG rings, i.e. $B \otimes_A^{\mathbf{L}} B \xrightarrow{\sim} B$.
(iii) The Yoneda embedding $Y_{\mathcal{C}} : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$ has a twice left adjoint.

We call a continuous functor $F : \mathcal{C} \to \mathcal{D}$ *strongly continuous* if its right adjoint is continuous. We denote by $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{dual}} \subset \mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{cont}}$ the (non-full) subcategory formed by dualizable categories and strongly continuous functors. We have a fully faithful embedding

$$\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{tr}} \hookrightarrow \mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{dual}}, \quad \mathcal{A} \mapsto \text{Mod-}\mathcal{A}.$$

Here $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{tr}}$ denotes the ∞-category of small pre-triangulated Karoubi complete DG categories.

Let $\mathcal{C}$ be a dualizable category. Choose a homological epimorphism $A \to B$, such that $\mathcal{C} \simeq \ker(\text{Mod-}A \to \text{Mod-}B)$. We would like to define

$$K^{\mathrm{cont}}(\mathcal{C}) := \mathrm{Fiber}(K(A) \to K(B)).$$

Note that we need to check that this does not depend on the choice of $A \to B$, and also we need to see the functoriality of $K^{\mathrm{cont}}(\mathcal{C})$ in $\mathcal{C}$ (with respect to strongly continuous functors).

The most natural way to do it is via the so-called continuous Calkin category. Define $\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})$ to be the Karoubi closure of the image of $\mathrm{Cone}(Y_{\mathcal{C}}^{LL} \to Y_{\mathcal{C}}) : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. This allows to define

$$K^{\mathrm{cont}} : \mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{dual}} \to \mathrm{Sp}, \quad K^{\mathrm{cont}}(\mathcal{C}) := \Omega K(\mathrm{Calk}^{\mathrm{cont}}(\mathcal{C})).$$

See [1, 3] for details.

2. Mittag-Leffler sequences of categories and the main result

We now define the notion of a Mittag-Leffler inverse sequence of dualizable categories.

Definition 4 (E.). Let $(\mathcal{C}_n)_{n\geq 0}$ be an inverse sequence of dualizable categories, so that the transition functors F_{nm}, $n \geq m$, are strongly continuous. Denote by F_{nm}^R the right adjoint functors. We say that $(\mathcal{C}_n)_n$ is a (secondary) Mittag-Leffler inverse system of DG categories if the following conditions hold:

1) For any $n \geq 0$, the inverse sequence $(F_{kn}F_{kn}^R)_{k\geq n}$ is essentially constant in the category $\mathrm{Fun}^{\mathrm{cont}}(\mathcal{C}_n, \mathcal{C}_n)$.

2) For any $n, m \geq 0$, the functor $(\varprojlim_{k\geq n,m} F_{km}F_{kn}^R) : \mathcal{C}_n \to \mathcal{C}_m$ is strongly continuous and has a left adjoint.

If the sequence $(\mathcal{C}_n)$ is given by $(\text{Mod-}A_n)_n$, where $A_0 \leftarrow A_1 \leftarrow \dots$ is an inverse sequence of DG rings, then conditions 1) and 2) are translated into the following:

1)' For any $n \geq 0$, the sequence $A_n \otimes_{A_k} A_n$ is essentially constant in A_n-Mod-A_n;

2)' For any $n, m \geq 0$, the A_n-A_m-bimodule $\varprojlim_{k\geq m,n} A_n \otimes_{A_k} A_m$ is perfect over A_n^{op} and over A_m.

Example 5. *One can check that the inverse sequence of DG rings* $(\mathbb{Z}[x]/x^n)_{n\geq 1}$ *(concentrated in degree zero) satisfies the conditions 1)' and 2)'. From this it is easy to deduce that for a noetherian commutative ring* R *and an ideal* $I \subset R$ *the inverse system* $(\text{Mod-}R/I^n)_{n\geq 1}$ *is pro-equivalent to a Mittag-Leffler system (here* Mod- *stands for the DG category of homotopically projective complexes of modules).*

It turns out that Mittag-Leffler sequences of categories behave extremely well with respect to inverse limits, so that the terminology is justified. This is expressed in the following results. We denote by $\varprojlim^{\mathrm{dual}}$ the inverse limit taken in $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{dual}}$ (it is quite different from the inverse limit in $\mathrm{dgcat}_{\mathbb{Z}}^{\mathrm{cont}}$). We ignore set-theoretic issues for simplicity.

Theorem 6 (E.). *Let* $(\mathcal{C}_n)_n$ *be a Mittag-Leffler sequence of dualizable categories.*

1) The functor $\varprojlim_n \mathcal{C}_n \to \varprojlim_n \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}_n)$ *is a homological epimorphism.*

2) We have a short exact sequence

$$0 \to \varprojlim_n{}^{\mathrm{dual}} \mathcal{C}_n \to \mathrm{Ind}(\varprojlim_n \mathcal{C}_n) \to \mathrm{Ind}(\varprojlim_n \mathrm{Calk}^{\mathrm{cont}}(\mathcal{C}_n)) \to 0.$$

Theorem 7 (E.). *Let* $(\mathcal{C}_n)_n$ *be a Mittag-Leffler sequence of dualizable categories. Then the natural map*

$$K^{\mathrm{cont}}(\varprojlim_n{}^{\mathrm{dual}} \mathcal{C}_n) \to \varprojlim_n K^{\mathrm{cont}}(\mathcal{C}_n)$$

is an equivalence of spectra.

Remark 8. *For a noetherian affine formal scheme* $\mathrm{Spf}(R_{\hat{I}})$ *we have a fully faithful strongly continuous functor*

$$\mathrm{Nuc}(R_{\hat{I}}) \hookrightarrow \widetilde{\mathrm{Nuc}}(R_{\hat{I}}) := \varprojlim_{n}{}^{\mathrm{dual}}\mathrm{Mod}\text{-}R/I^n.$$

It can be shown that it induces an equivalence on the continuous K-theory.

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On the index of G-invariant operators and cyclic homology

Victor Nistor

Recall that a linear operator $T : X \to Y$ acting between Banach spaces X and Y is a *Fredholm operator* if, by definition, the two linear spaces

$$\ker(T) := T^{-1}(0) \quad \text{and } \mathrm{coker}(T) := Y/TX$$

are finite dimensional. Then, its *(Fredholm) index* $\mathrm{ind}(T)$ is defined by

$$\mathrm{ind}(T) \ := \ \dim\ker(T) - \dim\mathrm{coker}(T) \in \mathbb{Z}\,. \tag{1}$$

Let us assume that P is a *pseudodifferential operator of order m on a compact, smooth manifold M acting from sections of a Hermitian vector bundle E to the sections of a Hermitial vector bundle F.* (This will be the case throughout this report.) If P is elliptic, then $P : H^s(M;E) \to H^{s-m}(M;F)$ is Fredholm [13, 17, 18, 19, 21] (see also the earlier work of Mihlin). Its index is then given by the Atiyah-Singer index formula. (Here H^s is the sth Sobolev space.) Fredholm operators have any applications in areas such as Hodge theory, Index theory, Differential Geometry, Spectral Theory, and Partial Differential Equations.

Let G be a compact Lie group. If our linear operator $T \in \mathcal{L}(X,Y)$ is also *G-invariant*, then both $\ker(T)$ and $\mathrm{coker}(T)$ will be G–modules and, as such, they decompose as a direct sum of their isotypical components, which are parameterized by $\widehat{G}$, the set of isomorphism classes of irreducible representations of G, namely:

$$\mathrm{ind}_G(T) \ := \ [\ker(T)] - [\mathrm{coker}(T)] \in R(G) \ := \ \Big\{ \sum_{\alpha\in\widehat{G}} k_\alpha[\alpha] \ \mid k_\alpha \in \mathbb{Z} \Big\} \simeq \mathbb{Z}^{(\widehat{G})}\,. \tag{2}$$

More precisely, let $m_\alpha(T) := (\dim\alpha)^{-1}\big[\dim\big(\ker(T)_\alpha\big) - \dim\big(\mathrm{coker}(T)_\alpha\big)\big]$, then

$$\mathrm{ind}_G(T) \ := \ \sum_{\alpha\in\widehat{G}} m_\alpha(T)\,[\alpha] \in R(T)\,. \tag{3}$$

Consider the case when $T = P$, our order m pseudodifferential operator. This leads us to the following problem.

Problem 1. *Determine the cofficients $m_\alpha(P)$.*

A formula for $\mathrm{ind}_G(P)(g)$, the value at $g \in G$ of the (character of the) G-equivariant index $\mathrm{ind}_G(P)$ of P is given by the Atiyah-Segal-Singer formula. Of course

$$\mathrm{ind}_G(P)(g) := \sum_{\alpha \in \widehat{G}} m_\alpha(P) \,\mathrm{Tr}(\alpha(g))\,, \tag{4}$$

from which, in principle, one could determine the coefficients $m_\alpha(P)$.

Let us take a closer look at the coefficients $m_\alpha(T)$. For any linear G-invariant map $L : V \to W$ of G-modules V and W, let

$$\pi_\alpha(L) : V_\alpha \to W_\alpha \tag{5}$$

be the restriction of L to the α-isotypical components of the corresponding modules. Then we also have

$$m_\alpha(T) = \frac{\mathrm{ind}\big(\pi_\alpha(T)\big)}{\dim \alpha}\,. \tag{6}$$

Again for $T = P$, our order m pseudodifferential operator, this then leads to the following problems.

Problem 2. *Given $\alpha \in \widehat{G}$, determine the index* $\mathrm{ind}\big(\pi_\alpha(T)\big)$.

Of course, it may happen that $\pi_\alpha(P)$ is Fredholm without P being so, in which case, it still makes sense to ask for the index of $\pi_\alpha(P)$. In any case, this leads to one of the main problem (and result) that we will discuss here, namely:

Problem 3. *Given $\alpha \in \widehat{G}$, determine necessary and sufficient conditions for $\pi_\alpha(P)$ to be Fredholm.*

In this note, we state a complete solution to the last problem and we discuss also the second problem. For the last problem, it is enough to assume that $E = F$. For all problems it is enough to assume that $m = 0$. It would be interesting to study these problems also for the case G non-compact, starting with the case of proper actions.

The results reported in this note have been published or accepted for publication [1, 2, 3, 4]. They are joint works with Alexandre Baldare, Moulay Benameur, Rémi Côme, and Matthias Lesch. The author thanks Alexandre Baldare, Moulay Benameur, Alain Connes, and Joachim Cuntz for useful discussions.

1. Statement of the main result

In order to state our solution to the last problem, we need to introduce several definitions (most of which are not new). We shall let S^*M denote the set of unit vectors in the cotangent space T^*M of M. If P is of order zero, then its principal symbol is determined by its restriction to S^*M and does not depend on the choice of metric on M, so we shall write

$$\sigma_0(P) \in \mathcal{C}^\infty(S^*M; \mathrm{End}(E))\,. \tag{7}$$

Of course, as in the case when there is no group G, our main result will involve the principal symbol. In addition to that, the main new ingredients specific to the G-equivariant case are the following:

(1) the *transverse cosphere bundle* $S^*_G M \subset S^*M$;
(2) the *principal orbit bundle* M_0; and
(3) the *space of "blocks"* Ξ (which, we will see, is closely related to primitive ideal spectrum of the algebra $\mathcal{C}^\infty(S^*_G M; \mathrm{End}(E))^G$).

1.1. **The transverse cotangent space $T^*_G M$ and $S^*_G M$.** The *first new ingredient* for the case of the action of a general group G is the transverse cotangent space $T^*_G M$ (due to Atiyah).

Definition 4. The *G-transverse cotangent space* of M is

$$T^*_G M \,:=\, \big\{\, \xi \in T^*M \mid\ \xi \text{ restricts to } 0 \text{ on the } \textbf{orbits } Gx \text{ of } G \,\big\}\,.$$

If $G = 1$, then, of course, $T^*_G M = T^*_1 M = T^*M$.

Theorem 5 (Atiyah '73, Atiyah-Singer)**.** *If $\sigma_m(P)$ is invertible on $T^*_G M \setminus 0$, then, for each $\alpha \in \widehat{G}$,* $\ker(P)_\alpha$ *and* $\mathrm{coker}(P)_\alpha$ *are finite dimensional.*

1.2. **The principal orbit bundle.** The *"second ingredient"* for general G is the *principal orbit bundle.* To define it, we need to introduce first the *stabilizer G_x* of a point x in a G-space:

$$G_x \,:=\, \{\, g \in G \mid\ g \cdot x = x \,\}\,. \tag{8}$$

Theorem 6 (Tom Dieck)**.** *Suppose that M/G is* ***connected.*** *Then, there exists a* dense open *subset $M_0 \subset M$ such that $\forall x \in M_0$ the stabilizers G_x are* minimal and conjugated.

The set M_0 is called the *principal orbit bundle.* A consequence is that, for all $x' \in M$, there exists $x \in M_0$ such that

$$G_x \subset G_{x'}\,.$$

Moreover, if $x, y \in M_0$, then G_x and G_y are conjugated.

From now on we assume that M/G is connected and we fix a minimal stabiliser subgroup $G_0 \subset G$.

1.3. **The action of the stabilizers and the "block decomposition".** Similarly, for $\xi \in T^*M$, we consider the stabilizer $G_\xi \,:=\, \{g \in G \mid\ g \cdot \xi = \xi\}$. We let

$$S^*_G M \,:=\, T^*_G M \cap S^*M \,:=\, T^*_G M \cap \{\|\xi\| = 1\} \tag{9}$$

and call it the *G-transverse cosphere bundle.* Using it, we can now introduce the *third ingredient,* that is the *space of blocks* Ξ:

$$\Xi \,:=\, \big\{\, (\xi, \rho) \in S^*_G M \times \widehat{G}_\xi \ \mid\ (E_\xi)_\rho \neq 0 \,\big\}\,. \tag{10}$$

The definition of the space of blocks is motivated by the following simple but important remark.

Remark 7. *Let $P \in \Psi^m(M; E)$ be a G-invariant pseudodifferential operator, as always in this report.*

(1) Its principal symbol $\sigma_m(P)$ is also G-invariant.

(2) G_ξ acts on E_ξ, and hence

$$\sigma_m(P)(\xi) \in \mathrm{End}(E_\xi)^{G_\xi} \simeq \bigoplus_{\rho \in \widehat{G}_\xi} \mathrm{End}(E_{\xi\rho})^{G_\xi},$$

a direct sum of matrix algebras, or blocks*, and these blocks are parametrized by Ξ, in the sense that* $\mathrm{End}(E_{\xi\rho})^{G_\xi} \neq 0$ *if, and only if,* $(\xi, \rho) \in \Xi$.

(3) Let us define

$$(11) \qquad \begin{cases} \widehat{\sigma}_m(P) : \Xi \to \cup_{(\xi,\rho)\in\Xi} \mathrm{End}(E_{\xi\rho})^{G_\xi} \\ \widehat{\sigma}_m(P)(\xi, \rho) := \pi_\rho\big(\sigma_m(P)(\xi)\big) \in \mathrm{End}(E_{\xi\rho})^{G_\xi}. \end{cases}$$

We thus see that P is elliptic if, and only if, the function $\widehat{\sigma}_0(P)(\xi, \rho)$ is invertible for all $(\xi, \rho) \in \Xi$.

1.4. **α-ellipticity and statement of the main result.** Let us fix $\alpha \in \widehat{G}$. We have seen in the previous subsection that $P : H^s(M; E) \to H^{s-m}(M; E)$ is Fredholm if, and only if, $\widehat{\sigma}_m(P)(\xi, \rho)$ is invertible for all $(\xi, \rho) \in \Xi$. Clearly, in general, a weaker condition on $\widehat{\sigma}_m(P)(\xi, \rho)$ will be needed in order to conclude that $\pi_\alpha(P)$ is Fredholm.

Notation 8. For each $\alpha \in \widehat{G}$, we let

$$\Xi^\alpha := \big\{ (\xi, \rho) \in \Xi \mid G_0 \subset G_\xi \text{ and } \mathrm{Hom}(\rho, \alpha)^{G_0} \neq 0 \big\}.$$

We can now state *the main result.*

Theorem 9 (Baldare–Côme–Lesch–Nistor)**.** *Let P be a G-invariant pseudodifferential operator. The restriction $\pi_\alpha(P)$ is Fredholm if, and only if, $\widehat{\sigma}_m(P)$ is invertible on Ξ^α.*

See [2, 3, 4].

2. Index and cyclic homology

A consequence of the last result is that $\mathrm{ind}(\pi_\alpha(P))$ (and hence also the multiplicity $m_\alpha(P)$ of $\alpha \in \hat{G}$ in the G-index $\mathrm{ind}_G(P) \in R(G)$) depends only on the restriction of $\sigma_m(P)$ to Ξ^α. Moreover, it is known [5, 6] that $m_\alpha(P) = \phi(\sigma_0(P)|_{\Xi^\alpha})$ for a suitable cyclic cocycle ϕ. Let $\mathcal{C}^\infty(\Xi^\alpha)$ be the algebra of restrictions to Ξ of functions in $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$, the algebra of G-invariant symbols.

Definition 10. Let A be a Fréchet, complex topological algebra with a fixed continuous Banach norm. Let $\overline{A}$ be its completion in this norm. We shall say that A is a *Connes algebras* if the following two maps (defined for $j \in \mathbb{Z}/2\mathbb{Z}$) are isomorphisms:

(1) $K_j(A) \to K_j(\overline{A})$ and

(2) $\mathrm{Ch} : K_j(A) \otimes \mathbb{C} \to \mathrm{HP}_j(A)$.

This definition implies a suitable choice of completion of the cyclic complex defining the (topological) periodic cyclic homology $\mathrm{HP}_j(A)$. Also, one has to make a suitable choice of a topological (so homotopy invariant) K-theory functor. The map Ch is the Chern–Connes–Karoubi character [5, 6, 7, 8, 14, 22].

Theorem 11 (Baldare–Benameur–Nistor). *The algebras $\mathcal{C}^\infty(S^*M; \mathrm{End}(E))^G$ and $\mathcal{C}^\infty(\Xi^\alpha)$ are Connes algebras.*

The proof is as in [1], in particular, it builds on the techniques and ideas developped in [15] and [20]. The framework is that of m-algebras [9, 10] and the proof uses excision in periodic cyclic homology [11, 12, 16]. The following proposition plays an important role in the proof.

Proposition 12. *Let X be a smooth, compact manifold with corners and $Y \subset \partial X$ be a union of closed faces of the boundary. Let $\mathcal{F} \to X$ be a smooth bundle of simple algebras. Let $\mathcal{C}_0^\infty(X, Y; \mathcal{F}) := \{f \in \mathcal{C}^\infty(X; \mathcal{F}) \mid f|_Y = 0\}$ and*

$$\mathcal{C}_\infty^\infty(X, Y; \mathcal{F}) := \{f \in \mathcal{C}^\infty(X) \mid f \text{ vanishes of infinite order on } Y\}.$$

Let $\mathcal{C}_\infty^\infty(X, Y; \mathcal{F}) \subset I \subset \mathcal{C}_0^\infty(X, Y; \mathcal{F})$ be a closed subalgebra of $\mathcal{C}_0^\infty(X, Y; \mathcal{F})$. Then I is a Connes algebra.

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Order-Preserving Automorphisms of Shubin Type Pseudodifferential Operators

Elmar Schrohe
(joint work with Robert Hesse, Ryszard Nest)

In 1976, J.J. Duistermaat and I.M. Singer proved the following statement [3]:

Theorem 1. *Let X, Y be C^∞ manifolds, $H^1(S^*X, \mathbb{C}) = 0$. Then every order-preserving isomorphism $i : L^\infty(X) \to L^\infty(Y)$ is of the form*

$$i(P) = A^{-1}PA$$

for some invertible elliptic properly supported Fourier integral operator $A : C^\infty(Y) \to C^\infty(X)$.

Duistermaat and Singer consider scalar-valued, classical, properly supported pseudodifferential operators on the (not necessarily closed) manifolds X and Y. They denote by $L^m(X)$, $m \in \mathbb{Z}$, the space of all such operators of order $\le m$ on X and by $L^\infty(X)$ the union over all m; correspondingly for Y.

Recall that an operator $T : C_c^\infty(Y) \to C^\infty(X)$ is properly supported, if on the support $\overline{\{(x,y) : K_T(x,y) \neq 0\}}$ of the Schwartz kernel K_T of T, the projections $(x,y) \mapsto x$ and $(x,y) \mapsto y$ are proper maps. Pseudodifferential operators on X with proper support extend to continuous maps $C^\infty(Y) \to C^\infty(X)$, so that they can indeed be composed. A pseudodifferential operator can always be made properly supported by adding an operator with smooth Schwartz kernel, i.e. without changing the terms in the asymptotic expansion of its symbol.

In an article published in 2017, Mathai and Melrose [5] took up the subject. Assuming that X and Y are closed manifolds, they established a corresponding result for pseudodifferential operators acting on sections of vector bundles over X and Y, respectively. Moreover, they showed that, as a consequence of Beals' characterization of pseudodifferential operators [1], the condition $H^1(S^*X, \mathbb{C}) = 0$ on the first cohomology group of the cosphere bundle of X is dispensable.

Results. In my talk, I reported on joint work in progress with R. Hesse (Hannover) and R. Nest (Copenhagen) on a corresponding assertion for Shubin type pseudodifferential operators.

A Shubin type pseudodifferential operator of order $m \in \mathbb{Z}$ on $\mathbb{R}^n$ is a pseudodifferential operators $P = \mathrm{op}\,(p)$, whose symbol p is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n$ satisfying the estimates

$$|D_\xi^\alpha D_x^\beta p(x,\xi)| \le C_{\alpha,\beta}(1+|x|+|\xi|)^{m-|\alpha|-|\beta|} \tag{1}$$

for all multi-indices α, β with suitable constants $C_{\alpha,\beta}$.

Such a symbol is called classical, if it has an asymptotic expansion

$$p \sim \sum_{j=0}^{\infty} p_{m-j}, \tag{2}$$

where p_{m-j} is a symbol of order $m-j$ that is positively homogeneous of degree $m-j$ in (x,ξ) for $|(x,\xi)| \ge 1$, i.e.

$$p_{m-j}(\lambda x, \lambda\xi) = \lambda^{m-j} p_{m-j}(x,\xi), \quad \lambda \ge 1, |(x,\xi)| \ge 1.$$

We call $\sigma(P) := p_m$ the principal symbol of P. Moreover, we denote by Ψ^m, $m \in \mathbb{Z}$, the space of Shubin type pseudodifferential operators of order $\le m$ on $\mathbb{R}^n$ and let $\Psi = \bigcup_m \Psi^m$, $\Psi^{-\infty} = \bigcap_m \Psi^m$. The class $\Psi^{-\infty}$ consists of integral operators with Schwartz kernel in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$. See [8] for more details.

One should note an immediate consequence of (1): In the asymptotic expansion formula for the symbol r of the composition $\mathrm{op}\, p \circ \mathrm{op}\, q$ of two Shubin type pseudodifferential operators:

$$r(x,\xi) \sim \sum_{|\alpha|=0}^{\infty} \frac{1}{\alpha!} \partial_\xi^\alpha p D_x^\alpha q$$

the order of $\partial_\xi^\alpha p D_x^\alpha q$ decreases by two units as $|\alpha|$ increases by one unit. In particular, for classical symbols, the expansions for the terms of even and odd order do not mix.

The result we propose is:

Theorem 2. *Let $n \ge 2$ and $i : \Psi \to \Psi$ be an order-preserving automorphism of $\Psi(\mathbb{R}^n)$ (i.e., $i(\Psi^m) = \Psi^m$). Then i is given by conjugation with an invertible elliptic Shubin type Fourier integral operator $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$:*

$$i(P) = A^{-1} P A.$$

The operator A is unique up to nonzero scalar multiples.

The concept of 'Shubin type' Fourier integral operators will be explained, below.

Background. Our motivation for this investigation comes from joint work with Anton Savin (RUDN, Moscow) [6], [7]. We studied algebras of operators that are finite sums

$$D = \sum_g P_g \Phi_g,$$

where the P_g are Shubin type pseudodifferential operators and $g \mapsto \Phi_g$ is a representation of a Lie group G by quantized canonical transformations Φ_g. In the cases at hand, we considered representations of the unitary group $U(n)$ by operators in the complex metaplectic group (identifying $\mathbb{R}^n \times \mathbb{R}^n$ with $\mathbb{C}^n$) and of $\mathbb{C}^n$ by Heisenberg-Weyl operators, so that the above operator algebras are of the form $\Psi \rtimes G$ for a subgroup G of $U(n)$ and $\mathbb{C}^n \rtimes U(n)$, respectively. The question to what extent G could be enlarged leads to the question what the automorphisms of Ψ are, and in this context the preservation of order is a natural requirement.

Ideas. Following roughly the same strategy as Duistermaat-Singer and Mathai-Melrose, there are five main steps in the proof.

(A) It is a consequence of Eidelheit's lemma, see [3, Lemma 3], that there exists an invertible continuous operator $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}(\mathbb{R}^n)$ unique up to a nonzero scalar multiple such that

$$i(P) = A^{-1}PA. \tag{3}$$

The continuity of A follows without any continuity assumption on i.

(B) The isomorphism i induces a canonical transformation

$$C : T^*\mathbb{R}^n \setminus \{0\} \to T^*\mathbb{R}^n \setminus \{0\},$$

i.e. a symplectomorphism that is homogeneous of degree 1 in (x, ξ) (note that in contrast to the situation in [3] and [5] we only delete the point 0, not the zero section). The principal symbols of P and $i(P)$ then satisfy

$$\sigma(i(P)) = \sigma(P) \circ C^{-1}, \quad P \in \Psi. \tag{4}$$

To see this, we observe the following: Since i is order-preserving, it induces an automorphism on $\Psi^0/\Psi^{-1} \cong C^\infty(\mathbb{S}^{2n-1})$ and thus also on the maximal ideals, which are the kernels of the point evaluation maps. Hence i induces a bijection

$$C : \mathbb{S}^{2n-1} \to \mathbb{S}^{2n-1}$$

that is smooth together with its inverse, since it maps smooth symbols to smooth symbols. We obtain (4) for $P \in \Psi^0$.

Next we recall that there exists an invertible operator $\Lambda \in \Psi^1$ with positive principal symbol λ. Its image under i then also is invertible of order 1; a spectral argument shows that its principal symbol, say μ, is also positive, see [5, p.19]. Moreover, every $P \in \Psi^1$ can be written in the form ΛQ for some $Q \in \Psi^0$. Considering next Ψ^1/Ψ^0 and using the fact that

$$\sigma([P_1, P_2]) = \frac{1}{i}\{\sigma(P_1), \sigma(P_2)\}, \quad P_1, P_2 \in \Psi,$$

one can show that C has an extension to a canonical transformation on $T^*\mathbb{R}^n \setminus \{0\}$ via $\mu = \lambda \circ C^{-1}$.

As a consequence, one obtains (4) for all $P \in \Psi^1$ and then, since the principal symbol map is multiplicative, for all P.

(C) There exists an elliptic Fourier integral operator F, which, near every point $(C(x_0,\xi_0),(x_0,\xi_0))$ in the graph of C is given by a Schwartz kernel K_F of the form

$$K_F(x,y) \equiv \int e^{i\varphi(x,y,\theta)} b(x,y,\theta)\, d\theta \bmod \mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

The phase ϕ is positively 2-homogenous in (x,y,θ) and describes the graph of C near $(C(x_0,\xi_0),(x_0,\xi_0))$, while b is an amplitude in the Shubin calculus. We call these Shubin type Fourier integral operators. The ellipticity implies that there exists another Shubin type Fourier integral operator $F^{\#}$, associated with the graph of C^{-1} such that

$$FF^{\#} - I \text{ and } F^{\#}F - I$$

are operators with Schwartz kernels in $\mathscr{S}(\mathbb{R}^n \times \mathbb{R}^n)$.

Under more global aspects, such operators have been considered by Helffer [4], see also [2] for a more restricted class. Helffer showed a Egorov type theorem for these operators. It implies that for P in Ψ^m

$$P - F^{\#} i(P) F \in \Psi^{m-2}.$$

Moreover, a Beals type argument shows that AF and $F^{\#}A^{-1}$ are (a priori not necessarily classical) Shubin type pseudodifferential operators.

(D) The above approximation of i can be improved. Following an iteration scheme devised by Duistermaat and Singer, we find an elliptic Shubin type pseudodifferential operator Q with parametrix $Q^{\#}$ such that

$$P - Q^{\#}F^{\#} i(P) F Q \in \Psi^{-\infty}. \tag{5}$$

(E) Combining (5) with (3) we see that

$$Q^{\#}F^{\#}A^{-1}PAFQ - P \in \Psi^{-\infty}.$$

Let $E = AFQ$ and $\tilde{E} = Q^{\#}F^{\#}A^{-1}$. Then $E\tilde{E} - I, \tilde{E}E - I \in \Psi^{-\infty}$ and

$$PE - EP, \tilde{E}P - P\tilde{E} \in \Psi^{-\infty}, \quad P \in \Psi^{\infty}.$$

We finally show that this implies that $E = cI + R$ for some $c \neq 0$ and $R \in \Psi^{-\infty}$. It is here that we make use of the assumption $n \geq 2$.

Composing the identity $AFQ = cI + R$ from the right with $Q^{\#}F^{\#}$ and noting that $FQQ^{\#}F^{\#} = I + S$ for some $S \in \Psi^{-\infty}$, we see that

$$A = c(Q^{\#}F^{\#} + RQ^{\#}F^{\#}) - AS.$$

Since $RQ^{\#}F^{\#}$ and AS are in $\Psi^{-\infty}$, A is an invertible Fourier integral operator of Shubin type.

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Characteristic classes in derived geometry

Ezra Getzler

In the work of Bott, Shulman and Stasheff [2], characteristic classes are realized as closed differential forms on the simplicial manifold

$$N_k\mathrm{GL}(N) = \mathrm{GL}(N)^k.$$

The complex of differential forms on a simplicial manifold is the total complex of the double complex

$$\Omega^{p,q}(X_\bullet) = \Omega^p(X_q)$$

with de Rham and Čech differentials. This is a filtered complex, with the *Hodge* filtration

$$F^p\Omega^*(X_\bullet) = \bigoplus_{p'\geq p} \Omega^{p',*-p'}(X_\bullet).$$

In his thesis [4], Shulman gave an explicit formula for the universal differential form on $N_\bullet\mathrm{GL}(n)$ associated to an invariant polynomial P on the Lie algebra $\mathfrak{gl}(n)$ of degree r. We start with the Maurer–Cartan one-form $g^{-1}dg \in \Omega^1(\mathrm{GL}(N), \mathfrak{gl}(N))$ with values in the Lie algebra. Let Δ^k be the k-simplex

$$\Delta^k = \{(t_0, \dots, t_k) \in [0,1]^k \mid t_0 + \cdots + t_k = 1\}.$$

On $\Delta^k \times \mathrm{GL}(N)^{k+1}$, we consider the connection 1-form

$$\theta_k = t_0 g_0^{-1}dg_0 + \cdots + t_k g_k^{-1}dg_k \in \Omega^1(\Delta^k \times \mathrm{GL}(N)^{k+1}, \mathfrak{gl}(N)),$$

with curvature

$$F_k = d\theta_k + \frac{1}{2}[\theta_k, \theta_k].$$

Integration of the differential form

$$P\big(-F_k/2\pi\big) \in \Omega^{2r}(\Delta^k \times \mathrm{GL}(N)^{k+1}).$$

over the k-simplex gives a basic form of degree $2r-k$ on $\mathrm{GL}(N)^{k+1}$, inducing a closed differential form on $N_\bullet\mathrm{GL}(N)$. Shulman's main result is that this form lies in $F^r\Omega^{2r}(\mathrm{GL}(N))$.

In our talk, we extended this story to derived geometry, at least in the special case $P(A) = \mathrm{Tr}(A^k)$, using a cosimplicial generalization of Connes's Chern character in negative cyclic homology. Consider a differential graded algebra A: in the application, this will be the endomorphisms $\mathrm{End}(V)$ of a finite-dimensional graded vector space V. We assume that A is finite-dimensional in each degree, and bounded below (with respect to the cohomological grading). Let C be a differential graded coalgebra: in the application, C will be the (unnormalized) simplicial chains on a simplicial set finite in each dimension.

The graded morphisms from C to A form a differential graded algebra, and we form the differential graded Lie algebra $[C, A]$ by truncation to dimensions ≥ 1. The Maurer-Cartan locus $\mathrm{MC}(C, A) = V(d\mu + \frac{1}{2}[\mu, \mu] \mid \mu \in [C, A]^1)$ of $[C, A]$ is an affine variety, the space of twisting cochains from C to A; this may be identified with the space of homomorphisms of differential graded algebras from the bar construction BC to A. The differential graded Lie algebra may be thought of as a derived scheme $\mathcal{MC}(C, A)$ thickening $\mathrm{MC}(C, A)$ — we call it the derived Maurer–Cartan locus. Its ring of functions $\mathcal{O}(\mathcal{MC}(C, A))$ is the Chevalley–Eilenberg algebra of cochains on the differential graded Lie algebra $[C, A]$.

Consider the cosimplicial category

$$[n] \mapsto [\![n]\!],$$

where $[\![n]\!]$ is the groupoid with objects $\{0, \ldots, n\}$, and one morphism between each two objects. Let $\mathbb{\Delta}^n$ be the nerve of $[\![n]\!]$: its k-simplices are paths $i_0 \ldots i_k$ on the n-simplex, where $i_\ell \in \{0, \ldots, n\}$. In the case $n = 0$, we have $\Delta^0 = \mathbb{\Delta}^0$, while for $n = 1$, the simplicial set $\mathbb{\Delta}^1$ plays the same rôle in the study of ∞-categories that Δ^1 plays in the study of ∞-groupoids (i.e. Kan complexes).

Applying the (unnormalized) chain functor, we obtain a cosimplicial differential graded coalgebra:

$$C^{\bullet,*} = C_{-*}(\mathbb{\Delta}^\bullet).$$

The derived stack of perfect complexes is the inductive limit over V of the simplicial derived schemes

$$V \mapsto \mathcal{MC}(C^\bullet, \mathrm{End}(V)).$$

Note that if V is a vector space concentrated in degree 0, then $\mathrm{MC}(C^\bullet, \mathrm{End}(V))$ is the nerve of the general linear group $\mathrm{GL}(V)$.

The (completed) de Rham complex of $\mathcal{MC}(C^\bullet, \mathrm{End}(V))$ is the product

$$\Omega^k(\mathcal{MC}(C^\bullet, \mathrm{End}(V))) = \prod_{q=0}^{\infty} \Omega^{k-q}(\mathcal{MC}(C^q, \mathrm{End}(V))).$$

Instead of constructing the Chern character as a differential form on $\mathcal{MC}(C^\bullet, \mathrm{End}(V))$, we construct a cycle in the negative cyclic complex of the cosimplicial algebra $\mathcal{O}(\mathcal{MC}(C^\bullet, \mathrm{End}(V)))$:

$$\mathrm{CN}_k(\mathcal{O}(\mathcal{MC}(C^\bullet, \mathrm{End}(V)))) = \prod_{q=0}^{\infty} \mathrm{CN}_{k+q}(\mathcal{MC}(C^q, \mathrm{End}(V))).$$

Since $\mathrm{End}(V)$ is finite-dimensional, there is a morphism of cosimplicial differential graded algebras

$$BC^\bullet \to \mathcal{O}(\mathcal{MC}(C^\bullet, \mathrm{End}(V))) \otimes \mathrm{End}(V),$$

familiar from the theory of derived representation schemes, and hence morphisms of negative cyclic chain complexes

$$\begin{aligned}\mathrm{CN}_*(BC^\bullet) &\to \mathrm{CN}_*(\mathcal{O}(\mathcal{MC}(C^\bullet, \mathrm{End}(V))) \otimes \mathrm{End}(V)) \\ &\to \mathrm{CN}_*(\mathcal{O}(\mathcal{MC}(C^\bullet, \mathrm{End}(V)))) \otimes \mathrm{CN}_*(\mathrm{End}(V)) \\ &\to \mathrm{CN}_*(\mathcal{O}(\mathcal{MC}(C^\bullet, \mathrm{End}(V)))).\end{aligned}$$

The second morphism is the cyclic Alexander–Whitney map of Hood and Jones [3] (for which an explicit formula may be found in Bauval [1]), and the third morphism is induced by the supertrace on $\mathrm{End}(V)$.

In our talk, we gave an explicit formula for the Chern character as a cycle in $\mathrm{CN}_0(BC^\bullet)$. This is the image of a cycle in the negative cyclic homology of the cosimplicial category $[n] \mapsto k[\![n]\!]$, calculated using homological perturbation theory. Details will appear in a forthcoming article.

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L-theory of C^*-algebras

Markus Land

(joint work with T. Nikolaus and M. Schlichting)

The purpose of this talk was to explain and expand on the following theorem.

Theorem 1. *Let A be a real C^*-algebra. Then there are canonical isomorphisms*

(1) $\mathrm{L}_0(A) \cong \mathrm{KO}_0(A)$,
(2) $\mathrm{L}_1(A) \cong \mathrm{KO}_1(A)/\eta = \mathrm{Coker}(\mathrm{KO}_0(A) \xrightarrow{\eta} \mathrm{KO}_1(A))$,
(3) $\mathrm{L}_2(A) \cong \mathrm{KO}_6(A)[\eta] = \mathrm{Ker}(\mathrm{KO}_6(A) \xrightarrow{\eta} \mathrm{KO}_7(A))$, *and*
(4) $\mathrm{L}_3(A) \cong \mathrm{KO}_7(A)$.

Since the L-groups are 4-periodic, i.e. for all $n \in \mathbb{Z}$ we have $\mathrm{L}_{n+4}(A) \cong \mathrm{L}_n(A)$, this describes all algebraic L-groups.

Here, a real C^*-algebra is an involutive Banach algebra over $\mathbb{R}$, satisfying

(1) the C^*-identity $||x^*x|| = ||x||^2$ for all $x \in A$, and the requirement that
(2) for all $x \in A$, the element $1 + x^*x$ is invertible (in the unitalisation of A if A does not have a unit).

Examples are norm and $*$-closed subalgebras of $\mathcal{B}(\mathcal{H})$, the bounded operators on a real Hilbert space $\mathcal{H}$ (in fact, all examples are of this kind), continuous C_2-equivariant $\mathbb{C}$-valued functions on a compact Hausdorff space X equipped with a C_2-action $C(X;\mathbb{C})^{C_2}$, where C_2 acts on $\mathbb{C}$ by complex conjugation (in fact, all commutative unital examples are of this kind) and group C^*-algebras $C^*_{(r)}G$ obtained by completing the real group ring $\mathbb{R}G$ in appropriate C^*-norms.

By $\mathrm{KO}_*(A)$ we denote the real topological K-groups of such a real C^*-algebra A. These can be constructed from the topological category of finitely generated projective A-modules and are 8-periodic by real Bott periodicity. More specifically, $\mathrm{KO}_*(A)$ is a graded module over the graded commutative ring $\mathrm{KO}_* = \mathrm{KO}_*(\mathbb{R})$ which is given by

$$\mathrm{KO}_* = \mathbb{Z}[\eta, x, \beta_{\mathbb{R}}^{\pm 1}]/(\eta^3, 2\eta, \eta x, x^2 = 4\beta_{\mathbb{R}}) \quad \text{with} \quad |\eta| = 1, |x| = 4, |\beta_{\mathbb{R}}| = 8.$$

I then spent some time indicating what the algebraic L-groups are. In fact, these are defined more generally for any ring R equipped with an involution. To begin, let us first consider $\epsilon = \pm 1$ and make the following purely algebraic definition. We define $W^\epsilon(R)$, the ϵ-symmetric Witt group of R, to be given by the abelian monoid of isomorphism classes of *ϵ-symmetric unimodular forms over* $\mathrm{Proj}(R)$ modulo the submonoid generated by *metabolic forms*. Here,

(1) An ϵ-symmetric form over $\mathrm{Proj}(R)$ consists of a pair (P, β) of a finitely generated projective R-module P and an element β of $\mathrm{Hom}_{R\otimes R}(P \otimes P, R)^{C_2}$ where C_2 acts on the Hom set by flipping the factors of $P \otimes P$ and acting via ϵ on R. Concretely, this means that $\beta(x, y) = \epsilon \cdot \overline{\beta(y, x)}$ and that $\beta(rx, sy) = r\beta(x, y)\overline{s}$, where $\overline{(-)}$ denotes the involution in R.
(2) Such a form (P, β) is called unimodular if the canonically associated map $\beta^\sharp \colon P \to P^\vee = \mathrm{Hom}_R(P, R)$, given by $x mapsto \beta(x, -)$, is an isomorphism.
(3) A unimodular form (P, β) is called metabolic if there exists a direct summand inclusion $\iota\colon L \to P$ such that $\iota^*(\beta) = 0$ and such that the induced sequence

$$0 \longrightarrow L \longrightarrow P \overset{\beta^\sharp}{\cong} P^\vee \longrightarrow L^\vee \longrightarrow 0$$

is exact.
(4) We note that $(P, \beta) \oplus (P, -\beta)$ is metabolic: As L we can choose the diagonal embedding of P into $P \oplus P$. Consequently, $W^\epsilon(R)$ is indeed an abelian group.

Now, The L-groups appearing in the above theorem are Ranicki's projective symmetric L-groups, which turn out to be a direct generalisation of the above defined algebraic Witt groups. Concretely, $\mathrm{L}_n(R)$ is given by the abelian monoid of isomorphism classes of *ϵ-symmetric n-dimensional unimodular forms over* $\mathrm{Perf}(R)$ modulo the submonoid generated by *metabolic forms*. Here,

(1) $\mathrm{Perf}(R)$ denotes the stable ∞-category of perfect R-modules, (this is equivalent to the localisation of finite chain complexes of finitely generated projective R-modules at quasi-isomorphisms). In this context,
(2) an n-dimensional ϵ-symmetric form is a pair (P, β) where P is a perfect module and β is an element of $H_n(\mathrm{Hom}_{R\otimes R}(P \otimes P, R)^{hC_2})$, where all tensors are to be read as derived tensor products, and the superscript hC_2 refers to homotopy fixed points of the analogous C_2-action on the Hom-complex.

(3) (P,β) is called unimodular if the canonical map $\beta^\sharp\colon P \to P^\vee[-n] = \mathrm{Hom}_R(P, R[-n])$ is an equivalence. Here $[-n]$ denotes the appropriate shift functor in $\mathrm{Perf}(R)$, and finally
(4) (P,β) is called metabolic if there exists a map $\iota\colon L \to P$ in $\mathrm{Perf}(R)$ and a null-homotopy $0 \overset{\alpha}{\simeq} \iota^*(\beta)$ such that the induced diagram

$$\begin{tikzcd} L \arrow[r] \arrow[d] & P \overset{\beta^\sharp}{\simeq} P^\vee[-n] \arrow[d] \\ 0 \arrow[r] & L^\vee[-n] \end{tikzcd}$$

is a pullback diagram.

Fact 1.

(1) As described above, the L-groups are canonically 4*-periodic. The isomorphism is implemented (on the level of the underlying object) by the 2-fold shift functor in* $\mathrm{Perf}(R)$*. We have* $\mathrm{L}_*(\mathbb{R}) = \mathbb{Z}[b^{\pm 1}]$ *with* $|b| = 4$.
(2) There is a canonical map $W^\epsilon(R) \to \mathrm{L}_{\epsilon-1}(R)$*. This map is an isomorphism if* $2 \in R^\times$*, in particular for (unital)* C^**-algebras.*
(3) If $2 \in R^\times$*, the odd L-groups can also be described purely algebraically in terms of linking forms or formations over* R*.*
(4) For any two rings R, R' *we have a canonical isomorphism* $\mathrm{L}_n(R \times R') \xrightarrow{\cong} \mathrm{L}_n(R) \times \mathrm{L}_n(R')$*. In particular, when we define*

$$\mathrm{L}_n(A) = \mathrm{Ker}(\mathrm{L}_n(A^+) \to \mathrm{L}_n(\mathbb{R}))$$

for possibly non-unital C^**-algebras* A*, we have not changed the definition of L-theory on unital algebras (up to isomorphism).*

Remark. *For our purposes, it is important to note that both the topological K-groups and the algebraic L-groups are in fact given by the homotopy groups of topological K-theory and algebraic L-theory* spectra $\mathrm{KO}(A)$ *and* $\mathrm{L}(A)$*, respectively. We shall also use the connective topological K-theory* $\mathrm{ko}(A) = \tau_{\geq 0}\mathrm{KO}(A)$ *obtained from the (periodic) spectrum* $\mathrm{KO}(A)$ *by setting all negative homotopy groups to be zero. Moreover, both* KO *and* L *are canonically lax symmetric monoidal when viewed as functors* $\mathrm{C}^*\mathrm{Alg} \to \mathrm{Sp}$*, the symmetric monoidal structure essentially being induced by tensor products of finitely generated projective modules and unimodular symmetric forms, respectively. The functor* ko *then inherits a canonical symmetric monoidal structure from* KO.

Some motivation. Let me briefly explain where some of the motivation for studying L-theory of C^*-algebras comes from.

(1) It is known that both the Baum–Connes conjecture and the L-theoretic Farrell–Jones conjecture imply the Novikov conjecture. Our original motivation was to study the precise relation between these conjectures. Since the BCC and the FJC are determined by the assembly maps for spectrum valued functors associated to $\mathrm{KO}(-)$ and $\mathrm{L}(-)$, this really amounts to

studying the relation between the two functors $\mathrm{KO}, \mathrm{L}\colon \mathrm{C}^*\mathrm{Alg} \to \mathrm{Sp}$. This relation was essentially worked out for complex C^*-algebras and for real C^*-algebras after inverting 2, i.e. for $\mathrm{KO}[\frac{1}{2}]$ and $\mathrm{L}[\frac{1}{2}]$ in [1], but the precise integral relation for real C^*-algebras was left open in loc. cit.

(2) As a second curious point, the signature operator gives rise to a genus on oriented bordism with values in $\mathrm{KO}[\frac{1}{2}]$ while the Sullivan–Ranicki orientation provides a genus with values in $\mathrm{L}[\frac{1}{2}]$. Understanding the precise relationship between these two genera requires to fix a canonical comparison between $\mathrm{KO}[\frac{1}{2}]$ and $\mathrm{L}[\frac{1}{2}]$, and in our new joint paper with Nikolaus and Schlichting, we discuss this relation as well.

Being homotopy theorists, it may come as no surprise that our approach to proving Theorem 1 above is by homotopy theoretic means. Indeed, we derive it from the following spectral version of it:

Theorem 2. *There is a unique lax symmetric monoidal transformation $\tau\colon \mathrm{ko} \to \mathrm{L}$. The induced map*

$$\mathrm{ko}(A) \otimes_{\mathrm{ko}(\mathbb{R})} \mathrm{L}(\mathbb{R}) \to \mathrm{L}(A)$$

is an equivalence for every C^*-algebra A.

Deriving Theorem 1 from Theorem 2 still requires an observation, namely a suitable presentation (in small degrees) of $\tau_{\geq 0}\mathrm{L}(\mathbb{R})$ as a $\mathrm{ko}(\mathbb{R})$-module spectrum.

To prove Theorem 2, we wish to use KK-theoretic tools. For this, I briefly recalled the relevant properties of KK-theory: Kasparov constructed an additive category KK and a functor $\mathrm{C}^*\mathrm{Alg} \to \mathrm{KK}$. Let us then consider the following situation:

$$\begin{tikzcd} \mathrm{C}^*\mathrm{Alg} \arrow[r] \arrow[dr, "F"'] & \mathrm{KK} \arrow[d, dashed, "\bar{F}"] \\ & \mathcal{C} \end{tikzcd}$$

Fact 2.

(1) The canonical restriction functor $\mathrm{Fun}(\mathrm{KK}, \mathcal{C}) \to \mathrm{Fun}(\mathrm{C}^*\mathrm{Alg}, \mathcal{C})$ *is fully faithful. In other words, either a functor F factors through* KK *(and if so in a unique way, denoted by $\bar{F}$) or it doesn't.*

(2) Higson then showed that if $\mathcal{C}$ is additive, and the functor F above is $\mathcal{K}$-stable and split-exact, then F factors through KK *and its induced functor $\bar{F}$ is again additive.*

(3) It was shown early on that KK *admits a canonical triangulated structure. One would like to then have a variant of Higson's result in case $\mathcal{C}$ is triangulated. However, a functor being triangulated is not a* property *of a functor so such an analog is more subtle to obtain. ∞-categories help in this respect. Based on various earlier results in KK-theory, Nikolaus and I showed in* [1] *that the ∞-categorical localisation* $\mathrm{KK}_\infty = \mathrm{C}^*\mathrm{Alg}[\mathrm{KK}_{eq}^{-1}]$ *of* $\mathrm{C}^*\mathrm{Alg}$ *at the* KK*-equivalences is a stable ∞-category. Moreover, we showed that if $\mathcal{C}$ as above is stable, and F is $\mathcal{K}$-stable a semi-exact, then*

$\bar{F}$ is an exact functor of stable ∞-categories. Here, semi-exact means that F sends short exact sequences with cpc split to fibre sequences in $\mathcal{C}$.

(4) Using this, we showed that $\mathrm{KO}(-) \simeq \mathrm{map}_{\mathrm{KK}_\infty}(\mathbb{R}, -)$. Thus, studying transformations out of K-theory is controlled by the Yoneda lemma. One result we proved using such KK*-theoretic tools is that there is a canonical equivalence $\mathrm{KO}[\frac{1}{2}] \simeq \mathrm{L}[\frac{1}{2}]$, and we constructed a map as in Theorem 2 for* complex *C^*-algebras.*

To use such KK-theoretic tools we then need to know that L-theory factors through KK. As indicated, our previous work [1] treated only the case of complex C^*-algebras, or needed to invert 2. Whether or not L-theory is (integrally) KK-invariant was left open in loc. cit. The following theorem settles this.

Theorem 3. *The functor $\mathrm{L}\colon \mathrm{C^*Alg} \to \mathrm{Sp}$ factors through an additive functor $\mathrm{KK} \to \mathrm{Sp}$. Moreover, there is a canonical equivalence $\mathrm{L}(-) \to \mathrm{ko}(-)^{tC_2}$ after 2-adic completion.*

Indeed, the functor $\mathrm{L}\colon \mathrm{KK} \to \mathrm{Sp}$ is *not* exact and the second part of Theorem 3 gives a formula for the failure of exactness. This formula is also used in our proof of Theorem 2, which in addition uses topological Grothendieck–Witt theory and a fibre sequence relating (connective) topological K-theory, topological Grothendieck–Witt theory, and L-theory.

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Riemann-Roch for $\overline{\mathrm{Spec}\mathbb{Z}}$

Alain Connes
(joint work with C. Consani)

The main result [5] is the arithmetic Riemann-Roch formula (1) for $\overline{\mathrm{Spec}\mathbb{Z}}$, the one point compactification of $\mathrm{Spec}\mathbb{Z}$. We endow $\overline{\mathrm{Spec}\mathbb{Z}}$ with a structure sheaf constructed as a subsheaf of the constant sheaf $\mathbb{Q}$ using the following generalization of abelian groups. It is obtained by viewing an abelian group A as a covariant functor HA from the category of finite pointed sets to the category of pointed sets which assigns to X the pointed set of A-valued divisors on X which are 0 on the base point. The functoriality is obtained by taking the sum over the preimage of a point.

Let D be an Arakelov divisor on $\overline{\mathrm{Spec}\mathbb{Z}}$. Then

$$\dim_{\mathbb{S}[\pm 1]} H^0(D) - \dim_{\mathbb{S}[\pm 1]} H^1(D) = \left\lceil \frac{\deg D + \log 2}{\log 3} \right\rceil' - \mathbf{1}_L. \tag{1}$$

On the (topological) right-hand side of (1), $\lceil x \rceil'$ denotes the odd function on $\mathbb{R}$ that agrees with the ceiling function on positive reals. The set $L \subset \mathbb{R}$ is the union, for $k \geq 0$, of the intervals $(\log \frac{3^k}{2}, \log \frac{3^k+1}{2})$: L has finite Lebesgue measure. The symbol $\mathbf{1}_L$ denotes the (degree) indicator function: this is zero unless $\deg D \in L$ in which case it is 1. The numbers $\dim_{\mathbb{S}[\pm 1]} H^i(D)$ on the (cohomological) left-hand side are integers providing the dimension of the cohomology of D, as a module over the spherical group ring $\mathbb{S}[\pm 1]$. The notion of dimension and the definition of the cohomologies are based on the universal arithmetic over $\mathbb{S}[\pm 1]$ which we developed using Segal's Gamma rings in [1, 2, 3, 4]. This new perspective allows us to parallel Weil's adelic proof of the Riemann-Roch formula for function fields including the use of Pontryagin duality. We comment on the comparison with the asymptotic Riemann-Roch formula of S. Lang [6] (foreseen by A. Weil in [7]) relating $\log \# H^0(D)$ and $\deg D + \log 2$, when $\deg D \to \infty$. The first novelty is that unlike $\log \# H^0(D)$, $\dim_{\mathbb{S}[\pm 1]} H^i(D)$ are non-negative integers. Secondly, the cohomological side of (1) implements explicitly (the dimension of) $H^1(D)$, thus it goes well beyond the arithmetic Riemann-Roch formula for $\mathbb{Q}$, where $H^1(D)$ is either undefined or set to equate $H^0(K - D)$, for an expected Serre duality. All the more: (1) is an exact formula and not an asymptotic statement. All these advancements derive from the new understanding of $\overline{\mathrm{Spec}\mathbb{Z}}$ as a curve over the absolute ring $\mathbb{S}[\pm 1]$. The third relevant fact is that on the topological side of the Riemann-Roch formula the traditional sum $\deg D + \log 2$ is now divided by $\log 3 \sim 1.09861$, and there also appears an exceptional set $L \subset \mathbb{R}$ of finite Lebesgue measure $|L| = \log \prod (1 + 3^{-n}) \sim 1.14099$. The Euler characteristic of D is a non-decreasing function of $\deg D$, while the topological side in (1) drops by 1 if $\deg D \in L$. Except for the set L, this topological side is invariant under Serre's duality which replaces D with $K - D$ ($\deg K = -2 \log 2$), since the function $\lceil x \rceil'$ is odd.

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On the K-theory of $\mathbb{Z}/p^n$

Achim Krause
(joint work with Ben Antieau, Thomas Nikolaus)

In recent work, we develop new methods to study the K-theory of rings such as $\mathbb{Z}/p^n$. Previously, these have been computed only up to degree $2p-2$, [3]. For the similar ring $\mathbb{F}_p[x]/x^n$, a complete computation is known [2]. We can make the following comparison:

	$\mathbb{F}_p[x]/x^n$	$\mathbb{Z}/p^n$
	$K_0 \cong \mathbb{Z}$	same
$i > 0$:	K_i finite torsion	same
tors. prime:	$K_{2i-1}[1/p] \cong \mathbb{Z}/(p^i - 1)$	same
to p	$K_{2i-2}[1/p] \cong 0$	same
p-power:	$K_{2i-2}(\mathbb{F}_p[x]/x^n; \mathbb{Z}_p) = 0$	$K_{2i-2}(\mathbb{Z}/p^n; \mathbb{Z}_p) \cong 0$ for $i < p$,
tors.		but $K_{2p-2}(\mathbb{Z}/p^n; \mathbb{Z}_p) = \mathbb{Z}/p$
	$\lvert K_{2i-1}(\mathbb{F}_p[x]/x^n; \mathbb{Z}_p)\rvert = p^{i(n-1)}$	$\frac{\lvert K_{2i-1}(\mathbb{Z}/p^n;\mathbb{Z}_p)\rvert}{\lvert K_{2i-2}(\mathbb{Z}/p^n;\mathbb{Z}_p)\rvert} = p^{i(n-1)}$

The mysterious part is thus entirely given by the p-power torsion, equivalently the p-complete K-theory of $\mathbb{Z}/p^n$. Our first result is an explicit algorithm to compute these groups:

Theorem 1. *For each i and n, there is an explicit 3-term cochain complex of the form*

$$\mathbb{Z}_p^{in-1} \to \mathbb{Z}_p^{2(in-1)} \to \mathbb{Z}_p^{in-1}$$

with $H^0 \cong 0$, $H^1 \cong K_{2i-1}(\mathbb{Z}/p^n; \mathbb{Z}_p)$, $H^2 \cong K_{2i-2}(\mathbb{Z}/p^n; \mathbb{Z}_p)$ (for $i > 1$).

The explicit description of the maps in this complex is in terms of prismatic cohomology, but can be made explicit enough to implement in a computer algebra system. We have done so, and obtain explicit tables (currently for $p = 2$, $i \leq 16$ and $n \leq 5$, to be extended). One striking pattern is that, while the pattern of vanishing of even K-groups observed for $\mathbb{F}_p[x]/x^n$ only holds for $\mathbb{Z}/p^n$ in degrees $< 2p-2$, it does actually restart in large degrees!

Theorem 2 (Even Vanishing Theorem)**.** *For i large enough ($i \geq \frac{p^2}{(p-1)^2}(p^n - 1)$), we have:*

$$K_{2i-2}(\mathbb{Z}/p^n; \mathbb{Z}_p) = 0$$
$$\lvert K_{2i-1}(\mathbb{Z}/p^n; \mathbb{Z}_p)\rvert = p^{i(n-1)}.$$

Remark 3. *More generally, our methods apply to rings of the form $\mathcal{O}_K/\mathfrak{m}^n$, where K is a p-adic number field, i.e. a finite extension of $\mathbb{Q}_p$. This generalizes both $\mathbb{Z}/p^n$ and $\mathbb{F}_p[x]/x^n$.*

Our approach is based on trace methods, which we quickly review. For any ring R, we have its Hochschild homology $\mathrm{HH}(R)$, which comes with extra structure, in the form of a degree $+1$ differential, the Connes operator. In terms of this structure, one defines refinements of HH, negative cyclic and periodic homology $\mathrm{HC}^-(R)$ and $\mathrm{HP}(R)$, with a canonical map $\mathrm{HC}^-(R) \to \mathrm{HP}(R)$.

We can think of Hochschild homology as being formed relative to $\mathbb{Z}$. In higher algebra, there is a deeper base: The initial commutative ring spectrum is given by the sphere spectrum $\mathbb{S}$. Viewing our ordinary ring R as a ring spectrum, one can form Hochschild homology relative $\mathbb{S}$, which is called topological Hochschild homology $\mathrm{THH}(R) = \mathrm{HH}(R/\mathbb{S})$. The analogue of the Connes operator is given by an action of S^1 on $\mathrm{THH}(R)$, and $\mathrm{TC}^-(R)$ and $\mathrm{TP}(R)$ can be defined in terms of that action. One genuinely new piece of structure is that there is a *cyclotomic Frobenius* map $\varphi : \mathrm{TC}^-(R;\mathbb{Z}_p) \to \mathrm{TP}(R;\mathbb{Z}_p)$ in addition to the canonical map. The equalizer of these two maps is denoted $\mathrm{TC}(R;\mathbb{Z}_p)$ and called (p-typical) topological cyclic homology[1]. There is a trace map $K(R) \to \mathrm{THH}(R)$ roughly taking every projective module to the trace of its identity, it lifts to a map $K(R;\mathbb{Z}_p) \to \mathrm{TC}(R;\mathbb{Z}_p)$, the cyclotomic trace. The situation is summarized in the following diagram:

$$\begin{tikzcd}
K(R;\mathbb{Z}_p) \arrow[r] & \mathrm{TC}(R;\mathbb{Z}_p) \arrow[d] & \\
& \mathrm{TC}^-(R;\mathbb{Z}_p) \arrow[r, shift left, "\varphi"] \arrow[r, shift right, "\mathrm{can}"'] \arrow[d] & \mathrm{TP}(R;\mathbb{Z}_p) \\
& \mathrm{THH}(R;\mathbb{Z}_p) &
\end{tikzcd}$$

Here $\mathrm{TC}(R;\mathbb{Z}_p)$ is the fiber of the difference of the two middle horizontal maps.

In good cases, the map $K_*(R;\mathbb{Z}_p) \to \mathrm{TC}_*(R;\mathbb{Z}_p)$ is an isomorphism in nonnegative degrees. A special case of [1] implies that this is the case for $R = \mathbb{Z}/p^n$. Thus, it suffices to understand $TC(\mathbb{Z}/p^n;\mathbb{Z}_p)$. The aforementioned computations of $K_*(\mathbb{F}_p[x]/x^n)$ in all degrees, and $K_*(\mathbb{Z}/p^n)$ in degrees $\leq 2p-2$, also follow this approach, but use topological methods to control THH, TC^- and TP. We will employ a more algebraic perspective.

In ordinary Hochschild homology, $\mathrm{HH}(R)$ is closely related to the algebraic differential forms $\Omega^*_{R/\mathbb{Z}}$. $\mathrm{HP}(R)$ incorporates the Connes operator, which corresponds to the de Rham differential, and is closely related to $H_{dR}(R)$. Finally, $\mathrm{HC}^-(R)$ interpolates between the two: It is closely related to $H_{dR}(R)$ with its Hodge filtration, which is the filtration obtained by truncating the de Rham complex (whose associated graded terms recover Ω^*).

Prismatic cohomology, recently discovered and developed in [4], [5], [6] plays an analogous role for TP and TC^-. Specifically, they construct a filtration on $\mathrm{TP}(R;\mathbb{Z}_p)$ and $\mathrm{TC}^-(R;\mathbb{Z}_p)$, whose i-th associated graded (shifted by $-2i$) is given by $\widehat{\Delta}_R\{i\}$ and $\mathcal{N}^{\geq i}\widehat{\Delta}_R\{i\}$. Here $\widehat{\Delta}_R$ is (Nygaard-completed) prismatic cohomology of R, $\{i\}$ denotes the Breuil-Kisin twist, and $\mathcal{N}^{\geq i}$ the i-th stage of the Nygaard filtration. Without going more into detail here what these things are precisely, crucially they are defined in purely algebraic terms.

[1] Unfortunately, the name is potentially confusing: This is not the direct generalisation of what is called cyclic homology in the ordinary setting!

From the fiber sequence

$$\mathrm{TC}(R;\mathbb{Z}_p) \longrightarrow \mathrm{TC}^-(R;\mathbb{Z}_p) \xrightarrow{\mathrm{can}-\varphi} \mathrm{TP}(R;\mathbb{Z}_p)$$

one obtains a filtration on $\mathrm{TC}(R;\mathbb{Z}_p)$, whose i-th associated graded (shifted by $-2i$) is the i-th syntomic complex $\mathbb{Z}_p(i)(R)$, defined as the fiber of

$$\mathrm{can} - \varphi : \mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_R\{i\} \to \widehat{\mathbb{\Delta}}_R\{i\}.$$

It turns out both terms here are concentrated in degree -1 in the case $R = \mathbb{Z}/p^n$. So the fiber is concentrated in degrees -1, -2, so they cannot interact in the spectral sequence recovering TC_*. We get $K_{2i-1}(\mathbb{Z}/p^n;\mathbb{Z}_p) \cong \mathrm{TC}_{2i-1}(\mathbb{Z}/p^n;\mathbb{Z}_p) \cong H^1(\mathbb{Z}_p(i)(R))$, and analogously for K_{2i-2} and H^2. This is the source for Theorem 1, alternatively one can phrase Theorem 1 as giving a equivalence between $\mathbb{Z}_p(i)(R)$ and the complex given there.

This description arises from the following steps:

(1) We describe a descent mechanism exhibiting $\widehat{\mathbb{\Delta}}_R$ as cosimplicial limit of relative variants. This is based on ideas in [7], [8], but depends crucially on extended functoriality of relative prismatic cohomology, which we found.
(2) Making the formalism of prismatic envelopes of [5] explicit, we obtain generators-and-relations descriptions for all terms involved in the cosimplicial diagram of step (1). This shows that the cochain complex obtained from step (1) has cohomological dimension 1, in total, this allows us to describe each of $\widehat{\mathbb{\Delta}}_R\{i\}$, $\mathcal{N}^{\geq i}\widehat{\mathbb{\Delta}}_R\{i\}$ as 2-term complex, and so the fiber $\mathbb{Z}_p(i)(R)$ as total complex of a square.
(3) A filtration argument allows us to replace the infinitely presented terms in the square by terms of finite type, explicitly giving

$$\mathbb{Z}_p(i)(R) \cong \mathrm{Tot}\left(\begin{array}{ccc} \mathbb{Z}_p^{in-1} & \longrightarrow & \mathbb{Z}_p^{in-1} \\ \downarrow & & \downarrow \\ \mathbb{Z}_p^{in-1} & \longrightarrow & \mathbb{Z}_p^{in-1} \end{array}\right)$$

The proof of Theorem 2 is based on further investigating when the two maps into the lower right corner are jointly surjective.

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Prolate spheroidal operator and Riemann zeta function

Henri Moscovici
(joint work with Alain Connes)

1.1. The differential operator called “prolate spheroidal” (because it first appeared from separation of variables in the Laplacian for the prolate spheroid) is given by the expression

$$(W_\lambda\xi)(x) = -\partial_x(\lambda^2 - x^2)\partial_x\xi(x) + (2\pi\lambda)^2x^2\xi(x), \quad \xi \in C^\infty(\mathbb{R}),\ \lambda > 0,\ \partial_x = \frac{d}{dx}.$$

It has the remarkable property of simultaneously commuting with the projection $P_\lambda(\xi) = \xi \mid [-\lambda, \lambda]$ and with the Fourier transform

$$\mathbb{F}(\xi)(x) = \int_{\mathbb{R}} e^{-2\pi ixy}\xi(y)dy, \qquad \xi \in L^2(\mathbb{R});$$

it therefore commutes with the projection $\widehat{P_\lambda} = \mathbb{F}^{-1}P_\lambda\mathbb{F}$ too, hence also with the integral operator

$$\left(\widehat{P}_\lambda P_\lambda\xi\right)(x) = \frac{1}{\pi}\int_{-\lambda}^{\lambda} \frac{\sin\big(2\pi\lambda(x-y)\big)}{x-y}\,\xi(y)dy.$$

In early 1960s the latter operator sat at the heart of the bandwidth concentration problem in signal processing, and its commutation with W_λ amounted to what David Slepian later [6] called the “lucky accident” which allowed him and his collaborators H. Pollak and H. Landau at Bell Labs to solve that problem “completely, easily and quickly” in 1961. Indeed, as solutions of a second-order ODE, the regular eigenfunctions of the restriction of $W_\lambda \mid [-\lambda, \lambda]$ were known, and they allowed computing the eigenvalues of $\widehat{P}_\lambda P_\lambda$. The same PSWFs (prolate spheroidal wave functions) played a key role in relation with the Riemann zeta function, in A. Connes semi-local trace formula [1, 2] and in his joint papers with C. Consani [3, 4].

Although a natural self-adjoint extension $W_{\rm sa}$ of W_λ (viewed as a differential operator on the entire real line with core the Schwartz space $\mathcal{S}(\mathbb{R})$) was introduced by Connes [1] in 1998, its full spectrum remained unexplored until recently, when its investigation carried out in [5] revealed another surprising feature of the prolate spheroidal operator: besides the positive eigenvalues (corresponding to the PSWFs and their Fourier transforms), $W_{\rm sa}$ admits a negative spectrum, whose ultraviolet behavior closely matches that of the squares of the zeros of the Riemann zeta function.

1.2. The precise statements of our main results are as follows.

Theorem 1 (**Self-adjoint extension).**
(i) *The closure* $W_{\min}$ *of* $W_\lambda : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ *has deficiency indices* $(4;4)$.
(ii) $W_{\min}$ *has a unique self-adjoint extension* W_{sa} *commuting with* P_λ *and* $\widehat{P}_\lambda$.
(iii) $\mathrm{Dom}W_{\mathrm{sa}}$ *consists of* $\xi \in \mathrm{Dom}W_{\max}$ *satisfying the boundary condition:*

$$\lim_{x\to\pm\lambda}(\lambda^2 - x^2)\partial_x\xi(x) = 0,$$

$$\lim_{x\to\pm\infty}\big(x\sin(2\pi\lambda x)\partial_x\xi^+ - \big(2\pi\lambda x\cos(2\pi\lambda x) - \sin(2\pi\lambda x)\big)\xi^+\big) = 0,$$
$$\lim_{x\to\pm\infty}\big(x\cos(2\pi\lambda x)\partial_x\xi^- + \big(2\pi\lambda x\sin(2\pi\lambda x) + \cos(2\pi\lambda x)\big)\xi^-\big) = 0,$$

where $\xi = \xi^+ + \xi^-$, $\xi^\pm \in \mathrm{Dom}(W^\pm_{\max})$, *with* $\pm$ *signifying restriction to* even, *resp.* odd *functions.*

(iv) $\mathrm{Spec}W_{\mathrm{sa}}$ *is discrete and unbounded on both sides; the positive eigenvalues of* W_{sa} *are double (with possibly finitely many exceptions), and the negative eigenvalues are simple.*

Denote by $S_{\lambda,\lambda}$ the *Sonin space*, which consists of the functions $\xi \in L^2(\mathbb{R})$ satisfying $P_\lambda(\xi) = 0 = \widehat{P_\lambda}(\xi)$.

Theorem 2 (**Sonin space).** *The eigenfunctions corresponding to the negative eigenvalues of* W_{sa} *belong to the Sonin space* $S_{\lambda,\lambda}$*; they generate it up to at most finite codimension.*

Let $W'_{\mathrm{sa}} = (I - P_\lambda)W_{\mathrm{sa}}$ on $\mathcal{H} = (I - P_\lambda)L^2(\mathbb{R})$ decomposed as $W'_{\mathrm{sa}} = {W'_{\mathrm{sa}}}^+ \oplus {W'_{\mathrm{sa}}}^-$ with respect to the orthogonal decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into even, resp. odd functions. By restriction of functions, $\mathcal{H}^\pm \cong L^2(\lambda, \infty)$.

Denoting $\nabla := p^{1/4}\partial_x p^{1/4}$, consider the Riccati equation

$$p^{1/2}(x)\partial w(x) + w(x)^2 = -q(x) + \left(\frac{p''(x)}{4} - \frac{p'(x)^2}{16p(x)}\right), \quad x \in (\lambda, \infty);$$

Theorem 3 (**Darboux factorization).** *The solutions of the Riccati equation are of the form* $w_z = \frac{\nabla u}{u}$ *with* $u = u_1 + zu_2$*, where* u_1, u_2 *linearly independent real solutions of* $(W_\lambda u)(x) = 0$, $x \in (\lambda, \infty)$, $z \in \mathbb{C}\setminus\mathbb{R}$*, and the map* $z \mapsto w_z$ *is a homeomorphism of* $\mathbb{C}\setminus\mathbb{R}$ *onto the space of solutions. Furthermore,* $W_\lambda|(\lambda,\infty) = (\nabla + w)(\nabla - w)$.

Theorem 4 (**Dirac-type operators).** *With* w *denoting a solution of the Riccati equation, let* $\not{D}_w$ *be the operator defined by*

$$\not{D}_w = \begin{pmatrix} 0 & \nabla + w \\ \nabla - w & 0 \end{pmatrix},$$

with domain

$$\mathrm{Dom}\not{D}_w := \left\{ \begin{pmatrix} \xi \\ \tilde{\xi} \end{pmatrix};\ \xi \in \mathrm{Dom}{W'_{\mathrm{sa}}}^+,\ (\nabla + w)(\tilde{\xi}) \in \mathrm{Dom}{W'_{\mathrm{sa}}}^+ \right\}.$$

Then

(i) $\not{D}_w^2 = \begin{pmatrix} {W'_{\rm sa}}^+ & 0 \\ 0 & {W'_{\rm sa}}^+ + 2\nabla w \end{pmatrix}$ *with the diagonal terms isospectral operators;* ${W'_{\rm sa}}^+$ *is self-adjoint and* ${W'_{\rm sa}}^+ + 2\nabla w$ *is quasi-Hermitian.*

(ii) $\mathrm{Spec}\not{D}_w = \{\pm\sqrt{\mu} \mid \mu \in \mathrm{Spec}{W'_{\rm sa}}^+\}$*, independently of* w*.*

Recall Riemann's estimate for the counting function of the zeros:

$$N(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) + O(\log E), \quad \text{as} \quad E \to \infty.$$

Theorem 5 (**Ultraviolet behavior of the Dirac spectrum).** $\mathrm{Spec}2\not{D}$ *is discrete, simple and contained in* $\mathbb{R} \cup i\mathbb{R}$*. The imaginary eigenvalues are symmetric under complex conjugation and the function* $N_{\not{D}}(E)$*, counting those of positive imaginary part less than* E*, has the asymptotic behavior*

$$N_{\not{D}}(E) \sim \frac{E}{2\pi}\left(\log\left(\frac{E}{2\pi}\right) - 1\right) - \frac{\log E}{2\pi} + O(1), \quad \textit{as} \quad E \to \infty.$$

1.3. **Final remark.** The Sonin space $S_{1,1}$ was shown in [3] to be the root of Weil's positivity at the Archimedean place, for test functions supported in the interval $[2^{-1/2}, 2^{1/2}]$. The operator $W_{\rm sa}$ commutes with P_λ and $\widehat{P}_\lambda$, and the above result suggests that its restriction to $S_{1,1}$ essentially captures the contribution of the Archimedean place to the Riemann zeta spectrum.

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Index theory and cyclic (co)homology with a view towards secondary index theory

Thomas Schick

(joint work with Paolo Piazza, Vito Zenobi)

The Chern character is a well known transformation from (topological) K-theory of (Fréchet) algebras to cyclic homology. There are explicit formulas for the Chern character of the algebraic index of an elliptic differential operator.

Given a smooth manifold M with fundamental group Γ, Higson and Roe [3] introduced a short exact sequence of C^*-algebras whose associated long exact sequence in K-theory. After natural identifications, this K-theory sequence is of the form $\cdots \to S_*^\Gamma(\widetilde{M}) \to K_*(M) \to K_*(C^*_{red}\Gamma) \to \cdots$ and combines information about topology (in terms of the K-homology of M), analytic index (living in the K-theory of the group C^*-algebra of Γ), and secondary structural information about geometric reasons for the vanishing of indices. Here, we refer in particular to positive scalar curvature metrics which imply that the index of the Dirac operator is zero, or to homotopy equivalence which imply that the difference of signature indices is zero.

Zenobi [5] gives a new definition of the Higson-Roe sequence as the K-theory sequence of (relative) algebras of pseudodifferential operators. These algebras are much smaller than the ones of Higson and Roe and therefore much more amenable to define invariants out of the K-theory groups.

In the talk, we report on some of the work achieved in [4]. There, we develop a corresponding theory of *relative* cyclic homology (in the form of de Rham homology) and obtain in the end a commutative diagram of Chern character maps

$$\begin{tikzcd}
\cdots \arrow[r] & K_{*-1}(C^*_{red}\Gamma) \arrow[r, "s"] \arrow[d, "Ch_\Gamma"] & S_*^\Gamma(\tilde{M}) \arrow[r, "c"] \arrow[d, "Ch_\Gamma^{del}"] & K_*^\Gamma(\tilde{M}) \arrow[r] \arrow[d, "Ch_\Gamma^{e}"] & \cdots \\
\cdots \arrow[r, "j_*"] & H_{*-1}(\mathcal{A}\Gamma) \arrow[r, "\iota"] & H_{*-1}^{del}(\mathcal{A}\Gamma) \arrow[r, "\delta"] & H_*^e(\mathcal{A}\Gamma) \arrow[r, "j_*"] & \cdots
\end{tikzcd}$$

Here $\mathcal{A}\Gamma$ is any dense and holomorphically closed Fréchet subalgebra of $C^*\Gamma$, The Chern character map is obtained via intermediate cyclic/de Rham homology groups which are relative, and uses the full force of the developed theory of relative non-commutative de Rham homology.

If Γ is hyperbolic we derive explicit formulas for the pairing of Ch_Γ^{del} of a secondary index and cyclic group cohomology (supported away from the identity element). Using this, we deduce lower bounds on the size of the moduli space of Riemannian metrics of positive scalar curvature.

Related work has been carried out independently by Deeley and Goffeng [2] and Chen, Wang, Xie and Yu [1].

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Topological cyclic homology

Lars Hesselholt

The organizers asked me to give an introduction to topological cyclic homology and to recent developments, and this talk was my attempt to do so.

Every cohomology theory on an ∞-category Geom of "geometric" objects should arise in three steps as follows. Firstly, we should define a functor

$$\mathrm{Cat}_\infty^{\mathrm{ex}} \xrightarrow{E} \mathcal{C}$$

from the ∞-category of small stable ∞-categories and exact functors to some stable ∞-category, and show that this functor is a localizing invariant in the following sense. The ∞-category $\mathrm{Cat}_\infty^{\mathrm{ex}}$ is not stable, but it is pointed by the ∞-category 0. Now, the statement that E is a localizing invariant[1] means that:

(1) $E(0) \simeq 0$;
(2) E takes bicartesian squares in $\mathrm{Cat}_\infty^{\mathrm{ex}}$ to (bi)cartesian squares in $\mathcal{C}$; and
(3) E takes exact functors that become equivalences after idempotent completion to equivalences.

We should also promote E to a lax symmetric monoidal functor, so that it induces a functor from the ∞-category of small stably symmetric monoidal ∞-categories to that of commutative algebras in $\mathcal{C}$,

$$\mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{ex}}) \xrightarrow{E} \mathrm{CAlg}(\mathcal{C}).$$

Secondly, we should define a functor

$$\mathrm{Geom} \xrightarrow{\mathcal{V}^{\otimes}} \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{ex}})$$

that to a geometric object X assigns a small stably symmetric monoidal ∞-category $\mathcal{V}(X)^{\otimes}$ of "modules" of some kind.[2] For instance, to a scheme X, we assign the small stably symmetric monoidal ∞-category $\mathrm{Perf}(X)^{\otimes}$ of complexes of perfect $\mathcal{O}_X$-modules. Thirdly, we should define a "movitic" filtration

$$\cdots \longrightarrow \mathrm{Fil}^{n+1}E(X) \longrightarrow \mathrm{Fil}^{n}E(X) \longrightarrow \mathrm{Fil}^{n-1}E(X) \longrightarrow \cdots$$

on the composite functor $E(X) \simeq E(\mathcal{V}(X))$. The nth graded piece $\mathrm{gr}^n E(X)$ is now the desired cohomology of X of weight n.

The final third step has so far been more an art than a science. However, a recent paper by Hahn–Raksit–Wilson [6] defines the "even" filtration on CAlg(Sp) and shows that it recovers the "motivic" filtration on p-adic topological cyclic homology (in all its variants) defined by Bhatt–Morrow–Scholze [3], as well as the refinements thereof on non-completed topological cyclic homology defined independently by

[1] There is an initial localizing invariant, whose target, by definition, is the stable ∞-category NMot of noncommutative motives, envisioned by Kontsevich. This initial localizing invariant is a "noncommutative algebraic cycles functor" and it is basically impossible to understand.

[2] In analytic situations, the requirement that $\mathcal{V}(X)^{\otimes}$ be small is too restrictive. Efimov's lecture at the conference concerned this issue.

Morin [8] and Bhatt–Lurie [2]. In the case, where $E = \mathrm{TP}$ is periodic topological cyclic homology, the resulting cohomology theory is prismatic cohomology.

Shifting gears, I discussed anima or animated sets and the fact that these are discrete objects, as opposed to anything resembling a topological space. This is much clarified by the condensed mathematics of Clausen–Scholze [4], which unite "analytic" or "continuous" phenomena and "animated" or "derived" phenomena in the ∞-category of condensed anima. So anima are discrete, but have inner symmetries that make them animated. Among anima, sets are the static anima, whereas among condensed sets, sets are the discrete condensed sets.

$$\begin{tikzcd}
 & \mathrm{Cond}(\mathrm{An}) \simeq \mathrm{An}(\mathrm{Cond}) \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathrm{Cond} \arrow[ur, hook, shift left, "\text{static}"] \arrow[dr, shift left] & & \mathrm{An} \arrow[ul, hook, shift left, "\text{discrete}"'] \arrow[dl, shift left] \\
 & \mathrm{Set} \arrow[ul, hook, shift left, "\text{discrete}"] \arrow[ur, hook, shift left, "\text{static}"'] &
\end{tikzcd}$$

The inclusions of static objects are right adjoints and preserve limits, but not colimits, whereas the inclusions of discrete objects are left adjoints and preserve colimits, but not limits.

The fact that all anima are discrete informs the definition of the six-functor formalism on An that encodes algebraic topology. In his thesis, Mann [7] has given a precise definition of a what a six-functor formalism is, along with an omnibus theorem for producing this structure starting from classes of "local isomorphisms" and "proper" maps. In the (trivial) case of anima, we declare *every* map to be a local isomorphism, and we say that a map is proper if its fibers are compact projective anima a.k.a. finite sets. So to a map of anima $f\colon T \to S$, we have four functors among ∞-categories of presheaves of spectra

$$\begin{tikzcd}
\mathrm{Sp}^S \arrow[r, "f^! \simeq f^*"] & \mathrm{Sp}^T \arrow[l, bend right, "f_!"'] \arrow[l, bend left, "f_*"]
\end{tikzcd}$$

with $f^! \simeq f^*$ the restriction along f and with $f_!$ and f_* the left and right Kan extensions along f. The ∞-categories Sp^S and Sp^T are generated under small colimits by compact objects. The "homology" functor $f_!$ preserves compact objects, but the "cohomology" functor f_* does not. Nikolaus–Scholze [10] show that there is an essentially unique map $f_* \to f_*^T$ to a "Tate cohomology" functor that takes compact objects to zero. Its fiber preserves colimits, and therefore, necessarily is of form $f_!(D_f \otimes -)$ for some "dualizing" object D_f.[3]

Cyclic homology is naturally formulated in the stable ∞-category of spectra with $U(1)$-action. The Lie group $U(1)$ is not relevant, only its underlying group

[3] The dualizing object D_f is not $f^!(1)$ and does not appear to have an interpretation within the six-functor formalism.

in anima, or equivalently, the pointed connected anima $s\colon 1 \to BU(1)$ such that

$$\begin{tikzcd} U(1) \arrow[r, "s'"] \arrow[d, "s'"] & 1 \arrow[d, "s"] \\ 1 \arrow[r, "s"] & BU(1) \end{tikzcd}$$

is a cartesian diagram of anima. Since Connes' category Λ is a model for $BU(1)$, we can use the ∞-category of (co)cyclic spectra as a model for the ∞-category of spectra with $U(1)$-action. Now, in the case of the unique map of anima

$$BU(1) \xrightarrow{f} 1,$$

we have $D_f \simeq S^{\mathfrak{u}(1)} \simeq S^1$. Hence, for $X \in \mathrm{Sp}^{BU(1)}$ a spectrum with $U(1)$-action, the defining fiber sequence for Tate cohomology takes the form

$$\Sigma f_!(X) \longrightarrow f_*(X) \longrightarrow f_*^T(X).$$

For example, if $X \simeq \mathrm{HH}(A/R)$, then this recovers Connes' sequence

$$\Sigma\, \mathrm{HC}(A/R) \xrightarrow{B} \mathrm{HC}^-(A/R) \xrightarrow{I} \mathrm{HP}(A/R)$$

with "boundary" map S.

To recall the Nikolaus–Scholze [10] definition of a cyclotomic spectrum, let p be a prime number. As $U(1)$ is abelian, the map $p\colon U(1) \to U(1)$ that to z assigns z^p is a group homomorphism, or equivalent, is induced by a map

$$BU(1) \xrightarrow{p} BU(1)$$

of pointed anima. Now, a cyclotomic spectrum is pair $(X, (\varphi_p))$ of a spectrum with $U(1)$-action X and a family, indexed by the set of prime numbers, of maps of spectra with $U(1)$-action $\varphi_p\colon X \to p_*^T(X)$ from X to its Tate cohomology along the map $p\colon BU(1) \to BU(1)$. From the cartesian square of anima

$$\begin{tikzcd} BC_p \arrow[r, "i"] \arrow[d, "g"] & BU(1) \arrow[d, "p"] \\ 1 \arrow[r, "s"] & BU(1) \end{tikzcd}$$

and base-change, we obtain the equivalence

$$s^* p_*^T(X) \simeq g_*^T i^*(X)$$

between the underlying spectrum of the spectrum with $U(1)$-action $p_*^T(X)$ and the Tate cohomology of the spectrum with C_p-action $i^*(X)$ underlying the spectrum with $U(1)$-action X. Nikolaus–Scholze organize cyclotomic spectra into a stably symmetric monoidal ∞-category CycSp, equipped with a conservative symmetric monoidal "forgetful" functor to $\mathrm{Sp}^{BU(1)}$.

Topological Hochschild homology is a localizing invariant

$$\mathrm{Cat}_\infty^{\mathrm{ex}} \xrightarrow{\mathrm{THH}} \mathrm{CycSp},$$

and topological cyclic homology is defined in [10] as the localizing invariant

$$\mathrm{Cat}_\infty^{\mathrm{ex}} \xrightarrow{\mathrm{TC}} \mathrm{Sp}$$

obtained by composing THH with the functor CycSp $\to$ Sp corepresented by the tensor unit. The construction of THH is given in [9], but the composite functor

$$\mathrm{CAlg}(\mathrm{Sp}) \xrightarrow{\mathrm{Perf}} \mathrm{CAlg}(\mathrm{Cat}_\infty^{\mathrm{ex}}) \xrightarrow{\mathrm{THH}} \mathrm{CAlg}(\mathrm{CycSp})$$

may be described more easily. We have the (multiplicative) adjunction

$$\mathrm{CAlg}(\mathrm{Sp}) \underset{s^*}{\overset{s_!}{\rightleftarrows}} \mathrm{CAlg}(\mathrm{Sp})^{BU(1)},$$

where the left Kan extension $s_!$ takes a commutative algebra in spectra R to the free commutative algebra in spectra with $U(1)$-action $s_!(R)$ that it generates. Base-change identifies the underlying commutative algebra in spectra with

$$s^*s_!(R) \simeq s'_!s'^*(R) \simeq R^{\otimes U(1)},$$

the colimit in commutative algebras in spectra of the constant diagram with value R indexed by the anima $U(1)$. Now, by Nikolaus–Scholze [10], we have

$$\mathrm{THH}(R) \simeq (s_!(R), (\varphi_p))$$

where, to define the Frobenius maps φ_p, we consider the diagram of pointed anima

$$\begin{tikzcd}
1 \arrow[dr, "t"] \arrow[drr, bend left, "s"] \arrow[ddr, bend right, "\mathrm{id}"'] & & \\
& BC_p \arrow[r, "i"] \arrow[d, "g"] & BU(1) \arrow[d, "p"] \\
& 1 \arrow[r, "s"] & BU(1).
\end{tikzcd}$$

The Nikolaus–Scholze “Tate diagonal” is a map of commutative algebras in spectra

$$R \xrightarrow{\delta} g_*^T t_!(R)$$

and this map is the primordial Frobenius.[4] It gives a map

$$R \xrightarrow{\delta} g_*^T t_!(R) \xrightarrow{\eta} g_*^T i^* i_! t_!(R) \simeq g_*^T i^* s_!(R) \simeq s^* p_*^T s_!(R)$$

[4] By abuse notation, I also write g_*^T for the functor between ∞-categories of commutative algebras induced by the lax symmetric monoidal functor g_*^T.

of commutative algebras in spectra, whose adjunct

$$s_!(R) \xrightarrow{\varphi_p} p_*^T s_!(R)$$

is the desired map of commutative algebras in spectra with $U(1)$-action.

Finally, I discussed the motivic filtrations of the negative and periodic versions of topological cyclic homology, which both are given by the Hahn–Raksit–Wilson even filtration [6]. The canonical map of commutative algebras in spectra

$$\mathrm{TC}^-(R) \simeq f_*(\mathrm{THH}(R)) \longrightarrow \mathrm{TP}(R) \simeq f_*^T(\mathrm{THH}(R))$$

induces a map of commutative algebras in filtered spectra

$$\mathrm{Fil}^\star_{\mathrm{ev}} \mathrm{TC}^-(R) \longrightarrow \mathrm{Fil}^\star_{\mathrm{ev}} \mathrm{TP}(R),$$

which, in turn, induces a map of commutative algebras in graded spectra

$$\mathrm{gr}^\star_{\mathrm{ev}} \mathrm{TC}^-(R) \longrightarrow \mathrm{gr}^\star_{\mathrm{ev}} \mathrm{TP}(R).$$

Replacing $\mathrm{TC}^-(R)$ and $\mathrm{TP}(R)$ by $\mathrm{HC}^-(R)$ and $\mathrm{HP}(R)$, Antieau [1] has identified the cofiber with the Hodge truncated derived de Rham cohomology of $R/\mathbb{Z}$,

$$\mathrm{gr}^n_{\mathrm{ev}} \mathrm{HC}^-(R) \longrightarrow \mathrm{gr}^n_{\mathrm{ev}} \mathrm{HP}(R) \longrightarrow L\Omega^{<n}_{R/\mathbb{Z}}.$$

By analogy, Morin [8] defines the Hodge truncated derived de Rham cohomology of $R/\mathbb{S}$ to be the cofiber

$$\mathrm{gr}^n_{\mathrm{ev}} \mathrm{TC}^-(R) \longrightarrow \mathrm{gr}^n_{\mathrm{ev}} \mathrm{TP}(R) \longrightarrow L\Omega^{<n}_{R/\mathbb{S}},$$

and finds a remarkable relation to earlier work with Flach [5] on special values of the zeta function $\zeta(X, s)$ defined by Serre [11] of a regular scheme X, proper over $\mathrm{Spec}(\mathbb{Z})$. This earlier work expresses[5] the special value $\zeta^*(X, n)$ in terms of:

(1) The determinant $\det(R\Gamma_{W,c}(X, \mathbb{Z}(n)))$ of Weil-étale cohomology.
(2) The determinant $\det(R\Gamma(X, L\Omega^{<n}_{X/\mathbb{Z}}))$ of derived DR cohomology of $X/\mathbb{Z}$.
(3) An archimedean correction factor, concocted from the Hodge cohomology of the generic fiber.

The remarkable discovery in [8] is that (2) and (3) precisely combine to give

$$\det(R\Gamma(X, L\Omega^{<n}_{X/\mathbb{S}}).$$

This is a quantitative statement to the effect that the sphere spectrum "knows" the archimedean place!

[5] Both Serre's definition of the zeta function and the Flach–Morin formula for its special values depend on conjectures that are very far from being proved.

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The Novikov conjecture, operator K theory, and diffeomorphism groups

Sherry Gong
(joint work with Jianchao Wu, Zhizhang Xie, Guoliang Yu)

The Novikov conjecture is a central conjecture in manifold topology that states that the higher signatures of closed oriented smooth manifolds are invariants under orientation-preserving homotopy equivalences. In the case of aspherical manifolds, the Novikov conjecture can be seen as an infinitesimal version of the Borel conjecture, which states that if two closed orientable aspherical manifolds are homotopy equivalent, then they are homeomorphic.

One powerful approach to the Novikov conjecture uses noncommutative geometry and the C^* algebras of groups. The key to this approach is that by a theorem of Kasparov, the Novikov conjecture follows from the rational strong Novikov conjecture, which states that the rational Baum-Connes assembly map is injective.

This approach has been used to show the Novikov conjecture for many classes of groups, including hyperbolic groups [1], groups coarsely embeddable into Hilbert spaces [5, 2, 4], and groups acting properly and isometrically on simply connected and non-positively curved manifolds [3].

Our results generalize the latter to groups acting on metric spaces that are not finite dimensional manifolds, but have a property that is a generalization of negative curvature.

In particular, let us define a *Hilbert-Hadamard space* to be a complete CAT(0) space with tangent cones isometrically embeddable into Hilbert spaces. A separable Hilbert-Hadamard space X is called *admissible* if there is a sequence of convex subsets isometric to finite-dimensional geodesically complete Riemannian manifolds, whose union is dense in X.

Then we show the following:

Theorem 1 (Jianchao Wu, Guoliang Yu, G.). *If a countable group G acts properly on an admissible Hilbert Hadamard space, then the rational strong Novikov conjecture holds for G.*

The main application of this result is to subgroups of the group of volume preserving diffeomorphisms of a compact manifold: Let N be a manifold with a volume form ω. Let $\mathrm{Diff}(N, \omega)$ denote the volume preserving diffeomorphisms of N. Consider the function $\lambda_+ : \mathrm{Diff}(N, \omega) \to \mathbb{R}_{\geq 0}$ given by

$$\lambda_+(\phi) = \left(\int \log\left(\|D\phi\|\right)^2 d\omega\right)^{1/2}$$

Let λ be the length function on $\mathrm{Diff}(N, \omega)$ given by

$$\lambda(\phi) = \max\{\lambda_+(\phi), \lambda_+(\phi^{-1})\}.$$

Then we say that a group $\Gamma \subset \mathrm{Diff}(N, \omega)$ is geometrically discrete if $\lambda(\gamma_i) \to \infty$ whenever $\gamma_i \to \infty$, that is for any $R > 0$, $\lambda(\gamma) > R$ except for a finite subset of Γ.

The main application of the above theorem is then the following:

Corollary 2. *The rational strong Novikov conjecture holds for any geometrically discrete countable subgroup of* $\mathrm{Diff}(N, \omega)$.

We extend this to subgroups of the group of (not necessarily volume preserving) diffeomorphisms. Given a probability measure μ and a metric g, we have a length function,

$$\begin{aligned} \lambda_{\mu,g} \colon \ \mathrm{Diff}(N) &\to [0, \infty)\,, \\ \varphi &\mapsto \left(\int_N \left(\left(\log \|D_x\varphi\|_g\right)^2 + \left(\log \left\|D_x\varphi^{-1}\right\|_g\right)^2\right) \mathrm{d}\,\mu\right)^{\frac{1}{2}} \ . \end{aligned}$$

We say that a group $\Gamma \subset \mathrm{Diff}(N)$ is μ-discrete if $\lambda_{\mu,g}(\gamma_i) \to \infty$ whenever $\gamma_i \to \infty$.

In order to generalize the above result to μ-discrete subgroups of $\mathrm{Diff}(N)$, we generalize the above theorem by considering a more general object than an admissible Hilbert Hadamard space, called an admissible continuous field of Hilbert Hadamard spaces, and show the following theorem.

Theorem 3 (Jianchao Wu, Zhizhang Xie, Guoliang Yu, G.). *If Γ is a countable group that acts isometrically and metrically properly on an admissible continuous field of Hilbert Hadamard spaces, then the rational strong Novikov conjecture holds for Γ.*

Finally, we deduce the following.

Corollary 4. *If Γ is a μ-discrete countable subgroup of $\mathrm{Diff}(N)$ for a regular Borel probability measure μ on N, then the rational strong Novikov conjecture holds for Γ.*

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What is de Rham cohomology?

BENJAMIN ANTIEAU

Work of Raksit [7] and of Moulinos–Robalo–Toën [6] gives a universal property for the HKR filtration on Hochschild homology and the Hodge filtration on Hodge-complete derived de Rham cohomology in terms of filtered derived commutative rings. It is possible to extend this result to the non-Hodge complete case and also to a closely related theory I call derived infinitesimal cohomology.

1. The discrete de Rham complex

Let k be a commutative ring and let R be a k-algebra. The **discrete de Rham complex** is the commutative differential graded k-algebra

$$\Omega^\bullet_{R/k}\colon R \xrightarrow{d} \Omega^1_{R/k} \xrightarrow{d} \Omega^2_{R/k} \to \cdots .$$

This **cdga** is **strict** because odd-degree elements square to zero.

The discrete de Rham complex admits a natural universal property. The forgetful functor

$$\mathrm{CAlg}_k \xleftarrow{A^\bullet \mapsto A^0} \mathrm{scdga_k}$$

admits a left adjoint: $\Omega^\bullet_{-/k}$. This tells us what the discrete de Rham complex is: it is the universal way of turning a commutative k-algebra into a strict cdga over k. This implies that the construction is not ad hoc. It also explains the precise structure on the discrete de Rham complex: it is a strict cdga over k.

There are major defects to using $\Omega^\bullet_{-/k}$ as the basis for a cohomology theory. First, the category of strict cdgas over k does not admit a 'nice' homotopy theoretical localization at quasi-isomorphisms, unless $\mathbf{Q} \subseteq k$. Second, the theory is poorly behaved for general singular commutative k-algebras. For example, for non-smooth k-algebras, the natural Hochschild–Kostant–Rosenberg map [4]

$$\Omega^*_{R/k} \to \mathrm{HH}_*(R/k)$$

typically fails to be an isomorphism. Or, for smooth commutative $\mathbf{C}$-algebras, there is a natural isomorphism $\mathrm{H}^*(\Omega^\bullet_{R/k}) \cong \mathrm{H}^*_{\mathrm{sing}}((\mathrm{Spec}R)(\mathbf{C}), \mathbf{C})$. This is typically not true in the singular case.

The solution to these problems is to use derived de Rham cohomology. This theory is constructed as the left Kan extension $\mathrm{L}\Omega_{-/k}$ in the diagram

$$\begin{tikzcd} \mathrm{CAlg}_k^{\mathrm{fppoly}} \arrow[r, "\Omega_{-/k}"] \arrow[d] & \mathrm{CAlg}(\mathrm{D}(k)) \\ \mathrm{CAlg}_k, \arrow[ur, dotted, "\mathrm{L}\Omega_{-/k}"] & \end{tikzcd}$$

where $\mathrm{CAlg}_k^{\mathrm{fppoly}}$ is the category of finitely presented polynomial algebra over k and $\Omega_{-/k}$ is the functor to $\mathbf{E}_\infty$-algebras over k which takes such a polynomial algebra to the $\mathbf{E}_\infty$-ring underlying the discrete de Rham complex.

If $k \subseteq \mathbf{Q}$, then $\Omega_{k[x_1,\ldots,x_n]/k} \simeq \mathbf{Q}$ for any n. It follows that the top functor in the left Kan extension diagram above is the constant functor on $\mathbf{Q}$. Therefore, $\mathrm{L}\Omega_{R/k} \simeq k$ for any R, as long as k contains $\mathbf{Q}$.

This motivates introduction of a certain completion of derived de Rham cohomology. For any commutative ring R, the de Rham complex $\Omega_{R/k}$, viewed as an $\mathbf{E}_\infty$-algebra over $\mathbf{Z}$, admits a Hodge filtration $\mathrm{F}^\star_{\mathrm{H}}\Omega_{R/k}$ where $\mathrm{F}^i_{\mathrm{H}}\Omega_{R/k} \to \Omega_{R/k}$ is the morphism in $\mathrm{D}(k)$ underlying $\Omega^{\bullet \geq i}_{R/k} \to \Omega^\bullet_{R/k}$. This makes $\mathrm{F}^\star_{\mathrm{H}}\Omega_{R/k}$ into an $\mathbf{E}_\infty$-algebra in $\mathrm{FD}(k) = \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathrm{D}(k))$, the stable ∞-category of *decreasing* filtered complexes.

This Hodge filtration left Kan extends to a Hodge filtration $\mathrm{F}^\star_{\mathrm{H}}\mathrm{L}\Omega_{R/k}$. By construction, $\mathrm{gr}^i_{\mathrm{H}}\mathrm{L}\Omega_{R/k} \simeq \mathrm{L}\Omega^i_{R/k}[-i]$, the derived ith exterior power of $\mathrm{L}_{R/k}$ placed in homological degree $-i$. The completion $\widehat{\mathrm{L}\Omega}_{R/k}$ defined as

$$\lim_i \frac{\mathrm{L}\Omega_{R/k}}{\mathrm{F}^i_{\mathrm{H}}\mathrm{L}\Omega_{R/k}}$$

is **Hodge-complete derived de Rham cohomology**.

A theorem of Bhatt [2], extending work of Hartshorne and Grothendieck, is that for a finite type $\mathbf{C}$-algebra R, one has an isomorphism

$$\mathrm{H}^*(\widehat{\mathrm{L}\Omega}_{R/k}) \equiv \mathrm{H}^*_{\mathrm{sing}}((\mathrm{Spec}R)(\mathbf{C}), \mathbf{C});$$

that is, Hodge-complete derived de Rham cohomology computes the singular cohomology of the space of complex points of $\mathrm{Spec}R$.

Question. What is derived de Rham cohomology?
As it stands, derived de Rham cohomology is given as a construction. What structure does it have? And, is it in some sense universal with respect to that structure? We are particularly interested in answers to this question that integrate the Hodge filtration. This talk explains the answer to these questions, closely following and somewhat extending [7].

2. Coherent cochain complexes

The stable ∞-category of filtered k-module spectra[1] is the functor ∞-category

$$\mathrm{FD}(k) = \mathrm{Fun}(\mathbf{Z}^{\mathrm{op}}, \mathrm{D}(k))$$

of arbitrary functors from the opposite of the ordered set of integers to $\mathrm{D}(k)$. Similarly, the stable ∞-category of graded complexes is the functor category

$$\mathrm{GrD}(k) = \mathrm{Fun}(\mathbf{Z}^{\mathrm{discrete}}, \mathrm{D}(k)),$$

where now $\mathbf{Z}^{\mathrm{discrete}}$ is viewed as a set.

These admit natural, pointwise t-structures where a filtered k-module spectrum $\mathrm{F}^\star M$ is (co)connective is if each is, and similarly for $\mathrm{GrD}(k)$. The hearts of these t-structures are

$$\mathrm{FMod}_k \quad \text{and} \quad \mathrm{GrMod}_k,$$

the abelian categories of filtered k-modules[2] and graded k-modules, respectively.

We can build filtered spectra out of graded spectra in three ways.

(1) The forgetful functor $\mathrm{FD}(k) \to \mathrm{GrD}(k)$ which sends a filtered object $\mathrm{F}^\star M$ to the graded object $\mathrm{F}^\star M$, i.e., by forgetting the transition maps, is monadic and realizes $\mathrm{FD}(k)$ as a category of modules over a graded ring spectrum. As what is needed to go from a graded ring spectrum to a filtered ring spectrum are the transition maps, it is possible to show that the associated ring spectrum is $k[t_{-1}]$, the discrete graded commutative ring where t has weight -1. So, $\mathrm{FD}(k) \simeq \mathrm{Mod}_{k[t_{-1}]}(\mathrm{GrD}(k))$. This is the **Rees construction**.

(2) Alternatively, we can take the associated graded functor $\mathrm{gr}^\star\colon \mathrm{FD}(k) \to \mathrm{GrD}(k)$. This functor factors through the completion $\mathrm{FD}(k) \to \widehat{\mathrm{FD}}(k)$. There is extra structure on the associated graded pieces

$$\mathrm{gr}^i M = \frac{\mathrm{F}^i M}{\mathrm{F}^{i+1} M}.$$

The fiber sequences

$$\mathrm{gr}^{i+1} M \to \frac{\mathrm{F}^i M}{\mathrm{F}^{i+1} M} \to \mathrm{gr}^i M$$

induce boundary maps

$$\cdots \to \mathrm{gr}^0 M \xrightarrow{d} \mathrm{gr}^1 M[1] \xrightarrow{d} \mathrm{gr}^2 M[2] \to \cdots .$$

Moreover, there are naturally present nullhomotopies for each composition d^2. Let $\mathbf{D}_-$ be the graded exterior algebra $k(0) \oplus k[-1](1)$. (A more natural description will appear later as the shearing down of the graded circle.) Then, it is possible to prove that the associated graded functor

[1] I will distinguish carefully between chain complexes or cochain complexes and their homotopy types. The latter are objects in the derived ∞-category $\mathrm{D}(k)$, a stable ∞-category whose homotopy category is the familiar triangulated derived category of k. Objects of $\mathrm{D}(k)$ are called $\mathbf{Z}$-module spectra.

[2] For us, the transition maps in a filtered k-module need not be injective.

$\widehat{\mathrm{FD}}(k) \to \mathrm{GrD}(k)$ factors through $\mathrm{Mod}_{\mathbf{D}_-}(\mathrm{GrD}(k)) \to \mathrm{GrD}(k)$ and that the associated map $\widehat{\mathrm{FD}}(k) \to \mathrm{Mod}_{\mathbf{D}_-}(\mathrm{GrD}(k))$ is an equivalence. We will call this functor the E_1-functor.

(3) Taking the notion of cochain complex suggested above more seriously, define, following Ariotta [1], a pointed category Ξ where the objects consist of the union of the integers and a basepoint $*$, which is initial and final. The morphisms between integers are given by

$$\mathrm{Hom}_{\Xi}(m,n) = \begin{cases} \{\mathrm{id}, *\} & \text{if } n = m, \\ \{d, *\} & \text{if } n = m-1, \\ \{*\} & \text{otherwise.} \end{cases}$$

In particular, $d^2 = *$ whenever this makes sense. Ariotta proves that there is an equivalence

$$\mathrm{Fun}_*(\Xi^{\mathrm{op}}, \mathrm{D}(k)) \simeq \widehat{\mathrm{FD}}(k).$$

The structure of a pointed functor from Ξ^{op} to $\mathrm{D}(k)$ is homotopy coherent way of constructing graded complexes with differentials d together with nullhomotopies for d^2 which are compatible to all higher homotopies. These are thus called **coherent cochain complexes**.

The **Beilinson t-structure** on filtered complexes introduced by Beilinson and studied in [3] is most elegantly seen from perspective (3) above. It is the pointwise t-structure on coherent complexes. The heart consists of the complete filtered complexes $\mathrm{F}^{\star}M$ such that $\mathrm{gr}^i M[i]$ is in the heart of the standard t-structure on $\mathrm{D}(k)$. The heart itself is the abelian category $\mathrm{Ch}^{\bullet}(k)$ of cochain complexes of k-modules.

A cochain complex $M^{\bullet}$ is viewed as a complete filtered k-module spectrum as follows. For each $i \in \mathbf{Z}$, let $M^{\geq i}$ be the k-module spectrum associated to the cochain complex $M^{\bullet \geq i}$. These assemble into a filtered k-module given by $i \mapsto M^{\geq i}$.

Another way to look at the Beilinson t-structure is that a (complete, exhaustive) filtered k-module spectrum $\mathrm{F}^{\star}M$ is in the heart if and only if it 'is' the stupid filtration associated to a cochain complex model for M.

3. Spectra with S^1-action

Let $\mathbf{T} = k[S^1]$ be the group algebra of the circle over k; it is an object of $\mathrm{D}(k)$ which computes the integral homology of S^1. It also admits the structure of a bicommutative bialgebra, encoding the multiplication arising from S^1 and the comultiplication present on chains on any space (or anima).

The ∞-category of k-module spectra with S^1-action can be realized either as the functor ∞-category $\mathrm{Fun}(S^1, \mathrm{D}(k))$ or as the module category $\mathrm{Mod}_{\mathbf{T}}(\mathrm{D}(k)) = \mathrm{D}(\mathbf{T})$. In the latter case it is important to note that the symmetric monoidal structure arises from the comultiplication on $\mathbf{T}$.

A classical understanding of Hochschild homology, in the commutative case, is a theorem of McClure–Schwänzl–Vogt [5], who proved[3] that the left adjoint of the forgetful functor

$$\mathrm{CAlg}(\mathrm{D}(k)) \leftarrow \mathrm{CAlg}(\mathrm{D}(\mathbf{T}))$$

is given by $\mathrm{HH}(-/k)$, Hochschild homology relative to k with its natural S^1-action. Here, $\mathrm{CAlg}(\mathrm{D}(k))$ for example refers to the ∞-category of commutative, or $\mathbf{E}_\infty$, algebra objects in $\mathrm{D}(k)$. This is a homotopy coherent of the notion of a commutative k-algebra. If k is a $\mathbf{Q}$-algebra, then $\mathrm{CAlg}(\mathrm{D}(k))$ is equivalent to the homotopy theory of commutative differential graded k-algebras and quasi-isomorphisms.

The Hochschild–Kostant–Rosenberg theorem [4] gives a tantalizing and well-studied connection between Hochschild homology and de Rham cohomology. It says that if R is a smooth commutative k-algebra, then $\mathrm{HH}_*(R/k) \cong \Omega^*_{R/k}$. Moreover, the S^1-action on $\mathrm{HH}(R/k)$ induces an operator (often called the "B-operator") $\mathrm{HH}_*(R/k) \to \mathrm{HH}_{*+1}(R/k)$ which is identified under the HKR isomorphism with the de Rham differential.

In order to make this connection more precise and expansive, we introduce derived commutative rings.

4. Derived commutative rings

Recall that an *animated commutative k-algebra* is an object of the ∞-category $\mathrm{sCAlg}_k[W^{-1}]$, the ∞-categorical localization of the 1-category of simplicial commutative k-algebras at the weak equivalences. The forgetful functor

$$\mathrm{D}(k)_{\geq 0} \leftarrow \mathrm{sCAlg}_k[W^{-1}] \colon U$$

admits a left adjoint LSym_k making the adjunction monadic. So, animated commutative rings are LSym_k-algebras in $\mathrm{D}(k)_{\geq 0}$.

The ∞-category of animated commutative rings admits a universal property: any functor $F\colon \mathrm{sCAlg}_k[W^{-1}] \to \mathcal{C}$ to an ∞-category $\mathcal{C}$ which preserves sifted colimits is left Kan extended from its restricted to finitely presented polynomial k-algebras. Conversely, any functor on finitely presented polynomial k-algebras can be uniquely extended to a functor on $\mathrm{sCAlg}_k[W^{-1}]$ which preserves sifted colimits. This is Quillen's philosophy of nonabelian derived functors.

A crucial observation, due to Mathew and Bhatt, is that the monad $U\mathrm{LSym}_k$ formally extends, via Goodwillie calculus methods, to a monad on the entire derived ∞-category; we write $U\mathrm{LSym}_k$ for this extension as well. At heart this extension exists because $U\mathrm{LSym}_k$ is a direct sum $\oplus_{r\geq 0}\mathrm{LSym}^r_k$ and each LSym^r_k, which computes the derived symmetric powers following Illusie, is polynomial in an appropriate sense.

By definition, a **derived commutative k-algebra** is a $U\mathrm{LSym}_k$-algebra in $\mathrm{D}(k)$. Let DAlg_k be the ∞-category of these algebras. It admits a forgetful functor to $\mathrm{CAlg}(\mathrm{D}(k))$; this forgetful functor preserves limits and colimits and

[3]They worked more generally in the context of $\mathbf{E}_\infty$-ring spectra and THH.

hence admits both left and right adjoints. The forgetful functor is an equivalence if k is a $\mathbf{Q}$-algebra.

As an example, note that if X is a topological space, then the $\mathbf{E}_\infty$-algebra $\mathrm{C}^\bullet(X,k)$, which is typically not a cdga, can be given the structure of a derived commutative k-algebra, by writing $\mathrm{C}^\bullet(X,k)\simeq \lim_X k$ and using that the forgetful functor preserves limits.

5. Putting it all together

The first point is that the theorem of McClure, Schwänzl, and Vogt extends to derived commutative rings. Namely, there is a good ∞-category of derived commutative rings with circle action $\mathrm{DAlg}(\mathrm{D}(\mathbf{T}))$ and Raksit proves in [7] that the left adjoint to the forgetful functor

$$\mathrm{DAlg}(k)\leftarrow \mathrm{DAlg}(\mathrm{D}(\mathbf{T}))$$

is given by $\mathrm{HH}(-/k)$. In particular, if R is a derived commutative k-algebra, then $\mathrm{HH}(R/k)$, defined a priori by considering R only as an $\mathbf{E}_\infty$-ring, admits a natural derived commutative ring structure and that the circle action preserves this structure. Moreover, this package has a natural universal property.

To bring filtrations into play, let $\mathbf{T}_{\mathrm{fil}}=\tau_{\geq\star}\mathbf{T}$, the Whitehead (or Postnikov) tower of $\mathbf{T}$. This is a filtered k-module spectrum with $\mathrm{F}^i(\mathbf{T}_{\mathrm{fil}})\simeq 0$ for $i>1$, $\mathrm{F}^1(\mathbf{T}_{\mathrm{fil}})\simeq k[1]$, and $\mathrm{F}^i(\mathbf{T}_{\mathrm{fil}})\simeq \mathbf{T}$ for $i\leq 0$.

There is a good notion of filtered derived commutative rings obtained by declaring that $\mathrm{LSym}^r_k\mathrm{ins}^iP\simeq \mathrm{ins}^{ir}\mathrm{LSym}_kP$ when P is a finitely presented projective k-module; here ins^iM denotes the filtered k-module with $\mathrm{F}^j\mathrm{ins}^iM=0$ for $j>i$ and M for $j\leq i$, with transition maps given by the identity for $j\leq i$. The free objects on ins^iP are in other words filtered polynomial rings on generators of weight i.

The notion of filtered derived commutative rings above is not the only possible notion, so I will call them the **infinitesimal filtered derived commutative rings**.

The dual object $\mathbf{T}_{\mathrm{fil}}^\vee$ naturally admits the structure of a grouplike $\mathbf{E}_\infty$-object in infinitesimal filtered derived commutative rings. This lets us make precise an ∞-category $\mathrm{DAlg}^{\mathrm{inf}}(\mathrm{FD}(\mathbf{T}_{\mathrm{fil}})$ of infinitesimal filtered derived commutative rings with $\mathbf{T}_{\mathrm{fil}}$-action.

Raksit proves that the natural forgetful functor $\mathrm{DAlg}_k\leftarrow \mathrm{DAlg}^{\mathrm{inf}}(\mathrm{FD}(\mathbf{T}_{\mathrm{fil}}))$, which forgets both the filtration and the filtered circle action, admits a left adjoint $\mathrm{HH}_{\mathrm{fil}}(R/k)$ which recovers the Hochschild–Kostant–Rosenberg filtration on Hochschild homology.

Raksit also introduces what I will call **crystalline complete filtered derived commutative rings** yielding an ∞-category $\mathrm{DAlg}^{\mathrm{crys}}(\widehat{\mathrm{FD}}(k))$. He proves that the gr^0-functor

$$\mathrm{DAlg}_k\leftarrow \mathrm{DAlg}^{\mathrm{crys}}(\widehat{\mathrm{FD}}(k))$$

admits a left adjoint given by $\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{R/k}$, Hodge-filtered, Hodge-complete derived de Rham cohomology.

Here is a souped up version of HKR proved by Raksit. It involves the **shearing down** operator on graded k-module spectra. This is the operation which takes a graded k-module spectrum $M(\star)$ to $M(\star)[-2\star]$. Shearing down induces a symmetric monoidal autoequivalence of $\mathrm{GrD}(k)$. The shearing down $\mathbf{T}_{\mathrm{gr}}[-2\star]$ of the associated graded of the filtered circle is precisely the graded Hopf algebra $\mathbf{D}_{-}$ encountered above. At heart, shearing is responsible for the difference between infinitesimal and crystalline filtered derived commutative rings.

Raksit proves that there is a composition of functors

$$\mathrm{DAlg}^{\mathrm{inf}}(\mathrm{FD}(\mathbf{T}_{\mathrm{fil}})) \xrightarrow{\mathrm{gr}^\star} \mathrm{DAlg}^{\mathrm{inf}}(\mathrm{GrD}(\mathbf{T}_{\mathrm{gr}})) \xrightarrow{[-2\star]} \mathrm{DAlg}^{\mathrm{crys}}(\mathrm{GrD}(\mathbf{D}_{-}))$$

$$\simeq \mathrm{DAlg}^{\mathrm{crys}}(\widehat{\mathrm{FD}}(k)),$$

which takes $\mathrm{HH}_{\mathrm{fil}}(R/k)$ to $\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{R/k}$. In other words,

$$\mathrm{HH}_{\mathrm{gr}}(R/k)[-2\star]$$

with its residual $\mathbf{T}_{\mathrm{gr}}[-2\star] \simeq \mathbf{D}_{-}$ action recovers $\mathrm{F}^\star_{\mathrm{H}}\widehat{\mathrm{L}\Omega}_{R/k}$.

Work in progress of my own aims to further elucidate the nature of crystalline filtered derived commutative rings. Here is a summary.

There is an extension of the crystalline LSym monad from the complete filtered derived category to all of $\mathrm{FD}(k)$. The left adjoint of $\mathrm{DAlg}_k \xleftarrow{\mathrm{gr}^0} \mathrm{DAlg}^{\mathrm{crys}}(\mathrm{FD}(k))$ is $\mathrm{F}^\star_{\mathrm{H}}\mathrm{L}\Omega_{-/k}$, i.e., *non-Hodge-complete* derived de Rham cohomology with its Hodge filtration.

There is a map of map of monads $\mathrm{LSym}^{\mathrm{inf}}_k \to \mathrm{LSym}^{\mathrm{crys}}_k$. In particular, every crystalline filtered derived commutative ring is infinitesimal. This in turns that the underlying $\mathbf{E}_\infty$-ring is derived.

There is a left adjoint to

$$\mathrm{DAlg}_k \xleftarrow{\mathrm{gr}^0} \mathrm{DAlg}^{\mathrm{inf}}(\mathrm{FD}(k)),$$

which I call derived infinitesimal cohomology $\mathrm{F}^\star_{\mathrm{H}}\mathbb{\Pi}_{R/k}$. The graded pieces of the Hodge filtration are no longer $\mathrm{L}\Lambda^i\mathrm{L}_{R/k} \simeq \mathrm{L}\Omega^i_{R/k}$, but rather $\mathrm{LSym}^i(\mathrm{L}_{R/k}[-1])$.

The map of monads from the infinitesimal to the crystalline induces a crystallization functor

$$\square\colon \mathrm{DAlg}^{\mathrm{inf}}(\mathrm{FD}(k)) \to \mathrm{DAlg}^{\mathrm{crys}}(\mathrm{FD}(k)).$$

In particular, playing around with adjoints, there is a canonical map

$$\mathrm{F}^\star_{\mathrm{H}}\mathbb{\Pi}_{R/k} \to \mathrm{F}^\star_{\mathrm{H}}\mathrm{L}\Omega_{R/k},$$

adjoint to an equivalence $\square\left(\mathrm{F}^\star_{\mathrm{H}}\mathbb{\Pi}_{R/k}\right) \simeq \mathrm{F}^\star_{\mathrm{H}}\mathrm{L}\Omega_{R/k}$.

An amusing example is that $\mathrm{F}^\star_{\mathrm{H}}\mathbb{\Pi}_{\mathbf{F}_p/\mathbf{Z}_p} \simeq p^\star\mathbf{Z}_p$, the p-adic integers with the p-adic filtration. The crystallization $\square(p^\star\mathbf{Z}_p)$ is thus $\mathrm{F}^\star_{\mathrm{H}}\mathrm{L}\Omega_{R/k}$, which can be identified up to p-completion with $\mathbf{Z}_p\langle x\rangle/(x-p)$ with the divided power filtration, where $\mathbf{Z}_p\langle x\rangle$ is the free divided power algebra on a generator x.

Finally, in the smooth case, infinitesimal cohomology recovers the cohomology of Grothendieck's infinitesimal site.

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On noncommutative crystalline cohomology

Boris Tsygan

A noncommutative generalization of differential forms and De Rham cohomology in characteristic zero is Hochschild and cyclic homology [1], [2], [6], [10]. One of its versions, periodic cyclic homology, generalizes crystalline cohomology in characteristic zero [4]. In positive or mixed characteristic, if one wants to generalize cohomology theories of schemes to noncommutative setting, to get the full extent of the theory one has to use topological methods such as [8]. However, in recent years it became apparent that significant part of the theory can be done using variations on the usual definitions (i.e. working with modules over a base ring rather than with spectra).

In the talk, we present four recent approaches along these lines. The first is due to Kaledin [5]. It is based on the construction of noncommutative Witt vectors and generalizes the Deligne-Illusie construction of crystalline cohomology.

The second is due to Petrov and Vologodsky [9] and the third is from [7]. Both emulate the classical approach, due to Grothendieck and Berthelot, of taking an $\mathbb{F}_p$-algebra and trying to lift it to a $\mathbb{Z}_p$-algebra, and applying some variation of the construction of the De Rham complex to it. In [9], this idea is realized by computing derived periodic cyclic homology, i.e. replacing an $\mathbb{F}_p$-algebra by a DG resolution which is flat as a $\mathbb{Z}_p$-module and computing the periodic cyclic homology of the latter. This requires a subtle final step of reducing the result by tensoring by $\mathbb{Z}_p$ over the derived periodic cyclic homology of $\mathbb{F}_p$. The subtlety is that the latter is not a ring. In [7], we lift an $\mathbb{F}_p$-algebra to a $\mathbb{Z}_p$-module with a multiplication that is associative only modulo p. The differential in the periodic cyclic complex is now square zero only modulo p. However, we show how it can be flattened and transformed into a square zero differential.

Finally, the fourth approach, by Cortiñas, Cuntz, Meyer, Mukherjee, and Tamme, is based on non-Archimedean topological and bornological methods and generalizes the construction of Monsky and Washnitzer.

We also discuss how the first three approaches can be compared. We note that all of them are based on the same idea: take some version of a complex computig periodic cyclic homology in characteristic p, and lift it to a complex over $\mathbb{Z}_p$. (In Kaledin's case, this is because noncommutative Hochschild-Witt complex deforms the limit of twisted Hochschild complexes of p^n-fold tensor powers of the algebra, and the latter compute the usual Hochschild homology). A better understanding of the full algebra of operations on Hochschild and cyclic complexes, probably involving higher Hochschild cochain complexes of Kontsevich and Vlassopoulos, should provide a common framework for the three constructions.

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Cyclic cocycles in the spectral action

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(joint work with Teun van Nuland)

The spectral action [2, 3] is one of the key instruments in the applications of noncommutative geometry to particle physics. With inner fluctuations [8] of a noncommutative manifold playing the role of gauge potentials, the spectral action principle yields the corresponding Lagrangians. Indeed, the asymptotic behavior of the spectral action for small momenta leads to experimentally testable field theories, by interpreting the spectral action as a classical action and applying the usual renormalization group techniques. In particular, this provides the simplest way known to geometrically explain the dynamics and interactions of the gauge bosons and the Higgs boson in the Standard Model Lagrangian as an effective field theory [4] (see also the textbooks [9, 14]). More general noncommutative manifolds (spectral triples) can also be captured by the spectral action principle, leading to models beyond the standard model as well. As shown in [11], if one restricts to the scale-invariant part, one may naturally identify a Yang–Mills term and a Chern–Simons term to elegantly appear in the spectral action. From the perspective of quantum field theory, the appearance of these field-theoretic action functionals sparks hope that we might find a way to go beyond the classical framework provided by the spectral action principle. It is thus a natural question whether we can also field-theoretically describe the full spectral action, without resorting to the scale-invariant part.

Motivated by this, we study the spectral action when it is expanded in terms of inner fluctuations associated to an arbitrary noncommutative manifold, without resorting to heat-kernel techniques. Indeed, the latter are not always available and an understanding of the full spectral action could provide deeper insight into how gauge theories originate from noncommutative geometry. Let us now give a more precise description of our setup.

We let $(\mathcal{A},\mathcal{H},D)$ be an finitely summable spectral triple. If $f:\mathbb{R}\to\mathbb{C}$ is a suitably nice function we may define the spectral action [3]:

$$\mathrm{Tr}(f(D)).$$

An inner fluctuation, as explained in [8], is given by a Hermitian universal one-form

$$A=\sum_{j=1}^{n}a_j db_j\in\Omega^1(\mathcal{A}), \tag{1}$$

for elements $a_j,b_j\in\mathcal{A}$. The terminology 'fluctuation' comes from representing A on $\mathcal{H}$ as

$$V:=\pi_D(A)=\sum_{j=1}^{n}a_j[D,b_j]\in\mathcal{B}(\mathcal{H})_{\mathrm{sa}}, \tag{2}$$

and fluctuating D to $D+V$ in the spectral action. The variation of the spectral action under the inner fluctuation is then given by

$$\mathrm{Tr}(f(D+V))-\mathrm{Tr}(f(D)). \tag{3}$$

As spectral triples can be understood as noncommutative spin$^{\mathrm{c}}$ manifolds (see [10]) encoding the gauge fields as an inner structure, one could hope that perturbations of the spectral action could be understood in terms of noncommutative versions of geometrical, gauge theoretical concepts. Hence we would like to express (3) in terms of universal forms constructed from A. To express an action functional in terms of universal forms, one is naturally led to cyclic cohomology. As it turns out, hidden inside the spectral action we will identify an odd (b,B)-cocycle $(\tilde{\psi}_1,\tilde{\psi}_3,\ldots)$ and an even (b,B)-cocycle $(\phi_2,\phi_4,\ldots)$ for which $b\phi_{2k}=B\phi_{2k}=0$, i.e., each Hochschild cochain ϕ_{2k} forms its own (b,B)-cocycle $(0,\ldots,0,\phi_{2k},0,\ldots)$. On the other hand, the odd (b,B)-cocycle $(\tilde{\psi}_{2k+1})$ is truly infinite (in the sense of [7]).

The key result is that for suitable $f:\mathbb{R}\to\mathbb{C}$ we may expand

$$\mathrm{Tr}(f(D+V)-f(D))=\sum_{k=1}^{\infty}\left(\int_{\psi_{2k-1}}\mathrm{cs}_{2k-1}(A)+\frac{1}{2k}\int_{\phi_{2k}}F^k\right), \tag{4}$$

in which the series converges absolutely. Here ψ_{2k-1} is a scalar multiple of $\tilde{\psi}_{2k-1}$, $F_t = tdA + t^2A^2$, so that $F = F_1$ is the curvature of A, and $\mathrm{cs}_{2k-1}(A) = \int_0^1 AF_t^{k-1}dt$ is a generalized noncommutative Chern–Simons form.

As already mentioned, a similar result was shown earlier to hold for the scale-invariant part $\zeta_D(0)$ of the spectral action. Indeed, Connes and Chamseddine [11] expressed the variation of the scale-invariant part in dimension ≤ 4 as

$$\zeta_{D+V}(0) - \zeta_D(0) = -\frac{1}{4}\int_{\tau_0}(dA + A^2) + \frac{1}{2}\int_{\psi}\left(AdA + \frac{2}{3}A^3\right),$$

for a certain Hochschild 4-cocycle τ_0 and cyclic 3-cocycle ψ.

Interestingly, a key role in our extension of this result to the full spectral action will be played by multiple operator integrals. It is the natural replacement of residues in this context, and also allows to go beyond dimension 4. For our analysis of the cocycle structure that appears in the full spectral action we take the Taylor series expansion as a starting point. In previous works this has been studied in great detail using multiple operator integrals, as traces thereof are multilinear extensions of the derivatives of the spectral action. This viewpoint is also taken in [12, 13], where multiple operator integrals are used to investigate the Taylor expansion of the spectral action. Indeed, multiple operator integrals can also be used to define cyclic cocycles, because of some known properties of the multiple operator integral that have been proved in increasing generality in the last decades (e.g., in [1, 5, 15, 12, 13]). In our [15] we have pushed these results even further, by proving estimates and continuity properties for the multiple operator integral when the self-adjoint operator has an s-summable resolvent, thereby supplying the discussion here with a strong functional analytic foundation.

There are two interesting possibilities for application of our main result and the techniques used to obtain it. The first application is to index theory. The analytically powerful multiple operator integration techniques used for the absolute convergence of our expansion also allow us to show that the found (b, B)-cocycles are *entire* in the sense of [6]. This makes it meaningful to analyze their pairing with K-theory, which is found to be trivial [15].

The second application is to quantization. In [16] we have taken a first step towards the quantization of the spectral action within the framework of spectral triples. Using the asymptotic expansion of the spectral action and some basic quantum field theoretic techniques, we have proposed a one-loop quantum effective spectral action and showed that it satisfies a similar expansion formula, featuring in particular a new pair of cyclic cocycles.

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How to compute the dimension of spectral triple? Alain Connes' question on spectral asymptotics

Fedor Sukochev

(joint work with D. Zanin)

Compact spectral triple is called d-dimensional if $(D^2+1)^{-\frac{d}{2p}} \in \mathcal{L}_{p,\infty}$ for every $p > 0$. Here, $\mathcal{L}_{p,\infty}$ is the collection of all infinitesimals of order $\frac{1}{p}$.

Locally compact spectral triple is called d-dimensional if $\pi(a)(D^2+1)^{-\frac{d}{2p}} \in \mathcal{L}_{p,\infty}$ for every $p > 0$ and for every $a \in \mathcal{A}$, where π is the $*$-representation of the algebra $\mathcal{A}$ from the spectral triple definition.

Computing the dimension of a non-compact spectral triple is quite an involved task. The most efficient device up to date is the so-called Cwikel-type estimate. Corresponding theory is developed by Cwikel [2], Birman and Solomyak [1], Weidl [4] and, in the most general shape, in the recent paper [3] by Levitina, Sukochev and Zanin. The cited results provide rather pleasing estimates in the pre-critical (i.e. $p > 2$) and post-critical (i.e. $p < 2$) case. However, the most difficult case $p = 2$ is still a challenge.

Physicists are also interested in estimates for the operators $M_f(1-\Delta)^{-\frac{d}{4}}$ in Euclidean setting (here, M_f is a multiplication operator by a function f) and relate them to Cwikel-Lieb-Rozenblum inequalities estimating the number of bound states (eigenvalues outside essential spectrum) of Schroedinger operators. This motivated a fundamental paper by Solomyak (1994), who proved that, on a d-dimensional torus with *even* d

$$\|(1-\Delta)^{-\frac{d}{4}}M_f(1-\Delta)^{-\frac{d}{4}}\|_{1,\infty} \leq c_d\|f\|_{L\log L}. \tag{1}$$

Here, $\|\cdot\|_{1,\infty}$ denotes weak l_1-quasi-norm and $L\log L$ is the so-called Zygmund space on $\mathbb{T}^d$ (i.e., Orlicz space with Orlicz function $t \to t\log(e+t)$).

Our aim is multi-fold

- to extend (1) to odd dimension. [DONE]
- to extend (1) to Euclidean setting. [Proved to be impossible. However, a satisfactory replacement is found.]
- to develop the technique which would allow proving the Solomyak-type results in NCG settings [This remains an open problem even for the non-commutative torus.]

We also discuss the spectral asymptotics of the operator $(1-\Delta)^{-\frac{d}{4}}M_f(1-\Delta)^{-\frac{d}{4}}$. During 2017 conference *"Noncommutative Geometry: State of the Art and Future Prospects"*, Alain Connes asked whether spectral asymptotics for such operators can be proved directly (that is, without involving heavy machinery of singular traces). The answer appears to be positive.

If $f \in L\log L(\mathbb{T}^d)$ and if Δ is the Laplacian on $\mathbb{T}^d$, then there exists a limit

$$\lim_{t\to\infty} t\mu\big(t,(1-\Delta)^{-\frac{d}{4}}M_f(1-\Delta)^{-\frac{d}{4}}\big) = c_d\|f\|_1.$$

If $f \in L\log L(\mathbb{R}^d)$ is such that

$$\int_{\mathbb{R}^d} |f(s)|\cdot\log(1+|s|)ds < \infty$$

and if Δ is the Laplacian on $\mathbb{R}^d$, then there exists a limit

$$\lim_{t\to\infty} t\mu\big(t,(1-\Delta)^{-\frac{d}{4}}M_f(1-\Delta)^{-\frac{d}{4}}\big) = c_d\|f\|_1.$$

Similar assertion is available for compact Riemannian manifolds. Let (X,g) be a d-dimensional Riemannian manifold and let ν_g and Δ_g be the Riemannian volume and the Laplace-Beltrami operator. If $f \in L\log L(X,\nu_g)$, then there exists a limit

$$\lim_{t\to\infty} t\mu\big(t,(1-\Delta_g)^{-\frac{d}{4}}M_f(1-\Delta_g)^{-\frac{d}{4}}\big) = c_d\|f\|_1.$$

As far as we know, those results appear to be the best possible up to date.

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Determinants, K_1(Var), and point counting

Inna Zakharevich

Let k be a field. The Grothendieck ring of varieties $K_0(\mathrm{Var}_k)$, is defined to have underlying group

$$K_0(\mathrm{Var}_k) := \begin{matrix}\text{free ab. gp gen by}\\ \text{varieties over } k\end{matrix} \Big/ \begin{matrix}\forall \text{ closed immersions } Y \hookrightarrow X\\ [X] = [Y] + [X \backslash Y].\end{matrix}$$

Here, a *variety* over k is a reduced separated scheme of finite type over k. In other words, the underlying group of the Grothendieck ring of varieties encodes the geometric process of cutting out a closed subvariety of X and declaring that X is equal to the subvariety plus the complement. The product structure is induced by the Cartesian product of varieties:

$$[X][Y] := [X \times Y].$$

This ring was first introduced in Grothendieck's letter to Serre [Groth, p174-175] discussing the conjectural abelian category of motives. The idea is that this ring is a first approximation to what the K_0 of this category should be, as it encodes the ways in which cohomological invariants split varieties.

In [Zak], the notion of a *Grothendieck spectrum of varieties* $K(\mathrm{Var}_k)$, a spectrum whose π_0 is $K_0(\mathrm{Var}_k)$ and whose higher homotopy groups are related to other decomposition invariants, is introduced. The spectrum is formed using algebraic K-theory and thus is functorial with respect to well-behaved combinatorial functors of categories. In particular, there are two immediate functors of interest:

$$\begin{matrix}\mathbf{FinSet} \to \mathbf{Var}_k \\ S \longmapsto \coprod_S \operatorname{Spec} k\end{matrix} \qquad \begin{matrix}\mathbf{Var}_k \to \mathbf{FinSet} \\ X \longmapsto X(k)\end{matrix}$$

Since the composition of these two functors is the identity on **FinSet**, a copy of the sphere spectrum splits off of $K(\mathrm{Var}_k)$, thereby showing that the higher homotopy groups of $K(\mathrm{Var}_k)$ are in general nontrivial. In [CWZ] it is further shown that when $|k| \equiv 3 \pmod 4$ the group $K_1(\mathrm{Var}_k) \not\equiv \pi_1\mathbb{S}$, by showing that an element represented by the automorphism $x \mapsto 1/x$ of $\mathbb{P}^1$ represents a nontrivial element in $K_1(\mathrm{Var}_k)/\pi_1\mathbb{S}$. However, two important questions are left unanswered:

(1) For any finite field k, is it always the case that $K_1(\mathrm{Var}_k) \not\cong \pi_1\mathbb{S}$?
(2) The spectrum $K(\mathrm{Var}_K)$ is an E_∞-ring spectrum [Cam], and thus its homotopy groups form a graded ring. Is the map

$$K_0(\mathrm{Var}_k) \otimes \pi_1\mathbb{S} \to K_1(\mathrm{Var}_k)$$

surjective? In other words, do there exist elements in $K_1(\mathrm{Var}_k)$ that cannot be represented by permutations of varieties?

If the answer ot the second question is yes, the answer to the first question must also be yes, and thus we focus on the second question.

In this talk we discuss a method for answering these questions using a souped-up version of point counting. The basic model of point counting is the functor on the right above, taking each variety to its set of k-points. However, a variety contains all of the information of points over all extensions of the base field. Instead of mapping to the category of finite sets, we instead map to the category of *almost-finite sets*, **AFSet**. These sets are introduced in [DS88]. An almost-finite set is a set S together with an action of $\widehat{\mathbf{Z}}$ (the profinite completion of the integers), satisfying the extra condition that every orbit is finite and the fixed-point set of any nontrivial subgroup is finite. The K-theory of almost-finite sets is well-defined, and is naturally equipped with a map

$$K(\mathbf{AFSet}) \to \prod_{n\geq 1} \Sigma^\infty_+ B\mathbf{Z}/n.$$

In particular, after applying π_1 we obtain a homomorphism

$$K_1(\mathbf{AFSet}) \to \prod_{n\geq 1} (\mathbf{Z}/2 \oplus \mathbf{Z}/n).$$

It turns out that whenever an element of $K_1(\mathbf{Var}_k)$ is in the image of the map in question (2) above, it must lie in the subgroup $\prod_{n\geq 1}(\mathbf{Z}/2 \oplus 1)$. Moreover, elements with nontrivial $\mathbf{Z}/n$-coordinate can be detected by analyzing permutations of points in degree-n extensions.

This leads to the following theorem answering question (2) in the affirmative, at least for fields of characteristic not equal to 2:

Theorem 1 (Zakharevich)**.** *Let k be a finite field and let $b = \mathrm{ord}_2(|k| - 1)$. Then the automorphism of $\mathbb{A}^1$ induced by scaling by a primitive 2^b-th root of unity is not in the image of the map*

$$K_0(\mathrm{Var}_k) \otimes \pi_1\mathbb{S} \to K_1(\mathrm{Var}_k).$$

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Nonarchimedean local and analytic cyclic homology

Devarshi Mukherjee

(joint work with Guillermo Cortiñas and Ralf Meyer)

Cyclic homology in the positive characteristic setting has witnessed intense activity in recent years. One motivation, particularly in the commutative setting, is to find better approximations to algebraic K-theory using trace methods. Our main motivation is in finding an invariant of noncommutative $\mathbb{F}_p$-algebras that specialises to a homotopy invariant de Rham theory in the commutative case, namely, *rigid* *cohomology*. Using such an invariant of $\mathbb{F}_p$-algebras, we then construct invariants of noncommutative topological $\mathbb{Z}_p$-algebras, that behave in a manner similar to periodic and local cyclic homology for locally convex topological $\mathbb{C}$-algebras.

Our construction is based on the Cuntz-Quillen approach to periodic cyclic homology (see [3]), which was established to prove excision and nilinvariance of the theory. Their approach replaces the usual $(B-b)$-bicomplex of a $\mathbb{Q}$-algebra R computing $\mathbb{HP}$ with the X-complex of the tensor algebra $\mathsf{T}(R) = \bigoplus_{n=0}^{\infty} R^{\otimes n}$. Now suppose we have a complete, torsionfree $\mathbb{Z}_p$-algebra D. In [2], we build a certain completed tensor algebra extension

$$0 \to \mathcal{J}(R) \to \mathcal{T}(R) \to R \to 0,$$

obtained by applying the tube algebra construction to the usual tensor algebra extension $0 \to \mathsf{J}(R) \to \mathsf{T}(R) \to R \to 0$. Here, completions take place in the category of complete bornological $\mathbb{Z}_p$-modules, which we use as our framework for homological algebra in functional analytic contexts, just as in the Archimedean setting (see [4]). The *analytic cyclic homology* of R is defined as the 2-periodic complex

$$\mathbb{HA}^{\mathrm{tf}}(R) = X(\mathcal{T}(R) \otimes \mathbb{Q}_p);$$

taking values in the derived category of the quasi-abelian category $\overleftarrow{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{Q}_p})}$. This functor satisfies homotopy invariance $\mathbb{HA}^{\mathrm{tf}}(R) \cong \mathbb{HA}^{\mathrm{tf}}(R \otimes \mathbb{Z}_p[t]^\dagger)$ with respect to the Monsky-Washnitzer algebra of overconvergent analytic functions on the unit disc. It also satisfies matricial stability, nilinvariance, and excision with respect to linearly split extensions of complete bornological algebras.

Next, in [5], we define analytic cyclic homology $\mathbb{HA}(A)$ for algebras A over the residue field $\mathbb{F}_p$. It is constructed by lifting such an $\mathbb{F}_p$-algebra to a complete, bornologically torsionfree $\mathbb{Z}_p$-module, whose reduction mod p is the *fine bornology*, that is the bornology generated by finitely generated $\mathbb{Z}_p$-submodules. A canonical lifting of this kind is the free $\mathbb{Z}_p$-module $\mathbb{Z}_p A$ with the fine bornology. This again induces the tensor algebra extension $0 \to I \to \mathbb{Z}_p\langle A\rangle \to A \to 0$, using which we can mimic the definition of analytic cyclic homology for torsionfree $\mathbb{Z}_p$-algebras. This process yields a chain complex valued functor

$$\mathbb{HA}\colon \{\mathbb{F}_p\text{-algebras}\} \to \mathsf{Der}(\overleftarrow{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{Q}_p})})$$

that again satisfies homotopy invariance with respect to polynomial homotopies, matricial stablility and excision for extensions of $\mathbb{F}_p$-algebras. Crucially, the functor we define is independent of choices of liftings to complete, torsionfree $\mathbb{Z}_p$-modules that reduce mod p to the fine bornology. In particular, we prove that if $0 \to pD \to D \to A \to 0$ is a dagger algebra lifting that reduces mod p to the fine bornology, then $\mathbb{HA}(A) \cong \mathbb{HA}^{\mathrm{tf}}(D)$.

We then address (in [6]) the technical condition concerning the class of algebras whose reductions mod p are the fine bornology. These are called *nuclear* bornological algebras, and their bornologies are described by collections of convergent power series with coefficients in $l^\infty(\mathbb{N}, \mathbb{Z}_p)$. Denote a torsionfree bornological $\mathbb{Z}_p$-module D with this bornology by D'. We show importantly that if D is a dagger algebra, then so is D' - with the additional property of inherting the fine bornology mod p. This change of bornology is used to define the *local cyclic homology* functor

$$\mathbb{HL}\colon \{\text{Dagger algebras}\} \to \mathsf{Der}(\overleftarrow{\mathsf{Ind}(\mathsf{Ban}_{\mathbb{Q}_p})}), \quad D \mapsto \mathbb{HA}(D');$$

which by construction and Theorem 5.9 in [5], only depends on its reduction mod p. In other words, we have $\mathbb{HL}(D) \cong \mathbb{HA}(D/pD)$. Finally, using mainly the formal properties of our functors, we compute local and analytic cyclic homology for Leavitt path algebras and smooth curves over $\mathbb{F}_p$; in the latter case, our theory coincides with Berthelot's rigid cohonology.

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Metric inequalities with scalar curvature via the Dirac operator

Rudolf Zeidler

(joint work with Simone Cecchini)

In classical Riemannian comparison geometry, positive lower bounds on sectional and Ricci curvature have concrete consequences such as global upper bounds on volumes and distances. In contrast, the study of positive *scalar curvature* from a geometric perspective is much more subtle. However, recently Gromov proposed a number of conjectures (compare [3, 4]) which—in specific situations—predict concrete metric inequalities under lower scalar curvature bounds reminiscent of classical comparison geometry. This includes Gromov's *width conjecture*:

Conjecture 1 ([4, 11.12, Conjecture C]). Let M be a closed connected manifold of dimension $n-1 \neq 4$ such that M does not admit a metric of positive scalar curvature. Let g be a Riemannian metric on $V = M \times [-1,1]$ of scalar curvature bounded below by $n(n-1) = \mathrm{scal}_{\mathbb{S}^n}$. Then

$$\mathrm{width}(V,g) \leq \frac{2\pi}{n},$$

where $\mathrm{width}(V,g) := \mathrm{dist}_g(\partial_- V, \partial_+ V)$ is the distance between the two boundary components of V with respect to g.

There are essentially two main tools to study scalar curvature. One is based on geometric measure theory and the use of minimal hypersurfaces going back to Schoen and Yau. The other, closer in spirit to methods of non-commutative geometry, uses the spinor Dirac operator $\mathcal{D}$ and the corresponding Schrödinger–Lichnerowicz formula $\mathcal{D}^2 = \nabla^*\nabla + \frac{\mathrm{scal}}{4}$ which connects spectral properties of the Dirac operator to scalar curvature.

Gromov's first result on Conjecture 1 was a proof for the torus and related manifolds via minimal hypersurface methods, and subsequently a general proof for $n \leq 7$ [3] (see also Räde [5]). The first main result given in the talk showed that Conjecture 1 can also be addressed via the Dirac method. More precisely, Conjecture 1 holds whenever M admits an obstruction to the existence of positive scalar curvature metrics based on a generalized index invariant associated to the spinor Dirac operator on the universal covering:

Theorem 2 ([6, 2, 7])**.** *Let M be a closed connected manifold of non-vanishing Rosenberg index $\alpha_{\pi_1 M}(M) \in \mathrm{KO}_*(\mathrm{C}^*\pi_1 M)$. Then for every Riemannian metric g on $V = M \times [-1, 1]$ of scalar curvature bounded below by $n(n-1) = \mathrm{scal}_{\mathrm{S}^n}$, we have $\mathrm{width}(V, g) < \frac{2\pi}{n}$. In particular, Conjecture 1 holds for all simply-connected manifolds M of dimension ≥ 5.*

The proof of this result is based on a modification of the spinor Dirac operator $\mathcal{D}$, essentially of the form $\mathcal{B} = \mathcal{D} + f(x)\epsilon$, where $x\colon V \to \mathbb{R}$ is a distance function, $f\colon \mathbb{R} \to \mathbb{R}$ a suitable real-valued function and ϵ the Clifford generator corresponding to an auxilliary one-dimensional vector space endowed with an inner-product of negative signature. Then there are two main ingredients, one topological, and one geometric. The topological is that—roughly speaking—the index of the operator $\mathcal{B}$ can be interpreted as a pairing between the Bott element dual to the embedding $M \subset V$ and the K-homological fundamental class of V, and thereby yields the index of M. The geometric ingredient is that $\mathcal{B}$ satisfies a version of the Schrödinger–Lichnerowicz formula such that its invertibility is governed by the dominant energy condition (DEC) of a suitable initial data set $(V, g, k = -\frac{2f(x)}{n} g)$ (in the sense of relativity). Finally, the function f may be chosen such that the DEC is satisfied strictly if the distance exceeds the threshold $2\pi/n$ and a lower scalar curvature bound $\geq n(n-1)$ is maintained, thereby leading to a contradiction to non-vanishing of the index.

It turns out that the function f used in this argument is closely related to the mean curvature in certain model warped product metrics. Taking this observation to its conclusion lead to the joint work with Cecchini [1], where we studied Conjecture 1 and related questions via boundary conditions on the mean curvature. This also allowed rigidity statements in certain situations:

Theorem 3 ([1])**.** *Let (V, g) be a* Riemannian spin band*, that is, a compact spin manifold V together with a decomposition $\partial V = \partial_- V \sqcup \partial_+ V$ into non-empty unions of components, such that $\hat{\mathrm{A}}(\partial_- V) \neq 0$ and $\mathrm{scal}_g \geq n(n-1)$. Then the following holds:*

(1) If $\mathrm{H}_g \geq -\tan(nl/2)$ *for some* $0 < l < \pi/n$*, then*

$$\mathrm{width}(V, g) = \mathrm{dist}_g(\partial_- V, \partial_+ V) \leq 2l.$$

(2) If, in addition, equality in the above estimate is attained, then V *is isometric to* $M \times [-l, l]$,

$$g = \cos(nx/2)^{2/n} g_M + \mathrm{d}x^2,$$

for some spin manifold (M, g_M) *that admits a parallel spinor.*

(3) In particular, $\mathrm{width}(V, g) < 2\pi/n$.

In addition, our results on the related *long neck problem* (compare [3, p. 87, *Long neck problem*]) were presented:

Theorem 4 ([1]). *Let* (M, g) *be a compact connected* n*-dim. Riemannian spin manifold with boundary such that* $\mathrm{scal}_g \geq n(n-1)$ *on* M*, where* $n \geq 2$ *is even. Let* $f\colon M \to \mathrm{S}^n$ *be a smooth area non-increasing map. Suppose that for some* $0 < l < \pi/n$ *the following estimates hold:*

- $\mathrm{dist}_g(\partial M, \mathrm{supp}(\mathrm{d}f)) \geq l$.
- $\mathrm{H}_g \geq -\tan(nl/2)$ *on* ∂M,

Then the mapping degree of f *is zero.*

One crucial observation in the latter two theorems is that as $l \to \pi/n$, the lower mean curvature bound in the hypotheses of both theorems tends to $-\infty$. In other words, as l approaches this threshold and assuming that there is a non-trivial index invariant or mapping degree, the mean curvature must explode somewhere at the boundary.

Finally, we discussed new extremality and rigidity results for annuli in space forms:

Theorem 5 ([1]). *Let* $n \geq 3$ *be odd,* $\kappa \in \mathbb{R}$ *and* (M_κ, g_κ) *be the* n*-dimensional simply connected space form of curvature* κ*. Consider the* annulus *around a basepoint* $p_0 \in M_\kappa$

$$\mathrm{A}_{t_-, t_+} = \{p \in M_\kappa \mid t_- \leq d_{g_\kappa}(p, p_0) \leq t_+\},$$

where $0 < t_- < t_+ < t_\infty$ *with* $t_\infty = \pi/\sqrt{\kappa}$ *if* $\kappa > 0$ *and* $t_\infty = +\infty$ *otherwise.*

Then any Riemannian metric g *on* A_{t_-, t_+} *such that*

- $g \geq g_\kappa$,
- $\mathrm{scal}_g \geq \mathrm{scal}_{g_\kappa} = \kappa n(n-1)$,
- $\mathrm{H}_g \geq \mathrm{H}_{g_\kappa} = \pm\, \mathrm{ct}_\kappa(t_\pm)$,

must satisfy $g = g_\kappa$.

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Interior Kasparov products for ϱ-classes on Riemannian foliated bundles

Vito Felice Zenobi

Let us consider a smooth closed manifold M along with a spin foliation $\mathcal{F}$. Let G be the monodromy groupoid over M associated to $(M, \mathcal{F})$. We can associate to G a long exact sequence in K-theory of C*-algebras

$$\cdots \to K_*(C^*(G \times (0,1))) \to K_*(G_{ad}^{[0,1)}) \to K_*(AG) \to \cdots \tag{1}$$

where AG is the Lie algebroid of G and $G_{ad}^{[0,1)} := AG \times \{0\} \to G \times (0,1)$ is the so-called adiabatic deformation of G.

Let g be a metric on AG and let D_g be the longitudinal Dirac operator associated to it. It's symbol defines a class $[\hat{\sigma}]$ in $K_*(AG)$ and, through the boundary map, its higher index class $\mathrm{Ind}(D_g)$ in $K_*(C^*(G))$, both independent of g.

If g has positive scalar curvature (psc), the Lichnerowicz formula implies that D_g is invertible and then that the index vanishes. So $\mathrm{Ind}(D_g)$ is an obstruction to the existence of longitudinal psc metrics on $(M, \mathcal{F})$.

When g is a longitudinal psc metric we can define a canonical lift $\varrho(g)$ of $[\hat{\sigma}]$, which lives in $K_*(G_{ad}^{[0,1)})$ and it is a well defined invariant on the set $P^+(M, \mathcal{F})$ of concordance classes of longitudinal psc metrics, see [2]. Recall that two psc metrics g_0 and g_1 are concordant if there exists a psc metric g on $(M, \mathcal{F}) \times (0,1)$, of product type near the boundary, whose restriction to $(M, \mathcal{F}) \times \{i\}$ is g_i, with $i = 0, 1$.

Let $\iota\colon \mathcal{F}_0 \to \mathcal{F}_1$ be an inclusion of spin foliations over a manifold M and let G and H the monodromy groupoid associate to $\mathcal{F}_0$ and $\mathcal{F}_1$, respectively. Moreover let us fix two metrics g_0 on $\mathcal{F}_0$ and $g_1 := g_0 \oplus g_N$ on $\mathcal{F}_1$, where g_N is a metric on the normal bundle of the inclusion. In [1], the authors constructed a lower shriek map $\iota_! \in KK_n\left(C^*(G), C^*(H)\right)$ and prove a product formula for the index classes of the following sorts:

$$\mathrm{Ind}(D_{g_0}) \otimes \iota_! = \mathrm{Ind}(D_{g_1}). \tag{2}$$

In [3] we extend the construction of the lower shriek maps given by Hilsum and Skandalis to adiabatic deformation groupoid C*-algebras: we construct an asymptotic morphism $(\iota_{ad}^{[0,1)})_! \in E_n\left(C^*(G_{ad}^{[0,1)}), C^*(G_{ad}^{[0,1)})\right)$, where G and H are the monodromy groupoids associated with $\mathcal{F}_0$ and $\mathcal{F}_1$ respectively. Furthermore, if both g_0 and g_1 are psc metrics and $\mathcal{F}_0$ and $\mathcal{F}_1$ are Riemannian foliated bundles, we prove the following interior Kasparov product formula for foliated ϱ-classes associated with longitudinal metrics of positive scalar curvature

$$\varrho(g_0) \otimes (\iota_{ad}^{[0,1)})_! = \varrho(g_1). \tag{3}$$

This formula implies that if g_1 and g_1' are non-concordant longitudinal psc metrics on $\mathcal{F}_1$, then g_0 and g_0' are non-concordant as longitudinal psc metrics on $\mathcal{F}_0$, which can be seen as an higher secondary version of Connes' Theorem. Two main problem stay open:

- proving formula (3) for general foliations, namely not only Riemannian foliated bundles;
- finding under which assumptions the product in (3) is injective; injectivity would imply interesting rigidity results, namely that if any two longitudinal psc metrics g_0 and g_0' are non-concordant on $\mathcal{F}_0$, then g_1 and g_1' remain non-concordant longitudinal psc metrics on $\mathcal{F}_1$ even if there is a priori more room to construct a concordance between them.

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Excision and the K-theory of pushouts

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(joint work with Markus Land)

It is a classical result of Suslin and Wodzicki that algebraic K-theory satisfies excision on C^*-algebras [9, 8] . Another important result is the theorem of Cuntz and Quillen asserting that periodic cyclic homology satisfies excision on $\mathbb{Q}$-algebras [3]. However, in general K-theory does not satisfy excision. Instead, one can measure the failure of excision in algebraic K-theory by trace methods (see [2, 6, 4, 5]). Another approach, which works more generally for any localizing invariant, is the following [7].

Theorem A. To any pullback square of ring spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \end{array} \tag{1}$$

one can canonically associate a ring spectrum $A' \odot_A^{B'} B$ together with a map $A' \odot_A^{B'} B \to B'$ and a commutative diagram of ring spectra

$$\begin{array}{ccc} A & \longrightarrow & B \\ \big\downarrow & & \big\downarrow \\ A' & \longrightarrow & A' \odot_A^{B'} B \end{array} \tag{2}$$

refining (1) in the obvious sense. Any localizing invariant F sends square (2) to a pullback square of spectra. Moreover, the underlying spectrum of $A' \odot_A^{B'} B$ is the relative tensor product $A' \otimes_A B$.

Thus, the failure of excision for a localizing invariant F on the square (1) is now translated into the failure of the map $F(A' \odot_A^{B'} B) \to F(B')$ being an equivalence. I indicated how this implies the excision results of Suslin–Wodzicki, Cuntz–Quillen, Cortiñas, and Geisser–Hesselholt. As an example, if $A = R$ is any ring, $A' = R[x]$, $B = R[y]$, $B' = R[x, x^{-1}]$ with the maps given by $x \mapsto x$, $y \mapsto x^{-1}$, the ring spectrum $A' \odot_A^{B'} B$ turns out to be a discrete ring isomorphic to the Toeplitz ring $R\langle x, y\rangle/(yx - 1)$.

Now let $A' \leftarrow A_0 \to B$ be a diagram of ring spectra. Put $M = B \otimes_{A_0} A'$ and $A := A' \times_M B$. It turns out that A is canonically a ring spectrum and also that the analog of Theorem A holds in this more general setting. In this case, we have a formula for the ring spectrum $A' \odot_A^M B$:

Theorem B. In the above situation, the canonical map $A' \amalg_{A_0} B \to A' \odot_A^M B$ is an equivalence of ring spectra. In particular, for any localizing invariant F, there is a pullback square of spectra

$$\begin{array}{ccc} F(A) & \longrightarrow & F(B) \\ \big\downarrow & & \big\downarrow \\ F(A') & \longrightarrow & F(A' \amalg_{A_0} B). \end{array}$$

As an illustration, I discussed the following example. Let R be a discrete ring, and let G and H be groups. Put $A_0 = R$, $A' = R[G]$, and $B = R[H]$. Then $M \cong R[G \times H]$ and for the pullback A we obtain that $\pi_0(A) = R$, $\pi_{-1}(A)$ is a free R-module, and all other homotopy groups of A vanish. The pushout is given by $R[G] \amalg_R R[H] = R[G * H]$. If R is regular, stably coherent, then a recent result of Burklund and Levy on the K-theory of coconnective ring spectra implies that the canonical map $K(R) \to K(A)$ is an equivalence [1]. Thus, in this situation, we obtain a cartesian square of spectra

$$\begin{array}{ccc} K(R) & \longrightarrow & K(R[H]) \\ \big\downarrow & & \big\downarrow \\ K(R[G]) & \longrightarrow & K(R[G * H]). \end{array}$$

This recovers a result of Waldhausen [10].

Finally, I indicated a proof of Theorems A and B.

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Algebraic K-theory of the second truncated Brown-Peterson spectrum

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(joint work with Gabriel Angelini-Knoll, Christian Ausoni, Dominic Leon Culver, John Rognes)

Let $BP\langle 2\rangle$ be a second truncated Brown-Peterson spectrum equipped with the E_3-ring structure of Hahn-Wilson [3]. Let TC denote topological cyclic homology. We outline a proof of the following theorem [1]:

Theorem 1. *Let $p \geq 7$. There is a preferred isomorphism*

$$\begin{aligned} V(2)_*TC(BP\langle 2\rangle) \quad &\cong \quad P(v_3)\otimes E(\partial,\lambda_1,\lambda_2,\lambda_3) \\ &\oplus \quad P(v_3)\otimes E(\lambda_2,\lambda_3)\otimes \mathbb{F}_p\{\Xi_{1,d}\mid 0<d<p\} \\ &\oplus \quad P(v_3)\otimes E(\lambda_1,\lambda_3)\otimes \mathbb{F}_p\{\Xi_{2,d}\mid 0<d<p\} \\ &\oplus \quad P(v_3)\otimes E(\lambda_1,\lambda_2)\otimes \mathbb{F}_p\{\Xi_{3,d}\mid 0<d<p\} \end{aligned}$$

of $P(v_3)\otimes E(\lambda_1,\lambda_2,\lambda_3)$-modules. This is a finitely generated and free $P(v_3)$-module on $12p+4$ explicit generators in degrees $-1\leq * \leq 2p^3+2p^2+2p-3$.

Here, $V(2)$ is the Smith-Toda complex, and $P(-)$ and $E(-)$ are the polynomial and exterior algebra over $\mathbb{F}_p$.

The theorem confirms the original, strong form of the chromatic redshift conjecture of Rognes [4] for $BP\langle 2\rangle$ at $p\geq 7$. It also determines the mod (p,v_1,v_2) homotopy of the algebraic K-theory of $BP\langle 2\rangle$.

One ingredient of the proof of Theorem 1 is the following proposition:

Proposition 2. *The images of the classes $\alpha_1 \in \pi_{2p-3}(S)$, $\beta_1' \in \pi_{2p^2-2p-1}V(0)$, $\gamma_1'' \in \pi_{2p^3-2p^2-1}V(1)$ and $v_3 \in \pi_{2p^3-2}V(2)$ in $V(2)_*THH(BP\langle 2\rangle)^{hS^1}$ are detected in the homotopy fixed point spectral sequence*

$$\begin{aligned} E^2 = \quad & P(t) \otimes V(2)_*THH(BP\langle 2\rangle) = P(t) \otimes E(\lambda_1, \lambda_2, \lambda_3) \otimes P(\mu_3) \\ & \Longrightarrow V(2)_*THH(BP\langle 2\rangle)^{hS^1} \end{aligned}$$

by $t\lambda_1$, $t^p\lambda_2$, $t^{p^2}\lambda_3$ and $t\mu_3$.

The statement about β_1' and γ_1'' can be proven from that of α_1 using homotopy power operations for E_2 ring spectra R

$$P^k : \pi_{2k-1}R \to V(0)_{2pk-1}R \tag{1}$$

$$P^k : V(0)_{2k-1}R \to V(1)_{2pk-1}R. \tag{2}$$

For E_∞ ring spectra the operations (1) were already constructed by Ausoni and Rognes in their work about the Adams summand $BP\langle 1\rangle$ [2]. The operations lift the Dyer-Lashof operations in mod p homology and satisfy a homotopy Cartan formula under additional assumptions.

Using Proposition 2 and the fact that the classes α_1, β_1', γ_1'' and v_3 map to zero in $V(2)_*THH(BP\langle 2\rangle)^{tC_p}$ one can determine the differentials in the spectral sequence

$$E^2 = E(u_1) \otimes P(t^{\pm}) \otimes V(2)_*THH(BP\langle 2\rangle) \Longrightarrow V(2)_*THH(BP\langle 2\rangle)^{tC_p}.$$

One gets that the Frobenius map

$$\varphi_p : V(2) \wedge THH(BP\langle 2\rangle) \to V(2) \wedge THH(BP\langle 2\rangle)^{tC_p}$$

is $(2p^2 + 2p - 3)$-coconnected. This provides the starting point to compute

$$V(2)_*THH(BP\langle 2\rangle)^{hC_{p^n}} \text{ and } V(2)_*THH(BP\langle 2\rangle)^{tC_{p^{n+1}}}$$

by an inductive argument. Passing to the limit, one gets to $V(2)_*THH(BP\langle 2\rangle)^{hS^1}$ and $V(2)_*THH(BP\langle 2\rangle)^{tS^1}$. One can then compute the maps

$$\mathrm{can}, \varphi_p^{hS^1} : V(2)_*THH(BP\langle 2\rangle)^{hS^1} \to V(2)_*THH(BP\langle 2\rangle)^{tS^1}$$

defining TC and prove Theorem 1.

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A bicategorical interpretation for crossed products and Cuntz-Pimsner algebras

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(joint work with Suliman Albandik, Alcides Buss, Camila Fabre Sehnem, and Chenchang Zhu)

Many interesting C*-algebras are defined as a crossed product for a group action on a C*-algebra. Here an action of a group G on a C*-algebra A is a homomorphism α from G to the automorphism group of A. For various purposes, we would like to consider more general group actions. For instance, in the study of the Farrell–Jones conjecture, a key tool are group actions up to homotopy, where the homomorphism condition $\alpha_g\alpha_h = \alpha_{gh}$ is replaced by a homotopy, and higher homotopies as well. These higher homotopies may be encoded succinctly by a principal bundle over the classifying space BG of G with fibre A. A drawback of these actions up to homotopy is that there is no way to define a crossed product C*-algebra for them. How can we generalise group actions so that the crossed product C*-algebra may still be defined? I have studied this question and its ramifications for several years, with various coauthors.

It is useful to start from the crossed product $B = A \rtimes G$ and observe some extra structure on it. Namely, it is graded by the group G. View $A \rtimes G$ as a completion of the space of finitely supported functions $G \to A$ and let $B_g \subseteq B$ be the subspace of functions supported in the singleton $\{g\}$. This is a closed subspace of B, and the *-algebra structure on B is defined so that $B_g \cdot B_h \subseteq B_{gh}$ and $B_g^* = B_{g^{-1}}$. When we remember only the Banach spaces B_g with the multiplication and involution maps that they inherit from B, then we get a *Fell bundle*. Given a Fell bundle $(B_g)_{g\in G}$, the direct sum $\bigoplus B_g$ carries a canonical *-algebra structure, and the C*-completion of $\bigoplus B_g$ is the analogue of the crossed product for a Fell bundle. Thus we may view Fell bundles as generalised group actions. Here the C*-algebra on which the action takes place is the unit fibre $A := B_e$ for the unit element $e \in G$.

Actually, it is better to view Fell bundles as generalised partial actions, where a partial action of G on A only gives isomorphisms between ideals in A, not on A itself. Each fibre B_g becomes a Hilbert A-bimodule using the multiplication maps $B_e \times B_g \to B_g$, $B_g \times B_e \to B_g$ and the inner products $x^*y, xy^* \in B_e$ for $x, y \in B_g$. Such a Hilbert bimodule gives a Morita–Rieffel equivalence between the two-sided ideals in A that are spanned by the inner products x^*y and xy^*, respectively. That is, it is a partial Morita–Rieffel equivalence on A. A Fell bundle is called *saturated* if these ideals are equal to A for all $g \in G$. Then each B_g is a Morita–Rieffel self-equivalence of A. The multiplication maps $B_g \times B_h \to B_{gh}$ in the Fell bundles are a crucial ingredient, which we should not forget. To encode these as well, we consider a bicategory that has C*-algebras as objects, Hilbert bimodules as arrows, and bimodule maps that are isometric for both inner products as 2-arrows. The composition of arrows is given by the balanced tensor product of Hilbert bimodules. A bicategory homomorphism from G, viewed as a bicategory to this bicategory of Hilbert bimodules is equivalent to a saturated Fell bundle over G (see [2]).

(The more general bicategory morphisms also allow partial actions, but we must add some technical extra conditions to the standard definition of a bicategory morphism to get exactly the right concept. The issue is to recover the involution of a Fell bundle from the inner products xy^* and x^*y that it defines. This only works under extra conditions.)

The crossed product for a group action is defined by a universal property, and an analogous universal property works for the "crossed product" C*-algebra of a Fell bundle. This universal property becomes that of a (bi)limit in a suitable bicategory. To understand such universal properties, we must enlarge our bicategory because we need *-homomorphisms such as the inclusion of A into its crossed products to be arrows. When we combine *-homomorphisms with Hilbert bimodules, we get C*-correspondences. A *C*-correspondence* $A \leftarrow B$ is a Hilbert B-module E with a nondegenerate *-homomorphism from A to the C*-algebra of adjointable operators on E. If we insist on Hilbert bimodules that are full on the left, then the *-homomorphism has values in the C*-algebra of compact operators on E. Such C*-correspondences are called *proper*.

In my earlier papers, I viewed a C*-correspondence as an arrow from A to B, but I have found since then that it is better to treat it as an arrow from B to A. A homomorphism from G to the C*-correspondence bicategory is still the same as a saturated Fell bundle because the equivalences in the C*-correspondence bicategory are exactly the Morita–Rieffel equivalences and a homomorphism from a group to a bicategory maps all arrows in G to equivalences.

Now it turns out that the crossed product C*-algebra B of a Fell bundle or group action satisfies the universal property of a limit in the C*-correspondence bicategory. The limit is universal for bicategorical cones. Such a cone under the homomorphism (B_g) consists of a C*-correspondence $F\colon B_e \leftarrow D$ and isomorphisms of correspondences $u_g\colon B_g \otimes_{B_e} F \to F$ for all $g \in G$, satisfying some coherence conditions. The isomorphism u_g is equivalent to a Toeplitz representation T_g of B_g by adjointable operators on F with the extra nondegeneracy property that the linear span of $T_g(b_g) \cdot x$ for $b_g \in B_g$, $x \in F$ is dense in F; this translates to u_g being surjective. It turns out that this is exactly the same as a representation of the Fell bundle on F by adjointable operators or, in other words, a C*-correspondence $B \leftarrow D$. It is remarkable that the limit in a bicategory becomes "bigger" than the original diagram. In contrast, if we take the limit for a group action on a set, we get the subset of fixed points, while the colimit is the set of orbits. In a bicategory, however, the cone comes with extra data – the unitaries u_g above – instead of extra conditions.

Now let us go beyond the group case. What is a homomorphism from, say, a monoid M to the C*-correspondence bicategory, and what would its limit look like? These questions are mostly answered in [1]. A homomorphism from M to the C*-correspondence bicategory is the same as a product system over M. (With the conventions used in [1], it is a product system over the opposite monoid instead, and the limit becomes a colimit instead.) Assume that the product system is proper, meaning that the left actions in the C*-correspondences are by compact operators. Then a cone over the diagram with summit D turns out to be the same as a Cuntz–Pimsner covariant representation of the product system by adjointable operators on a Hilbert D-module F. Thus the Cuntz–Pimsner algebra of the product system is a bicategorical limit. If the product system is not proper, then it is unclear whether a limit exists. Certainly, the Cuntz–Pimsner algebra is not a limit any more.

Already the case of the monoid of natural numbers is interesting. In this case, a homomorphism or product system is determined by a single C*-correspondence $E\colon A \leftarrow A$. Many interesting C*-algebras are defined as the Cuntz–Pimsner algebra of such a single C*-correspondence. This includes the C*-algebras of (regular) graphs, (regular) topological graphs, self-similar groups, or self-similar graphs. In these cases, the C*-algebra A is much simpler than the final C*-algebra, and the Cuntz–Pimsner algebra definition offers a lot of tools to study it. The case when the C*-correspondence is not proper is more complicated. The most interesting aspect in the construction of a Cuntz–Pimsner algebra is the Cuntz–Pimsner covariance condition, which only makes sense for an element of A that acts on E by a compact operator. To get the most interesting Cuntz–Pimsner algebra, Katsura proposed to ask the Cuntz–Pimsner covariance condition on the largest ideal in A that acts on E faithfully and by compact operators. This rather careful choice is not just coming from bicategory theory.

What bicategory theory can explain is the *relative Cuntz–Pimsner algebra* construction, where we specify the ideal on which we ask the Cuntz–Pimsner covariance condition as part of the data (see [3]). The limit construction for diagrams is right adjoint to the inclusion of constant diagrams. To get a more general construction, we may consider a larger subcategory of "nice" diagrams and consider a reflector to that subcategory. The Cuntz–Pimsner algebra carries a canonical gauge action by the circle group, which is equivalent to a grading by the group of integers. This grading is always "semi-saturated", so that it is determined by its subspaces of degree 0 and 1. These form another C*-algebra B and a Hilbert bimodule over B, which usually is not saturated. Thus the Cuntz–Pimsner algebra construction, together with the gauge action, may be interpreted as mapping a C*-correspondence $A \leftarrow A$ to a Hilbert bimodule $B \leftarrow B$. Hilbert bimodules form a subbicategory in the C*-correspondence bicategory. In [3], we also add an ideal in A to the data to be able to choose the ideal on which the Cuntz–Pimsner covariance condition is imposed. Then we carefully choose a bicategory of such decorated self-correspondences and a subbicategory, so that the Cuntz–Pimsner algebra construction is a reflector to that subbicategory.

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