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Calculus of Variations

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Abstract. The Calculus of Variations is at the same time a classical subject, with long-standing open questions which have generated exciting discoveries in recent decades, and a modern subject in which new types of questions arise, driven by mathematical developments and emergent applications. It is also a subject with a very wide scope, touching on interrelated areas that include geometric variational problems, optimal transportation, geometric inequalities and domain optimization problems, elliptic regularity, geometric measure theory, harmonic analysis, physics, free boundary problems, etc. The workshop balances the traditional interests of past conferences with new emerging perspectives.

Mathematics Subject Classification (2020): 49-xx, 35Axx, 35Bxx, 35Jxx, 53Cxx.

Introduction by the Organizers

The workshop *Calculus of Variations* organized by Lia Bronsard (McMaster University), László Székelyhidi (University of Leipzig), Yoshihiro Tonegawa (Tokyo Institute of Technology) and Tatiana Toro (University of Washington) was very well attended with 45 in-person participants, with broad geographic and gender representation from all continents. In this workshop we observed exactly what we hoped for: a mixture of theoretical and applied problems of interest to all participants. The large number of on-site participants allowed for a schedule where all talks were in-person. The carefully weighed and prepared hygiene measures by the MFO staff and a schedule allowing for ample free discussion time led to a productive, convivial atmosphere where participants interacted with new people, new directions and new techniques, and were also able to advance in current and new projects. Several talks included very recent, as yet unpublished results; a level of timeliness that is almost never observed in virtual talks.

On the applied problems related to the Calculus of Variations, Riccardo Cristoferi presented new results on phase transitions in heterogeneous materials (in collaboration with Irene Fonseca, Adrian Hagerty and Cristina Popovici). These heterogeneous materials are of importance in biology and material sciences. In particular, he presented a surprising result in which a first order asymptotic expansion in a two parameter Gibbs energy yields an anisotropic surface energy in contrast to the usual isotropic surface energy obtained for the classical Cahn-Hilliard/Van der Waals functional. This result is due to the heterogeneity of the material. Ihsan Topaloglu presented results for nonlocal variational problems on polygons, which represents a new direction for these problems. Among other results, he showed that the Wulff shape remains a minimizer for the nonlocal anisotropic liquid drop model with crystalline surface tensions. This is in contrast to previous results for anisotropic Gamow type energy where the Wulff shape is not even a critical point of the energy. Maria Westdickenberg presented a general L^1 method for convergence and metastability in 1-d Cahn-Hilliard type problems, which can be extended to the Mullins-Sekerka evolution problems in 2 and 3 space dimensions. Xavier Lamy presented impressive new estimates generalizing previous estimates by Lorent on the very difficult Aviles-Giga functional, and used it to prove stability estimates associated to a rigidity result for zero-energy states obtained by Jabin, Otto and Perthame. Antonin Chambolle presented new technical results on compactness for Griffith-type energies which will have important consequences in the understanding of fracture problems. We note that several talks were blackboard talks and this was very much appreciated as it leads to a slower pace and better understanding of the presentations.

We also had five shorter talks on Tuesday afternoon given by either post-docs or finishing graduate students. These short talks were very well received, as their format made their delivery very dynamic. Michael Novack presented new results, joint with Francesco Maggi, on "mesoscale" flatness criterion that applies to hypersurfaces for which neither blow-up and blow-down techniques can describe flatness at desired scales. This result is a key tool for the resolution of large volume exterior isoperimetric sets. Dominik Stantejsky presented his new results on the description of defects around a colloid in a nematic liquid crystal under an imposed magnetic field in appropriate asymptotic regimes. Using very sophisticated geometric measure theoretical tools, he was able (jointly with F. Alouges and A. Chambolle) to obtain the Γ-limit of the Landau-de Gennes functional in an appropriate regime and use it to describe the defects of minimizers. Andrew Colinet presented new technical results on Γ-convergence of the famous Ginzburg-Landau energy which hold up to boundary, where zero Dirichlet data are specified on either the tangential part or the normal part of the boundary. Hyunju Kwon presented the state-of-the-art in convex integration for the Euler equations and the difficulties in ensuring positivity of the local dissipation measure. Finally, André Guerra presented recent results on extension of Sobolev type inequalities for non-elliptic operators.

The research areas related to **minimal surfaces**, **geometric flows** and **free boundaries** are well represented in this workshop. In the areas related to minimal surfaces, Antonio De Rosa presented a new min-max existence theorem of an anisotropic 2D minimal surface which is smooth except possibly for one point in a closed 3D smooth Riemannian manifold, and Joaquim Serra presented the existence of infinitely many nonlocal minimal surfaces on any closed Riemannian manifold which resolves the nonlocal version of Yau's conjecture. Salvatore Stuvard presented the state-of-the-art on the fine properties of singular sets of area minimizing currents mod p, both for the hypersurface case and general codimensional case. Zihui Zhao presented the estimate on the Hausdorff measure of the singular set of a harmonic function in terms of Almgren's frequency function near the boundary. For the geometric flows, varieties of results on the network curvature flow were presented by Alessandra Pluda such as the asymptotic behaviors and restarting procedures. Felix Schulze presented the uniqueness result for tangent flow of rational Lagrangian mean curvature flow which allows the continuation of the flow past the singularity. For problems related to free boundary, Luca Spolaor presented a fine regularity result on the branch point in the 2D Bernoulli problem using quasi-conformal maps and the hodograph transformation. Mariana Smit Vega Garcia presented results on the fractional free boundary problem of Alt-Caffarelli-Friedman (ACF) type. Denis Kriventsov gave a new integral estimate on the deviation of the first eigenfunction of the Laplacian on the unit ball in terms of that of the eigenvalue, and explained its application to the ACF problem.

Various **regularity issues** as well as new **constraints on oscillatory behaviour** were also prevailing topics throughout the workshop. Xavier Lamy and Andrew Lorent presented, in very nicely complementing talks, their recent progress on the Aviles Giga conjecture and the fine-scale regularity and rigidity of elements in the Aviles Giga space. Filip Rindler highlighted in his talk the need to develop a new duality and slicing theory for the transport of currents. Daniel Faraco and André Guerra presented recent exciting progress on Morrey's conjecture concerning quasiconvexity, focussing of the Burkholder functional. Finally, Annalisa Massaccesi and Jonas Hirsch focussed on the role of geometric (non)-integrability (Hörmander vs Frobenius condition) on regularity, Annalisa in the geometric setting of submanifolds and Jonas in the setting of the De Giorgi-Nash-Moser theorem.

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Workshop: Calculus of Variations

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Abstracts

Min-max construction of anisotropic CMC surfaces

Antonio De Rosa
(joint work with Guido De Philippis)

The existence of closed minimal hypersurfaces in closed Riemannian manifolds is a well-known theorem, that has been proved by means of the min-max theory. More precisely, Almgren [4] showed the existence of stationary integral varifolds and Pitts [8] proved them to be smooth up to dimension 6 of the ambient manifold, relying on the curvature estimates for stable minimal hypersurfaces proved by Schoen, Simon, and Yau [10]. These results have been generalized to every dimension by Schoen and Simon [9]. Moreover, the existence of closed hypersurfaces with zero mean curvature has been extended by Zhou and Zhu [12] to the existence of constant mean curvature hypersurfaces in closed Riemannian manifolds.

Notwithstanding this wide literature for the area functional, the existence of closed anisotropic minimal hypersurfaces or constant anisotropic mean curvature hypersurfaces, referred to as anisotropic CMC hypersurfaces, was completely open. Given an n-dimensional closed Riemannian manifold M^n and a smooth elliptic positive function F on the Grassmannian bundle of $(n-1)$-planes, the anisotropic energy of any compact $(n-1)$-submanifold $\Sigma \subset M^n$ is defined as follows:

$$\mathbf{F}(\Sigma) := \int_\Sigma F(x, T_x\Sigma) d\mathcal{H}^{n-1}(x).$$

Choosing $F \equiv 1$ we get the classical $(n-1)$-dimensional area functional.

Allard [3] proved an anisotropic version of the curvature estimates of Schoen and Simon [9] and conjectured that in every closed Riemannian manifold there exist closed anisotropic minimal hypersurfaces, i.e. hypersurfaces with zero anisotropic mean curvature with respect to $\mathbf{F}$ [3]. However he observed that: *"there remains a considerable amount of work to do before this becomes feasible for general integrands"*[3, Page 288].

In joint work with De Philippis [5], we partially solved this problem for $n = 3$, proving the following first main result:

Theorem 1 ([5]). *If $n = 3$, there exists a nontrivial surface $\Sigma \subset M^3$ without boundary which is a critical point with respect to $\mathbf{F}$. Moreover there exists at most one singular point $p \in M^3$ for Σ, i.e. Σ is smooth embedded away from p. Furthermore one of the following properties holds:*

- (a) *there exists $R > 0$ such that Σ is smooth anisotropic stable in $B_R(x)$ for every $x \in M^3$;*
- (b) *Σ is anisotropic stable in $M^3 \setminus \{p\}$.*

We proved Theorem 1 in [5] via a compactness argument for anisotropic stable surfaces, applying the following second main result of our work [5]:

Theorem 2 ([5]). *If $n = 3$, for every $c \in \mathbb{R} \setminus \{0\}$ there exists a nontrivial surface $\Sigma \subset M^3$ without boundary which has constant anisotropic mean curvature c with respect to $\mathbf{F}$. Moreover there exists at most one singular point $p \in M^3$ for Σ, i.e. Σ is smooth almost embedded away from p. Furthermore one of the following properties holds:*

(a) *there exists $R > 0$ such that Σ is smooth anisotropic stable in $B_R(x)$ for every $x \in M^3$;*
(b) *Σ is anisotropic stable in $M^3 \setminus \{p\}$.*

Theorem 1 easily follows applying Theorem 2 to construct a sequence of anisotropic CMC surfaces Σ_j with anisotropic mean curvature $1/j$, smooth away from at most one point p_j. Indeed, a compactness theorem for anisotropic stable surfaces will guarantee the convergence of Σ_j (up to subsequences) to an anisotropic minimal surface which is smooth outside of at most one point p, where p is a subsequential limit of p_j.

Our proof of Theorem 2 in [5] follows the scheme developed by Almgren and Pitts [8] for the existence of isotropic minimal surfaces. This scheme consists of two parts: the existence and the regularity of the anisotropic CMC surface.

For the first part, we prove the existence of an anisotropic CMC rectifiable varifold with a similar strategy to the one used by Almgren and Pitts [8], that is by performing a pull-tight procedure and by proving a replacement property in annuli. However, we need to substitute Allard rectifiability theorem [1] with its anisotropic counterpart, that we proved with De Philippis and Ghiraldin in [6]. To apply the latter result, we have to employ a density lower bound for smooth anisotropic stable surfaces, proved by Allard in [3]. We show the integrality of the constructed anisotropic CMC varifold via the maximum principle and the anisotropic constancy theorem, proved in our joint work with De Philippis and Hirsch [7].

For the regularity part, there are several obstructions. To prove regularity in punctured balls, the main difficulty is showing that two consecutive replacements in concentric annuli glue smoothly at their interface. Indeed, due to the lack of monotonicity formula [2], it is not clear a priori if upper density estimates hold at the points of the interface and, if they hold, whether the blowups at these points are flat. This is the main reason we focus on the existence of constant non-zero anisotropic mean curvature surfaces in Theorem 2. The main advantage compared to the case of anisotropic minimal surfaces is that, borrowing ideas from the work of Zhou and Zhu [12], in the CMC setting we can construct multiplicity one replacements, containing just a 1-dimensional set of touching points with multiplicity 2. Hence we need to prove the smooth gluing at the points of the interface just in two cases:

Case 1: At the points of multiplicity one for the first replacement, we refine the construction of the second replacement, showing that its approximating sequence is regular up to the interface. Combining the stability of the sequence with our joint work with De Philippis and Hirsch [7] and with the work of White [11], we prove uniform boundary curvature estimates for the approximating sequence. Hence we can pass the sequence to the limit, to conclude that the second replacement is smooth up to the interface and glues smoothly with the first replacement.

Case 2: At the isolated points of multiplicity two for the first replacement, we show upper density estimates adapting [7]. The blowups are then proved to be planes with multiplicity two, exploiting the regularity in the points of multiplicity one and the maximum principle. Then by a graphicality argument we conclude that the two replacements glue smoothly.

The argument above allows us to show that the constructed anisotropic CMC surface is smooth and locally anisotropic stable away from finitely many points: the centers of the annuli of the aforementioned replacements. Refining Almgren-Pitts combinatorial lemma, by compactness we are able to remove all these isolated singularities, except one. This last singular point accounts for the index of the constructed CMC surface and cannot be removed by the Almgren-Pitts combinatorial lemma. We expect that this singularity is removable, by means of the geometric PDE and of the stability inequality outside of the singularity. However, the main obstruction in doing so is that this singular point may have infinite density, which does not allow the standard logaritmic cutoff argument or the blowup analysis at this point.

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Network flow: resolution of singularities and stability

Alessandra Pluda

(joint work with Jorge Lira, Rafe Mazzeo, Marco Pozzetta, and Mariel Sáez)

Geometric flows are parabolic partial differential equations that aim to deform a geometric object to simplify it (e.g. to reduce its topological complexity or to make it more symmetric). They have been applied to a variety of topological, analytical and physical problems, giving in some cases very fruitful results. Among other geometric equations the *Mean Curvature Flow* has been extensively studied. This flow can be regarded as the gradient flow of the area functional: an n–dimensional time dependent surface evolves with normal velocity given by its mean curvature. Nowadays the mean curvature flow of a smooth submanifold is deeply understood. The situation is different when we consider the evolution of generalized, possibly singular, submanifolds. The simplest example of motion by mean curvature of a set which is essentially singular is the *evolution of networks*: 1–dimensional connected sets composed of a finite number of curves that meet at junctions. This is the case of our interest.

Appearing initially in materials science as a model for the evolution of grain boundaries in polycrystals, this flow was later treated by Brakke [1] using varifold methods. A network of curves is a "nearly smooth" object, hence one is tempted to describe its evolution by a direct PDE approach, but doing so requires one to deal with the singular nature of the PDE at the vertices of the network. Interpreting the junctions as boundary points (free to move during the evolution), Bronsard and Reitich [2] proved a *short time existence* result with initial datum a *regular network* composed of three curves meeting at a triple junction forming equal angles. However, this class of networks, is not preserved by the flow: two triple junctions might coalesce while the curvature remains bounded [10], or an enclosed region bounded by a loop of curves in the network might collapse to a point [11].

We would like to know whether this limiting network can evolve past this singular time. Thus the motivation to understand how to start/restart the flow from a network of curves with *"irregular" multiple junctions* goes beyond the basic inherent interest in enlarging the class of admissible initial data. Keeping in mind that stable critical points of the length functional present only triple junctions whose unit tangent vectors at the junctions sum up to zero, we expect to see configurations of this kind for almost all times. We must, in particular, show precisely how a single multi–point gives birth to a cluster of triple junctions.

The short time existence for *irregular* networks [7, 8] reads as follow: let Γ_0 be an initial network of class C^2 and at least one interior vertex is irregular. Then there exists a time $T > 0$ and an evolving family of regular networks Γ_t, $0 < t < T$, with the property that $\Gamma_t \to \Gamma_0$ uniformly. Moreover the set of these possible flowouts is classified by the collection of all *"expanding soliton solutions"* of the flow at each interior vertex.

These results were first proved in [7], but our approach in [8] draws out some important features and precise information not accessible by the previous method. The inspiration for the constructions, that we briefly describe now, comes from the methods of geometric microlocal analysis.

Let $k \geq 3$ and suppose that p is a k–valent irregular vertex of our initial network Γ_0 where the curves $\gamma^{(1)}, \ldots, \gamma^{(k)}$ meet. Denoting by τ_j the unit tangent vector to γ_j at p, the incoming edges at p determine a fan of rays $\ell_j = \mathbb{R}^+\tau_j$ in $\mathbb{R}^2$ emanating from p. There exists at least one (typically many) expanding soliton solution for the network flow which has this fan as its initial condition [15]. Choose one of these solutions, and denote it by S_p. Having made this choice for every irregular interior vertex, we prove that there exists a unique solution Γ_t of the network flow, defined on some interval $0 < t < T$, whose combinatorics are the same as if we were to replace a small ball around each irregular vertex p with the corresponding choice of soliton S_p. This solution converges to Γ_0 as $t \to 0$.

One striking feature is that if there is a k–valent irregular vertex, then there exists more than one solution to the network flow emanating from the given initial configuration.

Once a solid short time existence result is established, the *long time behavior* of the evolution deserves to be investigated. Suppose that Γ_t is a maximal solution to the network flow in $[0, T)$. Then if T is finite, either the inferior limit of the length of at least one curve of the network $\mathcal{N}(t)$ is zero or the superior limit of the L^2–norm of the curvature of the network is $+\infty$ [12, 6]. There are explicit examples of all these behaviors [11, 16].

Now when $T = +\infty$ we want to understand whether Γ_t converges to a critical point of the length functional. Instead when the flow develops a singularity at a finite time, we aim at a deeper understanding of the singular limit of Γ_t as $t \to T^-$, by means of the analysis of *tangent flows* (sequences of space-time rescalings of the flow that exists for all times). One of the main difficulty is to understand whether the limits as $t \to \infty$ of such rescalings is unique, i.e., whether the tangent flow do not depend on the choice of the rescaling sequence. Hence, both to understand the singularities of the flow and its asymptotic behavior, the key point is the existence of a full limit critical point as t tends to $+\infty$ of certain gradient flows. This issue can be addressed by means of the so-called *Łojasiewicz–Simon* gradient inequalities introduced in the seminal works [9, 18]. Roughly speaking, an energy functional E satisfies a Łojasiewicz–Simon inequality in the neighborhood U of a critical point if a concave power of the difference in energy between the critical point and a point in U can be bounded from above by a norm of the gradient of E. It turns out that if the given energy E satisfies Łojasiewicz–Simon inequalities in neighborhoods of its critical points, then the gradient flow for E exists for all times and converges to a critical point as $t \to +\infty$ (see [3, 13, 14]).

The study of the network flow imposes new difficulties on the application of the aforementioned results carried out for flows of smooth submanifolds, because of the natural singularities of our evolving objects. In [16] we overcome these technical issues and we prove a Łojasiewicz–Simon inequality for the length functional of networks suitably close to minimal ones. This allows us to prove not only the smooth convergence of the network flow whenever it does not develop singularities, but also the stability of minimal networks: for initial data suitably close to minimal networks, the evolution exists for all times without singularities and it smoothly converges to a critical point.

The same method has been applied in [17] to prove the uniqueness of compact blowups of network flow. We plan to investigate the (much more difficult) case of possibly noncompact blowups, also exploiting the recent fundamental achievement on the same problem for mean curvature flow of hypersurfaces [5, 4].

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Phase separation in heterogeneous media

Riccardo Cristoferi
(joint work with Irene Fonseca, Likhit Ganedi, Giovanni Gravina, Adrian Hagerty, and Cristina Popovici)

Many of the new media needed, and sought for, by the applications are heterogeneous, with materials inclusions. Being able to describe and predict stable equilibrium configurations of phases is thus fundamental in order to take full advantages of such materials.

In this talk, we restrict our attention to the case where the media has a periodic microstructure. We work within the gradient theory of phase separations, namely using a Modica-Mortola type of energy. The periodic heterogeneity of the material is modeled by allowing the periodic potential and its wells to depend on the spatial position, also in a discontinuous way, to take into consideration the possibility of materials inclusions. On the other hand, the singular perturbation, is considered to be homogeneous (see [1] for the case where the singular perturbation is highly oscillatory, but the potential and wells are homogeneous). The Gibbs free energy we consider, for $u \in H^1(\Omega, \mathbb{R}^M)$, is the following:

$$\mathcal{F}_{\varepsilon,\delta}(u) := \int_\Omega \left[W\left(\frac{x}{\delta}, u(x)\right) + \varepsilon^2 |\nabla u(x)|^2 \right] dx.$$

Here, $\varepsilon > 0$ is the parameter related to phase separation, while $\delta > 0$ is the scale of heterogeneities of the material. The potential $W : \Omega \times \mathbb{R}^M \to [0, \infty)$ is such that $W(x,p) = 0$ if and only if $p \in \{z_1(x), \dots, z_k(x)\}$, for $\mathcal{L}^N$-a.e. $x \in \Omega$. The functions $x \mapsto W(x,p)$ and $x \mapsto z_i(x)$ are Q-periodic, with $Q := (0,1]^N$, and piecewise Lipschitz; the function $p \mapsto W(x,p)$ is locally Lipschitz, with quadratic growth at infinity, and with quadratic behavior close to the wells.

The goal is to understand the behavior of (mass constrained) minimizers of $\mathcal{F}_{\varepsilon,\delta}$, and it will be achieved by studying the properties of minimizers of each term in the asymptotic expansion by Γ-convergence of the energy. The one dimensional scalar case was studied in [2]. Different behaviors are expected according to the interplay between the scale of heterogeneities of the materials and that at which phase transitions take place. Moreover, an important role is also played by the behavior of the wells: whether or not they are constant will influence the asymptotic expansion.

The zeroth order term is a bulk energy, whose integrand depends on which of the regimes $\delta \ll \varepsilon$, $\delta \sim \varepsilon$, or $\varepsilon \ll \delta$ is considered. In all of the cases, the minimum of the limiting energy is zero. Moreover, the leading order of the energy of minimizing sequences is ε in the first two cases, and ε/δ in the latter.

The first order term is identified in three different cases.

The regime $\delta \ll \varepsilon^{3/2}$ was considered in [7], by assuming the wells to be constant. In this case, the first order term in the asymptotic expansion is the same as what is obtained by first sending $\delta \to 0$, and then sending $\varepsilon \to 0$. In particular, the limiting energy is the Γ-limit of the classical Modica-Mortola functional whose potential is given by the homogenization of the highly oscillatory potential in $\mathcal{F}_{\varepsilon,\delta}$. The reason why only the subregime $\delta \ll \varepsilon^{3/2}$ is treated seems to be just technical, since a Poincaré inequality is used in the proof. We are currently refining the argument in order to extend the analysis to the whole regime $\delta \ll \epsilon$.

On the other hand, when the wells are not constant, we expect a different behavior in the three subregimes $\delta \ll \varepsilon^{3/2}$, $\delta \sim \varepsilon^{3/2}$, and $\varepsilon^{3/2} \ll \delta \ll \varepsilon$. These should correspond to the cases where the energy of the microscopic oscillations is negligible, of the same order, and of a lower order than that of the phase separation, respectively.

The regime $\varepsilon \sim \delta$ was addressed in [4, 5], also assuming the wells of the potential to be constant. In this case, the functional in the first order asymptotic expansion is an anisotropic surface energy. This is in contrast with the isotropic surface energy peculiar to the classical Modica-Mortola functional. The anisotropy character of the limiting energy reflects the fact that the potential depends on the spatial variable. Its expression is given by an asymptotic cell formula.

Finally, the regime $\varepsilon \ll \delta$ was investigated in [3] by allowing the potentials and its wells to depend in a piecewise Lipschitz way on the spatial position. In this case, something interesting happens: at first order, phase separations take place at the microscopic level, but not at the macroscopic one. This is in sharp contrast with the character of the limiting energy of the Modica-Mortola functional and its variants, as well as of that of the cases considered above: indeed, in those cases, the fact that they are surface energies means that phase separation happens at the macroscopic scale. This peculiar feature of the limiting energy seems to be related to a phenomenon observed experimentally by biologists in [8]. In order to capture this peculiar situation mathematically, we consider the Γ-limit in the two-scale convergence: we separate the macroscopic scale, and the microscopic scale. In the latter, a change of variable allows us to work with a classical Modica-Mortola functional with a moving potential vanishing at some moving wells (but with no highly oscillatory variable). This situation was investigated in [6], and the result was extended to the case of weaker assumptions in [3]. The limiting functional at the microscopic level is an isotropic surface energy, with a space dependent density. On the other hand, at the macroscopic level, there is no penalization in passing from different microstructures from one point to another. Namely, the limiting energy is a bulk term in the macroscopic variable, and a surface energy in the microscopic variable.

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Nonlocal approximations of minimal surfaces and Yau's conjecture

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(joint work with Michele Caselli, Hardy Chan, Serena Dipierro, Enric Florit, and Enrico Valdinoci)

The existence and regularity of minimal hypersurfaces is one of the central problems in Riemannian geometry. As a paradigmatic example, S.-T. Yau raised, in 1982, the following delicate and influential question:

> Do all 3-manifolds contain infinitely many immersed minimal surfaces?

The efforts towards understanding this and related questions triggered important progress in the last few years.

Since many 3-manifolds do not contain any area-minimizing surfaces, one is lead to consider surfaces that are merely critical points of the area functional. Such surfaces are naturally constructed by min-max —i.e., mountain-pass— type methods and have finite Morse index.

Almgren-Pitts approach. The most standard method for constructing min-max minimal surfaces is that of Almgren and Pitts [7]. Building on it, Irie, Marques, and Neves [6] gave a positive answer to Yau's conjecture in the case of generic metrics. Soon after this paper, Song [8] was able to modify the method in order to establish the existence of infinitely many closed surfaces in *every* 3-manifold.

Allen-Cahn approach. Guaraco [5] proposed an alternative to Almgren-Pitts theory, later extended by Gaspar-Guaraco [4]. The idea is to find minimal surfaces as limits of (the zero level sets of) sequences, with $\epsilon \downarrow 0$, of critical points of the Allen-Cahn functional:

$$J_\epsilon(u) := \int_M \left(\epsilon|\nabla u|^2 + \frac{1}{\epsilon}(1-u^2)^2\right) dV_g,$$

where (M, g) is a closed 3-manifold, ∇ is the gradient, and V_g is the volume measure.

Heuristically, the level zero level sets of critical points of J_ϵ should converge to minimal surfaces as $\epsilon \downarrow 0$.

Now, critical points which solve the semilinear PDE (the Allen-Cahn equation), and the Palais-Smale condition holds so min-max methods become much simpler than in the Almgren-Pitts setting.

However, the difficult part comes when one tries to pass the critical points constructed for each $\epsilon > 0$ to the limit $\epsilon \downarrow 0$. Indeed, the nonlinearity in the PDE blows-up and the elliptic estimates become useless.

In the paper [3], Chodosh and Mantoulidis managed to exploit the finite Morse index property of the min-max critical points to be able to perform this delicate passage to the limit. Among other things, they re-proved the existence of infinitely many minimal surfaces in closed 3-manifolds (with generic metrics) and established, for the first time, the validity of the multiplicity one conjecture of Marques and Neves.

The most critical steps in their work are:

- Proving a (uniform in ϵ) curvature estimate for level sets of stable critical points of J_ϵ (with "area bound").
- Using delicate information on "interaction" between different sheets (Toda's system) in order to prove that, whenever multiple sheets converge towards the same minimal surface, this surface must be degenerate.

To achieve this, [3] builds on ideas developed by Wang and Wei in [9] to study finite Morse index solutions of Allen-Cahn in $\mathbb{R}^2$.

A new approach. In our forthcoming paper [1], finite Morse index nonlocal minimal surfaces in a manifold are defined and studied.

Surprisingly, such surfaces enjoy much better properties than their classical (local) counterparts.

For instance (see [1] for the precise statements) the collection of all nonlocal minimal surfaces of a given closed n-manifold with index bounded by k is compact in an extremely strong sense: they converge strongly in the natural norm used to define nonlocal minimal surfaces (in particular the "area" of the limit equals the limit of the "areas").

Moreover, in low dimensions ($n = 3, 4$) finite Morse index surfaces are compact in the strongest possible sense: as "C^2 submanifolds". Such compactness properties makes nonlocal minimal surfaces extremely well-suited to perform min-max constructions.

Indeed, [1] we establish the existence of infinitely many nonlocal minimal surfaces in any given close n-manifold (this can be regarded as the nonlocal analogue of Yau's conjecture).

Nonlocal minimal surfaces depend on a parameter $s \in (0, 1)$, and classical minimal surfaces are recovered as the limit $s \uparrow 1$. Thus, it is very natural to ask whether one can send $s \uparrow 1$ in the constructions from [1] to recover the classical Yau conjecture. As for Allen-Cahn, this can only be done if one can show some robust curvature bounds as $s \uparrow 1$ and understand the delicate information on interaction between layers that "survives" after taking the limit (the analogue of the Toda system).

Surprisingly, all this can be done — this is one of the main contributions of the paper [2] —, and actually the proofs turn out to be cleaner and less technical than in the Allen-Cahn case.

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Transport of currents

Filip Rindler

(joint work with Paolo Bonicatto and Giacomo Del Nin)

Transport phenomena are ubiquitous in physics and engineering: For instance, given a bounded (time-dependent) vector field $b_t = b(t,\cdot)\colon \mathbb{R}^d \to \mathbb{R}^d$, $t \in [0,1]$ (the time interval being $[0,1]$ throughout this work for reasons of simplicity only), the *transport equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}u_t + b_t \cdot \nabla u_t = 0, \tag{1}$$

describes the transport of *scalar fields* $u_t = u(t,\cdot)\colon \mathbb{R}^d \to \mathbb{R}$ (e.g., electrical potentials), while the *continuity equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t + \operatorname{div}(b_t\mu_t) = 0, \tag{2}$$

describes the transport of *densities*, or, more generally, of measures $\mu_t = \mu(t,\cdot)$ representing (possibly singular) mass distributions.

Another example of a transport phenomenon is the motion of *dislocations*, which constitutes the main mechanism of plastic deformation in solids composed of crystalline materials, e.g. metals [1, 9, 4] (also see [8] for an extensive list of further references). Dislocations are topological defects in the lattice of the crystal material which carry an orientation and a "topological charge", called the *Burgers vector*. If one considers *fields* $\tau_t = \tau(t, \cdot)\colon \mathbb{R}^3 \to \mathbb{R}^3$ of continuously-distributed dislocations (for a fixed Burgers vector) transported by a velocity field b_t, one obtains the *dislocation-transport* equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\tau_t + \operatorname{curl}(b_t \times \tau_t) = 0. \tag{3}$$

This equation follows more or less directly from the Reynolds transport theorem for 1-dimensional quantities. A theory of plasticity based on dislocation transport are the *field dislocation mechanics* developed by Acharya, see, for instance, [5] and [2], and the recent variational model in [8, 12, 11].

In all of the above equations, the case of "singular" objects being transported is just as natural as the case of fields. Besides the dislocations mentioned already, moving point masses, lines, or sheets are particularly relevant in fluid mechanics when considering concentrated vorticity. Intermediate-dimensional structures also appear in the setting of Ginzburg–Landau energies, even in the static case, see [3, 10, 7]. On the other hand, the continuum case corresponds to "fields" of such points, lines, membranes, etc., and should arise via homogenisation. However, the rigorous justification of this limit passage is missing in many cases, e.g., in the theory of dislocations, where it constitutes one of the most important open problems. This is partly due to the present lack of understanding of the singular versions of the equations above.

The starting point for the present work is the observation that the transport equation (1), the continuity equation (2), the dislocation-transport equation (3), as well as several other transport-type equations, are all special cases of the *geometric transport equation*

$$\frac{\mathrm{d}}{\mathrm{d}t}T_t + \mathcal{L}_{b_t}T_t = 0 \tag{GTE}$$

for families of normal or integral k-currents $t \mapsto T_t \in \mathrm{N}_k(\mathbb{R}^d)$, $t \in [0,1]$. We understand this equation in a weak sense, that is, for every $\psi \in \mathrm{C}_c^\infty((0,1))$ and every smooth k-form $\omega \in \mathscr{D}^k(\mathbb{R}^d)$ it needs to hold that

$$\int_0^1 \langle T_t, \omega\rangle \psi'(t) - \langle \mathcal{L}_{b_t}T_t, \omega\rangle \psi(t)\,\mathrm{d}t = 0. \tag{4}$$

Here, we define the *Lie derivative* $\mathcal{L}_{b_t}T_t$ of T_t with respect to b_t as the current given by

$$\mathcal{L}_{b_t}T_t := -b_t \wedge \partial T_t - \partial(b_t \wedge T_t),$$

which is obtained by duality via Cartan's formula for differential forms.

When considering the transport of a family of currents $t \mapsto T_t$, it turns out that another notion of solution is, in a sense, more natural than the weak solutions considered above, namely the *space-time solutions*. This concept builds on the theory of space-time currents, introduced in [12], and can be explained, in the case of integral currents, as follows: Let S be a $(1+k)$ integral current in $[0,1] \times \mathbb{R}^d$.

Denote by $S|_t$ the slice of S at time t (with respect to the time projection $\mathbf{t}(t,x) := t$) and by

$$S(t) := \mathbf{p}_*(S|_t)$$

its pushforward under the spatial projection $\mathbf{p}(t,x) := x$. Standard theory gives that $S(t)$ is an integral k-current in $\mathbb{R}^d$ and that the orienting map $\vec{S} \in \mathrm{L}^\infty(\|S\|; \bigwedge_k(\mathbb{R}\times\mathbb{R}^d))$ (with $|\vec{S}| = 1$ $\|S\|$-a.e.) decomposes orthogonally as

$$\vec{S} = \xi \wedge \vec{S}|_t, \qquad \xi \perp \vec{S}|_t,$$

where $\vec{S}|_t$ is the orienting map of the slice $S|_t$, and

$$\xi(t,x) := \frac{\nabla^S \mathbf{t}(t,x)}{|\nabla^S \mathbf{t}(t,x)|}.$$

Here, $\mathbf{t}(t,x) := t$ is the temporal projection and $\nabla^S \mathbf{t}$ is its approximate gradient with respect to S, i.e., the projection of $\nabla \mathbf{t}$ onto the approximate tangent space $\mathrm{Tan}_{(t,x)} S$ (more precisely, "S" should be replaced by the $\mathcal{H}^{1+k}$-rectifiable carrier set of S here, but we consider this to be implicit).

We can now define the *geometric derivative* of S as the (normal) change of position per time of a point travelling on the current being transported, that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t,x) := \frac{\mathbf{p}(\xi(t,x))}{|\mathbf{t}(\xi(t,x))|} = \frac{\mathbf{p}(\xi(t,x))}{|\nabla^S \mathbf{t}(t,x)|} = \mathbf{p}\left(\frac{\nabla^S \mathbf{t}(t,x)}{|\nabla^S \mathbf{t}(t,x)|^2}\right)$$

for $\|S\|$-a.e. (t,x). Clearly, the geometric derivative exists only outside the *critical set*

$$\mathrm{Crit}(S) := \big\{(t,x) \in [0,1]\times\mathbb{R}^d : \nabla^S \mathbf{t}(t,x) = 0\big\},$$

which is related to Sard's theorem and which turns out to play a major role in this work.

We then say that a space-time current S as above is a *space-time solution* of (GTE) if

$$\frac{\mathrm{d}}{\mathrm{d}t} S(t,x) = b(t,x) \qquad \text{for } \|S\|\text{-a.e. } (t,x). \tag{5}$$

In fact, this is the approach taken in the modelling of dislocation movements contained in [8, 12, 11]. There, the geometric derivative can be identified with the (normal) dislocation velocity, which is a key quantity in any theory of plasticity driven by dislocation motion.

One can see without too much effort that space-time solutions give rise to weak solutions: The projected slices $S(t) := \mathbf{p}_*(S|_t)$ of an integral $(1+k)$-current S satisfying (5) solve (GTE). The converse question, that is, when a collection of currents $\{T_t\}_t$ solving (GTE) can be realised as the slices of a space-time current lies much deeper.

The work [6] develops a general theory of the geometric transport equation (GTE) in the case of transported integral (sometimes only normal) k-currents, including the case of intermediate dimensions ($k \neq 0, d$). We prove in particular the following theorems:

- *Existence & Uniqueness Theorem*: In the case where the driving vector field is assumed smooth, we show the existence and uniqueness of a path of integral currents solving the geometric transport equation.
- *Disintegration Structure Theorem:* For a space-time current, this theorem details the structure of its slices. In particular, it clarifies the role of "critical points" of the currents, which turn out to be central.
- *Rectifiability Theorem:* In a sense a converse to the disintegration structure theorem, this result shows that if a time-indexed collection of boundaryless integral k-currents satisfies suitable continuity conditions, then these currents are in fact the slices of a space-time integral current. In this sense the path is "space-time rectifiable" (i.e., rectifiable of dimension $1+k$).
- *Advection Theorem:* We show that a boundaryless space-time current satisfies the *negligible criticality condition* (meaning that critical points are negligible for the mass measure of the current) if and only if its slices are advected by some vector field.
- *Weak and Strong Rademacher-type Differentiability Theorems:* These results show that a time-indexed family of boundaryless integral currents, satisfying suitable Lipschitz-continuity (or even absolute continuity) assumptions, is a solution to the geometric transport equation for some driving vector field.

We call the last results "Rademacher-type" theorems because they show the differentiability of Lipschitz-continuous evolutions of currents with respect to the time variable. However, the notion of derivative is not the classical one (e.g., in the Gateaux-sense with respect to the linear structure of the space of normal currents), but the geometric one defined above.

On a technical side we point out the connection to the following open problem: For boundaryless integral currents, there are two possible definitions of the homogeneous (boundaryless) Whitney flat norm:

$$\mathbb{F}(T) := \inf\bigl\{\mathbf{M}(Q) : Q \in \mathrm{N}_{k+1}(\mathbb{R}^d),\, T = \partial Q\bigr\},$$
$$\mathbb{F}_{\mathrm{I}}(T) := \inf\bigl\{\mathbf{M}(Q) : Q \in \mathrm{I}_{k+1}(\mathbb{R}^d),\, T = \partial Q\bigr\},$$

meaning that in $\mathbb{F}(T)$ we use normal test currents and in $\mathbb{F}_{\mathrm{I}}(T)$ we use integral test currents. It is known that for $k \in \{0, d-2, d-1, d\}$ these two flat norms coincide, but for other k their equivalence seems to be unknown. A number of our results (e.g., the Rademacher-type differentiability theorems) are quite sensitive to the type of flat norm used and we need to carefully distinguish them in this work.

Finally, we record the phenomenon that, unlike in the classical theory of BV-functions on a time interval, a path of integral k-currents may have a diffusely-concentrated or fractal structure *in space only*, while being regular in time. This phenomenon is closely related to Sard's theorem and to the presence of critical points smeared out in time. We give an explicit example of such behaviour in the form of the "Flat Mountain", which is a path of integral currents that is Lipschitz in time, but which has many critical points and the geometric differentiability fails.

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(Quasi)-conformal methods in some 2-dimensional free boundary problems

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(joint work with Guido De Philippis and Bozhidar Velichkov)

We consider the two-phase functional J_{TP} defined, for every open set $D \subset \mathbb{R}^d$ and every function $u : D \to \mathbb{R}$, as

$$\text{(TP)} \qquad J_{\text{TP}}(u, D) := \int_D |\nabla u|^2\, dx + \lambda_+^2 |\Omega_u^+ \cap D| + \lambda_-^2 |\Omega_u^- \cap D|,$$

where the constants $\lambda_+ > 0$ and $\lambda_- > 0$ are given and fixed, and the two phases

$$\Omega_u^+ = \{u > 0\} \qquad \text{and} \qquad \Omega_u^- = \{u < 0\}$$

are the positivity sets of the functions $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$. A function $u : D \to \mathbb{R}$ is a *local minimizer of J_{TP} in D* if

$$J_{\text{TP}}\big(u, \Omega\big) \le J_{\text{TP}}\big(v, \Omega\big),$$

for all open sets Ω and functions $v : D \to \mathbb{R}$ such that $\overline{\Omega} \subset D$ and $v = u$ on $D \setminus \Omega$. Alt, Caffarelli and Friedman introduced this functional in [1] and proved, among other things, existence and Lipschitz regularity of the minimizers. Our first main theorem is the following $C^{1,\alpha}$ regularity result for the so-called *free boundary* of a minimizer u, that is $\partial\Omega_{u^+} \cup \partial\Omega_{u^-}$. Moreover we set the notation:

$$\mathcal{C}_2(u_+, u_-) := \partial\Omega_u^+ \cap \partial\Omega_u^- \cap B_1 \qquad \text{and} \qquad \mathcal{O}_\pm := \Big(\partial\Omega_u^\pm \cap B_1\Big) \setminus \mathcal{C}_2(u_+, u_-)\,.$$

Theorem 1 (Regularity of the free boundary)**.** *Let $u : D \to \mathbb{R}$ be a local minimizer of J_{TP} in the open set $D \subset \mathbb{R}^d$. Then, each of the sets $\partial\Omega_u^+ \cap D$ and $\partial\Omega_u^- \cap D$ can be decomposed as a disjoint union of a regular and a (possibly empty) singular part*

$$\partial\Omega_u^\pm \cap D = \mathrm{Reg}(\partial\Omega_u^\pm) \cup \mathrm{Sing}(\partial\Omega_u^\pm),$$

with the following properties.

(1) $\mathrm{Reg}(\partial\Omega_u^\pm)$ *is a relatively open subset of $\partial\Omega_u^\pm \cap D$ and is locally the graph of a $C^{1,\eta}$-regular function, for some $\eta > 0$. Moreover, $\mathcal{C}_2(u_+, u_-) \cap D \subset$* $\mathrm{Reg}(\partial\Omega_u^\pm)$.

(2) $\mathrm{Sing}(\partial\Omega_u^\pm)$ *is a closed subset of $\partial\Omega_u^\pm \cap D$ of Hausdorff dimension at most $d-5$. Precisely, there is a critical dimension $d^* \in [5, 7]$ such that* $\mathrm{Sing}(\partial\Omega_u^\pm)$ *is a closed $(d - d^*)$-rectifiable subset of $\partial\Omega_u^\pm \cap D$ with locally finite $\mathcal{H}^{d-d^*}$ measure.*

Next we want to investigate the set of *branch points* $\mathcal{B}_2(u_+, u_-)$, that is

$$\mathcal{B}_2(u_+, u_-) = \big\{x \in \mathcal{C}_2(u_+, u_-) \ : \ B_r(x) \cap \mathcal{O}_\pm \neq \emptyset \ \text{ for every } \ r > 0\big\}. \tag{1}$$

By Theorem 1, in dimension $d = 2$, $\partial\Omega_u^\pm$ are locally parametrized by two $C^{1,\alpha}$ curves. Precisely, suppose that $(0, 0) \in \mathcal{C}_2(u_+, u_-)$, then there is an interval $\mathcal{I}_\rho := (-\rho, \rho)$ and two $C^{1,\alpha}$-regular functions $f_\pm : \mathcal{I}_\rho \to \mathbb{R}$, such that

$$f_+ \geq f_- \quad \text{on} \quad \mathcal{I}_\rho \qquad \text{and} \qquad f_+(0) = f_-(0) = \partial_x f_+(0) = \partial_x f_-(0) = 0\,,$$

and, up to rotations and translations,

$$\begin{cases} u(x,y) > 0 & \text{for} \quad (x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \quad \text{such that} \quad y > f_+(x); \\ u(x,y) = 0 & \text{for} \quad (x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \quad \text{such that} \quad f_-(x) \leq y \leq f_+(x); \\ u(x,y) < 0 & \text{for} \quad (x,y) \in \mathcal{I}_\rho \times \mathcal{I}_\rho \quad \text{such that} \quad y < f_-(x). \end{cases} \tag{2}$$

Thus, in the square $\mathcal{I}_\rho \times \mathcal{I}_\rho$, the one-phase parts $\mathcal{O}_+$ and $\mathcal{O}_-$ of the free boundary are the union of $C^{1,\alpha}$ (actually analytic) graphs over a countable family of disjoint open intervals:

$$\mathcal{O}_\pm := \bigcup_{i\in\mathbb{N}} \Gamma_\pm^i\,,$$

where, for every $i \in \mathbb{N}$, there is an open interval $\mathcal{I}_i \subset \mathcal{I}_\rho$ such that

$$\Gamma_\pm^i = \big\{(x, f_\pm(x)) \,:\, x \in \mathcal{I}_i\big\}. \tag{3}$$

Definition 1 (Symmetric solutions of the two-phase problem)**.** In dimension $d = 2$, we will say that a continuous function $u : B_1 \to \mathbb{R}$ is a *symmetric minimizer of the two-phase problem* if u satisfies

$$\mathcal{H}^1(\Gamma_+^i) = \mathcal{H}^1(\Gamma_-^i) \quad \text{for every} \quad i \in \mathbb{N} \quad \text{such that} \quad \overline{\mathcal{I}_i} \subset \mathcal{I}_\rho\,. \tag{4}$$

Theorem 2 (Cuspidal points)**.** *Let $u : B_1 \subset \mathbb{R}^2 \to \mathbb{R}$ be a symmetric minimizer of the two-phase problem. Then the branching set $\mathcal{B}(u_+, u_-)$ is locally finite. Moreover if $z_0 \in \mathcal{B}(u_+, u_-)$, then we have one of the following possibilities:*

(1) z_0 *is an isolated point of* $\mathcal{C}_2(u_+, u_-)$ *and, in a neighborhood of* z_0*, the free boundaries* $\partial\Omega_u^+$ *and* $\partial\Omega_u^-$ *are analytic graphs meeting only in* z_0*;*

(2) z_0 *is an endpoint of a non-trivial arc in* $\mathcal{C}_2(u_+, u_-)$*, and there are an interval* $\mathcal{I}_\rho = (-\rho, \rho)$ *a constant* $k \in \mathbb{N}$*,* $k \geq 3$*, and an analytic function* $\phi : \mathcal{I}_\rho \to \mathbb{R}$ *such that* $\phi(0) \neq 0$ *and, up to setting* $z_0 = 0$ *and changing the coordinates,*

$$f_+(x) - f_-(x) = \begin{cases} x^{k/2}\,\phi(|x|^{\frac{1}{2}}) & \text{if } x \leq 0 \\ 0 & \text{if } x \geq 0\,. \end{cases} \tag{5}$$

Furthermore we can construct local minimizers with any such behavior.

Open problems:

- The critical dimension d^* is the first dimension, for which there exists a one-homogeneous non-negative local minimizer of the one-phase functional with a singular free boundary. It is a famous open question to determine wether d^* is equal to 5, 6 or 7 (see [2, 6, 5]).
- Can one give a more precise description of the singular set $\mathrm{Sing}(\partial\Omega_u^\pm)$?
- Can one remove the symmetric assumption in Theorem 2?
- The proof of Theorem 2 is based on the use of (Quasi)-conformal maps. Is it possible to give a frequency function proof? This might allow to extend the result to higher dimensions.
- Is it possible to estimate the number of branching points in Theorem 2 based on the energy (or some other quantity)?

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Stability of the vortex in micromagnetics and related models

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(joint work with Elio Marconi)

Two dimensional vector fields which are of unit length, divergence-free and tangent to the domain boundary, arise in scaling limits of several variational models. For instance, the Aviles-Giga energy (used to describe smectic liquid crystals, blistering, pattern formation, etc) is given by

$$E_\epsilon(m;\Omega) = \int_\Omega \left(\frac{\epsilon}{2}|\nabla m|^2 + \frac{1}{2\epsilon}(1-|m|^2)^2 \right) dx,$$

for domains $\Omega \subset \mathbb{R}^2$ and maps $m\colon \Omega \to \mathbb{R}^2$ satisfying $\nabla\cdot m = 0$ in Ω and $m\cdot n_{\partial\Omega} = 0$ on $\partial\Omega$. Compactness of bounded energy sequences $E_\epsilon(m_\epsilon)$ [3, 2] ensures that their limits $m = \lim m_\epsilon$ satisfy

$$|m| = 1, \quad \nabla \cdot m \quad \text{in } \Omega, \quad m \cdot n_{\partial\Omega} = 0 \text{ on } \partial\Omega, \tag{1}$$

and this is also true for some micromagnetics models [9, 1]. At the heart of long-standing open questions about the above variational models, is understanding the structure of solutions of (1) that can arise as limits $m = \lim m_\epsilon$ with $E_\epsilon(m_\epsilon) \le C$.

Zero-energy states, that is, limits $m = \lim m_\epsilon$ with $E_\epsilon(m_\epsilon;\Omega) \to 0$, have been characterized by Jabin, Otto and Perthame [4]: the domain must be a disk, and the vector field a radial vortex $m_* = \pm ix/|x|$. In the reported work, we prove stability estimates associated to this rigidity result: under the normalizing condition $\mathcal{H}^1(\partial\Omega) = 2\pi$, we show

$$\int_{\partial\Omega} \left| n_{\partial\Omega} - \frac{x}{|x|} \right|^2 d\mathcal{H}^1(x) \le C \liminf_{\epsilon\to 0} \min E_\epsilon(\cdot;\Omega), \tag{2}$$

where the constant $C > 0$ depends on the $C^{1,1}$ regularity of $\partial\Omega$, and the minimum of $E_\epsilon(\cdot;\Omega)$ is taken over all admissible maps m (for any of the aforementioned models). This is sharp, in the sense that the left-hand side of (2) could not be bounded by a higher power of the limit energy. As corollaries, we also deduce bounds on $\operatorname{dist}(\partial\Omega, \partial D_1)$ and $\|m - m_*\|_{L^p(\Omega)}$, which are, however, probably not sharp. This generalizes previous estimates by Lorent [5], which are valid for all $\epsilon > 0$ (not just in the limit), but are far from sharp and require convexity of the domain and more restrictive boundary conditions. The methods rely on a Lagrangian formulation introduced by Marconi in [8, 7, 6].

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Burkholder Functional, restricted quasiconvexity and Energy Integrands in NonLinear Elasticity

Daniel Faraco
(joint work with Kari Astala, André Guerra, Aleksis Koski, and Jan Kristensen)

In 1952, Morrey coined the notion of quasiconvexity to investigate the scope of the direct method of the calculus of variations for vectorial variational problems of the type

$$I(\varphi) = \int_\Omega f(D\varphi(x))dx.$$

Here $\varphi : \Omega \to \mathbb{R}^m$ and $f : M^{n\times m} \to \mathbb{R}$. The notion of quasiconvexity is almost equivalent to the lower semicontinuity of I, and it says that for any matrix $A \in M^{n\times m}$

$$\int_\Omega (f(A + D\varphi(x)) - f(A))dx \geq 0.$$

Considering simple deformations which take two values (planar waves), shows that quasiconvexity implies convexity along rank-one directions, i.e. rank-one convexity. A celebrated counterexample of V. Šverák [11] proves that if $n \geq 3$, superposition of three planar waves already shows that rank-one convexity does not imply quasiconvexity. In dimension 2, the example does not work [9] and indeed three waves cannot provide a counterexample [10]. Indeed, in two dimensions, there are partial positive results [8, 7, 5] in the direction that rank-one convexity might imply quasiconvexity.

In this project, we investigate whether rank-one convexity might imply quasiconvexity at least for special functionals with symmetries and natural constraints. The most natural symmetries are isotropies, homogeneity and the most natural constraint is penalizing negative determinants. Perhaps the most famous rank-one convex functional with those symmetries is the Burkholder functional:

$$B_p(A) = ((p-2)|A|^2 - p\det(A))|A|^{p-2}.$$

This functional, introduced by Donald Burkholder in the context of martingale inequalities, is central in the theory of quasiconformal mappings. Its quasiconvexity would imply determining the precise L^p norm of the Beurling-Ahlfors transform.

In a joint work with K. Astala (Helsinki), A. Guerra (IAS), A. Koski (Madrid) and J. Kristensen (Oxford) we have shown that if we declare

$$\hat{B}_p(A) = \begin{cases} B_p(A) & \text{if } B_p(A) < 0 \\ \infty & \text{otherwise,} \end{cases}$$

then the following strong version of quasiconvexity holds:

Theorem 1. *$\hat{B}_p$ is closed $W^{1,p}$-quasiconvex.*

The proof combines holomorphic deformations [3], the extremality of the Burkholder functional among rank-one convex functionals [4, 6], Beltrami operators as projectors [1] and the theory of gradient Young measures. A limiting argument shows that the functional [12] $W(A) = K_A - \log K_A + \log(\det(A))$ also satisfies the quasiconvexity inequality for mappings with integrable distortion. The corresponding lower semicontinuity argument also holds.

Theorem 2. *W is lower semicontinuous with respect to sequences with $K_A \in L^{1+\epsilon}$ for $\epsilon > 0$.*

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Asymptotic limit of the Landau-de Gennes model for nematic liquid crystals around an inclusion

Dominik Stantejsky

(joint work with François Alouges and Antonin Chambolle)

We study the formation and shape of singularities in nematic liquid crystals under the presence of an external magnetic field $\mathbf{H} = h\mathbf{e}_3$. Using the Landau-de Gennes model, we explain the appearance of point and line defects that are observed around a colloidal particle immersed into a liquid crystal (the so called *Saturn ring phenomenon*), see e.g. [5]. More precisely, we want to find $Q-$tensor fields on Ω, the domain outside the particle, which minimize the Landau-de Gennes energy defined by

$$\mathcal{E}_{\eta,\xi}(Q) = \int_\Omega \frac{1}{2}|\nabla Q|^2 + \frac{1}{\xi^2}f(Q) + \frac{1}{\eta^2}g(Q)\,\mathrm{d}x\,,$$

where the Dirichlet term models elastic forces in the liquid crystal, f models the bulk interaction and g incorporates the effects of the external magnetic field. We consider the asymptotic regime in which the parameters η, ξ vanish with the coupling condition $\eta|\ln(\xi)| \to \beta \in (0,\infty)$ for $\eta, \xi \to 0$, corresponding to large particle and weak magnetic fields. In this regime both point and line singularities occur and we derive an effective energy $\mathcal{E}_0$ describing the formation and transition between different singularities by using tools from [1, 4].

In a first work, we deal with the physically relevant case of a spherical particle [2]. We derive a limit energy in the spirit of $\Gamma-$convergence, stated for sets of finite perimeter on the particle surface. Studying the limit problem, we explain the transition between the point (dipole) and line singularity (Saturn ring configuration) by varying the parameter β and show the occurrence of a hysteresis phenomenon.

In [3], we then extend our result to the general case of a closed and sufficiently regular particle with surface $\mathcal{M}$. We show quantitatively that the close-to-minimal energy is asymptotically concentrated on lines and surfaces nearby or on the particle. We obtain the asymptotic energy $\mathcal{E}_0$ stated for two dimensional $\pi_1(\mathbb{R}P^2)-$valued flat chains given by

$$\mathcal{E}_0(T) = C_{\mathcal{M}} + 4s_*c_*\int_{\mathcal{M}} |\nu\cdot\mathbf{e}_3|\,\mathrm{d}\mu_{T|_{\mathcal{M}}} + 4s_*c_*\mathbb{M}(T|_\Omega) + \frac{\pi}{2}s_*^2\beta\,\mathbb{M}(\partial T + \Gamma)\,,$$

where $C_{\mathcal{M}}$ is a constant depending only on $\mathcal{M}$, ν is the outward unit normal vector field on $\mathcal{M}$, $\mathbb{M}$ denotes the mass norm, $\Gamma = \{\omega \in \mathcal{M} \,:\, \nu(\omega)\cdot\mathbf{e}_3 = 0\}$ is a line and s_*, c_* are constants depending on f and g. We discuss regularity of minimizers and optimality conditions for the limit energy and show how this result generalizes the previous work [2].

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Zeroth order up to the boundary Γ-convergence for the Ginzburg-Landau energy

Andrew Colinet

(joint work with Stanley Alama and Lia Bronsard)

In the study of the Ginzburg-Landau functional, E_ε, defined for $0 < \varepsilon \leq 1$, on an open, bounded subset Ω of $\mathbb{R}^n$ or of a compact manifold M^n, where $n \geq 2$, and $u \in W^{1,2}(\Omega;\mathbb{C})$ by

$$E_\varepsilon(u) := \int_\Omega \Big\{ \frac{1}{2}|\nabla u|^2 + \frac{1}{4\varepsilon^2}(|u|^2-1)^2 \Big\}, \tag{1}$$

a topic of considerable interest is the limiting behaviour of (1) as $\varepsilon \to 0^+$ when degree restrictions are in place. The pioneering work of [4] provided a detailed first order description of the limiting behaviour of local minimizers of (1). They showed, among other things, that the leading (zeroth) order energy diverges to order $|\log(\varepsilon)|$ and computed the next order renormalized energy.

More recently, there has been a lot of work to use the tools of Γ-convergence to study the limiting behaviour of (1) in the interior of Ω. To do this, the Jacobian $Ju := \det(\nabla u)$ is used to encode concentration of u about a vortex. Geometric norms are then used to measure the concentration of u about a vortex, which permits us to conclude corresponding results about energy. Examples of such results are [8] and [2] which consider bounded open subsets of Euclidean space as well as [6] which considers a smooth, compact, orientable 2-dimensional Riemannian manifold without boundary. There is also work, see [5], to demonstrate convergence, to zeroth order, to prescribed critical points of (1) which may not be local minimums. Interest in this last result is due to an association of the Ginzburg-Landau energy, up to a factor of $|\log(\varepsilon)|$, with the Hausdorff measure of the limiting vortex set.

However, despite the considerable amount of work put towards understanding Γ-convergence in the interior of Ω, not much has been shown for convergence up to the boundary. In [7] a counterexample was constructed to show that, in the absence of some sort of boundary restriction, a complete Γ-convergence result up to the boundary would not be possible. On the other hand, for suitable Dirichlet data it was shown in [3] that a Γ-convergence result up to the boundary is possible.

In a forthcoming work [1] we prove an up to the boundary Γ-convergence result intermediate between the results of [7] and [3] in which we specify Dirichlet data on either the tangential part or the normal part. Specifically, we demonstrate:

Theorem 1. *Suppose $\Omega \subseteq \mathbb{R}^2$ is open, bounded, connected and that $\partial\Omega$ is $C^{2,1}$ with $b+1$ connected components $(\partial\Omega)_i$ for $i = 0, 1, \ldots, b$, where $b \geq 0$.*

(i) If $\{u_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq W^{1,2}(\Omega;\mathbb{C})$ all have either zero tangential part or zero normal part and $E_\varepsilon(u_\varepsilon) \leq C|\log(\varepsilon)|$, for $C > 0$, then there exists a subsequence, which we still label with ε, as well as $M_1, M_{2,j} \in \mathbb{N} \cup \{0\}$ for $j = 0, 1, \ldots, b$ and

$$
\begin{gathered}
a_i \in \Omega \text{ for } i = 1, 2, \ldots, M_1, \\
c_{jk} \in (\partial\Omega)_j \text{ for } k = 1, 2, \ldots, M_{2,j},\ j = 0, 1, \ldots, b, \\
d_i, d_{jk} \in \mathbb{Z} \setminus \{0\} \text{ for } i = 1, 2, \ldots, M_1,\ k = 1, 2, \ldots, M_{2,j},\ j = 0, 1, \ldots, b,
\end{gathered} \tag{2}
$$

such that

$$
\left\| Ju_\varepsilon - \pi \sum_{i=1}^{M_1} d_i \delta_{a_i} - \frac{\pi}{2} \sum_{j=0}^{b} \sum_{k=1}^{M_{2.j}} d_{jk} \delta_{c_{jk}} \right\|_{(C^{0,\alpha}(\overline{\Omega}))^*} \longrightarrow 0^+ \tag{3}
$$

for each $\alpha \in (0, 1]$. In addition, we have

$$
\sum_{i=1}^{M_1} d_i + \frac{1}{2} \sum_{j=0}^{b} \sum_{k=1}^{M_{2,j}} d_{jk} = \chi_{\text{Euler}}(\Omega) \tag{4}
$$

as well as

$$
\frac{1}{2} \sum_{k=1}^{M_{2,j}} d_{jk} \in \mathbb{Z} \quad \text{for each } j = 0, 1, \ldots, b. \tag{5}
$$

(ii) Suppose that $\{u_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq W^{1,2}(\Omega;\mathbb{C})$ all have either zero tangential part or zero normal part and satisfy (3) for points and integers as in (2) and for some $\alpha \in (0, 1]$. In addition, suppose that these integers satisfy (4) and (5). Then

$$
\liminf_{\varepsilon\to 0^+} \frac{E_\varepsilon(u_\varepsilon)}{|\log(\varepsilon)|} \geq \pi \sum_{i=1}^{M_1} |d_i| + \frac{\pi}{2} \sum_{j=0}^{b} \sum_{k=1}^{M_{2,j}} |d_{jk}|. \tag{6}
$$

(iii) For all points and integers as in (2) satisfying (4) and (5) there exists a sequence $\{u_\varepsilon\}_{\varepsilon\in(0,1]} \subseteq W^{1,2}(\Omega;\mathbb{C})$ such that (3) holds as well as

$$
\lim_{\varepsilon\to 0^+} \frac{E_\varepsilon(u_\varepsilon)}{|\log(\varepsilon)|} = \pi \sum_{i=1}^{M_1} |d_i| + \frac{\pi}{2} \sum_{j=0}^{b} \sum_{k=1}^{M_{2,j}} |d_{jk}|.
$$

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A mesoscale flatness criterion and its application to exterior isoperimetry

Michael Novack

(joint work with Francesco Maggi)

Flatness criteria based on blow-up and blow-down techniques are basic tools in the Calculus of Variations and Geometric Measure Theory. A prototypical blow-up type result is the classical theorem of Allard [1], which describes the regularity of a surface M at scales that are sufficiently small with respect to the inverse mean curvature and for which M has small area excess. Conversely, the large-scale flatness, or flatness "at infinity," of an exterior minimal surface can be addressed via blow-down arguments following the work of Allard-Almgren [2] and Simon [4] on the uniqueness of tangent cones. In [3] we prove a "mesoscale flatness criterion" that applies to hypersurfaces for which neither blow-ups nor blow-downs are sufficient to satisfactorily describe flatness at the desired scales. This theorem is the key tool in the resolution of large volume exterior isoperimetric sets and has potential for further applications.

Consider the class $\mathcal{M}_{\Lambda,R}$ of hypersurfaces $M \subset \mathbb{R}^{n+1} \setminus B_R$ with $|H_M|_\infty \le \Lambda$ on $B_{1/\Lambda} \setminus B_R$ and boundary $\partial M \subset \partial B_R$. The control quantity is the monotone decreasing area deficit

$$\delta_{M,\Lambda,R}(r) := \omega_n - \frac{\mathcal{H}^n(M \cap (B_r \setminus B_R))}{r^n} + \frac{1}{n\, r^n} \int_{\partial M} x \cdot \nu_M^{\mathrm{co}}\, d\mathcal{H}^{n-1} - \Lambda \int_R^r \frac{\mathcal{H}^n(M \cap (B_\rho \setminus B_R))}{\rho^n}\, d\rho\,.$$

Here we use the term "deficit" instead of the more usual "excess" since, at variance with the blow-up and blow-down scenarios described above, the sign of $\delta_{M,\Lambda,R}$ may change.

Theorem 1. (Mesoscale Flatness Criterion) *If $n \geq 2$, $\sigma > 0$, $\Gamma > 0$, then there exists $\varepsilon_0 = \varepsilon_0(n,\sigma,\Gamma)$ such that if $M \in \mathcal{M}_{\Lambda,R}$,*

$$\mathcal{H}^{n-1}(\partial M) \leq \Gamma\, R^{n-1}\,, \qquad \sup_{\rho\in(R,1/\Lambda)} \frac{\mathcal{H}^n(M\cap(B_\rho\setminus B_R))}{\rho^n} \leq \Gamma\,,$$

and there exists a mesoscale $s \in [R/\varepsilon_0, \varepsilon_0/\Lambda]$ such that

$$\max\{|\delta_{M,\Lambda,R}(s/2)|, |\delta_{M,\Lambda,R}(s)|\} \leq \varepsilon_0\,,$$

then there exists a hyperplane Π such that $M\cap(B_{S_*/2}\setminus B_{s/2})$ corresponds to the graph over Π of a C^1-function with norm bounded by σ, where

$$S_* = \min\Big\{\frac{\varepsilon_0}{\Lambda}, R_*\Big\} \quad \textit{and} \quad R_* = \sup\{\rho \geq s : |\delta_{M,\Lambda,R}(\rho)| \leq \varepsilon_0\}\,.$$

If M is a hypersurface containing the origin, so that, formally speaking, $R = 0$, and the tangent cone of M there is a plane, Theorem 1 reduces to Allard's theorem [1]. Similarly, if $\Lambda = 0$ and the exterior minimal hypersurface M has a planar tangent cone at infinity, we recover the exterior blow-down results stated in [5, 6]. In particular, Theorem 1 can be viewed as a general framework containing as special cases the blow-up and blow-down flatness criteria for hypersurfaces with planar tangent cones. At the same time, however, as demonstrated by the following application to exterior isoperimetry, Theorem 1 really points in a different direction than these classical results, since it also pertains to situations where neither blow-up nor blow-down limits make sense. The proof is a development of ideas originating in the theory of minimal surfaces with isolated singularities [2], and so Theorem 1 should generalize to higher co-dimension and minimal cones that are smooth away from the origin.

Next, we examine the exterior relative isoperimetric problem

$$\mathcal{I}_{W^c}(v) = \inf\big\{P(E;\mathbb{R}^{n+1}\setminus W) : |E| = v,\, E\subset\mathbb{R}^{n+1}\setminus W\big\}$$

outside a compact obstacle W in the large volume regime $v \to \infty$. To leading order in v, the minimum energy is evidently the perimeter of a ball of volume v. Also, for large v, one expects to see a minimizer E_v with boundary ∂E_v that, far from W, is a normal graph over a ball of volume v, and close to W is a graph over an asymptotically planar minimal surface. Unfortunately, it cannot be concluded using standard tools (for example the stability of Allard's criterion with respect to area convergence) that these domains of resolution overlap. Thus the initial energetic and geometric descriptions fail to achieve two natural goals:

(i) obtaining an asymptotic expansion for $\mathcal{I}_{W^c}(v)$ as $v \to \infty$;
(ii) reconciling the small- and large-scale descriptions of ∂E_v.

The answers to these questions depend on a variational problem which is a Plateau-type problem in $\mathbb{R}^{n+1}\setminus W$ with free boundary along W and at infinity. We consider admissible pairs (F,ν) with $\nu\in\mathbb{S}^n$ and $F\subset\mathbb{R}^{n+1}$ a set of locally finite perimeter in $\mathbb{R}^{n+1}\setminus W$ such that, for some $\alpha,\beta\in\mathbb{R}$,

$$\partial F \subset \big\{x : \alpha < x\cdot\nu < \beta\big\}\,,$$
$$\mathbf{p}_{\nu^\perp}(\partial F) = \nu^\perp := \big\{x : x\cdot\nu = 0\big\}\,,$$

where $\mathbf{p}_{\nu^\perp}(x) = x - (x \cdot \nu)\,\nu$, $x \in \mathbb{R}^{n+1}$. For such a pair, we compute the limit

$$\mathrm{res}_W(F,\nu) = \limsup_{R\to\infty} \omega_n\, R^n - P(F;\mathbf{C}_R^\nu \setminus W)\,; \tag{1}$$

here $\mathbf{C}_R^\nu = \{x \in \mathbb{R}^{n+1} : |\mathbf{p}_{\nu^\perp}(x)| < R\}$, and the right hand side of (1) is decreasing in R. The "isoperimetric residue" $\mathcal{R}(W)$ of W is then defined by maximizing res_W over admissible pairs (F,ν), so that

$$\mathcal{R}(W) = \sup_{(F,\nu)} \mathrm{res}_W(F,\nu)\,.$$

Theorem 2. (Resolution of Large Volume Exterior Isoperimetric Sets) *If $W \subset \mathbb{R}^{n+1}$ is compact, then*

$$\mathcal{I}_{W^c}(v) = (n+1)\omega_{n+1}^{1/(n+1)} v^{n/(n+1)} - \mathcal{R}(W) + \mathrm{o}(1) \quad \text{as } v \to \infty\,.$$

Furthermore, there exists positive R_1, R_2, and v_0, all depending on W, such that for any $v \geq v_0$ and minimizer E_v of $\mathcal{I}_{W^c}(v)$, there exists an optimal pair (F,ν) for res_W *such that ∂E_v is a normal graph over ∂F on $B_{R_2 v^{1/(n+1)}} \setminus B_{R_1}$.*

The proof of Theorem 2, in particular the resolution of ∂E_v, utilizes the mesoscale flatness criterion to obtain graphicality over ∂F between R_1 and $R_2 v^{1/(n+1)}$. Now the graphicality of ∂E_v over a large sphere is valid outside $B_{R_0(v)v^{1/(n+1)}}$, where $R_0(v) \to 0^+$. Thus, we eventually have $R_2 > R_0(v)$, so that the domains of resolution for ∂E_v overlap as desired. The asymptotic expansion for $\mathcal{I}_{W^c}(v)$ relies on fine properties of optimizers for res_W. The derivation of these properties requires the blow-down result corresponding to the $\Lambda = 0$ case of Theorem 1 and the extraction of sharp decay information towards hyperplane blowdown limits.

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Euler flows and strong Onsager conjecture

Hyunju Kwon

The incompressible 3D Euler equations have total kinetic energy conservation for smooth (spatially periodic) solutions. In the recent resolution of the Onsager conjecture by Isett [1], below certain threshold Holder regularity, Euler flows with total kinetic energy dissipation have been constructed. In this talk, I'll discuss a strong Onsager conjecture: the existence of Holder continuous Euler flows with total kinetic energy dissipation and satisfying the local energy inequality.

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Sobolev inequalities and quasiconcavity

André Guerra

(joint work with Bogdan Raiţă and Matthew Schrecker)

The classical Sobolev inequality asserts that, for each $1 \leq p < n$, we have

$$\|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|Dv\|_{L^p(\mathbb{R}^n)} \quad \text{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m).$$

To which extent can D be replaced by another linear homogeneous first order operator $\mathcal{P}$ in this inequality? The answer is:

Theorem 1 (Elliptic operators)**.** *The estimate*

$$\|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\mathcal{P}v\|_{L^p(\mathbb{R}^n)} \quad \textit{for all } v \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^m) \tag{1}$$

holds if and only if:

(a) *Calderón–Zygmund:* $1 < p < n$ *and* $\mathcal{P}$ *is elliptic;*
(b) *Bourgain–Brezis* [3]*:* $p = 1$ *and* $\mathcal{P}$ *is elliptic and canceling.*

The special canceling condition in (b) is due to the fact that the Calderón–Zygmund theory breaks down when $p = 1$; in our context, this is made particularly clear by a remarkable result of Ornstein [10]. Cancellation is an algebraic condition on $\mathcal{P}$ introduced in general by Van Schaftingen [14], who established the analogue of Theorem 1 case (b) when $\mathcal{P}$ has higher order.

We are interested in investigating whether (1) can hold for *non-elliptic operators*, assuming that we only test with fields satisfying a compensating pointwise constraint. Thus, we look for *nonlinear* inequalities of the form

$$\|v\|_{L^{p^*}(\mathbb{R}^n)} \leq C\|\mathcal{P}v\|_{L^p(\mathbb{R}^n)} \quad \text{for } v \in C_c^\infty(\mathbb{R}^n, \mathcal{C}), \tag{2}$$

where $\mathcal{C} \subset \mathbb{R}^m$ is a closed cone. We emphasize that, in (2), the nonlinearity is in the function space $C_c^\infty(\mathbb{R}^n, \mathcal{C})$, due to the pointwise constraint.

A simple necessary condition for (2) to hold is that

$$\mathcal{C} \cap \Lambda_{\mathcal{P}} = \{0\}, \qquad \text{where } \Lambda_{\mathcal{P}} := \bigcup_{\xi \neq 0} \ker \mathcal{P}(\xi)$$

and $\mathcal{P}(\xi)$ denotes the symbol of $\mathcal{P}$. The set $\Lambda_{\mathcal{P}}$ is known as the *wave cone* of $\mathcal{P}$ and it is the set of directions along which $\mathcal{P}$ is not elliptic: note that $\Lambda_{\mathcal{P}} = \{0\}$ if and only if $\mathcal{P}$ is elliptic. The wave cone is a classical object in the Compensated Compactness theory of Murat and Tartar [12] and in fact Tartar's framework for problems in continuum mechanics [13] was in part the motivation for considering inequality (2).

When $\mathcal{C}$ is linear, (2) reduces to (1). The proof of Theorem 1 relies on Fourier methods, which become unavailable in the nonlinear case (they can, however, be used if $\mathcal{C}$ is only approximately linear [1]), thus completely new methods are needed. The goal of the talk is to give three natural examples of inequalities of the type (2), proved in [8].

Our first example is concerned with the cone of K-quasiconformal matrices:

Theorem 2 (Curl in quasiconformal matrices)**.** *Let $K \geq 1$ and take*

$$\mathcal{C} = Q_2(K) := \{A \in \mathbb{R}^{2\times 2} : |A|^2 \leq K \det A\}, \qquad \mathcal{P} = \mathrm{Curl}.$$

Then (2) *holds if and only if* $1 < p < \frac{2K}{2K-1}$.

Note that, when $K = 1$, $Q_2(1)$ is a linear space and $\frac{2K}{2K-1} = 2$: thus Theorem 2 reduces to Theorem 1 in this case. However, for $K > 1$, we have $\frac{2K}{2K-1} < 2$. Thus, in the nonlinear setting, (2) does not hold in the full range of exponents $1 < p < n$, and there is a *maximal gain of integrability* depending on the cone and on the operator:

Definition 1. The maximal gain of integrability $q_{\max} = q_{\max}(\mathcal{P}, \mathcal{C})$ is the smallest exponent such that if $q > q_{\max}$ then the following inequality fails:

$$\|v\|_{L^q(\mathbb{B}^n)} \leq C\|\mathcal{P}v\|_{L^\infty(\mathbb{B}^n)} \qquad \text{for all } v \in C_c^\infty(\mathbb{B}^n, \mathcal{C}).$$

In fact, we have $q_{\max}(\mathrm{Curl}, Q_2(K)) = \frac{2K}{K-1} = (\frac{2K}{2K-1})^*$.

The next two examples are in some sense dual to each other. For $k = 1, \ldots, n$ and $A \in \mathrm{Sym}_n$ we consider the k-th symmetric polynomial on the eigenvalues:

$$F_k(A) := \sum_{i_1 < \cdots < i_k} \lambda_{i_1}(A) \ldots \lambda_{i_k}(A).$$

The Gårding cones and their polars are defined by

$$\Gamma_k := \{F_1 \geq 0, \ldots, F_k \geq 0\}, \qquad \Gamma_k^* := \{B \in \Gamma_n : \langle A, B\rangle \geq 0 \text{ for all } A \in \Gamma_k\}.$$

We have $\{\mathrm{tr} \geq 0\} = \Gamma_1 \supset \cdots \supset \Gamma_n = (\Gamma_n)^* \supset \cdots \supset \Gamma_1^* = \mathbb{R}_+ \mathrm{Id}$.

Theorem 3 (Curl in symmetric matrices)**.** *Let $k = 2, \ldots, n$, $p^* > k$ and $\mathcal{C} \subset \mathrm{int}\,\Gamma_k$. Then*

$$\|A\|_{L^{\frac{k}{k-1}}(\mathbb{B}^n)} \leq C\|\mathrm{Div}\, A\|_{L^p(\mathbb{B}^n)} \quad \textit{for all } A \in C_c^\infty(\mathbb{B}^n, \mathcal{C})$$

and the restriction to $p^* > k$ is necessary unless $k = n$. Moreover $q_{\max}(\mathrm{Curl}, \Gamma_k) = k$.

Theorem 4 (Div in symmetric matrices). *Let $k = 2, \dots, n$, $p^* > \frac{k}{k-1}$ and $\mathcal{C} \subset \operatorname{int} \Gamma_k^*$. Then*

$$\|A\|_{L^{\frac{k}{k-1}}(\mathbb{B}^n)} \leq C \|\operatorname{Div} A\|_{L^p(\mathbb{B}^n)} \quad \text{for all } A \in C_c^\infty(\mathbb{B}^n, \mathcal{C})$$

and the restriction to $p^* > \frac{k}{k-1}$ is necessary unless $k = n$. Moreover $q_{\max}(\operatorname{Div}, \Gamma_k^*) = \frac{k}{k-1}$.

When $k = n$, Theorem 4 was first proved by Serre [11] and de Rosa–Tione [5].

Theorems 2 to 4 are intimately connected to *quasiconcavity* in the sense of Morrey [9] (see also [7]) of an appropriate functional $G_{\mathcal{C}} \colon \mathcal{C} \to \mathbb{R}$. The functional $G_{\mathcal{C}}$ is 1-homogeneous and vanishes on ∂C and, for the above examples, we consider

$$G_{Q_2(K)} := (K \det A - |A|^2)^{\frac{K-1}{2K}} |A|^{\frac{1}{K}}, \qquad G_{\Gamma_k} := F_k^{1/k}, \qquad G_{\Gamma_k^*} := \inf_{A \in \Gamma_k} \frac{\langle A, \cdot \rangle}{F_k^{1/k}(A)}.$$

Theorem 5 (Quasiconcavity). *For the above examples, $(G_{\mathcal{C}})^q \colon \mathcal{C} \to \mathbb{R}$ is $\mathcal{P}$-quasiconcave if and only if $1 \leq q \leq q_{\max}(\mathcal{P}, \mathcal{C})$.*

The functional associated with $Q_2(K)$ was introduced by Burkholder [4] and its Curl-quasiconcavity in the same cone was proved in joint work with Astala, Faraco, Koski and Kristensen [2]. We also refer the reader to [6] for results related to quasiconvexity of the functionals associated with Γ_k.

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Quantitative boundary unique continuation and the estimate of the singular set

Zihui Zhao
(joint work with Carlos Kenig)

The classical unique continuation theorem states: if a harmonic function u vanishes at a point to infinite order (that is, near that point u decays to zero faster than polynomials of any integer degree), then u must vanish everywhere in a connected set containing the point. This is a fundamental property of harmonic functions, as well as solutions to a large class of elliptic and parabolic PDEs. In the same spirit, mathematicians are interested in *quantitative* unique continuation results, which are to use the local information about the growth rate of a harmonic function to study its global properties. In particular, we are interested in studying, for a non-trivial harmonic function u, how big its singular set $\mathcal{S}(u) := \{x : u(x) = 0 = |\nabla u(x)|\}$ can be.

The study of the size of the singular set *at the boundary* is related to a classical question asked by L. Bers: considering a domain Ω in $\mathbb{R}^n$ (with $n \geq 3$) and a harmonic function u in Ω, if both u and ∇u vanish on a boundary set with positive surface measure, does it follow that u must vanish everywhere in the domain? In general, the answer can be no (by a counter-example of Bourgain and Wolff in [1]) unless one assumes a priori that $u \equiv 0$ on a relative open set of the boundary. There have been many attempts to answer this question in a more and more general class of domains, and so far the best known result is the following theorem by Tolsa:

Theorem 1 (Theorem 1.1 in [8])**.** *Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, B be a ball centered in $\partial\Omega$, and suppose that $\Sigma := B \cap \partial\Omega$ is a Lipschitz graph with slope at most τ_0 (τ_0 is a positive constant depending only on the dimension n). Let $u \in C(\overline{\Omega})$ be a harmonic function in Ω. Suppose that u vanishes on Σ and that*

$$\mathcal{H}^{n-1}(\{x \in \Sigma : \nabla u(x) = 0\}) > 0.$$

Then $u \equiv 0$ in Ω.

It is still an open question whether the same statement holds if we only assume Ω is a Lipschitz domain (and remove the smallness assumption on the Lipschitz constant).

On the other hand, it has been observed that the singular set of a harmonic function (in the interior) is $(n-2)$-dimensional, see for example [4, 3]; moreover, the work of Cheeger, Naber and Valtorta in [2, 7] give quantitative estimates of the $(n-2)$-dimensional size of the singular set. Inspired by this, my collaborator and I set out to give a fine estimate of the size of the singular set *at the boundary*, as is in the setting of Bers and the above theorem. Roughly speaking we prove the following theorem:

Theorem 2 (Theorem 1.1 in [5]). *Let $\Omega \subset \mathbb{R}^n$ be a Dini domain, and B be a ball centered in $\partial\Omega$. Let $u \in C(\overline{\Omega})$ be a non-trivial harmonic function in $\Omega \cap 5B$ such that $u \equiv 0$ on $\partial\Omega \cap 5B$. Then the singular set $\mathcal{S}(u) := \{x \in \overline{\Omega} : u(x) = 0 = |\nabla u(x)|\}$ satisfies that $\mathcal{S}(u) \cap B$ is $(n-2)$-rectifiable, and*

$$\mathcal{H}^{n-2}(\mathcal{S}(u) \cap B) \leq C,$$

where C depends only on the dimension n and the upper bound of the growth rate of u in $5B$ (or more precisely, the modified frequency function of u in the ball $5B$).

It is worth remarking that the class of Dini domains is the optimal class of domains for which one can make sense of the $\mathcal{H}^{n-2}$-measure of the singular set.

In a follow-up work, we also show that centered at every boundary point, the harmonic function has an asymptotic expansion as follows:

Theorem 3 (Theorem 1.1 in [6]). *Under the same assumption as above, we have that for every $x \in \partial\Omega \cap B$, there exists $r > 0$ such that*

$$u(y) = P_N(y-x) + \psi(y-x) \quad \text{ in } B_r(x) \cap \Omega,$$

where P_N is a non-trivial homogeneous harmonic polynomial of degree $N \in \mathbb{N}$, the error term ψ satisfies

$$|\psi(z)| \leq C|z|^N \theta(|z|), \quad |\nabla\psi(z)| \leq C|z|^{N-1}\theta(|z|),$$

and $\theta(r) \to 0$ as $r \to 0$, with a decay rate determined by the Dini parameter of the domain Ω.

Compared to the interior case, one of the difficulties in proving such expansion is that u merely has C^1-regularity at the boundary.

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Regularity and structure of singular sets of area minimizing currents modulo p

Salvatore Stuvard

(joint work with Camillo De Lellis, Jonas Hirsch, Andrea Marchese, and Luca Spolaor)

Introduced by Federer and Fleming in 1960, *integral currents* provide a natural class of generalized surfaces to formulate and solve the oriented Plateau's problem: prove the existence of m-dimensional oriented surfaces of least area (mass) spanning a given boundary in Euclidean space $\mathbb{R}^{m+n}$ or in an $(m+n)$-dimensional Riemannian manifold Σ^{m+n}. The achievements of the corresponding theory are tremendous, both in the direction of showing *existence* of solutions for arbitrary boundary data, and, perhaps more importantly, in that of proving optimal *partial regularity* results for such solutions. It is by now a well known fact, for instance, that the interior singularities of mass minimizing integral currents (that is, those points where the current does not coincide, locally, with an integer multiple of a smooth oriented boundaryless submanifold) are rather rigid, to the extent that m-dimensional mass minimizing integral currents in $\mathbb{R}^{m+1}$ are necessarily smooth in the interior whenever $m \leq 6$.

Looking from a different perspective, the very same regularity results mentioned above suggest a possible drawback of setting Plateau's problem in the framework of integral currents, in that several examples of (possibly singular) minimal surfaces which also naturally appear as soap films are not admissible competitors, in the sense of integral currents, for their own boundary datum, even though they may have smaller area than the integral current solution. Classical instances of this phenomenon include minimal Möbius strips (and in general all unoriented minimal surfaces), or the triple junction solution to the Steiner problem for the vertices of an equilateral triangle in the plane, or the singular surface spanning two parallel circles and consisting of two pieces of catenoids joined at 120 degree angles to a disc floating at mid-height.

To obtain a class of area minimizing generalized surfaces with sufficient complexity to describe such singular geometries, Ziemer [11] and Federer [5] suggested to relax the homological notion of boundary in use in the classical theory of currents and recast Plateau's problem in the framework of *integral currents modulo* p, where $p \geq 2$ is an integer. Roughly speaking, two integral flat chains Z and Z' are said to be *equivalent modulo* p, written $Z = Z' \,\mathrm{mod}(p)$, if the integral flat chain $Z - Z'$ is a multiple of p. Direct methods allow then to prove the following statement: given a smooth submanifold $\Sigma^{m+n} \subset \mathbb{R}^d$ (complete, boundaryless), and T_0 an m-dimensional integral current $\mathrm{mod}(p)$ in Σ, there exists an integer rectifiable current T in Σ minimizing the mass functional among all rectifiable currents S such that $\partial S = \partial T_0 \,\mathrm{mod}(p)$. Simply put, this relaxed notion of spanning in the sense of currents $\mathrm{mod}(p)$ allows surfaces in the competition class for Plateau's problem to exhibit additional boundary other than the prescribed one, as long as the former is weighted with a multiplicity that is a multiple of p. This additional boundary, whose points can be treated as *interior singular points* for the current, is *optimally placed* in order to guarantee mass minimization, so that the regularity of mass minimizing currents mod(p) can be regarded as a *geometric free boundary problem.* Our goal is to investigate the size and structure of the set of interior singular points.

More precisely, let T be m-dimensional and minimizing mod(p) in Σ^{m+n}, and let $\mathrm{Sing}(T)$ denote the set of interior points x of (the support of) T such that T is *not*, locally at x, mod(p)-equivalent to an integer multiple of a smooth submanifold without boundary. The relevant problems are then to provide estimates for the (Hausdorff) dimension of $\mathrm{Sing}(T)$ and, if possible, to understand its structure as well as the local behavior of the regular part of T in a neighborhood of $\mathrm{Sing}(T)$. In this direction, partial results have been obtained in the case $p = 2$ (Federer [6]), $p = 3$ for 2-dimensional surfaces in $\mathbb{R}^3$ (Taylor [8]), $p = 4$ and $p \in 2\mathbb{Z} + 1$ for a general dimension m and codimension $n = 1$ (White [9, 10]). In [3], we have proved the following sharp Hausdorff dimension estimate for the singular set valid for any values of the modulus p, the dimension m, and the codimension n.

Theorem 1 ([3, Theorem 1.4, Theorem 1.9, and Corollary 1.10])**.**
Let $p \geq 2$, and let T be m-dimensional and mass minimizing mod(p) *in Σ^{m+n}. Then,* $\dim_{\mathcal{H}}(\mathrm{Sing}(T)) \leq m - 1$. *If p is an odd integer, then* $\mathrm{Sing}(T)$ *is countably $(m-1)$-rectifiable, and $\mathcal{H}^{m-1}(\mathrm{Sing}(T) \cap K) < \infty$ for every compact K. In this case, there is a choice of an orientation $\vec{\xi}$ on* $\mathrm{Sing}(T)$ *such that*

$$\partial T = p\,\vec{\xi}\,\mathcal{H}^{m-1} \llcorner \mathrm{Sing}(T) \qquad \text{in } U$$

for any open set U with compact closure in the complement of the boundary mod(p) *of T.*

The proof follows the blueprint of Almgren's proof of the partial regularity of mass minimizing integral currents in codimension higher than one, as revisited by De Lellis and Spadaro. A very interesting fact is that, in the mod(p) framework, Almgren's scheme is needed also in codimension $n = 1$, in order to deal with the possible presence of branch singularities when p is an even integer. The study, carried out in [4] of the singularities of *special Q-valued functions* with $Q = p/2$ plays a pivotal role towards this goal.

When the codimension is $n = 1$, our understanding of the singularities of special Q-valued functions improves remarkably. In turn, we can say much more on the structure of $\mathrm{Sing}(T)$ and on how the regular part of T approaches it.

Theorem 2 ([1, Theorem 1.3] and [2, Theorem 1.3])**.** *For $p \geq 2$, if T is m-dimensional and mass minimizing* mod(p) *in Σ with $\dim \Sigma = m + 1$, then there exists a closed set $\mathcal{S} \subset \mathrm{Sing}(T)$ with $\dim_{\mathcal{H}}(\mathcal{S}) \leq m - 2$ such that $\mathcal{R} := \mathrm{Sing}(T) \setminus \mathcal{S}$ is a smooth submanifold of dimension $(m-1)$. For every $x \in \mathcal{R}$ there is a neighborhood $U \ni x$ with the property that the regular part of T in U consists of precisely p smooth minimal hypersurfaces with transverse intersection along the common boundary* $\mathrm{Sing}(T) \cap U$.

The proof of Theorem 2 is based on an effective combination of Almgren's scheme with the techniques introduced by Simon in [7]. Both Theorem 1 and Theorem 2 are optimal.

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Neck pinches along the Lagrangian mean curvature flow of surfaces

Felix Schulze

(joint work with Jason D. Lotay and Gábor Székelyhidi)

The question of the existence of special Lagrangian submanifolds is an important problem in complex and symplectic geometry. Special Lagrangians play a central role in the Strominger–Yau–Zaslow conjecture [12] on mirror symmetry, and are of interest in the variational problem of finding area-minimizing Lagrangians, studied extensively by Schoen–Wolfson [9]. Smoczyk [11] showed that the mean curvature flow preserves the class of Lagrangian submanifolds in Calabi–Yau manifolds, and so a natural expectation is that a suitable Lagrangian can be deformed into a special Lagrangian using the flow. The Thomas–Yau conjecture [14], motivated by mirror symmetry [13], predicts that this is indeed the case, assuming that the initial Lagrangian satisfies a certain stability condition. More recently Joyce [3] formulated a detailed conjectural picture relating singularity formation along the Lagrangian mean curvature flow to Bridgeland stability conditions on Fukaya categories.

It was shown by Neves [8] that singularities are, in a sense, unavoidable along the Lagrangian mean curvature flow, even if the initial Lagrangian is a small

Hamiltonian perturbation of a special Lagrangian. At the same time, Neves [6] shows that for the flow of zero Maslov Lagrangians any tangent flow at a singular point is a union of special Lagrangian cones. This means that Type I singularities – which are typically easier to analyse – do not exist. In this work we study the simplest kind of singularities, called neck pinches in [3, Conjecture 3.16], in the two-dimensional case. Our main result is the following, which we state in the setting of a compact ambient Calabi–Yau surface, though it also works in $\mathbb{C}^2$. Note that here, and throughout, we allow our Lagrangians to be immersed, which is important in the context of Lagrangian mean curvature flow.

Theorem 1. *Let X be a compact Calabi–Yau surface, and $L \subset X$ a zero Maslov, rational Lagrangian. Let L_t be the mean curvature flow starting from L for $t \in [0, T)$, where T is the first finite singular time. Let $(\mathbf{x}_T, T)$ be a singular point, and suppose that a tangent flow at $(\mathbf{x}_T, T)$ is given by the transverse union of two multiplicity one planes. The tangent flow at $(\mathbf{x}_T, T)$ is then unique.*

In Theorem 1 the assumption is that one tangent flow is given by a union of multiplicity one transverse planes $P_1 \cup P_2$, with corresponding Lagrangian angles θ_1, θ_2. We note here that by Neves [7, Corollary 4.3] the flow cannot form a singularity unless $\theta_1 = \theta_2$. Therefore throughout the article we will only be concerned with the case when P_1 and P_2 have the same Lagrangian angle.

The uniqueness of tangent flows is a fundamental problem for analysing the singularities of mean curvature flow, and there have been several important results in this direction recently [10, 2, 1]. A major new difficulty in Theorem 1 is that it is the first example of uniqueness for a tangent flow that is singular. The proof crucially exploits several aspects of the Lagrangian setting, and does not apply in the general setting of mean curvature flow.

Theorem 1 allows us to analyse the behavior of the flow at the singularity. First, we have the following, showing that the flow can be continued past the singular time if all singularities at time T are modelled on two transverse, multiplicity one planes. Recall that the grading of a zero Maslov Lagrangian corresponds to a global choice of function representing the Lagrangian angle.

Theorem 2. *Suppose that X, L are as in Theorem 1 and assume that at each singular point $(\mathbf{x}, T)$ a tangent flow is a static union of two multiplicity one, transverse planes. Then L_t converges to an immersed Lagrangian C^1-submanifold L_T in the sense of currents as $t \to T$, and the flow can be restarted as a smooth, zero Maslov, rational Lagrangian mean curvature flow with initial condition L_T. Furthermore, the extended flow is smooth (together with its grading) through the singular time, away from the singular points.*

A slight extension of the ideas involved in proving uniqueness of the tangent flow, combined with the classification of low-entropy ancient solutions to Lagrangian mean curvature flow [4], also allows us to show that if the tangent flow is given by the union of two transverse planes, then close to the singularity the flow looks like the two transverse planes, desingularized by a Lawlor neck which "pinches off". This has the following consequence.

Theorem 3. *Suppose that X, L are as in Theorem 1. For $t < T$ sufficiently close to the singular time we can write L_t as a graded self-connected sum of an immersed Lagrangian M at a self-intersection point.*

If M is not connected, then we can write it is a graded connected sum $M = M_1 \# M_2$ and the following holds:

$$\mathrm{vol}(L) > \left| \int_{M_1} \Omega \right| + \left| \int_{M_2} \Omega \right|, \tag{1}$$

where Ω is the holomorphic volume form on X. If in addition L is almost calibrated, then we also have

$$\phi(M_1), \phi(M_2) \subset (\inf_L \theta, \sup_L \theta), \tag{2}$$

where $\phi(M_i)$ is the "cohomological" Lagrangian angle of M_i.

This result provides some evidence for Thomas–Yau's Conjecture 7.3 in [14]. Indeed, their conjecture states that if the flow has a finite time singularity, then L can be decomposed into a graded connected sum $M_1 \# M_2$ satisfying the conditions in (1). Our result shows that this is one of the possible scenarios when the tangent flow at the first singular time is given by two transverse planes. In particular, the decomposition as a graded connect sum is guaranteed if L is a sphere.

Note that Joyce's conjectural picture [3] predicts that other singularities could still form, notably those with tangent flows given by two static planes meeting along a line. It is an important problem to understand what we can say about the flow in the presence of such singularities and some progress towards this was made in the authors' previous work [5]. An optimistic expectation is that for a generic initial surface, the only tangent flows that appear at singularities are of these two types, i.e. two multiplicity one planes meeting either at a point or along a line.

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Nonlocal variational problems on polygons

Ihsan Topaloglu
(joint work with Marco Bonacini and Riccardo Cristoferi)

In [1] we consider the nonlocal isoperimetric problem

$$\inf\Big\{\mathcal{E}_\gamma(E) \,:\, |E|=1\Big\}, \tag{1}$$

among sets of finite perimeter $E \subset \mathbb{R}^d$ with given volume 1, where the energy functional $\mathcal{E}_\gamma$ is defined as

$$\mathcal{E}_\gamma(E) \,=\, \int_{\partial^* E} \psi(\nu_E)\,\mathrm{d}\mathcal{H}^{d-1} + \gamma \int_E \int_E \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha}$$

for $\gamma > 0$, $\alpha \in (0,d)$. We are interested in surface energies determined by *crystalline* surface tensions ψ, whose Wulff shapes (which are the corresponding isoperimetric regions) are given by convex polyhedra.

This minimization problem was recently introduced by Rustum Choksi, Robin Neumayer, and the author in [5] as an extension of the classical liquid drop model of Gamow [7, 4] to the anisotropic setting. In the *anisotropic* liquid drop model the competition is not only between the attractive and repulsive forces, but also between the anisotropy in the surface energy and the isotropy of the Riesz-like interaction energy. As in the isotropic case, the problem admits a minimizer when γ is sufficiently small and fails to have minimizers for large values of γ. However when ψ is smooth, and different than the Euclidean norm, its Wulff shape W_ψ is not a critical point of the energy $\mathcal{E}_\gamma(E)$ for any $\gamma > 0$, whereas in the isotropic case the ball is the unique global minimizer for $\gamma > 0$ sufficiently small.

In contrast, in [1], we prove that, for a wide class of crystalline surface tensions, where the Wulff shape of ψ enjoys particular symmetry properties, the corresponding isoperimetric set W_ψ *remains* as the minimizer of the nonlocal isoperimetric problem for small values of $\gamma > 0$. To this end, let $\mathscr{P}_n$, $n \geq 3$, be the class of open, convex polygons $\mathcal{P} \subset \mathbb{R}^2$ with n sides $L_1, \ldots, L_n$ and unit area $|\mathcal{P}| = 1$, which are reflection symmetric with respect to the bisectors of all angles. Then our first main result states the minimality of polygons in $\mathscr{P}_n$.

Theorem 1. *Let $\mathcal{P} \in \mathscr{P}_n$ and let ψ be a surface energy density whose Wulff shape is $\mathcal{P}$. Then there exists $\bar{\gamma} > 0$, depending on $\mathcal{P}$ and α, such that for all $\gamma < \bar{\gamma}$ the polygon $\mathcal{P}$ is the unique (up to translations) solution to* (1).

The proof of Theorem 1 follows by the combination of three main ingredients: (a) the stability of the Wulff inequality; (b) the fact that any solution to (1) is an ω-minimizer of the anisotropic perimeter and in turn, if γ is sufficiently small, it is a polygon with sides parallel to those of $\mathcal{P}$; and (c) the following quadratic upper bound for variations within the class $\mathscr{C}(\mathcal{P}, \varepsilon)$ where the sides of competitors are parallel to those of $\mathcal{P}$ and at distance at most ε.

Theorem 2 (Quadratic bound)**.** *Let $\mathcal{P} \in \mathscr{P}_n$. There exists $\varepsilon_0 > 0$ and $c_0 > 0$ (depending on the polygon $\mathcal{P}$ and on α) such that for every $\widetilde{\mathcal{P}} \in \mathscr{C}(\mathcal{P}, \varepsilon_0)$ one has the quadratic estimate*

$$\left| \int_{\mathcal{P}} \int_{\mathcal{P}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha} - \int_{\widetilde{\mathcal{P}}} \int_{\widetilde{\mathcal{P}}} \frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^\alpha} \right| \leq c_0 |\mathcal{P} \triangle \widetilde{\mathcal{P}}|^2 \,.$$

For small γ, minimizers of $\mathcal{E}_\gamma$ are always obtained by perturbations of the Wulff shape of the surface energy, whose sides are translated parallel to themselves. In our result we exhibit an explicit class of Wulff shapes which remain global minimizers for $\gamma > 0$. However, we cannot prove that polygons in this class are exactly those with this global minimality property. It is an open problem to classify the crystalline anisotropies whose Wulff shapes remain the global minimizers of $\mathcal{E}_\gamma$ for $\gamma > 0$ sufficiently small which would require us to characterize the critical points of the nonlocal energy with respect to the restricted class of variations.

This question naturally led us to study a class of nonlocal repulsive energies of generalized Riesz-type

$$\mathcal{V}(E) = \int_E \int_E K(|x-y|)\,\mathrm{d}x\,\mathrm{d}y$$

on polygons, where the kernel $K \geq 0$ is strictly decreasing and locally integrable.

The energy $\mathcal{V}$ (in any dimension) is uniquely maximized by the ball under volume constraint, as a consequence of Riesz's rearrangement inequality. Moreover, at least in the case of Riesz kernels, balls are characterized as the unique critical points for the energy under volume constraint. This was proved in a series of contributions (see e.g. [9]) via moving plane methods, and in full generality for Riesz kernels in [8] via a continuous Steiner symmetrization argument. In [2], we investigate the same two questions in a discrete setting, namely where the energy is evaluated on polygons with a fixed number of sides. In our first result we show that among triangles and quadrilaterals the regular polygon is the unique maximizer of the Riesz-type energy $\mathcal{V}$.

Theorem 3. *The equilateral triangle is the unique (up to rigid movements) maximizer of $\mathcal{V}$ in $\mathscr{P}_3$ under area constraint, and the square is the unique (up to rigid movements) maximizer of $\mathcal{V}$ in $\mathscr{P}_4$ under area constraint.*

The second main question that we address is whether the regular N-gon is characterized by the stationarity conditions, as it is the case for the ball. In order to state precisely this result, we need to fix some notation. Given two points $P, Q \in \mathbb{R}^2$, we denote by $\overline{PQ} = \{tP + (1-t)Q : t \in [0,1]\}$ the segment joining P and Q. For $N \geq 3$, let $\mathcal{P} \in \mathscr{P}_N$ be a polygon with N vertices $P_1, \dots, P_N$. For notational convenience we also identify $P_0 = P_N$, $P_{N+1} = P_1$. For $i \in \{1, \dots, N\}$ we let ℓ_i be the length of the side $\overline{P_i P_{i+1}}$, and M_i be the midpoint of the side $\overline{P_i P_{i+1}}$. Denoting by $v_{\mathcal{P}}(x) = \int_{\mathcal{P}} K(|x-y|)\,\mathrm{d}y$ the potential associated with the polygon, we then consider the following two conditions:

$$\begin{aligned}(2)\quad \frac{1}{\ell_i}\int_{\overline{P_iP_{i+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) \\ = \frac{1}{\ell_j}\int_{\overline{P_jP_{j+1}}} v_{\mathcal{P}}(x)\,\mathrm{d}\mathcal{H}^1(x) \quad \text{for all } i,j \in \{1,\dots,N\},\end{aligned}$$

which corresponds to the criticality condition for the energy $\mathcal{V}$ under an area constraint, when sides are translated parallel to themselves, and

$$\begin{aligned}(3)\quad \int_{\overline{P_iM_i}} v_{\mathcal{P}}(x)|x-M_i|\,\mathrm{d}\mathcal{H}^1(x) \\ = \int_{\overline{P_{i+1}M_i}} v_{\mathcal{P}}(x)|x-M_i|\,\mathrm{d}\mathcal{H}^1(x) \quad \text{for all } i \in \{1,\dots,N\},\end{aligned}$$

which corresponds to the criticality condition for the energy $\mathcal{V}$ under an area constraint, when a side is rotated around its midpoint. Our second result is the following.

Theorem 4. *If $\mathcal{P} \in \mathscr{P}_3$ obeys condition* (3)*, then $\mathcal{P}$ is an equilateral triangle. If $\mathcal{P} \in \mathscr{P}_4$ obeys conditions* (2) *and* (3)*, then $\mathcal{P}$ is a square.*

The proof of this theorem uses a reflection argument inspired by [6] as well as an argument based on a continuous symmetrization, inspired by an idea of Carrillo, Hittmeir, Volzone, and Yao [3]. We prove that the conditions (2) and (3) enforce the property of being equilateral, thus reducing the proof to the class of rhombi; then in a second step we prove that the polygon has to be also equiangular, using a reflection argument.

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An L^1 method for convergence and metastability

Maria G. Westdickenberg
(joint work with Sarah Biesenbach, Felix Otto, Sebastian Scholtes, and Richard Schubert)

We present a novel method developed for the 1-d Cahn Hilliard equation in [6] and extended in [1]. In the first part of the talk, we consider the one-dimensional, fourth-order Cahn–Hilliard equation

$$u_t = -\big(u_{xx} - G'(u)\big)_{xx} \qquad t > 0, \qquad x \in \mathbb{R}, \tag{1}$$

where G is a double-well potential with nondegenerate absolute minima at ± 1; a canonical choice is $G(u) = \frac{1}{4}(1-u^2)^2$.

The Cahn–Hilliard equation has a gradient flow structure with energy and dissipation given by

$$E(u) = \int \frac{1}{2}u_x^2 + G(u)\,dx, \qquad D := \int \Big(\big(G'(u) - u_{xx}\big)_x\Big)^2\,dx. \tag{2}$$

The so-called "centered kink" v minimizes the energy subject to ± 1 boundary conditions at $\pm\infty$ and is normalized so that $v(0) = 0$. We call its energy $e_* := E(v)$. For any $a \in \mathbb{R}$, the kink $v_a := v(\cdot - a)$ is also an energy minimizer, so that there is a whole continuum of minima.

We are interested in optimal convergence rates for initial data that is an order-one L^1 perturbation of a kink. It turns out that it is useful to work in terms of the L^2-closest kink $v_c(x) = v(x-c)$, the shift c, and the associated L^1 distance

$$V := \int |u - v_c|\,dx.$$

Further defining the energy-gap $\mathcal{E} := E(u) - E(v)$, we are able to establish a Nash-type estimate

$$\mathcal{E} \lesssim D^{\frac{1}{3}}(V+1)^{\frac{4}{3}},$$

from which an elementary ODE argument yields

$$\mathcal{E} \lesssim \frac{\bar{V}^2 + 1}{t^{\frac{1}{2}}} \qquad \text{for } t \in [0,T], \qquad \text{where } \bar{V} := \sup_{t \le T} V. \tag{3}$$

Hence it remains "only" to deduce that V remains bounded. For this, we use a duality argument inspired by [5] together with decay estimates for the linear equation on a domain with a (subcritical) moving boundary and a buckling argument.

It is important to notice that the conserved quantity

$$\int (u_0 - v)\, dx = 0 \qquad \Rightarrow \qquad \int (u(t) - v)\, dx = 0 \ \text{ for all } t > 0,$$

which uniquely determines *to which kink the solution converges* in the long-time limit. Here and below, we normalize so that this is the centered kink.

Our result [6] says that under the conditions

$$\mathcal{E}_0 \leq 2e_* - \varepsilon \ \text{ for some } \varepsilon > 0 \quad \text{and} \qquad V_0 < \infty,$$

there holds

$$V \lesssim V_0 + 1, \quad \mathcal{E} \lesssim \min\left\{\mathcal{E}_0, \frac{V_0^2 + 1}{t^{\frac{1}{2}}}\right\}, \quad \text{and} \quad c^2 \lesssim V_0^2 + 1. \tag{4}$$

Previous results include [2], [3], and [4]. All three results are perturbative in the sense that they assume initial data satisfying a smallness assumption in terms of some sufficiently small parameter δ.

In the second part of the talk, we present the work from [1] on order-one disturbances of "bump-like" initial conditions, both on the line and on the one-dimensional torus. On the torus, the bump w can be defined as

$$w = \arg\min E \ \text{ subject to } \frac{1}{2L}\int_{-L}^{L} (w + 1) = m \in (-1, 1).$$

The important difference in this case is that the conserved quantity

$$\int (u_0 - w)\, dx = 0 \qquad \Rightarrow \qquad \int (u(t) - w)\, dx = 0 \ \text{ for all } t > 0$$

does not determine the long-time limit, since also

$$\int (w - w_a)\, dx = 0 \ \text{ for all } a \in \mathbb{R}.$$

Hence a new question in this setting is whether the shift c of the L^2-closest bump converges at all. We are able to show that the initial algebraic relaxation leads into an exponential regime and the shift converges to a point c_* and that its distance from its starting position is controlled by the initial L^1 distance to the bump. The necessary duality argument must be modified to control additional error terms.

On the line, there is no bump-like minimum of finite energy. We consider initial data that is L^1-close to a bump constructed by pasting together a kink and an anti-kink pair. This state is *metastable* in the sense that we establish algebraic relaxation even though in the longtime limit, the solution converges to $\bar{u} \equiv -1$ (as mass leaks to infinity).

In addition we announced joint work with Felix Otto and Richard Schubert, which has not yet appeared, in which we apply the L^1 framework to the (geometrically nontrivial) setting of the Mullins–Sekerka evolution in 2 and 3 space dimensions.

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Stability for Faber-Krahn inequalities and the ACF formula

Dennis Kriventsov

(joint work with Mark Allen and Robin Neumayer)

The Faber-Krahn inequality states that the first Dirichlet eigenvalue $\lambda_1(\Omega)$ of the Laplacian of a domain Ω is greater than or equal to that of a ball $B(x)$ of the same volume (and if equality holds, then the domain is a translate of a ball). Similar inequalities are available on other manifolds where balls minimize perimeter over sets of a given volume. I will present a new sharp stability theorem for such inequalities: if the eigenvalue of a set is close to a ball, then the first eigenfunction u_Ω of that set must be close to the first eigenfunction of a ball, with the closeness quantified in an optimal way,

$$\inf_x \int |u_\Omega - u_{B(x)}|^2 \leq C[\lambda_1(\Omega) - \lambda_1(B(x))].$$

This stability estimate is different (and usually stronger) than the previously best available one, which has $\inf_x |\Omega \triangle B(x)|^2$ on the left. Indeed, there are both "linear" and "nonlinear" perturbations of a ball for which the difference of eigenfunctions is a higher-order measure of closeness than the symmetric difference. The proof is based on combining a linear stability theorem in this topology with a reduction argument involving solving a Bernoulli-type free boundary problem with a critically perturbed free boundary condition. The latter requires showing the existence of a *smooth* solution, which is possible here by obtaining some nonlinear estimates and developing a suitable regularity theory. The result is also valid on the sphere and hyperbolic space.

I will also explain an application of this to the behavior of the Alt-Caffarelli-Friedman monotonicity formula, which has implications for free boundary problems with multiple phases. The formula states that for a pair of continuous subharmonic functions $u, v \geq 0$ with disjoint support ($uv = 0$), the quantity

$$J_x(r) = \frac{1}{r^4} \int_{B_r(x)} \frac{|\nabla u(y)|^2 dy}{|x-y|^{n-2}} \int_{B_r(x)} \frac{|\nabla v(y)|^2 dy}{|x-y|^{n-2}}$$

is nondecreasing. There is also a rigidity property: if $J_0(r) \equiv J_0(0+) > 0$ is constant, then $u(y) = \alpha(y \cdot \nu)_+$ and $v(y) = \beta(y \cdot \nu)_-$ for some numbers $\alpha, \beta > 0$ and unit vector ν. It follows that if one performs blow-ups like $u(x + r\cdot)/r$ at a point where $J_x(0+) > 0$, along subsequences they will converge to these types of two-plane functions. A natural question then is whether or not the limiting function obtained is independent of the subsequence. While the answer in general is no, we give sharp estimates for how close a function must be to a two-plane function at scale r in terms of $J_x(r)/J_x(0+)$, as well as a Dini-type condition on $J_x(r)$ which guarantees the existence of unique blow-ups. The key ingredient in the proofs is the above Faber-Krahn stability theorem (for domains on the sphere). I will also briefly mention forthcoming work considering the rectifiability of the set $\{x : J_x(0+) > 0\}$.

Formal statements, proofs, and further discussion can be found in the two papers [1, 2].

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Separation of a lower dimensional free boundary for almost minimizers

Mariana Smit Vega Garcia

(joint work with Mark Allen)

Given $0 < s < 1$ and $n \geq 2$, denote a point $x \in \mathbb{R}^n$ by (x', x_n), where $x' = (x_1, \ldots, x_{n-1})$. We write $B_r(x_0) = \{x \in \mathbb{R}^n \ : \ |x - x_0| < r\}$, and given $\Omega \subset \mathbb{R}^n$ open, we define $\Omega' = \Omega \cap (\mathbb{R}^{n-1} \times \{0\})$.

We start by recalling one of the definitions of the fractional Laplacian $(-\Delta_{x'})^s$: given $u \in L^1(\mathbb{R}^{n-1}, (1 + |x'|^{n-1+2s})^{-1})$,

$$(-\Delta_{x'})^s u(x') = C_{n-1,s} \text{p.v.} \int_{\mathbb{R}^{n-1}} \frac{u(x') - u(x' + z')}{|z'|^{n-1+2s}} dz',$$

where $C_{n-1,s}$ is a normalization constant. Non-local problems involving the fractional Laplacian have received a surge of attention in recent years. In particular, the discovery of the localization through the Caffarelli-Silvestre extension procedure (see [2]) has lead to incredible progress in the field. Such localization procedure involves the following Poisson kernel

$$P(x', x_n) = C_{n-1,a} \frac{|x_n|^{1-a}}{(|x'|^2 + |x_n|^2)^{\frac{n-a}{2}}}, \qquad (x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}_+ = \mathbb{R}^n_+,$$

where $a = 1 - 2s \in (-1, 1)$. One then considers the following convolution, still denoted by u:

$$u(x', x_n) = u * P(\cdot, x_n) = \int_{\mathbb{R}^{n-1}} u(z')P(x' - z', x_n)dz', \qquad (x', x_n) \in \mathbb{R}^n_+.$$

Then $u(x', x_n)$ solves the following Cauchy problem:

$$L_a u = \operatorname{div}(|x_n|^a \nabla u) = 0 \text{ in } \mathbb{R}^n_+,$$
$$u(x', 0) = u(x') \text{ on } \mathbb{R}^{n-1},$$

where $\nabla = \nabla_{x',x_n}$ is the full gradient. The operator L_a is known as the Caffarelli-Silvestre extension operator. One can recover $(-\Delta_{x'})^s u$ as the fractional normal derivative on $\mathbb{R}^{n-1}$:

$$(-\Delta_{x'})^s u(x') = -C_{n-1,a} \lim_{x_n \to 0+} x_n^a \partial_{x_n} u(x', x_n), \qquad x' \in \mathbb{R}^{n-1},$$

understood in the sense of traces. Denoting the even reflection of u in the x_n-variable to all of $\mathbb{R}^n$ still by u, that is,

$$u(x', x_n) = u(x', -x_n), \qquad x' \in \mathbb{R}^{n-1}, x_n < 0,$$

then $u(x')$ is s-fractional harmonic in an open set $U \subset \mathbb{R}^{n-1}$ if, and only if, $u(x', x_n)$ satisfies

$$L_a u = 0 \text{ in } \Omega = \mathbb{R}^n_+ \cup (U \times \{0\}) \cup \mathbb{R}^n_-. \tag{1}$$

We have $L_a u = 0$ in $\mathbb{R}^n_\pm$, so (1) is equivalent to

$$L_a u = 0 \text{ in } B_r(x_0)$$

for any ball $B_r(x_0)$ centered at $x_0 \in U \times \{0\}$ such that $B_r(x_0) \Subset \Omega$, or, equivalently, $B'_r(x_0) \Subset U$. Noticing that solutions of this equation are minimizers of the weighted Dirichlet energy $\int_{B_r(x_0)} |x_n|^a |\nabla v|^2$, one concludes that a function $u \in L^1(\mathbb{R}^{n-1}, (1 + |x'|^{n-1+2s})^{-1})$ is s-fractional harmonic in U if, and only if, its even reflected Caffarelli-Silvestre extension $u(x', x_n)$ is in $W^{1,2}_{\text{loc}}(\Omega, |x_n|^a)$ and for any ball $B_r(x_0)$ with $x_0 \in U$ and $B'_r(x_0) \Subset U$, we have

$$\int_{B_r(x_0)} |x_n|^a |\nabla u|^2 \leq \int_{B_r(x_0)} |x_n|^a |\nabla v|^2,$$

for any $v \in u + W^{1,2}_0(B_r(x_0), |x_n|^a)$. This will serve as a motivation for our definition of almost minimizers.

More precisely, we are interested in almost minimizers of the energy functional

$$J_\Omega(u) = \int_\Omega |x_n|^a |\nabla u|^2 + \int_{\Omega'} \left(\lambda^+ \chi_{\{u>0\}} + \lambda^- \chi_{\{u<0\}}\right) d\mathcal{H}^{n-1}, \tag{2}$$

where $\lambda^\pm$ are positive constants. To define almost minimizers, let $r_0 > 0$ and $\omega : (0, r_0) \to [0, \infty)$ be a gauge function, that is, non-decreasing with $\omega(0+) = 0$. We say that $u \in L^1(\mathbb{R}^{n-1}, (1+|x'|^{n-1+2s})^{-1})$ is an almost minimizer of the energy (2) in an open set $U \subset \mathbb{R}^{n-1}$ with gauge function ω if its reflected Caffarelli-Silvestre extension $u(x', x_n) \in W^{1,2}_{\text{loc}}(\Omega, |x_n|^a)$, where $\Omega = \mathbb{R}^n_+ \cup (U \times \{0\}) \cup \mathbb{R}^n_-$, and for any ball $B_r(x_0)$ with $x_0 \in U \times \{0\}$ and $0 < r < r_0$ such that $B'_r(x_0) \Subset U$, we have

$$J_{B_r(x_0)}(u) \le (1 + \omega(r)) J_{B_r(x_0)}(v)$$

for any $v \in u + W^{1,2}_0(B_r(x_0), |x_n|^a)$.

We will assume $\omega(r) = \kappa r^\alpha$ for some $\kappa > 0$ and $\alpha \in (0, 1]$.

As happens with minimizers in [1], we prove that almost minimizers have optimal $C^{0,s}$ regularity. We also prove that that the two free boundaries, $\Gamma^+ = \partial\{u(\cdot, 0) > 0\}$ and $\Gamma^- = \partial\{u(\cdot, 0) < 0\}$ cannot touch.

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Differential inclusions, entropies and the Aviles Giga functional

Andrew Lorent

(joint work with Xavier Lamy and Guanying Peng)

We start with an introduction to differential inclusions in $\mathbb{R}^{2\times 2}$ and state Šverák's regularity theorem for differential inclusions into connected elliptic smooth submanifolds of $\mathbb{R}^{2\times 2}$. We then give a brief outline of the Aviles Giga functional and its Γ-converence conjecture [1].

After this we discuss joint work with Guanying Peng [6] on a generalization of the Jabin-Otto-Perthame rigidity result for zero energy solutions, [3]. We replaced the hypothesis in [3] of all entropies vanishing with the weaker hypothesis of only the Jin-Kohn entropies vanishing - this hypothesis is (for simply connected domains) equivalent to the existence of a differential inclusion into the set

$$K := \left\{P\left(e^{it}\right) : t \in [0, 2\pi)\right\}$$

where $P\left(e^{it}\right) = \left[e^{it}\right]_c + 3^{-1}\left[e^{3ti}\right]_a$ and

$$\left[e^{it}\right]_c := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}, \text{ and } \left[e^{it}\right]_a := \begin{pmatrix} \cos(t) & \sin(t) \\ \sin(t) & -\cos(t) \end{pmatrix}.$$

We were able to use Šverák's methods [8] to show that if $w : \Omega \to \mathbb{R}^2$ with $Dw \in K$ a.e. then $Dw \in B^{\frac{1}{3}}_{4,loc,\infty}$. So if $m : \Omega \to \mathbb{S}^1$ is divergence free in the distributional sense and if the two Jin-Kohn entropies vanish then there must exist some $w : \Omega \to \mathbb{R}^2$ such that $Dw(x) = P(m(x))$ and m inherits the $B^{\frac{1}{3}}_{4,loc,\infty}$ regularity which ultimately allowed us to show m is rigid in the sense of Jabin-Otto-Perthame.

Later Xavier Lamy, Guanying Peng and myself [4] considered the problem of regularity of the differential inclusion $Dw \in K$ a.e. By construction there must exists some $m : \Omega \to \mathbb{S}^1$ such that $Dw(x) = P\left(m(x)\right)$ a.e.. The issue being that there is apriori no reason for $\operatorname{div}(m) = 0$ in $\mathcal{D}'(\Omega)$. We were able to show rigidity of the differential inclusion (in the sense that m must be rigid in the sense of Jabin-Perhame-Otto) by gaining a contradiction from the assumption that $\operatorname{div}(m) \neq 0$ in $\mathcal{D}'(\Omega)$. This was done by heavily using the construction of a wide class of entropies for the Aviles Giga function achieved in [2]. This construction is in a sense a linear construction of an entropy from a scalar valued function. The introduction of this linearity allowed the question of non-vanishing of $\operatorname{div}(m)$ to be transformed into a question about the boundedness of the Hilbert transform on C^0. This was one of the key ideas of the proof of rigidity of Dw. This result is to our knowledge the first regularity result for non elliptic differential inclusions and it naturally suggests the following problems.

Problem 1. (Stability of approximate differential inclusions into K). Does there exist constants $C > 0$ and $\alpha \in (0,1)$ such that if $w \in H^1\left(B_1; \mathbb{R}^2\right)$

$$\inf_{A \in K} \int_{B_{\frac{1}{2}}} |Dw - A|^2 \, dx \leq C \left(\int_{B_1} \operatorname{dist}^2\left(Dw, K\right) dx \right)^{\alpha}.$$

This would represent the first quantitative rigidity result for non-elliptic differential inclusions and would be a useful tool in the study of the Aviles Giga conjecture. This was our original motivation for the study of the differential inclusion into K. As a step towards this goal we recently established the following [5].

Theorem 1. *Let $S \subset \mathbb{R}^{2\times 2}$ be a smooth, compact and connected 1-manifold without rank-one connections, that is elliptic in the sense that there exists $C_* > 0$ such that*

$$|M - M'|^2 \leq C_* \det(M - M') \qquad \forall M, M' \in S. \tag{1}$$

Then for any $u \in H^1(B_1; \mathbb{R}^2)$ we have

$$\inf_{M \in S} \int_{B_{1/2}} |Du - M|^2 \, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, S) \, dx, \tag{2}$$

for some constant $C = C(S) > 0$.

Problem 2. (Rigidity of differential inclusions into non-elliptic sets). Let $\Pi_1, \Pi_2 \subset \mathbb{R}^2$ be two convex sets and let $\gamma_1, \gamma_2 : \mathbb{S}^1 \to \mathbb{R}^2$ be arclength parametrisations of $\partial\Pi_1, \partial\Pi_2$. Let $n \in \mathbb{N}, n \geq 3$, suppose Π_1, Π_2 are such that

$$\sup_t |\gamma_1''(t)|^2 < n^2 \inf_t |\gamma_2''(t)|^2 . \tag{3}$$

For any $n \in \mathbb{N}$ the set $\Gamma_n := \left\{ \left[\gamma_1(e^{it})\right]_c + n^{-1} \left[\gamma_2(e^{int})\right]_a \right\}$ forms a non-elliptic set and a small neighborhood around every point does not contain Rank-1 connections.

Is it the case that for every Γ_3 there exists some $R = R(\Gamma_3) > 0$ such that if $w \in W^{1,2}\left(\Omega : \mathbb{R}^2\right)$ with $Dw(x) \in \Gamma_3 \cap B_R(M)$ a.e. (for some $M \in K$) then Dw is rigid? More generally is it true for Γ_n where $n \in \mathbb{N}$ and $n \geq 3$? It is true for all $1d$ non-ellptic sets without Rank-1 connections?

Problem 3. (Threshold regularity of the eikonal equation). A weak consequence of a proof of the Aviles Giga conjecture would be a proof of the following.

Conjecture 1. *Let* $ENT := \left\{ \Phi : \mathbb{S}^1 \to \mathbb{R}^2 : \frac{d}{dt}\left(\Phi\left(e^{it}\right)\right) \cdot e^{it} = 0 \right\}$. *Suppose* $m : \Omega \to \mathbb{S}^1$ *satisfies* $\mathrm{div}(m) = 0$ *in* $\mathcal{D}'(\Omega)$ *and*

$$\mathrm{div}\left(\Phi\left(m\right)\right) \in L^p_{loc}\left(\Omega\right) \text{ for every } \Phi \in ENT \tag{4}$$

then m *is rigid.*

It was shown in [7] that if $p \in \left(1, \frac{4}{3}\right)$, then $m : \Omega \to \mathbb{S}^1$ with $\mathrm{div}(m) = 0$ in $\mathcal{D}'(\Omega)$ satisfying (4) is equivalent to $m \in B^{\frac{1}{3}}_{3p,loc,\infty}$. The primary result of [2] is that for the more general hypothesis of (4) with $L^p_{loc}(\Omega)$ replaced with $\mathcal{M}_{loc}(\Omega)$, this is equivalent to $m \in B^{\frac{1}{3}}_{3,loc,\infty}$. Since the latter case includes all finite limits of Aviles Giga energy it is clear solutions of the eikonal equation $m \in B^{\frac{1}{3}}_{3,loc,\infty}$ can have line singularities, whereas a proof of Conjecture 1 would show that for solutions of the eikonal equation $m \in B^{\frac{1}{3}}_{q,loc,\infty}$ with $q > 3$ they can only have a discrete set of point singularities. Thus threshold regularity of the eikonal equation on the Besov scale occurs exactly at $m \in B^{\frac{1}{3}}_{3,loc,\infty}$.

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Frobenius theorem for weak submanifolds

Annalisa Massaccesi
(joint work with Giovanni Alberti, Andrea Merlo, and Eugene Stepanov)

1. Introduction

In this talk I focus on the developments of the Frobenius Theorem (see Theorem 1 in this report) for two weak notions of surfaces: tangency subsets of a (rather) smooth surface, on the one hand, and normal currents on the other hand.

Theorem 1 (Frobenius)**.** *Take a C^1 distribution V of k-planes in $\mathbb{R}^n$.*

(i) *If $S \subset \mathbb{R}^n$ is a k-dimensional surface tangent to V (that is, $\mathrm{Tan}_x S = V(x)$ at every $x \in S$), then V must be involutive at every point of S. This means that, if $V = \mathrm{span}(v_1, \ldots, v_k)$, then the Lie bracket $[v_i, v_j](x)$ belongs to $V(x)$ for any $i, j = 1, \ldots, k$, for any $x \in S$.*

(ii) *If V is everywhere involutive, then $\mathbb{R}^n$ can be locally foliated with k-dimensional surfaces which are tangent to V. Namely, the distribution is integrable.*

In our case, the interesting statement is (i), which concerns the degree of rigidity of the property of integrability for a distribution and which leaves some room for generalizations to weak surfaces in the following sense:

(i) If V is non-involutive, is it possible to infer the triviality of the tangent object?

(ii) Vice versa, if V is integrable in some broad sense, is V necessarily involutive?

Toy models for these questions are those concerning the relationship between the existence of a potential for a prescribed vectorfield (corresponding to integrable) and its being irrotational (corresponding to involutive, respectively).

For the sake of readability, from now on I consider the problem of 2-dimensional surfaces in $\mathbb{R}^3$, with the distribution V being spanned by two vectorfields X, Y of class C^1.

2. Frobenius theorem for tangency subsets of surfaces

Given a non-involutive distribution $V \in C^1$, we prove the following "rigidity" result.

Theorem 2. *If S is a $C^{1,\alpha}$-surface, with $\alpha \in [0,1]$, and the tangency set $E = \{x \in S : \mathrm{Tan}_x S = V(x)\}$ has finite $(1-\alpha)$- perimeter (i.e., the characteristic function χ_E belongs to the fractional Sobolev space $W^{1,\alpha}$), then $|E| = 0$.*

This result interpolates [4] for $\alpha = 1$ and [6] for $\alpha = 0$. The latter paper contains an important hint at the role of the regularity of the boundary of the tangency set in the matter of rigidity for the Frobenius theorem: in fact, if $\alpha = 0$ the assumption on the tangency set is to have finite perimeter. Another important concept in [6] is superdensity. More precisely, a point $x \in \mathbb{R}^2$ is said to be a superdensity point for a set $E' \ni x$ when $\lim_{r\downarrow 0} r^{-3}|B_r(x) \setminus E'| = 0$.

Theorem 2 is a consequence of the fact that, at the (many) superdensity points of E, the involutivity relation must be fulfilled.

To conclude this section I recall that in [2] we produce counterexamples to the "rigidity" of integrability. More precisely, if $\alpha, \beta \in (0, 1)$ and $2\beta > 1 + \alpha$, then for any distribution V (even a non-involutive one) there exists a $C^{1,\alpha}$-surface such that the boundary of the tangency set E has regularity $1 - \beta$ and the tangency set E has positive measure. I wish to underline how the construction of such counterexamples produces a degeneration of the boundary of the tangency set, as it happens in [1] for the construction of a potential for a prescribed gradient with non-trivial curl.

3. Frobenius theorem for currents

Given a distribution of planes $V = \mathrm{span}\{X, Y\}$ of class C^1 one can prove, as in [9], that $T = (X \wedge Y)\mathcal{L}^3$, where $\mathcal{L}^3$ stands for the 3-dimensional Lebesgue measure, is a locally normal current (i.e., the mass of the current and the mass of its boundary are locally finite). Hence the integrability of a (possibly non-involutive) distribution V in the sense of normal currents does not imply involutivity.

Nonetheless, in [7] we prove that, under the stronger assumption that there exists an integral current tangent to a given distribution V, involutivity is necessary at almost every point in the closure of the support of the current. This result is closely related to another "rigid" feature of integral currents: if $X \wedge Y$ is the orientation of a current and Z is the orientation of its boundary, then $Z \in \mathrm{span}\{X, Y\}$ at almost every point of the boundary. This is what we call (see [7] and [8]) the "geometric property" of the boundary for a current.

For normal currents the geometric property of the boundary fails. More precisely, in [3] a complete characterization of this phenomenon is given.

Theorem 3. *A normal current $T = (X \wedge Y)\mu$ tangent to a distribution V of class C^1 satisfies the geometric property of the boundary for μ'_s-a.e. x and μ'_a-a.e. x at which V is involutive, where the measure $\mu' = \mu'_s + \mu'_a$ associated with the boundary ∂T is decomposed in an absolutely continuous part μ'_a and a singular part μ'_s with respect to μ.*

To complete the picture, it is possible to show (see [3]) that even the production of a normal current with a non-involutive orientation, such as the one in [9], has a describable and significant effect.

Theorem 4. *If $T = (X \wedge Y)\mu$ is a normal 2-current tangent to a non-involutive distribution V of class C^1, then $\mu \ll \mathcal{L}^3$.*

As an open problem, we seek a direct, constructive proof of Theorem 4 which does not require the powerful characterization of $\mathcal{A}$-free measures in [5].

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On the De Giorgi–Nash–Moser theorem for hypoelliptic operators of second order

Jonas Hirsch

(joint work with Helge Dietert)

The "original" setting of the De Giorgi–Nash–Moser theorem, considers an elliptic/parabolic equation of second order with measurable coefficients on a cylindrical domain $C_r = (-r^2, 0] \times B_r \subset \mathbb{R}^{n+1}, r > 0$

$$(\partial_t - \sum_{i,j=1}^n \partial_i a^{ij} \partial_j) u = - \sum_{i=1}^n \partial_i f^i + g \,, \tag{1}$$

where $\lambda \leq a^{ij} \leq \Lambda$ is uniformly elliptic but merely measurable.

The "classical" result provides uniform boundedness, a weak Harnack inequality and Hölder regularity of solutions to (1) under the appropriate integrability conditions on f, g:

Theorem 1 ("classical" uniform boundedness, [4, 13, 10])**.** *Let $u \in H^1(C_1)$ be a sub-solution of* (1) *with $|f|^2, |g| \in L^q(C_1)$ for some $q > n+2$ then*

$$\sup_{x \in C_{\frac{1}{2}}} u^+(x) \lesssim ||u^+||_{L^1(C_1)} + ||f||_{L^{2q}(C_1)} + ||g||_{L^q(C_1)} \,.$$

Theorem 2 ("classical" weak Harnack inequaltiy, [12])**.** *For any $\mu > 0$ there is $\sigma, \epsilon > 0$ with the property that for any non-negative super-solution $u \in H^1(C_2)$ of* (1) *with*

$$|\{(x,t) \in C_{\frac{1}{2}} : u(x) > 1, t < -1\}| > \mu, \qquad ||f||_{L^{2q}(C_2)} + ||g||_{L^q(C_2)} < \epsilon$$

one has

$$u(x) > \sigma \qquad \forall (x,t) \in C_{\frac{1}{2}}\,.$$

The hypoelliptic elliptic analog replaces the classical spatial derivatives by a family of smooth vectorfields: One considers

$$\left(X_0 - \sum_{i,j=1}^{m} X_i^t a^{ij} X_j\right) u = -\sum_{i=1}^{m} X_i^t f^i + g, \tag{2}$$

where $\lambda \leq a^{ij} \leq \Lambda$ is uniformly elliptic but merely measurable and we have used the notation $X_i^t = -X_i^* = X_i + \operatorname{div}(X_i)$. Furthermore the given family of smooth vectorfields $X_0, X_1, \ldots, X_m$ satisfies the following additional assumptions:

(1) their Lie algebra span $\mathbb{R}^{n+1}$ at every point, i.e. satisfy the Hörmander condition;
(2) $X_0 = \partial_t + \tilde{X}_0$ and $\tilde{X}_0, X_1, \ldots, X_m$ have no t direction, i.e. $\langle \tilde{X}_0, \partial_t \rangle = 0$ and $\langle X_i, \partial_t \rangle = 0$ for $i = 1, \ldots, m$.

Our approach shows the uniform boundedness and the weak Harnack inequality for general hypoelliptic operators i.e.

Theorem 3 (uniform boundedness)**.** *There is $R > 0, n^* > n$ such that for any sub-solution $u \in H_{hyp}(C_R^X)$ of (2) with $|f|^2, |g| \in L^q(C_R^X)$ for some $q > n^*$ then*

$$\sup_{x \in C_1^X} u^+(x) \lesssim ||u^+||_{L^1(C_R^X)} + ||f||_{L^{2q}(C_R^X)} + ||g||_{L^q(C_R^X)}\,,$$

where C_R^X are cylinders adapted to the vectorfield X_0.

Theorem 4 ("classical" weak Harnack inequaltiy)**.** *There is $R > 0$ such that for any $\mu > 0$ there is $\sigma, \epsilon, R_0 > 0$ with the property that any non-negative super-solution $u \in H^1(C_{R_0}^X)$ of (2) with*

$$|\{(x,t) \in C_1^X : u(x) > 1, t < -\frac{3}{2}\}| > \mu, \qquad ||f||_{L^{2q}(C_{R_0}^X)} + ||g||_{L^q(C_{R_0}^X)} < \epsilon$$

one has

$$u(x) > \sigma \qquad \forall (x,t) \in C_{\frac{1}{2}}^X\,.$$

The novelty is the avoidance of a "general" Sobolev embedding and a "quantitative" Poincare inequality. In simple terms, we rather merge the iterative scheme proposed by De Girogi, with the regularity theory developed by Hörmander and the maximum principle of Bony developed for the smooth setting. Applied to the classical non-degenerate setting, our approach suggests that the classical De Giorgi-Nash-Moser theorem can be seen as a "perturbation" of the heat equation.

One should remark that we are not the first to extend the classical result to the hypoelliptic setting. So far, most obtained results had been specific to specific equations, e.g. the Kolmogorov equation. Without claiming any completeness, results can be found in [1, 6, 7, 9, 14] or miss the first order part X_0 c.f. [3] for the sub-Riemannian setting.

The presented results are about work in progress.

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Compactness results for Griffith-type energies

Antonin Chambolle

(joint work with Vito Crismale)

This talk was about the compactness bounded or minimizing sequences for Griffith-type energies, which are typically of the form:

$$\mathcal{E}(u) = \int_\Omega |e(u)|^2 dx + \mathcal{H}^{d-1}(J_u) \tag{Gr}$$

where $u : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$ is a displacement, $e(u) = (\nabla u + (\nabla u)^T)/2$ its symmetrized gradient (infinitesimal strain), J_u its jump set and $\mathcal{H}^{d-1}$ the Hausdorff $(d-1)$-dimensional measure (in practice, for applications, $d \in \{2, 3\}$).

Together with Vito Crismale, we had shown in [4] a compactness result, in the space "$GSBD^2(\Omega)$" (the energy space for (Gr), see [7]): if $\mathcal{E}(u_n)$ is bounded, then there exist a subsequence (u_{n_k}), a limiting function $u \in GSBD^2(\Omega)$, a set F with finite perimeter such that

$$\begin{cases} u_{n_k}(x) \to u(x) & \text{a.e. in } \Omega \setminus F\,, \\ |u_{n_k}(x)| \to \infty & \text{a.e. in } F. \end{cases}$$

In addition, letting $u' = u\chi_{\Omega\setminus F}$, we have

$$\int_\Omega |e(u')|^2 dx \le \liminf_n \int_\Omega |e(u_n)|^2 dx \quad \text{and} \quad \mathcal{H}^{d-1}(J_{u'}) \le \liminf_n \mathcal{H}^{d-1}(J_{u_n}),$$

allowing to deduce easily the existence of minimizers to the energy $\mathcal{E}$ in (Gr).

We discussed, in this talk, extensions of this result, and in particular a recent result with V. Crismale. First, Almi and Tasso [1] have given an alternative proof of this result, valid also when the bound is in "GBD" (roughly speaking, with growth 1, whereas our previous result was valid in $GSBD^p$, $p > 1$, that is with a L^p-bound on $e(u)$). Next, Friedrich [8] has shown a much more precise compactness result in $GSBV^p$, which pushed us to improve the former result in $GSBD^p$, $p > 1$, in [5]. In the latter reference, we describe better what happens in F, and more precisely, we show, for $p \in (1, \infty)$:

Theorem 1. *Let (u_n) be a sequence in $GSBD^p(\Omega)$ with:*

$$\sup_n \int_\Omega |e(u_n)|^p dx + \mathcal{H}^{d-1}(J_{u_n}) < +\infty.$$

Then there exist a subsequence (u_{n_k}), a Caccioppoli partition $(E^i)_{i\ge1}$, and sequences of infinitesimal rigid motions $a_k^i(x) = A_k^i x + b_k^i$, $A_k^i \in \mathbb{R}^{d\times d}$, skew-symmetric and $b_k^i \in \mathbb{R}^d$, such that:

$$u_{n_k} - \sum_i a_i^i \chi_{E_i}$$

converges pointwise a.e. in Ω.

The proof of this result, which was quite involved, required a delicate result of [2] (inspired by [6] where it was shown in dimension $d = 2$), which showed that a $GSBD^p$ function with small jump set coincides with a $W^{1,p}$ Sobolev function up to a small set of small perimeter. The result presented in this talk consisted in:

- a simpler proof of the theorem, not relying on [2], but rather only on the rigidity result of [3];
- an extension to the case of linear growth (GBD, $p = 1$), as in [1].

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